Symmetries and Conservation Laws in Classical and Quantum Mechanics

1. Classical Mechanics

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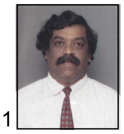
We describe the connection between continuous symmetries and conservation laws in classical mechanics. This is done at successively more sophisticated levels, bringing out important features at each level: the Newtonian[1], the Euler[2]–Lagrange[3], and the Hamiltonian phase-space forms of mechanics. The role of the Action Principle is emphasised, and many examples are given.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1. Introduction

It is generally well known that the description and consequences of symmetry are important and beautiful components of both classical mechanics (CM) and quantum mechanics (QM). The connection between the ten basic 'Galilean' conservation laws in Newtonian mechanics and fundamental space-time symmetries was first shown by G Hamel in 1904. A few years later, in 1918, Emmy Noether[4] brought out the connection between the invariances of variational principles under groups of continuous transformations and conservation theorems. Depending on the age of the reader, these may seem to be fairly recent advances, or else ancient knowledge.

In this two-part article, we shall review these topics using the notations appropriate for systems with a finite number of degrees of freedom; the generalization to (classical) field theory is quite elementary, in principle. In Part 1, we consider CM in successively more sophisticated versions, and explore in each the definition, description and consequences of symmetry. In Part 2, we shall consider the transition to QM. We emphasize the

[1] See Isaac Newton, *Resonance*, Vol.11, No.12, 2006.

[2] See Leonhard Euler, *Resonance,* Vol.2, No.5, 1997.

[3] See Joseph Louise Lagrange, *Resonance,* Vol.11, No.4, 2006.

[4] See Amalie Emmy Noether, *Resonance,* Vol.3, No.9, 1998.

general theory, and look at both similarities and differences between classical and quantum mechanics as far as symmetry is concerned.

2. Symmetry in Classical Mechanics

In recalling familiar material, we make a step-by-step progression from elementary considerations to a comprehensive formalism. Boldface symbols will denote vectors, as usual.

(i) Newton's First Law of Motion states that an isolated material body (of sufficiently small size) maintains a state of rest or of uniform motion in a straight line. Here, the concept of inertial frames of reference, the validity of Euclidean geometry for physical three-dimensional space, and the uniform flow of absolute time are all assumed. We can see that in the absence of external forces, space is homogeneous, and the momentum of the body is a constant of the motion (COM).

Keywords

Classical mechanics, canonical transformations, constants of motion, dynamical symmetry, generators.

(ii) Next, moving on to a system of two bodies or 'particles', we have Newton's Second Law, the equation of motion (EOM), together with his Third Law relating the forces the particles exert on each other:

$$\mathbf{p}_1 = m_1\,\dot{\mathbf{r}}_1\,,\;\; \mathbf{p}_2 = m_2\,\dot{\mathbf{r}}_2\,.$$
$$\dot{\mathbf{p}}_1 = \mathbf{F}_{12}\,,\;\; \dot{\mathbf{p}}_2 = \mathbf{F}_{21}\,.$$
$$\mathbf{F}_{12} = -\mathbf{F}_{21} \Rightarrow \mathbf{p}_1 + \mathbf{p}_2 \text{ is conserved (it is a COM).} \tag{1}$$

This is the first nontrivial instance of a conservation theorem, in which the crucial role of the Third Law (entirely Newton's contribution) should be appreciated.

Conservation of linear momentum is a consequence of the translation invariance of the dynamics.

(iii) If, in **(ii)** above, the two forces arise from an inter-particle potential, we have

$$\mathbf{F}_{12} = -\nabla_1\,V(\mathbf{r}_1 - \mathbf{r}_2), \quad \mathbf{F}_{21} = -\nabla_2\,V(\mathbf{r}_1 - \mathbf{r}_2). \tag{2}$$

The Third Law then holds because of the translation invariance of the potential. We see that momentum conservation is connected to a symmetry, namely, *translation invariance*. The proper understanding and appreciation of angular momentum came somewhat later, in Euler's work. (Recall, though, that Kepler's[5] Second Law already amounts to the conservation of angular momentum.)

[5] See Johannes Kepler, *Resonance*, Vol.14, No.12, 2009.

(iv) Let us consider, next, the Lagrangian form of classical mechanics. We have a system with n degrees of freedom described by generalized coordinates $q_r\,(r = 1, 2, \cdots, n)$, and are given a Lagrangian $L(q, \dot{q})$. Here, and in all that follows, we use q and $\dot{q}$ as short-hand for the sets of variables $\{q_r\}$ and $\{\dot{q}_r\}$, where r runs from 1 to n. Explicit time dependence in L can be included. We further assume that L is a nonsingular Lagrangian, so that no constraints arise in passing to the Hamiltonian formulation. (This means that we can invert the defining equations $p_r = \partial L/\partial \dot{q}_r$ to find the $\dot{q}$'s as functions of the q's and p's, and thus eliminate them to write the Hamiltonian as a function of the q's and p's. In technical terms, this requires that the $(n \times n)$ matrix whose (ij)th element is $\partial^2 L/\partial \dot{q}_i\, \partial \dot{q}_j$ be a non-singular matrix.) The Euler–Lagrange EOM are equivalent to the simplest form of the Action Principle:

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{q}_r}\right) - \frac{\partial L}{\partial q_r} = 0 \quad (r = 1, 2, \cdots, n)$$

$$\Leftrightarrow \delta \int_{t_1}^{t_2} L(q, \dot{q})\mathrm{d}t = 0, \quad \delta q_r(t_1) = \delta q_r(t_2) = 0,$$

$$(r = 1, 2, \cdots, n). \qquad (3)$$

In *Figure* 1 we depict the kind of variations in configuration space trajectories involved in this form of the Action Principle. In this framework, a COM is any function $f(q, \dot{q}, t)$ whose *total* time derivative vanishes identically:

$$\text{EOM} \Rightarrow \frac{\mathrm{d}}{\mathrm{d}t} f(q, \dot{q}, t) = 0\,. \qquad (4)$$

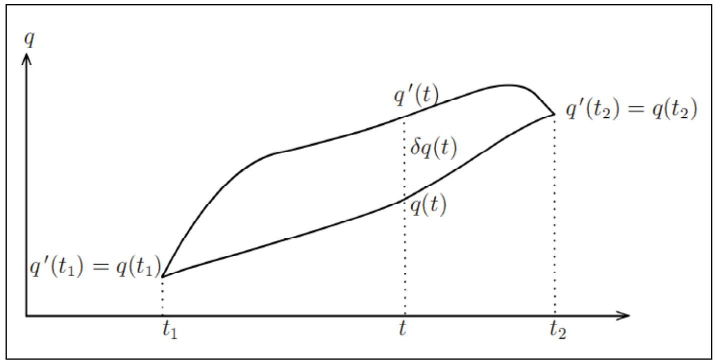

Figure 1. Action Principle without end-point variation.

