

# A Class of $N$ -body Problems With Nearest And Next-to-Nearest Neighbour Interactions

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## Abstract

We obtain the exact ground state and a part of the excitation spectrum in one dimension on a line and the exact ground state on a circle in the case where  $N$  particles are interacting via nearest and next-to-nearest neighbour interactions. Further, using the exact ground state, we establish a mapping between these  $N$ -body problems and the short-range Dyson models introduced recently to model intermediate spectral statistics. Using this mapping we compute the one- and two-point functions of a related many-body theory and show the absence of long-range order in the thermodynamic limit. However, quite remarkably, we prove the existence of an off-diagonal long-range order in the symmetrized version of the related many-body theory. Generalization of the models to other root systems is also considered. Besides, we also generalize the model on the full line to higher dimensions. Finally, we consider a model in two dimensions in which all the states exhibit novel correlations.

**Keywords :** Exactly solvable models, banded random matrices, off-diagonal long-range order

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# Contents

1	Introduction	3
2	N-body problem in one dimension on a line	5
3	N-body problem in one-dimension with periodic boundary condition	13
4	Some exact results for the many-body problem	15
5	Off-diagonal long-range order	18
6	The $B_N$ model in one dimension	24
7	$BC_N$ model in one dimension with periodic boundary condition	27
8	$N$ -body problem in $D$ -dimensions	28
9	Short-range model in two dimensions with novel correlations	29
10	Summary	32

# 1 Introduction

In recent years, the Calogero-Sutherland type  $N$ -body problems [1, 2] in one dimension have attracted considerable attention not only because they are exactly solvable [3] but also due to their relationship with (1+1)-dimensional conformal field theory, random matrix theory [4] etc. In particular, the connections between exactly solvable models [5] and random matrix theory [6] have been very fruitful. For example, by mapping these models to random matrices from an orthogonal, unitary or symplectic Gaussian ensemble, Sutherland [2] was able to obtain all static correlation functions of the corresponding many body theory. The key point of this model is the pairwise long-range interaction among the  $N$  particles. One may add here that the family consisting of exactly solvable models, related to fully integrable systems, is quite small [3] and their importance lies in the fact that their small perturbations describe wide range of physically interesting situations. Further, recent developments [7] relating equilibrium statistical mechanics to random matrix theory owing to non-integrability of dynamical systems has made the pursuit of unifying seemingly disparate ideas a very important theme. The results presented in this paper belong to the emerging intersection of several frontiers like quantum chaos, random matrix theory, many-body theory and equilibrium statistical mechanics [8].

The universality in level correlations in linear (Gaussian) random matrix ensembles agrees very well with those in chaotic quantum systems [9] as also in many-body systems like nuclei [6]. On the other hand, random matrix theory was connected to the world of exactly solvable models when the Brownian motion model was presented by Dyson [10], and later on, by the works on level dynamics [11]. However, there are dynamical systems which are neither chaotic nor integrable - the so-called pseudo-integrable systems [12]. It is known that the spectral statistics of such systems are “non-universal with a universal trend” [13]. In particular, for Aharonov-Bohm billiards, the level spacing distribution is linear for small spacing and it falls off exponentially for large spacing [14]. Similar features are numerically observed for the Anderson model in three dimensions at the metal-insulator transition point [15]. To understand these statistical features, and in the context of random banded matrices, a new random matrix model (which has been called as the short-range Dyson model in [16]) was introduced [17, 18] wherein the energy levels are treated as in the Coulomb gas model with the difference that only nearest neighbours interact. This new model explains features of intermediate statistics [16] in some polygonal billiards.

In view of all this it is worth enquiring if one can construct an  $N$ -body problem which is exactly solvable and which is connected to the short-range

Dyson model (SRDM)? If possible, then using this correspondence one can hope to calculate the correlation functions of the corresponding many-body theory and see if the system exhibits long-range order and/or off-diagonal long-range order.

The purpose of this paper is to present two such models in one dimension, one on a line and the other on a circle. We obtain the exact ground state and a part of the excitation spectrum on a line and the exact ground state on a circle in case the  $N$  particles are interacting via nearest and next-to-nearest neighbour interactions [19]. Further, in both the cases we show how the norm of the ground state wave function is related to the joint probability density function of the eigenvalues of short-range Dyson models. Using this mapping, we obtain one- and two-point functions of a related many-body theory in the thermodynamic limit and prove the absence of long-range order in the system. However, quite remarkably, we prove the existence of an off-diagonal long-range order in the symmetrized version of the corresponding many-body theory [20].

We also extend this work in several different directions. For example, we consider an  $N$ -body problem with nearest and next-to-nearest neighbour interaction in an arbitrary number of dimensions  $D$  and show that the ground state and a part of the excitation spectrum can still be obtained analytically. We also obtain a part of the bound state spectrum in one dimension (both on a full line and on a circle) by replacing the root system  $A_{N-1}$  by  $BC_N, D_N$  etc. Besides, we also consider a model in two dimensions for which novel correlations are present in the ground as well as the excited states.

The plan of the paper is the following. In Sec.II we consider an  $N$ -body problem on a line characterized by the Hamiltonian (throughout this paper we shall use  $\hbar = m = 1$ )

$$H = -\frac{1}{2} \sum_{i=1}^N \frac{\partial^2}{\partial x_i^2} + g \sum_{i=1}^{N-1} \frac{1}{(x_i - x_{i+1})^2} - G \sum_{i=2}^{N-1} \frac{1}{(x_{i-1} - x_i)(x_i - x_{i+1})} + V \left( \sum_{i=1}^N x_i^2 \right) \quad (1)$$

with  $G \geq 0$  while  $g > -1/4$  to prevent the collapse that a more attractive inversely quadratic potential would cause. We show that the ground state and at least a part of the excitation spectrum can be obtained if

$$g = \beta(\beta - 1), \quad G = \beta^2, \quad V = \frac{\omega^2}{2} \sum_{i=1}^N x_i^2. \quad (2)$$

Note that with the above restriction on  $G$  and  $g$ ,  $\beta \geq 1/2$ . Further we also point out the connection between the norm of the ground state wave function

and the joint probability distribution function for eigenvalues in SRDM. In Sec. III we consider another  $N$ -body problem, but this time on a circle characterized by the Hamiltonian

$$H = -\frac{1}{2} \sum_{i=1}^N \frac{\partial^2}{\partial x_i^2} + g \frac{\pi^2}{L^2} \sum_{i=1}^N \frac{1}{\sin^2[\frac{\pi}{L}(x_i - x_{i+1})]} - G \frac{\pi^2}{L^2} \sum_{i=1}^N \cot \left[ (x_{i-1} - x_i) \frac{\pi}{L} \right] \cot \left[ (x_i - x_{i+1}) \frac{\pi}{L} \right], \quad (x_{N+1} = x_1), \quad (3)$$

(where again  $G \geq 0$  while  $g > -1/4$ ) and obtain the exact ground state in case  $g$  and  $G$  are again as related by eq. (2). Further, we also point out the connection between the norm of the ground state wave function and the joint probability distribution function for eigenvalues of short-range circular Dyson model (SRCDM). Using this connection, in Secs. IV and V we obtain several exact results for the corresponding many-body theory in the thermodynamic limit. In particular, in Sec. IV we calculate the two-particle correlation functions of a related many-body theory in the thermodynamic limit and prove the absence of long-range order in the system. In Sec. V we consider the symmetrized version of the model considered in Sec. III and show the existence of an off-diagonal long-range order in the bosonic system in the thermodynamic limit. In Sec. VI we consider the  $BC_N$  generalization of the model (1) and obtain the exact ground state of the system. In Sec. VII we consider the  $BC_N$  generalization of the model (3) and obtain the exact ground state of the system. In Sec. VIII we consider a generalization of the model (1) to higher dimensions and obtain the ground state and a part of the excitation spectrum. In Sec. IX we consider a variant of the model (1) in two dimensions and obtain the ground state as well a class of excited states all of which have a novel correlation built into them. Finally, in Sec. X we summarize the results obtained and point out several open questions.

## 2 N-body problem in one dimension on a line

Let us start from the Hamiltonian (1) and restrict our attention to the sector of configuration space corresponding to a definite ordering of the particles, say

$$x_i \geq x_{i+1}, \quad i = 1, 2, \dots, N-1. \quad (4)$$

On using the ansatz

$$\psi = \phi \prod_{i=1}^{N-1} (x_i - x_{i+1})^\beta, \quad (5)$$

in the corresponding Schrödinger equation  $H\psi = E\psi$ , it is easily shown that, provided  $g$  and  $G$  are related to  $\beta$  by eq. (2),  $\phi$  satisfies the equation

$$-\frac{1}{2} \sum_{i=1}^N \frac{\partial^2 \phi}{\partial x_i^2} - \beta \sum_{i=1}^{N-1} \frac{1}{(x_i - x_{i+1})} \left( \frac{\partial \phi}{\partial x_i} - \frac{\partial \phi}{\partial x_{i+1}} \right) + (V - E)\phi = 0. \quad (6)$$

Following Calogero we start from  $\phi$  as given by eq. (6) and assume that

$$\phi = P_k(x)\Phi(r). \quad (7)$$

where  $r^2 = \sum_{i=1}^N x_i^2$ . The function,  $\Phi$  satisfies the equation

$$\Phi''(r) + [N + 2k - 1 + 2(N - 1)\beta] \frac{1}{r} \Phi'(r) + 2[E - V(r)]\Phi(r) = 0, \quad (8)$$

provided  $P_k(x)$  is a homogeneous polynomial of degree  $k$  ( $k = 0, 1, 2, \dots$ ) in the particle-coordinates and satisfies generalized Laplace equation

$$\left[ \sum_{i=1}^N \frac{\partial^2}{\partial x_i^2} + 2\beta \sum_{i=1}^{N-1} \frac{1}{(x_i - x_{i+1})} \left( \frac{\partial}{\partial x_i} - \frac{\partial}{\partial x_{i+1}} \right) \right] P_k(x) = 0. \quad (9)$$

We shall discuss few solutions of the Laplace equation (9) below.

Let us now specialize to the case of the oscillator potential i.e.  $V(r) = \frac{\omega^2}{2}r^2$ . In this case, eq. (8) is the well known radial equation for the oscillator problem in more than one dimension and its solution is

$$\Phi(r) = \exp(-\omega r^2/2) L_n^a(\omega r^2), \quad n = 0, 1, 2, \dots \quad (10)$$

where  $L_n^a(x)$  is the associated Laguerre polynomial while the energy eigenvalues are given by

$$E_n = \left[ 2n + k + \frac{N}{2} + (N - 1)\beta \right] \omega = E_0 + (2n + k)\omega, \quad (11)$$

with  $a = \frac{E}{\omega} - 2n - 1$ . Few comments are in order at this stage.

1. For large  $N$ , the energy  $E$  is proportional to  $N$  so that

$$\lim_{N \rightarrow \infty} \frac{E}{N} = \left( \beta + \frac{1}{2} \right) \omega, \quad (12)$$

i.e., the system has a good thermodynamic limit. In contrast, notice that the long-ranged Calogero model does not have a good thermodynamic limit since in that case for large  $N$ ,  $E/N$  goes like  $N$ .

2. The spectrum can be interpreted as due to noninteracting bosons (or fermions) plus  $(n, k)$ - independent (but  $N$ -dependent) shift.

