A Review of Nadir Point Estimation Procedures Using Evolutionary Approaches: A Tale of Dimensionality Reduction

Kalyanmoy Deb and Kaisa Miettinen

Abstract Estimation of the nadir objective vector is an important task, particularly for multi-objective optimization problems having more than two conflicting objectives. Along with the ideal point, nadir point can be used to normalize the objectives so that multi-objective optimization algorithms can be used more reliably. The knowledge of the nadir point is also a pre-requisite to many multiple criteria decision making methodologies. Moreover, nadir point is useful for an aid in interactive methodologies and visualization softwares catered for multi-objective optimization. However, the computation of exact nadir point for more than two objectives is not an easy matter, simply because nadir point demands the knowledge of extreme Pareto-optimal solutions. In the past few years, researchers have proposed several nadir point estimation procedures using evolutionary optimization methodologies. In this paper, we review the past studies and reveal an interesting chronicle of events in this direction. To make the estimation procedure computationally faster and more accurate, the methodologies were refined one after the other by mainly focusing on increasingly lower dimensional subset of Pareto-optimal solutions. Simulation results on a number of numerical test problems demonstrate better efficacy of the approach which aims to find only the extreme Pareto-optimal points compared to its higher-dimensional counterparts.

Kalyanmoy Deb
Department of Mechanical Engineering, Indian Institute of Technology Kanpur, PIN 208016, India, e-mail: deb@iitk.ac.in
Also Finland Distinguished Professor, Helsinki School of Economics, Finland

Kaisa Miettinen
Department of Mathematical Information Technology, P.O. Box 35 (Agora), FI-40014 University of Jyväskylä, Finland, e-mail: kaisa.miettinen@jyu.fi

1 Introduction

A reliable and accurate estimation of the nadir point in multi-objective optimization is an important task from a number of reasons. First, along with the ideal objective vector, the nadir objective vector can be used to *normalize* objective functions [14], a matter often desired for an adequate functioning of multi-objective optimization algorithms in the presence of objective functions with different magnitudes. Second, the nadir objective vector is a pre-requisite for finding preferred Pareto-optimal solutions in different interactive algorithms, such as the *guess* method [3] (where the idea is to maximize the minimum weighted deviation from the nadir objective vector), or it is otherwise an integral part of an interactive method like the NIMBUS method [14, 15]. Third, the knowledge of nadir and ideal objective values helps the decision-maker in adjusting her/his expectations on a realistic level by providing the range of each objective and can then be used to aid in specifying preference information in interactive methods in order to focus on a desired region.

Despite the long-term efforts by researchers, estimation of nadir point is still an open matter for research. Recently, researchers have suggested different ways to employ an evolutionary multi-objective optimization (EMO) procedure for this purpose. Since an EMO methodology works with a number of points in each iteration, its operators can be designed to focus its search towards a number of Pareto-optimal solutions simultaneously in a single simulation. This flexibility makes an EMO procedure a potential tool for arriving at important Pareto-optimal points for estimating the nadir point. In this paper, we review the existing EMO methodologies from the point of view of dimensionality of the target solutions and discuss advantages and disadvantages of these methodologies. We argue that an EMO method of finding the extreme Pareto-optimal points (instead of intermediate Pareto-optimal points) is a computationally faster approach and the modification of obtained extreme points by a local search may provide accuracy in the estimation of the nadir point.

Motivations for estimating the nadir point led the MCDM researchers dealing with methodologies to suggest procedures for approximating the nadir point using a so-called *payoff table* [1], since 1971. This involves computing the individual optimum solutions, constructing a payoff table by evaluating other objective values at these optimal solutions, and estimating the nadir point from the worst objective values from the table. This procedure may not guarantee a true estimation of the nadir point for more than two objectives. Moreover, the estimated nadir point can be either an over-estimation or an under-estimation of the true nadir point. For example, Isserman and Steuer [11] have demonstrated these difficulties for finding a nadir point using the payoff table method even for linear problems and emphasized the need of using a better method. Among others, Dessouky et al. [8] suggested three heuristic methods and Korhonen et al. [13] another heuristic method for this purpose. Let us point out that all these methods suggested have been developed for linear multi-objective problems where all objectives and constraints are linear functions of the variables.

In [9], an algorithm for deriving the nadir point is proposed based on subproblems. In other words, in order to find the nadir point for an M-objective problem, Pareto-optimal solutions of all $\binom{M}{2}$ bi-objective optimization problems must first be found. Such a requirement may make the algorithm computationally impractical beyond three objectives, although Szczepanski and Wierzbicki [16] implemented the above idea using EAs and showed applications up to three and four objective linear optimization problems. It must be emphasized that although the determination of the nadir point depends on finding the worst objective values in the set of Pareto-optimal solutions, even for linear problems, this is a difficult task [2].

