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Complex Periodic Potentials with a Finite Number of Band Gaps

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Abstract: We obtain several new results for the complex generalized associated Lamé potential

$$V(x) = a(a+1)m\ \mathrm{sn}^2(y,m) + b(b+1)m\ \mathrm{sn}^2(y+K(m),m)$$

$$+f(f+1)m\ \mathrm{sn}^2(y+K(m)+iK'(m),m) + g(g+1)m\ \mathrm{sn}^2(y+iK'(m),m)\ ,$$

where $y \equiv x - \frac{K(m)}{2} - \frac{iK'(m)}{2}$, $\mathrm{sn}(y,m)$ is a Jacobi elliptic function with modulus parameter m, and there are four real parameters a, b, f, g. First, we derive two new duality relations which, when coupled with a previously obtained duality relation, permit us to relate the band edge eigenstates of the 24 potentials obtained by permutations of the four parameters a, b, f, g. Second, we pose and answer the question: how many independent potentials are there with a finite number "a" of band gaps when a, b, f, g are integers? For these potentials, we clarify the nature of the band edge eigenfunctions. We also obtain several analytic results when at least one of the four parameters is a half-integer. As a by-product, we also obtain new solutions of Heun's differential equation.

1 Introduction.

In a recent paper [1], hereafter referred to as I, we discussed the generalized associated Lamé (GAL) potential given by

$$
\begin{aligned}
\hat{V}(x) &= a(a+1)m\ \mathrm{sn}^2(x,m) + b(b+1)m\ \mathrm{sn}^2(x+K(m),m) \\
&\quad + f(f+1)m\ \mathrm{sn}^2(x+K(m)+iK'(m),m) + g(g+1)m\ \mathrm{sn}^2(x+iK'(m),m) \\
&= a(a+1)m\ \mathrm{sn}^2(x,m) + b(b+1)m\ \frac{\mathrm{cn}^2(x,m)}{\mathrm{dn}^2(x,m)} + f(f+1)\frac{\mathrm{dn}^2(x,m)}{\mathrm{cn}^2(x,m)} + g(g+1)\frac{1}{\mathrm{sn}^2(x,m)}\ ,
\end{aligned}
\tag{1}
$$

which involves four real parameters a, b, f, g. Here, sn (x,m), cn (x,m), dn (x,m) are Jacobi elliptic functions with elliptic modulus parameter m $(0 \le m \le 1)$. They are doubly periodic functions with periods $[4K(m), i2K'(m)]$, $[4K(m), 2K(m)+i2K'(m)]$, $[2K(m), i4K'(m)]$ respectively [2], where $K(m) \equiv \int_0^{\pi/2} d\theta[1-m\sin^2\theta]^{-1/2}$ denotes the complete elliptic integral of the first kind, and $K'(m) \equiv K(1-m)$. From now on, unless essential, we will not explicitly display the modulus parameter m as an argument of Jacobi elliptic functions. It may be noted here that the four terms in the GAL potential (1) correspond to complex translations of the independent variable x by $0, K(m), K(m)+iK'(m), iK'(m)$. Although the GAL potential is real, it does have singularities on the real axis coming from the zeros of the Jacobi elliptic functions sn (x) and cn (x) in the last two terms. One way to avoid these singularities is to make a complex change of variables $y = ix + \beta$, with β being an arbitrary real, non-zero constant, chosen so as to avoid the singularities arising from the zeros of Jacobi elliptic functions on the real axis [3]. This procedure was used in I, and we studied many of the properties of the resulting PT-invariant complex periodic potential [4]. However, consistent with the practice in the mathematics literature, in this paper we use an alternative approach of avoiding singularities by simply translating the independent variable x by an arbitrary non-zero amount in the complex plane. In fact, for simplicity, from symmetry considerations, we take the translated variable to be $x - \frac{K(m)}{2} - \frac{iK'(m)}{2}$. Note that the potential still is a PT-invariant complex periodic potential. One very important consequence of this choice is that the energy eigenvalues which we shall obtain here will be opposite in sign from the values obtained in I. This point should be kept in mind while comparing the results from the two papers. Thus explicitly, we consider the potential

$$
\begin{aligned}
V(x) &= \hat{V}(y) \\
&= a(a+1)m\ \mathrm{sn}^2(y,m) + b(b+1)m\ \mathrm{sn}^2(y+K(m),m) \\
&\quad + f(f+1)m\ \mathrm{sn}^2(y+K(m)+iK'(m),m) + g(g+1)m\ \mathrm{sn}^2(y+iK'(m),m) \\
&= a(a+1)m\ \mathrm{sn}^2(y,m) + b(b+1)m\ \frac{\mathrm{cn}^2(y,m)}{\mathrm{dn}^2(y,m)} + f(f+1)\frac{\mathrm{dn}^2(y,m)}{\mathrm{cn}^2(y,m)} + g(g+1)\frac{1}{\mathrm{sn}^2(y,m)} \\
&\equiv [a,b,f,g]\ , \qquad (2)
\end{aligned}
$$

where

$$
y = x - \frac{K(m)}{2} - \frac{iK'(m)}{2}\ . \qquad (3)
$$

It may be noted here that in I we used the notation $[a(a+1), b(b+1), f(f+1), g(g+1)]$ to denote this potential. However, for consistency with the prevailing practice in the mathematics literature, in this paper we use the notation $[a,b,f,g]$. In this notation, ordinary Lamé potentials are denoted by $[a,0,0,0]$, and associated Lamé (AL) potentials are denoted by $[a,b,0,0]$. Note that the potential (2) remains unchanged when any one or more of the parameters a,b,f,g change to $-a-1, -b-1, -f-1, -g-1$ respectively.

There is one important point which permits us to construct many supersymmetric partner potentials corresponding to a given potential [5]. Although this point was previously made in I, it is worth restating here. Normally, in supersymmetric quantum mechanics [6], given a potential $V_-(x)$, the ground state wave function $\psi_0(x)$ is used to construct the superpotential $W(x) = -\psi_0'(x)/\psi_0(x)$, which then yields the supersymmetric (SUSY) partner potential $V_+(x) = W^2 + W'$. If one uses any excited state wave function $\psi(x)$ of $V_-(x)$ to construct a superpotential $W(x)$, then the original potential $V_-(x)$ is recovered correctly (by construction), but the corresponding partner potential $V_+(x)$ turns out to be singular on the real x-axis due to the zeros of the excited state wave function $\psi(x)$. However, for the complex potential (2), the singularities are not on the real axis [7]. Thus in this case one could also use any of the excited state wave functions to obtain the superpotential $W(x)$ and hence discover several supersymmetric partner potentials with the same energy spectrum.

In I we showed that several GAL potentials with specific integer values of a,b,f,g have a finite number of band gaps. Further, looking at the symmetry of these potentials, we conjectured that all GAL potentials with integer values of a, b, f, g also have a finite number of band gaps. Some results of this type are available in the mathematics literature, and it is worthwhile to present a brief review of what is already known about the GAL potential.

The GAL potential (2), expressed in terms of Weierstrass functions, was discussed in a brief note by Darboux in 1882 [8], as well as in two subsequent articles in 1914 and 1915. He mentioned that some results had already been presented by Hermite in 1872 in unpublished lectures at Ecole Polytechnique. In 1883, Sparre [9] wrote two long papers on the GAL potential expressed in terms of Jacobi elliptic functions. Unfortunately, the mathematics community was largely unaware of these papers, and the GAL potential was "rediscovered" by Treibich and Verdier in 1990 [10]. In the current mathematics literature, the GAL potential, expressed in terms of Weierstrass functions, is known as the Treibich-Verdier potential. Several workers have shown that when a, b, f, g are integers, these potentials have a finite number of band gaps. In particular, the number of band gaps p is given by [11]

$$p = \frac{1}{2}\,\max\Big[2\,\max\,[a,b,f,g],\ 1+N-\Big\{1+(-1)^N\Big\}\Big\{\min\,[a,b,f,g]+\frac{1}{2}\Big\}\Big]\ , \tag{4}$$

where $N = a+b+f+g$. It was also shown that finite band gap potentials are solutions of higher order KdV equations [12]. In recent times, several people have also discussed the connection between Heun's equation and the Treibich-Verdier potential [13]. Finally, some authors have studied even more general potentials with a finite number of band gaps [14].

In I using supersymmetry we showed that the band edge eigenvalues of the Lamé and the AL potentials $[2,0,0,0], [3,0,0,0], [a,(a-3),0,0]$ are the same as those of the GAL potentials $[1,1,1,0], [2,1,1,1], [a,a-1,1,0]$ respectively. We had also conjectured in I that for integer a, b, Lamé and AL potentials have the same band edge eigenvalues as some GAL potentials - the explicit relationship being:

$$\begin{aligned}
[a,0,0,0] &\equiv \left[\frac{a}{2},\frac{a}{2},\frac{a}{2},\frac{a-2}{2}\right]\ , \quad a = \text{even integer}\ ,\\
[a,0,0,0] &\equiv \left[\frac{a+1}{2},\frac{a-1}{2},\frac{a-1}{2},\frac{a-1}{2}\right]\ , \quad a = \text{odd integer}\ ,\\
[a,b,0,0] &\equiv \left[\frac{a+b}{2},\frac{a+b}{2},\frac{a-b}{2},\frac{a-b-2}{2}\right]\ , \quad a+b = \text{even integer}\ ,\\
[a,b,0,0] &\equiv \left[\frac{a+b+1}{2},\frac{a+b-1}{2},\frac{a-b-1}{2},\frac{a-b-1}{2}\right]\ , \quad a+b = \text{odd integer}\ .
\end{aligned} \tag{5}$$

Note that we are using the notation "$\equiv$" to denote "same band edge eigenvalues", but not identical potentials. It is interesting to note that recently, Takemura [15] has verified the conjectures expressed in (5). Further, more generally, he has proved that the GAL potential $[a,b,f,g]$ for integer a,b,f,g has the same band edge eigenvalues as another GAL potential, with the explicit relationship depending on whether $N \equiv a+b+f+g$ is an even or an odd integer. If N is an even integer, the relationship is

$$[a,b,f,g] \equiv \left[\frac{a+b+f-g}{2}, \frac{a+b+g-f}{2}, \frac{a+f+g-b}{2}, \frac{b+f+g-a}{2}\right] , \tag{6}$$

while if N is an odd integer, then the relationship is

$$[a,b,f,g] \equiv \left[\frac{a+b+f+g+1}{2}, \frac{a+b-f-g-1}{2}, \frac{a+f-b-g-1}{2}, \frac{a+g-b-f-1}{2}\right] . \tag{7}$$