Thus, along any solution of the EOM – a physical trajectory in configuration space – the COM has an unchanging numerical value (see *Figure* 2). There is as yet no link to any symmetry, but we move in that direction via two simple examples. This will set the stage for a general formulation.

(v) Suppose the Lagrangian $L(q, \dot{q})$ has no dependence on a particular generalized coordinate q_k (but does, of course, depend on $\dot{q}_k$), for some k. This coordinate is then said to be 'cyclic'. The corresponding EOM reads:

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{q}_k}\right) = 0 \quad \Leftrightarrow \quad p_k = \frac{\partial L}{\partial \dot{q}_k} \quad \text{is a COM.} \qquad (5)$$

We can express this as an invariance property of the Lagrangian. The cyclic nature of q_k means (trivially) that

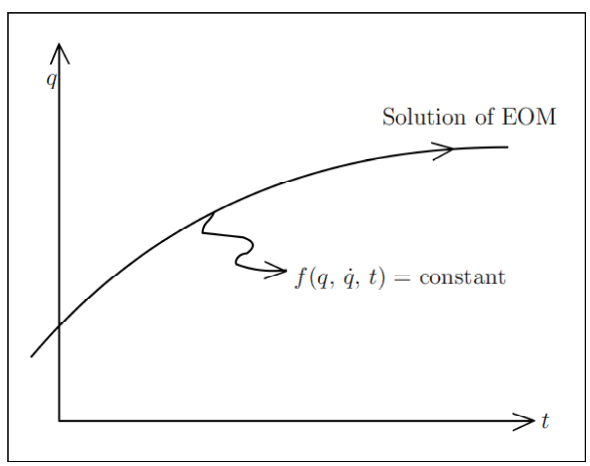

Figure 2. COM in Lagrangian mechanics.

$L(q, \dot{q})$ is invariant under (infinitesimal) translations or shifts in q_k which are time-independent. We can therefore express (5) as follows:

$$\text{If} \quad \delta q_r = \varepsilon \delta_{rk} \Rightarrow \delta L = 0, \text{ then} \quad p_k = \text{COM}. \tag{6}$$

The Kronecker delta ensures that only the kth generalized coordinate is shifted. ε is a small (infinitesimal) time-independent parameter, so that $\delta \dot{q}_r = 0$.

(vi) We have said that an explicit time dependence in the Lagrangian can always be permitted. In case there is none, we can say 'time is cyclic', even though time is not treated as a generalized coordinate. The Lagrangian then enjoys invariance under translation in time, and this leads to a COM, which is just the Hamiltonian. Using the summation convention for repeated indices (here and in all that follows), we have

$$\begin{aligned}\frac{\partial L}{\partial t} = 0 \quad \Rightarrow \quad \frac{\mathrm{d}}{\mathrm{d}t} L(q, \dot{q}) &= \frac{\partial L}{\partial q_r}\, \dot{q}_r + \frac{\partial L}{\partial \dot{q}_r}\, \ddot{q}_r \\ &= \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{q}_r}\right) \dot{q}_r + \frac{\partial L}{\partial \dot{q}_r}\, \ddot{q}_r \quad \text{(using the EOM)} \\ &= \frac{\mathrm{d}}{\mathrm{d}t}\,(p_r\, \dot{q}_r) \quad (\text{recall that } p_r \equiv \partial L / \partial \dot{q}_r). \end{aligned} \tag{7}$$

Hence

$$\frac{\partial L}{\partial t} = 0 \text{ together with the EOM} \Rightarrow \; p_r\, \dot{q}_r - L(q, \dot{q}) \text{ is a COM.} \tag{8}$$

As you know, this is the Hamiltonian of the system. Its numerical value is just the total energy of the system, in the familiar situations.

Time translation invariance of the Lagrangian leads to a constant of the motion that is just the Hamiltonian.

(vii) We now move towards a general treatment of the connection between infinitesimal ($\Rightarrow$ continuous) symmetries and conservation laws in CM, covering it in

If the Lagrangian is unchanged under a point transformation, there arises a constant of the motion that is linear in the generalized momenta.

two steps. An infinitesimal *point transformation* in Lagrangian mechanics is a small change made in each generalized coordinate at each time, of the form

$$\delta q_r = \varepsilon \phi_r(q), \quad r = 1,\, 2,\, \cdots, n\,. \tag{9}$$

Here ε is an infinitesimal parameter, and in all calculations we need only to keep terms up to first order in ε. The functions ϕ_r depend on the q's alone (hence the name 'point transformation'), and they specify the transformation: we view it as a geometrical point-to-point mapping in configuration space. By the rules of Lagrangian mechanics, as in the working out of the Action Principle, we have

$$\delta \dot{q}_r = \frac{\mathrm{d}(\delta q_r)}{\mathrm{d}t} = \varepsilon\, \frac{\mathrm{d}\phi_r(q)}{\mathrm{d}t} = \varepsilon\, \frac{\partial \phi_r(q)}{\partial q_l}\, \dot{q}_l\,. \tag{10}$$

Hence these increments depend on the q's, and linearly on the $\dot{q}$'s. If the Lagrangian happens to be unchanged under this transformation, then, combining this property with the EOM leads to a COM that is linear in the canonical momenta, as follows:

$$\begin{aligned} \delta L &= \frac{\partial L}{\partial q_r}\;\delta q_r + \frac{\partial L}{\partial \dot{q}_r}\;\delta \dot{q}_r = \frac{\partial L}{\partial q_r}\;\delta q_r + p_r \frac{\mathrm{d}(\delta q_r)}{\mathrm{d}t} = 0 \\ &\Rightarrow \frac{\mathrm{d}p_r}{\mathrm{d}t}\,\delta q_r + p_r \frac{\mathrm{d}(\delta q_r)}{\mathrm{d}t} = 0 \quad \text{using the EOM} \\ &\Rightarrow G(q,p) = p_r\,\phi_r(q) = \text{COM}. \end{aligned} \tag{11}$$

We have introduced the symbol G for a generic COM, regarded as a function in phase space. Thus,

- the invariance of the Lagrangian under an infinitesimal point transformation leads, when combined with the EOM, to a COM linear in the momenta.

