A group theoretic approach to detecting gravitational waves from asymmetric rotating neutron stars

S. V. Dhurandhar[1] and B. Krishnan[2,3]

[1] *Inter University Centre Astronomy & Astrophysics, Pune - 411 007, India.*

[2] *Max-Planck-Institut für Gravitationsphysik, Albert Einstein Institut, Am Mühlenberg 1, D-14476 Golm, Germany.*

[3] *Max-Planck-Institut für Gravitationsphysik, Albert-Einstein-Institut, Callinstr. 38, D-30167 Hannover, Germany.*

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Abstract. The era of advanced ground based interferometric detectors of gravitational waves (GW) has arrived. These detectors are expected to go line in six years or so from now and which will have requisite sensitivity for detecting and observing astrophysical gravitational wave sources. In this article we will focus on a specific GW source, the GWs emitted by an isolated rotating neutron star/pulsar, and describe a novel approach to address this highly computationally intensive problem. We will describe how the symmetries in the model can be potentially used to reduce the computational effort.

1. Introduction

We dedicate this article to Prof. P. C. Vaidya who was an eminent Indian relativist and a fantastic teacher.

Gravitational waves (GW) can be described as a space-time warpage which travels with the speed of light. Astrophysical GW will bring us information complementary to that obtained from electromagnetic observations, because the GW have very different properties. GW are produced in compact, dense, high velocity regions of matter in the universe and are not easily scattered, unlike electromagnetic waves. Therefore, it seems reasonable to expect a revolution in understanding of our universe akin to the one brought on by radio astronomy in the last half a century. Exciting times lie ahead for GW astronomy.

Currently, the large scale detectors, LIGO of the US, Virgo of Italy and France, the Japanese TAMA detector and the UK-German GEO detector [1] have achieved impressive sensitivities. In many cases, they have infact surpassed their proposed design sensitivities. Now efforts are on to construct advanced detectors which will have the requisite sensitivity for detecting astrophysical sources of gravitational waves with a reasonable event rate. A large scale detector LCGT in Japan [2] has been recently funded and the plan of LIGO collaborating with Australia (LIGO-Australia) has been approved by the NSF. On the Indian front, during the past year or so, an initiative in gravitational wave astronomy has begun on the experimental front - there has already been expert effort on data analysis and waveform computations for the past twenty years - and an Indian consortium involving researchers from several Indian leading institutions has been formed - the IndIGO consortium.

Several types of GW sources have been envisaged which could be directly observed by Earth-based detectors, such as the burst sources, examples are binary systems of neutron stars and/or black holes and supernovae explosions; stochastic backgrounds of radiation, either of primordial or astrophysical origin and continuous wave sources – e.g. rapidly rotating asymmetrical neutron stars, where a weak deterministic signal is continuously present in the data stream. In this article we will focus on this last type of source, namely, continuous wave signals and describe a novel approach based on group theory which could speed up the data analysis.

Continuous wave sources pose one of the most computationally intensive problems in GW data analysis [4, 5, 6]. A rapidly rotating asymmetrical neutron star is a source of continuous gravitational waves. There are some astrophysical systems known from electromagnetic observation which might be promising sources of continuous GWs. Surveys for continuous GWs have so for not led to a direct detection, but the searches are now become astrophysically interesting. For example, data from the LIGO detectors has been used to set an upper limit on the emission of continuous GWs from the pulsar in the Crab nebula which beats the known limit based on the observed spindown rate of the pulsar [3]. These searches for known systems are not computationally intensive since they target a known sky position, frequency and spindown rates. On the other hand, blind all-sky and broad-band searches for previously unknown neutron stars are a different matter altogether. Long integration times, typically of the order of a few months or years are needed to build up sufficient signal power. Earth's motion Doppler modulates the signal, and this Doppler modulation depends on the direction of the GW source. Thus, coherent extraction of the signal whose direction and frequency is unknown is impossibly computationally expensive. The parameter space here is very large and a blind survey requires large computational resources. Several such searches have been carried out using data from ground based interferometric detectors [7, 8, 9, 10, 11, 12]. The results of this paper might help in addressing this computational challenge of covering this large parameter space.

