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Collective dynamics of delay-coupled limit cycle oscillators

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Abstract. Coupled limit cycle oscillators with instantaneous mutual coupling offer a useful but idealized mathematical paradigm for the study of collective behavior in a wide variety of biological, physical and chemical systems. In most real-life systems however the interaction is not instantaneous but is delayed due to finite propagation times of signals, reaction times of chemicals, individual neuron firing periods in neural networks etc. We present a brief overview of the effect of time-delayed coupling on the collective dynamics of such coupled systems. Simple model equations describing two oscillators with a discrete time-delayed coupling as well as those describing linear arrays of a large number of oscillators with time-delayed global or local couplings are studied. Analytic and numerical results pertaining to time delay induced changes in the onset and stability of amplitude death and phase-locked states are discussed. A number of recent experimental and theoretical studies reveal interesting new directions of research in this field and suggest exciting future areas of exploration and applications.

Keywords. Coupled oscillators; time delay; Hopf bifurcation; amplitude death, synchronization, phase locking.

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1. Introduction

A system of coupled non-linear oscillators is capable of displaying a rich variety of collective states many of which resemble cooperative phenomena seen in real-life situations. This has motivated a great amount of theoretical and experimental interest in mathematical models based on coupled oscillator systems and in their application to a wide variety of physical and biological problems [1–6]. Examples of such applications include interactions of arrays of Josephson junctions [7,8], semiconductor lasers [9,10], charge density waves [11], phase locking of relativistic magnetrons [12], Belousov–Zhabotinskii reactions in coupled Brusselator models [2,13–15] and neural oscillator models for circadian pacemakers [16]. One of the most common occurrences of cooperative phenomena, that was first highlighted by Winfree [1] in a simple model of weakly coupled limit cycle oscillators, is that of frequency entrainment or synchronization of the diverse frequencies of the oscillator assembly to a single common frequency [17,18]. This happens in a spontaneous and abrupt fashion once the coupling strength exceeds a critical threshold. Real-life examples of such behavior abound in nature, e.g., the synchronous flashing of a swarm of fire flies, chirping of crickets in unison, and electrical synchrony in cardiac cells. When the coupling between the oscillators is comparable to the attraction to their individual limit cycles, other interesting phenomena can occur [19–22] which involve the amplitudes of the individual oscillators. For example, if the coupling is sufficiently strong and the spread in the natural frequencies of the oscillators sufficiently broad, the oscillators can suffer an amplitude quenching or death [5,23,24]. Such behavior has been observed in experiments of coupled chemical oscillator systems, e.g. coupled Belousov–Zhabotinskii reactions carried out in coupled tank reactors [25]. Other collective phenomena that these coupled oscillator models display include partial synchronization, phase trapping, large-amplitude Hopf oscillations and even chaotic behavior [24,26] – all of which have been discussed widely in the literature.

In most of the model calculations, the interaction time between oscillators is taken to be zero, i.e. the coupling is assumed to be instantaneous. This leads to considerable mathematical simplification and makes both analytical and numerical investigations of the system a lot easier. The approximation is also often justified on physical grounds in cases where the interaction time is much shorter than other natural time-scales in the system, such as the individual oscillator periods, and the collective oscillation periods. However, as is well-known from the study of single-delay differential equations, finite time delay can introduce fundamental changes in the nature of solutions and bring about novel effects that are absent in a non-delayed system. Despite the vast and well-developed literature on delay-differential equations and the considerable recent developments in the field of coupled oscillator research there have been surprisingly very few attempts at incorporating time-delay effects in coupled oscillator systems. Some notable exceptions are the work of Schuster and Wagner [27], Niebur *et al* [28], Nakamura *et al* [29] and Kim *et al* [30], who in the past have looked at time-delay effects in the context of phase-only coupled oscillator models and found interesting effects like the existence of higher frequency states and changes in the onset conditions, and nature of synchronization. More recently, we have investigated a variety of model systems starting from a simple case of just two oscillators with a discrete time-delayed coupling to a large number of oscillators with time-delayed global, and local couplings. In this report we present a brief overview of the effect of time-delayed coupling on the collective dynamics of such coupled systems based primarily on the analytic and numerical results obtained by us in [31–34]. Time delay is found to introduce significant changes in the character and onset properties of the various collective regimes such as amplitude death and phase locked states. Some of the results are novel and somewhat surprising – such as time delay induced death in an assembly of identical oscillators – and may have important applications. The physical reason for the existence of time-delayed couplings can be due to a variety of factors such as finite propagation speeds of signals, for example, or from finite processing times in synapses, finite reaction times in chemical processes and so on. Time delay is ubiquitous in most physical and biological systems like optical bistable devices [35], electromechanical systems [36], predator–prey models [37], and physiological systems [38,39]. With growing recognition of the significance and ubiquity of time delay in various systems, there is now a considerable increase in the number of investigations on this topic. Many of these experimental and theoretical studies reveal interesting new directions of research in this field and suggest exciting future areas of exploration and applications. In the spirit of evolving a 'perspective' we provide a brief description of a few of these works.

The paper is organized as follows: Section 2 is devoted to a discussion of the basic two-oscillator model and a comparison of the collective states in the absence and presence of time-delayed coupling. The principal findings of the two-oscillator model are further explored in §3 on an extended model of N oscillators with two different kinds of delayed couplings, namely global and local (nearest neighbor). The final section is devoted to a discussion on open problems as well as some recent experimental and theoretical work which point towards exciting new areas of research and applications in the field.