The ground state eigenvalue and eigenfunction of the model is thus given by ( $n = k = 0$ )

$$E_0 = \left[ (N-1)\beta + \frac{N}{2} \right] \omega, \quad (13)$$

$$\psi_0 = \exp\left(-\frac{\omega}{2} \sum_{i=1}^N x_i^2\right) \prod_{i=1}^{N-1} (x_i - x_{i+1})^\beta. \quad (14)$$

A neat way of proving that we have indeed obtained the ground state can be given using the method of supersymmetric quantum mechanics [21]. To this end, we define the operators

$$\begin{aligned} Q_i &= \frac{d}{dx_i} + \omega x_i + \beta \left[ \frac{1}{(x_{i-1} - x_i)} - \frac{1}{(x_i - x_{i+1})} \right], \quad (i = 2, 3, \dots, N-1), \\ Q_1 &= \frac{d}{dx_1} + \omega x_1 - \beta \frac{1}{x_1 - x_2}, \\ Q_N &= \frac{d}{dx_N} + \omega x_N + \beta \frac{1}{x_{N-1} - x_N}, \end{aligned} \quad (15)$$

and their Hermitian conjugates  $Q_i^+$ . It is easy to see that the  $Q$ 's annihilate the ground state as given by eq. (14). Further, the Hamiltonian (1) can be written in terms of these operators as

$$H - E_0 = \frac{1}{2} \sum_{i=1}^N Q_i^+ Q_i, \quad (16)$$

where  $E_0$  is as given by eq. (13). Now since the operator on the right hand side is nonnegative and annihilates the ground state wavefunction as given by eq. (14), hence  $E_0$  as given by eq. (13) must be the ground state energy of the system.

On rewriting  $\psi_0$  in terms of a new variable

$$y_i \equiv \sqrt{\frac{\omega}{\beta}} x_i, \quad (17)$$

one finds that the probability distribution for  $N$  particles is given by

$$\psi_0^2 = C \exp\left(-\beta \sum_{i=1}^N y_i^2\right) \prod_{i=1}^{N-1} (y_i - y_{i+1})^{2\beta} \quad (18)$$

where  $C$  is the normalization constant. We now observe that for  $\beta = 1, 2, 4$ , this  $\psi^2$  can be identified with the joint probability density function for the

eigenvalues of SRDM with Gaussian orthogonal, unitary or symplectic ensembles respectively. We can therefore borrow the well-known results for these ensembles [18, 17] and obtain exact results about a many-body theory defined in the limit,  $N \rightarrow \infty$ ,  $\omega \rightarrow 0$ ,  $N\omega = \text{finite}$  which defines the density of the system. For example, as  $N \rightarrow \infty$ , the one-point function tends to a Gaussian for any  $\beta$  [17] and is given by

$$R_1(x) = \frac{N}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{x^2}{2\sigma^2}\right), \quad (19)$$

where  $\sigma^2 = \frac{(\beta+1)}{\omega}$ . Other results about the many-body theory will be discussed in Secs. IV and V.

Finally, let us discuss the polynomial solutions to the Laplace equation (9). So far, we have been able to obtain solutions in the following cases:

(i)  $k = 2, N \geq 2$  (ii)  $k = 3, N \geq 3$  (iii)  $k = 4, N \geq 4$  (iv)  $k = 5, N \geq 5$  (v)  $k = 6, N \geq 6$ .

Besides we have also obtained solutions for  $k = 4, 5, 6$  in case  $N = 3$ , and for  $k = 5, 6$  in case  $N = 4$ . We find that for  $k \geq 3$ , the demand that there be no pole in  $P_k(x)$  alone does not require  $P_k(x)$  to be completely symmetrical polynomial. However, for  $k = 3, 4$  and  $N = 3, 4$  it turns out that solution to Laplace eq. (9) exists only if  $P_k(x)$  is a completely symmetric polynomial. We suspect that this may be true in general. On assuming completely symmetric  $P_k(x)$  we find that in all the above cases we have a one-parameter family of solutions. In particular the various solutions are as follows (it is understood that the particle indices  $i, j, k, \dots$  are always unequal unless mentioned otherwise).

(i)  $k = 2, N \geq 2$

$$P_k(x) = a \sum_{i=1}^N x_i^2 + b \sum_{i<j}^N x_i x_j, \quad (20)$$

with  $\beta$  given by

$$\beta = \frac{aN}{(N-1)(b-2a)}. \quad (21)$$

(ii)  $k = 3, N \geq 3$

$$P_k(x) = a \sum_{i=1}^N x_i^3 + b \sum_{i,j=1}^N x_i^2 x_j + c \sum_{i<j<k}^N x_i x_j x_k, \quad (22)$$

where  $c = 3(b-a)$  and  $\beta$  is given by

$$\beta = \frac{3a + (N-1)b}{(N-1)(b-3a)}. \quad (23)$$



(iii)  $k = 4, N \geq 4$

$$\begin{aligned}
P_k(x) &= a \sum_{i=1}^N x_i^4 + b \sum_{i,j=1}^N x_i^3 x_j + c \sum_{i<j}^N x_i^2 x_j^2 \\
&+ d \sum_{i,j<k}^N x_i^2 x_j x_k + e \sum_{i<j<k<l}^N x_i x_j x_k x_l, \tag{24}
\end{aligned}$$

where

$$\begin{aligned}
e &= 6(c - 2a), \quad d = b + 2c - 4a, \\
(N + 4)b + 2(N - 2)c - 4(N - 2)a + 2(N - 1)(2a + b - c)\beta &= \mathbf{(25)}
\end{aligned}$$

and  $\beta$  is given by

$$\beta = \frac{6a + (N - 1)c}{(N - 1)(b - 4a)}. \tag{26}$$

(iv)  $k = 5, N \geq 5$

$$\begin{aligned}
P_k(x) &= a \sum_{i=1}^N x_i^5 + b \sum_{i,j=1}^N x_i^4 x_j + c \sum_{i,j=1}^N x_i^3 x_j^2 + d \sum_{i,j<k}^N x_i^3 x_j x_k \\
&+ e \sum_{k,i<j}^N x_i^2 x_j^2 x_k + f \sum_{i,j<k<l}^N x_i^2 x_j x_k x_l + g \sum_{i<j<k<l<m}^N x_i x_j x_k x_l x_m, \tag{27}
\end{aligned}$$

where

$$\begin{aligned}
e &= 5c - 5a - 3b, \quad d = b + 2c - 5a, \\
f &= 12c - 15a - 9b, \quad g = 30(c - a - b), \\
(5N - 7)c - 3(N - 4)b - 5(N - 2)a \\
&+ (N - 1)(5a + 3b - 2c)\beta = 0, \tag{28}
\end{aligned}$$

and  $\beta$  is given by

$$\beta = \frac{10a + (N - 1)c}{(N - 1)(b - 5a)}. \tag{29}$$

(iv)  $k = 6, N \geq 6$

$$\begin{aligned}
P_k(x) = & a \sum_{i=1}^N x_i^6 + b \sum_{i,j=1}^N x_i^5 x_j + c \sum_{i,j=1}^N x_i^4 x_j^2 + d \sum_{i,j < k}^N x_i^4 x_j x_k + e \sum_{i < j}^N x_i^3 x_j^3 \\
& + f \sum_{i,j,k=1}^N x_i^3 x_j^2 x_k + g \sum_{i,j < k < l}^N x_i^3 x_j x_k x_l + h \sum_{i < j < k}^N x_i^2 x_j^2 x_k^2 + p \sum_{i < j, < k < l}^N x_i^2 x_j^2 x_k x_l \\
& + q \sum_{i,j < k < l < m}^N x_i^2 x_j x_k x_l x_m + r \sum_{i < j < k < l < m < n}^N x_i x_j x_k x_l x_m x_n, \tag{30}
\end{aligned}$$

where

$$\begin{aligned}
3e = & 4b - 2c + 6a + f, \quad d = b + 2c - 6a, \quad g = 2f + 2c - 4b - 6a, \\
h = & 2f + 9a - 4b - c, \quad p = 5f + 18a - 8b - 6c, \\
q = & 6(2f + 9a - 4b - 3c), \quad r = 30(f + 6a - 2b - 2c), \\
& (5N - 9)f - 2(4N - 15)b + 18(N - 5)a - 6(N - 5)c \\
& + (N - 1)(8b + 6c - 2f - 18a)\beta = 0, \\
& 14b - 2c + 6a + (N - 1)f + 2(N - 1)(3a + 2b - c)\beta = 0, \tag{31}
\end{aligned}$$

and  $\beta$  is given by

$$\beta = \frac{15a + (N - 1)c}{(N - 1)(b - 6a)}. \tag{32}$$

It would be nice if one can find solutions for higher values of  $k$  and further check if solutions exist (if at all) only if  $P_k(x)$  is a completely symmetric polynomial. While we are unable to prove it, we suspect that, subject to the solutions of the Laplace equation for higher  $k$ , we have obtained the complete spectrum for this problem.

Finally it is worth enquiring if the bound state spectrum of the Hamiltonian (1) can also be obtained in case the oscillator potential is replaced by any other potential. It turns out that as in the Calogero case [22], in this case also the answer to the question is in affirmative. In particular, if instead the  $N$  particles are interacting via the  $N$ -body potential as given by

$$V(x_1, x_2, \dots, x_N) = -\alpha \sum_{i=1}^N \frac{1}{\sqrt{\sum_i x_i^2}}, \tag{33}$$

then also (most likely the entire) discrete spectrum can be obtained. This is because, after using the ansatz (7), eq. (8) is essentially the radial Schrödinger equation for an attractive Coulomb potential and it is well known that the only two problems which are analytically solvable for all partial waves are

the Coulomb and the oscillator potentials. In particular the solution of (8) is then given by (note  $r^2 = \sum_{i=1}^N x_i^2$ )

$$\Phi(r) = \exp(-\sqrt{2|E|r})L_n^b(2\sqrt{2|E|r}), \quad (34)$$

and the corresponding energy eigenvalues are

$$E_{n,k} = -\frac{\alpha^2}{2\left[n+k+\frac{N}{2}-1+(N-1)\beta\right]^2}, \quad (35)$$

when  $b = N+2k-3+2(N-1)\beta$ . It may be noted that whereas in the oscillator case the spectrum is linear in  $\beta$ , it is  $(-E)^{-1/2}$  which is linear in  $\beta$  in the case of the Coulomb-like potential. Secondly, as in any oscillator (Coulomb) problem, the energy depends on  $n$  and  $k$  only through the combination  $2n+k$  ( $n+k$ ).

Is there any underlying reason why one is able to obtain the discrete spectrum for the  $N$ -body problem with either the oscillator or the Coulomb-like potential (33)? Following [23] it is easily shown that in both the cases one can write down an underlying  $SU(1,1)$  algebra. Further, since the many-body potential  $W$  in (1) is a homogeneous function of the coordinates of degree -2, i.e. it satisfies

$$\sum_{l=1}^N x_l \frac{\partial W}{\partial x_l} = -2W, \quad (36)$$

hence, following the arguments of [23], one can also establish a simple algebraic relationship between the energy eigenstates of the  $N$ -body problem (1) with the Coulomb-like potential (33) and the harmonic oscillator potential.

It may be noted that the Hamiltonian (1) is not completely symmetric in the sense that whereas all other particles have two neighbours, particle 1 and  $N$  have only one neighbour. Can one make it symmetric so that all particles will be treated on the same footing? One possible way is to add some extra terms in  $H$ . For example, consider

$$H_1 = H + H', \quad (37)$$

where  $H$  is as given by eq. (1) while  $H'$  has the form

$$H' = \frac{g}{(x_N - x_1)^2} - G \left[ \frac{1}{(x_N - x_1)(x_1 - x_2)} + \frac{1}{(x_{N-1} - x_N)(x_N - x_1)} \right]. \quad (38)$$

Clearly, by adding these extra terms, the problem has become cyclic invariant for any  $N$  while for  $N = 3$  it is identical to the Calogero problem and hence

is in fact completely symmetric under the interchange of any two of the three particle coordinates. It may be noted that in the thermodynamic limit, these extra terms are irrelevant.