In this article we will consider the simple model of an isolated rapidly rotating neutron star and ignore the spindown. We will show here, how the group theory and other algebraic methods can be used to elegantly formulate the problem by exploiting the symmetries in the physical model and potentially map the road for a fast algorithm by saving on the computational costs. In this endeavour, we will make use of the *stepping around the sky method* - a method proposed by Schutz twenty years ago [4], which gives a useful framework for this approach. There have been a host of methods proposed, notably the Hough transform, the stack and slide, resampling methods etc. [13] which reduce the computational cost over the straight forward search over the sky directions, frequency and spindown parameters. Although these methods significantly reduce the computational burden, it is not reduced to the point where the search can be performed with the current computer resources available in reasonable time. Therefore, it becomes necessary to explore novel approaches which may go towards addressing and ultimately solving this problem. This article presents one such attempt.

2. The formulation of the problem: Modulation operators

Consider a *barycentric frame* in which the isolated neutron star is at rest or moving with uniform velocity. Ignoring spindowns the signal in this frame is assumed to be a pure sinusoid - monochromatic of constant frequency say f. The detector however, takes part in an accelerated motion - in general a superposition of simple harmonic motions of varying amplitudes and phases. In some approximation this motion can also be considered as superposition of circular motions - cycles and epicycles. Thus the signal in the detector is not a pure sinusoid but is modulated by a Doppler correction. The Doppler correction depends on the direction to the source relative to the motion of the detector. If the direction to the source and the frequency is unknown, one must scan over all directions in the sky and the frequency. From astrophysical considerations usually the maximum frequency f_{max} is taken to be 1 kHz. The *stepping* method gives a direct way for obtaining the Fourier transform in the barycentric frame of the demodulated signals connecting two different directions say $\hat{\mathbf{n}}$ and $\hat{\mathbf{n}}'$.

The signal expected is so weak that one typically needs to integrate the signal for several months or an year before one could build up a significant signal to noise ratio (SNR). So the observation time needed is $T \sim 10^7$ sec or more and if the maximum frequency to be scanned is taken about a kHz, then the number of samples in a data train are $N \sim 10^{10}$. Since the detector orbits the Sun in this time, the 'aperture' of the 'telescope' is the diameter D of the Earth's orbit; $D \sim 3 \times 10^8$ km while the minimum GW wavelength is $\lambda \sim 300$ km corresponding to a frequency of 1 kHz. Thus the resolution is $\Delta\theta = \lambda/D \sim 10^{-6}$ radian, fraction of an arc second - a Fourier transform of such a signal spreads into a million Fourier bins, and consequently the signal is lost in the noise of the detector. One therefore needs to demodulate the signal first and then take its Fourier transform in order to collect the signal power in a single frequency bin. This means one needs to scan or demodulate over $4\pi/\Delta\theta^2 \sim 10^{13}$ directions in the sky. So even this naive calculation gives the number of operations for the search to be $N_{\rm ops} \sim 10^{25}$ if one were to perform the FFT of the data train after demodulating in each direction. A machine having a speed of few teraflops would need several thousand years to perform the analysis!

Moreover, this estimate excludes overheads, ignores spindown parameters etc. Including these would increase the cost of the search by several orders of magnitude.

Let the motion of the detector be described in general by $\mathbf{R}(t)$ in the barycentric frame (X, Y, Z); for circular motion $\mathbf{R}(t) = R(\cos\Omega t\hat{\mathbf{X}} + \sin\Omega t\hat{\mathbf{Y}})$ - we take the detector motion in the (X, Y) plane. We will treat $\mathbf{R}(t)$ generally, except later, when we consider the specific case of circular motion.

