

## Hyperbolic unit groups and quaternion algebras

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**Abstract.** We classify the quadratic extensions  $K = \mathbb{Q}[\sqrt{d}]$  and the finite groups  $G$  for which the group ring  $\mathfrak{o}_K[G]$  of  $G$  over the ring  $\mathfrak{o}_K$  of integers of  $K$  has the property that the group  $\mathcal{U}_1(\mathfrak{o}_K[G])$  of units of augmentation 1 is hyperbolic. We also construct units in the  $\mathbb{Z}$ -order  $\mathcal{H}(\mathfrak{o}_K)$  of the quaternion algebra  $\mathcal{H}(K) = \left(\frac{-1, -1}{K}\right)$ , when it is a division algebra.

**Keywords.** Hyperbolic groups; quaternion algebras; free groups; group rings; units.

### 1. Introduction

The finite groups  $G$  for which the unit group  $\mathcal{U}(\mathbb{Z}[G])$  of the integral group ring  $\mathbb{Z}[G]$  is hyperbolic, in the sense of Gromov [8], have been characterized in [13]. The main aim of this paper is to examine the hyperbolicity of the group  $\mathcal{U}_1(\mathfrak{o}_K[G])$  of units of augmentation 1 in the group ring  $\mathfrak{o}_K[G]$  of  $G$  over the ring  $\mathfrak{o}_K$  of integers of a quadratic extension  $K = \mathbb{Q}[\sqrt{d}]$  of the field  $\mathbb{Q}$  of rational numbers, where  $d$  is a square-free integer  $\neq 1$ . Our main result (Theorem 4.7) provides a complete characterization of such group rings  $\mathfrak{o}_K[G]$ .

In the integral case the hyperbolic unit groups are either finite, hence have zero end, or have two or infinitely many ends (see Theorem I.8.32 of [4] and [13]); in fact, in this case, the hyperbolic boundary is either empty, or consists of two points, or is a Cantor set. In particular, the hyperbolic boundary is not a (connected) manifold. However, in the case we study here, it turns out that when the unit group is hyperbolic and non-abelian it has one end, and the hyperbolic boundary is a compact manifold of constant positive curvature (see Remark after Theorem 4.7).

Our investigation naturally leads us to study units in the order  $\mathcal{H}(\mathfrak{o}_K)$  of the standard quaternion algebra  $\mathcal{H}(K) = \left(\frac{-1, -1}{K}\right)$ , when this algebra is a division algebra. We construct units, here called Pell and Gauss units, using solutions of certain diophantine quadratic equations. In particular, we exhibit units of norm  $-1$  in  $\mathcal{H}(\mathfrak{o}_{\mathbb{Q}[\sqrt{-7}]})$ ; this construction, when combined with the deep work in [5], helps to provide a set of generators for the full unit group  $\mathcal{U}(\mathcal{H}(\mathfrak{o}_{\mathbb{Q}[\sqrt{-7}]}))$ .

The work reported in this paper corresponds to the first chapter of the third author's PhD thesis [17], where analogous questions about finite semi-groups (see [10]) and RA-loops (see [14]) have also been studied.

## 2. Preliminaries

Let  $\Gamma$  be a finitely generated group with identity  $e$  and  $S$  a finite symmetric set of generators of  $\Gamma$ ,  $e \notin S$ . Consider the Cayley graph  $\mathcal{G} = \mathcal{G}(\Gamma, S)$  of  $\Gamma$  with respect to the generating set  $S$  and  $d = d_S$  the corresponding metric (see chap. 1.1 of [4]). The induced metric on the vertex set  $\Gamma$  of  $\mathcal{G}(\Gamma, S)$  is then the word metric: for  $\gamma_1, \gamma_2 \in \Gamma$ ,  $d(\gamma_1, \gamma_2)$  equals the least non-negative integer  $n$  such that  $\gamma_1^{-1}\gamma_2 = s_1s_2 \dots s_n$ ,  $s_i \in S$ . Recall that in a metric space  $(X, d)$ , the Gromov product  $(y \cdot z)_x$  of elements  $y, z \in X$  with respect to a given element  $x \in X$  is defined to be

$$(y \cdot z)_x = \frac{1}{2}(d(y, x) + d(z, x) - d(y, z)),$$

and that the metric space  $X$  is said to be *hyperbolic* if there exists  $\delta \geq 0$  such that for all  $w, x, y, z \in X$ ,

$$(x \cdot y)_w \geq \min\{(x \cdot z)_w, (y \cdot z)_w\} - \delta.$$

The group  $\Gamma$  is said to be hyperbolic if the Cayley graph  $\mathcal{G}$  with the metric  $d_S$  is a hyperbolic metric space. This is a well-defined notion which depends only on the group  $\Gamma$ , and is independent of the chosen generating set  $S$  (see [8]).

A map  $f: X \rightarrow Y$  between topological spaces is said to be *proper* if  $f^{-1}(C) \subseteq X$  is compact whenever  $C \subseteq Y$  is compact. For a metric space  $X$ , two proper maps (rays)  $r_1, r_2: [0, \infty[ \rightarrow X$  are defined to be equivalent if, for each compact set  $C \subset X$ , there exists  $n \in \mathbb{N}$  such that  $r_i([n, \infty[)$ ,  $i = 1, 2$ , are in the same path component of  $X \setminus C$ . Denote by  $\text{end}(r)$  the equivalence class of the ray  $r$ , by  $\text{End}(X)$  the set of the equivalence classes  $\text{end}(r)$ , and by  $|\text{End}(X)|$  the cardinality of the set  $\text{End}(X)$ . The cardinality  $|\text{End}(\mathcal{G}, d_S)|$  for the Cayley graph  $(\mathcal{G}, d_S)$  of  $\Gamma$  does not depend on the generating set  $S$ ; we thus have the notion of the number of ends of the finitely generated group  $\Gamma$  (see [4, 8]).

We next recall some standard results from the theory of hyperbolic groups:

1. Let  $\Gamma$  be a group. If  $\Gamma$  is hyperbolic, then  $\mathbb{Z}^2 \not\rightarrow \Gamma$ , where  $\mathbb{Z}^2$  denotes the free Abelian group of rank 2 (Corollary III.Γ 3.10(2) of [4]).
2. An infinite hyperbolic group contains an element of infinite order (Proposition III.Γ 2.22 of [4]).
3. If  $\Gamma$  is hyperbolic, then there exists  $n = n(\Gamma) \in \mathbb{N}$  such that  $|H| \leq n$  for every torsion subgroup  $H < \Gamma$  (Theorem III.Γ 3.2 of [4] and Chapter 8, Corollaire 36 of [7]).

These results will be used freely in the sequel. In view of (1) above, the following observation is quite useful.