This conclusion can also be arrived at in an elegant manner from the extended form of the Action Principle, in which we allow variations that do *not* vanish at the initial and final times. For brevity, let us write (in honour of Euler and Lagrange!)

$$(\mathrm{EL})_r \equiv \frac{\partial L}{\partial q_r} - \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{q}_r}\right). \qquad (12)$$

We develop the argument starting with the change in the action under general infinitesimal changes in the q's at each instant of time as well as in the initial and final times t_1 and t_2.

$$\delta \int_{t_1}^{t_2} L(q,\dot{q})\mathrm{d}t = \left[L\delta t\right]_{t_1}^{t_2} + \int_{t_1}^{t_2} \delta L(q,\dot{q})\,\mathrm{d}t$$

$$= \Big[p_r \delta q_r + L\delta t\Big]_{t_1}^{t_2} + \int_{t_1}^{t_2} (\mathrm{EL})_r\, \delta q_r(t)\,\mathrm{d}t, \qquad (13)$$

for any δt_1, δt_2 and $\delta q_r(t)$. In the first term on the right-hand side, we may express $\delta q_r(t_1)$ and $\delta q_r(t_2)$ in terms of $\Delta q_r(t_1)$ and $\Delta q_r(t_2)$, the *total* variations at the end points, using

$$\Delta q_r(t_i) = \delta q_r(t_i) + \dot{q}_r(t_i)\delta t_i$$
$$\text{or} \quad \delta q_r(t_i) = \Delta q_r(t_i) - \dot{q}_r(t_i)\delta t_i\,, \qquad (14)$$

where $i = 1$, 2. Further, putting $H = p_r\dot{q}_r - L$, we get

$$\delta \int_{t_1}^{t_2} L(q,\dot{q})\mathrm{d}t = \Big[p_r \Delta q_r - H\delta t\Big]_{t_1}^{t_2} + \int_{t_1}^{t_2} (\mathrm{EL})_r\, \delta q_r(t)\,\mathrm{d}t, \qquad (15)$$

for any δt_1, δt_2 and $\delta q_r(t)$. Now using the EOM (i.e., $(\mathrm{EL})_r = 0$ for each r), this reduces to

$$\delta \int_{t_1}^{t_2} L(q,\dot{q})\mathrm{d}t = \Big[p_r \Delta q_r - H\delta t\Big]_{t_1}^{t_2},$$
$$\text{for any } \delta t_1,\ \delta t_2 \text{ and } \delta q_r(t). \qquad (16)$$

Finally, using $\delta q = \varepsilon\,\phi(q)$, $\delta L = 0$ and setting $\delta t_1 = \delta t_2 = 0$, we have

$$0 = \Big[\varepsilon\, p_r\, \phi_r(q)\Big]_{t_1}^{t_2} \Rightarrow G(q,p) = p_r\, \phi_r(q) = \mathrm{COM}\,. \quad (17)$$

We can view this as a derivation of the result (11) from a global rather than local point of view in time. For this limited purpose, it is adequate to use the extended form of the Action Principle but with $\delta t_1 = \delta t_2 = 0$. The meaning of the two kinds of variations $\delta q(t)$ and $\Delta q(t)$ that appear in the discussion above is illustrated schematically in *Figure* 3.

(viii) When we translate the result above into the Hamiltonian or phase space language, new and crucial aspects enter the picture. Let us recall some important quantities and expressions in phase space mechanics. Given two functions $f(q,p)$ and $g(q,p)$, their Poisson Bracket (PB) is a third function given by

$$\{f,\, g\}(q,\, p) = \frac{\partial f}{\partial q_r}\frac{\partial g}{\partial p_r} - \frac{\partial f}{\partial p_r}\frac{\partial g}{\partial q_r}. \quad (18)$$

The canonical PBs satisfied by the q's and p's are of course

$$\{q_r\,,\, q_s\} = 0,\ \{p_r\,,\, p_s\} = 0,\ \{q_r\,,\, p_s\} = \delta_{rs}\,. \quad (19)$$

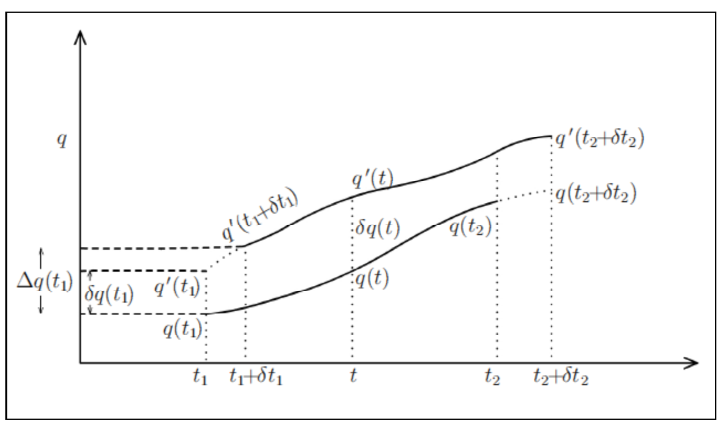

Figure 3.
The variations δq(t), Δq(t).

PBs satisfy the familiar properties of antisymmetry, bilinearity and the Jacobi identity. Next, a canonical transformation (CT) is a phase space mapping

$$q,\, p \to Q(q,p),\, P(q,p) \tag{20}$$

such that all PB relations are maintained. Lastly, an infinitesimal CT is one in which Q, P differ infinitesimally from q, p:

$$Q_r = q_r + \delta q_r\,, \quad P_r = p_r + \delta p_r\,, \tag{21}$$

where δq_r and δp_r are functions of q, p. The assertion is that such a CT is always of the form

$$\delta q_r = \varepsilon\{q_r, G(q,p)\}\,, \quad \delta p_r = \varepsilon\{p_r, G(q,p)\}, \tag{22}$$

where ε is a small parameter, and $G(q,p)$ (determined up to an additive constant) is the *generator* of the transformation. This statement is valid whether or not the infinitesimal CT arises from a symmetry. We now show that an infinitesimal point transformation *symmetry*, when regarded as a phase space transformation in the natural manner, turns out to be an infinitesimal CT with the COM as its generator, in the sense of equation (22).

The increments in the dynamical variables under an infinitesimal CT must necessarily have a very special form that involves the generator of the transformation.