He has also shown that if a,b,f,g are all half-integers and their sum is an even integer, then the band edge energy eigenvalues of the potential $[a,b,f,g]$ are the same as two other GAL potentials where also all four parameters are half-integers with their sum being an even integer. In particular, the explicit relationships are

$$\begin{aligned}
&\left[a=k+\frac{1}{2}, b=l+\frac{1}{2}, f=n+\frac{1}{2}, g=p+\frac{1}{2}\right] \\
&\quad\equiv \left[\frac{k+l+n+p+3}{2}, \frac{k+l-n-p-1}{2}, \frac{k+n-l-p-1}{2}, \frac{k+p-n-l-1}{2}\right] \\
&\quad\equiv \left[\frac{k+l+n-p+1}{2}, \frac{k+l+p-n+1}{2}, \frac{k+n+p-l+1}{2}, \frac{l+n+p-k+1}{2}\right] .
\end{aligned} \tag{8}$$

The above mentioned results, when combined with our work in I, raise several questions. For example, while it is clear from the relation (4) that all the 24 potentials obtained by permutation of the four parameters a,b,f,g have the same number of band gaps, what is the precise relationship between the band edge eigenstates of these 24 potentials? Second, a very interesting question is to ask how many independent GAL potentials are there with say a band gaps. Third, is it possible to further generalize the results of Takemura [15] ? Besides, what is the nature of the band edge eigenfunctions for a general GAL potential in case a,b,f,g are all integers? Finally, in view of the connection between the GAL potentials and Heun's equation [9, 13], it is worth enquiring about the implications of these results in the context of solutions to Heun's equation.

In this paper, we address all the above-raised issues. Further, we also consider GAL potentials in which at least one of the four parameters is a half-integer while the other parameters are arbitrary numbers and show that both relations (6) and (7) are valid in all these cases. As a consequence, we conjecture that the relations (6) and (7) are in fact simultaneously valid even when all four parameters a, b, f, g are integers with their band edge eigenvalues being the same as the quasi-exactly solvable (QES) eigenvalues of potentials where all four parameters are half-integers (with their sum being an odd integer). Further, knowing the QES eigenvalues of the GAL potentials $[a, 1/2, f, g], [a, b, 1/2, g], [a, b, f, 1/2]$ (where a, b, f, g are arbitrary numbers), and using the connection between the Schrödinger equation for the GAL potentials and Heun's equation, we show that the corresponding eigenfunctions can be obtained by directly solving the algebraic form of Heun's equation.

The plan of this paper is as follows. In Sec. 2, we derive new duality relations and examine some consequences. In particular, using these duality relations we obtain the precise connection between the energy eigenvalues and eigenfunctions of the 24 potentials obtained by permuting the four parameters a, b, f, g. In Sec. 3, we discuss the GAL potential (2) in some detail when the four parameters a, b, f, g are all integers. First, we find various GAL potentials which are related to each other in the sense that they have identical band edge energy eigenvalues. We also find a large number of self-isospectral potentials as well as self-dual but non-self-isospectral potentials. Using all these results, we obtain the number of independent potentials with say a band gaps. We also clarify the nature of the band edge eigenfunctions for these potentials. In Sec. 4 we discuss the GAL potential when either one or more of the four parameters take half-integer values while the remaining parameters are arbitrary. In this case, in general one expects to obtain QES mid-band states, by which we mean any energy state lying inside an energy band and thus not a band edge. Quite remarkably, by generalizing Takemura's results [15] we obtain GAL potentials which have the same band edge eigenvalues as the mid-band energy values of the potentials with one or more of the parameters being half-integers. In Sec. 5 we discuss the GAL potential in case the parameters a, b, f, g take arbitrary values and obtain the corresponding GAL potentials with the same band edge and mid-band energy values when $a + b + f + g$ [or any other combination obtained by replacing one or more of these parameters by $-a-1, -b-1, -f-1, -g-1$ respectively] is an even integer. In Sec. 6 we discuss the implications of all these results in the context of Heun's equation. In particular, we show that in view of the connection between the different GAL potentials, given a periodic solution of Heun's equation, one immediately obtains four periodic and three quasi-periodic solutions of the same equation. We also show that in many cases, knowing the QES energy values of a GAL potential, it is much easier to solve the algebraic form of Heun's equation and obtain the corresponding eigenfunctions of the GAL potential. Finally, in Sec. 7 we summarize the results obtained in this paper and spell out some open problems.

2 Duality Relations for GAL Potentials.

In this paper our main focus is on the Schrödinger equation

$$-\frac{d^2}{dx^2}\psi(x) + \hat{V}(x)\psi(x) = E\psi(x) \ , \tag{9}$$

where $\hat{V}(x)$ is the GAL potential given by eq. (1), and we have chosen units with $\hbar = 2m = 1$. Displaying parameters more explicitly, eq. (9) states that the potential $\hat{V}(a,b,f,g,m;x)$ has band edge eigenvalues $E(a,b,f,g,m)$ and eigenfunctions $\psi(a,b,f,g,m;x)$. Of course, this means that the translated potential $V(x) \equiv V(a,b,f,g,m;x) = \hat{V}(a,b,f,g,m;x - \frac{K(m)}{2} - \frac{iK'(m)}{2})$ given in eq. (2) has the same eigenvalues $E(a,b,f,g,m)$ and translated eigenfunctions $\psi(a,b,f,g,m;x - \frac{K(m)}{2} - \frac{iK'(m)}{2})$.

First we want to show that the band edge eigenstates of the 24 potentials [obtained via permutations of the 4 parameters, a,b,f,g in $V(x)$] are all related, so that once the band edge eigenstates of any one permutation are known, the complete band edge eigenstates of all 24 potentials are also known. Actually, these relations are valid for both band edges as well as mid-band states.

The first relation is simple - in view of the invariance of the Schrödinger eq. (9) under the translation $x \to x + K(m)$ followed by the interchanges $a \leftrightarrow b$ and $f \leftrightarrow g$, one gets [1]

$$E(b,a,g,f,m) = E(a,b,f,g,m) \ , \quad \psi(b,a,g,f,m;x) \ \propto \ \psi(a,b,f,g,m;x+K(m)) \ . \tag{10}$$

Similarly, the translations $x \to x + iK'(m)$ and $x \to x + K(m) + iK'(m)$ followed by suitable interchanges of parameters, yield

$$E(g,f,b,a,m) = E(a,b,f,g,m) \ , \quad \psi(g,f,b,a,m;x) \ \propto \ \psi(a,b,f,g,m;x+iK'(m)) \ , \tag{11}$$

$$E(f,g,a,b,m) = E(a,b,f,g,m) \ , \quad \psi(f,g,a,b,m;x) \ \propto \ \psi(a,b,f,g,m;x+K(m)+iK'(m)) \ . \tag{12}$$

Thus, once we obtain the eigenvalues and eigenfunctions of a given potential $[a,b,f,g]$, then we immediately know the eigenvalues and eigenfunctions of three other potentials: $[b,a,g,f]$, $[g,f,b,a]$ and $[f,g,a,b]$. Note that relations (10), (11), (12) all involve the same modulus parameter m.

We now derive three remarkable duality relations, which connect the energy states of different GAL potentials, and involve changes in the modulus parameter from m to $1-m$, $1/m$ and $-m/(1-m)$.

Duality Relation I: This was already derived in I and is given by

$$\begin{aligned} &E(a,b,f,g,m) = [a(a+1)+b(b+1)+f(f+1)+g(g+1)] - E(a,g,f,b,1-m) \ , \\ &\psi(a,b,f,g,m;x) \ \propto \ \psi(a,g,f,b,1-m;ix+K'(m)+iK(m)) \ . \end{aligned} \tag{13}$$

Duality Relation II: Using the formulas [2]

$$\mathrm{sn}(x,m) = \frac{1}{\sqrt{m}}\mathrm{sn}(\sqrt{m}x,\frac{1}{m}) \ \ , \ \mathrm{cn}(x,m) = \mathrm{dn}(\sqrt{m}x,\frac{1}{m}) \ , \ \mathrm{dn}(x,m) = \mathrm{cn}(\sqrt{m}x,\frac{1}{m}) \ , \tag{14}$$

and redefining a new variable $z=\sqrt{m}x$, the Schrödinger eq. (9) takes the form

$$\begin{aligned} -\frac{d^2}{dz^2}\psi(z) + \Bigg[\frac{a(a+1)}{m}\mathrm{sn}^2(z,\frac{1}{m}) + \frac{f(f+1)}{m}\frac{\mathrm{cn}^2(z,\frac{1}{m})}{\mathrm{dn}^2(z,\frac{1}{m})} + b(b+1)\frac{\mathrm{dn}^2(z,\frac{1}{m})}{\mathrm{cn}^2(z,\frac{1}{m})} \\ + g(g+1)\frac{1}{\mathrm{sn}^2(z,\frac{1}{m})}\Bigg]\psi(z) = \frac{E(m)}{m}\psi(z) \ . \end{aligned} \tag{15}$$

On comparing eqs. (9) and (15) we obtain the duality relation

$$E(a,b,f,g,m) = mE(a,f,b,g,\frac{1}{m}) \ , \ \psi(a,b,f,g,m;x) \propto \psi(a,f,b,g,\frac{1}{m};\sqrt{m}x) \ . \tag{16}$$

Duality Relation III: We again start from the Schrödinger equation (9) and now use the formulas [2]

$$\begin{aligned} \mathrm{sn}(x,m) &= \frac{\mathrm{sn}[\sqrt{1-m}\ x,\frac{-m}{1-m}]}{\sqrt{1-m}\ \mathrm{dn}[\sqrt{1-m}\ x,\frac{-m}{1-m}]} \ , \\ \mathrm{cn}(x,m) &= \frac{\mathrm{cn}[\sqrt{1-m}\ x,\frac{-m}{1-m}]}{\mathrm{dn}[\sqrt{1-m}\ x,\frac{-m}{1-m}]} \ , \\ \mathrm{dn}(x,m) &= \frac{1}{\mathrm{dn}[\sqrt{1-m}\ x,\frac{-m}{1-m}]} \ . \end{aligned} \tag{17}$$