The modulation operator which has a matrix representation in the time domain and Fourier domain (or any other convenient basis), can be computed from the key defining equation:

$$s'(t') = s(t)\,, \tag{1}$$

where we define the detector time coordinate t', which is infact a retarded or advanced time, given by $t' = t - \mathbf{R}(t)\cdot\hat{\mathbf{n}}/c$ and related to the barycentric time coordinate t. (c is the velocity of light or more appropriately gravitational wave). From our assumptions, the signal in the barycentric frame can be taken to be monochromatic. So after demodulation a Fourier transform is all that is necessary to extract the signal from the detector noise. Of course for computing the modulation/demodulation operators, the specific form the signal is not relevant. However, a specific model of the signal helps in fixing ideas. Note that the retarded time contains the unit vector $\hat{\mathbf{n}}$ describing the direction of the source as a parameter and therefore the modulation operator too is dependent on $\hat{\mathbf{n}}$. We express the modulation operators in the Fourier domain. Since the detector data are also stored in the Fourier domain, this domain is appropriate. Then we have the following relations:

$$s'(\hat{\mathbf{n}}; f') = \int df\ M(\hat{\mathbf{n}}; f', f)\ s(f), \quad s(\hat{\mathbf{n}}; f) = \int df' M^{-1}(\hat{\mathbf{n}}; f, f')\ s'(f')\,, \tag{2}$$

where now the matrices $M(\hat{\mathbf{n}})$ and $M^{-1}(\hat{\mathbf{n}})$ are given by the integrals:

$$M(\hat{\mathbf{n}}; f', f) = \int dt'\ e^{2\pi i(ft(t') - f't')}, \quad M^{-1}(\hat{\mathbf{n}}; f, f') = \int dt\ e^{2\pi i(f't'(t) - ft)}\,. \tag{3}$$

These are explicit representations of the matrices M and M^{-1} respectively in the Fourier domain, where the (un)primed variable corresponds to the ('barycentric) detector frame respectively. In the real world, the frequencies take discrete set of values and the integrals are to be interpreted as sums. M may be called the modulation operator, while M^{-1} may be called the demodulation operator.

In the abstract form, the signal at the detector coming from the direction $\hat{\mathbf{n}}$ is related to the signal in the barycentric frame by the equation:

$$\mathbf{s}'(\hat{\mathbf{n}}) = M(\hat{\mathbf{n}})\ \mathbf{s}\,, \tag{4}$$

where $M(\hat{\mathbf{n}})$ is the modulation operator. Note that in the all sky all frequency search we do not know $\hat{\mathbf{n}}$. Therefore we need to scan over the directions, a trial demodulation is performed for some general direction $\hat{\mathbf{n}}'$ given by $\hat{\mathbf{n}}' = (\sin\theta'\cos\phi', \sin\theta'\sin\phi', \cos\theta')$, which is not necessarily $\hat{\mathbf{n}}$ and so we have:

$$\mathbf{s}_{\rm trial}(\hat{\mathbf{n}}'; \hat{\mathbf{n}}) = M^{-1}(\hat{\mathbf{n}}')\ \mathbf{s}'(\hat{\mathbf{n}})\,. \tag{5}$$

If $\hat{\mathbf{n}}' \neq \hat{\mathbf{n}}$, then the demodulation is incorrect and we must try again with a different $\hat{\mathbf{n}}'$ until we get to $\hat{\mathbf{n}}$ or atleast get close enough. If $\hat{\mathbf{n}}' = \hat{\mathbf{n}}$, we must observe a peak in Fourier domain. Using these formulae we can now *step directly* to a direction $\hat{\mathbf{n}}'$ as follows:

$$\begin{aligned} \mathbf{s}_{\rm trial}(\hat{\mathbf{n}}'; \hat{\mathbf{n}}) &= M^{-1}(\hat{\mathbf{n}}')\ \mathbf{s}'(\hat{\mathbf{n}})\,, \\ &= M^{-1}(\hat{\mathbf{n}}')\ M(\hat{\mathbf{n}})\ \mathbf{s}\,, \\ &\equiv Q(\hat{\mathbf{n}}', \hat{\mathbf{n}})\ \mathbf{s}\,, \end{aligned} \tag{6}$$

where the *stepping* operator is defined by:

$$Q(\hat{\mathbf{n}}', \hat{\mathbf{n}}) = M^{-1}(\hat{\mathbf{n}}')\ M(\hat{\mathbf{n}})\,. \tag{7}$$