*Lemma 2.1.* *Let  $A$  be a unital ring whose additive group is torsion free, and let  $\theta_1, \theta_2 \in A$  be two 2-nilpotent commuting elements which are  $\mathbb{Z}$ -linearly independent. Then  $\mathcal{U}(A)$  contains a subgroup isomorphic to  $\mathbb{Z}^2$ .*

*Proof.* Set  $u = 1 + \theta_1$  and  $v = 1 + \theta_2$ . It is clear that  $u, v \in \mathcal{U}(A)$  and both have infinite order. If  $1 \neq w \in \langle u \rangle \cap \langle v \rangle$ , then there exists  $i, j \in \mathbb{Z} \setminus \{0\}$ , such that,  $u^i = w = v^j$ . Since  $u^i = 1 + i\theta_1$  and  $v^j = 1 + j\theta_2$ , it follows that  $i\theta_1 - j\theta_2 = 0$  and hence  $\{\theta_1, \theta_2\}$  is  $\mathbb{Z}$ -linearly dependent, a contradiction. Hence  $\mathbb{Z}^2 \simeq \langle u, v \rangle \subseteq \mathcal{U}(A)$ .  $\square$

Let  $C_n$  denote the cyclic group of order  $n$ ,  $S_3$  the symmetric group of degree 3,  $D_4$  the dihedral group of order 8, and  $Q_{12}$  the split extension  $C_3 \rtimes C_4$ . Let  $K$  be an algebraic number field and  $\mathfrak{o}_K$  its ring of integers. The analysis of the implication for torsion subgroups  $G$  of a hyperbolic unit group  $\mathcal{U}(\mathbb{Z}[\Gamma])$  leading to Theorem 3 of [13] is easily seen to remain valid for torsion subgroups of hyperbolic unit groups  $\mathcal{U}(\mathfrak{o}_K[\Gamma])$ . We thus have the following:

**Theorem 2.2.** *A torsion group  $G$  of a hyperbolic unit group  $\mathcal{U}(\mathfrak{o}_K[\Gamma])$  is isomorphic to one of the following groups:*

1.  $C_5, C_8, C_{12}$ , an Abelian group of exponent dividing 4 or 6;
2. a Hamiltonian 2-group;
3.  $S_3, D_4, Q_{12}, C_4 \rtimes C_4$ .

We denote by  $\mathcal{H}(K) = \left(\frac{a,b}{K}\right)$  the generalized quaternion algebra over  $K$ :  $\mathcal{H}(K) = K[i, j: i^2 = a, j^2 = b, ji = -ij = :k]$ . The set  $\{1, i, j, k\}$  is a  $K$ -basis of  $\mathcal{H}(K)$ . Such an algebra is a totally definite quaternion algebra if the field  $K$  is totally real and  $a, b$  are totally negative. If  $a, b \in \mathfrak{o}_K$ , then the set  $\mathcal{H}(\mathfrak{o}_K)$ , consisting of the  $\mathfrak{o}_K$ -linear combinations of the elements  $1, i, j$  and  $k$ , is an  $\mathfrak{o}_K$ -algebra. We denote by  $N$  the norm map  $\mathcal{H}(K) \rightarrow K$ , sending  $x = x_1 + x_i i + x_j j + x_k k$  to  $N(x) = x_1^2 - ax_i^2 - bx_j^2 + abx_k^2$ .

Let  $d \neq 1$  be a square-free integer,  $K = \mathbb{Q}[\sqrt{d}]$ . Let us recall the basic facts about the ring of integers  $\mathfrak{o}_K$  (see, for example, [11] or [15]). Set

$$\vartheta = \begin{cases} \sqrt{d}, & \text{if } d \equiv 2 \text{ or } 3 \pmod{4} \\ (1 + \sqrt{d})/2, & \text{if } d \equiv 1 \pmod{4}. \end{cases}$$

Then  $\mathfrak{o}_K = \mathbb{Z}[\vartheta]$  and the elements  $1, \vartheta$  constitute a  $\mathbb{Z}$ -basis of  $\mathfrak{o}_K$ . If  $d < 0$ , then

$$\mathcal{U}(\mathfrak{o}_K) = \begin{cases} \{\pm 1, \pm \vartheta\}, & \text{if } d = -1, \\ \{\pm 1, \pm \vartheta, \pm \vartheta^2\}, & \text{if } d = -3, \\ \{\pm 1\}, & \text{otherwise.} \end{cases} \quad (1)$$

If  $d > 0$ , then there exists a unique unit  $\epsilon > 1$ , called the *fundamental unit*, such that

$$\mathcal{U}(\mathfrak{o}_K) = \pm \langle \epsilon \rangle. \quad (2)$$

We need the following:

**PROPOSITION 2.3**

*Let  $K = \mathbb{Q}[\sqrt{d}]$ , with  $d \neq 1$  a square-free integer, be a quadratic extension of  $\mathbb{Q}$ , and  $u \in \mathcal{U}(\mathfrak{o}_K)$ . Then  $u^i \equiv 1 \pmod{2}$ , where*

$$i = \begin{cases} 1 & \text{if } d \equiv 1 \pmod{8}, \\ 2 & \text{if } d \equiv 2, 3 \pmod{4}, \\ 3 & \text{if } d \equiv 5 \pmod{8}. \end{cases}$$

*Proof.* The assertion follows immediately on considering the prime factorization of the ideal  $2\mathfrak{o}_K$  (see Theorem 1, p. 236 of [3]).  $\square$

### 3. Abelian groups with hyperbolic unit groups

#### PROPOSITION 3.1

Let  $R$  be a unitary commutative ring,  $C_2 = \langle g \rangle$ . Then  $u = a + (1 - a)g$ ,  $a \in R \setminus \{0, 1\}$  is a non-trivial unit in  $\mathcal{U}_1(R[C_2])$  if, and only if,  $2a - 1 \in \mathcal{U}(R)$ .

*Proof.* Let  $C_2 = \langle g \rangle$  and suppose that  $u = a + (1 - a)g$ ,  $a \in R \setminus \{0, 1\}$  is a non-trivial unit in  $R[C_2]$  having augmentation 1. Let  $\rho: R[C_2] \rightarrow M_2(R)$  be the regular representation. Clearly  $\rho(u) = \begin{pmatrix} a & 1-a \\ 1-a & a \end{pmatrix}$ . Since  $u$  is a unit, it follows that  $2a - 1 = \det \rho(u) \in \mathcal{U}(R)$ .

Conversely, let  $a \in R \setminus \{0, 1\}$  be such that  $e = 2a - 1 \in \mathcal{U}(R)$ . It is then easy to see that  $u = a + (1 - a)g$  is a non-trivial unit in  $R[C_2]$  with inverse  $v = ae^{-1} + (1 - ae^{-1})g$  (Proposition I of [9]).  $\square$

#### PROPOSITION 3.2

The unit group  $\mathcal{U}(\mathfrak{o}_K[C_2])$  is trivial if, and only if,  $K = \mathbb{Q}$  or an imaginary quadratic extension of  $\mathbb{Q}$ , i.e.,  $d < 0$ .

*Proof.* It is clear from the description (1) of the unit group of  $\mathfrak{o}_K$  that the equation

$$2a - 1 = u, \quad a \in \mathfrak{o}_K \setminus \{0, 1\}, \quad u \in \mathcal{U}(\mathfrak{o}_K) \quad (3)$$

does not have a solution when  $K = \mathbb{Q}$  or  $d < 0$ .

Suppose  $d > 1$  and  $\epsilon$  is the fundamental unit in  $\mathfrak{o}_K$ . In this case we have  $\mathcal{U}(\mathfrak{o}_K) = \pm\langle \epsilon \rangle$ . By Proposition 2.3,  $\epsilon^i \in 1 + 2\mathfrak{o}_K$  for some  $i \in \{1, 2, 3\}$ . Consequently eq. (3) has a solution and so, by Proposition 3.1,  $\mathcal{U}(\mathfrak{o}_K[C_2])$  is non-trivial.  $\square$

**Theorem 3.3.** Let  $\mathfrak{o}_K$  be the ring of integers of a real quadratic extension  $K = \mathbb{Q}[\sqrt{d}]$ ,  $d > 1$  a square-free integer,  $\epsilon > 1$  the fundamental unit of  $\mathfrak{o}_K$  and  $C_2 = \langle g \rangle$ . Then

$$\mathcal{U}_1(\mathfrak{o}_K[C_2]) \cong \langle g \rangle \times \left\langle \frac{1 + \epsilon^n}{2} + \frac{1 - \epsilon^n}{2}g \right\rangle \cong C_2 \times \mathbb{Z},$$

where  $n$  is the order of  $\epsilon \pmod{2\mathfrak{o}_K}$ .