2. Two-oscillator model

The basic features of the collective states of a coupled limit-cycle oscillator system are well-illustrated by first studying the solutions for just two coupled oscillators without time delay. A convenient model oscillator for this purpose is the so-called Stuart–Landau oscillator that has the generic form

$$\dot{Z}(t) = (a + i\omega - |Z(t)|^2)Z(t), \tag{1}$$

where Z is a complex amplitude and ω and a are real constants. Rewriting eq. (1) in polar coordinates (i.e. $Z = r\exp(i\theta)$), we have

$$\dot{r} = r(a - r^2), \tag{2}$$

$$\dot{\theta} = \omega. \tag{3}$$

From (2) it is easy to see that for $a < 0$, the system has a single fixed point $r = 0$ which is linearly stable and corresponds to a state of no oscillation. For $a > 0$, the origin $r = 0$ becomes linearly unstable and the system acquires an oscillatory state – a limit-cycle oscillation with amplitude $r = \sqrt{a}$ and frequency ω. Such a change of behavior from a stable fixed point to a stable limit-cycle state is termed a Hopf bifurcation and the point in parametric space where this occurs ($a = 0$ in this case) is called the bifurcation point. Since the bifurcation to the limit-cycle state occurs after the critical point (as one moves in the positive direction from $a < 0$ to $a > 0$), this is classified as a supercritical Hopf bifurcation. Consider now a situation where two such oscillators are linearly coupled to each other. A detailed mathematical study of the various solutions that arise from such a case was undertaken by Aronson *et al* [23] who examined the existence and stability of amplitude death, phase-locked, phase-drift, and other solutions. For the purpose of illustration and to introduce the basic equations for later sections, we review here briefly the three solutions that emerge from diffusive coupling between two such oscillators with $a = 1$ so that individually each oscillator is in the regime of limit-cycle oscillations. The model equations are

$$\dot{Z}_1(t) = (1 + i\omega_1 - |Z_1(t)|^2)Z_1(t) + K[Z_2(t) - Z_1(t)], \quad (4)$$

$$\dot{Z}_2(t) = (1 + i\omega_2 - |Z_2(t)|^2)Z_2(t) + K[Z_1(t) - Z_2(t)], \quad (5)$$

where Z_j is the complex amplitude of the jth oscillator. Each oscillator when uncoupled has a stable limit cycle of unit amplitude $|Z_j| = 1$ with angular frequency ω_j. $K \geq 0$ is a measure of the linear coupling strength between the two oscillators. The coupled equations represent the interaction between weakly non-linear oscillators (that are near a supercritical Hopf bifurcation) whose coupling strength is comparable to the attraction of the limit cycles. It is important then to retain both the phase and amplitude response of the oscillators. These equations have a variety of stationary and non-stationary solutions which depend on the strength of the coupling parameter K and the frequency mismatch between the oscillators $\Delta = |\omega_1 - \omega_2|$. For extremely weak coupling ($K \to 0$) and large Δ, the oscillators behave independently and the long-term behavior is a non-stationary incoherent state in which the relative phase of the two oscillators moves through all phases. Such a state is called a phase-drift state. With increasing coupling strength, two important classes of stationary solutions emerge. One of them is amplitude death in which the oscillators pull each other off their limit cycles and collapse into the origin ($|Z_1| = |Z_2| = 0$) as $t \to \infty$. The other collective state is called *frequency locking* or *mutual entrainment* in which the two oscillators synchronize to a common frequency and the time asymptotic state is one of coherent or collective oscillation.

Amplitude death corresponds to the stability of the stationary states $Z_1 = Z_2 = 0$. A linear stability analysis around this state results in the following condition for amplitude death to occur, relating coupling strength with the frequency dispersion: $1 < K < \gamma(\Delta) = \frac{1}{2}(1 + \frac{\Delta^2}{4})$. Region I in figure 1A represents this boundary. The phase-locked solutions can be described by the representation, $(r_1(t), r_2(t), \theta_1(t), \theta_2(t)) = (R_1, R_2, \Omega t - \alpha/2, \Omega t + \alpha/2)$ where $R_{1,2}$ are constants. Substituting such a form in (4) and (5), one obtains the two possible equilibrium solutions that are given by (i) $R_1^2 = R_2^2 = \rho^2$ which is called the symmetric equilibrium and (ii) $R_1^2 + R_2^2 = 1 - K$, the asymmetric case. Both these equilibria have been studied in detail by Aronson *et al* [23]. The symmetric phase-locked equilibria are given by $\rho_\pm^2 = 1 - K \pm \sqrt{K^2 - \Delta^2/4}$, $\Omega = (\omega_1 + \omega_2)/2$ and $\alpha_\pm$, where $\alpha_+ = \sin^{-1}(\Delta/(2K))$, and $\alpha_- = \pi - \sin^{-1}(\Delta/(2K))$. Of these two symmetric equilibria, the set given by $(\rho_+^2, \Omega, \alpha_+)$ is found to be stable, whereas the solution $(\rho_-^2, \Omega, \alpha_-)$ is unstable. The asymmetric phase-locked solutions turn out to be unstable. Thus in the absence of any delay in the coupling the only stable phase-locked equilibrium is the one where the two oscillators are synchronized to the mean frequency and have identical amplitudes. Region II in figure 1A shows the stable phase-locked region. Region III corresponds to phase-drift solutions.

The condition on amplitude death boundaries implies that identical oscillators ($\Delta = 0$) cannot exhibit amplitude death. Note that the boundaries of the three regions meet at a highly degenerate point at $(K, \Delta) = (1, 2)$. Another distinguishing feature of this model is that the critical curves are independent of the mean frequency $\bar{\omega}$. In fact one can set $\bar{\omega} = 0$ (which is equivalent to transforming to a frame rotating at the mean frequency) and carry out the same analysis without any loss of generality. This property follows from the original symmetry of the coupled equations.