We can again start from the ansatz (5) (but with  $N - 1$  replaced by  $N$ ) in the Schrödinger equation  $H_1\psi = E\psi$  and using eq. (2) we find that  $\phi$  again satisfies eq. (6) but with  $N - 1$  in the second term being replaced by  $N$ . On further using the substitution as given by eq. (7) one finds that  $\Phi$  satisfies eq. (8) but with the coefficient of the  $2\beta$  term being  $N$  instead of  $N - 1$  while  $P_k(x)$  is again a homogeneous polynomial of degree  $k$  ( $k=0,1,2,\dots$ ) in the particle coordinates, which now satisfies instead of eq. (9)

$$\left[ \sum_{i=1}^N \frac{\partial^2}{\partial x_i^2} + 2\beta \sum_{i=1}^N \frac{1}{(x_i - x_{i+1})} \left( \frac{\partial}{\partial x_i} - \frac{\partial}{\partial x_{i+1}} \right) \right] P_k(x) = 0, \quad (39)$$

with  $x_{N+1} = x_1$ .

How do the solutions to the Laplace eqs. (9) and (39) compare? For  $N = 3$ , eq. (39) is identical to that of Calogero and for this case Calogero has already obtained the solutions for any  $k$ . For  $N > 3$  and for  $k \geq 3$ , the demand that there be no pole in  $P_k(x)$  alone does not require  $P_k(x)$  to be completely symmetrical polynomial. However, for  $k = 3, 4$  and  $N = 4$  it again turns out that solution to Laplace eq. (39) exists only if  $P_k(x)$  is a completely symmetric polynomial. We suspect that this may be true in general. On assuming completely symmetric  $P_k(x)$  we have been able to obtain a two-parameter family of solutions in case  $k=3,4,5,6$  and  $N \geq k$  (note that for eq. (9) we have obtained only one-parameter family of solutions). As an illustration, the solution for  $N \geq 4$  and  $k = 4$  is given by (it is understood that the particle indices  $i, j, k, \dots$  are always unequal)

$$\begin{aligned} P_k(x) = & a \sum_{i=1}^N x_i^4 + b \sum_{i,j=1}^N x_i^3 x_j + c \sum_{i<j}^N x_i^2 x_j^2 \\ & + d \sum_{i,j<k}^N x_i^2 x_j x_k + e \sum_{i<j<k<l}^N x_i x_j x_k x_l, \end{aligned} \quad (40)$$

where

$$e = 2(2a + 2d - 2b - c), \quad (41)$$

$$6a + (N - 1)c + \beta[8a - 2b + (2c - d)(N - 2)] = 0, \quad (42)$$

$$6b + (N - 2)d + 2\beta[2(N - 1)b - 2(N - 4)a + (N - 4)c - 2(N - 1)d] = 0. \quad (43)$$

Solution to the new  $\Phi$  equation can be easily written down in case  $V(r) = \frac{\omega^2 r^2}{2}$  or if it is given by eq. (33). For example, it is easily checked that in the

former case the solution is again given by eq. (11) but the energy eigenvalues are now given by

$$E_{n,k} = [2n + k + \frac{N}{2} + N\beta]\omega = E_0 + (2n + k)\omega. \quad (44)$$

Similarly, in the later case, the solution is as given by eq. (35) except that in the term containing  $\beta$ ,  $N - 1$  must be replaced by  $N$ .

Apart from these two potentials, where we have obtained the entire bound state spectrum, there are several other potentials which are quasi-exactly solvable. For example, for the potential

$$V\left(\sum x_i^2\right) = A \sum_{i=1}^N x_i^2 - B \left(\sum_{i=1}^N x_i^2\right)^2 + C \left(\sum_{i=1}^N x_i^2\right)^3, \quad (45)$$

it is easily shown that the ground state energy and eigenfunctions are

$$E = -\frac{B}{4\sqrt{C}}[N + 2(N - 1)\beta], \quad (46)$$

$$\psi_0 = \exp\left[-\frac{\sqrt{C}}{4}\left(\sum_{i=1}^N x_i^2\right)^2 + \frac{B}{4\sqrt{C}}\sum_{i=1}^N x_i^2\right] \prod_{i=1}^{N-1} (x_i - x_{i+1})^\beta, \quad (47)$$

provided  $A, B, C$  are related by

$$A = \frac{B^2}{4C} - [N + 2 + 2(N - 1)\beta]\sqrt{C}. \quad (48)$$

It is worth enquiring if the probability distribution for  $N$  particles corresponding to (47) can be mapped to some matrix model. In this context let us point out that the corresponding (long-ranged) Calogero problem was in fact mapped to the matrix model corresponding to branched polymers [24]. So far as we are aware of, answer to this question is not known in our case.

### 3 N-body problem in one-dimension with periodic boundary condition

Soon after the seminal papers of Calogero [1] and Sutherland [2] where they considered an  $N$ -body problem on full line, Sutherland [25] also considered an  $N$ -body problem with long-ranged interaction and with periodic boundary condition. He obtained the exact ground state energy and showed that the corresponding  $N$ -particle probability density function is related to the random matrix with circular ensemble [25]. Using the known results for the

random matrix theory [6], he was able to obtain the static correlation functions of the corresponding many body theory. It is then natural to enquire if one can also obtain the exact ground state of an  $N$ -body problem with nearest and next-to-nearest neighbour interaction with periodic boundary condition (PBC). Further, one would like to enquire if the corresponding  $N$ -particle probability density can be mapped to some known matrix model. The hope is that in this case one may be able to obtain the correlation functions of a related many-body theory in the thermodynamic limit. We now show that the answer to the question is in the affirmative.

Let us start from the Hamiltonian (3). We wish to find the ground state of the system subject to the periodic boundary condition (PBC)

$$\psi(x_1, \dots, x_i + L, \dots, x_N) = \psi(x_1, \dots, x_i, \dots, x_N). \quad (49)$$

For this, we start with a trial wave function of the form

$$\Psi_0 = \prod_{i=1}^N \sin^\beta \left[ \frac{\pi}{L}(x_i - x_{i+1}) \right], (x_{N+1} = x_1). \quad (50)$$

In this section, we restrict the coordinates  $x_i$  to the sector  $L \geq x_1 \geq x_2 \geq \dots \geq x_N \geq 0$ , so that eq. (50) makes sense even for noninteger  $\beta$ . The extension to the full configuration space will be made in Sec. 5. On substituting eq. (50) in the Schrödinger equation for the Hamiltonian (3), we find that it is indeed a solution provided  $g$  and  $G$  are again related to  $\beta$  by eq. (2). The corresponding ground state energy turns out to be

$$E_0 = \frac{N\beta^2\pi^2}{L^2}. \quad (51)$$

The fact that this is indeed the ground state energy can be neatly proved by using the operators [34]

$$Q_i = \frac{d}{dx_i} + \beta \frac{\pi}{L} \left[ \cot(x_{i-1} - x_i) - \cot(x_i - x_{i+1}) \right], \quad (52)$$

and their Hermitian conjugates  $Q_i^+$ . It is easy to see that the  $Q_i$ 's annihilate the ground state as given by eq. (50). The Hamiltonian (3) can be rewritten in terms of these operators as

$$H - E_0 = \frac{1}{2} \sum_i Q_i^+ Q_i, \quad (53)$$

where  $E_0$  is as given by eq. (51). Hence  $E_0$  must be the ground state energy of the system.

Thus unlike the Calogero-Sutherland type of models, our models (both of Sec. II and here) have good thermodynamic limit, i.e., the ground state energy per particle ( $= E_0/N$ ) is finite as  $N \rightarrow \infty$ .

Having obtained the exact ground state, it is natural to enquire if the corresponding  $N$ -particle probability density can be mapped to the joint probability distribution of some SRCDM so that we can obtain some exact results for the corresponding many-body theory. It turns out that indeed the square of the ground-state wave function is related to the joint probability distribution function for the SRCDM from where we conclude that the density is a constant if  $0 \leq x \leq \frac{N}{L}$ , and zero outside. Other exact results for the many-body theory will be discussed in the next two sections.

## 4 Some exact results for the many-body problem

The square of the ground-state wavefunction of the many-body problem introduced in Sec.II (Sec.III) can be identified with the joint probability distribution function of eigenvalues of the SRDM (SRCDM). Using SRCDM, Pandey [17] and Bogomolny et al. [18] have shown that for any  $\beta$ , the two-point correlation function has the form

$$R_2^{(\beta)}(s) = \sum_{n=1}^{\infty} P^{(\beta)}(n, s), \quad (54)$$

where  $s$  is the separation of two levels (or distance between two particles in the many-body theory considered here) and

$$P^{(\beta)}(n, s) = \frac{(\beta + 1)^{n(\beta+1)}}{\Gamma[n(\beta + 1)]} s^{n(\beta+1)-1} e^{-(\beta+1)s}. \quad (55)$$

From this expression it is not very easy to compute  $R_2(s)$  for arbitrary  $\beta$ . However, it is easy to obtain the Laplace transform of  $R_2(s)$  for any  $\beta$ . In particular, if

$$g_2(t) = \int_0^{\infty} R_2(s) e^{-ts} ds, \quad (56)$$

then

$$g_2(t) = \sum_{n=1}^{\infty} g(n, t), \quad (57)$$

where  $g(n, t)$  is the Laplace transform of  $P(n, s)$ , i.e.,

$$g(n, t) = \int_0^{\infty} P(n, s) e^{-ts} ds. \quad (58)$$

On using  $P^{(\beta)}(n, s)$  as given by eq. (55) in eq. (58) it is easily shown that

$$g^{(\beta)}(n, t) = \left( \frac{\beta + 1}{t + \beta + 1} \right)^{(\beta+1)n}. \quad (59)$$

Hence

$$g_2^{(\beta)}(t) = \sum_{n=1}^{\infty} g^{(\beta)}(n, t) = \frac{1}{\left(\frac{t+\beta+1}{\beta+1}\right)^{\beta+1} - 1}, \quad (60)$$

from which one has to compute  $R_2^{(\beta)}(s)$  by the Laplace inversion.

For integer  $\beta$ , it is possible to perform the Laplace inversion by making use of the fact that

$$\frac{1}{x^n - 1} = \frac{1}{n} \sum_{k=0}^{n-1} \frac{e^{2ik\pi/n}}{x - e^{2ik\pi/n}}, \quad (61)$$

yielding

$$R_2^{(\beta)}(s) = \sum_{k=0}^{\beta} \Omega^k e^{(\beta+1)s(\Omega^k - 1)} \quad (62)$$

where

$$\Omega = e^{2\pi i/(\beta+1)}. \quad (63)$$

For  $\beta = 1$ , which corresponds to the orthogonal ensemble, the result is already known [17, 18] :  $R_2^{(1)}(s) = 1 - e^{-4s}$ .

It is interesting to mention that  $R_2^{(1)}(s)$  agrees very well with some of the pseudo-integrable billiards (e.g., the  $\frac{\pi}{3}$ -rhombus billiard). It is important here to note that for rhombus billiards [16], the Hamiltonian matrix has elements which fall in their magnitude away from the principal diagonal. Thus, beyond a certain bandwidth, the elements are insignificant and the matrix is effectively banded. Immediately then, the results of banded matrices become applicable. Although there seems to be good agreement of the results from this random matrix theory as shown in [16, 18], in [16] it is also shown that there are other polygonal billiards for which  $R_2^{(1)}(s)$  is not an appropriate correlator. It is possible that for different bandwidths, and, by an inclusion of interactions beyond nearest neighbours in the short-range Dyson model, a family of random matrices result. This may, eventually, explain the entire family of systems exhibiting intermediate spectral statistics.