Consider a point transformation symmetry of the Lagrangian leading to the COM $G(q,p)$ in (11). Since this is linear in the canonical momenta, the canonical PBs (19) immediately yield (recall (9))

$$\delta q_r = \varepsilon\, \phi_r(q) = \varepsilon\{q_r, G(q,p)\}. \tag{23}$$

Note that this is a kinematic fact, independent of the EOM. We thus have the 'first half' of an infinitesimal CT in phase space, the first of equations (22). This result can now be extended to the momenta, too, by defining and calculating their changes as follows:

$$\delta p_r = \delta\left(\frac{\partial L(q,\dot q)}{\partial \dot q_r}\right) = \frac{\partial^2 L}{\partial \dot q_r\, \partial q_s}\,\delta q_s + \frac{\partial^2 L}{\partial \dot q_r\, \partial \dot q_s}\,\delta \dot q_s$$

$$= \varepsilon\frac{\partial^2 L}{\partial \dot q_r\, \partial q_s}\,\phi_s(q) + \varepsilon\frac{\partial^2 L}{\partial \dot q_r\, \partial \dot q_s}\,\frac{\partial \phi_s(q)}{\partial q_l}\dot q_l\,, \tag{24}$$

using (9) and (10) for δq_s and $\delta \dot{q}_s$. The expression on the right-hand side involves only the q's and $\dot{q}$'s (and no higher derivatives), and hence is kinematically expressible in terms of the q's and p's. Further, since the point transformation is taken to be a symmetry of the Lagrangian, we have:

$$\begin{aligned} \delta L &= \frac{\partial L}{\partial q_s}\delta q_s + \frac{\partial L}{\partial \dot{q}_s}\delta \dot{q}_s = 0 \\ &\Rightarrow \frac{\partial L}{\partial q_s}\phi_s + \frac{\partial L}{\partial \dot{q}_s}\frac{\partial \phi_s}{\partial q_l}\dot{q}_l = 0. \end{aligned} \tag{25}$$

Differentiating this equation with respect to $\dot{q}_r$ and using the result in (24), we get (after a little bit of algebra, which you will find instructive to work out)

$$\delta p_r = -\varepsilon\, p_s \frac{\partial \phi_s}{\partial q_r} = \varepsilon \{p_r, G(q,p)\}. \tag{26}$$

This is the 'second half' of an infinitesimal CT, the second of equations (22). Thus, we have the important result:

- An infinitesimal point transformation symmetry of the Lagrangian appears in phase space as an infinitesimal CT, with the associated COM as its generator. *This is a kinematical fact that does not require the use of the EOM.*
- The EOM are needed only to show that the generator of the CT is a COM.

(ix) We come, now, to the last natural generalization of the symmetry↔COM connection in the Lagrangian formalism. We extend (9) and consider an infinitesimal change in the q's of the form

$$\delta q_r = \varepsilon \phi_r(q,\, \dot{q}), \tag{27}$$

permitting a dependence on the generalized velocities $\dot{q}$ as well. We then define the changes in the generalized velocities $\{\dot{q}_r\}$, in the Lagrangian spirit, as

$$\delta \dot{q}_r = \frac{\mathrm{d}}{\mathrm{d}t}\, \delta q_r = \varepsilon \frac{\mathrm{d}}{\mathrm{d}t}\, \phi_r(q,\, \dot{q}). \tag{28}$$

Hence $\delta \dot{q}_r$ may now involve the accelerations $\{\ddot{q}_l\}$ as well, in general. If, under these changes, the Lagrangian changes by a total time derivative,

$$\delta L(q, \dot{q}) = \varepsilon\, \frac{\mathrm{d}}{\mathrm{d}t} F(q,\, \dot{q}), \tag{29}$$

where $F(q,\, \dot{q})$ is a function of the indicated variables that is 'local in time', then we have a *dynamical symmetry* of the Lagrangian. The following three consequences can be established quite easily, either by using the differential EOM directly, or (more elegantly) via the extended Action Principle:

(a) $G(q, p) = p_r \phi_r(q, \dot{q}) - F(q,\, \dot{q}) = \text{COM}$, not necessarily linear in p, (30a)

(b) $\delta q_r \,=\, \varepsilon\{q_r,\, G(q,\, p)\}$, without using the EOM, (30b)

(c) $\delta p_r \,=\, \varepsilon\{p_r,\, G(q,\, p)\}$, using the EOM, in general. (30c)

Thus:

- The most general infinitesimal symmetry of a Lagrangian is a dynamical symmetry, characterized by equations (27)–(29).
- It leads to a COM, given by the function G of (30a).
- When recast in phase space language, the symmetry appears as an infinitesimal CT, with the COM as the generator.

The most general continuous symmetry of a Lagrangian is a dynamical symmetry, the associated constant of the motion being the generator of a CT.

In all the quantities above, namely, ϕ_r, L, F and G, we can permit explicit time dependence without any difficulty.

We invite the reader to convince herself that the argument leading from an infinitesimal symmetry to a COM can now be expressed in a very economical or compact form, as follows. Given a Lagrangian $L(q,\, \dot{q},\, t)$, the following implications hold good:

(a) The Euler–Lagrange EOM are obeyed $\Leftrightarrow$ the variation in the Lagrangian, for *arbitrary* infinitesimal variations $\delta q_r(t)$ at each instant t, is of the form

$$\delta L = \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{q}_r}\delta q_r\right). \tag{31}$$

(b) An infinitesimal transformation $\delta q_r = \varepsilon\, \phi_r(q, \dot{q}, t)$ of *specified functional form* is a (dynamical) *symmetry transformation* $\Leftrightarrow$ there exists a function $F(q, \dot{q}, t)$ such that

$$\delta L = \varepsilon\, \frac{\mathrm{d}}{\mathrm{d}t} F(q, \dot{q}, t). \tag{32}$$

(c) Using the δq_r of a *symmetry* transformation in statement (a), we conclude that

$$\text{EOM plus infinitesimal symmetry} \Rightarrow G = p_r\, \phi_r - F \text{ is a COM.} \tag{33}$$

Classical Hamiltonian time evolution is the gradual unfolding of a family of CTs; similarly, every continuous symmetry can also be viewed as a sequence of CTs.

The simplicity and naturalness of the argument should now be evident.

Just as the evolution in time of a classical Hamiltonian system can be pictured as the continuous unfolding of a family of CTs, so also every symmetry in the sense described above is a continuous family of CTs. Moreover, every such symmetry preserves the EOM, as is most easily seen from the Action Principle. Therefore it acts as a mapping of the set of all solutions of the EOM onto itself.

We remark that the case of no explicit time-dependence in the Lagrangian, considered in **(vi)** above, is itself an example of dynamical symmetry. In that case equations (27)–(29) become, respectively,

$$\delta q_r = \varepsilon\, \dot{q}_r\,,\ \ \delta \dot{q}_r = \varepsilon\, \ddot{q}_r\,,\ \ \text{so that } \delta L = \varepsilon\, \frac{\mathrm{d}L}{\mathrm{d}t}\,. \qquad (34)$$

Then (30a) directly leads to (8), namely, the conservation of the quantity $p_r\, \dot{q}_r - L(q, \dot{q})$.