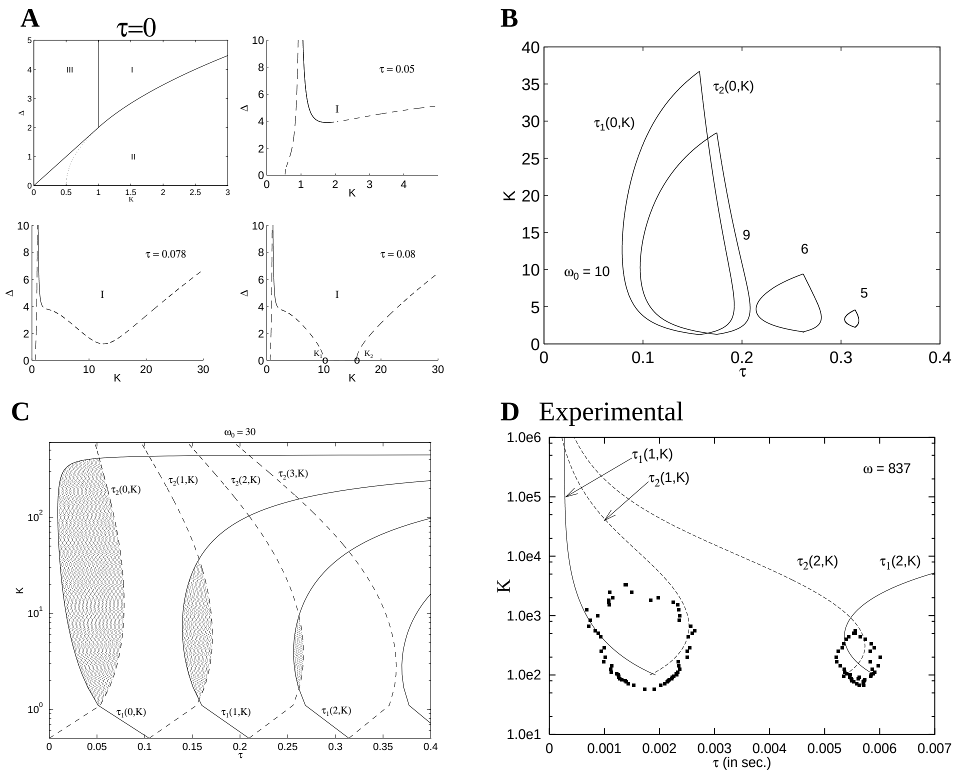

Figure 1. Two delay-coupled oscillators, theoretical and experimental. (**A**) The first panel shows phase diagram of Aronson *et al* [23] of two coupled oscillators without time delay in the plane of coupling strength (K) and the frequency mismatch (Δ). The other panels show evolution of the amplitude death (region I) boundaries as τ is increased. $\omega_0 = 10$. (**B**) Amplitude death (death islands) occurring for identical oscillators for finite time delay at different ω_0. (**C**) Multiply connected death islands occurring for $\omega_0 = 30$ [31,32]. (**D**) Experimental boundaries of multiple death islands [33].

2.1 *Time-delay effects*

Time-delay effects on the collective states of the two-coupled oscillator model discussed above, were studied in [31] by investigating the following set of equations which are a generalization of eqs (4) and (5) to incorporate time-delayed coupling:

$$\dot{Z}_1(t) = (1 + i\omega_1 - |Z_1(t)|^2)Z_1(t) + K[Z_2(t-\tau) - Z_1(t)], \tag{6}$$

$$\dot{Z}_2(t) = (1 + i\omega_2 - |Z_2(t)|^2)Z_2(t) + K[Z_1(t-\tau) - Z_2(t)], \tag{7}$$

where τ is a discrete and constant delay time. The time-delay parameter has been introduced in the argument of the coupling oscillator (e.g. Z_2 in (6)) to physically account for the fact that its phase and amplitude information is received by oscillator Z_1 only after a finite time τ. Note that the origin $Z_j = 0$ $(j = 1, 2)$ is still a fixed point of the system. The characteristic equation for its linear stability in terms of the eigenvalue λ is now transcendental and is given by

$$\lambda^2 - 2(a + i\bar{\omega})\lambda + (b_1 + ib_2) + c\mathrm{e}^{-2\lambda\tau} = 0. \tag{8}$$

In general it is not possible to solve the above equation analytically, except for some special cases, and a numerical solution is necessary to obtain the critical curves in the $(K - \Delta)$ space. Typical results for $\bar{\omega} = 10$ and different values of τ are shown in figure 1A. From the series of diagrams one notices that with the introduction of finite τ the point $(K = 1, \Delta = 2)$ no longer has a degenerate character and the critical curves begin separating and distorting in a continuous manner. The amplitude death region grows in size as the value of τ is increased from 0 and for a critical value of $\tau = \tau_c$ it touches the $\Delta = 0$ axis and the region of death on the axis lies between the two points of intersection K_1 and K_2. This death region has a finite width $K_2 - K_1$ for a range of values of the delay parameter τ. This phenomenon of 'death of identical oscillators' is one of the most interesting and important results of [31] and is an effect purely induced by the temporal delay in the coupling of the oscillators. The phenomenon is not restricted to just the two-oscillator model but persists even for larger systems and for both global and local couplings. The death region exists for a range of τ values after which the bifurcation curve lifts up from the $\Delta = 0$ line and starts moving upward. In fact the region of amplitude death for identical oscillators can be analytically quantified from the original eigenvalue equation (8) which can be solved exactly when one sets $\omega_1 = \omega_2 = \omega_0$. The equation simplifies to

$$\lambda = 1 - K + i\omega_0 \pm K\mathrm{e}^{-\lambda\tau}. \tag{9}$$

Let $\lambda = \alpha + i\beta$, where α and β are real. In order to obtain the critical curves one sets $\alpha = 0$ and considers both the equations arising out of '+' and '−' signs in eq. (9). By choosing the correct signs in the inversion of the resultant cosine function, the following two sets of critical curves that bound amplitude death region are obtained:

$$\tau_1 \equiv \tau_1(n, K) = \frac{n\pi + \cos^{-1}(1 - 1/K)}{\omega - \sqrt{2K - 1}}, \tag{10}$$

$$\tau_2 \equiv \tau_2(n, K) = \frac{(n + 1)\pi - \cos^{-1}(1 - 1/K)}{\omega + \sqrt{2K - 1}}, \tag{11}$$

where $n = 0, 1, \ldots, \infty$. One further needs to know the nature of the transition of pairs of eigenvalues as it crosses these curves. For this it is necessary to evaluate $\mathrm{d}\alpha/\mathrm{d}\tau$ on each of these curves and examine its sign. Setting $\lambda = \alpha + i\beta$ in eq. (9) and differentiating with respect to τ, it is straightforward to get

$$\left.\frac{\mathrm{d}\alpha}{\mathrm{d}\tau}\right|_{\alpha=0} = c_1\beta(\beta - \omega_0)$$
$$= \begin{cases} c_1\sqrt{2K - 1}(\omega_0 + \sqrt{2K - 1}), & \text{if } \beta = \beta_+ \\ -c_1\sqrt{2K - 1}(\omega_0 - \sqrt{2K - 1}), & \text{if } \beta = \beta_- \end{cases}, \tag{12}$$

where $c_1 = [(1 \pm K\tau)^2 + (K\tau \sin(\beta\tau))^2]^{-1}$, which is a real positive constant and $\beta \equiv \beta_\pm = \omega_0 \pm \sqrt{2K-1}$. From the above equation it is easily seen that