*Proof.* Let  $u \in \mathcal{U}_1(\mathfrak{o}_K[C_2])$  be a non-trivial unit. Then, there exists  $a \in \mathfrak{o}_K$  such that,  $2a - 1 = \pm\epsilon^m$  for some non-zero integer  $m$ . Since  $n$  is the order of  $\epsilon \pmod{2\mathfrak{o}_K}$ ,  $m = nq$  with  $q \in \mathbb{Z}$ . We thus have

$$\begin{aligned} u &= a + (1 - a)g \\ &= \frac{1 \pm \epsilon^m}{2} + \frac{1 \mp \epsilon^m}{2}g \end{aligned}$$

$$\begin{aligned}
&= \frac{1 \pm \epsilon^{nq}}{2} + \frac{1 \mp \epsilon^{nq}}{2} g \\
&= \left( \frac{1 + \epsilon^n}{2} + \frac{1 - \epsilon^n}{2} g \right)^q \quad \text{or} \quad g \left( \frac{1 + \epsilon^n}{2} + \frac{1 - \epsilon^n}{2} g \right)^q.
\end{aligned}$$

Hence  $\mathcal{U}_1(\mathfrak{o}_K[C_2]) \cong \langle g \rangle \times \left\langle \frac{1+\epsilon^n}{2} + \frac{1-\epsilon^n}{2} g \right\rangle \cong C_2 \times \mathbb{Z}$ .  $\square$

As an immediate consequence of the preceding analysis, we have:

**COROLLARY 3.4**

*If  $K$  is a quadratic extension of  $\mathbb{Q}$ , then  $\mathcal{U}_1(\mathfrak{o}_K[C_2])$  is a hyperbolic group.*

**COROLLARY 3.5**

*Let  $G$  be a non-cyclic elementary Abelian 2-group. Then  $\mathcal{U}_1(\mathfrak{o}_K[G])$  is hyperbolic if, and only if,  $\mathfrak{o}_K$  is imaginary.*

*Proof.* Suppose  $\mathfrak{o}_K$  is real. Since  $G$  is not cyclic, there exist  $g, h \in G$ ,  $g \neq h$ ,  $o(g) = o(h) = 2$ . By Theorem 3.3,  $\mathcal{U}_1(\mathfrak{o}_K[\langle g \rangle]) \cong C_2 \times \mathbb{Z} \cong \mathcal{U}_1(\mathfrak{o}_K[\langle h \rangle])$ . Since  $\langle g \rangle \cap \langle h \rangle = \{1\}$ ,  $\mathcal{U}_1(\mathfrak{o}_K[\langle g \rangle]) \cap \mathcal{U}_1(\mathfrak{o}_K[\langle h \rangle]) = \{1\}$ . Therefore  $\mathcal{U}_1(\mathfrak{o}_K)$  contains an Abelian group of rank 2, so it is not hyperbolic. Conversely, if  $\mathfrak{o}_K$  is imaginary, then, proceeding by induction on the order  $|G|$  of  $G$ , we can conclude that  $\mathcal{U}_1(\mathfrak{o}_K[G])$  is trivial, and hence is hyperbolic.  $\square$

For an Abelian group  $G$ , we denote by  $r(G)$  its torsion-free rank. In order to study the hyperbolicity of  $\mathcal{U}_1(\mathfrak{o}_K[G])$ , it is enough to determine the torsion-free rank  $r(\mathcal{U}_1(\mathfrak{o}_K[G]))$ . Since  $\mathcal{U}(\mathfrak{o}_K[G]) \cong \mathcal{U}(\mathfrak{o}_K) \times \mathcal{U}_1(\mathfrak{o}_K[G])$ , we have  $r(\mathcal{U}_1(\mathfrak{o}_K[G])) = r(\mathcal{U}(\mathfrak{o}_K[G])) - r(\mathcal{U}(\mathfrak{o}_K))$ . If  $K$  is an imaginary extension, then  $r(\mathcal{U}(\mathfrak{o}_K[G])) = r(\mathcal{U}_1(\mathfrak{o}_K[G]))$ , whereas if  $K$  is a real quadratic extension, then  $r(\mathcal{U}(\mathfrak{o}_K)) = 1$ , and therefore

$$r(\mathcal{U}_1(\mathfrak{o}_K[G])) = r(\mathcal{U}(\mathfrak{o}_K[G])) - 1.$$

We note that

$$\mathbb{Q}[C_n] \cong \bigoplus \sum_{d|n} \mathbb{Q}[\zeta_d],$$

where  $\zeta_d$  is a primitive  $d$ -th root of unity, and therefore, for any algebraic number field  $L$ ,

$$L[C_n] \cong \bigoplus \sum_{d|n} L \otimes_{\mathbb{Q}} \mathbb{Q}[\zeta_d].$$

We say that two groups are commensurable with each other when they contain finite index subgroups isomorphic to each other. Since the unit group  $\mathcal{U}(\mathfrak{o}_L[C_n])$  is commensurable with  $\mathcal{U}(\Lambda)$ , where  $\Lambda = \bigoplus \sum_{d|n} \mathfrak{o}_{L \otimes_{\mathbb{Q}} \mathbb{Q}[\zeta_d]}$ , we essentially need to compute the torsion-free rank of  $\mathfrak{o}_{K \otimes_{\mathbb{Q}} \mathbb{Q}[\zeta_d]}$  for the needed cases.

**PROPOSITION 3.6**

*Let  $K = \mathbb{Q}[\sqrt{d}]$ , with  $d$  a square-free integer  $\neq 1$ . The table below shows the torsion-free rank of the groups  $\mathcal{U}_1(\mathfrak{o}_K[C_n])$ ,  $n \in \{2, 3, 4, 5, 6, 8\}$ .*

$n$	$r(\mathcal{U}_1(\mathfrak{o}_K[C_n]))$	$n$	$r(\mathcal{U}_1(\mathfrak{o}_K[C_n]))$
2	0 if $d < 0$	3	1 if $d < 0, d \neq -3$
	1 if $d > 1$		0 if $d = -3$
			1 if $d > 1$
4	1 if $d < -1$	5	6 if $d < 0$
	0 if $d = -1$		2 if $d = 5$
	2 if $d > 1$		6 if $d \in \mathbb{Z}^+ \setminus \{1, 5\}$
6	2 if $d < -3$	8	4 if $d < -1$
	0 if $d = -3$		1 if $d = -1$
	3 if $d > 1$		4 if $d = 2$
	5 if $d > 2$		

In all the cases, the computation is elementary and we omit the details.

**Theorem 3.7.** *If  $K = \mathbb{Q}[\sqrt{d}]$ , with  $d$  a square-free integer  $\neq 1$ , then*

1.  $\mathcal{U}_1(\mathfrak{o}_K[C_3])$  is hyperbolic;
2.  $\mathcal{U}_1(\mathfrak{o}_K[C_4])$  is hyperbolic if, and only if,  $d < 0$ ;
3. for an Abelian group  $G$  of exponent dividing  $n > 2$ , the group  $\mathcal{U}_1(\mathfrak{o}_K[G])$  is hyperbolic if, and only if,  $n = 4$  and  $d = -1$ , or  $n = 6$  and  $d = -3$ ;
4.  $\mathcal{U}_1(\mathfrak{o}_K[C_8])$  is hyperbolic if, and only if,  $d = -1$ ;
5.  $\mathcal{U}_1(\mathfrak{o}_K[C_5])$  is not hyperbolic.

