

Phase transitions in crowd dynamics of resource allocation

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We define and study a class of resources allocation processes where gN agents, by repeatedly visiting N resources, try to converge to optimal configuration where each resource is occupied by at most one agent. The process exhibits a phase transition, as the density g of agents grows, from an absorbing to an active phase. In the latter, even if the number of resources is in principle enough for all agents ($g < 1$), the system never settles to a frozen configuration. We recast these processes in terms of zero-range interacting particles, studying analytically the mean field dynamics and investigating numerically the phase transition in finite dimensions. We find a good agreement with the critical exponents of the stochastic fixed-energy sandpile. The lack of coordination in the active phase also leads to a non-trivial faster-is-slower effect.

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I. INTRODUCTION

A general question that naturally arises in the study of social systems is how collective behaviors can emerge from the interactions among individuals. In spite of the inherent complexity of these phenomena, simplified mathematical models that assume simple automatic responses of individuals to stimuli can reproduce non-trivial effects in the observed behavior [1]. Statistical mechanics had a certain success in the use of coarse-grained models of physical systems to connect the microscopic dynamics with the macroscopic behavior, and its techniques and concepts are starting to be fruitfully applied to understand social dynamics [2–5]. In particular, the phenomena observed in the crowd dynamics, from pedestrians flows [6] to vehicular traffic [7], have been recently subject to quantitative measures. Observations range from bottleneck oscillations, lanes and stripes formation, intermittent flows, waves, turbulence [8], faster-is-slower [9] and freezing-by-heating effects [10]. Interestingly, this extremely broad class of collective phenomena is explained at semi-quantitative level with the use of models of interacting particles or granular fluids. This shows that even if interactions among individuals in these settings are rather simplified and mechanical, they still lead to a certain level of collective coordination.

In this paper we show how a crowd dynamics in a resources allocation context can give rise to a phase tran-

sitions between an active and an absorbing phase. The class of models we discuss is inspired by the Kolkata Paise Restaurant (KPR) problem [11–14], which we generalize and recast in the broader context of zero-range interacting particle models [15].

The KPR is a repeated game played by a large number (gN , g being a real number) of agents with equally shared past information. Every evening they try to get the best service in one of the N restaurants of commonly agreed ranks, each providing food for one person only. It serves as a paradigm for a problem of resource utilization, where agents learn from their actions to maximize the effective utilization of available resources. At unit density ($g = 1$), a simple random choice algorithm [11] leads (in unit convergence time) to a occupation of a fraction of $f = 1 - e^{-1} \simeq 0.63$ only of the resources, which of course falls much short of a fully efficient usage of them. As seen in various earlier studies [12–14], apparently smarter strategies fail to yield better results. However, a stochastic strategy [14] that maintains a naïve tendency to stick to an agent’s past choice, with probability decreasing with the crowd size, leads to an efficient utilization fraction f of about 0.796 [14] and it converges to the above within a time (iteration) independent of N . An extension of this strategy to the Minority game [16] problem (with knowledge of both the sign and size of the population difference among two alternative choices), gives the optimal solution (with fluctuation bounded by $\log N$ and convergence time bounded by $\log(\log N)$ [17]). Our focus will be on the nature of the collective behavior rather than on the efficiency of individual behavior. For the latter, we refer to applications of game theory and adaptive learning, as e.g. in [18–22]. We note, in passing, that the relation between the degree of individual

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rationality and the efficiency of collective allocation can be a non-trivial one, as shown e.g. in Refs. [23, 24].

Here we will study the general problem with N restaurants and gN agents, where g , the density, is a fixed external parameter of the dynamics. Recasting the problem in terms of zero-range interacting particles, we report a phase transition from a frozen phase with satisfied agents to an active phase with unsatisfied ones at a critical density g_c and perform extensive numerical simulations as well as some analytical calculations to understand its features, finding a good agreement with the exponents of stochastic fixed-energy sandpiles [25]. The study of the relaxation properties of the frozen phase reveals an interesting faster-is-slower effect: This consists in the general observation that, from vehicular traffic to logistic, a higher level of coordination can arise from strategies which involve a slower dynamics (e.g. waiting), that however speed up performances.