$$\left.\frac{\mathrm{d}\alpha}{\mathrm{d}\tau}\right|_{\alpha=0} \begin{cases} < 0 \ \text{ on } \tau_1 \ \text{ if } K < f(\omega_0) \\ > 0 \ \text{ on } \tau_1 \ \text{ if } K > f(\omega_0) \\ > 0 \ \text{ on } \tau_2 \end{cases}, \tag{13}$$

where $f(\omega_0) = (1+\omega_0^2)/2$. Thus on a τ_1 curve, a pair of eigenvalues transits to the left-half plane provided the coupling strength is smaller than $f(\omega_0)$ and to the right side if the coupling strength is greater than $f(\omega_0)$. On a τ_2 branch of the critical curves, however, a pair of eigenvalues always crosses into the right-half plane of the complex plane. Thus for a finite region of amplitude death to exist in the K–τ plane it needs to be bounded by appropriate branches of the τ_1 and τ_2 curves and condition $K < f(\omega_0)$ should hold. For $K > f(\omega_0)$, there would be no amplitude death region at all. The region enclosed by the intersection of $\tau_1(0, K)$ and $\tau_2(0, K)$ forms a region of amplitude death in the $K - \tau$ space and can be labeled as a death island. Physically such an island represents a region in phase space where for a given value of ω and at a fixed K one moves (by varying τ) from an unstable region (corresponding to phase-locked states) into a stable region as one crosses the left boundary of the island to emerge again into an unstable region as one crosses the right boundary of the island. The size of the primary death island is found to be a function of ω_0. Figure 1B displays the island sizes for different values of ω_0. The size of the island decreases with decreasing frequency and vanishes below a certain threshold. Time delay also introduces multiple connectivity and in this model one finds higher order islands for $\omega_0 > 14.438$ as shown in figure 1C for $\omega_0 = 30$.

We next take a look at the effect of time delay on the phase-locked states. Using the ansatz discussed earlier for the description of phase-locked states, the corresponding polar form equations for (6,7) are

$$(1 - K - R_1^2)R_1 + KR_2 \cos(\alpha - \Omega\tau) = 0, \tag{14}$$

$$(1 - K - R_2^2)R_2 + KR_1 \cos(\alpha + \Omega\tau) = 0, \tag{15}$$

$$\Omega = \omega_1 + K\frac{R_2}{R_1}\sin(\alpha - \Omega\tau), \tag{16}$$

$$\Omega = \omega_2 - K\frac{R_1}{R_2}\sin(\alpha + \Omega\tau). \tag{17}$$

Since this is a set of transcendental equations, with an infinite number of solutions, one immediate consequence of time delay is the appearance of a richer fare of phase-locked equilibria. Although it is difficult to obtain analytic forms for these solutions, it is easy to track these solutions numerically. Such an analysis has been carried out in [32] and a set of typical results are displayed in figure 2A for $\tau = 0.4084$. The number of these coherent states increases as a function of K and τ. The stability of these states was also examined in [32] by carrying out a linear perturbation analysis of eqs (4) and (5) around the phase-locked solutions. Numerical solution of the stability conditions shows that all these higher frequency states are stable (except for small bands of unstable regions which are indicated

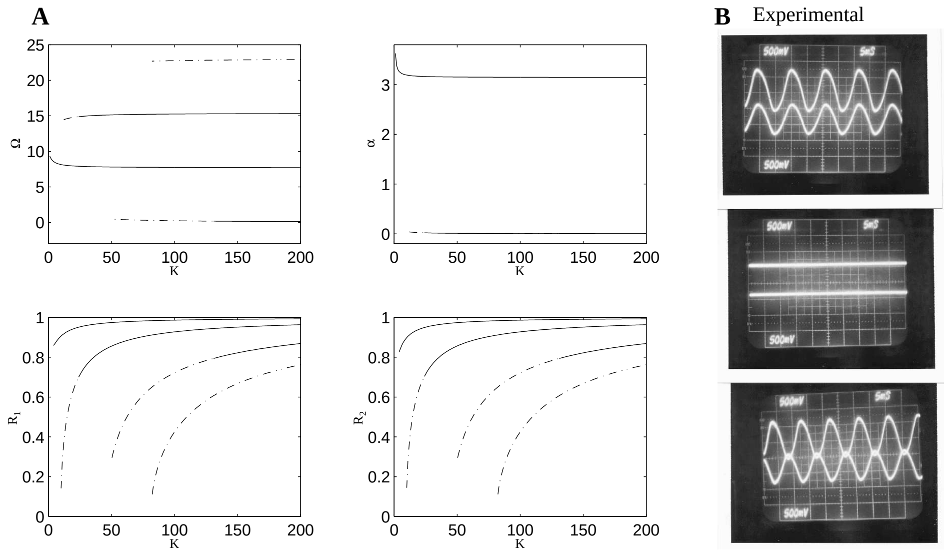

Figure 2. The phase-locked solutions, theoretical and experimental. (**A**) The common frequency Ω, the phase difference α and the amplitudes of the individual oscillators plotted as a function of the coupling strength K for $\tau = 0.4084$, $\Delta = 1$, and $\bar{\omega} = 10$. The oscillators have the phase difference α, either around 0 (i.e. in-phase synchronization) or around π (anti-synchronization). The unstable portions of the solutions are plotted as dashed lines [32]. (**B**) Experimental realization of in-phase, death, and antiphase locked solutions [33].

by dashed curves in figure 2A). Thus these higher frequency states are genuine collective states of the system that are physically accessible.

In summary, analytical and numerical studies of the simple two-oscillator model show that time delay introduces significant changes in the onset and nature of the characteristic collective regimes of the death state and phase-locked states. The region of death state is expanded to include the case of identical oscillators ($\Delta = 0$) for certain values of τ and the spectrum of phase-locked states acquires higher frequency branches in the presence of finite time delay.

3. *N*-Oscillator models

When the number of oscillators N is greater than 2 they can be mutually coupled in a variety of ways. The most common models are those that adopt a global coupling (where each oscillator is coupled to every other one in the system) or some form of a local coupling, e.g. nearest-neighbor coupling. The effect of time delay on both these models have been carried out in [32] and [34] respectively and we will briefly highlight their main conclusions.