Since an estimation of the nadir objective vector necessitates information about the whole Pareto-optimal surface, any procedure of estimating this point should involve finding Pareto-optimal solutions. This makes the task more difficult compared to finding the ideal point [13]. Since EMO algorithms are potential for finding an approximate set of the entire or a part of the Pareto-optimal surface, they stand as viable candidates for this task. A couple of recent studies [5, 16] have demonstrated a promise in this direction. Another motivation for using an EMO procedure is that nadir point estimation is to be made only once in a problem before beginning the actual decision making process. So, even if the proposed procedure uses somewhat large computational effort (one of the criticisms made often against an evolutionary optimization method), a reliable and accurate methodology for estimating the nadir point is desired.

2 Dimensional Decomposition of Nadir Point Estimation Procedures

In this section, we review the existing evolutionary optimization based nadir point estimation procedures from a point of view of the dimensionality of the target set for the evolutionary optimization procedure. The nadir point can be estimated from any of the following scenarios: (i) the entire Pareto-optimal surface is known, (ii) the *critical edges* of the Pareto-optimal surface (boundaries of Pareto-optimal surface responsible for locating the nadir point) are known, or (iii) only the *critical extreme* Pareto-optimal points (extreme points of the Pareto-optimal surface responsible for locating the nadir point) are known. Interestingly, at least one procedure is already suggested for each of the above tasks and we describe them here.

2.1 'Surface-to-Nadir': Computing Solutions from Entire Pareto-optimal Surface

A naive and simple-minded idea comes from finding a representative set of the entire Pareto-optimal surface with an EMO approach. Although the idea is intuitive, the difficulties of this method are many: (i) it needs an exponentially higher number of points to cover the entire Pareto-optimal surface as the number of objectives increase, (ii) to estimate the nadir point accurately, it must find extreme Pareto-optimal points accurately, thereby deserving special attention for the search of the extreme points, (iii) it often requires a diversity parameter specifying the minimum desirable distance between any two obtained points, hence making the procedure sensitive to a parameter. An earlier study [16] has shown the effect of the diversity parameter on the obtained accuracy of the estimated nadir point. Further, EMO methodologies have shown to not work well in finding a well-distributed set of solutions on the entire Pareto-optimal surface for more than four objectives [7], thereby making EMO methodologies difficult to apply in practical scenarios.

2.2 'Edge-to-Nadir': Computing Edge Solutions of Pareto-optimal Surface

Since intermediate (non-extreme) Pareto-optimal solutions do not usually contribute in determining the location of the nadir point, one may try to find only critical edges (boundaries responsible for a true estimate of the nadir point) of the Pareto-optimal surface. One way to do this would be to solve $\binom{M}{2}$ pair-wise objective combinations and collect the corresponding solutions together by computing the missing objectives [9, 16]. The dominated points can then be deleted and the nadir point can be estimated. However, although this procedure requires relatively smaller computational effort than that in the 'Surface-to-Nadir' approach, there are still some difficulties: (i) the accuracy of the procedure depends on the diversity parameter used to find a distributed set of solutions on the critical edges of the Pareto-optimal surface, (ii) many pair-wise optimizations may find the same boundary (or a part of them) repeatedly, thereby wasting computational efforts, (iii) such a technique may require to find multi-modal Pareto-optimal solutions (solutions having identical efficient f_i-f_j solutions but differing in at least one other objective, for example) and may need to employ a lexicographic procedure to find the true extreme Pareto-optimal points.

2.3 'Extreme-point-to-Nadir': Computing Objective-wise Worst Pareto-optimal Points

It is also intuitive to realize that even most of the intermediate edge points do not help in estimating the nadir point. It is then quite tempting to develop a procedure which will find only the extreme Pareto-optimal points, so that the nadir point can be constructed from these points. A couple of recent studies [5, 16] suggested such procedures using an EMO approach and showed their potential for the purpose. However, evolutionary optimization algorithms are not guaranteed to find the exact optimal solutions. Hence, an EMO designed to find the extreme Pareto-optimal points may not be able to exactly locate the extreme points, thereby making only an approximate estimation of the nadir point. A recent study [6] suggested the use of a local search procedure based on the reference point approach [14, 17] on the approximate extreme solutions obtained by the modified NSGA-II procedure [5]. This study used a heuristic weight fixation scheme which may face difficulties in certain problems.

This is because the task of a local search in locating an extreme Pareto-optimal point is more involved than the usual task of finding a locally optimal solution in single-objective optimization. Consider Figure 1, in which the outcome of a typical EMO procedure is usually a near-extreme solution, $\mathbf{z}_{\mathrm{EMO}}$ (the figure may indicate an exaggeration of an actual EMO outcome). Usually, such a solution need not even be a Pareto-optimal solution. The task of the local search is *not* to find any arbitrary Pareto-optimal point (say $\mathbf{z}_P$) close to the EMO point, but to find the true extreme Pareto-optimal point ($\mathbf{z}_E$) corresponding to the objective function f_j for which the EMO point was found to be the worst. It is not a straightforward task to get the this point ($\mathbf{z}_E$) from the EMO point ($\mathbf{z}_{\mathrm{EMO}}$) directly in every scenario using a single-level heuristic optimization.