*Proof.* Proposition 3.6 gives us the torsion-free rank

$$r := r(\mathcal{U}_1(\mathfrak{o}_K[C_n]))$$

for  $n \in \{2, 3, 4, 5, 8\}$ . The group  $\mathcal{U}_1(\mathfrak{o}_K[C_n])$  is hyperbolic if, and only if,  $r \in \{0, 1\}$ . Thus, it only remains to consider the case (3).

Suppose  $n = 6$  and  $\mathcal{U}_1(\mathfrak{o}_K[G])$  is hyperbolic. We, hence, have  $r \in \{0, 1\}$ . If  $G$  is cyclic, then, by Proposition 3.6, we have  $d = -3$ . If  $G$  is not cyclic, then  $G \cong C_2^l \times C_3^m$ ,  $l, m \geq 1$ . Since  $\mathfrak{o}_K[C_3] \hookrightarrow \mathfrak{o}_K[G]$ , it follows that  $d = -3$ .

Conversely, if  $n = 6$  and  $d = -3$ , then, proceeding by induction on  $|G|$ , it can be proved that  $\mathcal{U}_1(\mathfrak{o}_K[G])$  is hyperbolic.

The case  $n = 4$  can be handled similarly. □

### PROPOSITION 3.8

*If  $K = \mathbb{Q}[\sqrt{d}]$ , with  $d$  square-free integer  $\neq 1$ , then  $\mathcal{U}_1(\mathfrak{o}_K[C_{12}])$  is not hyperbolic.*

*Proof.* Since  $K[C_{12}] \cong K \otimes_{\mathbb{Q}}[\mathbb{Q}[C_{12}]] \cong K \otimes_{\mathbb{Q}}[\mathbb{Q}[C_3 \times C_4]] \cong K[C_3 \times C_4]$ , we have the immersions  $\mathfrak{o}_K[C_3] \hookrightarrow \mathfrak{o}_K[C_{12}]$  and  $\mathfrak{o}_K[C_4] \hookrightarrow \mathfrak{o}_K[C_{12}]$ . Therefore,  $r(\mathcal{U}_1(\mathfrak{o}_K[C_{12}])) \geq r(\mathcal{U}_1(\mathfrak{o}_K[C_3])) + r(\mathcal{U}_1(\mathfrak{o}_K[C_4]))$ .

Suppose  $\mathcal{U}_1(\mathfrak{o}_K[C_{12}])$  is hyperbolic. Then, since  $r(\mathcal{U}_1(\mathfrak{o}_K[C_{12}])) < 2$ , we have, by Proposition 3.6,  $d \in \{-3, -1\}$ . We also have

$$\begin{aligned} K[C_3 \times C_4] &\cong (K[C_3])[C_4] \cong (K \oplus K[\sqrt{-3}])[C_4] \\ &\cong K[C_4] \oplus (K[\sqrt{-3}])[C_4] \\ &\cong 2K \oplus K[\sqrt{-1}] \oplus 2K[\sqrt{-3}] \oplus K[\sqrt{-3} + \sqrt{-1}]. \end{aligned}$$

Set  $\mathbb{L} = \mathbb{Q}[\sqrt{-3} + \sqrt{-1}]$  and suppose  $d = -3$ . Then  $\mathfrak{o}_K[C_{12}] \hookrightarrow 4\mathfrak{o}_K \oplus 2\mathfrak{o}_{\mathbb{L}}$  and  $r(\mathcal{U}(\mathfrak{o}_{\mathbb{L}})) = 1$ . Thus  $r(\mathcal{U}(\mathfrak{o}_K[C_{12}])) = 2$ , and we have a contradiction.

Analogously, for  $d = -1$ ,  $\mathfrak{o}_K[C_{12}] \hookrightarrow 3\mathfrak{o}_K \oplus 3\mathfrak{o}_{\mathbb{L}}$  and so  $r(\mathcal{U}(\mathfrak{o}_K[C_{12}])) = 3$ . Since the extensions are non-real, we have that  $r(\mathcal{U}_1(\mathfrak{o}_K[C_{12}])) = r(\mathcal{U}(\mathfrak{o}_K[C_{12}])) \geq 2$ , and, hence, we again have a contradiction.

We conclude that  $\mathcal{U}_1(\mathfrak{o}_K[C_{12}])$  is not hyperbolic.  $\square$

#### 4. Non-Abelian groups with hyperbolic unit groups

Theorem 2.2 classifies the finite non-Abelian groups  $G$  for which the unit group  $\mathcal{U}_1(\mathbb{Z}[G])$  is hyperbolic. These groups are:  $S_3$ ,  $D_4$ ,  $Q_{12}$ ,  $C_4 \rtimes C_4$ , and the Hamiltonian 2-group, where  $Q_{12} = C_3 \rtimes C_4$ , with  $C_4$  acting non-trivially on  $C_3$ , and also on  $C_4$  (see [13]).

Jespers, in [12], classified the finite groups  $G$  which have a normal non-Abelian free complement in  $\mathcal{U}(\mathbb{Z}[G])$ . The group algebra  $\mathbb{Q}[G]$  of these groups has at most one matrix Wedderburn component which must be isomorphic to  $M_2(\mathbb{Q})$ .

*Lemma 4.1.* *Let  $G$  be a group and  $K$  a quadratic extension. If  $M_2(K)$  is a Wedderburn component of  $K[G]$ , then  $\mathbb{Z}^2 \hookrightarrow \mathcal{U}_1(\mathfrak{o}_K[G])$ . In particular,  $\mathcal{U}_1(\mathfrak{o}_K[G])$  is not hyperbolic.*

*Proof.* The ring  $\Gamma = M_2(\mathfrak{o}_K)$  is a  $\mathbb{Z}$ -order in  $M_2(K)$  and

$$X = \{e_{12}, e_{12}\sqrt{d}\} \subset \Gamma$$

is a set of commuting nilpotent elements of index 2, where  $e_{ij}$  denotes the elementary matrix. The set  $\{1, \sqrt{d}\}$  is a linearly independent set over  $\mathbb{Q}$ , and hence so is  $X$ . Therefore, by Lemma 2.1,  $\mathbb{Z}^2 \hookrightarrow \mathcal{U}_1(\Gamma) \subset \mathcal{U}_1(\mathfrak{o}_K[G])$ , and so,  $\mathcal{U}_1(\mathfrak{o}_K[G])$  is not hyperbolic.  $\square$

#### COROLLARY 4.2

*If  $G \in \{S_3, D_4, Q_{12}, C_4 \rtimes C_4\}$ , then  $\mathcal{U}_1(\mathfrak{o}_K[G])$  is not hyperbolic.*

*Proof.* We have that  $K[G] \cong K \otimes_{\mathbb{Q}} (\mathbb{Q}[G])$ . For each of the groups under consideration,  $M_2(\mathbb{Q})$  is a Wedderburn component of  $\mathbb{Q}[G]$ ; it therefore follows that  $M_2(K)$  is a Wedderburn component of  $K[G]$ . The preceding lemma implies that  $\mathcal{U}_1(\mathfrak{o}_K[G])$  is not hyperbolic.  $\square$

If  $H$  is a non-Abelian Hamiltonian 2-group, then  $H = E \times Q_8$ , where  $E$  is an elementary Abelian 2-group and  $Q_8$  is the quaternion group of order 8. Since  $Q_8$  contains a cyclic subgroup of order 4, it follows, by Theorem 3.7, that if  $\mathcal{U}_1(\mathfrak{o}_K[Q_8])$  is hyperbolic, then  $\mathfrak{o}_K$  is not real.