On defining a new variable $z = \sqrt{1-m}\ x$, the Schrödinger eq. (9) takes the form

$$
\begin{aligned}
&-\frac{d^2}{dz^2}\psi(z) + \Big[-\frac{m\ b(b+1)}{1-m}\mathrm{sn}^2(z, \frac{-m}{1-m}) - \frac{m\ a(a+1)}{1-m}\frac{\mathrm{cn}^2(z,\frac{-m}{1-m})}{\mathrm{dn}^2(z,\frac{-m}{1-m})} + f(f+1)\frac{\mathrm{dn}^2(z,\frac{-m}{1-m})}{\mathrm{cn}^2(z,\frac{-m}{1-m})} \\
&\qquad + g(g+1)\frac{1}{\mathrm{sn}^2(z,\frac{-m}{1-m})}\Big]\psi(z) \\
&= \frac{1}{1-m}E\psi(z) - \frac{m}{1-m}\Big[a(a+1)+b(b+1)+f(f+1)+g(g+1)\Big]\psi(z).
\end{aligned}
\tag{18}
$$

Comparing eqs. (9) and (18), one gets the duality relation

$$
\begin{aligned}
&E(a,b,f,g,m) = (1-m)E(b,a,f,g,\frac{-m}{1-m}) + m\Big[a(a+1)+b(b+1)+f(f+1)+g(g+1)\Big]\ , \\
&\psi(a,b,f,g,m;x) \propto \psi(b,a,f,g,\frac{-m}{1-m};\sqrt{1-m}\ x)\ .
\end{aligned}
\tag{19}
$$

Using the three duality relations [eqs. (13), (16), (19)] along with the translation results [eqs. (10), (11), (12)], it is easily shown that once the eigenstates of a given potential are known, we can immediately obtain the energy eigenstates of all the 24 potentials obtained by permuting the 4 parameters a, b, f, g. Hence, out of these 24 potentials, there is only one independent potential and without loss of any generality, throughout this paper we only consider the potential $[a, b, f, g]$ with $a \geq b \geq f \geq g$ (unless stated otherwise).

The duality relations are very powerful and have many interesting consequences. For example, we find that for arbitrary integer values of a, f, the potential $[a, 0, f, 0]$ has only a finite number of band gaps. This follows because, eq. (16) gives

$$
E(a,0,f,0,m) = mE(a,f,0,0,\frac{1}{m})\ ,
\tag{20}
$$

so that both the potentials $[a, f, 0, 0]$ and $[a, 0, f, 0]$ must have the same number of band edges. Since one knows [16] that the AL potential $[a, f, 0, 0]$ has a finite number of band gaps, the same statement holds for the potential $[a, 0, f, 0]$.

3 Independent GAL Potentials with a Band Gaps $[a, b, f, g =$ integers$]$.

In this section, we want to answer the following interesting question: given a potential of the form $[a, b, f, g]$ with $a \geq b \geq f \geq g \geq 0$, how many independent GAL potentials are there with exactly a band gaps?

To begin, for a given integer a, let us calculate the total number of possible GAL potentials with $a \geq b \geq f \geq g \geq 0$. The number of such potentials is

$$\sum_{b=0}^{a} \sum_{f=0}^{b} \sum_{g=0}^{f} 1 = (a+1)(a+2)(a+3)/6 \ . \tag{21}$$

Now due to the Landen transformations [2, 17], a potential $[a, a, b, b]$ with $a \neq b$ is essentially the same as the potential $[a, b, 0, 0]$ and hence not really distinct. There are a such potentials. Similarly, the potential $[a, a, a, a]$ is related to the potential $[a, 0, 0, 0]$ and hence not distinct. This follows from the relations [2]

$$\mathrm{sn}^2(x) = \frac{1 - \mathrm{cn}(2x)}{1 + \mathrm{dn}(2x)} \ , \quad m\mathrm{sn}^2(x + iK'(m), m) = \frac{1}{\mathrm{sn}^2(x, m)} \ , \tag{22}$$

$$\mathrm{sn}^2(x,m)+\mathrm{sn}^2(x+K(m),m)+\mathrm{sn}^2(x+iK'(m),m)+\mathrm{sn}^2(x+K(m)+iK'(m),m) = 4\mathrm{sn}^2(2x+iK'(m)) \ . \tag{23}$$

Thus the number of distinct GAL potentials of the form $[a, b, f, g]$ with $a \geq b \geq f \geq g \geq 0$ is given by

$$N_{dist} = a(a+1)(a+5)/6 \ . \tag{24}$$

In view of the formula (4) for the number of band gaps, it is obvious that these distinct potentials are of two types - there are those with exactly a band gaps and those with more than a band gaps. Using eqs. (6) and (4) it follows that in case $a+b+f+g$ is an even integer and $a+g \geq b+f$, then the potential has a band gaps, while if $a+g < b+f$, then it has $(a+b+f-g)/2$ band gaps. Similarly, using eqs. (7) and (4) it follows that in case $a+b+f+g$ is an odd integer and $a \geq b+f+g$, then the potential has a band gaps, while if $a < b+g+f$, then it has $(a+b+f+g+1)/2$ band gaps.

It is now straightforward to count the number of independent GAL potentials $[a, b, f, g]$ with a band gaps and show that

$$\begin{aligned}
N_a &= \frac{1}{18}[a^3 + 9a^2 + 6a + 2] \ , \quad a = 1 \ (\mathrm{mod}\ 3) \ ; \\
N_a &= \frac{1}{18}[a^3 + 9a^2 + 6a - 2] \ , \quad a = 2 \ (\mathrm{mod}\ 3) \ ; \\
N_a &= \frac{a}{18}[a^2 + 9a + 6] \ , \quad a = 0 \ (\mathrm{mod}\ 3) \ .
\end{aligned} \tag{25}$$

Several comments are in order at this stage.

1. There is only 1 independent potential with one band gap but there are 3 independent potentials with 2 band gaps and 7 independent potentials with 3 band gaps. This implies [12] that while there is only

one independent KdV equation of 3rd order, there should be 3 such equations of 5th order and 7 such equations of 7th order. It is worth pointing out that indeed there are 3 independent equations of 5th order called Sawada-Kotera and Kaup-Kupershmidt equations. It may be interesting to explicitly obtain all the 7 independent KdV equations of 7th order.

2. The number of independent potentials of the form $[a,b,f,g]$ having more than a band gaps is obtained by subtracting eq. (25)) from eq. (24):

$$N_{>a} = \frac{1}{18}(a+2)(2a^2+5a-1), \quad a = 1 \ (\mathrm{mod}\ 3) \ ;$$
$$N_{>a} = \frac{1}{18}(a+1)(2a^2+7a+2), \quad a = 2 \ (\mathrm{mod}\ 3) \ ;$$
$$N_{>a} = \frac{1}{18}a(a+3)(2a+3), \quad a = 0 \ (\mathrm{mod}\ 3) \ . \tag{26}$$

For example, for $a = 2$, there are 4 potentials of the form $[a,b,f,g]$ with more than 2 band gaps:

$$N_{>2} = [2,2,2,0], \ [2,2,2,1], \ [2,2,1,0], \ [2,1,1,1] \ . \tag{27}$$

For $a = 3$, there are 9 potentials with more than 3 band gaps:

$$N_{>3} = [3,3,3,0], \ [3,3,3,1], \ [3,3,3,2], \ [3,3,2,0], \ [3,3,2,1], \ [3,3,1,0], \ [3,2,1,1], [3,2,2,0], [3,2,2,2] \ . \tag{28}$$

3. If $a+b+f+g$ is an even integer and further if $a+g = b+f$, then both sides of eq. (6) are identical and the corresponding potential is self-dual. Similarly, if $a+b+f+g$ is an odd integer and $a = b+f+g+1$, then both sides of eq. (7) are identical, and one has a self-dual potential. As an illustration, the AL potential $[a, a-1, 0, 0]$ is self-dual, a fact we already knew from ref. [18]. But we now get a large number of additional self-dual GAL potentials, like $[2,1,1,0]$, $[4,2,2,0]$, and $[4,2,1,0]$, for example.

4. A related question is, out of the above self-dual potentials how many are also self-isospectral [18, 19]? It is easy to see that a choice of eigenfunction yielding self-isospectral potentials is

$$\psi = \frac{\mathrm{dn}^a(x,m)\mathrm{sn}^b(x,m)}{\mathrm{cn}^b(x,m)} \ , \tag{29}$$

and that the corresponding self-isospectral potential has the form

$$[a, a-1, b, b-1]\ , \quad b > 0\ ; \quad [a, a-1, 0, 0]\ , \quad b = 0\ . \tag{30}$$

The fact that the AL potentials of the form $[a, a-1, 0, 0]$ are self-isospectral was established many years ago [18]. However, what is new is the realization that the potentials $[a, a-1, b, b-1]$ are also self-isospectral for arbitrary integer values of a, b. Some examples are $[1, 0, 0, 0]$, $[2, 1, 0, 0]$, $[3, 2, 2, 0]$, $[3, 2, 1, 1]$, and $[3, 2, 0, 0]$. Thus out of all the self-dual potentials, those which are of form (30) are also self-isospectral, while the rest are self-dual but not self-isospectral. The number of self-dual potentials N_{sd} which are not self-isospectral is

$$\begin{aligned} N_{sd} &= \frac{1}{3}(a-1)^2\ , \quad a = 1, 4, 7, \ldots\ ; \\ N_{sd} &= \frac{1}{3}a(a-2)\ , \quad a = 2, 3, 5, 6, \ldots\ , \end{aligned} \tag{31}$$

whereas the number of self-isospectral potentials is $N_{si} = a$. As an illustration, the self-isospectral potentials with two band gaps are $[2, 1, 0, 0]$ and $[2, 1, 1, 0]$ while there are no self-dual, non-self-isospectral potentials with two band gaps. On the other hand, the self-isospectral potentials with three band gaps are $[3, 2, 2, 1]$, $[3, 2, 1, 0]$, and $[3, 2, 0, 0]$, while the only self-dual, non-self-isospectral potential with three band gaps is $[3, 1, 1, 0]$.