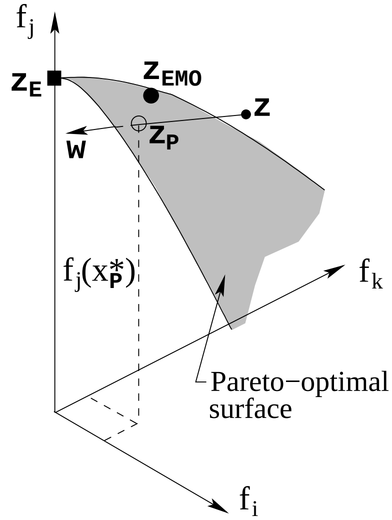

Fig. 1 The local search procedure is illustrated.

In this study, we replace the heuristic local search procedure by using a two-level reference point based approach to improve the accuracy of locating extreme points. In the outer-level optimization task, a combination of a reference point ($\mathbf{z}$) and a weight vector ($\mathbf{w}$) is the set of decision variables and the objective function evaluation involves another (lower-level) optimization task. For the lower-level optimization, original variable vector ($\mathbf{x}$) is the variable and the augmented achievement scaralizing function with supplied ($\mathbf{z}$, $\mathbf{w}$) by the outer-level solution is optimized. The starting solution is the EMO solution ($\mathbf{x}_{\mathrm{EMO}}$) for this optimization task. At the optimal solution ($\mathbf{x}_P^*$, which corresponds to the efficient vector, $\mathbf{z}_P$) to this task, the value of the critical objective function $f_j(\mathbf{x}_P^*)$ is computed and is used as the objective value of ($\mathbf{z}$, $\mathbf{w}$) solution of the upper-level optimization problem. Thus, the outer-level optimization searches for $(\mathbf{z},\mathbf{w})$ for which the above-computed objective function has its maximum value, thereby finding the desired extreme point. The starting solution for the outer-level optimization can be $(\mathbf{z}_{\mathrm{EMO}},\mathbf{w}_0)$, where $\mathbf{w}_0$ is a vector with all entries equal to $1/M$ (M is the number of objectives). During the optimization, $\mathbf{z}$ is restricted to lie within a hyperbox around the EMO point ($\mathbf{z}_{\mathrm{EMO}}$) and $\mathbf{w}$ is restricted to lie within the range [0.001, 1] in each dimension. In the following, we present the overall procedure:

Step 1: Compute ideal and worst objective vectors by minimizing and maximizing each function individually. They are needed in computing the termination criterion for the EMO procedure.

Step 2: Apply extremized-crowded NSGA-II approach [5] to find a set of non-dominated extreme points. Iterations are continued till a termination criterion is met. Say, at the end of this simulation, P non-dominated near-extreme points

($\mathbf{x}_{\mathrm{EMO}}^{(j)}$ for $j=1,2,\ldots,P$) are found. Identify the best and worst objective vectors $\mathbf{f}^{\min}$ and $\mathbf{f}^{\max}$ from these P solutions.

Step 3: Apply a local search procedure from each near-extreme solution $\mathbf{x}_{\mathrm{EMO}}^{(j)}$ (having objective vector $\mathbf{f}_{\mathrm{EMO}}^{(j)}$) by using the two-level reference point approach (described below) to find the corresponding extreme solution $\mathbf{y}^{*(j)}$. A point $\mathbf{x}_{\mathrm{EMO}}^{(j)}$ is used for local search if at least one of its objective values (say k-th objective) matches to that in the maximum objective vector $\mathbf{f}^{\max}$ and the point is declared as a critical extreme point. The outer-level optimization uses a combination of reference point and weight vector $(\mathbf{z},\mathbf{w})$ as the decision variable vector and maximizes an objective function which is computed by an inner-level optimization (given in equation 2):

$$\begin{array}{ll} \text{Maximize}_{(\mathbf{z},\mathbf{w})} & f_k^{*(j)}(\mathbf{z},\mathbf{w}), \\ \text{subject to} & \mathbf{w} \in [0.001,1]\mathbf{1}, \\ & \mathbf{z} \in \mathbf{f}_{\mathrm{EMO}}^{(j)} + [-0.5,1.5](\mathbf{f}^{\max}-\mathbf{f}^{\min}), \end{array} \tag{1}$$

where $\mathbf{1}$ is vector of ones. The optimal objective value $f_k^{*(j)}(\mathbf{z},\mathbf{w})$ depends on the current reference point $\mathbf{z}$ and weight vector $\mathbf{w}$ and is the optimal objective function values to the following inner-level optimization problem involving the augmented achievement scalarizing function:

$$\begin{array}{ll} \text{Minimize}_{(\mathbf{y}^{(j)})} & \max_{i=1}^{M} w_i \left(\frac{f_i(\mathbf{y}^{(j)})-z_i}{f_i^{\max}-f_i^{\min}}\right) + \rho \sum_{m=1}^{M} w_m \left(\frac{f_m(\mathbf{y}^{(j)})-z_m}{f_m^{\max}-f_m^{\min}}\right), \\ \text{subject to} & \mathbf{y}^{(j)} \in \mathscr{S}, \end{array} \tag{2}$$