The paper is organized as follows: In Sec. II, we introduce the class of models under study. Next, in Sec. III we present the results of numerical simulations and in Sec. IV we provide the analytic treatment for the mean field case. We conclude with a summary and some discussions in Sec. V.

II. THE MODELS

Inspired by the KPR problem [11, 14] with N individuals competing for N restaurants (each serving food to one person each), we propose a more generic and fundamental stochastic occupation problem with exclusion.

The rank ordering among the restaurants, as in the original KPR problem [11, 14] is dropped here and we consider in general gN agents, where g , the density, is an external fixed parameter of the dynamics. We will refer from now on to individuals as particles and restaurants as the sites or nodes of an underlying graph. In these terms the original problem is defined in a fully connected geometry.

A particle moves from the site i it occupies to a randomly chosen neighbor j with a rate that depends only on the number of particles that is present on it $v(n_i)$. This is by definition a *zero range process* (see [15] for a review) which allows us to conclude that the stationary probability distribution of the number of particles per site factorizes in terms of single site functions. Given the nature of the problem, we will consider models with $v(1) = 0$, i.e. costumers are happy while alone, and $v(n+1) \geq v(n)$, i.e. repelling particles. Given this expression for the rates, at low densities ($g < 1$) there are dynamically frozen configurations with sites filled by single particles. On the other hand, for high densities ($g > 1$) a finite fraction of sites – that we shall call ‘active’ – will have multiple occupancy. We will show that there is a transition between these two phases that occurs at a certain density $g_c \leq 1$. Notice that, in principle, the process is ergodic so every configuration of is accessible. Therefore, for $g \leq 1$, the

process will sooner or later visit an absorbing configuration where $n_i \leq 1$ for all sites i . Still, when N is large enough and $g > g_c$, frozen configurations will be visited extremely rarely, over time-scales which are much longer than those which are accessible to numerical simulations. We choose as the order parameter the steady state density of active sites ρ_a (density of sites having $n > 1$). So the absorbing phase corresponds to $\rho_a = 0$ whereas, above some density g_c the steady state shows a persistent dynamics with a non-zero value of the order parameter ($\rho_a > 0$).

We will analyze in particular two models:

$$(A) \ v(n) = 1 - \frac{1}{n}$$

$$(B) \ v(n) = (1 - p)\theta(n - 1).$$

We will refer from now on to a parallel dynamics in which a simultaneous update of the nodes is done at each time step, i.e. agents’ actions are simultaneous akin to a repeated game setting.

It will be pointed out later that a sequential update in which at each step a randomly chosen particle jumps with some probability leads here to the trivial result that $g_c = 1$. This corresponds to the limit $p \rightarrow 1$ of model B.

The model A implements the stochastic crowd avoiding strategy of the original KPR problem[14]. Here, if the site k has $n_k \geq 1$ particles, each of them stay back with probability $1/n_k$ in the next time step, otherwise it jumps to any of the neighboring sites.

In the model B an external parameter p is introduced that represents the ‘patience’ of costumers to overcrowded conditions. Here, if the site k has $n_k \geq 1$ particles, each of them stays with probability p in the next time step, otherwise it jumps to any of the neighboring sites.

The model B is in practice a kind of fixed energy sandpile [25], but the study of its dynamics as a function of the parameter p will reveal an interesting faster-is slower effect related to the relaxation time of the frozen phase.

We finally point out that a waiting choice can be *rational* from the point of view of game theory: the agents in overcrowded sites could wait simply because they expect that others are leaving them alone.

In the next section we report a detailed analysis of the phase transition of both models with the use of numerical simulations in a fully connected geometry, a 1d chain and a 2d square lattice. Then, we perform analytical calculations for the mean field case.