#### PROPOSITION 4.3

*If  $G$  is a Hamiltonian 2-group of order greater than 8, then  $\mathcal{U}_1(\mathfrak{o}_K[G])$  is not hyperbolic.*

*Proof.* Let  $G = E \times Q_8$  with  $E$  elementary Abelian of order  $2^n > 1$ . We then have  $K[G] = K[E \times Q_8] \cong K \otimes_{\mathbb{Q}} (\mathbb{Q}[E \times Q_8]) \cong K \otimes_{\mathbb{Q}} (\mathbb{Q}[E])[Q_8] \cong K \otimes_{\mathbb{Q}} (2^n \mathbb{Q})[Q_8] \cong (2^n K)[Q_8]$ . If  $d = -1$ , it is well-known that  $KQ_8$  has a Wedderburn component isomorphic to  $M_2(K)$  and hence, by Lemma 4.1,  $\mathcal{U}_1(\mathfrak{o}_K[Q_8])$  is not hyperbolic. If  $d < -1$ , then, by Proposition 3.6,  $r(\mathcal{U}_1(\mathfrak{o}_K[C_4])) = 1$ . Since  $C_4$  is a subgroup of  $Q_8$ , it follows that  $\mathcal{U}_1((2^n \mathfrak{o}_K)[C_4])$  embeds into  $\mathcal{U}_1(\mathfrak{o}_K[G])$ . Thus, since  $\mathcal{U}_1(\prod_{2^n} \mathfrak{o}_K[C_4])$  has rank  $2^n \geq 2$ ,  $\mathcal{U}_1(\mathfrak{o}_K[G])$  is not hyperbolic.  $\square$

In view of the above Proposition, it follows that  $Q_8$  is the only Hamiltonian 2-group for which  $\mathcal{U}_1(\mathfrak{o}_K[G])$  can possibly be hyperbolic, and in this case  $\mathfrak{o}_K$  is the ring of integers of an imaginary extension. By Lemma 4.1,  $K[Q_8]$  can not have a matrix ring as a Wedderburn component. Since  $\mathbb{Q}[Q_8] \cong 4\mathbb{Q} \oplus \mathcal{H}(\mathbb{Q})$ , we have  $K[Q_8] \cong K \otimes_{\mathbb{Q}} (4\mathbb{Q} \oplus \mathcal{H}(\mathbb{Q})) \cong 4K \oplus \mathcal{H}(K)$ ; hence  $K[Q_8]$  must be a direct sum of division rings, or equivalently, has no non-zero nilpotent elements. In particular,  $\mathcal{H}(K)$  is a division ring.

**Theorem 4.4.** *Let  $K = \mathbb{Q}[\sqrt{d}]$ , with  $d$  square-free integer  $\neq 1$ . Then  $K[Q_8]$  is a direct sum of division rings if, and only if, one of the following holds:*

- (i)  $d \equiv 1 \pmod{8}$ ;
- (ii)  $d \equiv 2, \text{ or } 3 \pmod{4}$ , or  $d \equiv 5 \pmod{8}$ , and  $d > 0$ .

*Proof.* The assertion follows from Theorem 2.3 of [1]; Theorem 1, p. 236 of [3] and Theorem 3.2 of [16].  $\square$

#### COROLLARY 4.5

*If  $K = \mathbb{Q}[\sqrt{d}]$ , where  $d$  is a negative square-free integer, then the group  $\mathcal{U}_1(\mathfrak{o}_K[Q_8])$  is not hyperbolic if  $d \not\equiv 1 \pmod{8}$ .*

Let  $\mathbb{H}: \mathbb{C} \times ]0, \infty[$  be the upper half-space model of three-dimensional hyperbolic space and  $\text{Iso}(\mathbb{H})$  its group of isometries. In the quaternion algebra  $\mathcal{H} := \mathcal{H}(-1, -1)$  over  $\mathbb{R}$ , with its usual basis, we may identify  $\mathbb{H}$  with the subset  $\{z + rj : z \in \mathbb{C}, r \in \mathbb{R}^+\}$ . The group  $PSL(2, \mathbb{C})$  acts on  $\mathbb{H}$  in the following way:

$$\begin{aligned} \varphi: PSL(2, \mathbb{C}) \times \mathbb{H} &\longrightarrow \mathbb{H} \\ (M, P) &\longmapsto \begin{pmatrix} a & b \\ c & d \end{pmatrix} P := (aP + b)(cP + d)^{-1}, \end{aligned}$$

where  $(cP + d)^{-1}$  is calculated in  $\mathcal{H}$ . Explicitly,  $MP = M(z + rj) = z^* + r^*j$ , with

$$z^* = \frac{(az + b)(\bar{c}\bar{z} + \bar{d}) + a\bar{c}r^2}{|cz + d|^2 + |c|^2r^2} \quad \text{and} \quad r^* = \frac{r}{|cz + d|^2 + |c|^2r^2}.$$

Let  $K$  be an algebraic number field and  $\mathfrak{o}_K$  its ring of integers. Let

$$SL_1(\mathcal{H}(\mathfrak{o}_K)) := \{x \in \mathcal{H}(\mathfrak{o}_K) : N(x) = 1\},$$

where  $N$  is the norm in  $\mathcal{H}(K)$ . Clearly the groups  $\mathcal{U}(\mathcal{H}(\mathfrak{o}_K))$  and  $\mathcal{U}(\mathfrak{o}_K) \times SL_1(\mathcal{H}(\mathfrak{o}_K))$  are commensurable. Consider the subfield  $F = K[i] \subset \mathcal{H}(K)$  which is a maximal subfield in  $\mathcal{H}(K)$ . The inner automorphism  $\sigma$ ,

$$\begin{aligned} \sigma: \mathcal{H}(K) &\longrightarrow \mathcal{H}(K) \\ x &\longmapsto jxj^{-1}, \end{aligned}$$

fixes  $F$ . The algebra  $\mathcal{H}(K) = F \oplus Fj$  is a crossed product and embeds into  $M_2(\mathbb{C})$  as follows:

$$\begin{aligned} \Psi: \mathcal{H}(K) &\hookrightarrow M_2(\mathbb{C}) \\ x + yj &\longmapsto \begin{pmatrix} x & y \\ -\sigma(y) & \sigma(x) \end{pmatrix}. \end{aligned} \tag{4}$$

This embedding enables us to view  $SL_1(\mathcal{H}(\mathfrak{o}_K))$  and  $SL_1(\mathcal{H}(K))$  as subgroups of  $SL(2, \mathbb{C})$  and hence  $SL_1(\mathcal{H}(K))$  acts on  $\mathbb{H}$ .



## PROPOSITION 4.6

Let  $K = \mathbb{Q}[\sqrt{d}]$ ,  $d \equiv 1 \pmod{8}$  a square-free negative integer, and  $\mathfrak{o}_K$  its ring of integers. Then  $\mathcal{U}(\mathcal{H}(\mathfrak{o}_K))$  and  $\mathcal{U}(\mathfrak{o}_K[Q_8])$  are hyperbolic groups.