where $\mathscr{S}$ is the feasible variable space restricted by the original constraints and variable bounds. To this problem, search is performed in the original decision variable space. The solution $\mathbf{y}^{*(j)}$ to this inner-level optimization problem determines the optimal objective vector $f_k^{*(j)}$, which is used in the outer-level optimization problem. The outer-level optimization is initialized with the EMO solution $\mathbf{z}_{(0)} = \mathbf{f}_{\mathrm{EMO}}^{(j)}$ and $\mathbf{w}_{(0)} = (1/M)\mathbf{1}$. The inner-level optimization is initialized with the EMO solution $\mathbf{y}_{(0)}^{(j)} = \mathbf{x}_{\mathrm{EMO}}^{(j)}$. Resulting optimal solution for the two-level local search is $\mathbf{y}_{\mathrm{LS}}^{(j)}$ with an objective vector $\mathbf{f}_{\mathrm{LS}}^{(j)}$ and corresponding reference point and weight vectors are $\mathbf{z}_{\mathrm{LS}}^{(j)}$ and $\mathbf{w}_{\mathrm{LS}}^{(j)}$, respectively. Step 3 is repeated for all P EMO solutions.

Step 4: Finally, construct the nadir point from the worst objective values of extreme Pareto-optimal points ($\mathbf{f}_{\mathrm{LS}}^{(j)}$), $j=1,2,\ldots,P$) obtained by the local search procedure.

The use of augmented achievement scalarizing function does not allow the inner-level optimization to converge to a weak Pareto-optimal solution. But, in certain problems, the approach may only allow to find an extreme *proper* Pareto-optimal solution [14] depending on the value of the parameter ρ. Alternatively, it is possible to use a lexicographic formulation of the achievement scalarizing function to guarantee Pareto optimality [14].

3 Results on Numerical Test Problems

In this section, we present simulation results of 'Extreme-point-to-Nadir' approach and compare its performance with the other two procedures for which results are borrowed from the original study [16]. For all simulations using the 'Extreme-point-to-Nadir' approach, we have used the following parameter values. Details of this procedure are given in [5]. Population size (N) is proportional to number of variables (n), as $N = 20n$. Crossover and mutation probabilities are 0.9 and $1/n$, respectively. The distribution index for simulated binary crossover operator (SBX) [4] is 10 and the same for polynomial mutation operator [4] is 50. The NSGA-II procedure is terminated when the change in normalized distance metric (computed as $(ND_{\max} - ND_{\min})/ND_{\text{avg}}$) is less than 0.0001. The quantities $ND_{\max}$, $ND_{\min}$, and ND_{avg} are maximum, minimum and average normalized distance (ND) metric value (defined below) over the past 50 generations:

$$ND = \sqrt{\frac{1}{M}\sum_{i=1}^{M}\left(\frac{z_i^{\text{est}} - z_i^*}{z_i^w - z_i^*}\right)^2}, \tag{3}$$

where z_i^{est}, z_i^w, z_i^* are the estimated nadir point, worst objective point and ideal point, respectively. The parameter ρ for the augmented scalarizing function is set to 10^{-5}.

3.1 Problem SZ1

We borrow the first two problems from a recent study [16] which applied the first two nadir point estimation methodologies ('Surface-to-Nadir' and 'Edge-to-Nadir' approaches). The first problem is as follows:

$$\begin{aligned} &\text{Minimize} \left\{ \begin{array}{l} f_1(\mathbf{x}) = -(100 - 7x_1 - 20x_2 - 9x_3), \\ f_2(\mathbf{x}) = -(4x_1 + 5x_2 + 3x_3), \\ f_3(\mathbf{x}) = -x_3, \end{array} \right\}, \\ &\text{subject to } 1\tfrac{1}{2}x_1 + x_2 + 1\tfrac{3}{5}x_3 \le 9, \\ &\qquad\qquad\; x_1 + 2x_2 + x_3 \le 10, \\ &\qquad\qquad\; x_i \ge 0, \quad i = 1,2,3. \end{aligned} \tag{4}$$

The previous study [16] reported the true nadir point to be $\mathbf{z}^{\text{nad}} = (-3.6364, 0, 0)^T$. Figure 2 shows a sketch of the feasible objective space and the corresponding Pareto-optimal surface (shaded region). The 'Surface-to-Nadir' approach first finds a set of well-distributed points on the entire Pareto-optimal surface and then constructs the nadir point from the obtained points. After 120,000 *solution evaluations* (total number of solutions evaluated by the during the entire optimization process),

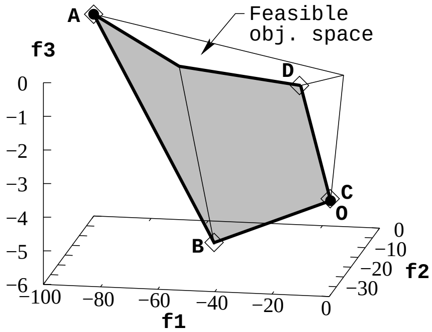

Fig. 2 Feasible objective space and Pareto-optimal surface for problem SZ1.