III. RESULTS FROM NUMERICAL SIMULATIONS

We measure the times required to reach the steady state below and above g_c . Below g_c , the order parameter ρ_a reaches a value $\rho_a = 0$ in the steady state. Above g_c , order parameter ρ_a evolves to a stationary state and fluctuates around a mean value $\rho_a^0 (> 0)$. The system has

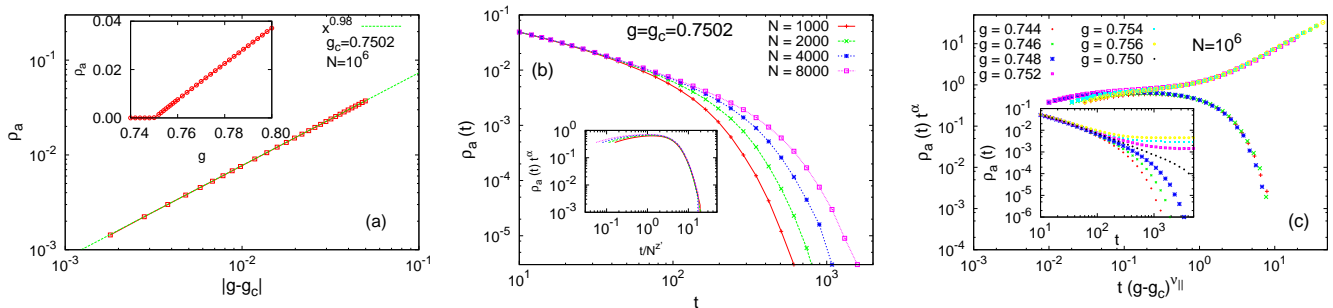


FIG. 1. Simulation results for mean field case, $g_c = 0.7502 \pm 0.0002$. (a) Variation of steady state density ρ_a of active sites versus $g - g_c$, fitting to $\beta = 0.98 \pm 0.02$. The inset shows the variation of ρ_a with density g . (b) relaxation to absorbing state near critical point for different system sizes, the inset showing the scaling collapse giving estimates of critical exponents $\alpha = 1.00 \pm 0.01$ and $z' = 0.50 \pm 0.01$. (c) Scaling collapse of $\rho_a(t)$. The inset shows the variation of $\rho_a(t)$ versus time t for different densities g . The estimated critical exponent is $\nu_{\parallel} = 1.00 \pm 0.01$. The system sizes N are mentioned.

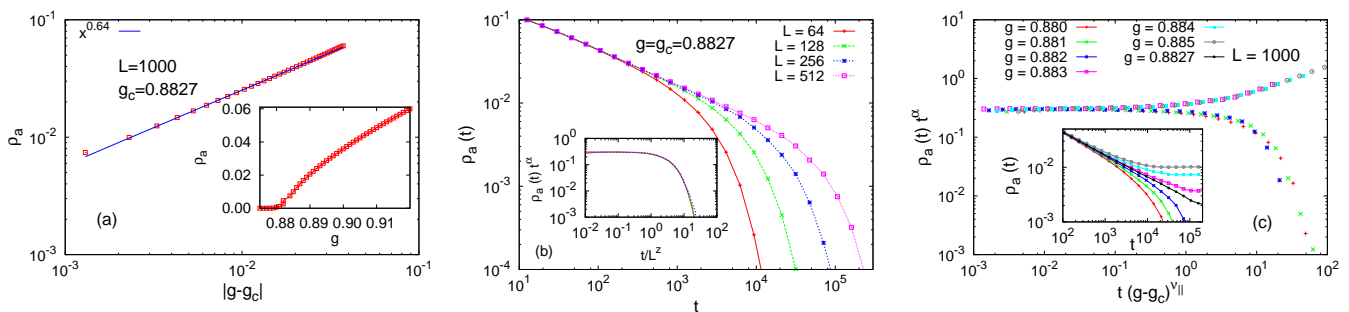


FIG. 2. Simulation results for 2-d case, $g_c = 0.8827 \pm 0.0002$. (a) Variation of steady state density ρ_a of active sites versus $g - g_c$, fitting to $\beta = 0.68 \pm 0.01$. The inset shows the variation of ρ_a with density g . (b) relaxation to absorbing state near critical point for different system sizes, the inset showing the scaling collapse giving estimates of critical exponents $\alpha = 0.42 \pm 0.01$ and $z = 1.65 \pm 0.02$. (c) Scaling collapse of $\rho_a(t)$. The inset shows the variation of $\rho_a(t)$ versus time t for different densities g . The estimated critical exponent is $\nu_{\parallel} = 1.24 \pm 0.01$. The simulations are done for square lattices of linear size L ($N = L^2$).