It is important to appreciate the depth of these results. We have defined an infinitesimal symmetry of a Lagrangian as a transformation (27) that changes the Lagrangian at most by a total time derivative (or preserves the action functional up to end-point terms). Such a transformation will preserve the EOM and will lead to a COM. But then the COM $G(q, p)$ plays a new role: it generates the symmetry transformation in phase space as a CT. Thus, there are two roles for each COM:

(a) $G(q, p)$ is constant, in numerical value, along each solution of the EOM.

(b) $G(q, p)$ generates a symmetry as an infinitesimal CT, mapping each solution of EOM onto another, generally different, solution.

In the second role above, it is the algebraic form rather than the numerical value that is important. By EOM, we mean here the complete Hamiltonian system of equations

Each constant of the motion generates an infinitesimal CT that maps a solution of the equations of motion to another solution.

$$\dot{q}_r = \{q_r,\, H(q, p)\} = \frac{\partial H}{\partial p_r},$$
$$\dot{p}_r = \{p_r,\, H(q, p)\} = -\frac{\partial H}{\partial q_r}\,. \qquad (35)$$

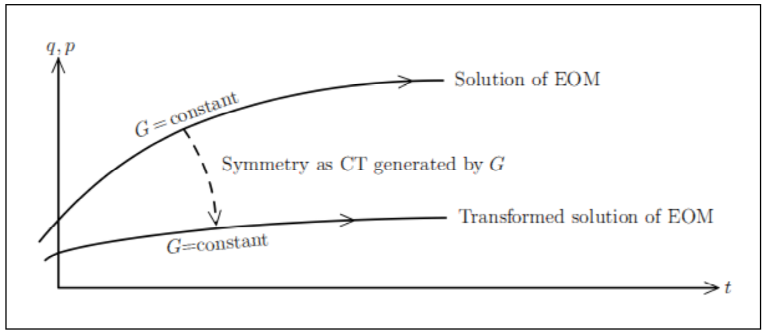

Figure 4. Roles of COM in Hamiltonian mechanics.

In the spirit of *Figure* 2, we can depict the situation as in *Figure* 4, except that the trajectories shown now are phase space trajectories. We need only add that in the case of a single transformation involving just one COM $G(q, p)$, the numerical values of $G(q, p)$ are the same on both the original and the transformed solutions of the EOM.

At this point, it is helpful to summarise what we have learnt about symmetry in classical mechanics in the form of two tables. In *Table* 1, a cross indicates 'not necessarily' or 'not meaningful'. Clearly, it is only in the Hamiltonian formalism that we have a complete and well-rounded picture. *Table* 2 describes the transformations, their COMs, etc., in more detail. Of the two rows in this table, the first is a simple special case, while the second is the most general case in the framework of classical mechanics.

Table 1. Consequences and properties of infinitesimal symmetry transformations.

Form of dynamics	Maps solutions of EOM to solutions	Leads to COM	Symmetry is a CT	COM generates symmetry
Newtonian	$\surd$	$\times$	$\times$	$\times$
Lagrangian	$\surd$	$\surd$	$\times$	$\times$
Hamiltonian	$\surd$	$\surd$	$\surd$	$\surd$

Type of symmetry	Behaviour of Lagrangian	COM	Symmetry as CT	Remarks
$\delta q_r = \varepsilon\phi_r(q)$	$\delta L = 0$	$G(q,p) = p_r\,\phi_r(q)$	kinematic; EOM not used	geometric; point transformation; COM linear in p
$\delta q_r = \varepsilon\,\phi_r(q,\dot{q},t)$	$\delta L = \varepsilon\, dF(q,\dot{q},t)/dt$	$G(q,p) = p_r\,\phi_r(q,\dot{q},t) - F(q,\dot{q},t)$	dynamic; may need EOM, in general	dynamic symmetry; G nonlinear in p

Table 2. Structure of symmetry transformations and their COMs.

3. Examples

We turn now to some examples of systems in which the generalized coordinates undergo infinitesimal variations under symmetry transformations. These variations can be of different types, as mentioned in *Section* 2. The aim is to identify the COMs in each case.

3.1 System of Particles Moving in a Potential due to Newtonian Forces

We begin with a system of n point masses moving in a potential that depends only on the inter-particle distances–for instance, the case of particles interacting among themselves via two-body conservative Newtonian forces. The Lagrangian for such a system is

$$L = \frac{1}{2}\sum_{j=1}^{n} m_j \dot{\mathbf{r}}_j^{\,2} - V(\{r_{jk}\}), r_{jk} = |\mathbf{r}_j - \mathbf{r}_k| \qquad (j,\, k = 1,\, 2, \cdots, n). \qquad (36)$$

(a) Consider the variation $\delta\mathbf{r}_j = \varepsilon\mathbf{a}$, where ε is infinitesimal in magnitude and $\mathbf{a}$ is a constant vector. It is obvious that r_{jk} and $\dot{\mathbf{r}}_j$, and hence L, remain unchanged. From (11), we then have $G(\mathbf{r},\mathbf{p}) = \mathbf{a}\cdot\sum_1^n \mathbf{p}_j =$ COM. Since $\mathbf{a}$ is constant and arbitrary, the total linear momentum $\mathbf{P} = \sum_1^n \mathbf{p}_j$ is a COM.

(b) Next, consider the variation $\delta \mathbf{r}_j = \varepsilon\,(\mathbf{n} \times \mathbf{r}_j)$, where $\mathbf{n}$ is any constant unit vector. This is just the change in $\mathbf{r}_j$ under an infinitesimal rotation of the coordinate axes about the direction of the unit vector $\mathbf{n}$. Comparing this with the general form $\delta \mathbf{r} = \varepsilon\, \boldsymbol{\phi}(\mathbf{r})$, we have $\boldsymbol{\phi}(\mathbf{r}_j) = \mathbf{n} \times \mathbf{r}_j$. The variation now depends on the generalized coordinates $\mathbf{r}_j$. Under this infinitesimal change,

$$\delta \dot{\mathbf{r}}_j = \frac{\mathrm{d}}{\mathrm{d}t}(\delta \mathbf{r}_j) = \varepsilon\, \mathbf{n} \times \dot{\mathbf{r}}_j \quad \text{and} \quad \delta r_{jk} = 0. \tag{37}$$