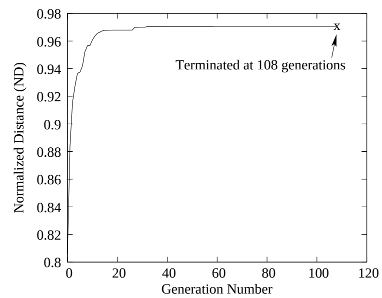

Fig. 3 Normalized distance metric for problem SZ1.

the point $(-5.06,0,0)^T$ was declared as an estimate of the nadir point in [16]. The 'Edge-to-Nadir' approach finds the Pareto-optimal edges corresponding to pair-wise minimizations of objectives. In this problem, all three pairs of objectives will find representative points on the edges shown with a thick line. The nadir point is then estimated to be $(-4.78,0,0)^T$ [16]. Even after 360,000 solution evaluations, the resulting estimate using the 'Edge-to-Nadir' approach was $(-4.36,0,0)^T$, having 20% higher value in the first objective from the true value. Although the problem is linear and has only three variables, the above two evolutionary methodologies seem to have faced difficulties in finding the true nadir point in this problem. We now employ the 'Extreme-point-to-Nadir' approach with the proposed local search procedure.

In Step 1, we find the ideal point by minimizing each objective using Matlab's `fmincon()` code, which employs the sequential quadratic method with BFGS as a unconstrained optimization procedure and cubic search as a line search procedure. Three minimizations provide $\mathbf{z}^* = (-100,-31,-5.625)^T$ as the ideal point, requiring 28, 16, and 16 solution evaluations, respectively. We also need the worst point for terminating Step 2. The point $\mathbf{z}^w = (0,0,0)^T$ is found with 28, 24, 28 solution evaluations.

In Step 2, we apply the extremized crowded NSGA-II with a parameter setting as described above. Figure 3 shows the variation of the normalized distance metric with generation. The algorithm is terminated after generation 108 and total solution evaluations needed are $60 \times 109 = 6,540$. Four solutions are found at the end of the simulation and are presented in Table 1 and in Figure 2 with diamonds. Notice, how the modified NSGA-II finds non-dominated near-extreme points (A, B, C, and D) for the entire Pareto-optimal surface, without finding any intermediate points. From the table with four obtained solutions, we observe the worst and best objective vectors as $\mathbf{f}^{\max} = (-3.7878,0.0000,0.0000)^T$ and $\mathbf{f}^{\min} = (-100.0000,-30.9050,-5.6207)^T$, respectively. Interesting to note that $\mathbf{f}^{\max}$ is already close to the true nadir point

Table 1 Four solutions found by the extremized crowded NSGA-II for Problem SZ1. 'LS' stands for results after local search.

j	$\mathbf{f}^{(j)}_{\mathrm{EMO}}$	$\mathbf{x}^{(j)}_{\mathrm{EMO}}$
1	$(-12.4541, -30.9050, -0.0052)^T$	$(4.0065, 2.9727, 0.0052)^T$
2	$(-49.3921, -16.8739, -5.6207)^T$	$(0.0029, 0.0001, 5.6207)^T$
3	$(-3.7878, -26.8347, -3.5788)^T$	$(0.0434, 3.1850, 3.5788)^T$
4	$(-100.0000, 0.0000, 0.0000)^T$	$(0.0000, 0.0000, 0.0000)^T$

j	k	$\mathbf{f}^{(j)}_{\mathrm{LS}}$	$\mathbf{x}^{(j)}_{\mathrm{LS}}$
1	–	No worst objective value	–
2	–	No worst objective value	–
3	1	$(-3.6364, -26.8182, -3.6364)^T$	$(0, 3.1818, 3.6364)^T$
4	2,3	$(-100, 0, 0)^T$	$(0, 0, 0)^T$

$\mathbf{z}^{\mathrm{nad}} = (-3.6364, 0, 0)^T$. Now, we employ the local search procedure from the two solutions corresponding to the worst objective vector $\mathbf{f}^{\max}$.

In Step 3, we observe that solution 1 (point D) and solution 2 (point B) are not one of the worst solutions, so we ignore these points from further consideration. In fact, these two solutions exist in the NSGA-II final population because they correspond to the minimum value of objectives f_2 and f_3, respectively. Solution 3 (point C) corresponds to the worst of objective f_1 (with $k = 1$) and hence will be subjected to a local search in the hope of improving it to reach the true extreme (worst) Pareto-optimal point corresponding to objective f_1. The resulting solution (point O) is shown in the table. This optimization requires 204 solution evaluations. The corresponding optimal weight vector and reference point are found to be $\mathbf{w}^{(3)}_{\mathrm{LS}} = (0.0010, 1, 1)^T$ and $\mathbf{z}^{(3)}_{\mathrm{LS}} = (140.5305, -41.9543, -6.3891)^T$, respectively. It is interesting to observe from $\mathbf{f}^{(3)}_{\mathrm{EMO}}$ and $\mathbf{z}^{(3)}_{\mathrm{LS}}$ how the two-level local search procedure finds a large value of the first objective in $\mathbf{f}^{(3)}_{\mathrm{LS}}$ by keeping the other two objective values close to the NSGA-II point and uses a relatively small value of weight for the first objective to allow the search of the achievement scalarizing function almost along the $-f_1$ direction to locate the extreme point.