persistent dynamics in this phase. The evolution of the order parameter is exponential away from g_c , and can be expressed as

$$\rho_a(t) = \rho_a^0 \left[1 - e^{-t/\tau} \right] \quad (1)$$

for $g > g_c$, and

$$\rho_a(t) \propto e^{-t/\tau} \quad (2)$$

for $g < g_c$, where τ is the relaxation time. We will denote the asymptotic value of the order parameter as ρ_a hereafter. Near the critical point ($g - g_c \rightarrow 0_+$), we find $\rho_a \sim (g - g_c)^\beta$ where β is the order parameter exponent, and $\tau \sim (g - g_c)^{-\nu_{\parallel}}$. Generally $\rho_a(t)$ obeys a scaling form

$$\rho_a(t) \sim t^{-\alpha} \mathcal{F} \left(\frac{t}{\tau} \right); \quad \tau \sim (g - g_c)^{-\nu_{\parallel}} \sim L^z, \quad (3)$$

where α and z are dynamic exponents and L denotes size of the system. We then get $\beta = \nu_{\parallel} \alpha$ by comparing Eq. (1), Eq. (2) and Eq. (3) when t/τ is a constant for $t \rightarrow \infty$. We study numerically the time variation of $\rho_a(t)$ and measure the exponents by fitting to the above scaling relation.

A. Model A

1. Mean Field case

For the mean field case, we have studied systems of $N = 10^6$ sites, averaging over 10^3 initial conditions. We get $g_c = 0.7502 \pm 0.0002$. The scaling fits of $\rho_a(t)$ for different g values (see Fig. 1) give $\beta = 0.98 \pm 0.02$, $z' = 0.50 \pm 0.01$ (if we assume $N = L^4$ and using Eqn. (3), we get a relation $z = 4z'$ and therefore $z = 2.0 \pm 0.04$), $\nu_{\parallel} = 1.00 \pm 0.01$, $\alpha = 1.00 \pm 0.01$. It may be noted that these independently estimated exponent values satisfy the scaling relation $\beta = \nu_{\parallel} \alpha$ well.

2. Lattice cases

We studied the same dynamics in 1-d and 2-d. For a linear chain in 1-d, we took $N = L = 10^4$ and averaged over 10^3 initial conditions. For 2-d we consider square lattice ($N = L^2$) with $L = 1000$ and averaging over 10^3 initial conditions. Periodic boundary condition were employed in both cases.

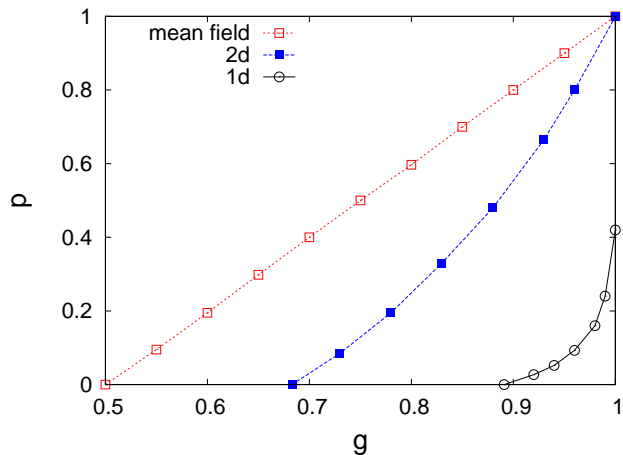


FIG. 3. Phase diagram for the generalized model in the (g, p) plane, showing the phase boundaries separating the active and absorbing phases in 1-d, 2-d and mean field cases. The active phases are on the right of the phase boundaries while the absorbing phases are on the left in the respective cases. The system sizes are $N = 10^5$ for mean field, 1000×1000 for 2-d, and 10^4 for 1-d.

- (a) The model is defined in 1-d as follows: The particles are allowed to hop only to their nearest neighbor sites, and each particle can choose either left or right neighbor randomly. We find $g_c = 1$ and hence the phase transition is not very interesting.
- (b) In the 2-d version of the model, we consider square lattices and the particles are allowed to choose one of the 4 nearest neighbors randomly. For $N = 1000 \times 1000$, we get $g_c = 0.88 \pm 0.01$, $\beta = 0.68 \pm 0.01$, $z = 1.65 \pm 0.02$, $\nu_{\parallel} = 1.24 \pm 0.01$ and $\alpha = 0.42 \pm 0.01$ (Fig. 2). It may be noted that these independently estimated exponent values do not fit very well with the scaling relation $\beta = \nu_{\parallel}\alpha$. However, this type of scaling violation is also observed in many active-absorbing transition cases [26].