Coming back to the two-point correlation function, depending on if  $\beta$  is an odd or an even integer,  $R_2(s)$ , as given by eq. (62), can be written in



a closed form which shows that  $R_2(s)$  is indeed real and further, it clearly exhibits oscillations for large  $s$ . In particular, it is easily shown that

$$R_2(\beta = 2p + 1, s) = 1 - e^{-2(2p+2)s} + 2e^{-(2p+2)s} \sum_{m=1}^p e^{(2p+2)s \cos(\frac{m\pi}{p+1})} \cos \left[ \frac{m\pi}{p+1} + (2p+2)s \sin(\frac{m\pi}{p+1}) \right], \quad (64)$$

$$R_2(\beta = 2p, s) = 1 + 2e^{-(2p+1)s} \sum_{m=1}^p e^{(2p+1)s \cos(\frac{2m\pi}{2p+1})} \cos \left[ \frac{2m\pi}{2p+1} + (2p+1)s \sin(\frac{2m\pi}{2p+1}) \right]. \quad (65)$$

For illustration, we give below explicit expressions for  $\beta = 2, 3, 4$

$$\begin{aligned} R_2^{(2)}(s) &= 1 - 2e^{-\frac{9s}{2}} \cos \left( \frac{3\sqrt{3}s}{2} - \frac{\pi}{3} \right); \\ R_2^{(3)}(s) &= 1 - e^{-8s} - 2e^{-4s} \sin(4s); \\ R_2^{(4)}(s) &= 1 + 2e^{5s(-1+\cos(2\pi/5))} \cos \left[ \frac{2\pi}{5} + 5s \sin \left( \frac{2\pi}{5} \right) \right] \\ &\quad + 2e^{5s(-1+\cos(4\pi/5))} \cos \left[ \frac{4\pi}{5} + 5s \sin \left( \frac{4\pi}{5} \right) \right]. \end{aligned} \quad (66)$$

In Fig. 1, we have plotted  $R_2^{(\beta)}(s)$  as a function of  $s$  for  $\beta = 1, 2, 3, 4$ . These results show that, for integer  $\beta$ , there is no long-range order in the corresponding many-body theory.

Similarly, if  $\beta$  is half-integral, i.e.,  $\beta = (2n+1)/2$  then it is easily shown that

$$R_2^{((2n+1)/2)}(s) = \frac{1}{2} \sum_{k=0}^{2n} \Omega^{2k} e^{-\frac{2n+1}{2}s(1-\Omega^{2k})} \left[ 1 + \operatorname{erf} \left( \sqrt{\frac{(2n+1)s}{2}} \Omega^k \right) \right], \quad (67)$$

where  $\Omega$  is as given by eq. (63).

For arbitrary  $\beta$ , however, we are unable to perform the Laplace inversion and hence we do not have a closed expression for  $R_2(s)$ . However, one can numerically calculate it by using eqs. (54) and (55). In Fig. 2, we have plotted  $R_2^{(\beta)}(s)$  as a function of  $s$  for  $\beta = 1, 4/3, 3/2, 5/3, 2, 7/3, 5/2$ . From this figure it is clear that even for fractional  $\beta$ , there is no long-range order.

## 5 Off-diagonal long-range order

So far, nothing has been specified regarding the statistical character of the particles involved in the N-body problem of Sec. III. We now do that by first symmetrizing the Hamiltonian, that is by rewriting it as

$$H = -\frac{1}{2} \sum_{i=1}^N \frac{\partial^2}{\partial x_i^2} + \sum_{P \in S_N} \Theta(x_{P(1)} - x_{P(2)}) \dots \Theta(x_{P(N-1)} - x_{P(N)}) W(x_{P(1)}, \dots, x_{P(N)}) \quad (68)$$

where  $\theta$  is the step function and  $W(x_1, \dots, x_N)$  is the N-body potential of eq. (3). Next, relying on the solution given in eq. (50), we introduce the (not normalized) wave function:

$$\psi_N(x_1, \dots, x_N) = \varepsilon_P \phi_N(x_{P(1)}, \dots, x_{P(N)}), \quad (69)$$

where P is the permutation in  $S_N$  such that  $1 > x_{P(1)} > x_{P(2)} > \dots > x_{P(N)} > 0$ ,  $\varepsilon_P = 1$  ( $\varepsilon_P = \text{sign}(P)$ ) in the N-boson (N-fermion) case and

$$\phi_N(x_1, \dots, x_N) = \prod_{n=1}^N | \sin \pi(x_n - x_{n+1}) |^\beta; \quad (x_{N+1} = x_1), \quad (70)$$

(we have set the scale factor  $L$  equal to 1). Primitively, the function (69) is defined on the hypercube  $[0, 1]^N$ . The following properties of  $\psi_N$  are easily verified, provided that  $\beta \geq 2$ :

1. In the bosonic case,  $\psi_N$  can be continued to a multi-periodic function in the whole space  $\mathcal{R}^N$  (or equivalently on the torus  $T^N$ ):

$$\psi_N(x_1, \dots, x_i+1, \dots, x_N) = \psi_N(x_1, \dots, x_i, \dots, x_N); \quad (i = 1, \dots, N), \quad (71)$$

which belongs to  $C^2$  (i.e. is twice continuously differentiable). Owing to this property and the results of Sec.3,  $\psi_N$  then obeys the Schrödinger equation (with Hamiltonian (3) and energy as given by eq. (51)) not only in the sector  $x_1 > x_2 > \dots > x_N$  but every where. Thus,  $\psi_N$  describes the ground state wave function of the N-boson system. Moreover, it is translation invariant (on  $\mathcal{R}^N$ ):

$$\psi_N(x_1 + a, x_2 + a, \dots, x_N + a) = \psi_N(x_1, x_2, \dots, x_N); \quad \forall a \in \mathcal{R}. \quad (72)$$

2. In the fermionic case, the continuation by periodicity is possible only for **odd**  $N$ , in which case eq. (71) still holds with  $\psi_N \in C^2$ . For even  $N$  on the contrary, enforcing the periodicity (71) leads to a discontinuous function  $\psi_N$ , so that the Schrödinger equation is no longer satisfied on the configuration space  $T^N$ .

Therefore, in the following we shall implicitly restrict ourselves to odd values of  $N$  when treating fermions. The translation invariance (72) then remains valid.

We are interested in the one-particle reduced density matrix, given by

$$\rho_N(x-x') = \frac{N}{C_N} \int_0^1 dx_1 \dots \int_0^1 dx_{N-1} \psi_N(x_1, \dots, x_{N-1}, x) \psi_N(x_1, \dots, x_{N-1}, x'), \quad (73)$$

where  $C_N$  stands for the squared norm of the wave function:

$$C_N = \int_0^1 dx_1 \dots \int_0^1 dx_N |\psi_N(x_1, \dots, x_N)|^2. \quad (74)$$

That the R.H.S. of eq. (73) defines a (periodic) function of  $(x - x')$  is an easy consequence of eqs. (71) and (72). The normalization of  $\rho_N$  is such that  $\rho_N(0) = N$ , the particle density. Further, the function  $\rho_N(\xi)$  is manifestly of positive type on the  $U(1)$  group, which implies that its Fourier coefficients,

$$\rho_N^{(n)} = \int_0^1 d\xi e^{-2i\pi n\xi} \rho_N(\xi); \quad (n = 0, \pm 1, \pm 2, \dots), \quad (75)$$

are non-negative (Bochner's theorem). In fact, this directly appears if one writes their explicit expression

$$\begin{aligned} \rho_N^{(n)} &= \frac{N}{C_N} \int_0^1 dx_1 \dots \int_0^1 dx_{N-1} \psi_N(x_1, \dots, x_{N-1}, 0) X \\ &X \int_0^1 dx e^{2i\pi n x} \psi_N(x_1, \dots, x_{N-1}, x), \end{aligned} \quad (76)$$

in the form (obtained by using the periodicity property):

$$\rho_N^{(n)} = \frac{N}{C_N} \int_0^1 dx_1 \dots \int_0^1 dx_{N-1} \left| \int_0^1 dx e^{2i\pi n x} \psi_N(x_1, \dots, x_{N-1}, x) \right|^2. \quad (77)$$

In the **bosonic** case, since the function  $\rho_N$  is not only of positive type but also **positive** (like  $\psi_N$ ), eq. (75) shows us that

$$\rho_N^{(0)} \geq \rho_N^{(n)}; \quad (n = \pm 1, \pm 2, \dots). \quad (78)$$

In the fermionic case, eq. (78) is not necessarily true (because  $\psi_N$  changes sign on  $T^N$ ) and it is not an easy matter to determine the largest Fourier

coefficient. Notice that the coefficients  $\rho_N^{(n)}$ , which physically represent the expectation values of the number of particles having momentum  $k_n = 2\pi n$  in the ground state, are nothing but the eigenvalues of the one-particle reduced density matrix (diagonal in the  $k_n$  representation). According to the Onsager-Penrose criterion [26], no condensation can occur in the system (at least for Bose particles) if the largest of these eigenvalues is not an extensive quantity in the thermodynamic limit, that is, if

$$\lim_{N \rightarrow \infty} \frac{\rho_N^{(0)}}{N} = 0. \quad (79)$$

For Fermi particles, this criterion is not sufficient, and one has to look also at the largest eigenvalue of the two-particle reduced density matrix [28]. Since we are presently unable to determine the largest eigenvalue of  $\rho_N$  itself in the fermionic case, we shall not discuss extensively the latter here. Nevertheless, we shall look for the large  $N$  behaviour of  $\rho_N^{(0)}$  for bosons and fermions at a time, as this does not require much extra work and can give some indications in the fermionic case too. Let us write:

$$\frac{\rho_N^{(0)}}{N} = \frac{A_N}{C_N}, \quad (80)$$

where  $C_N$  is given by eq. (74) and

$$A_N = \int_0^1 dx_1 \dots \int_0^1 dx_{N-1} \psi_N(x_1, \dots, x_{N-1}, 0) \int_0^1 dx \psi_N(x_1, \dots, x_{N-1}, x), \quad (81)$$

(the expression (76) of  $\rho_N^{(0)}$  is more convenient than (77) for our purpose). Because of the special form (69)-(70) of the wave function, the computation of the squared norm  $C_N$  is already not a trivial task, in sharp contrast to the case of  $N$  free, impenetrable particles. As a consequence, the (mainly algebraic) method introduced long ago by Lenard [27] to deal with the latter case does not apply here, and we have to resort to another device. For conciseness, we introduce the notation:

$$S(x_n - x_{n-1}) = |\sin \pi(x_n - x_{n+1})|^\beta, \quad (82)$$

and define:

$$S_2(\Delta) = \int_0^\Delta dx S(x) S(\Delta - x); \quad (0 \leq \Delta \leq 1). \quad (83)$$

Our starting point will be the following representations of  $C_N$  and  $A_N$ :

$$C_N = (N-1)! \frac{1}{2\pi} \int_{-\infty}^{\infty} dx e^{-ix} \tilde{F}(x)^N, \quad (84)$$

$$A_N = (N-1)! \frac{1}{2\pi} \int_{-\infty}^{\infty} dx e^{-ix} \tilde{F}(x)^{N-3} \left[ \tilde{F}(x) \tilde{G}(x) + \eta_N \tilde{H}(x)^2 \right], \quad (85)$$

where

$$\begin{aligned} \tilde{F}(x) &= \int_0^1 d\Delta e^{i\Delta x} S(\Delta)^2, \\ \tilde{G}(x) &= \int_0^1 d\Delta e^{i\Delta x} S_2(\Delta)^2, \\ \tilde{H}(x) &= \int_0^1 d\Delta e^{i\Delta x} S(\Delta) S_2(\Delta)^2, \end{aligned} \quad (86)$$

and

$$\eta_N = \begin{array}{ll} (N-2) & \text{for bosons} \\ -1 & \text{for fermions.} \end{array} \quad (87)$$

The representations (84)-(87) follow from the convolution structure of the expressions (74) and (81) of  $C_N$  and  $A_N$ , when written in terms of appropriate variables. Their proof is given in the Appendix. Our aim is to extract from them the large  $N$  behaviour of  $C_N$  and  $A_N$ . Their form is especially suited for that purpose, because the integrands in eqs. (84) and (85) are entire functions, as polynomial combinations of Fourier transforms of functions with compact support (eq. (86)). Indeed, we are then allowed to, first, shift the integration path and then apply the residue theorem to meromorphic pieces of the integrands. However, it turns out that the calculations needed for arbitrary (integer) values of  $\beta$  are quite cumbersome. So, in order to keep the argument clear enough, we shall content ourselves to present below these calculations in the simplest case, namely  $\beta = 1$  (recall that, strictly speaking, this value is not allowed), being understood that similar results are obtained for all integers  $\beta \geq 2$ .