Next, we consider solution 4 (point A), which corresponds to the worst of both objectives f_2 and f_3 (with $k = 2$ and 3). Thus, we maximize a normalized sum of both these objectives ($\sum_{j=2}^{3} \frac{(f_j(\mathbf{x}) - f_j^{\min})}{f_j^{\max} - f_j^{\min}}$) in the inner-loop of local search method. The same point $\mathbf{f}^{(4)}_{\mathrm{LS}} = (-100, 0, 0)^T$ is found in only 20 solution evaluations. Corresponding optimal weight vector and reference point are $\mathbf{w}^{(4)}_{\mathrm{LS}} = (0.3333, 0.0010, 0.0010)^T$ and $\mathbf{z}^{(4)}_{\mathrm{LS}} = (-148.1061, -15.4525, -2.8103)^T$, respectively.

In Step 4, we collate these points ($\mathbf{f}^{(3)}_{\mathrm{LS}}$ and $\mathbf{f}^{(4)}_{\mathrm{LS}}$) and declare the estimated nadir point as $(-3.6364, 0, 0)^T$, which is identical to the exact nadir point. Total number of solution evaluations needed by all steps of the procedure is 6,904, of which about 95% evaluations are needed by the EMO procedure alone. The computation needed by this ‘Extreme-point-to-Nadir’ approach is only about 6% of that needed by the other two approaches and importantly the ‘Extreme-point-to-Nadir’ approach also finds a more accurate result. This study demonstrates how the task of finding the nadir point can become computationally faster and accurate if the focus is made in finding extreme points, rather than on the entire Pareto-optimal surface or on the edges of the Pareto-optimal surface.

3.2 Problem SZ2

Next, we consider the second numerical optimization problem of [16]:

$$\begin{aligned} \text{Minimize} & \left\{ \begin{array}{l} 9x_1 + 19.5x_2 + 7.5x_3 \\ 7x_1 + 20x_2 + 9x_3 \\ -4x_1 - 5x_2 - 3x_3 \\ -x_3 \end{array} \right\}, \\ \text{subject to} \; & 1.5x_1 - x_2 + 1.6x_3 \le 9, \\ & x_1 + 2x_2 + x_3 \le 10, \\ & x_i \ge 0, \quad i = 1,2,3. \end{aligned} \tag{5}$$

The true nadir point for this problem is $\mathbf{z}^{\text{nad}} = (94.5, 96.3636, 0, 0)^T$. The earlier study [16] obtained a close point $(94.4998, 95.8747, 0, 0)^T$ using the ’Edge-to-Nadir’ approach. This study required a total of 120,000 solution evaluations. In the following, we show the results of ‘Extreme-point-to-Nadir’ approach on this problem.

In Step 1 of the procedure, we find the ideal and worst objectives values: $\mathbf{z}^* = (0, 0, -31, -5.625)^T$ and $\mathbf{z}^w = (97.5, 100, 0, 0)^T$, respectively. This requires (12+12+24+28)=76 and (28+12+16+16)=72 solution evaluations, respectively.

Thereafter, in Step 2, we apply the extremized crowded NSGA-II procedure using a population size of 60 and initializing the population around $x_i \in [0, 10]$ for all three variables. The NSGA-II run is terminated at generation 315 with the prescribed termination criterion, thereby requiring a total of 50×316 or 12,640 solution evaluations. Solutions obtained are tabulated in Table 2. The minimum and

Table 2 Extremized crowded NSGA-II and local search method on problem SZ2.

j	$\mathbf{x}_{\text{EMO}}^{(j)}$	Objective vector, $\mathbf{f}_{\text{EMO}}^{(j)}$
1	$(0.0001, 0, 5.6249)^T$	$(42.1879, 50.6249, -16.8752, -5.6249)^T$
2	$(0.0001, 3.1830, 3.6336)^T$	$(89.3219, 96.3635, -26.8164, -3.6336)^T$
3	$(3.9980, 2.9998, 0.0003)^T$	$(94.4810, 87.9854, -30.9920, -0.0003)^T$
4	$(0, 0, 0)^T$	$(0, 0, 0, 0)^T$

j	k	$\mathbf{w}_{\text{LS}}^{(j)}$	$\mathbf{z}_{\text{LS}}^{(j)}$	Extreme point, $\mathbf{f}_{\text{LS}}^{(j)}$
1	–	No worst objective value		
2	2	$(1.0, 1.0, 0.7, 0.8)^T$	$(183.8, 192.7, -26.8, -3.6)^T$	$(89.3182, 96.3636, -26.8182, -3.6364)^T$
3	1	$(0.3, 0.3, 0.2, 0.3)^T$	$(189.0, 184.4, -31.0, 5.6)^T$	$(94.5000, 88.0000, -31.0000, 0.0000)^T$
4	3,4	$0.25(1, 1, 1, 1)^T$	$(0, 0, 0, 0)^T$	$(0, 0, 0, 0)^T$

maximum objective vectors are: $\mathbf{f}^{\min} = (0.0000, 0.0000, -30.9920, -5.6249)^T$ and $\mathbf{f}^{\max} = (94.4810, 96.3635, 0.0000, 0.0000)^T$, respectively. Notice that the maximum vector is close to the true nadir point mentioned above. We shall now investigate whether the proposed local search is able to improve this point to find the exact nadir point.