This was the approach suggested by Schutz so that one may directly *step* from the direction $\hat{\mathbf{n}}$ to the direction $\hat{\mathbf{n}}'$. This formulation was expected to enhance the efficiency of the search. Our approach builds upon this formulation. The goal of our work is towards using the symmetries in the problem and capturing them in the algebra to efficiently step in the sky. The symmetry is made manifest via the language of group theory. We outline the group theory below in the next section and show how it can be applied to the idealistic case of the circular orbit.

3. Modulation matrices as representations of the translation group in $\mathcal{R}^3$

We show that the matrices M, M^{-1} and the stepping kernel Q are elements of the same group which can be identified with the group $\mathcal{G}$ of translations in $\mathcal{R}^3$. For the relevant group theory see [14].

Consider the observation time interval $[0, T]$. We need to somewhat modify and enlarge our set of transformations from those we have considered before. In order that we have a group, we must consider all vectors $\mathbf{a} \in \mathcal{R}^3$ instead of just unit vectors we have considered in defining the matrices $M(\hat{\mathbf{n}})$; so that we now consider (i) the enlarged set $M(\mathbf{a})$ - the previous matrices were only confined to the unit sphere, a subset of $\mathcal{R}^3$ or more precisely, they were defined on the projective space of directions - it is better to deal with the full space $\mathcal{R}^3$, rather than the quotient space of directions, (ii) we replace $\mathbf{R}(t)$ by $\Delta\mathbf{R}(t) = \mathbf{R}(t) - \mathbf{R}(0)$, and assume that the motion of the detector is periodic and T is an integral number of periods. This period could be a day or about a year when the detector comes back to its initial location approximately. These modifications and conditions are required in order to have a 1-1 and onto map of $[0, T]$ into itself for the group action. Note that replacing $\mathbf{R}$ by $\Delta\mathbf{R}$ does not change the Doppler modulation profile, which is dependent only on the detector velocity $\mathbf{V}$, because $\dot{\mathbf{R}} = \Delta\dot{\mathbf{R}} \equiv \mathbf{V}$; it means a translation of the origin to the initial location of the detector $\mathbf{R}(0)$, or a (cancellation of) constant phase factor in the Fourier domain. We define the group action on $[0, T]$ by:

$$g_{\mathbf{a}}(t) = t - \frac{\Delta\mathbf{R} \cdot \mathbf{a}}{c}, \quad g_{\mathbf{a}} \in \mathcal{G}\,. \tag{8}$$

Note that $g_{\mathbf{a}}(0) = 0$, because $\mathbf{R}$ has been replaced by $\Delta\mathbf{R}$. Also because of the periodicity assumed $g_{\mathbf{a}}(T) = T$, since $\Delta\mathbf{R}(T) = \mathbf{R}(T) - \mathbf{R}(0) = 0$. Also taking the derivative of this equation we get:

$$\frac{dg_{\mathbf{a}}}{dt} = 1 - \frac{\mathbf{V} \cdot \mathbf{a}}{c}\,. \tag{9}$$

As long as $|\mathbf{a}| \sim 1$ and since $V/c \sim 10^{-4}$ or less in our context, we notice that $g_{\mathbf{a}}$ is a monotonic function of t. This proves that the function $g_{\mathbf{a}} : [0, T] \longrightarrow [0, T]$ is both 1-1 and onto.