Hence $\mathbf{r}_j \cdot \delta \mathbf{r}_j = 0$ and $\dot{\mathbf{r}}_j \cdot \delta \dot{\mathbf{r}}_j = 0$, so that L remains unchanged up to first order in ε, i.e.,

$$L(\mathbf{r}_j + \delta \mathbf{r}_j, \dot{\mathbf{r}}_j + \delta \dot{\mathbf{r}}_j) \simeq L(\mathbf{r}_j, \dot{\mathbf{r}}_j) \quad \text{or} \quad \delta L = 0. \tag{38}$$

The COM is therefore

$$G(\mathbf{r}, \mathbf{p}) = \sum_j \mathbf{p}_j \cdot \boldsymbol{\phi}(\mathbf{r}_j) = \sum_j \mathbf{p}_j \cdot (\mathbf{n} \times \mathbf{r}_j)$$
$$= \sum_j \mathbf{n} \cdot (\mathbf{r}_j \times \mathbf{p}_j) = \mathbf{n} \cdot \sum_j (\mathbf{r}_j \times \mathbf{p}_j). \tag{39}$$

Since $\mathbf{n}$ is constant and arbitrary, the total orbital angular momentum $\mathbf{L} = \sum_j (\mathbf{r}_j \times \mathbf{p}_j)$ is a COM.

(c) Third, consider the explicitly time-dependent variation $\delta \mathbf{r}_j = -\varepsilon\, \mathbf{v}\, t$, where $\mathbf{v}$ is a constant vector. (Obviously, this variation simulates the effect of shifting to a mutually inertial frame moving with a uniform velocity $\varepsilon\, \mathbf{v}$ with respect to the original frame of reference.) As before, using the rule given by (10), we have

$$\frac{\mathrm{d}}{\mathrm{d}t} \delta \mathbf{r}_j = \delta \frac{\mathrm{d}\mathbf{r}_j}{\mathrm{d}t} = \delta \dot{\mathbf{r}}_j = -\varepsilon\, \mathbf{v}. \tag{40}$$

The change in the Lagrangian up to first order in ε turns out to be

$$L(\mathbf{r}_j + \delta \mathbf{r}_j\,,\, \dot{\mathbf{r}}_j + \delta \dot{\mathbf{r}}_j) \simeq L(\mathbf{r}_j\,,\, \dot{\mathbf{r}}_j) - \varepsilon \frac{\mathrm{d}}{\mathrm{d}t} \sum_j m_j\, \mathbf{v} \cdot \mathbf{r}_j\,, \tag{41}$$

so that

$$\delta L = -\varepsilon \frac{\mathrm{d}}{\mathrm{d}t} \sum_j m_j \, \mathbf{v} \cdot \mathbf{r}_j \, . \tag{42}$$

The COM can now be written down by looking at the general form in (22) and deducing G, or else by using (29) and (30a). We get

$$\begin{aligned} G(\mathbf{r}, \mathbf{p}) &= -t \, \mathbf{v} \cdot \sum_j \mathbf{p}_j + \mathbf{v} \cdot \sum_j m_j \, \mathbf{r}_j \\ &= \mathbf{v} \cdot \Big[\sum_j m_j \, \mathbf{r}_j - \mathbf{P}t \Big]. \end{aligned} \tag{43}$$

Since $\mathbf{v}$ is constant and arbitrary, $\sum_j m_j \, \mathbf{r}_j - \mathbf{P}t$ is a COM. If M is the total mass and $\mathbf{R}$ is the position vector of the centre of mass of the system of particles, the COM can also be expressed, apart from the overall constant factor M, as $\mathbf{R} - \mathbf{P}t/M$. You will recognise that this is nothing but the *initial* position vector of the centre of mass. The numerical values of the three components of this COM are therefore given by the coordinates of the centre of mass at $t = 0$.

A final remark is in order here. The Lagrangian L in (36) does not have any explicit t-dependence. Hence the corresponding Hamiltonian $H = \sum_1^n \mathbf{p}_j \cdot \dot{\mathbf{r}}_j - L$ is a COM. Together with the three components each of $\mathbf{P}$, $\mathbf{L}$ and $\mathbf{R} - \mathbf{P}t/M$, this makes a total of 10 independent COMs. These are called the Galilean invariants. For a *general* potential $V(\{r_{jk}\})$, these are the *only* independent COMs (in the form of smooth functions of the dynamical variables and time) that the system possesses. In the language of the theory of dynamical systems, the system is, in general, non-integrable, and the dynamics is generically chaotic.

For a system of non-relativistic particles interacting via a potential that depends on the inter-particle distances alone, the 10 Galilean invariants are the only independent constants of the motion.

3.2 Relativistic Free Particle

Consider a relativistic free particle of rest mass m moving with a velocity $\dot{\mathbf{r}}(t)$ relative to an inertial frame S. Its Lagrangian is

$$L = -m\,c^2\,\sqrt{1-\dot{\mathbf{r}}^2(t)/c^2}\,. \tag{44}$$

The time argument of the instantaneous velocity $\dot{\mathbf{r}}$ has been indicated explicitly with good reason, as will become clear shortly. Under an infinitesimal Lorentz boost $\varepsilon\,\mathbf{v}$ to a frame S', the space-time coordinates $(\mathbf{r},\,t)$ of the particle transform to $(\mathbf{r}',\,t')$, where

$$\mathbf{r}'(t') \simeq \mathbf{r}(t) - \varepsilon\,\mathbf{v}\,t,\;\; t' \simeq t - \varepsilon\,\mathbf{v}\cdot\mathbf{r}(t)/c^2. \tag{45}$$

The crucial point is that we can regard the difference $\delta\mathbf{r}(t) \equiv \mathbf{r}'(t) - \mathbf{r}(t)$ (note especially the time argument of $\mathbf{r}'$ in this definition!) as a variation of the coordinate $\mathbf{r}$ in the frame S. The variations $\delta\mathbf{r}(t)$ and $\delta\dot{\mathbf{r}}\,(t)$ are then given, to first order in ε, by

$$\left.\begin{aligned} \delta\mathbf{r}(t) &= -\varepsilon\,[\mathbf{v}\,t - (\mathbf{v}\cdot\mathbf{r})\dot{\mathbf{r}}/c^2]\,, \\ \delta\dot{\mathbf{r}}(t) &= -\varepsilon\,[\mathbf{v} - (\mathbf{v}\cdot\dot{\mathbf{r}})\,\dot{\mathbf{r}}/c^2 - (\mathbf{v}\cdot\mathbf{r})\,\ddot{\mathbf{r}}/c^2]\,. \end{aligned}\right\} \tag{46}$$