We observe that the objective values of the first solution does not correspond to any element of $\mathbf{f}^{\max}$. Thus, in Step 3, we employ the two-level local search procedure only from the other three solutions. Resulting solutions and corresponding $\mathbf{z}^{(j)}_{\mathrm{LS}}$ and $\mathbf{w}^{(j)}_{\mathrm{LS}}$ vectors are shown in the table. For solutions 2 and 3, we maximize objectives f_2 and f_1, respectively. Since solution 4 is worst with respect to both objectives f_3 and f_4, we maximize the sum of normalized objectives, as described for the previous problem. The solution evaluations required till convergence for each of the three optimizations are 204, 25 and 20, respectively.

From the obtained local search solutions (last column in the table), we estimate the nadir point as $(94.5000, 96.3636, 0, 0)^T$, which is identical to the true nadir point for this problem. The total number of solution evaluations is 13,037. This is only about 10% of the total solution evaluations needed in [16]. Moreover, our approach finds the exact nadir point, whereas [16] could not find the exact nadir point even with about 10 times more solution evaluations.

3.3 Problem KM

Next, we consider a three-objective optimization problem, which provides difficulty for the payoff table method to estimate the nadir point. This problem was used in [12]:

$$\begin{aligned}
&\text{Minimize} \left\{ \begin{array}{l} -x_1 - x_2 + 5 \\ \frac{1}{5}(x_1^2 - 10x_1 + x_2^2 - 4x_2 + 11) \\ (5 - x_1)(x_2 - 11) \end{array} \right\}, \\
&\text{subject to } 3x_1 + x_2 - 12 \le 0, \\
&\qquad\qquad\; 2x_1 + x_2 - 9 \le 0, \\
&\qquad\qquad\; x_1 + 2x_2 - 12 \le 0, \\
&\qquad\qquad\; 0 \le x_1 \le 4, \quad 0 \le x_2 \le 6.
\end{aligned} \tag{6}$$

Individual minimizations of objectives identify the vector $\mathbf{z}^* = (-2, -3.1, -55)^T$ as the ideal objective vector. This requires a total of (18 + 37 + 9)=64 solution evaluations. The maximization of the objectives leads to the worst objective vector $\mathbf{z}^w = (5, 4.6, -14.25)^T$ with (12+18+18)=48 solution evaluations. The payoff table method finds $(5, 2.2, -14.25)^T$ as the wrongly estimated nadir point from these minimization results. Another study [10] used a grid-search strategy (computationally possible due to the presence of only three objectives) of creating a number of feasible solutions systematically and constructing the nadir point from the solutions obtained. The estimated nadir point was $(5, 4.6, -14.25)^T$. We now employ the 'Extreme-point-to-Nadir' approach with the proposed local search procedure.

As described above, Step 1 of the approach finds $\mathbf{z}^* = (-2,-3.1,-55)^T$ and $\mathbf{z}^w = (5,4.6,-14.25)^T$. In Step 2 of the approach, we employ the extremized crowded NSGA-II and find four non-dominated extreme solutions, as shown in the second column of Table 3. It is interesting to note that the fourth solution is not

Table 3 Extremized crowded NSGA-II and local search method on Problem KM.

j	$\mathbf{x}^{(j)}_{\text{EMO}}$	Objective vector, $\mathbf{f}^{(j)}_{\text{EMO}}$	k	Extreme point, $\mathbf{f}^{(j)}_{\text{LS}}$
1	$(0,0)^T$	$(5,2.2,-55)^T$	1	$(5,2.2,-55)^T$
2	$(3.511,1.466)^T$	$(0.023,-3.100,-14.194)^T$	3	$(0,-3.1,-14.25)^T$
3	$(0,6)^T$	$(-1,4.6,-25)^T$	2	$(-1,4.6,-25)^T$
4	$(2.007,4.965)^T$	$(-1.973,-0.050,-18.060)^T$	–	No worst objective value

needed to estimate the nadir point, but the extremized principle keeps this extreme solution corresponding to f_1 to possibly eliminate spurious solutions which may otherwise stay in the population and provide a wrong estimate of the nadir point. The simulation is terminated after 135 generations, thereby requiring $40 \times 136 = 5,440$ solution evaluations. At the end of Step 2, the estimated nadir point is $\mathbf{z}^{\text{nad}} = (5,4.6,-14.212)^T$, which seems to disagree somewhat on the third objective value with that found by the grid-search strategy.