*Proof.* Observe that  $SL_1(\mathcal{H}(\mathfrak{o}_K))$  acts on the space  $\mathbb{H}$  and, hence, is a discrete subgroup of  $SL_2(\mathbb{C})$  (see Theorem 10.1.2, p. 446 of [6]). The quotient space  $Y := \mathbb{H}/SL_1(\mathcal{H}(\mathfrak{o}_K))$  is a Riemannian manifold of constant curvature  $-1$  and, since  $\mathbb{H}$  is simply connected, we have that  $SL_1(\mathcal{H}(\mathfrak{o}_K)) \cong \pi_1(Y)$ . Since  $d \equiv 1 \pmod{8}$ ,  $\mathcal{H}(K)$  is a division ring and, therefore, co-compact and  $Y$  is compact (see Theorem 10.1.2, item (3) of [6]). Hence  $SL_1(\mathcal{H}(\mathfrak{o}_K))$  is hyperbolic (see Example 2.25.5 of [2]). Since  $\mathcal{U}(\mathcal{H}(\mathfrak{o}_K))$  and  $\mathcal{U}(\mathfrak{o}_K) \times SL_1(\mathcal{H}(\mathfrak{o}_K))$  are commensurable and  $\mathcal{U}(\mathfrak{o}_K) = \{-1, 1\}$ , it follows that  $\mathcal{U}(\mathcal{H}(\mathfrak{o}_K))$  is hyperbolic. Since  $\mathcal{U}(\mathfrak{o}_K[Q_8]) \cong \mathcal{U}(\mathfrak{o}_K) \times \mathcal{U}(\mathfrak{o}_K) \times \mathcal{U}(\mathfrak{o}_K) \times \mathcal{U}(\mathfrak{o}_K) \times \mathcal{U}(\mathcal{H}(\mathfrak{o}_K))$  and  $\mathcal{U}(\mathfrak{o}_K) \cong C_2$ , we conclude that  $\mathcal{U}(\mathfrak{o}_K[Q_8])$  is hyperbolic.  $\square$

Combining the results in the present and the preceding section, we have the following main result.

**Theorem 4.7.** Let  $K = \mathbb{Q}[\sqrt{d}]$ , with  $d$  square-free integer  $\neq 1$ , and  $G$  a finite group. Then  $\mathcal{U}_1(\mathfrak{o}_K[G])$  is hyperbolic if, and only if,  $G$  is one of the groups listed below and  $\mathfrak{o}_K$  (or  $K$ ) is determined by the corresponding value of  $d$ :

1.  $G \in \{C_2, C_3\}$  and  $d$  arbitrary;
2.  $G$  is an Abelian group of exponent dividing  $n$  for:  $n = 2$  and  $d < 0$ , or  $n = 4$  and  $d = -1$ , or  $n = 6$  and  $d = -3$ .
3.  $G = C_4$  and  $d < 0$ .
4.  $G = C_8$  and  $d = -1$ .
5.  $G = Q_8$  and  $d < 0$  and  $d \equiv 1 \pmod{8}$ .

*Remark.* If the group  $\mathcal{U}(\mathfrak{o}_K[Q_8])$  is hyperbolic, then the hyperbolic boundary  $\partial(\mathcal{U}(\mathfrak{o}_K[Q_8])) \cong \mathbb{S}^2$ , the Euclidean sphere of dimension 2, and  $\text{End}(\mathcal{U}(\mathfrak{o}_K[Q_8]))$  has one element (see Example 2.25.5 of [2]). Note that if  $\mathcal{U}(\mathbb{Z}[G])$  is an infinite non-Abelian hyperbolic group, then  $\partial(\mathcal{U}(\mathbb{Z}[G]))$  is totally disconnected and is a Cantor set. So, in this case,  $\mathcal{U}(\mathbb{Z}[G])$  has infinitely many ends and also is a virtually free group, (Theorem 2 of [13] and §3 of [8]). However, if  $\mathcal{U}(\mathfrak{o}_K[G])$  is a non-Abelian hyperbolic group, then  $\mathcal{U}(\mathfrak{o}_K[G])$  is an infinite group which is not virtually free, it has one end and  $\partial(\mathcal{U}(\mathbb{Z}[G]))$  is a smooth manifold.

## 5. Pell and Gauss units

When the algebra  $\mathcal{H}(K)$  is isomorphic to  $M_2(K)$ , it is known how to construct the unit group of a  $\mathbb{Z}$ -order up to a finite index. Nevertheless, if  $\mathcal{H}(K)$  is a division ring, this is a highly non-trivial task; see [5], for example. In this section we study a construction of units of  $\mathcal{U}(\mathcal{H}(\mathfrak{o}_K))$  in the case when the quaternion algebra  $\mathcal{H}(K)$  is a division ring.

In the sequel,  $K = \mathbb{Q}[\sqrt{-d}]$  is an imaginary quadratic extension with  $d$  a square-free integer congruent to  $7 \pmod{8}$ , and  $\mathfrak{o}_K$  the ring of integers of the field  $K$ . Note that  $s(K)$ , the *stufe* of  $K$ , is 4, the quaternion algebra  $\mathcal{H}(K)$  is a division ring and  $\mathcal{U}(\mathfrak{o}_K) = \{\pm 1\}$ . Thus, if  $u = u_1 + u_i i + u_j j + u_k k \in \mathcal{U}(\mathcal{H}(\mathfrak{o}_K))$ , then its norm  $N(u) = u_1^2 + u_i^2 + u_j^2 + u_k^2 = \pm 1$ ; furthermore, if any of the coefficients  $u_1, u_i, u_j, u_k$  is zero, then  $N(u) = 1$ ,  $s(K)$  being 4.

The representation of  $u$ , given by (4), is

$$[u] := \Psi(u) = \begin{pmatrix} u_1 + u_i i & u_j + u_k i \\ -u_j + u_k i & u_1 - u_i i \end{pmatrix} \in M_2(\mathbb{C}).$$

Denote by  $\chi_u$  the characteristic polynomial of  $[u]$ , and by  $m_u$  its minimal polynomial. The degree  $\partial(\chi_u)$  of  $\chi_u$  is 2 and therefore  $\partial(m_u) \leq 2$ . If  $\partial(m_u) = 1$ , then  $m_u(X) = X - z_0$ ,  $z_0 \in \mathbb{C}$ , and therefore  $u = z_0$ . Note that the characteristic polynomial is  $\chi_u(X) = X^2 - \text{trace}([u])X + \det([u])$ , where  $\text{trace}([u]) = u_1 + u_i i + \sigma(u_1 + u_i i) = 2u_1$  and  $\det([u]) = \pm 1$ :

$$\chi_u(X) = X^2 - 2u_1 X \pm 1.$$

### PROPOSITION 5.1

Let  $u = u_1 + u_i i + u_j j + u_k k \in \mathcal{U}(\mathcal{H}(\mathfrak{o}_K))$ . Then the following statements hold:

1.  $u^2 = 2u_1 u - N(u)$ .
2. If  $N(u) = 1$ , then  $u$  is a torsion unit if, and only if,  $u_1 \in \{-1, 0, 1\}$  and the order of  $u$  is 1, 2, or 4.
3. If  $N(u) = -1$ , then order of  $u$  is infinite.

*Proof.*

(1) is obvious.