3.1 *Global coupling*

A system of N globally coupled limit cycle oscillators with a linear time-delayed coupling can be described by the following set of model equations:

$$\dot{Z}_j(t) = (1 + i\omega_j - |Z_j(t)|^2)Z_j(t) + \frac{K'}{N}\sum_{k=1}^{N}[Z_k(t-\tau) - Z_j(t)] - \frac{K'}{N}[Z_j(t-\tau) - Z_j(t)], \tag{18}$$

where $j = 1, \ldots, N$, $K' = 2K$ is the coupling strength and τ is the delay time. The last term on the RHS has been introduced to remove the self-coupling term. This formulation is a generalization of the mean field model that has been studied extensively in the past [5,19,24,26] without time delay. The eigenvalue equation for the stability of the origin can be expressed as a product of two factors,

$$\left[\prod_{k=1}^{N}(i\omega_k - \mu - f)\right]\left[1 + f\sum_{j=1}^{N}\frac{1}{i\omega_j - \mu - f}\right] = 0. \tag{19}$$

The first factor represents the continuous spectrum of the system whereas the second factor describes the discrete spectrum. In general, it is not possible to solve the characteristic equation (19) analytically and for large N the numerical tracking of all the eigenvalues is also an arduous task. There are two interesting limits however, in which the analysis gets considerably simplified. If the N oscillators have identical frequencies then it is possible to obtain exact algebraic relations for the critical curves marking the amplitude death region. Another interesting limit is the $N \to \infty$ limit, often called the thermodynamic limit, where it is once again possible to obtain some exact analytic results for the bifurcation diagram. We will restrict ourselves to discussing the case of identical oscillators in view of the novel result of the $N = 2$ model which showed amplitude death for $\Delta = 0$. The interesting question to ask is whether such a phenomenon exists for the arbitrary N case. For a set of identical oscillators the frequency distribution of the system is a delta function,

$$g(\omega) = \delta(\omega - \omega_0), \tag{20}$$

where ω_0 is the natural frequency of each oscillator. With this assumption the eigenvalues are explicitly given by

$$\lambda = \left\{1 - K'd + i\omega_0 + K'd\mathrm{e}^{-\lambda\tau},\ \ 1 - K'd + i\omega_0 - \frac{K'}{N}\mathrm{e}^{-\lambda\tau}\right\}, \tag{21}$$

in which the second eigenvalue has a degeneracy of $N - 1$. Considering both the eigenvalue equations and following a similar procedure as described for the $N = 2$ case, one can obtain the following set of critical curves:

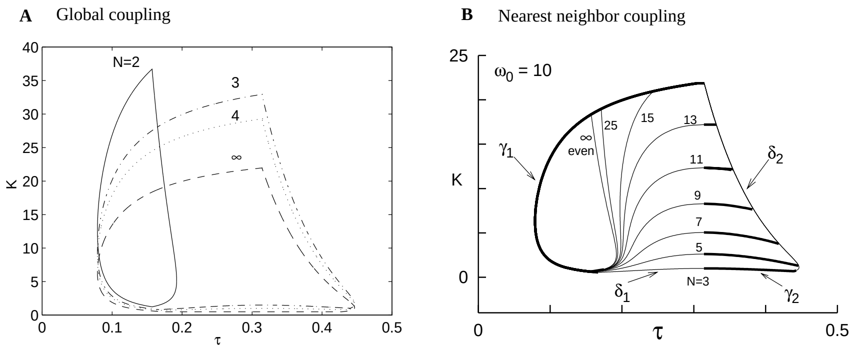

Figure 3. Death islands for global and nearest-neighbor couplings ($\omega_0 = 10$). (**A**) Death island regions for oscillators with global coupling [31,32]. (**B**) Death island regions for oscillators with nearest-neighbor coupling. All even number of oscillators have a single death island region that is independent of the number of oscillators. The odd number of oscillators are bounded by four curves when $N \leq 13$, and two curves otherwise. These two curves merge in the infinite limit with the curves that represent the even number of oscillators [34].

$$\tau_a(n, K) = \frac{2n\pi + \cos^{-1}\left[1 - \frac{1}{K'd}\right]}{\omega_0 - \sqrt{2K'd - 1}}, \tag{22}$$

$$\tau_b(n, K) = \frac{2(n+1)\pi - \cos^{-1}\left[1 - \frac{1}{K'd}\right]}{\omega_0 + \sqrt{2K'd - 1}}, \tag{23}$$

$$\tau_c(n, K) = \frac{2(n+1)\pi - \cos^{-1}\left[\frac{1-K'd}{K'(1-d)}\right]}{\omega_0 - \sqrt{[K'(1-d)]^2 - (K'd-1)^2}}, \tag{24}$$

$$\tau_d(n, K) = \frac{2n\pi + \cos^{-1}\left[\frac{1-K'd}{K'(1-d)}\right]}{\omega_0 + \sqrt{[K'(1-d)]^2 - (K'd-1)^2}}. \tag{25}$$

Note that N enters as a parameter (through $d = 1 - 1/N$) in these sets of curves and the family of curves is now four instead of the two found for the $N = 2$ case. In fact for $N = 2$, the curves $\tau_a(n, K)$ and $\tau_c(n, K)$ combine to give $\tau_1(n, K)$ and $\tau_b(n, K)$ and $\tau_d(n, K)$ combine to give $\tau_2(n, K)$ of the preceding section. Figure 3A displays some typical death island regions for various values of N as obtained from the critical curves (22)–(25) with $n = 0$. The sizes of the islands are seen to vary as a function of N and approach a saturated size as $N \to \infty$. The existence of these regions independently confirmed by direct numerical solution of the coupled oscillator equations, demonstrates that the amplitude death phenomenon for identical oscillators happens even in the case of an arbitrarily large number of oscillators. They also display multiple connectedness of the death region for higher values of ω_0 as was seen for the $N = 2$ case.