5. Clearly all potentials with $a + g > b + f$ or $a > b + f + g + 1$ depending on if $a + b + f + g$ is an odd or an even integer, have partner potentials as given by eqs. (6) or (7) respectively. Now out of these, some are SUSY partner potentials while the rest are merely partner potentials. So let us count both types of potentials. Now if two GAL potentials are SUSY partners, then one of their eigenfunctions must be related to each other by $\psi_{II} = \psi_I^{-1}$. Further, these two eigenfunctions must have the form

$$\mathrm{dn}^{\alpha}(x)\mathrm{cn}^{\beta}(x)\mathrm{sn}^{\gamma}(x)\ . \tag{32}$$

For such eigenfunctions, we know from I that $b = -\alpha, f = -\beta, g = -\gamma$ and $a + b + f + g = 0$. From here it is easy to show that the potential $[a, b, f, g]$ with a band gaps has a SUSY GAL partner provided either $a + g = b + f + 2$ or $a = b + f + g + 3$ depending on if $a + b + f + g$ is an even or an odd integer respectively. Hence, all potentials of the form $[a, b, f, g]$ with a band gaps and satisfying $a + g > b + f + 2$ or $a > b + g + f + 3$ have merely partner potentials of the form (6) or (7) respectively. It is worth emphasizing that while these partner potentials have the same band edge eigenvalues, none of them are SUSY partner potentials. For example, $[2, 0, 0, 0]$ and $[1, 1, 1, 0]$ are SUSY partner GAL potentials with two band gaps. Similarly $[3, 1, 0, 0]$ and $[2, 2, 1, 0]$ are SUSY partner GAL potentials with three band gaps. On the other hand, $[4, 0, 0, 0]$ and $[2, 2, 2, 1]$ are merely partner potentials with four band gaps.

6. One can count the number of potentials (N_{su}) of the form $[a, b, f, g]$ with a band gaps having another GAL potential as its SUSY partner and it is easy to show that

$$N_{su} = \frac{1}{3}[a^2 - 1] \ , \quad a \neq 0 \ (\mathrm{mod}\ 3) \ ;$$
$$N_{su} = \frac{1}{3}a^2 \ , \quad a = 0 \ (\mathrm{mod}\ 3) \ . \tag{33}$$

Finally, it is not difficult to show that the number of potentials of the form $[a, b, f, g]$ with a band gaps and having merely a (non-SUSY) partner potential of the form as given by eqs. (6) and (7) respectively is given by

$$N_{nsu} = \frac{1}{18}(a-1)(a^2 - 2a - 2) \ , \quad a = 1 \ (\mathrm{mod}\ 3) \ ;$$
$$N_{nsu} = \frac{1}{18}(a-2)(a^2 - a - 2) \ , \quad a = 2 \ (\mathrm{mod}\ 3) \ ;$$
$$N_{nsu} = \frac{1}{18}a^2(a-3) \ , \quad a = 0 \ (\mathrm{mod}\ 3) \ . \tag{34}$$

7. On adding the number of self-dual (but non-self-isospectral), self-isospectral, SUSY partner and (non-SUSY) merely partner potentials as given by eqs. (30) to (34) respectively, as expected, we find that the number of independent potentials with a band gaps is as given by eq. (25).

8. One obvious interesting question is whether there are non-GAL potentials with a finite number of band gaps. The answer to this question is yes. In particular, since the general form of the eigenfunction for any GAL potential is of the form [1]

$$\psi_{GAL}(x) = \mathrm{dn}^{-b}(x)\mathrm{cn}^{-f}(x)\mathrm{sn}^{-g}(x) \sum_{k=0}^{N} A_k \mathrm{sn}^{2k}(x) \ , \tag{35}$$

it follows that non-GAL potentials of the form

$$V_{+}(x) = V_{GAL}(x) - 2\frac{d^2}{dx^2}\ln\psi_{GAL}(x) \ , \tag{36}$$

are also finite gap potentials. In this context it is worth mentioning that some people [14] have recently obtained potentials with a finite number of band gaps which are more general than the GAL potentials. It is not clear if those potentials and the potentials (36) have any overlap.

3.1 The Nature of Band Edge Eigenstates

In I we showed that if $a+b+f+g=2n$, then $n+1$ QES states can be obtained for GAL potentials (2) and they are of the form given in eq. (35). On using this key result as well as the fact that the GAL potential $[a,b,f,g]$ remains unchanged when any one (or more) of the four parameters a,b,f,g is changed to $-a-1,-b-1,-f-1,-g-1$ respectively, it is easy to specify the nature of the band edge eigenfunctions for any potential $[a,b,f,g]$ with a band gaps. The nature as well as the number of eigenstates crucially depend on whether $a+b+f+g$ is an even or an odd integer - so we will discuss these situations separately.

$a+b+f+g =$ **even integer:**

In this case it is easy to show that one has $(a+b+f+g+2)/2$ eigenstates of the form

$$\mathrm{sn}^{-g}(x)\mathrm{cn}^{-f}(x)\mathrm{dn}^{-b}(x)F_{(a+b+f+g)/2}[\mathrm{sn}^2(x)] \ , \tag{37}$$

$(a+b-f-g)/2$ eigenstates of the form

$$\mathrm{sn}^{g+1}(x)\mathrm{cn}^{f+1}(x)\mathrm{dn}^{-b}(x)F_{(a+b-f-g-2)/2}[\mathrm{sn}^2(x)] \ , \tag{38}$$

$(a+f-b-g)/2$ eigenstates of the form

$$\mathrm{sn}^{g+1}(x)\mathrm{cn}^{-f}(x)\mathrm{dn}^{b+1}(x)F_{(a+f-b-g-2)/2}[\mathrm{sn}^2(x)] \ , \tag{39}$$

and $(a+g-b-f)/2$ eigenstates of the form

$$\mathrm{sn}^{-g}(x)\mathrm{cn}^{f+1}(x)\mathrm{dn}^{b+1}(x)F_{(a+g-b-f-2)/2}[\mathrm{sn}^2(x)] \ . \tag{40}$$

If instead $b+g > a+f$, then one has $(b+f-a-g)/2$ eigenstates of the form

$$\mathrm{sn}^{g+1}(x)\mathrm{cn}^{-f}(x)\mathrm{dn}^{-b}(x)F_{(b+f-a-g-2)/2}[\mathrm{sn}^2(x)] \ . \tag{41}$$

$a+b+f+g$ **= odd integer:**

In this case it is easy to show that one has $(a+b+f-g+1)/2$ eigenstates of the form

$$\mathrm{sn}^{g+1}(x)\mathrm{cn}^{-f}(x)\mathrm{dn}^{-b}(x)F_{(a+b+f-g-1)/2}[\mathrm{sn}^2(x)]\ , \tag{42}$$

$(a+b+g-f+1)/2$ eigenstates of the form

$$\mathrm{sn}^{-g}(x)\mathrm{cn}^{f+1}(x)\mathrm{dn}^{-b}(x)F_{(a+b+g-f-1)/2}[\mathrm{sn}^2(x)]\ , \tag{43}$$

$(a+f+g-b+1)/2$ eigenstates of the form

$$\mathrm{sn}^{-g}(x)\mathrm{cn}^{-f}(x)\mathrm{dn}^{b+1}(x)F_{(a+f+g-b-1)/2}[\mathrm{sn}^2(x)]\ , \tag{44}$$

$(a-b-f-g-1)/2$ eigenstates of the form

$$\mathrm{sn}^{g+1}(x)\mathrm{cn}^{f+1}(x)\mathrm{dn}^{b+1}(x)F_{(a-b-f-g-3)/2}[\mathrm{sn}^2(x)]\ , \tag{45}$$

If instead $b+f+g>a-1$ then one has $(b+f+g-a+1)/2$ eigenstates of the form

$$\mathrm{sn}^{-g}(x)\mathrm{cn}^{-f}(x)\mathrm{dn}^{-b}(x)F_{(b+f+g-a-1)/2}[\mathrm{sn}^2(x)]\ , \tag{46}$$

Here $F_n[\mathrm{sn}^2(x)]$ denotes a polynomial of order n in $\mathrm{sn}^2(x)$. It is worth pointing out that not only the QES band edge eigenvalues are identical for the two partner potentials as given either by eq. (6) or by (7), even the nature of the band edge eigenfunctions in the two cases is also similar. For example, the potential $[(a+b+f-g)/2,(a+b+g-f)/2,(a+f+g-b)/2,(b+f+g-a)/2]$ as given by eq. (6) has the same band edge eigenvalues and further the corresponding eigenfunction is simply obtained from eqs. (37) to (41) by replacing a,b,f,g with $(a+b+f-g)/2,(a+b+g-f)/2,(a+f+g-b)/2,(b+f+g-a)/2$ respectively. Exactly the same is also true about the equivalent potential given by eq. (7) in case $a+b+f+g$ is an odd integer and the corresponding eigenfunctions are exactly as given by eqs. (42) to (46) but with the replacement of a,b,f,g by $(a+b+f+g+1)/2,(a+b-f-g-1)/2,(a+f-b-g-1)/2,(a+g-b-f-1)/2$ respectively. It follows from here that irrespective of whether $a+b+f+g$ is an odd or an even integer, there are precisely a bound bands, same a number of band gaps and $2a+1$ number of band edges all of which are analytically known in principle, beyond which there is a continuum band extending up to

$E = \infty$. Further, irrespective of whether $a+b+f+g$ is an odd or an even integer, if $a+b$ is an even (odd) integer, then there are $a+b+1$ band edges of period $2K(4K)$ and $a-b$ band edges of period $4K(2K)$.

Thus in general the band structure of the GAL potentials is unusual in that if $a+b$ is an even (odd) integer, then b band gaps of period $4K(2K)$ must be of zero width, i.e. there must be b doubly degenerate states of period $4K(2K)$. Unfortunately, till today we do not know either the eigenvalue or the nature of the eigenfunction of even one of these doubly degenerate states. One exception is the case of pure Lamé (and their GAL partners) potentials, i.e. when $b = f = g = 0$, as in that case depending on if a is even or odd integer, one has $a+1$ band edges of period $2K(4K)$ and a band edges of period $4K(2K)$ and the band structure is normal one, with no doubly degenerate states.