B. Model B

1. Mean field case

For the mean field case, we have studied for $N = 10^6$, averaging over 10^3 initial condition. We numerically investigate the phase diagram and the universality classes of the transition. In mean field case, the phase boundary seems to be linear starting $g_c = 1/2$ for $p = 0$ and ending at $g_c = 1$ for $p = 1$ (Fig. 3), obeying $g_c = \frac{1}{2}(1+p)$. In this case, for $p = 0$, we find the critical point to be $g_c = 1/2$, and this is similar to the fixed energy sandpiles [25]. The critical exponents are the same along the phase boundary and they match with those of model A.

2. Lattice cases

We studied the same dynamics in 1-d and 2-d. For a linear chain in 1-d, here also we took $N = L = 10^4$ and average over 10^3 initial condition. For 2-d we took 1000×1000 square lattice with $L = 1000$ and averaging over 10^3 initial conditions.

- (a) For 1-d, for the case $p = 0$, we find $g_c = 0.89 \pm 0.01$, with $\beta = 0.42 \pm 0.01$, $z = 1.55 \pm 0.02$, $\nu_{\parallel} = 1.90 \pm 0.02$ and $\alpha = 0.16 \pm 0.01$ (Fig. 4). The phase boundary in (g, p) is nonlinear: it starts from $g_c = 0.89 \pm 0.01$ at $p = 0$ (Fig. 4) to $p = 0.43 \pm 0.03$ at $g = 1$ (Fig. 3). Thus, one can independently define a model at unit density ($g = 1$) and calculate the critical probability p_c for which the system goes from an active to an absorbing phase.
- (b) For 2-d, for the case $p = 0$, we find $g_c = 0.683 \pm 0.002$, with $\beta = 0.67 \pm 0.02$, $z = 1.55 \pm 0.02$, $\nu_{\parallel} = 1.20 \pm 0.03$ and $\alpha = 0.42 \pm 0.01$. The phase boundary seems nonlinear, from $g_c = 0.683 \pm 0.002$ for $p = 0$ (Fig. 3) extending to $g_c = 1$ at $p = 1$.

		Model A	Model B	Manna
β	1D		0.42 ± 0.01	0.382 ± 0.019
	2D	0.68 ± 0.01	0.67 ± 0.02	0.639 ± 0.009
	MF	0.98 ± 0.02	0.99 ± 0.01	1
z	1D		1.55 ± 0.02	1.393 ± 0.037
	2D	1.65 ± 0.02	1.55 ± 0.02	1.533 ± 0.024
	MF	2.00 ± 0.04	2.0 ± 0.04	2
α	1D		0.16 ± 0.01	0.141 ± 0.024
	2D	0.42 ± 0.01	0.42 ± 0.01	0.419 ± 0.015
	MF	1.00 ± 0.01	1.00 ± 0.01	1
ν_{\parallel}	1D		1.90 ± 0.02	1.876 ± 0.135
	2D	1.24 ± 0.01	1.20 ± 0.03	1.225 ± 0.029
	MF	1.00 ± 0.01	1.00 ± 0.01	1

TABLE I. Comparison of critical exponents of this model with those of the conserved Manna model [27].

IV. ANALYTICAL TREATMENT OF THE MODELS IN MEAN FIELD CASE

In the mean field case the particles can jump from a site to *any* other, all with equal probability, or, equivalently, the underlying graph is fully connected. From the theory of zero range processes we know that the stationary probability distribution is factorized in terms of single site quantities. If ρ_n^i is the probability that a site i has n particles, the average rate of outgoing particles is $\langle u^i \rangle = \sum_n \rho_n^i v(n)n$. By symmetry, we can work under the hypothesis that $\rho_n^i = \rho_n \forall i$, we have simply $\langle u^i \rangle = \langle u \rangle = \sum_n n v(n) \rho_n$, that at stationarity is equal to the average number of incoming particles, since the density is fixed. A sequential dynamics would consist