(2) Suppose  $N(u) = 1$  and  $u$  is a torsion unit of order  $n$ , say. If  $X^2 - 2u_1 X + \eta(u) = (X - \zeta_1)(X - \zeta_2)$ , then  $\zeta_i$ ,  $i = 1, 2$  are roots of unity and  $\zeta_1 \zeta_2 = 1$ . It follows that  $2u_1 = \zeta_1 + \zeta_2$  is a real number. Since  $u \in \mathcal{H}(\mathfrak{o}_K)$  and  $\{1, \vartheta\}$  is an integral basis of  $\mathfrak{o}_K$ , it follows that  $u_1 \in \mathbb{Z}$ . From the equality  $2u_1 = \zeta_1 + \zeta_2$ , we have  $2|u_1| = |\zeta_1 + \zeta_2| \leq 2$ , and therefore  $u_1 \in \{-1, 0, 1\}$ . If  $u_1 = 0$ , then  $u^2 = -1$  and therefore  $o(u) = 4$ . If  $u_1 = \pm 1$ , then  $\chi_u(X) = X^2 \mp 2X + 1 = (X \mp 1)^2$ , and therefore  $0 = \chi_u(u) = (u \mp 1)^2 \in \mathcal{H}(K)$ ; hence  $u = \pm 1$ .

(3) If  $N(u) = -1$ , then  $u^2 = 2u_1 u + 1$ ,  $(u^2)_1 = 2u_1^2 + 1$ ,  $\eta(u^2) = 1$ . If  $u$  were a torsion unit, then, by (2) above,  $(u^2)_1 \in \{-1, 0, 1\}$ . If  $(u^2)_1 = 0$ , then  $1/2 = -u_1^2 \in \mathfrak{o}_K$ , which is not possible. If  $(u^2)_1 = 1$ , then  $u_1 = 0$ , and therefore  $u^2 = 1$  yielding  $u = \pm 1$  which is not the case, because  $N(u) = -1$ . Finally, if  $(u^2)_1 = -1$ , then  $u_1^2 = -1$  which implies that  $\sqrt{-1} \in K$  which is also not the case, because  $\mathcal{H}(K)$  is a division ring. Hence  $u \in \mathcal{U}(\mathfrak{o}_K)$  is an element of infinite order.  $\square$

Let  $\xi \neq \psi$  be elements of  $\{1, i, j, k\}$ . Suppose

$$u := m\sqrt{-d}\xi + p\psi, \quad p, m \in \mathbb{Z}, \tag{5}$$

is an element in  $\mathcal{H}(\mathfrak{o}_K)$  having norm 1. Then

$$p^2 - m^2 d = 1, \tag{6}$$

i.e.,  $(p, m)$  is a solution of the Pell's equation  $X^2 - dY^2 = 1$ . Let  $\mathbb{L} := \mathbb{Q}[\sqrt{d}]$ . Equation (6) implies that  $\epsilon = p + m\sqrt{d}$  is a unit in  $\mathfrak{o}_{\mathbb{L}}$ . Conversely, if  $\epsilon = p + m\sqrt{d}$  is a unit of norm 1 in  $\mathfrak{o}_{\mathbb{L}}$ , then, necessarily,  $p^2 - m^2 d = 1$ , and, therefore, for any choice of  $\xi, \psi$  in  $\{1, i, j, k\}$ ,  $\xi \neq \psi$ ,

$$m\sqrt{-d}\xi + p\psi \quad (7)$$

is a unit in  $\mathcal{H}(\mathfrak{o}_K)$ . In particular,

$$u_{(\epsilon, \psi)} := p + m\sqrt{-d}\psi, \quad \psi \in \{i, j, k\} \quad (8)$$

is a unit in  $\mathcal{H}(\mathfrak{o}_K)$ .

With the notations as above, we have:

**PROPOSITION 5.2**

1. If  $1 \notin \text{supp}(u)$ , the support of  $u$ , then  $u$  is a torsion unit.
2. If  $\epsilon = p + m\sqrt{d}$  is a unit in  $\mathfrak{o}_{\mathbb{L}}$ , then

$$u_{(\epsilon, \psi)}^n = u_{(\epsilon^n, \psi)}$$

for all  $\psi \in \{i, j, k\}$  and  $n \in \mathbb{Z}$ .

*Proof.* If  $1 \notin \text{supp}(u)$ , then  $u_1 = 0$ ; therefore, by Proposition 5.1,  $u$  is torsion unit.

Let  $\mu = A + B\sqrt{d}$  and  $\nu = C + D\sqrt{d}$ , be units in  $\mathfrak{o}_{\mathbb{L}}$ . Then  $u_{(\mu, \psi)} = A + B\sqrt{-d}\psi$  and  $u_{(\nu, \psi)} = C + D\sqrt{-d}\psi$  are units in  $\mathcal{H}(\mathfrak{o}_K)$ . We have

$$\mu\nu = AC + dBD + (AD + BC)\sqrt{d}.$$

Also  $u_{(\mu, \psi)}u_{(\nu, \psi)} = (AC + dBD) + (AD + BC)\sqrt{-d}\psi = u_{(\mu\nu, \psi)}$ . It follows that we have  $u_{(\epsilon, \psi)}^n = u_{(\epsilon^n, \psi)}$  for all  $\psi \in \{i, j, k\}$  and  $n \in \mathbb{Z}$ .  $\square$

The units (7) constructed above are called *2-Pell units*.

**PROPOSITION 5.3**

Let  $L = \mathbb{Q}[\sqrt{2d}]$ ,  $2d$  square-free,  $\xi, \psi, \phi$  pairwise distinct elements in  $\{1, i, j, k\}$  and  $p, m \in \mathbb{Z}$ . Then the following are equivalent:

- (i)  $u := m\sqrt{-d}\xi + p\psi + (1-p)\phi \in \mathcal{U}(\mathcal{H}(\mathfrak{o}_K))$ .
- (ii)  $\epsilon := (2p-1) + m\sqrt{2d} \in \mathcal{U}(\mathfrak{o}_L)$ .

*Proof.* If  $u$  is a unit in  $\mathcal{H}(\mathfrak{o}_K)$ , then  $N(u) = -m^2d + p^2 + (1-p)^2 = 1$ , i.e.,  $2p^2 - 2p - m^2d = 0$ , and thus  $(2p-1)^2 - m^2d = 1$ . Consequently,  $\epsilon = (2p-1) + m\sqrt{2d}$  is invertible in  $\mathfrak{o}_L$ . The steps being reversible, the equivalence of (i) and (ii) follows.  $\square$

The units constructed above are called *3-Pell units*. We shall next determine units of the form  $u = m\sqrt{-d} + (m\sqrt{-d})i + pj + qk$ , with  $m, p, q \in \mathbb{Z}$  and  $N(u) = -2m^2d + p^2 + q^2 = 1$ . Set  $p + q =: r$  and consider the equation

$$2p^2 - 2pr - 2m^2d + r^2 - 1 = 0. \quad (9)$$

**Theorem 5.4.** *If  $r = 1$ , then equation (9) has a solution in  $\mathbb{Z}$ , and for each such solution,  $u = m\sqrt{-d} + (m\sqrt{-d})i + pj + qk$  is a unit in  $\mathcal{H}(\mathfrak{o}_K)$  of norm 1.*

*Proof.* Viewed as a quadratic equation in  $p$ , (9) has real roots

$$p = \frac{1 \pm \sqrt{1 + 4m^2d}}{2}.$$

To obtain a solution in  $\mathbb{Z}$ , we need the argument under the radical to be a square; we thus need to solve the diophantine equation

$$X^2 - 4dY^2 = 1. \quad (10)$$

Let  $\epsilon = x + y\sqrt{d}$ , with  $x, y \in \mathbb{Z}$ , be a unit in  $\mathfrak{o}_{\mathbb{L}}$  having infinite order. Replacing  $\epsilon$  by  $\epsilon^2$ , if necessary, we can assume that  $y$  is even. We then have  $x^2 - y^2d = 1$ , and so  $x$  must be odd. Taking  $m = y/2$  and  $p = \frac{1 \pm x}{2}$ , we obtain a solution of (10) in  $\mathbb{Z}$ . Clearly, for such a solution, the element  $u$  lies in  $\mathcal{H}(\mathfrak{o}_K)$  and has norm 1.  $\square$

Using Gauss' result which states that a positive integer  $n$  is a sum of three squares if, and only if,  $n$  is not of the form  $4^a(8b - 1)$ , where  $a \geq 0$  and  $b \in \mathbb{Z}$ , it is easy to see that, for every integer  $m \equiv 2 \pmod{4}$ , the integers  $m^2d - 1$  and  $m^2d + 1$  can be expressed as sums of three squares. We can thus construct units  $u = m\sqrt{-d} + pi + qj + rk \in \mathcal{H}(\mathfrak{o}_K)$  having prescribed norm 1 or  $-1$ ; we call such units *Gauss units*.