As an illustration, consider the GAL potentials with two band gaps. As seen above, there are 3 distinct potentials with 2 band gaps out of which we have already discussed the band structure of the two potentials $[2,0,0,0]$ and $[2,1,0,0]$ [16, 18]. Thus it would be interesting to know the band edges and the band structure of the remaining potential with two band gaps, i.e. $[2,1,1,0]$. Since $a+b=3$, it follows from the above discussion that in this case there must be 4 band edges of period $4K$ and 1 band edge of period $2K$. Using Table 4 of I it is easily seen that the eigenstate with period $2K$ is given by

$$\psi = \mathrm{dn}^2(x)\mathrm{sn}(x)\mathrm{cn}^{-1}(x)\ , \quad E = 9m\ , \tag{47}$$

while out of the 4 band edges of period $4K$, one eigenstate has the form

$$\psi = \mathrm{dn}^{-1}(x)\mathrm{sn}(x)\mathrm{cn}^2(x)\ , \quad E = 9\ . \tag{48}$$

The three other eigenstates of period $4K$ have the form

$$\psi = \mathrm{dn}^{-1}(x)\mathrm{cn}^{-1}(x)[A + B\mathrm{sn}^2(x) + D\mathrm{sn}^4(x)]\ , \tag{49}$$

and the corresponding three eigenvalues satisfy the cubic equation

$$r^3 + 8(1+m)r^2 + 80mr + 64m(1+m) = 0\ , \quad E = -r + 1 + m\ . \tag{50}$$

Further, it is clear that there must be one doubly degenerate state of period $2K$ whose eigenvalue and eigenfunctions are not known analytically.

4 GAL Potentials [with at least one parameter $a, b, f, g =$ half-integer].

So far, we have discussed GAL potentials when all the four parameters a, b, f, g take integer values. We have seen that these are problems with a finite number of band gaps. We now consider the case when at least one of the parameters a, b, f, g is a half-integer. In general, all such problems have an infinite number of band gaps and one has only a few QES states.

4.1 $a =$ half-integer:

As mentioned in the introduction, we conjecture that the relations (6) and (7) are simultaneously valid when at least one of the four parameters a, b, f, g is a half-integer. In particular, in case $a = k + \frac{1}{2}$ and b, f, g are arbitrary numbers, we assert that $k+1$ QES energy values are identical for three GAL potentials, that is

$$
\begin{aligned}
&[a = k + \frac{1}{2}, b, f, g] \\
&\equiv \left[\frac{2(k+b+f+g)+3}{4}, \frac{2(k+b-f-g)-1}{4}, \frac{2(k+f-b-g)-1}{4}, \frac{2(k+g-b-f)-1}{4}\right] \\
&\equiv \left[\frac{2(k+b+f-g)+1}{4}, \frac{2(k+b+g-f)+1}{4}, \frac{2(k+f+g-b)+1}{4}, \frac{2(b+f+g-k)-1}{4}\right] . \quad (51)
\end{aligned}
$$

This relation needs some clarification. What is being conjectured here is that there are $k+1$ QES mid-band energy values of the potential with $a = k + 1/2$ and b, f, g being arbitrary numbers, which are the same as the band edge energy eigenvalues of the two other potentials given in (51). For these two potentials, we can always obtain $k+1$ QES band edges, since for both of them the sum of the four parameters characterizing the potentials is $2k$ [1].

We have explicitly verified our conjecture in the following cases: (i) Lamé potentials with $a = 1/2, 3/2$ (and $b = f = g = 0$); (ii) AL potentials [16] with $a = 1/2, 3/2$, $b = 1, 2, 3$ (and $f = g = 0$); (iii) GAL potentials with $a = 1/2, 3/2$, and either b or f or g is arbitrary while the remaining two parameters take any integer values. For example, we know that the mid-band state of the $a = 1/2$ Lamé potential is at $(1+m)/4$. Using eq. (51), we predict that both the potentials $[\frac{3}{4}, -\frac{1}{4}, -\frac{1}{4}, -\frac{1}{4}]$ and $[\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, -\frac{1}{4}]$ must have a QES band edge eigenvalue at $E = (1+m)/4$. Using Table 4 of I it is easily checked that this is indeed so, with the corresponding band edge eigenfunctions being $\psi = [\mathrm{dn}(x)\mathrm{cn}(x)\mathrm{sn}(x)]^{1/4}$ and

$\psi = \mathrm{dn}^{-1/4}(x)\mathrm{cn}^{-1/4}(x)\mathrm{sn}^{3/4}(x)$ respectively. We have similarly checked the equivalence in all the above-mentioned cases. For example, using the results [16] for the AL potential $[3/2, 1, 0, 0]$, we find (and verify using Table 4 of I) that the GAL potentials $[7/4, 3/4, -1/4, -1/4]$ and $[5/4, 5/4, 1/4, -1/4]$ have QES band edges at $E = (13+5m)/4 \pm \sqrt{9-9m+m^2}$. The corresponding band edge eigenfunctions are also easily written down. Similarly, using the results in I for the mid-band states of the GAL potentials we verify that the potentials $[t/2+1, t/2-1, (1-t)/2, -(1+t)/2]$ and $[(1+t)/2, -(1-t)/2, 1-t/2, t/2]$ indeed have a QES band edge eigenvalue $E = t^2 + 9m/4$, for any non-integer t.

Thus one is fairly confident that the remarkable relationship (51) is indeed valid. We now turn around and use (51) to predict new results for the mid-band states of GAL potentials with half-integer values of a in case b, f, g take arbitrary values. As an illustration, we predict that the potential $[1/2, b, f, g]$, where b, f, g are arbitrary numbers must have a QES state at energy

$$E = (b+1/2)^2 + (f+1/2)^2 m \ , \tag{52}$$

while the potential $[3/2, b, f, g]$ with b, f, g being arbitrary numbers must have two QES states with energies

$$\begin{aligned} E &= [(b+1/2)^2+1] + [1+(f+1/2)^2]m \\ &\pm 2\sqrt{[(b+1/2)+(f+1/2)m]^2 - (b+f+g+3/2)(b+f-g+1/2)m} \ . \end{aligned} \tag{53}$$

Perhaps some clarification is required as to when the QES state is a band edge and when it is a mid-band state. We believe that only those states correspond to band edges for which $a+b+f+g$ [or any other combination obtained by changing one or more of these parameters to $-a-1, -b-1, -f-1, -g-1$ respectively] is equal to an even integer (including zero). All other QES states should correspond to mid-band states. We thus believe that the QES energy values as given above for the potentials $[1/2, b, f, g]$ and $[3/2, b, f, g]$ with arbitrary b, f, g are in most cases the energies for the mid-band states of these potentials. In Sec. 6 we shall obtain the QES eigenstates corresponding to some of these QES eigenvalues by using the connection of the GAL potential problem and Heun's equation.

We can also predict the QES energy values when $a = 5/2$ and b, f, g are arbitrary numbers, by computing the band edges of either of the two potentials given in (51). In this way, we predict that the mid-band energy values for the three QES states of the GAL potential $[5/2, b, f, g]$ are solutions of the cubic equation

$$r^3 + 2[1 + 6b + (1 + 6f)m]r^2 + 4[2(4b^2 - 1) + 2(4f^2 - 1)m^2 + (4b^2 + 4f^2 - 4g^2 + 24bf$$
$$+8b + 8f - 2g + 3)m]r + 8m[2b + 1 + (2f + 1)m][4(b + f)^2 - (2g + 1)^2] = 0 \ , \tag{54}$$

where $r = -E + (3/2 - b)^2 + (3/2 - f)^2 m$. For the special cases (i) $b = f = g = 0$ as well as (ii) $b = 1, f = g = 0$, it is easily checked that eq. (54) agrees with well known results [16], thereby providing a powerful check on our calculations. Generalization to higher half-integer values is straightforward (in principle) and it is easy to see that energy values for $a + 1/2$ QES mid-band states can be predicted (at least in principle) when b, f, g are arbitrary numbers and a is a half-integer.

4.2 $a, b =$ half-integers:

Let us now discuss the case when both a and b are half-integers while f and g are arbitrary numbers using our conjecture that eqs. (6) and (7) are both simultaneously valid. In particular, for $a = k + \frac{1}{2},\ b = l + \frac{1}{2}$ while f, g are any numbers, we assert that the three potentials

$$\begin{aligned}
&[a = k + \frac{1}{2}, b = l + \frac{1}{2}, f, g] \\
&\equiv \left[\frac{k + l + f + g + 2}{2}, \frac{k + l - f - g}{2}, \frac{k + f - l - g - 1}{2}, \frac{k + g - l - f - 1}{2}\right] \\
&\equiv \left[\frac{k + l + f - g + 1}{2}, \frac{k + l + g - f + 1}{2}, \frac{k + f + g - l}{2}, \frac{l + f + g - k}{2}\right] \ ,
\end{aligned} \tag{55}$$

have the same $k + l + 2$ QES energy values. This means that the QES mid-band energy values of the potential with $a = k + 1/2, b = l + 1/2$ are the same as the band edge eigenvalues of the two other potentials in (55). For these two potentials, we can obtain $k + l + 2 = a + b + 1$ QES states since, for both of them, the sum of the four numbers characterizing the potentials is either $2k$ or $2l$. This is possible because the GAL potential $[a, b, f, g]$ remains unchanged when any one (or more) of the four parameters a, b, f, g changes to $[-a - 1, -b - 1, -f - 1, -g - 1]$ respectively.