For  $\beta = 1$ ,  $S(\Delta) = \sin \pi \Delta$ , and eq. (86) gives, after reductions:

$$\begin{aligned} \tilde{F}(x) &= \frac{2\pi^2}{i} \frac{1 - e^{ix}}{x(x^2 - 4\pi^2)}, \\ \tilde{G}(x) &= \frac{4\pi^4}{i} \frac{5x^2 - 4\pi^2}{x^3(x^2 - 4\pi^2)^3} + e^{ix} R^{(-1)}(x), \\ \tilde{H}(x) &= -\frac{4\pi^3}{i} \frac{1}{x(x^2 - 4\pi^2)^2} + e^{ix} R^{(-2)}(x), \end{aligned} \quad (88)$$

where  $R^{(n)}(x)$  is a generic notation for rational functions behaving like  $x^n$  when  $x \rightarrow \infty$ , and the precise form of which will be eventually of no importance. This produces, for the functions to be integrated in eqs. (84) and (85):

$$\tilde{F}(x)^N = \left(\frac{2\pi^2}{i}\right)^N \left[ \frac{1}{[x(x^2 - 4\pi^2)]^N} + \sum_{n=1}^N e^{inx} R_n^{(-3N)}(x) \right], \quad (89)$$

$$\begin{aligned} & \tilde{F}(x)^{N-3} [\tilde{F}(x)\tilde{G}(x) + \eta_N \tilde{H}(x)^2] \\ &= i \left( \frac{2\pi^2}{i} \right)^N \left\{ \frac{(5 + 2\eta_N)x^2 - 4\pi^2}{[x(x^2 - 4\pi^2)]^{N+1}} + \sum_{n=1}^{N+1} e^{inx} R_n^{(-3N-1)}(x) \right\}. \end{aligned} \quad (90)$$

Let us stress again that these functions, when analytically continued, are holomorphic in the whole complex plane (the poles appearing in the first term are exactly canceled by the remaining ones).

We consider first  $C_N$ , now given by

$$C_N = (N-1)! \left( \frac{2\pi^2}{i} \right)^N \frac{1}{2\pi} \int_{-\infty}^{\infty} dx e^{-ix} \left\{ \frac{1}{[x(x^2 - 4\pi^2)]^N} + \sum_{n=1}^N e^{inx} R_n^{(-3N)}(x) \right\}. \quad (91)$$

Since the function within the curly bracket is an entire one, we can shift the integration path to  $I \equiv \{z = x + ia \mid x \in \mathcal{R}\}$ . Let us choose  $a > 0$ . Then, by Cauchy theorem

$$\int_I dz e^{-iz} \sum_{n=1}^N e^{inz} R_n^{(-3N)}(z) = 0. \quad (92)$$

Indeed, the integrand is holomorphic above  $I$  and is bounded there by  $\text{const.} |z|^{-3N}$ , which allows us to close the integration path at infinity in the **upper** complex plane. We end up with

$$C_N = (N-1)! \left( \frac{2\pi^2}{i} \right)^N \frac{1}{2\pi} \int_I dz \frac{e^{-iz}}{z^N (z^2 - 4\pi^2)^N}. \quad (93)$$

Similarly, we are allowed to close the integration path at infinity in eq. (93), but this time in the **lower** complex plane. The integrand has now poles at  $z = 0, \pm 2\pi$ , and applying the residue theorem leads to explicit expressions for  $C_N$ . Unfortunately, these expressions turn out to appear as (finite) sums with alternating signs, the terms of which become very close to each other for large  $N$ . They are therefore useless for determining the asymptotic behaviour of  $C_N$ , and we have to proceed differently. Let us write

$$\begin{aligned} \int_I dz \frac{e^{-iz}}{z^N (z^2 - 4\pi^2)^N} &= \frac{1}{(N-1)!} \frac{d^{N-1}}{d\alpha^{N-1}} \Big|_{\alpha=4\pi^2} \int_I dz \frac{e^{-iz}}{z^N (z^2 - \alpha)} \\ &= \frac{-2i\pi}{(N-1)!} \frac{d^{N-1}}{d\alpha^{N-1}} \Big|_{\alpha=4\pi^2} [R_+(\alpha) + R_-(\alpha) + R_0(\alpha)], \end{aligned} \quad (94)$$

where  $R_{\pm}(\alpha)$  and  $R_0(\alpha)$  are the residues of the last integrand at  $z = \pm\sqrt{\alpha}$  and  $z = 0$  respectively. They are readily computed, assuming first that  $N = 2M + 1$  is odd:

$$\begin{aligned} R_+(\alpha) + R_-(\alpha) &= \frac{\cos \sqrt{\alpha}}{\alpha^{M+1}} = \sum_{r=0}^{\infty} \frac{(-1)^r}{(2r)!} \alpha^{r-M-1}, \\ R_0(\alpha) &= - \sum_{r=0}^M \frac{(-1)^r}{(2r)!} \alpha^{r-M-1}. \end{aligned} \quad (95)$$

Hence

$$R_+(\alpha) + R_-(\alpha) + R_0(\alpha) = (-1)^{M+1} \sum_{s=0}^{\infty} \frac{(-1)^s}{(2M+2s+2)!} \alpha^s \quad (96)$$

Using eqs. (93), (94) and (96) we then obtain

$$\begin{aligned} C_N &= \left(\frac{2\pi^2}{i}\right)^N (-1)^{M+1} (-i) \frac{d^{N-1}}{d\alpha^{N-1}} \Big|_{\alpha=4\pi^2} \sum_{s=0}^{\infty} \frac{(-1)^s}{(2M+2s+2)!} \alpha^s \\ &= (2\pi^2)^N \sum_{n=0}^{\infty} \frac{(N+n-1)!}{n!(3N+2n-1)!} (-4\pi^2)^n. \end{aligned} \quad (97)$$

The result is exactly the same for even  $N$ . It suffices now to observe that the last series alternates in sign and is decreasing to deduce

$$C_N = (2\pi^2)^N \frac{(N-1)!}{(3N-1)!} \left[1 + O\left(\frac{1}{N}\right)\right]. \quad (98)$$

Our procedure for evaluating  $A_N$  is quite similar, and we give below only the main steps. From eqs. (85) and (90) we get

$$\begin{aligned} A_N &= (N-1)! \left(\frac{2\pi^2}{i}\right)^N \frac{i}{2\pi} \int_I dz e^{-iz} \frac{(5+2\eta_N)z^2-4\pi^2}{z^{N+1}(z^2-4\pi^2)^{N+1}} \\ &= \frac{1}{N} \left(\frac{2\pi^2}{i}\right)^N \frac{i}{2\pi} \frac{d^N}{d\alpha^N} \Big|_{\alpha=4\pi^2} \int_I dz e^{-iz} \left[ \frac{5+2\eta_N}{z^{N-1}(z^2-\alpha)} - \frac{4\pi^2}{z^{N+1}(z^2-\alpha)} \right], \end{aligned} \quad (99)$$

and, after computing the residues at  $z = \pm\sqrt{\alpha}$  and  $z = 0$ , we get

$$\begin{aligned} A_N &= \frac{(-2\pi^2)^N}{N} \frac{d^N}{d\alpha^N} \Big|_{\alpha=4\pi^2} \sum_{s=0}^{\infty} \left[ \frac{5+2\eta_N}{(N+2s)!} - \frac{4\pi^2}{(N+2s+2)!} \right] (-\alpha)^s \\ &= \frac{(2\pi^2)^N}{N} \sum_{n=0}^{\infty} \frac{(N+n)!}{n!} \left[ \frac{5+2\eta_N}{(3N+2n)!} - \frac{4\pi^2}{(3N+2n+2)!} \right] (-4\pi^2)^n. \end{aligned} \quad (100)$$

Again, the last series alternates in sign and decreases, which entails

$$A_N = (5+2\eta_N)(2\pi^2)^N \frac{(N-1)!}{(3N)!} \left[1 + O\left(\frac{1}{N}\right)\right]. \quad (101)$$

Finally, using eqs. (80), (98), (101) and (87) we obtain

$$\frac{\rho_N^{(0)}}{N} = \frac{5+2\eta_N}{3N} \left[1 + O\left(\frac{1}{N}\right)\right] = \begin{cases} \frac{2}{3} \left[1 + O\left(\frac{1}{N}\right)\right] & \text{for bosons} \\ \frac{1}{N} \left[1 + O\left(\frac{1}{N}\right)\right] & \text{for fermions} \end{cases} \quad (102)$$

The same procedure applies for all integer values of  $\beta$ , although the algebra becomes quite involved. The general result for bosons (and for any integer  $\beta$ ) is:

$$\lim_{N \rightarrow \infty} \frac{\rho_N^{(0)}}{N} = \frac{(\beta!)^4 [(3\beta+1)!]^2}{[(2\beta)!]^2 [(2\beta+1)!]^3}. \quad (103)$$

Our method does not adapt straight forwardly to the case of non-integer values of  $\beta$ , but there is clearly no reason to expect a different outcome for such intermediate values. Therefore, the Onsager-Penrose criterion (79) is **not** met for bosons, and we reach the conclusion that Bose-Einstein condensation is **possible** in the bosonic version of the  $N$ -body model discussed in Sect.3.

In the fermionic version, the result (102) is not conclusive, as explained after eq. (79). It only points (not too surprisingly) to the absence of quantum phase in the system.

## 6 The $B_N$ model in one dimension

Subsequent to the seminal work of Calogero and Sutherland for the  $A_{N-1}$  system, the entire bound state spectrum of the Calogero model was obtained for  $BC_N, D_N$  root systems [3]. It is then natural to enquire if in our case, can one at least obtain the exact ground state and radial excitation spectrum in the  $BC_N$  or  $D_N$  case? We now show that the answer to this question is in the affirmative.

Consider the  $BC_N$  Hamiltonian,

$$\begin{aligned}
H = & -\frac{1}{2} \sum_{i=1}^N \frac{\partial^2}{\partial x_i^2} + V \left( \sum_{i=1}^N x_i^2 \right) + g \sum_{i=1}^{N-1} \left[ \frac{1}{(x_i - x_{i+1})^2} + \frac{1}{(x_i + x_{i+1})^2} \right] \\
& -G \sum_{i=2}^{N-1} \left[ \left( \frac{1}{x_{i-1} - x_i} - \frac{1}{x_{i-1} + x_i} \right) \left( \frac{1}{x_i - x_{i+1}} + \frac{1}{x_i + x_{i+1}} \right) \right] \\
& +g_1 \sum_{i=1}^N \frac{1}{x_i^2}, \tag{104}
\end{aligned}$$

of which  $B_N, C_N$  and  $D_N$  are the special cases. We again restrict our attention to the sector of configuration space corresponding to a definite ordering of the particles as given by eq. (4).