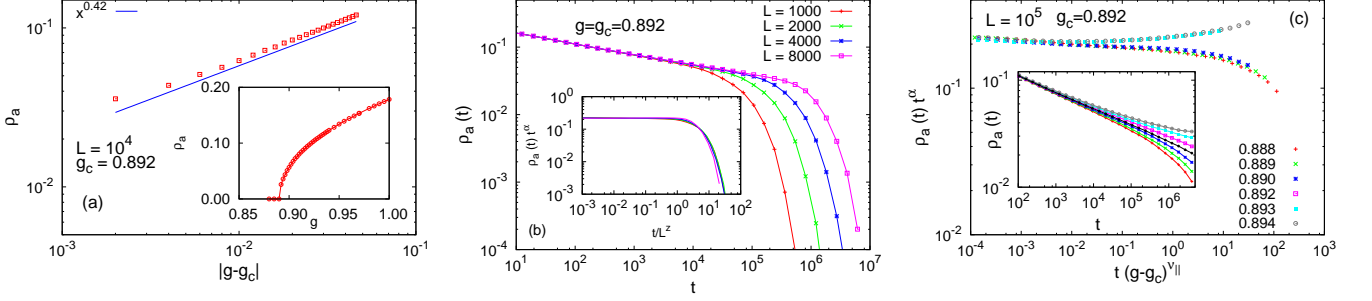


FIG. 4. Simulation results for the case $p = 0$ in 1-d, $g_c = 0.892 \pm 0.001$. (a) Variation of steady state density ρ_a of active sites versus $g - g_c$, fitting to $\beta = 0.42 \pm 0.01$. The inset shows the variation of ρ_a with density g . (b) relaxation to absorbing state near critical point for different system sizes L , the inset showing the scaling collapse giving estimates of critical exponents $\alpha = 0.15 \pm 0.01$ and $z = 1.40 \pm 0.02$. (c) Scaling collapse of $\rho_a(t)$. The inset shows the variation of $\rho_a(t)$ versus time t for different densities g . The estimated critical exponent is $\nu_{\parallel} = 1.90 \pm 0.02$. The simulations are done for linear chains of size L ($= N$).

of sequences of one-particle jump events and the master equation has the form:

$$\begin{aligned} \dot{\rho}_k(t) = & -(\langle u \rangle + kv(k))\rho_k(t) \\ & + \theta(k)\langle u \rangle\rho_{k-1}(t) + (k+1)v(k+1)\rho_{k+1}(t). \end{aligned} \quad (4)$$

Multiplying by s^k , summing over k we have the equation for the characteristic function $G(s) = \sum_k \rho_k s^k$:

$$\dot{G} = -(1-s)(\langle u \rangle G(s) - \sum_{n=1}^{\infty} nv(n)\rho_n s^{n-1}) \quad (5)$$

from which we have a self-consistent formula for the stationary solution:

$$G(s) = \frac{1}{\langle u \rangle} \sum_{n=1}^{\infty} nv(n)\rho_n s^{n-1}. \quad (6)$$

For the case (A) the solution of Eq.(6) is

$$G(s) = \theta(1-g)(1-g+sg) + s\theta(g-1)e^{-(g-1)(1-s)} \quad (7)$$

For the case (B) the solution of Eq.(6) is

$$G(s) = \theta(1-g)(1-g+sg) + \theta(g-1)\frac{e^{sx}-1}{x}(g-x), \quad (8)$$

where x is the solution of $g = \frac{xe^x}{e^x-1}$. In both cases below $g_c = 1$ the activity is zero, i.e. the system falls into the absorbing state till this is present. The order parameter or the fraction of active sites as a function of g is depicted in Fig. 5. On the other hand, the parallel dynamics of the model B with $p = 0$ is particularly simple to analyze, since the number of particles on top of each site follows the discrete evolution equation:

$$n_{t+1} = \begin{cases} n^i & \text{if } n(t) \neq 1, \\ n^i + 1 & \text{if } n(t) = 1. \end{cases}$$