3.2 *Local coupling*

We next discuss the amplitude death phenomena in a system of N identical limit cycle oscillators that are locally coupled to each other with a time delay and where the coupling is of the nearest-neighbor kind. The appropriate set of model equations are then

$$\frac{\partial Z_j}{\partial t} = (1 + i\omega_0 - |\psi_j|^2)Z_j + K[Z_{j+1}(t-\tau)Z_j(t)] \\ +K[Z_{j-1}(t-\tau) - Z_j(t)], \tag{26}$$

where the notation is as before. For determining the amplitude death region, one can once again carry out a linear perturbation analysis about the origin. Assuming the oscillator configuration to be in the form of a chain (so that periodic boundary conditions can be applied) the eigenvalue equation takes the form

$$\prod_{j=1}^{N} \left(\lambda + 2K - 1 - i\omega_0 - K\mathrm{e}^{-\lambda\tau}U_j - K\mathrm{e}^{-\lambda\tau}U_j^{N-1}\right) = 0,$$

where $U_j = \mathrm{e}^{i2\pi(j-1)/N}$ are the Nth roots of unity. Since $U_j + U_j^{N-1} = U_j + U_j^{-1} = 2\cos[(j-1)2\pi/N]$, the above equation is further reduced to

$$\prod_{j=1}^{N} \left(\lambda + 2K - 1 - i\omega_0 - 2K\cos[(j-1)2\pi/N]\mathrm{e}^{-\lambda\tau}\right) = 0. \tag{27}$$

The complete set of eigenvalue equations includes the second set obtained by considering the conjugate equation of the above. Note that for $\tau = 0$ the above eigenvalue equation (27) always admits at least one unstable eigenvalue, namely $\lambda = 1 + i\omega_0$. Hence identical oscillators that are locally coupled cannot have an amplitude death state in the absence of time delay. To determine the amplitude death regions for finite values of τ one follows the same standard procedure discussed for the global coupling case. There are however a few interesting differences to take note of. If the number of oscillators is a multiple of 4, there are some eigenvalue equations (i.e. factors of eq. (27) that have no dependence on τ when $R_j = 2\cos[(j-1)2\pi/N] = 0$. For example, consider the case of $N = 4$ and $j = 2, 4$. Then, the eigenvalue equation becomes: $\lambda = 1 - 2K \pm i\omega_0$. For this equation, the only criticality is given by $K = 1/2$. The stable region lies on the side of the parameter space that obeys $K > 1/2$. For other values of R_j, the death island boundaries can be derived by setting the real part of the eigenvalue to zero, and appropriately choosing the signs of the multiple curves that result. The final expressions for the critical curves in (τ, K) plane are

$$\tau_a(n,K) = \begin{cases} \frac{2n\pi - \cos^{-1}[(2K-1)/KR_j]}{\omega_0 + \sqrt{K^2R_j^2 - (2K-1)^2}}, & R_j > 0 \\ \frac{(2n+1)\pi - \cos^{-1}[(2K-1)/K|R_j|]}{\omega_0 + \sqrt{K^2R_j^2 - (2K-1)^2}}, & R_j < 0 \end{cases}, \tag{28}$$

$$\tau_b(m,K) = \begin{cases} \frac{2m\pi+\cos^{-1}[(2K-1)/KR_j]}{\omega_0-\sqrt{K^2R_j^2-(2K-1)^2}}, & R_j > 0 \\ \frac{(2m+1)\pi+\cos^{-1}[(2K-1)/K|R_j|]}{\omega_0-\sqrt{K^2R_j^2-(2K-1)^2}}, & R_j < 0 \end{cases}. \tag{29}$$

n and m are whole numbers. A detailed analysis of these curves that includes determination of useful bounds on K for ordering and finding the degeneracies of the critical curves is given in [34]. The death islands in $K-\tau$ space for different values of N are shown in figure 3B. A striking and curious result is that the size and shape of the death island is determined by the odd or even property of the number of oscillators. For N even there is a single death region. When N is odd the boundary of the death region depends on the value of N. As N increases, the area of the death region decreases. For $N\to\infty$ the area of the death island for N odd decreases and approaches, as a limit, the boundary for N even.

Finally, we briefly consider the phase-locked states emerging from the locally coupled model. Unlike the global coupling case, the equilibrium states will now have a spatial dependence due to the geometry of the coupling. For periodic boundary conditions plane-wave states are one possibility and these have been studied in detail in [34]. Their spectrum can be obtained by assuming plane-wave solutions of eq. (26) of the form

$$Z_j = R\mathrm{e}^{i(jka+\omega t)}, \tag{30}$$

where a is the distance between any two adjacent oscillators and k is the wave number such that $-\pi\le ka\le\pi$. The values of ka, which define the phase difference between adjoining oscillators, are further constrained by the periodicity conditions inherent in a closed chain configuration. Since Z_{N+1} must be identical to Z_1 and Z_0 must be identical to Z_N, one must satisfy the condition $\mathrm{e}^{iNka}=1$. This implies that $Nka = 2m\pi$, $m = 0,1,\ldots,N-1$, that is,

$$ka = m\frac{2\pi}{N}, \quad m = 0,1,\ldots,N-1. \tag{31}$$

These discrete values of ka are one of the defining properties of the various phase-locked states of the coupled set of oscillators. Further characteristics of these states are described by a dispersion relation which can be obtained by substituting (30) in eq. (26) to give, $i\omega = 1 + i\omega_0 - R^2 + 2K\left[\cos(ka)\mathrm{e}^{-i\omega\tau} - 1\right]$. This leads to

$$\omega = \omega_0 - 2K\sin(\omega\tau)\cos(ka), \tag{32}$$

$$R^2 = 1 - 2K + 2K\cos(\omega\tau)\cos(ka). \tag{33}$$

Using $R^2 > 0$ as the defining condition for the possibility of having a phase-locked state, one now needs to solve the above set of transcendental equations to determine ω, R^2 and the corresponding k for given values of τ, K and ω_0. In general this is difficult to do, even numerically, but some special cases have been examined in [34]. Figure 4A illustrates some of the existence regions of plane-wave states for these special cases. Apart from these special cases, the general existence regions are complicated functions of ka, K and τ. They need to be determined

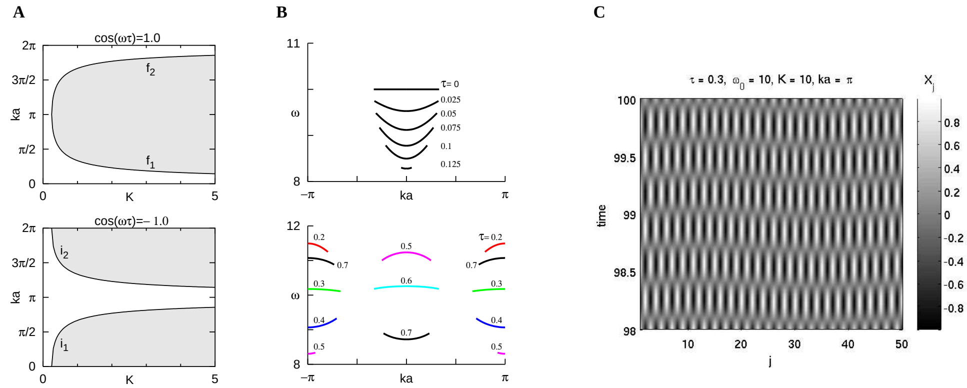

Figure 4. (**A**) Allowed (unshaded) and forbidden (shaded) wave modes in the presence of time delay for two values of $\cos(\omega\tau)$. (**B**) Dispersion relation between allowed wave numbers and the corresponding frequency shown as τ is gradually increased. $K = 1$ and $\omega_0 = 10$. A range of τ values is forbidden. (**C**) A numerical example of out-of-phase state shown for $N = 50$ oscillators by plotting the level of the real component of Z_j as a function of the oscillator number j [34].