*Example.* In [5], all units exhibited in  $\mathcal{H}(\mathfrak{o}_{\mathbb{Q}[\sqrt{-7}]})$  are of norm 1. We present some units of norm  $-1$  in this ring. The previous theorem guarantees the existence of integers  $p, q, r$ , such that

$$u = 6\sqrt{-7} + pi + qj + rk$$

is a unit of norm  $-1$ . Indeed,

$$(p, q, r) \in \{(\pm 15, \pm 5, \pm 1), (\pm 13, \pm 9, \pm 1), (\pm 11, \pm 11, \pm 3)\},$$

and the triples obtained by permutation of coordinates, are all possible integral solutions. In [5], the authors have constructed a set  $S$  of generators of the group  $SL_1(\mathcal{H}(\mathfrak{o}_{\mathbb{Q}[\sqrt{-7}]})$ . If  $v_0$  is a unit of  $\mathcal{H}(\mathfrak{o}_{\mathbb{Q}[\sqrt{-7}]})$  having norm  $-1$ , then clearly  $\langle v_0, S \rangle = \mathcal{U}(\mathcal{H}(\mathfrak{o}_{\mathbb{Q}[\sqrt{-7}]})$ . Thus, for example, taking  $v_0 = 6\sqrt{-7} + 15i + 5j + k$ , we have

$$\mathcal{U}(\mathcal{H}(\mathfrak{o}_{\mathbb{Q}[\sqrt{-7}]}) = \langle v_0, S \rangle. \quad (11)$$

The set  $\{1, \frac{1+\sqrt{-7}}{2}\}$  is an integral basis of  $R = \mathbb{Z}[\frac{1+\sqrt{-7}}{2}]$ . Consider units of the form

$$\frac{m + \sqrt{-d}}{2} \pm \left( \frac{m - \sqrt{-d}}{2} \right) i + pj.$$

These are neither Pell nor Gauss units. Those of norm  $\pm 1$ , are solutions of the equation

$$m^2 + 2p^2 = \pm 2 + d \quad (12)$$

in  $\mathbb{Z}$ . The main result of [5] states that if  $d = 7$ , then the units of norm 1 of the above type, together with the trivial units  $i$  and  $j$ , generate the group  $SL_1(\mathcal{H}(R))$ .

For  $d \equiv 7 \pmod{8}$ , there are no units of norm  $-1$  of the above type, since, in this case, the equation  $m^2 + 2p^2 = -2 + d$  has no solution in  $\mathbb{Z}$ , as can be easily seen working module 8.

In case  $d \neq 7$ , we give some more examples of negative norm units of the form  $\frac{m+\sqrt{-d}}{2} \pm \left(\frac{m-\sqrt{-d}}{2}\right)i + pj$ .

If  $d = 15$ , then eq. (12) becomes  $m^2 + 2p^2 = 17$ ; the pairs  $(m, p) \in \{(3, 2), (3, -2), (-3, 2), (-3, -2)\}$  are its integral solutions. For  $m = 3$  either  $p = 2$  or  $p = -2$  and so there are 8 units. Each coefficient of  $u$  is distinct, hence for each solution  $(m, p)$  there are  $3!$  units with the same support, thus there are 36 different units for a given fixed support. By Proposition 5.1, all these units have infinite order if  $u_1 \notin \{-1, 0, 1\}$ . If  $1 \in \text{supp}(u)$ , then either  $\{i, j\} \subset \text{supp}(u)$  or  $\{i, k\} \subset \text{supp}(u)$ , or  $\{j, k\} \subset \text{supp}(u)$ . Therefore there are 108 of these units and, for example,

$$\frac{3 + \sqrt{-15}}{2} + \left(\frac{3 - \sqrt{-15}}{2}\right)j - 2k$$

is one of them.

If  $1 \notin \text{supp}(u)$ , then  $u$  is a torsion unit, so there are 36 torsion units of this type. One of them is the unit

$$\left(\frac{-3 - \sqrt{-15}}{2}\right)i + \left(\frac{-3 + \sqrt{-15}}{2}\right)j + 2k,$$

of order 4.

For  $d = 31$ , we obtain  $m^2 + 2p^2 = 33$  whose solutions in  $\mathbb{Z}$  are:  $(m, p) \in \{(1, 4), (1, -4), (-1, 4), (-1, -4)\}$ .

As another example of a unit of norm  $-1$  in a quaternion algebra, we may mention that, in  $\mathcal{H}(\mathfrak{o}_{\mathbb{Q}[\sqrt{-23}]})$ ,  $u = 5\sqrt{-23} + 23i + 6j + 3k$  is a unit of norm  $-1$ .

We next exhibit some Gauss units of norm 1. For  $\mathcal{H}(\mathfrak{o}_{\mathbb{Q}[\sqrt{-15}]})$ , there exist  $p, q, r$ , such that  $u = 10\sqrt{-15} + pi + qj + rk$  is a unit of norm 1. In fact,  $(36, 14, 3), (36, 13, 6), (32, 21, 6), (30, 24, 5)$  are some of the possible choices for  $(p, q, r)$ . For  $\mathcal{H}(\mathfrak{o}_{\mathbb{Q}[\sqrt{-23}]})$ ,  $u = 2\sqrt{-23} + 8i + 5j + 2k$  is a unit of norm 1. It is interesting to note that  $u = 3588\sqrt{-23} + 12168i + 12167j$  is a Gauss unit, although 4 divides 3588.

We conclude with the following result:

**Theorem 5.5.** *Let  $K = \mathbb{Q}[\sqrt{-d}]$ ,  $0 < d \equiv 7 \pmod{8}$  and  $\mathfrak{o}_K$  the ring of integers of  $K$ . If  $\epsilon = p + m\sqrt{d}$  is a unit in  $\mathbb{Z}[\sqrt{d}]$ , and  $x := u_{(\epsilon, \psi)}$ ,  $y := u_{(\epsilon, \psi')}$  are two 2-Pell units in  $\mathcal{U}(\mathcal{H}(\mathfrak{o}_K))$ , where  $\psi$  and  $\psi' \in \{i, j, k\}$  and  $\psi \neq \psi'$ , then there exists a natural number  $m$  such that  $\langle x^m, y^m \rangle$  is a free group of rank 2.*

*Proof.* By Proposition 4.6,  $\mathcal{U}(\mathcal{H}(\mathfrak{o}_K))$  is a hyperbolic group. In view of Proposition III.Γ. 3.20 of [4], there exists a natural  $m$ , such that,  $\langle x^m, y^m \rangle$  is a free group of rank at most 2. However, Proposition 5.2, item (2) ensures that  $\langle x \rangle \cap \langle y \rangle = \{1\}$ . Therefore,  $\langle x^m, y^m \rangle$  has rank at least 2, and hence  $\langle x^m, y^m \rangle$  is a free group of rank 2. □

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