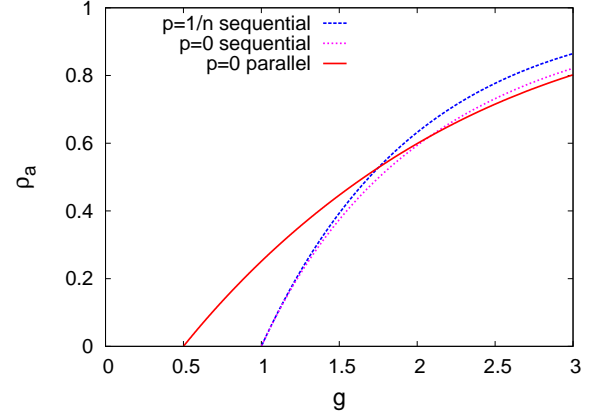


FIG. 5. Behavior of order parameter for sequential and parallel dynamics. For sequential dynamics, $g_c = 1$ for both models. For the model with $p = 0$, the parallel dynamics gives $g_c = \frac{1}{2}$.

Then, we have the self consistent equation for the stationary state:

$$\rho_n = e^{-g+\rho_1} \left[(1-\rho_1)\frac{(g-\rho_1)^n}{n!} + \theta(n)\rho_1\frac{(g-\rho_1)^{n-1}}{(n-1)!} \right]. \quad (9)$$

From we which we can work out the characteristic function

$$G(s) = (1-\rho_1 + s\rho_1)e^{-(g-\rho_1)(1-s)} \quad (10)$$

From $G'(0) = \rho_1$, we end up with an equation for $x = g - \rho_1$, that, apart from the solution $x = 0$, has the form

$$g = \frac{x(1+xe^{-x})}{1-e^{-x}+xe^{-x}}. \quad (11)$$

Finally, the order parameter can be calculated consistently once we know $G(s)$ from x (Eq. (8)), since $\rho_a = 1 - G(0) - G'(0)$ and its behavior as function of g is depicted in Fig. 5.

A. Approximate analysis of the critical point and faster-is-slower effect

To get insights on the value of the critical point as well as on the time to reach a frozen configuration below $g = 1$, we will analyze the dynamical stability of the frozen phase probing it with a simple perturbation of the form:

$$\rho_0 = 1 - g + \delta \quad (12)$$

$$\rho_1 = g - 2\delta \quad (13)$$

$$\rho_2 = \delta \quad (14)$$

i.e. we pick a fraction 2δ of particles and move them to already filled sites, neglecting the case in which some of them choose the same site. This perturbation evolves according to the equation [28]

$$\delta(t+1) = \delta(t) - (1 - g + \delta(t))(1 - e^{-2v(2)\delta(t)}) + \delta(t)v(2)^2 e^{-2v(2)\delta(t)} \quad (15)$$

whose solution scales, for large enough times, like

$$\delta(t) \propto \left[2v(2) \left(1 - \frac{v(2)}{2} - g \right) \right]^t \quad (16)$$

with a relaxation time

$$\tau = - \frac{1}{\log \left[1 - 2v(2) \left(1 - \frac{v(2)}{2} - g \right) \right]} \quad (17)$$

that diverges in $g_c = 1 - \frac{v(2)}{2}$. This value coincides in a very good approximation with the numerical values of the critical point both for the model A, $g_c = 3/4$ and B, $g_c = \frac{1}{2}(1 + p)$.

Moreover, for the model B the relaxation time at fixed density as a function of p

$$\tau(p) = - \frac{1}{\log \left[1 - 2(1-p) \left(\frac{1+p}{2} - g \right) \right]} \quad (18)$$

has a minimum, optimal value $\tau^* = -\frac{1}{\log[g(2-g)]}$ in $p^* = g$, as it is shown in Fig. 6. This is a very simple example of the faster-is-slower effect, by which agents who choose a strategy which imply longer waiting times allow for a faster collective coordination. Note that waiting can be rational at the individual level, if an agent expects that others are leaving her site.