numerically by simultaneous solution of eqs (32) and (33). To demonstrate the constraint imposed by (32), figure 4B shows a plot of the solution (ω vs. ka) for various values of τ and for a fixed value of ω_0 and K. When $\tau = 0$, the allowed range of τ is given by abs$(ka)< \cos^{-1}(1 - 1/2K)$. So for $K = 1$, the phase-locked patterns that have wave numbers less than $\pi/3$ are allowed, and all of them have identical frequencies. As τ is increased the frequency of oscillation decreases for small τ, and the dispersion relation acquires a non-linear parabolic character. As τ is further increased, depending on the actual value of K, there are bands in τ values where no modes exist. The shrinking and disappearance of the dispersion curve at $ka = 0$ beyond $\tau = 0.125$ up to $\tau = 0.2$ in the top panel of figure 4B illustrates this phenomenon. One also notices from the bottom panel of figure 4B that at higher values of τ the dispersion curves become discontinuous and have bands of forbidden ka regions. A linear stability analysis [34] reveals that time delay expands the stability domain of plane-wave states in the (ω, k) space as compared to the no-delay case and provides for a richer spectrum of states. As a specific case in point, the out-of-phase oscillation state (where each oscillator is π out of phase with its neighbor) is always unstable in the absence of time delay but can get stabilized for certain values of τ. A numerical example of such an out-of-phase state is shown in figure 4C where the level of the real part of Z_j is shown as a function of the oscillator number j by simulating $N = 50$ oscillators.

4. Perspective

As the preceding sections demonstrate, time-delayed coupling can have subtle and often profound effects on the collective dynamics of coupled oscillator systems. Can

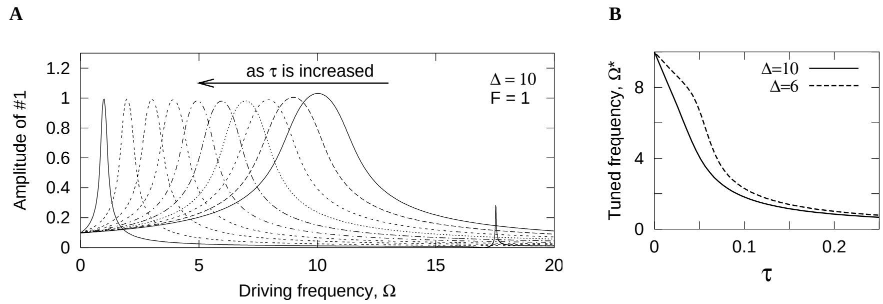

Figure 5. (**A**) Amplitude response of the first oscillator when both the coupled oscillators given in eqs (6) and (7) are each driven by $Fe^{i\Omega t}$. As τ is increased a different tuning frequency is selected. (**B**) The tuned frequency (at maximum response) is shown as a function of τ [40].

these effects be observed on real systems? A number of recent experiments have addressed this issue and carried out laboratory experiments to investigate time-delay effects discussed above. Observations on the phenomenon of amplitude death were reported by Herrero *et al* in [41] for a pair of optothermal oscillators that were thermally coupled and for which the occurrence of death for strong couplings and its relation to Hopf bifurcations of the uncoupled and coupled systems were experimentally verified. However, the experimental results were not conclusive about the role of delay in the death phenomenon although they tried to compare their results with those of a delay-based mathematical model. In a novel experiment reported by Takamatsu *et al* [42], time-delay effects were studied in a living coupled oscillator system by looking at the time variation of the thickness of the plasmodium of the slime mold (*physarum polycephalum*). Micro-fabricated tube-like structures were artificially inserted such that the oscillations in thickness of one part of the slime mold could influence neighboring regions of the organism through these structures. The dimensions of the tubes were varied to control the strength of the coupling and the amount of time delay. By varying these two parameters the authors were able to observe various synchronized states of a time-delayed two-oscillator system – in particular the in-phase and the out-of-phase states. The delay-induced amplitude death state was not observed presumably because they could not achieve the required parameter regime. Time delay induced amplitude death was unambiguosly demonstrated in [33] by looking at the output signals of two non-linear electronic circuits (variants of the so-called Chua circuits [43]) that were linearly coupled with a digital delay line. Figure 2B, taken from [33], shows experimental results for the time evolution of the oscillator voltages as a function of the delay parameter for a fixed value of the coupling strength, $K = 1000\ \mathrm{s}^{-1}$, and oscillator frequency $\omega = 837\ \mathrm{s}^{-1}$. The identical oscillators are seen to acquire an in-phase locked state (for $\tau = 0.514$ ms), death (for $\tau = 2$ ms) and an anti-phase locked state (for $\tau = 4.428$ ms). A series of such observations were recorded for a fixed frequency and at different fixed values of K to explore amplitude death over a large parameter space in $K - \tau$. Figure 1D displays experimental observations of death islands (marked with square dots) for the common frequency $\omega = 837\ \mathrm{s}^{-1}$. The solid and dashed lines are analytic curves obtained from a linear stability analysis of the origin for model equations that describe the coupled circuit system and are close to the model two-oscillator model equations discussed in §2. As can be seen there is close agreement between the experimental results and theoretical predictions including the existence of a secondary (higher order) death island. The experiments also charted the domains of existence of multiple phase-locked states and the variation of their frequencies as a function of τ. Experimental investigations of time-delay effects in coupled oscillator systems is a relatively unexplored field and many of the novel collective states for large N systems remain to be experimentally verified.