After some algebra, the change δL in L works out to

$$\delta L = -\varepsilon\,m\,\frac{\mathrm{d}}{\mathrm{d}t}\Big[(\mathbf{v}\cdot\mathbf{r})\,\sqrt{1-\dot{\mathbf{r}}^2/c^2}\,\Big]. \tag{47}$$

Since this is in the form of the total derivative of a function with respect to the time, we have an associated COM. Using the well-known relations

$$\mathbf{p} = m\,\dot{\mathbf{r}}/\sqrt{1-\dot{\mathbf{r}}^2/c^2} \quad \text{and} \quad \dot{\mathbf{r}} = c\mathbf{p}/\sqrt{\mathbf{p}^2 + m^2\,c^2}, \tag{48}$$

this COM is given by

$$\begin{aligned} G &= -\mathbf{p}\cdot\big[\mathbf{v}\,t - (\mathbf{v}\cdot\mathbf{r})\,\dot{\mathbf{r}}\,/c^2\big] + m\,(\mathbf{v}\cdot\mathbf{r})\,\sqrt{1-\dot{\mathbf{r}}^2/c^2} \\ &= \mathbf{v}\cdot\big[-\mathbf{p}\,t + \mathbf{r}\,(\mathbf{p}\cdot\dot{\mathbf{r}})/c^2 + m\,\mathbf{r}\sqrt{1-\dot{\mathbf{r}}^2/c^2}\,\big] \\ &= \mathbf{v}\cdot[(E/c^2)\mathbf{r} - \mathbf{p}\,t\,], \end{aligned} \tag{49}$$

where we have used the relation $E = (\mathbf{p}^2\, c^2 + m^2\, c^4)^{1/2}$ in the last line. As before, since G is a COM for any constant vector $\mathbf{v}$, we may conclude that the combination $(E/c^2)\,\mathbf{r}\, - \mathbf{p}\,t$ is a COM. In the limit $c \to \infty$ (implying that $|\dot{\mathbf{r}}|/c \to 0$ or $|\mathbf{p}|/(mc) \to 0$), this COM reduces to the Galilean invariant $(m\,\mathbf{r} - \mathbf{p}t)$ of a free nonrelativistic particle.

3.3 The Two-Dimensional Isotropic Harmonic Oscillator

Consider the two-dimensional *isotropic* harmonic oscillator, namely, a particle of mass m moving in a plane in the circularly symmetric potential $\frac{1}{2}m\omega^2 r^2$. Let $q_\alpha\, (\alpha = 1,\, 2)$ be the two Cartesian coordinates in this plane. The Lagrangian is

$$L = \tfrac{1}{2}m\, \dot{q}_\alpha\, \dot{q}_\alpha - \tfrac{1}{2}m\, \omega^2\, q_\alpha\, q_\alpha\,, \qquad (50)$$

where the repeated index α is to be summed over, as usual. This system has an interesting symmetry, with associated COMs. A comprehensive way of finding them is as follows.

Consider the group $U(2)$ of unitary (2×2) matrices, and consider an element u of this group that is infinitesimally close to the identity matrix. Then u is of the form $u \simeq I + i\varepsilon h$, where h is a hermitian (2×2) matrix. In terms of matrix elements,

$$u_{\alpha\beta} \simeq \delta_{\alpha\beta} + i\varepsilon h_{\alpha\beta}\,, \quad \text{where} \quad h^*_{\alpha\beta} = h_{\beta\alpha}\,. \qquad (51)$$

But $h_{\alpha\beta}$ itself can be written as the sum of a real symmetric part and a pure imaginary antisymmetric part, according to

$$h_{\alpha\beta} = \frac{(h_{\alpha\beta} + h^*_{\alpha\beta})}{2} + i\,\frac{(h_{\alpha\beta} - h^*_{\alpha\beta})}{2i} \equiv s_{\alpha\beta} + i\, a_{\alpha\beta}\,, \qquad (52)$$

where $s_{\alpha\beta} = s_{\beta\alpha}$ and $a_{\alpha\beta} = -a_{\beta\alpha}$. (Hence $a_{11} = a_{22} = 0$.) Now, observe that the variations

$$\delta q_\alpha = \varepsilon\big(a_{\alpha\beta}\, q_\beta - \omega^{-1} s_{\alpha\beta}\, \dot{q}_\beta\big), \quad \delta \dot{q}_\alpha = \varepsilon\big(a_{\alpha\beta}\, \dot{q}_\beta - \omega^{-1} s_{\alpha\beta}\, \ddot{q}_\beta\big) \tag{53}$$

lead to a dynamical symmetry, because the corresponding change in L is given by a total time derivative:

$$\delta L = \varepsilon\, \frac{\mathrm{d}F}{\mathrm{d}t}, \text{ where } F = \frac{m}{2}\big(\omega\, q_\alpha\, q_\beta - \omega^{-1}\dot{q}_\alpha\, \dot{q}_\beta\big)\, s_{\alpha\beta}. \tag{54}$$

The corresponding COM is $\varepsilon\, G = p_\alpha\, \delta q_\alpha - \varepsilon\, F$, where

$$\begin{aligned} G &= p_\alpha\, \big(a_{\alpha\beta}\, q_\beta - \omega^{-1} s_{\alpha\beta}\, \dot{q}_\beta\big) - \frac{m}{2}\big(\omega\, q_\alpha\, q_\beta - \omega^{-1}\, \dot{q}_\alpha\, \dot{q}_\beta\big) s_{\alpha\beta} \\ &= a_{12}\,(p_1\, q_2 - p_2\, q_1) - \frac{s_{\alpha\beta}}{2}\, \left(\frac{p_\alpha\, p_\beta}{m\omega} + m\omega\, q_\alpha\, q_\beta\right). \end{aligned} \tag{55}$$

But the matrix element a_{12} and the three matrix elements s_{11} , s_{22} and $s_{12}\,(= s_{21})$ are independent and arbitrary real numbers. Therefore (55) implies that we have four COMs, namely,

$$\left.\begin{aligned} c_1 &= (m\omega)^{-1} p_1^2 + m\omega\, q_1^2, \\ c_2 &= (m\omega)^{-1} p_2^2 + m\omega\, q_2^2, \\ c_3 &= p_1\, q_2 - p_2\, q_1, \\ c_4 &= (m\omega)^{-1} p_1\, p_2 + m\omega\, q_1\, q_2\,. \end{aligned}\right\} \tag{56}$$

Some constants of the motion are purely kinematical in origin, while others represent dynamical symmetry.