As an illustration, consider the potential $[3/2, 1/2, f, g]$ with f, g being arbitrary numbers. It is then easily shown that this potential must have 3 QES mid-band states at

$$E_1 = 4 + (g + 1/2)^2 m \ , \tag{56}$$

$$E_{2,3} = 2 + [(f+1/2)^2 + 1]m \pm \sqrt{[2-(2f+1)m]^2 + [(2g+1)^2 - (2f-1)^2 m]} \ . \tag{57}$$

Note that out of the $k+l+2$ QES states, the energies for the $k+1$ states can also be obtained by considering the previous case of $a = k+1/2$ and b, f, g arbitrary and putting $b = l+1/2$ at the end of the calculation. As an illustration, consider the case $[3/2, b, f, g]$ where b is any arbitrary number. As shown in the last section, there are two QES energies given by eq. (53). On putting $b = 1/2$ in eq. (53) we find that the two QES energy values of $[3/2, 1/2, f, g]$ are precisely as given by eq. (57). For the special case $f = g = 0$, these eigenvalues agree with well-known results for the AL potential [16]. Similarly, consider the case $[5/2, b, 0, 0]$. As shown in the previous subsection, there are three QES energy values as given by eq. (54). On putting $b = 1/2$, it is easily seen that the three eigenvalues are $E = (1+9m/4), (1+25m/4), (9+m/4)$, in agreement with the eigenvalues obtained by us previously [16].

4.3 a, b, f = half-integers:

We shall now discuss the case when three out of the four parameters (say a, b, f) are half-integers while g is any number (not a half-integer), and obtain the energies of the QES mid-band states. Our argument is again based on the assertion that relations (6) and (7) are simultaneously valid. In particular, we assert that for $a = k+\frac{1}{2}, b = l+\frac{1}{2}, f = n+\frac{1}{2}$ the three potentials

$$\begin{aligned} & [a = k+\frac{1}{2}, b = l+\frac{1}{2}, f = n+\frac{1}{2}, g] \\ \equiv & \left[\frac{2(k+l+n+g)+5}{4}, \frac{2(k+l-n-g)-1}{4}, \frac{2(k+n-l-g)-1}{4}, \frac{2(l+n-k-g)-1}{4}\right] \\ \equiv & \left[\frac{2(k+l+n-g)+3}{4}, \frac{2(k+l+g-n)+1}{4}, \frac{2(k+n+g-l)+1}{4}, \frac{2(l+n+g-k)+1}{4}\right] \end{aligned} \tag{58}$$

have identical $k+l+n+3$ QES energy values. What is being asserted here is that the $k+l+n+3$ QES mid-band energies of the potential with half-integer values of a, b, f and arbitrary g are the same as the band edge energy eigenvalues of the two other potentials in (58). This happens because for arbitrary values of g, for the two potentials

$$\left[\frac{2(k+l+n+g)+5}{4}, \frac{2(k+l-n-g)-1}{4}, \frac{2(k+n-l-g)-1}{4}, \frac{2(l+n-k-g)-1}{4}\right] , \tag{59}$$

$$\left[\frac{2(k+l+n-g)+3}{4}, \frac{2(k+l+g-n)+1}{4}, \frac{2(k+n+g-l)+1}{4}, \frac{2(l+n+g-k)+1}{4}\right] , \tag{60}$$

we can always obtain $k+l+n+3 = a+b+f+3/2$ QES band edges, since for both potentials the sum of the four numbers characterizing the potentials is either $2k$ or $2l$ or $2n$. This is possible because the GAL potential $[a, b, f, g]$ remains unchanged when any one (or more) of the four parameters a, b, f, g change to $-a-1, -b-1, -f-1, -g-1$ respectively. Thus we conjecture that the potential $[k+1/2, l+1/2, n+1/2, g]$, for arbitrary g has $k+l+n+3$ QES mid-band states. We now make a number of predictions for the energy eigenvalues of the mid-band states for GAL potentials of the form $[k+1/2, l+1/2, n+1/2, g]$.

1. We predict that the potential $[1/2, 1/2, 1/2, g]$ has three QES mid-band states with energy values

$$E_1 = (1+m) \ , \ E_2 = 1 + (g+1/2)^2 m \ , \ E_3 = (g+1/2)^2 + m \ . \tag{61}$$

2. The potential $[3/2, 1/2, 1/2, g]$ has four QES mid-band states with energy values

$$E_1 = (4+m) \ , \ E_2 = (g+1/2)^2 + 4m \ , \ E_{3,4} = 2(1+m) \pm \sqrt{4(1-m)^2 + (2g+1)^2 m} \ . \tag{62}$$

3. The potential $[3/2, 3/2, 1/2, g]$ has five QES mid-band states with energy values

$$\begin{aligned} E_1 &= (g+1/2)^2 + 4m \ , \ E_{2,3} = 5 + 2m \pm \sqrt{4(2-m)^2 + (2g+1)^2 m - 4m} \ , \\ E_{4,5} &= 5 + [(g+1/2)^2 + 1]m \pm \sqrt{[4-(2g+1)m]^2 - (2g-3)^2 m + 4m} \ . \end{aligned} \tag{63}$$

4. The potential $[3/2, 3/2, 3/2, g]$ has six QES mid-band states with energy values

$$\begin{aligned} E_{1,2} &= (5+m) \pm \sqrt{16(1-m)^2 + (2g+1)^2 m} \ , \\ E_{3,4} &= 5 + [(g+1/2)^2 + 1]m \pm \sqrt{16 - (2g+1)^2 m(1-m)} \ , \\ E_{5,6} &= [(g+1/2)^2 + 1] + 5m \pm \sqrt{16m^2 + (2g+1)^2(1-m)} \ . \end{aligned} \tag{64}$$

5. One can readily obtain some QES mid-band energy values when (i) k is an arbitrary integer while l and/or n are either 0 or 1 ; (ii) k, l are arbitrary integers with $n = 0$ or 1 and g being any arbitrary number. It would be nice to prove (or disprove) these conjectures and more importantly, try to obtain the corresponding energy eigenfunctions. We shall have something to say about this point when we discuss the implications of these results in the context of Heun's equation.

6. Note that out of the $k+l+n+3$ QES states, the energy values for $k+l+2$ states can also be obtained by considering the previous case [$a = k+1/2, b = l+1/2, f, g =$arbitrary] and putting $f = n+1/2$ at the end of the calculation. As an illustration, consider the case $[3/2, 1/2, f, g]$ where f, g are arbitrary numbers. As shown in the last subsection, there are three QES energy values and they are given by eqs. (56) and (57). On putting $f = 1/2$ in these equations, we find that three (out of four) QES eigenvalues of $[3/2, 1/2, 1/2, g]$ as given by eq. (63) are correctly obtained.

4.4 $a, b, f, g =$ half-integers:

Finally, let us consider the case when all four parameters are half-integers. The potential is of the form $[k+1/2, l+1/2, n+1/2, p+1/2]$, where k, l, n, p are integers and we take $k \geq l \geq n \geq p$. We shall discuss the two cases when the sum $k+l+n+p$ is an even or an odd integer separately.

4.4.1 $k+l+n+p =$ even integer:

As shown in I, for the GAL potential (2), QES energies are obtained when the sum of the four parameters is an even integer including zero. In this case, as already shown by Takemura [15], both the relations (6) and (7) are simultaneously valid. We would like to assert here that in this case, in general there should be $k+n+l+p+4$ QES states, since the sum of the four numbers characterizing the potentials is either $2k$ or $2l$ or $2n$ or $2p$. This is possible because the GAL potential $[a, b, f, g]$ remains unchanged when any one (or more) of the four parameters a, b, f, g change to $-a-1, -b-1, -f-1, -g-1$ respectively. Unfortunately, in the several specific cases that we have examined, we find that some of the eigenvalues simply get repeated. Thus the true number of QES states may be much less than $k+l+n+p+4$. For example, consider the case of $[5/2, 1, 2, 1/2, 1/2]$. While naively we expect six QES states, we only find three QES levels at $E = (1+m), (1+9m), (9+m)$. Similarly, for the potential $[9/2, 1/2, 1/2, 1/2]$ while naively we expect 8 QES states, we only find four QES levels at $E = (1+m),\ (1+25m),\ (25+m),\ 9(1+m)$.

4.4.2 $k + l + n + p =$ odd integer:

Let us now discuss perhaps the most intriguing case when all four parameters are half-integers and their sum is an odd integer. While it is clear from I that no QES band edges can be obtained when $k+l+n+p$ is an odd integer, it is not obvious whether mid-band states can be obtained in this case. In fact we shall now obtain energy values for several QES mid-band states for these potentials. As argued in the introduction, while Takemura has shown that relations (6) or (7) are valid when $a + b + f + g$ is an even or an odd integer respectively (and all are integers), we conjecture that irrespective of whether $a + b + f + g$ is an odd or an even integer, both the relations are always valid. Using eqs. (6) and (7), it then follows that irrespective of whether $a + b + f + g$ is an even or an odd integer, they always have a partner potential of the form $[k + 1/2, l + 1/2, n + 1/2, p + 1/2]$ where k, l, n, p are all integers and their sum is an odd integer. We thus conjecture that when $a + b + f + g$ is an even integer, then the QES mid-band energy values of the potential $[(a + b + f + g + 1)/2, (a + b - f - g - 1)/2, (a + f - b - g - 1)/2, (a + g - b - f - 1)/2]$ are the same as the band edge eigenvalues of the potential $[a, b, f, g]$. Similarly, when $a + b + f + g$ is an odd integer, then we conjecture that the QES mid-band energy values of the potential $[(a + b + f - g)/2, (a + b + g - f)/2, (a + f + g - b)/2, (b + f + g - a)/2]$ are the same as the band edge eigenvalues of the potential $[a, b, f, g]$. This is rather remarkable. Since the band edges of the GAL potentials $[a, b, f, g]$ with integer a, b, f, g are all known (at least in principle), hence one has a prediction for $k+l+n+p+4$ QES mid-band states of the GAL potentials of the type $[k + 1/2, l + 1/2, n + 1/2, p + 1/2]$ when $k + l + n + p$ is an odd integer. To be precise, we predict that the potential $[k+1/2, l+1/2, n+1/2, p+1/2]$ has $k+l+n+p+4$ QES mid-band states in case $k+l+n+p$ is an odd integer and these eigenvalues are identical to the band edges of the potential $[(k+l+n+p+3)/2, (k+l-n-p-1)/2, (k+n-l-p-1)/2, (k+p-l-n-1)/2]$. As an illustration, knowing the band edges of the Lamé potential $2m\mathrm{sn}^2(x)$ we predict that the three QES mid-band state energy values of the potential $[1/2, 1/2, 1/2, -1/2]$ must be at $E = m, 1, 1 + m$. Similarly, we predict that the five mid-band QES energies of the potential $[3/2, 1/2, 1/2, 1/2]$ are at

$$E_2 = 1 + m \ , \ E_3 = (1 + 4m) \ , \ E_4 = (4 + m) \ , \ E_{1,5} = 2(1 + m) \pm 2\sqrt{1 - m + m^2} \ , \qquad (65)$$

The validity of our conjecture and in a way the consistency of our whole approach can be checked by extrapolating the results obtained in the previous subsection. In particular, in the last subsection we have discussed the case when the potential is of the form $[k+1/2, l+1/2, n+1/2, g]$ where g is any arbitrary number and have seen that in that case one obtains energies for $k+l+n+3$ QES mid-band states. On choosing $g = p+1/2$ and choosing k, l, n, p such that their sum is an odd integer, we have verified in many cases the validity of our conjectures.