On the theoretical front, several recent investigations of time-delay effects in coupled systems have uncovered further novel effects through alternative representations of time delays and also by exploring various other models. Atay [44] showed that the parameter space of amplitude death for the two-oscillator model of §2 is enhanced when the oscillators are connected with time delays distributed over an interval rather than concentrated at a point. Distributed delays provide for a more realistic model for the description of larger physical systems where the delay parameter can be space- or time-dependent or in biological systems where memory effects are important. Atay [44] proposed a simple mathematical prescription to incorporate distributed delays by modifying the coupling term in eq. (6) from $K[Z_2(t-\tau) - Z_1(t)]$ to $K[\int_0^\infty f(\tau')Z_2(t-\tau')\mathrm{d}\tau' - Z_1(t)]$ (and analogously in eq. (7)). $f(\tau')$ represents the distribution of delays. For $f(\tau') = 1/2\alpha$ if $|\tau-\tau'| \leq \alpha$ and zero otherwise, he showed that distributed delays enlarge and merge death islands in the parameter space. For α larger than a threshold value the death region can become unbounded. The results are not limited to the simple distribution model but occur even with a small spread of delays, for different distribution functions, and an arbitrary number of oscillators.

Synchronization is one of the most widely studied phenomenon in coupled oscillator systems. As demonstrated here with simple models, significant modifications in the characteristics of synchronized states can take place in the presence of time-delayed coupling. This sensitivity to time delay can in fact be exploited to achieve a desired synchronized state often in a very efficient manner. For example, recent studies [45] on a network of neuronal oscillators with time-delayed coupling reveal an enhancement of neural synchrony by time delay. In other words, if coupling is time-delayed, stable synchronized states can be achieved with lower coupling strengths. Such enhanced neural synchrony by delay may have important implications, e.g., in understanding synchronization of distant neurons and information processing in the brain. Rosenblum and Pikovsky [46] propose a technique for controlling coherent collective oscillations in ensembles of globally coupled units (self-sustained oscillators or maps). They showed that a time-delayed feedback in the mean field can, depending on the parameters, enhance or suppress the self-synchronization in the population. The authors suggest a few possible applications of their technique in neuroscience, such as suppression of rhythmical brain activity associated with Parkinson's disease and feedback enhancement of oscillations in cases of failure of cardiac or neural pacemakers. Synchronization due to a weak global coupling with time delay has also been studied for a semiconductor laser array both in the cw and self-pulsing regimes [47]. Time delay is shown to induce in-phase synchronization in all dynamical regimes and in some cases bring about local extinction of self-pulsing in the array. The existence of multistability in the presence of time delay offers a useful paradigm for understanding mechanisms of memory storage and temporal pattern recognition in the nervous system. Kim *et al* [30] studied the dynamic behavior of coupled oscillators with time-delayed interaction under a pinning force. They showed that with intermediate delay times the system shows multistability of synchronized and de-synchronized states. In the synchronized states there are multiple states with different collective frequencies which show hysteretic behavior as the pinning force is varied. By associating these various states with stages of perception the authors suggest a possible mechanism for the perception of ambiguous or reversible figures. Time-delay studies have also been extended to higher dimensional systems (e.g. two- or three-dimensional lattices of oscillators) and to networks of coupled maps with various topologies. For a set of two-dimensional coupled phase oscillators it is found that distance-dependent time delays can induce various patterns including traveling rolls, square-like and rhombus-like patterns, spirals, and targets [48]. These spatio-temporal patterns highlight once again the sensitivity of synchrony to non-local interactions and the role of time delay in controlling their existence domains. Similar studies have recently been carried out for networks of coupled maps where the connections between units involve time delays and the nature of synchronized dynamics has been compared for various connection topologies (e.g. scale free, random and regular) [49]. It is found that connection delays can be conducive to synchronization and lead to the emergence of a wide range of new collective behavior which the individual units are incapable of producing in isolation. In the theoretical models discussed in §§2 and 3 we have considered coupled systems that have no external driving forces. As is well-known, an external periodic driving force can induce synchronized oscillations in the system with a frequency equal to the frequency of the driving force. Such a phenomenon was studied in [40] for a time-delay coupled system, namely eqs (6) and (7), with an external periodic driving force of the form $F\mathrm{e}^{i\Omega t}$ affecting each oscillator. The amplitude response curve as a function of Ω showed interesting behavior in the amplitude death region of the undriven system. In particular it displayed a maximum response at the internal eigenfrequency. This eigenfrequency is however controlled by the actual value of τ. Hence, for such a system the time-delay parameter τ is a useful knob which can be varied to optimally tune the coupled oscillators to an external natural frequency at a maximum gain. The accessible range of frequencies is a function of several parameters including τ and intrinsic frequency. In figure 5A a set of tuning curves are shown as the time delay (τ) between the oscillators is increased, and in figure 5B the tuned frequency (at which maximum response is found) is plotted as a function of τ. Driven systems with time-delayed coupling remain another vast area for future explorations.

Finally, we would like to briefly touch upon some open problems in this area and possible future directions of research. As is evident from the discussion above, finite time-delay effects influence the nature of collective dynamics in several ways such as changing the onset and stability characteristics of equilibrium states, creating new equilibrium states that are unavailable to individual oscillator units and providing a parametric tool for the control and enhancement of cooperative phenomena. The influence is not limited to just time stationary states but also extends to incoherent states and time-dependent states such as chaotic dynamics, Hopf oscillations etc. In fact, the nature of transition between chaotic and unsynchronized states is to date a poorly understood and open problem in the study of coupled oscillator systems. Time-delay effects which provide a sensitive probe by its subtle influence on the behavior of the order parameter can prove useful to further explore this problem [32]. The effect of non-linear coupling and its interaction with time delay [50] is an unexplored area. The stability of the unsynchronized states (both with and without time delay effects) is itself an important mathematical challenge with interesting connections to phenomena like Landau damping in plasmas [3]. The collective dynamics of systems with non-local couplings is an emerging area of research and shows interesting novel states where the oscillators evolve into distinct spatial domains consisting of coherent and incoherent states [51]. These states, named as 'chimera' states by Abrams and Strogatz [52], and their further exploration with time-delay effects offer exciting new possibilities. Finally, the myriad applications of time-delay effects, such as feedback and dynamic control schemes, and the prospect of improved modeling of complex systems in the biological and physical sciences will continue to stimulate considerable further research in this field.

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