Further, if $\mathbf{a} = \mathbf{0}$, the identity element of $\mathcal{G}$, then this corresponds to the identity transformation $g_{\mathbf{0}}(t) = t$. Now consider two elements $\mathbf{a}, \mathbf{b} \in \mathcal{G}$. We then have:

$$\begin{aligned} g_{\mathbf{a}}(g_{\mathbf{b}}(t)) &= t - \frac{\Delta\mathbf{R} \cdot \mathbf{a}}{c} - \frac{\mathbf{b}}{c} \cdot \Delta\mathbf{R}\left(t - \frac{\Delta\mathbf{R} \cdot \mathbf{a}}{c}\right) \\ &= t - \frac{\Delta\mathbf{R}(t)}{c} \cdot (\mathbf{a} + \mathbf{b}) + \text{relativistic corrections}\,, \end{aligned} \tag{10}$$

where we have performed a Taylor expansion. If $|\mathbf{a}|, |\mathbf{b}| \sim 1$ then we may drop the relativistic corrections which then gives $g_{\mathbf{a}}(g_{\mathbf{b}}(t)) = g_{\mathbf{a}+\mathbf{b}}(t) = (g_{\mathbf{a}} g_{\mathbf{b}})(t)$. Therefore $\mathcal{G}$ has a valid action on $[0, T]$ within this approximation.

As our further discussions will show this extension to arbitrary vectors in $\mathcal{R}^3$ is most useful. It has the following properties:

- When $\mathbf{a}$ is a unit vector, one then actually gets the modulation relating the detector frame to the barycentric frame; that is $M(\mathbf{a}) = M(\hat{\mathbf{n}})$.
- $\mathbf{a}$ can be chosen to be the step from the direction $\hat{\mathbf{n}}$ to the direction $\hat{\mathbf{n}}'$ or related to $\Delta\hat{\mathbf{n}} = \hat{\mathbf{n}}' - \hat{\mathbf{n}}$ in which case we get the stepping kernel q or $Q(\hat{\mathbf{n}}', \hat{\mathbf{n}})$.
- When $\mathbf{a}$ is infinitismal, then the step is infinitismal and the operator is then related in a straight forward way to the metric on the signal parameter space [15, 16, 5, 17].

We now describe the representation of $\mathcal{G}$. It is basically described by generalising Eq.(1) to arbitrary vectors $\mathbf{a} \in \mathcal{R}^3$. The equation for the signals is generalised to:

$$s_{\mathbf{a}}(t_{\mathbf{a}}) = s(t)\,. \tag{11}$$

Here $s_{\mathbf{a}}$ is the signal at the 'detector', $t_{\mathbf{a}} = g_{\mathbf{a}}(t)$ is the time at the 'detector' given by

Eq. (8) and s and t are the corresponding quantities in a special reference frame which we may call the 'barycentric' frame in keeping with earlier literature.

Here the carrier space is the infinite dimensional vector space $\mathcal{V}$ of continuous real valued functions, namely, the signals $s(t)$ on the interval $[0, T]$. The function s is now mapped to the function $s_{\mathbf{a}}$ by the operator which we call $M(\mathbf{a})$ by the rule:

$$s_{\mathbf{a}}(t_{\mathbf{a}}) \equiv (M(\mathbf{a})s)(t_{\mathbf{a}}) = s(t) \equiv s(g_{\mathbf{a}}^{-1}(t_{\mathbf{a}}))\,. \tag{12}$$

Thus $M(\mathbf{a})$ is an operator $M(\mathbf{a}) : \mathcal{V} \longrightarrow \mathcal{V}$ which is linear and maps the element $s \in \mathcal{V}$ to $s_{\mathbf{a}} = M(\mathbf{a})s \in \mathcal{V}$. Such groups are called transformation groups. The matrices $M(\mathbf{a})$ form a representation of $\mathcal{G}$.

4. The case of circular motion

We consider the problem of circular motion of the detector - essentially a toy problem - and investigate how the group theory works in this simplified case. The representation group then is the Euclidean group in two dimensions denoted by $E(2)$. Without loss of generality, by choosing the coordinates apprpriately, the detector motion is given by:

$$\mathbf{R}(t) = R[\cos(\Omega t)\ \hat{\mathbf{X}} + \sin(\Omega t)\ \hat{\mathbf{Y}}]\,, \tag{13}$$

where $0 \leq t \leq T$. We also assume that the detector goes through exactly one rotation, i.e., $\Omega T = 2\pi$.