We thus predict that the potentials $[3/2, 1/2, 1/2, 1/2]$, $[5/2, 1/2, 1/2, -1/2]$ and $[3/2, 3/2, 1/2, -1/2]$ have exactly the same (five) QES mid-band energies as the band edge energy eigenvalues of the potentials $[2, 0, 0, 0]$, $[2, 1, 1, 0]$, $[2, 1, 0, 0]$ respectively. It is worth observing that there are exactly three ways of obtaining $k+l+n+p = 1$ given that $k \geq l \geq n \geq p$ and that while k, l, n are non-negative integers, p is ≥ -1. It may be noted here that in case n is also -1 then the corresponding partner potentials as given by eqs. (6) and (7) are potentials of the type $[a, a, b.b]$ and similarly if both $n = l = -1$ then the partner potentials are of type $[a, a, a, a]$ which as explained in Sec. 2, are not included in our analysis. Thus, put another way, the problem of finding the number of independent potentials with say a band gaps reduces to finding four integers k, l, n, p with $k \geq l \geq n \geq p$ (with $n \geq 0, p \geq -1$) such that their sum equals $2a-3$.

5 GAL Potentials $[a, b, f, g =$ arbitrary numbers$]$.

So far, we have discussed the cases when the four parameters a, b, f, g take integer values or at least one of them is half-integer. Now we want to extend this discussion to the general case when a, b, f, g take arbitrary values.

As seen in Sec. 4, when either one, two or three of the parameters are half-integer while the remaining parameters are arbitrary, then clearly the corresponding partners indeed correspond to the case where a, b, f, g are arbitrary numbers. We therefore conjecture that the relations (6) and (7) are valid even when the four parameters a, b, f, g take any arbitrary values, the only restriction being that either $a+b+f+g$ or any other combination obtained by replacing one or more of these parameters by $-a-1, -b-1, -f-1, -g-1$ respectively is a non-negative even integer. Further, in that case the two partner potentials have the same band-edge eigenvalues as the QES mid-band energies of the potentials where either one, two or three of the parameters are half-integers.

A few illustrative examples are in order here. Consider the potential $[4/5, 2/5, 2/5, 2/5]$. In this case while $a + b + f + g = 2$, no other combination characterizing relations (38) to (46) gives an even integer. Using eq. (6) we find that this potential has a GAL partner $[3/5, 3/5, 3/5, 1/5]$. Using Table 4 of I it is easily shown that both these potentials have two (identical) QES band edge energy eigenvalues

$$E = \frac{26}{25}(1 + m) \pm \frac{2}{5}\sqrt{1 - m + m^2} \ . \tag{66}$$

If instead we consider the potential $[17/5, 8/5, 7/5, 6/5]$, then only $a + f - b - g$ is an integer and the corresponding GAL partner potential is $[13/5, 12/5, 11/5, 2/5]$ and both have one (identical) QES energy. Of course it can happen that a, b, f, g are such that more than one of the relations (37) to (46) are satisfied. In that case one has more QES band edge eigenvalues. For example, if the potential parameters are such that $a+b-g-f-1, a+f-b-g-1$ as well as $a+g-f-b-1$ are nonnegative integers, then using eqs. (38) to (40) it is easily seen that the number of QES energy eigenvalues is equal to $(3a-b-f-g)/2$. One illustration of this is the potential $[11/5, 1/5, 1/5, 1/5]$ which has three QES energies $(1+m), 144/25+m, 1+(144/25)m$. We might add here that the discussion above is valid even if the numbers a, b, f, g are irrational numbers but such that $a + b + f + g$ or any other combination obtained by replacing one or more of a, b, f, g to $-a - 1, -b - 1, -f - 1, -g - 1$ respectively is an even integer (including zero).

We would like to restate here that when all four parameters a, b, f, g are integers, one has a finite number of band gaps. In all other cases one expects to have an infinite number of bands and band gaps out of which only a few are QES states.

6 Implications for Heun's Equation.

Heun's equation, a second order linear differential equation with four regular singular points has been extensively discussed in the mathematics literature [20, 21, 22]. The intimate connection between Heun's equation and GAL potentials is well known [9]. In recent years, this equation has also proved very useful in the context of a number of physical problems, like quasi-exactly solvable systems [23], sphaleron stability [24], Calogero-Sutherland models [25], higher dimensional correlated systems [26], Kerr-de Sitter black holes [27], and finite lattice Bethe ansatz systems [28].

The canonical form of Heun's equation is given by [20]

$$\left[\frac{d^2}{dx^2} + (\frac{\gamma}{x} + \frac{\delta}{x-1} + \frac{\epsilon}{x-c})\frac{d}{dx} + \frac{\alpha\beta x - q}{x(x-1)(x-c)}\right]G(x) = 0 \ , \tag{67}$$

where $\alpha, \beta, \gamma, \delta, \epsilon, q, c$ are parameters, except that $c \neq 0, 1$ and the first five parameters are related by

$$\gamma + \delta + \epsilon = \alpha + \beta + 1 \ . \tag{68}$$

The four regular singular points of eq. (67) are located at $x = 0, 1, c$ and the point at infinity.

If we make the transformation $x = \mathrm{sn}^2(y, m)$, then Heun's equation takes the form [20]

$$\begin{aligned} &F''(y) + [(1-2\epsilon)m\frac{\mathrm{sn}(y,m)\mathrm{cn}(y,m)}{\mathrm{dn}(y,m)} + (1-2\delta)\frac{\mathrm{sn}(y,m)\mathrm{dn}(y,m)}{\mathrm{cn}(y,m)} + (2\gamma-1)\frac{\mathrm{cn}(y,m)\mathrm{dn}(y,m)}{\mathrm{sn}(y,m)}]F'(y) \\ &-[4mq - 4\alpha\beta m\mathrm{sn}^2(y,m)]F(y) = 0 \ , \end{aligned} \tag{69}$$

where $c = 1/m$, $G(x) \equiv F(y)$. The periodic solutions of eq. (69) correspond to the polynomial solutions of eq. (67) while the quasi-periodic solutions correspond to non-polynomial solutions of (67).

The interesting point is that after a transformation, the Schrödinger equation (9) for the GAL potential (2) is in fact Heun's eq. (69). In particular, let us start from the Schrödinger equation (9) for the GAL potential (2). On substituting

$$\psi(y) = \mathrm{dn}^{-b}(y)\mathrm{cn}^{-f}(y)\mathrm{sn}^{-g}(y)\phi(y) \ , \tag{70}$$

one can show that $\phi(y)$ satisfies the differential equation

$$\begin{aligned} &\phi''(y) + 2[mb\frac{\mathrm{sn}(y,m)\mathrm{cn}(y,m)}{\mathrm{dn}(y,m)} + f\frac{\mathrm{sn}(y,m)\mathrm{dn}(y,m)}{\mathrm{cn}(y,m)} - g\frac{\mathrm{cn}(y,m)\mathrm{dn}(y,m)}{\mathrm{sn}(y,m)}]\phi'(y) \\ &-[R - Qm\mathrm{sn}^2(y,m)]\phi(y) = 0 \ , \end{aligned} \tag{71}$$

where

$$R = -E + m(g+b)^2 + (f+g)^2 \ , \quad Q = (b+f+g)(b+f+g-1) - a(a+1) \ . \tag{72}$$

Thus once we obtain solutions of the Schrödinger equation for the GAL potential (2), then we can immediately write the solutions for the periodic form of Heun's eq. (69) and the solutions of the original Heun's eq. (67) with the identification

$$\begin{aligned} &\gamma = \frac{1}{2} - g \ , \ \delta = \frac{1}{2} - f \ , \ \epsilon\frac{1}{2} - b \ , \\ &\alpha + \beta = \frac{1}{2} - (b+f+g) \ , \ 4\alpha\beta = Q \ , \ 4mq = R \ , F(y) \equiv \phi(y) \ . \end{aligned} \tag{73}$$

We now make a crucial observation. From eq. (72), it follows that if under any transformation, the parameters b_1, f_1, g_1 change to b_2, f_2, g_2 and the energy E remains invariant, then the corresponding values of R are related by

$$R_1 - m(b_1+g_1)^2 - (f_1+g_1)^2 = R_2 - m(b_2+g_2)^2 - (f_2+g_2)^2 \ . \tag{74}$$

Making use of eq. (74) and the connection between GAL potentials as given by eqs. (6) and (7), we can obtain interesting relations for Heun's equation. Using the fact that the two GAL potentials given by eq. (6) have the same band edge energy eigenvalues and the eigenfunctions for both the partners as are given by eqs. (37) to (41), one can obtain the connection between the two corresponding solutions of Heun's equation. For example, consider the solution (37) and the corresponding solution of the GAL partner obtained by the above substitution. Using eq. (74) it then follows that corresponding to a given periodic (i.e. polynomial) solution of Heun's equation with parameter set $(\alpha, \beta, \delta, \epsilon, \gamma, q)$ there always exists another periodic solution with the *same* q provided the other parameters change as follows:

$$\gamma \to \alpha \ , \ \alpha \to \gamma \ , \ \beta \to \beta \ , \ \epsilon \to 1+\beta-\delta \ , \delta \to 1+\beta-\epsilon \ . \tag{75}$$