The COM c_3 arises as the coefficient of a_{12}. But δL has no a_{12} term. Therefore c_3 is a *kinematical* COM. You will recognise it as the orbital angular momentum of the particle about the centre of force (the origin). The other three COMs represent *dynamical* symmetry. c_1 and c_2 may be interpreted (apart from a constant factor) as the individual energies of two independent harmonic oscillators, which is another way of looking at the two-dimensional oscillator.

We can choose specific combinations of the COMs above to express them in a more compact form. Define a complex column vector a and its Hermitian conjugate $\mathsf{a}^\dagger$ according to

$$\mathsf{a} = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}, \quad \mathsf{a}^\dagger = \begin{pmatrix} a_1^* & a_2^* \end{pmatrix},$$
$$\text{where} \quad a_\alpha = \frac{p_\alpha - i\, m\, \omega\, q_\alpha}{\sqrt{m\, \omega}}, \ (\alpha = 1, 2). \tag{57}$$

The COMs can then be re-defined as

$$\left. \begin{array}{ll} s_0 = c_1 + c_2 = \mathsf{a}^\dagger \mathsf{a}, & s_1 = c_1 - c_2 = \mathsf{a}^\dagger \sigma_3 \mathsf{a}, \\ s_2 = 2c_4 = \mathsf{a}^\dagger \sigma_1 \mathsf{a}, & s_3 = -2c_3 = \mathsf{a}^\dagger \sigma_2 \mathsf{a}, \end{array} \right\} \tag{58}$$

in terms of the Pauli matrices $\sigma_i\,,\ i = 1, 2, 3$.

3.4 The Kepler Problem and the Laplace–Runge–Lenz Vector

The Lagrangian of a particle of mass m moving in the Coulomb potential $V(r) = -\,\kappa\,/r$ is

$$L = \frac{1}{2} m\, \dot{\mathbf{r}}^{\,2} + \frac{\kappa}{r}, \tag{59}$$

where the constant $\kappa \gtrless 0$ according as the inverse-square force is attractive or repulsive. The Hamiltonian $H = \mathbf{p}^2/(2m) - \,\kappa/r$ is a COM because L does not have explicit t-dependence. The three components of the orbital angular momentum $\mathbf{L} = \mathbf{r} \times \mathbf{p}$ are COMs because the potential $V(r)$, and hence L, are rotationally invariant. Over and above these, this system has another vector-valued COM, the well-known Laplace–Runge–Lenz vector. We may deduce this as follows.

The Kepler problem has a special vector constant of the motion, the Laplace–Runge–Lenz vector.

Let $\mathbf{a}$ be an arbitrary vector of infinitesimal magnitude ($|\mathbf{a}| << 1$). Consider the virtual displacement in $\mathbf{r}$ (see, e.g., Saletan and Cromer 1971) given by

$$\begin{aligned}\delta\mathbf{r} &= \mathbf{a}\times(\dot{\mathbf{r}}\times\mathbf{r}) + \mathbf{r}\times(\dot{\mathbf{r}}\times\mathbf{a}) \\ &= 2(\mathbf{a}\cdot\mathbf{r})\,\dot{\mathbf{r}} - (\mathbf{a}\cdot\dot{\mathbf{r}})\,\mathbf{r} - (\mathbf{r}\cdot\dot{\mathbf{r}})\,\mathbf{a}. \qquad (60)\end{aligned}$$

Hence the variation in the velocity, $\delta\dot{\mathbf{r}} = \mathrm{d}(\delta\mathbf{r})/\mathrm{d}t$, is

$$\delta\dot{\mathbf{r}} = (\mathbf{a}\cdot\dot{\mathbf{r}})\,\dot{\mathbf{r}} - \dot{\mathbf{r}}^2\,\mathbf{a} + 2(\mathbf{a}\cdot\mathbf{r})\,\ddot{\mathbf{r}} - (\mathbf{a}\cdot\ddot{\mathbf{r}})\,\mathbf{r} - (\mathbf{r}\cdot\ddot{\mathbf{r}})\,\mathbf{a}. \quad (61)$$

The incremental change in the Lagrangian turns out to be the total time derivative of a function $F(\mathbf{r},\dot{\mathbf{r}})$, namely,

$$\delta L = \frac{\mathrm{d}F}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t}\left\{m\left[(\mathbf{a}\cdot\mathbf{r})\,\dot{\mathbf{r}}^2 - (\mathbf{a}\cdot\dot{\mathbf{r}})\,(\mathbf{r}\cdot\dot{\mathbf{r}})\right] + (\mathbf{a}\cdot\mathbf{r})\,\frac{\kappa}{r}\right\} \quad (62)$$

The associated (dynamical) COM is then

$$\begin{aligned}G &= \mathbf{p}\cdot\delta\mathbf{r} - F \qquad (63)\\ &= \mathbf{p}\cdot\delta\mathbf{r} - m\left[(\mathbf{a}\cdot\mathbf{r})\,\dot{\mathbf{r}}^2 - (\mathbf{a}\cdot\dot{\mathbf{r}})\,(\mathbf{r}\cdot\dot{\mathbf{r}})\right] - (\mathbf{a}\cdot\mathbf{r})\,\frac{\kappa}{r}\\ &= m^{-1}\,\mathbf{a}\cdot\left[\mathbf{p}\times(\mathbf{r}\times\mathbf{p}) - \kappa m\,\mathbf{e}_r\right]\\ &= m^{-1}\,\mathbf{a}\cdot\left[(\mathbf{p}\times\mathbf{L}) - \kappa m\,\mathbf{e}_r\right],\end{aligned}$$

where $\mathbf{e}_r = \mathbf{r}/r$ is the unit vector in the radial direction. Since $\mathbf{a}$ is an arbitrary constant vector, the conserved quantity is $(\mathbf{p}\times\mathbf{L}) - \kappa m\,\mathbf{e}_r$, the Laplace–Runge–Lenz vector. Note that it is a COM for *either* sign of the constant κ. In the more familiar case of planetary motion (in an *attractive* inverse-square force), the orbits in physical space are ellipses. It is not hard to see that the Laplace–Runge–Lenz vector is in the direction of the semi-major axis of the ellipse in each case. Its conservation implies that *there is no precession of the orbits in a pure inverse-square force field.*

The conservation of the Laplace–Runge–Lenz vector implies that the elliptical orbits in an attractive inverse square force field do not precess.

In Part 2, we shall extend the foregoing discussion to the case of QM, and also comment further on some group-theoretical and algebraic aspects of COMs vis-à-vis the generation of symmetry transformations.

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