We start with the ansatz

$$\psi = P_{2k}(x)\phi(r) \left( \prod_{i=1}^N (x_i^2)^{\gamma/2} \right) \prod_{i=1}^{N-1} (x_i^2 - x_{i+1}^2)^\beta, \tag{105}$$

where  $r^2 = \sum_{i=1}^N x_i^2$ . On substituting it in the Schrödinger equation for the  $B_N$ -Hamiltonian (104) we find that  $\phi$  satisfies

$$\Phi''(r) + [N+4k-1+2N\gamma+4(N-1)\beta] \frac{1}{r} \Phi'(r) + 2[E - V(r)] \Phi(r) = 0, \tag{106}$$



provided  $g$  and  $G$  are again related to  $\beta$  by eq. (2) while  $g_1$  is related to  $\gamma$  by

$$g_1 = \frac{\gamma}{2}(\gamma - 1). \quad (107)$$

Here,  $P_{2k}(x)$  is a homogeneous polynomial of degree  $2k$  ( $k = 0, 1, 2, \dots$ ) in the particle-coordinates and satisfies the generalized Laplace equation

$$\left[ \sum_{i=1}^N \frac{\partial^2}{\partial x_i^2} + 2\gamma \sum_{i=1}^N \frac{1}{x_i} \frac{\partial}{\partial x_i} + 4\beta \sum_{i=1}^{N-1} \frac{1}{(x_i^2 - x_{i+1}^2)} \left( x_i \frac{\partial}{\partial x_i} - x_{i+1} \frac{\partial}{\partial x_{i+1}} \right) \right] P_{2k}(x) = 0. \quad (108)$$

Let us now specialize to the case of the oscillator potential, i.e.,  $V(r) = \frac{\omega^2}{2}r^2$ . In this case, (106) is the well known radial equation for the oscillator problem in more than one dimension and its solution is

$$\Phi(r) = \exp(-\omega r^2/2) L_n^a(\omega r^2), \quad n = 0, 1, 2, \dots \quad (109)$$

where  $L_n^a(x)$  is the associated Laguerre polynomial while the energy eigenvalues are given by

$$E_n = \left[ 2n + 2k + \frac{N}{2} + N\gamma + 2(N-1)\beta \right] \omega, \quad (110)$$

with  $a = \frac{E}{\omega} - 2n - 1$ . The exact ground state is obtained from here when  $n = k = 0$ . The fact that  $n = k = 0$  gives the exact ground state energy of the system can be easily shown *a la*  $A_{N-1}$  case by the method of supersymmetric quantum mechanics. It may be noted that for large  $N$ , the energy  $E$  is proportional to  $N$  so that like the  $A_{N-1}$  case, the  $B_N$  model also has a good thermodynamic limit. In contrast, notice that the long-ranged  $B_N$  Calogero model does not have a good thermodynamic limit.

Are there homogeneous polynomial solutions of eq. (108) of degree  $2k$  ( $k \geq 1$ )? While we are unable to answer this question for any  $k$ , at least for small values of  $k$  ( $k > 0$ ) there does not seem to be any solution to eq. (108). For example, we have failed to find any polynomial solution of degree 2, 4 and 6. Thus it appears that unlike the  $A_{N-1}$  case, in the  $BC_N$  case one is only able to obtain the ground state and radial excitations over it.

Proceeding in the same way, the energy eigenvalues and eigenfunctions in the case of the Coulomb-like potential (33) are

$$E = -\frac{\alpha^2}{2 \left[ n + 2k + \frac{N-1}{2} + N\gamma + 2(N-1)\beta \right]^2} \quad (111)$$

$$\Phi = e^{-\sqrt{2|E|r}} L_n^b(2\sqrt{2|E|r}) \quad (112)$$

where  $b = N - 2 + 4k + 2N\gamma + 4(N - 1)\beta$ . Again, so far we have been able to obtain solutions only in case  $k=0$ .

As in Sec.II, in the  $BC_N$  Hamiltonian (104), all the particles are not being treated on the same footing. Again, one possibility is to add extra terms. Consider for example,

$$H_1 = H + H', \quad (113)$$

where  $H$  is as given by eq. (104) while  $H'$  has the form

$$\begin{aligned} H' &= g \left[ \frac{1}{(x_N - x_1)^2} + \frac{1}{(x_N + x_1)^2} \right] \\ &- G \left[ \left( \frac{1}{x_N - x_1} - \frac{1}{x_N + x_1} \right) \left( \frac{1}{x_1 - x_2} + \frac{1}{x_1 + x_2} \right) \right. \\ &\left. + \left( \frac{1}{x_{N-1} - x_N} - \frac{1}{x_{N-1} + x_N} \right) \left( \frac{1}{x_N - x_1} + \frac{1}{x_N + x_1} \right) \right]. \quad (114) \end{aligned}$$

One can now run through the arguments as given above and show that the eigenstates for both the oscillator and Coulomb-like potentials have the same form as given above except that in the term multiplying  $\beta$ ,  $N-1$  gets replaced by  $N$  at all places including in the Laplace eq. (108). However, now we find that there are indeed solutions to the Laplace eq. (108) (with  $N-1$  replaced by  $N$ ). In particular, the solution for any  $N(\geq 4)$  and  $k = 4$  is given by

$$P_{k=4}(x) = a \sum_{i=1}^N x_i^4 + b \sum_{i<j}^N x_i^2 x_j^2, \quad (115)$$

where

$$\frac{b}{a} = -2 \left[ \frac{3 + 8\beta + 2\gamma}{N - 1 + 2(N - 1)\gamma + 4(N - 2)\beta} \right]. \quad (116)$$

As in the  $A_{N-1}$  case, we again find that even though the Laplace eq. (108) is only invariant under cyclic permutations, the solution is in fact invariant under the permutation of any two coordinates. It will be interesting to try to find solutions for higher values of  $k$  and study the full degeneracy of the spectrum.

Besides these two, one can obtain a part of the spectra including the ground state for several other potentials but we shall not discuss them here.

## 7 $BC_N$ model in one dimension with periodic boundary condition

Following the work of Sutherland [25] on the  $A_{N-1}$  root system, the exact ground state as well as the excitation spectrum was also obtained in the case of the  $BC_N, D_N$  root systems [3]. It is then worth enquiring if, in our case, one can also obtain the ground state and the excitation spectrum. As a first step in that direction, we shall obtain the exact ground state of the  $BC_N$  model with periodic boundary condition.

The Hamiltonian for the  $BC_N$  case is given by

$$\begin{aligned}
 H = & -\frac{1}{2} \sum_{i=1}^N \frac{\partial^2}{\partial x_i^2} + g \frac{\pi^2}{L^2} \sum_{i=1}^N \left[ \frac{1}{\sin^2 \frac{\pi}{L}(x_i - x_{i+1})} + \frac{1}{\sin^2 \frac{\pi}{L}(x_i + x_{i+1})} \right] \\
 & + g_1 \frac{\pi^2}{L^2} \sum_i \frac{1}{\sin^2 \frac{\pi}{L} x_i} + g_2 \frac{\pi^2}{L^2} \sum_i \frac{1}{\sin^2 \frac{2\pi}{L} x_i} - G \frac{\pi^2}{L^2} \sum_{i=1}^N \left[ \cot \frac{\pi}{L}(x_{i-1} - x_i) \right. \\
 & \left. - \cot \frac{\pi}{L}(x_{i-1} + x_i) \right] \left[ \cot \frac{\pi}{L}(x_i - x_{i+1}) + \cot \frac{\pi}{L}(x_i + x_{i+1}) \right]. \quad (117)
 \end{aligned}$$

We again restrict our attention to the sector of the configuration space corresponding to a definite ordering of the particles as given by eq. (2). For this case, we start with a trial wave function of the form

$$\Psi_0 = \prod_{i=1}^N \sin^\gamma \theta_i \prod_{i=1}^N (\sin^2 2\theta_i)^{\gamma_1/2} \prod_{i=1}^N [\sin^2(\theta_i - \theta_{i+1})]^{\beta/2} \prod_{i=1}^N [\sin^2(\theta_i + \theta_{i+1})]^{\beta/2}, \quad (118)$$

( $\theta_i = \pi x_i/L$ ) and substitute it in the Schrödinger equation for the Hamiltonian (117). We find that it is indeed a solution provided  $g$  and  $G$  are again related to  $\beta$  by eq. (2) while  $g_1, g_2$  are related to  $\gamma, \gamma_1$  by

$$g_1 = \frac{\gamma}{2}[\gamma + 2\gamma_1 - 1], \quad g_2 = 2\gamma_1(\gamma_1 - 1). \quad (119)$$

The corresponding ground state energy turns out to be

$$E_0 = \frac{N\pi^2}{2L^2}(\gamma + \gamma_1 + 2\beta)^2. \quad (120)$$

The fact that this is indeed the ground state energy can be easily proved as in Secs. II and III.

## 8 $N$ -body problem in $D$ -dimensions

Having obtained some results for the  $N$ -body problem (1) in one dimension, we study generalization to higher dimensions. Let us consider the following model in  $D$ -dimensions :

$$\begin{aligned}
 H &= -\frac{1}{2} \sum_{i=1}^N \vec{\nabla}_i^2 + g \sum_{i=1}^{N-1} \frac{1}{(\vec{r}_i - \vec{r}_{i+1})^2} \\
 &- G \sum_{i=2}^{N-1} \frac{(\vec{r}_{i-1} - \vec{r}_i) \cdot (\vec{r}_i - \vec{r}_{i+1})}{(\vec{r}_{i-1} - \vec{r}_i)^2 (\vec{r}_i - \vec{r}_{i+1})^2} + V \left( \sum_{i=1}^N r_i^2 \right). \quad (121)
 \end{aligned}$$

On using the ansatz,

$$\psi = \left( \prod_{i=1}^{N-1} |\vec{r}_i - \vec{r}_{i+1}|^\beta \right) \phi(r), \quad r^2 = \sum_{i=1}^N r_i^2, \quad (122)$$

in the Schrödinger equation for the Hamiltonian (121), it can be shown that  $\phi(r)$  satisfies

$$\phi''(r) + [DN - 1 + 2(N - 1)\beta] \frac{1}{r} \phi'(r) + 2(E - V(r))\phi(r) = 0, \quad (123)$$

provided  $g$  and  $G$  are related to  $\beta$  by

$$g = \beta^2 + (D - 2)\beta, \quad G = \beta^2. \quad (124)$$

Equation (123) is easily solved in the case of the oscillator potential (i.e.,  $V(r) = \frac{\omega^2}{2}r^2$ ) yielding the energy eigenstates as

$$\phi(r) = \exp\left(-\frac{\omega}{2}r^2\right) L_n^b(\omega r^2), \quad (125)$$

$$E_n = \left[ 2n + (N - 1)\beta + \frac{DN}{2} \right] \omega. \quad (126)$$

Here  $b = \frac{E}{\omega} - 2n - 1$ . It may be noted that as in all other higher dimensional many-body problems, one has only obtained a part of the energy eigenvalue spectrum which however includes the ground state. In particular, the ground state energy eigenvalue and eigenfunction is given by

$$E_0 = \left[ (N - 1)\beta + \frac{DN}{2} \right] \omega, \quad (127)$$

$$\psi_0 = \exp\left(-\frac{\omega}{2} \sum_{i=1}^N r_i^2\right) \prod_{i=1}^{N-1} |\vec{r}_i - \vec{r}_{i+1}|^\beta. \quad (128)$$

As expected, for  $D = 1$  these results go over to those obtained in Sec. II. The fact that this is indeed the ground state energy can be easily proved by using again a supersymmetric formulation [21].