To investigate if any further improvement is possible, we proceed to Step 3 and apply three local searches, each started with one of the first three solutions presented in Table 3, as these three solutions correspond to the worst value of one of the objectives. The minimum and maximum objective vectors from these solutions are: $\mathbf{f}^{\text{min}} = (-1,-3.1,-55)^T$ and $\mathbf{f}^{\text{max}} = (5,4.6,-14.194)^T$, respectively. Solution 1 from the table corresponds to the worst value of the first objective ($k = 1$). Thus, the outer optimization run maximizes objective f_1. This optimization took 487 solution evaluations. The table clearly shows that solution 2 obtained by NSGA-II was not a Pareto-optimal point. The local search approach starting from this solution is able to find a better solution $(0,-3.1,-14.25)^T$, requiring a total of 198 solution evaluations. This shows the importance of employing the local search approach in obtaining exact extreme points. The third solution could not be improved any further, since it is already the desired extreme point with respect to f_2 with $k = 2$, but the optimization requires 786 solution evaluations to terminate with the prescribed conditions.

The nadir point estimated by the combination of extremized crowded NSGA-II and the local searches is $(5,4.6,-14.25)^T$, which is identical to that obtained by the grid search strategy [10]. Overall, the ‘Extreme-point-to-Nadir’ approach required 7,023 solution evaluations to estimate the nadir point exactly to this non-linear problem, for which the EMO procedure required about 77% of the total computations.

4 Conclusions

Recent studies have shown that evolutionary multi-objective optimization (EMO) procedures are potential for the estimation of nadir point. In this paper, we have reviewed three such implementations which seemed to vary according to the dimension of the desired target set. By comparing the number of solution evaluations of these procedures, we have concluded that the 'Extreme-point-to-Nadir' approach which directly focuses to find extreme Pareto-optimal points is a computationally faster approach and requires an order of magnitude less solution evaluations. The accuracy of the EMO procedure has also been improved by using a two-level local search procedure and with a marginal increase in the computational effort. Similar results are observed on other problems (which we could not provide here due to space restrictions). The local search based 'Extreme-point-to-Nadir' approach seems to be a promising procedure for making a reliable and accurate estimate of the nadir point in a multi-objective optimization problem.

Acknowledgements

Authors acknowledge the FiDiPro support from the Academy of Finland (grant 118319) and Foundation of Helsinki School of Economics. The research is also partially funded by the Jenny and Antti Wihuri Foundation.

References

1. R. Benayoun, J. de Montgolfier, J. Tergny, and P. Laritchev. Linear programming with multiple objective functions: Step method (STEM). *Mathematical Programming*, 1(3):366–375, 1971.
2. H. P. Benson. Existence of efficient solutions for vector maximization problems. *Journal of Optimization Theory and Applications*, 26(4):569–580, 1978.
3. J. T. Buchanan. A naive approach for solving MCDM problems: The GUESS method. *Journal of the Operational Research Society*, 48(2):202–206, 1997.
4. K. Deb. *Multi-objective optimization using evolutionary algorithms*. Chichester, UK: Wiley, 2001.
5. K. Deb, S. Chaudhuri, and K. Miettinen. Towards estimating nadir objective vector using evolutionary approaches. In *Proceedings of the Genetic and Evolutionary Computation Conference (GECCO-2006)*, pages 643–650. New York: The Association of Computing Machinery (ACM), 2006.
6. K. Deb, K. Miettinen, and S. Chaudhuri. Estimating nadir objective vector: Hybrid of evolutionary and local search. Technical Report Electronic Working Paper W-440, Helsinki School of Economics, Finland, 2008.
7. K. Deb and D. Saxena. Searching for Pareto-optimal solutions through dimensionality reduction for certain large-dimensional multi-objective optimization problems. In *Proceedings of the World Congress on Computational Intelligence (WCCI-2006)*, pages 3352–3360, 2006.
8. M. I. Dessouky, M. Ghiassi, and W. J. Davis. Estimates of the minimum nondominated criterion values in multiple-criteria decision-making. *Engineering Costs and Production Economics*, 10:95–104, 1986.
9. M. Ehrgott and D. Tenfelde-Podehl. Computation of ideal and nadir values and implications for their use in MCDM methods. *European Journal of Operational Research*, 151:119–139, 2003.

10. P. Eskelinen, 2007. Personal Communication.
11. H. Iserman and R. E. Steuer. Computational experience concerning payoff tables and minimum criterion values over the efficient set. *European Journal of Operational Research*, 33(1):91–97, 1988.
12. K. Klamroth and K. Miettinen. Integrating approximation and interactibe decision making in multicrietria optimization. *Operations Research*, 56:222–234, 2008.
13. P. Korhonen, S. Salo, and R. Steuer. A heuristic for estimating nadir criterion values in multiple objective linear programming. *Operations Research*, 45(5):751–757, 1997.
14. K. Miettinen. *Nonlinear Multiobjective Optimization.* Kluwer, Boston, 1999.
15. K. Miettinen and M. M. Mäkelä. Synchronous approach in interactive multiobjective optimization. *European Journal of Operational Research*, 170(3):909–922, 2006.
16. M. Szczepanski and A. P. Wierzbicki. Application of multiple crieterion evolutionary algorithm to vector optimization, decision support and reference point approaches. *Journal of Telecommunications and Information Technology*, 3:16–33, 2003.
17. A. P. Wierzbicki. The use of reference objectives in multiobjective optimization. In G. Fandel and T. Gal, editors, *Multiple Criteria Decision Making Theory and Applications*, pages 468–486. Berlin: Springer-Verlag, 1980.