B. Analysis of the finite size effects on the time to reach the absorbing state.

Let's call π the weight of the frozen configuration for given N sites and $K = gN$ particles, its inverse can be an estimator of the time required to reach the absorbing state. We have:

$$\pi = \sum_{frozen} P(\vec{n}) \quad (19)$$

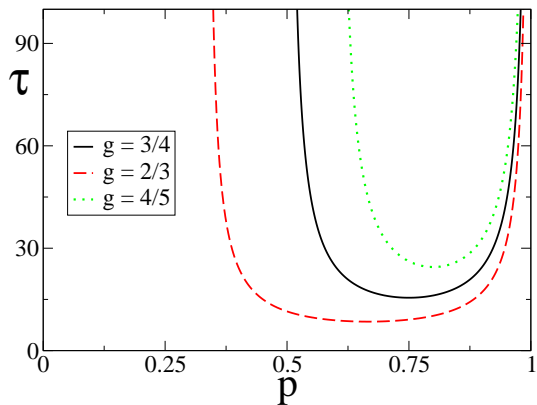


FIG. 6. Relaxation time τ as a function of the waiting probability p for the model B for a few values of g . It diverges at $p = 1$ and $p = 2g - 1$ and has an optimal minimum at $p^* = g$, when $\tau = \tau^*$.

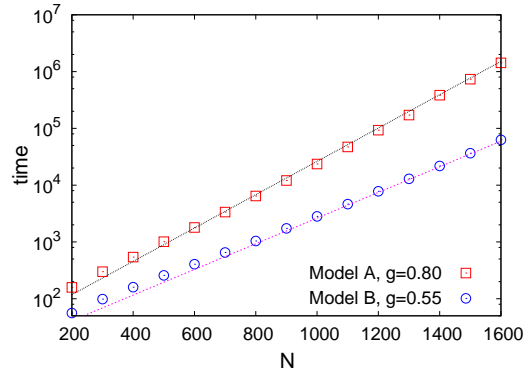


FIG. 7. Exponential dependency of timescale to reach the absorbing state, with the system size in mean field cases for Model A at $g = 0.55$ and Model B at $g = 0.80$. The data are averaged over 10^4 realizations.

Since the process is zero-range, at stationarity $P(\vec{n}) = \prod_i^N p_i(n_i)$, then, if the graph is homogeneous we can leave the dependency on the site to get

$$\pi = \binom{N}{K} \rho_1^K \rho_0^{N-K} \quad (20)$$

Using Stirling approximation we have finally

$$\log \pi \simeq -N \left[g \log \frac{g}{\rho_1} + (1-g) \log \left(\frac{1-g}{\rho_0} \right) \right] \quad (21)$$

i.e. it is exponentially decreasing with the system size. The Fig. 7 shows some typical numerical results for the average time to reach an absorbing configuration as a function of the system size.

V. SUMMARY AND DISCUSSIONS

The dynamics of social systems is recently gaining more and more insights borrowing concepts and techniques

from statistical mechanics and by developing some toy models. The toy model paradigm [29] helps us to understand complex emergent behavior of such socio-economic systems. In this paper we show how a crowd dynamics in a resources allocation game gives rise to a phase transition between an active and a frozen phase, as the density varies. In this respect, we have defined and studied a class of models, where gN agents compete among themselves to get the best service from N restaurants of same rank, generalizing the ‘Kolkata Paise Restaurant’ problem. In the original problem, where density $g = 1$ was far from its critical value g_c , the relaxation time τ , given by Eqn. (3), never showed any $L = N^{1/d}$ dependence. We recast these models in terms of zero-range interacting particles in order to get analytical insights on the systems’ behavior. As long as $g \leq 1$, absorbing frozen configurations are present, and that can be reachable or not, depends on the underlying dynamics. We found the existence of a critical point g_c above which the agents are

unable to find frozen configurations. In the case in which the agents are moving if and only if they are unsatisfied (model B) with $p = 0$, they fail to reach satisfactory configurations if the density is above $g_c = 1/2$. Strategies where agents wait longer (higher p) speed up the convergence, increasing g_c and decreasing the time to reach saturation configurations (faster-is-slower effect). We investigated numerically the phase transition in finite dimensions finding a good agreement with the exponents of stochastic fixed-energy sandpile (Table. I) [25, 27, 30]. Thus, we have a simple model for resource allocation, which is solvable, and shows a variety of interesting features including phase transitions as in well known models. Further investigations are needed in order to understand the emergence of waiting strategies, i.e. how they could depend on agents’ beliefs or learning processes, as well as it would interesting the study the case of heterogeneous agents (with different strategies) and/or restaurants (with different ranks).

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