At this point it is worth asking if the probability distribution for  $N$  particles (at least for some  $D(> 1)$ ) can be mapped to some known random matrix ensemble? In this context we recall that in the case of the Calogero-type model, it has been shown that in two space dimensions  $|\psi_0|^2$  can be mapped to complex random matrix [31]. Using this identification one was able to calculate all the correlation functions of the corresponding many-body theory and show the absence of long-range order but the presence of an off-diagonal long-range order in that theory. Unfortunately, so far as we are aware of, answer to this question is unknown in this particular case. We hope that at least in the case of two space dimensions, where  $|\psi_0|^2$  for our model is given by

$$|\psi_0(z_i)|^2 = C \exp\left(-\omega \sum_{i=1}^N |z_i|^2\right) \prod_{i=1}^{N-1} |z_i - z_{i+1}|^{2\beta}, \quad (129)$$

$|\psi_0|^2$  can be mapped to some variant of the short-range Dyson model.

Finally, we observe that the ground state and a class of excited states can also be obtained in  $D$ -dimensions in case the oscillator potential is replaced by the  $N$ -body Coulomb-like potential  $V(r) = -\alpha/\sqrt{\sum \mathbf{r}_i^2}$ , because the resulting equation (123) is essentially the radial equation for the Coulomb potential. In particular, the energy eigenvalues and eigenfunctions are given by

$$E_n = -\frac{\alpha^2}{2\left[n + \frac{DN-1}{2} + (N-1)\beta\right]^2}, \quad (130)$$

$$\psi_n = \exp(-\sqrt{2|E|r}) L_n^{b'}(2\sqrt{2|E|r}) \left(\prod_{i=1}^{N-1} |\mathbf{r}_i - \mathbf{r}_{i+1}|^\beta\right), \quad (131)$$

where  $b' = DN - 2 + 2(N-1)\beta$ . It may again be noted that whereas the ground state energy is linear in  $\beta$  in the oscillator case, it is not so in the case of the Coulomb-like  $N$ -body potential.

## 9 Short-range model in two dimensions with novel correlations

Few years back, Murthy et al. [32] considered a model in two dimensions with two-body and three-body long-ranged interactions and obtained the

exact ground state and a class of excited states. The interesting feature of this model was that all these states had a built-in novel correlation of the form  $|X_{ij}|^g$  where

$$X_{ij} = x_i y_j - x_j y_i. \quad (132)$$

It is then natural to enquire if one can construct a model in two dimensions and obtain ground and few excited states of the system all of which would have a built-in short-range correlation of the form

$$X_{j,j+1} = x_j y_{j+1} - y_j x_{j+1}. \quad (133)$$

We now show that this is indeed possible. Let us consider the following Hamiltonian

$$H = -\frac{1}{2} \sum_{i=1}^N \vec{\nabla}_i^2 + \frac{\omega^2}{2} \sum_{i=1}^N r_i^2 + g \sum_{i=1}^{N-1} \frac{\vec{r}_i^2 + \vec{r}_{i+1}^2}{X_{i,i+1}^2} - G \sum_{i=2}^{N-1} \frac{\vec{r}_{i-1} \cdot \vec{r}_{i+1}}{X_{i-1,i} X_{i,i+1}} \quad (134)$$

where  $X_{i,i+1}$  is as given by eq. (133). We start with the ansatz

$$\psi(x_i, y_i) = \left[ \prod_{i=1}^{N-1} X_{i,i+1}^\beta \right] \exp\left(-\frac{\omega}{2} \sum_i r_i^2\right) \phi(x_i, y_i). \quad (135)$$

On substituting the ansatz in the Schrödinger equation  $H\psi = E\psi$ , one finds that  $\phi$  satisfies the equation

$$\left[ -\frac{1}{2} \sum_{i=1}^N \vec{\nabla}_i^2 + \omega \sum_{i=1}^N \vec{r}_i \cdot \dot{\vec{\nabla}}_i + \beta \sum_{i=1}^{N-1} \frac{1}{X_{i,i+1}} \left( x_{i+1} \frac{\partial}{\partial y_i} - y_{i+1} \frac{\partial}{\partial x_i} + y_i \frac{\partial}{\partial x_{i+1}} - x_i \frac{\partial}{\partial y_{i+1}} \right) \right] \phi = \left( E - [N + 2(N-1)\beta]\omega \right) \phi, \quad (136)$$

provided  $g$  and  $G$  are related by (2). It is interesting to note that even though we are considering the novel correlation model in two-dimensions, the relationship between  $g$  and  $G$  is as in the case of our one-dimensional model. We do not know if this has any deep significance.

We conclude from here that  $\psi$ , as given by eq. (135), with  $\phi$  being a constant is the ground state of the system with the corresponding ground state energy being

$$E_0 = [N + 2(N-1)\beta]\omega. \quad (137)$$

Let us remark that, like the relationship between coupling constants, the ground state energy too has essentially the same form as that of the one-dimensional short-range  $A_{N-1}$  model as given by eq. (13). That one has indeed obtained the ground state can be proved as before.

As in other many-body problems in two and higher dimensions, we are unable to find the complete excited-state spectrum. However, a class of excited states can be obtained from (136). To that end we introduce the complex coordinates

$$z = x+iy, z^* = x-iy, \partial \equiv \frac{\partial}{\partial z} = \frac{1}{2} \left( \frac{\partial}{\partial x} - i \frac{\partial}{\partial y} \right), \partial^* \equiv \frac{\partial}{\partial z^*} = \frac{1}{2} \left( \frac{\partial}{\partial x} + i \frac{\partial}{\partial y} \right). \quad (138)$$

In terms of these coordinates, the differential eq. (136) takes the form

$$\left[ -2 \sum_{i=1}^N \partial_i \partial_i^* + 2\beta \sum_{i=1}^{N-1} \frac{\left( z_{i+1} \partial_i - z_i \partial_{i+1} + z_i^* \partial_{i+1}^* - z_{i+1}^* \partial_i^* \right)}{\left( z_i z_{i+1}^* - z_i^* z_{i+1} \right)} + \omega \sum_{i=1}^N \left( z_i \partial_i + z_i^* \partial_i^* \right) - (E - E_0) \right] \phi = 0. \quad (139)$$

Now it is readily proved shown that the Hamiltonian  $H$  commutes with the total angular momentum operator  $L = \sum_{i=1}^N (z_i \partial_i - z_i^* \partial_i^*)$ , so that one can classify solutions according to their angular momentum:  $L\phi = l\phi$ .

On defining  $t = \omega \sum_i z_i z_i^*$  and let  $\phi \equiv \phi(t)$  it is easily shown that  $\phi(t)$  satisfies

$$t\phi''(t) + \left[ \frac{E_0}{\omega} - t \right] \phi'(t) + \left( \frac{E - E_0}{2\omega} \right) \phi(t) = 0, \quad (140)$$

where  $E_0$  is as given by eq. (137). Hence the allowed solutions with  $l = 0$  are

$$E = E_0 + 2n\omega, \quad \phi(t) = L_n^{\frac{E_0}{\omega}-1}(t). \quad (141)$$

Solutions with angular momentum  $l > 0$  or  $l < 0$  can similarly be obtained by introducing  $t_z = \omega \sum_i z_i^2$  or  $t_{z^*} = \omega \sum_i (z_i^*)^2$ . For example, let  $\phi = \phi(t_z)$ . Then eq. (139) reduces to

$$2\omega t_z \frac{d\phi}{dt_z} = (E - E_0)\phi. \quad (142)$$

This is the well known Euler equation whose solutions are just monomials in  $t_z$ . The solution is given by  $\phi(t_z) = t_z^m (m > 0)$ , and hence the angular momentum  $l = 2m$  while the energy eigenvalues are  $E = E_0 + 2m\omega = E_0 + l\omega$ . Further, we can combine these solutions with the  $l = 0$  solutions obtained above and obtain a tower of excited states. For example, let us define  $\phi(z_i, z_i^*) = \phi_1(t)\phi_2(t_z)$ , where  $\phi_1$  is a solution with  $l = 0$ , while  $\phi_2$  is

the solution with  $l > 0$ . On using  $\phi_2(t_z) = t_z^m$  it is easily shown that  $\phi_1$  again satisfies a confluent hypergeometric equation,

$$t\phi_1''(t) + \left[ \frac{E_0}{\omega} + 2m - t \right] \phi_1'(t) + \left( \frac{E - E_0}{2\omega} + m \right) \phi_1(t) = 0. \quad (143)$$

Hence the energy eigenvalues are given by  $E - E_0 = (2n_r + 2m)\omega$ . One may repeat the procedure to obtain exact solutions for a tower of states with  $l < 0$ .

## 10 Summary

In this paper we have discussed an  $N$ -body problem in one dimension and presented its exact ground state on a circle and most likely the entire spectrum on a real line. There are several similarities as well as differences between the model discussed here and Calogero-Sutherland (CSM) type of models and it might be worthwhile to compare the salient features of the two.

1. Whereas in CSM the interaction is between all neighbours, in our case the interaction is only between nearest and next-to-nearest neighbours. Note however that in both the cases it is an inverse square interaction.
2. Whereas in CSM (in one dimension) there is only two-body interaction, both two- and three-body interactions are required in our model for partial (or possibly exact) solvability on a real line.
3. Whereas the complete bound state spectrum is obtained in the Sutherland model (periodic potential) or if there is external harmonic or Coulomb-like  $N$ -body potential as given by eq. (33) and in the case of both  $A_{N-1}$  and  $BC_N$  root systems, it is not clear if this is so in our case even though it is likely that this may be so in the  $A_{N-1}$  case.
4. Whereas our system, both on a line and on a circle, has good thermodynamic limit (i.e.  $E/N$  is finite for large  $N$ ), CSM does not have good thermodynamic limit in either case and  $E/N$  diverges like  $N$  for large  $N$ .
5. In both the cases, the norm of the ground state wavefunction can be mapped to the joint probability density function of the eigenvalues of some random matrix. Using this correspondence, in both the cases, one is able to calculate one- and two-point functions. However, whereas in the CSM this is possible only at three values of the coupling (corresponding to orthogonal, unitary or symplectic random matrices), in our



case the correlation functions can be computed analytically for any integral or half-integral values of the coupling while numerically it can be done for any positive  $\beta$ .

6. In the CSM case with an external potential of the form

$$V\left(\sum_i x_i^2\right) = A \sum_{i=1}^N x_i^2 + B \left(\sum_i x_i^2\right)^2 + C \left(\sum_i x_i^2\right)^3 \quad (144)$$

it has been shown [24] that the norm of the ground state wave function can be mapped to a random matrix corresponding to branched polymers. It is not known if a similar mapping is possible in our case.