Similarly, on considering the other three periodic solutions as given by eqs. (38) to (40) and the corresponding solutions of the GAL partner potential with the same energy, we find that corresponding to a given periodic solution of Heun's equation, there exist the following three (periodic) solutions with the change of parameters given by (note that $R = 4mq$ and $c = 1/m$)

$$\gamma \to 1+\beta-\epsilon \ , \ \alpha \to \delta \ , \ \beta \to \beta \ , \ \epsilon \to 1+\beta-\gamma \ , \delta \to \alpha \ , \ q \to q - \beta(\delta-\alpha) \tag{76}$$

$$\epsilon \to \alpha \ , \ \alpha \to \epsilon \ , \ \beta \to \beta \ , \ \gamma \to 1+\beta-\delta \ , \delta \to 1+\beta-\gamma \ , \ q \to q - \beta(\epsilon-\alpha)c \ , \tag{77}$$

$$\gamma \to 1+\beta-\gamma \ , \ \alpha \to 1+\beta-\alpha \ , \ \beta \to \beta \ , \ \epsilon \to 1+\beta-\epsilon \ , \delta \to 1+\beta-\delta \ , q \to q+\beta[(\alpha-\delta)+(\alpha-\epsilon)c] \ . \tag{78}$$

Thus given a periodic solution of Heun's equation, one immediately has 4 other periodic solutions as given by eqs. (75) to (78). We have checked that if instead we consider the two partner GAL potentials given by eq. (7) and consider the corresponding eigenfunctions given by eqs. (42) to (46) (and those of the corresponding GAL partner potentials with the same energy), then we again obtain the *same* relations [eqs. (75) to (78)]. As an additional check, we have looked at the partner GAL potentials as given by eqs. (51), (59), (60), (55) and (8) and in all these cases we get back the relations (75) to (78), which to the best of our knowledge, are new results. Several comments are in order:

1. In Sec. 4 we have shown that the QES mid-band energy values of the GAL potential $[a = k + 1/2, b, f, g]$ are the same as the QES energies of the two GAL potentials given in (51). What does this imply in the context of Heun's equation? It is easily shown that as a consequence of the discussion in Sec. 4, given a periodic solution of Heun's equation with the set of parameters $\alpha, \beta, \gamma, \delta, \epsilon, q$, one has a corresponding quasi-periodic solution with changed parameters:

$$\gamma \to 2-\alpha\ ,\ \alpha \to 1+\gamma-\alpha\ ,\ \beta \to 1+\beta-\alpha\ ,\ \epsilon \to 1+\beta-\delta\ , \delta \to 1+\beta-\epsilon\ ,$$
$$q \to q+(\alpha-1)[(1+\beta-\delta)+(1+\beta-\epsilon)c]\ . \tag{79}$$

   An additional check on this relation is obtained by using the connection between the mid-band states of the GAL potentials $[a = k+1/2, b = l+1/2, f, g]$ and $[a = k+1/2, b = l+1/2, f = n+1/2, g]$ and the QES energies of the potentials (55), (59) and (60) and we again obtain the *same* connection between the periodic and quasi-periodic solutions of Heun's equation.

2. Using the results in Sec. 4 regarding the case when three of the four parameters a, b, f, g are half-integers, we also obtain two more relations connecting the periodic and quasi-periodic solutions of Heun's equation. In particular, given a periodic solution of Heun's equation with the set of parameters $\alpha, \beta, \gamma, \delta, \epsilon, q$, it implies the following two quasi-periodic solutions of Heun's equation:

$$\gamma \to \alpha\ ,\ \alpha \to 1+\alpha-\epsilon\ ,\ \beta \to \delta\ ,\ \epsilon \to 1+\delta-\beta\ , \delta \to 1+\beta-\epsilon\ , q \to q+\alpha(\beta-\delta)\ , \tag{80}$$

$$\gamma \to \alpha\ ,\ \alpha \to 1+\alpha-\delta\ ,\ \beta \to \epsilon\ ,\ \epsilon \to 1+\beta-\delta\ , \delta \to 1+\epsilon-\beta\ , q \to q+4\alpha(\beta-\epsilon)c\ . \tag{81}$$

3. Needless to say that if instead, a quasi-periodic solution of Heun's equation is given, then by inverting eqs. (79) to (81), we immediately obtain three periodic solutions of Heun's equation.

So far we have discussed how the connections between different GAL potentials can help in finding new solutions of Heun's equation. It may happen that in some cases it may be simpler to solve the algebraic

Heun's equation (67) rather than its periodic variant. We now show that this is indeed so in the case of several quasi-periodic mid-band eigenfunctions. Consider for example the GAL potentials when either b or f or g is 1/2 while the other three parameters are arbitrary. Using arguments of section IV, we can easily obtain the eigenvalues for mid-band states for these potentials. We shall now show that using these eigenvalues we can easily solve the algebraic form of Heun's eq. (67) and hence using the connection as explained above, obtain the eigenfunctions for the mid-band states of these GAL potentials.

Consider the GAL potential $[a, 1/2, f, g]$ where a, f, g are arbitrary numbers, Note that using the relations (6), (7) and Table 4 of I, it is easily shown that the QES mid-band eigenvalue of the GAL potential $[a, 1/2, f, g]$ is at $E = (a+1/2)^2 + m(g+1/2)^2$. On using the connection formulas (72) and (73) it is easily shown that the corresponding parameters for Heun's eq. (67) are

$$\gamma = 1/2 - g \ , \ \delta = 1/2 - f \ , \ \epsilon = 0 \ , \ \alpha = (a - f - g + 1/2)/2 \ ,$$

$$\beta = -(a + f + g + 1/2)/2 \ , \ q = (a + f + g + 1/2)(f + g - a - 1/2)(c/4) \ . \tag{82}$$

Remarkably, for these parameters, it is straightforward to obtain the solution of the algebraic Heun's eq. (67) and show that

$$G(x) = F[(a - f - g + 1/2)/2, -(a + f + g + 1/2)/2, 1/2 - g; x] \ , \tag{83}$$

where $F(a, b, c; x)$ is the hypergeometric function. The corresponding mid-band state eigenfunction for the GAL potential $[a, 1/2, f, g]$ is then immediately written down. We have verified that this is indeed the correct eigenfunction in the following cases (i) $a = f = g = 0$ (ii) a integral, $f = g = 0$ (iii) a arbitrary while f, g are integral.

Similarly, for the GAL potential $[a, b, 1/2, g]$, with arbitrary a, b, g, the QES mid-band energy eigenvalue is $E = (g+1/2)^2+(a+1/2)^2 m$ and proceeding as above, it is easily shown that the solution of the algebraic Heun's eq. (67) is given by

$$G(x) = F[(a - b - g + 1/2)/2, -(a + b + g + 1/2)/2, 1/2 - g; mx] \ . \tag{84}$$

And finally, for the GAL potential $[a, b, f, 1/2]$, with arbitrary a, b, f, the QES energy is $E = (f + 1/2)^2 + (b + 1/2)^2 m$ and proceeding as above, it is easily shown that the solution of the algebraic Heun's eq. (67) is given by

$$G(x) = F[(a-b-f+1/2)/2, -(a+b+f+1/2)/2, 1/2-b; (1-mx)/(1-m)] \ . \tag{85}$$

On using the solutions (83) to (85) it follows that for the potential $[a, 1/2, 1/2, 1/2]$ one knows three QES mid-band energy eigenstates. In the special case when $a = 2k+3/2$ these eigenstates are the mid-band QES eigenstates for the potential $[2k+3/2, 1/2, 1/2, 1/2]$ where the sum of the four parameters characterizing the potential is an odd integer.

7 Summary and Open Questions.

In this paper, we have addressed many issues regarding GAL potentials with a number of choices for the parameters a, b, f, g. The most interesting case is when all the four parameters are integers. This is a potential with a finite number of band gaps. We have been able to count the number of independent GAL potentials with a given number of band gaps and completely specify the nature of the band edge eigenfunctions. We have introduced the new concept of self-dual potentials which are not self-isospectral. We are also able to specify how many of the independent potentials with a given number of band gaps have supersymmetric partner potentials and how many have non-supersymmetric partner potentials. Finally, using the results for the GAL potentials, we have shown that given any one periodic solution of Heun's equation, one can obtain four more periodic solutions.

We have also discussed several issues related with GAL potentials when one or more of the parameters take half-integer values. In particular, while nothing is known so far about GAL potentials when three of the parameters take half-integer values, we have been able to obtain the QES energy values for several of these potentials. Further, using these eigenvalues and the algebraic form of Heun's equation, we have also been able to obtain the corresponding eigenfunctions for potentials of the form $[a, 1/2, f, g], [a, b, 1/2, g], [a, b, f, 1/2]$ where a, b, f, g are arbitrary numbers. The key point to make while addressing these questions is that the relations (6) and (7) are not only valid when the four parameters a, b, f, g are integers but also when one or more of these parameters take half-integer values. This in turn immediately implies that these relations are also valid when the four parameters a, b, f, g take arbitrary values so long as either their sum $a+b+f+g$ [or one or more of the combinations obtained by changing one or more of the parameters to $-a-1, -b-1, -f-1, -g-1$ respectively] is a nonnegative even integer. We have also conjectured that both relations (6) and (7) are simultaneously valid when a, b, f, g are integers and that the energy eigenvalues for the band edges of these potentials are the same as mid-band QES energy values of GAL potentials in which all four parameters are half-integers and their sum is an odd integer. Finally, using these results we have also shown that given a periodic solution of Heun's equation, one can immediately obtain three quasi-periodic solutions of the same equation.

This work raises several issues which we have not been able to address satisfactorily:

1. Can one explicitly write down all seven KdV equations of seventh order?

2. What are the QES eigenfunctions for GAL potentials when one or more of the parameters is half-integral ($\geq 3/2$) while the remaining parameters are arbitrary?

3. The problem when two of the four parameters are half-integers needs further study. In particular, it is still not clear how many QES energy eigenvalues can be obtained, in general, in that case.

4. When the sum of all the four parameters is an even integer, it is clear that the QES states correspond to band edges. However, a complete understanding is still lacking regarding the number of QES states for various values of a, b, f, g. Further, when the sum of the four parameters is an odd integer, the form of the QES eigenfunctions is not clear when the half-integer parameters are $> 1/2$.

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