We need to adapt the Fourier representation of M to a finite time interval $[0, T]$. The integral must range only over the interval $[0, T]$. Then the frequencies take discrete set of values, namely, $f \longrightarrow n$ and $f' \longrightarrow n'$ and are explicitly given by the relations, $f = \frac{n}{T}$, $f' = \frac{n'}{T}$ where n, n' are integers. The Hilbert space here is $L_2[0, T]$ with the scalar product of two functions f_1 and f_2 defined by:

$$(f_1, f_2) = \frac{1}{T} \int_0^T f_1(t)\ f_2^*(t)\ dt\,. \tag{14}$$

The functions $e^{2\pi i n t/T}, -\infty < n < \infty$ form a complete orthonormal basis for the Hilbert space $L_2[0, T]$. With the convention followed earlier we consider the time $0 \leq t' \leq T$ and write:

$$M(\mathbf{a}; n', n) = (M(\mathbf{a})\ e^{2\pi i n \frac{t'}{T}},\ e^{2\pi i n' \frac{t'}{T}})\,. \tag{15}$$

But from the definition of M, we have $M(\mathbf{a})s(t') = s(t)$, where $t' = t - \mathbf{R} \cdot \mathbf{a}/c$; thus the effect of M is to replace t' by t in the function. We then have,

$$\begin{aligned} M(\mathbf{a}; n', n) &= (e^{2\pi i n \frac{t}{T}},\ e^{2\pi i n' \frac{t'}{T}}) \\ &= \frac{1}{T} \int_0^T dt'\ e^{2\pi i n \frac{t}{T} - 2\pi i n' \frac{t'}{T}}\,. \end{aligned} \tag{16}$$

The matrices $M(\mathbf{a}; n', n)$ form a representation of the group.

We now obtain an explicit expression for $M(\mathbf{a}; n', n)$, where we choose $\mathbf{a}$ to be the unit vector $\hat{\mathbf{n}} = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$. Making the subsitutions:

$$\psi = \Omega t', \quad \Omega T = 2\pi, \quad n = \frac{2\pi f}{\Omega} = fT, \quad n' = \frac{2\pi f'}{\Omega} = f'T, \quad \beta = \frac{R\Omega}{c}\,, \tag{17}$$

and we obtain:

$$\begin{aligned} M(\hat{\mathbf{n}}; n', n) &= \frac{1}{2\pi}\int_0^{2\pi} d\psi \; e^{i(n-n')\psi + in\beta\sin\theta\cos(\psi-\phi)} \\ &\equiv e^{i\chi(n-n')} \; J_{n-n'}\left(n\beta\sin\theta\right)\,, \end{aligned} \tag{18}$$

where $\chi = \phi + \pi/2$, the translated azimuthal angle.

The modulation matrices take a simple form in this specific case. From the form of $M(\hat{\mathbf{n}}; n', n)$ it is evident that when applied to the data vector x_n, the search in χ can be performed by a fast Fourier transform; the stepping in the azimuthal parameter is done in an efficient way. If there are B samples of the χ parameter, then the search over χ for a given θ and frequency n' can be performed in order of $B\log_2 B$ number of operations. It may be possible to further reduce this computational cost by applying methods such as singular value decomposition of the matrix M to obtain a speed up over other search parameters. Encouraging results have been obtained in this direction.

5. Conclusion

We have shown how the problem of the all sky all frequency search for asymmetric rotating neutron stars can be efficiently performed by capturing the symmetries of the model in a group theoretic framework and group representation theory. As an example, we have considered the simple case of a complete circular orbit of the detector and shown how it is amenable to the group theoretic approach. We have also indicated how this could lead to a gain in computational efficiency. Work is underway to extend this idea to the case of general elliptic orbits and to eventually implement this in a practical gravitational wave search pipeline.

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