7. A multi-species generalization of CSM has been done [33], it is not clear if a similar generalization is possible in our case or not.
8. Generalization to  $D$ -dimensions ( $D > 1$ ) is possible in CSM as well as in our model and in both the cases one is able to obtain only a partial spectrum including the ground state. In both the cases, both two- and three-body interactions are required. Whereas our system has a good thermodynamic limit in any dimension  $D$ , the CSM does not have a good thermodynamic limit in any dimension. However, whereas the norm of the ground state wave function can be mapped to complex random matrices in the CSM case in two dimensions [31], no such mapping has so far been possible in our case for  $D > 1$ .
9. Model with novel correlations is possible in two dimensions in both the cases [32] but unlike CSM, our system has a good thermodynamic limit.
10. In the CSM, it has been possible to obtain the entire spectrum algebraically by using supersymmetry and shape invariance [34]. It would be nice if similar thing can also be done in our model. Further, in the CSM, one has also written down the supersymmetric version of the model [35]. It would be worth enquiring if a similar thing can also be done in our model.
11. In the CSM type models, one knows the various exactly solvable problems in which the  $N$ -particles interact pairwise by two body interaction [36]. The question one would like to ask in our context is: what are the various exactly solvable problems in one dimension in which the  $N$  particles have only nearest- and next-to-nearest neighbour interactions?

12. In the CSM, not only one- and two-point but even n-point correlation functions are known. It would be nice if the same is also possible in the present context.
13. *A la* Haldane-Shastry spin models [37], can we also construct spin models in the context of our model?
14. Unlike CSM, in our case the off-diagonal long-range order is nonzero in the bosonic version of the many-body theory in one dimension. Note however that the off-diagonal long-range order is nonzero in the CSM in two dimensions.

## Appendix

### 1. Proof of the representation (84) of $C_N$

By construction, the square of the wave function (69) is a symmetrical function of all its arguments, so that we can write eq. (74) as well:

$$C_N = N! \int_0^1 dx_1 \int_0^{x_1} dx_2 \dots \int_0^{x_{N-1}} dx_N |\psi_N(x_1, \dots, x_N)|^2, \quad (145)$$

where the particle coordinates are now properly ordered. We are thus allowed to substitute  $\phi_N$  for  $\psi_N$  in (145) and obtain from eqs. (70) and (82)

$$C_N = N! \int_0^1 dx_1 \int_0^{x_1} dx_2 \dots \int_0^{x_{N-1}} dx_N \prod_{n=1}^N S(x_n - x_{n+1})^2. \quad (146)$$

Changing the integration variables  $(x_1, x_2, \dots, x_N)$  to  $(\Delta_1, \Delta_2, \dots, \Delta_{N-1}, x_N)$ , where

$$\Delta_n = x_n - x_{n+1}; \quad (n = 1, \dots, N - 1), \quad (147)$$

one easily gets

$$\begin{aligned} C_N &= N! \int_0^1 d\Delta_1 \int_0^{1-\Delta_1} d\Delta_2 \dots \int_0^{1-\Delta_1-\dots-\Delta_{N-2}} d\Delta_{N-1} X \\ &X \int_0^{1-\sum_{p=1}^{N-1} \Delta_p} dx_N \prod_{n=1}^{N-1} S(\Delta_n)^2 S(x_N - x_1)^2. \end{aligned} \quad (148)$$

Since  $x_N - x_1 = -\sum_{p=1}^{N-1} \Delta_p$  is in fact independent of  $x_N$  in the new set of variables, eq. (148) becomes, using also  $S(-x) = S(1-x)$ :

$$\begin{aligned} C_N &= N! \int_0^1 d\Delta_1 \int_0^{1-\Delta_1} d\Delta_2 \dots \int_0^{1-\Delta_1-\dots-\Delta_{N-2}} d\Delta_{N-1} X \\ &X \left(1 - \sum_{p=1}^{N-1} \Delta_p\right) \prod_{n=1}^{N-1} S(\Delta_n)^2 S\left(1 - \sum_{p=1}^{N-1} \Delta_p\right)^2. \end{aligned} \quad (149)$$

It is now convenient to introduce the extra variable

$$\Delta_N = 1 - \sum_{p=1}^{N-1} \Delta_p, \quad (150)$$

and to recast eq. (149) in the form

$$\begin{aligned} C_N &= N! \int_0^1 d\Delta_1 \int_0^1 d\Delta_2 \dots \int_0^1 d\Delta_{N-1} \int_0^1 d\Delta_N \delta\left(1 - \sum_{p=1}^N \Delta_p\right) X \\ &X \Delta_N \prod_{n=1}^N S(\Delta_n)^2 \\ &= N! \int_0^1 d\Delta_1 \dots \int_0^1 d\Delta_N \delta\left(1 - \sum_{p=1}^N \Delta_p\right) \frac{1}{N} \sum_{m=1}^N \Delta_m \prod_{n=1}^N S(\Delta_n)^2 \\ &= (N-1)! \int_0^1 d\Delta_1 \dots \int_0^1 d\Delta_N \delta\left(1 - \sum_{p=1}^N \Delta_p\right) \prod_{n=1}^N S(\Delta_n)^2 \end{aligned} \quad (151)$$

In the second equality, we have used the fact that, apart from the factor  $\Delta_N$ , the integrand and the integration range are completely symmetrical in the variables  $(\Delta_1, \dots, \Delta_N)$ . Finally, the integration over these variables factorizes after introducing the representation

$$\delta\left(1 - \sum_{p=1}^N \Delta_p\right) = \frac{1}{2\pi} \int_{-\infty}^{\infty} dx e^{-ix\left(1 - \sum_{p=1}^N \Delta_p\right)}, \quad (152)$$

and interchanging the  $x$ - and  $\Delta$ -integrations. This produces eq. (84).

## 2. Proof of the representation (85) of $A_N$

Proceeding along the same lines, we first put the expression (81) of  $A_N$  in the form

$$\begin{aligned} A_N &= (N-1)! \int_0^1 dx_1 \int_0^{x_1} dx_2 \dots \int_0^{x_{N-2}} dx_{N-1} \phi_N(x_1, \dots, x_{N-1}, 0) X \\ &X R_N(x_1, \dots, x_{N-1}), \end{aligned} \quad (153)$$

where

$$\begin{aligned}
R_N(x_1, \dots, x_{N-1}) &= \int_0^{x_{N-1}} dx \phi_N(x_1, \dots, x_{N-1}, x) \\
&\pm \int_{x_{N-1}}^{x_{N-2}} dx \phi_N(x_1, \dots, x, x_{N-1}) + \dots + \int_{x_1}^1 dx \phi_N(x, x_1, \dots, x_{N-1}) \\
&= \int_0^{x_{N-1}} dx \phi_N(x_1, \dots, x_{N-1}, x) + \int_{x_1}^1 dx \phi_N(x, x_1, \dots, x_{N-1}) \\
&+ \sum_{p=1}^{N-2} \nu_p \int_{x_{p+1}}^{x_p} dx \phi_N(x_1, \dots, x_p, x, x_{p+1}, \dots, x_N). \tag{154}
\end{aligned}$$

Here,  $\nu_p = 1$  ( $\nu_p = (-1)^p$ ) for bosons (fermions) and we have used the restriction to odd  $N$  in the second case. Thanks to the periodicity and the cyclic symmetry of  $\phi_N$ , the first two terms in the last expression above can be collected to give

$$\int_{x_1-1}^{x_{N-1}} dx \phi_N(x, x_1, \dots, x_{N-1}).$$

Hence  $R_N$  becomes (with  $x_N = x_1 - 1$ )

$$\begin{aligned}
R_N(x_1, \dots, x_{N-1}) &= \sum_{p=1}^{N-1} \nu_p \int_{x_{p+1}}^{x_p} dx \phi_N(x_1, \dots, x_p, x, x_{p+1}, \dots, x_{N-1}) \\
&= \sum_{p=1}^{N-1} \nu_p \prod_{n=1}^{N-1} S(x_n - x_{n+1}) \int_{x_{p+1}}^{x_p} dx S(x - x_{p+1}) S(x_p - x), \quad (n \neq p) \\
&= \sum_{p=1}^{N-1} \nu_p \sum_{n=1}^{N-1} S(x_n - x_{n+1}) S_2(x_p - x_{p+1}), \quad (n \neq p) \tag{155}
\end{aligned}$$

according to the definition (83). We also have:

$$\phi_N(x_1, \dots, x_{N-1}, 0) = \prod_{m=1}^{N-2} S(x_m - x_{m+1}) S(x_{N-1}) S(x_1). \tag{156}$$

Inserting eqs. (155) and (156) in eq. (153) and introducing as before the new integration variables  $\Delta_n \equiv x_n - x_{n+1}$  ( $n = 1, \dots, N-2$ ) and  $x_{N-1}$ , we obtain

$$A_N = (N-1)! \int_0^1 d\Delta_1 \int_0^{1-\Delta_1} d\Delta_2 \dots \int_0^{1-\Delta_1-\dots-\Delta_{N-3}} d\Delta_{N-2} X$$

$$\begin{aligned}
& X \int_0^{\Delta_{N-1}} dx_{N-1} \prod_{m=1}^{N-2} S(\Delta_m) S(x_{N-1}) S(x_{N-1} - \Delta_{N-1}) X \\
& X \sum_{p=1}^{N-1} \nu_p \prod_{n=1}^{N-1} S(\Delta_n) S_2(\Delta_p); \quad (n \neq p) \tag{157}
\end{aligned}$$

where  $\Delta_{N-1} = 1 - \sum_{p=1}^{N-2} \Delta_p$ . The integration over  $x_{N-1}$  gives the factor  $S_2(\Delta_{N-1})$  in place of  $S(x_{N-1})S(x_{N-1} - \Delta_{N-1})$ , so that

$$\begin{aligned}
A_N &= (N-1)! \int_0^1 d\Delta_1 \dots \int_0^1 d\Delta_{N-1} \delta(1 - \sum_{p=1}^{N-1} \Delta_p) \left[ \prod_{m=1}^{N-2} X \right. \\
& X \left. S(\Delta_m) S_2(\Delta_{N-1}) \sum_{p=1}^{N-1} \nu_p \prod_{n=1}^{N-1} S(\Delta_n) S_2(\Delta_p) \right], \quad (n \neq p) \tag{158}
\end{aligned}$$

On taking into account the complete symmetry of the integration measure, one finds that the square bracket in eq. (158) can be replaced by

$$\begin{aligned}
[\dots] &= \prod_{m=1}^{N-2} S(\Delta_m)^2 S_2(\Delta_{N-1})^2 + \eta_N \prod_{m=1}^{N-3} S(\Delta_m)^2 X \\
& X [S(\Delta_{N-2}) S_2(\Delta_{N-2})] [S(\Delta_{N-1}) S_2(\Delta_{N-1})], \tag{159}
\end{aligned}$$

where  $\eta_N$  is as defined in eq. (87). Finally, one obtains the factorization of the multiple integral in eq. (158) by using again the representation (152) of the  $\delta$  measure (with  $(N-1)$  in place of  $N$ ). This entails eq. (85).

A last remark may be in order. Alternative, equivalent forms of the representations (84) and (85) would be obtained by relying on Fourier expansions instead of Fourier integrals, that is by considering the integrands in eqs. (149) and (157) not as functions with compact supports  $[0, 1]^N \subset \mathcal{R}^N$ , resp.  $[0, 1]^{N-1} \subset \mathcal{R}^{N-1}$ , but as periodic functions (this would amount to modifying eq. (152) accordingly). It turns out however that the resulting representations of  $C_N$  and  $A_N$  (as Fourier series) are much less convenient for the explicit or asymptotic evaluations of these quantities.

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## Figure Legends

**Fig. 1** The two-point correlation function for four integer values of  $\beta$  (from left to rightmost are increasing values from 1 to 4) shows clearly an absence of long-range order.

**Fig. 2** The two-point correlation function for some fractional values of  $\beta$  plotted along with  $\beta$  equal to 1 and 2. From left to rightmost are increasing values from 1,  $4/3$ ,  $3/2$ ,  $5/3$ , 2,  $7/3$ , and  $5/2$ . Thus, even for fractional values, there is no long-range order.





