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CHARACTERIZATION OF UNITARY PROCESSES WITH INDEPENDENT INCREMENTS

UN CIG JI†, LINGARAJ SAHU*, AND KALYAN B. SINHA*‡

Abstract. In this paper, we study unitary Gaussian processes with independent increments with which the unitary equivalence to a Hudson - Parthasarathy evolution systems is proved. This gives a generalization of results in [11] and [12] in the absence of the stationarity condition.

1. Introduction

In the framework of the theory of quantum stochastic calculus developed by the work of Hudson and Parthasarathy,consider the (HP) quantum stochastic differential equations (qsde)

$$dV_t = \sum_{\mu,\nu\geq 0} V_t L^\mu_\nu(t)\Lambda^\nu_\mu(dt),\ V_0 = 1_{\mathbf{h}\otimes\Gamma}, \tag{1.1}$$

(where the coefficients $L^\mu_\nu(t)\ :\ \mu,\ \nu\ \geq 0$ are bounded operator-valued locally bounded functions on $\mathbb{R}_+$ in the initial Hilbert space $\mathbf{h}$ and Λ^ν_μ are the fundamental processes in the symmetric Fock space $\Gamma = \Gamma_{sym}(L^2(\mathbb{R}_+,\mathbf{k}))$ with respect to a fixed orthonormal basis (in short 'ONB') $\{E_j : j \geq 1\}$ of the noise Hilbert space $\mathbf{k}$) ([2]). The conditions for existence and uniqueness of a solution $\{V_t\}$ were studied by Hudson and Parthasarathy and others when the coefficient operators $\{L^\mu_\nu(t)\}$ are constants ([6, 8, 10]). In particular, in the absence of the conservation martingale, the equation takes the form

$$dV_t = \sum_j \{V_t L_j(t)a^\dagger(dt) - V_t L^*_j(t)a(dt)\} + V_t G(t)dt$$

with the formal unitarity condition:

$$\sum_j L^*_j(t)L_j(t) + 2Re\ G(t) = 0$$

for almost every $t \geq 0$, in analogy with the case when L^μ_ν are constants. The existence and unitarity of the solution V for the time dependent case will be proven here in theorem 5.1.

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In a series of earlier work ([11, 12]) it has been shown that unitary evolutions on $\mathbf{h}\otimes\mathcal{H}$ with stationary, independent increments and satisfying a Gaussian condition (where $\mathbf{h}$ and $\mathcal{H}$ are separable Hilbert spaces) with bounded or possibly unbounded generator (in the second case, one needs some further conditions) are unitarily isomorphic to the solutions of qsde of the type (1.1) with time independent coefficients.

In this article we are interested in the characterization of unitary evolutions with only independent increments on $\mathbf{h}\otimes\mathcal{H}$ and with the assumption that the expectation evolution relative to a distinguished vector in $\mathcal{H}$ is Lifshitz in the time variable.

The article is organized as follows: Section 2 is meant for recalling some preliminary ideas and fixing some notations on linear operators on Hilbert spaces and Section 3 collects some results associated with Hilbert space and properties of evolutions. The main results of section 3 are proved in the Appendix. Section 3 also contain the description of the unitary processes with independent increments and the assumptions on them. Section 4 is dedicated to the construction of a Hilbert space, called the noise space and operator coefficients associated with them. The HP evolution system and its minimality are discussed in Section 5 and consequently the unitary equivalence of the solution with the unitary process is proven.

2. Notation and Preliminaries

We assume that all Hilbert spaces in this article are complex separable with inner products which are anti-linear in the first variable. For each Hilbert spaces $\mathcal{H}$ and $\mathcal{K}$ we denote the Banach spaces of all bounded linear operators from $\mathcal{H}$ to $\mathcal{K}$ and all trace class operators on $\mathcal{H}$ by $\mathcal{B}(\mathcal{H},\mathcal{K})$ and $\mathcal{B}_1(\mathcal{H})$, respectively, and the trace on $\mathcal{B}_1(\mathcal{H})$ by $\mathrm{Tr}(\cdot)$. We note that for each $h\in\mathcal{H}$, there exists a unique operator $F_h\in\mathcal{B}(\mathcal{K},\mathcal{H}\otimes\mathcal{K})$ such that

$$F_h k = h\otimes k \text{ for all } k\in\mathcal{K}. \tag{2.1}$$

Let $\mathbf{h}$ and $\mathcal{H}$ be two Hilbert spaces with orthonormal bases $\{e_j : j\geq 1\}$ and $\{\zeta_j : j\geq 1\}$, respectively. For each $A\in\mathcal{B}(\mathbf{h}\otimes\mathcal{H})$ and $u,v\in\mathbf{h}$ we define a linear operator $A(u,v)\in\mathcal{B}(\mathcal{H})$ by

$$\langle\xi_1, A(u,v)\xi_2\rangle = \langle u\otimes\xi_1, A\ v\otimes\xi_2\rangle, \forall\xi_1,\xi_2\in\mathcal{H}$$

and read off the following properties (for the proof, see Lemma 2.1 in [11]):

Lemma 2.1. *Let $A,B\in\mathcal{B}(\mathbf{h}\otimes\mathcal{H})$. Then for any $u,v,u_i,v_i\in\mathbf{h}$ $(i=1,2)$ we have*

(i) $A(\cdot,\cdot):\mathbf{h}\times\mathbf{h}\mapsto\mathcal{B}(\mathcal{H})$ *is a jointly continuous sesqui-linear map, and if* $A(u,v)=B(u,v)$ *for all* $u,v\in\mathbf{h}$, *then* $A=B$,
(ii) $A(u,v)=F_u^*AF_v,\ \|A(u,v)\|\leq\|A\|\|u\|\|v\|$ *and* $A(u,v)^*=A^*(v,u)$,
(iii) $A(u_1,v_1)B(u_2,v_2)=\left[A\left(|v_1><u_2|\otimes 1_{\mathcal{H}}\right)B\right](u_1,v_2)$,
(iv) $AB(u,v)=\sum_{j\geq 1}A(u,e_j)B(e_j,v)$, *where the series converges strongly,*
(v) $0\leq A(u,v)^*A(u,v)\leq\|u\|^2A^*A(v,v)$,

(vi) *for any* $\xi_1, \xi_2 \in \mathcal{H}$ *we have*

$$\begin{aligned}\langle A(u_1,v_1)\xi_1, B(u_2,v_2)\xi_2\rangle &= \sum_{j\geq 1}\langle u_2\otimes\zeta_j, [B(|v_2><v_1|\otimes|\xi_2><\xi_1|)A^*]\, u_1\otimes\zeta_j\rangle \\ &= \langle v_1\otimes\xi_1, [A^*(|u_1><u_2|\otimes 1_{\mathcal{H}})B]\, v_2\otimes\xi_2\rangle.\end{aligned}$$

For each $A \in \mathcal{B}(\mathbf{h}\otimes\mathcal{H})$ and $\epsilon \in \mathbb{Z}_2 = \{0,1\}$, we define an operator $A^{(\epsilon)} \in \mathcal{B}(\mathbf{h}\otimes\mathcal{H})$ by

$$A^{(\epsilon)} := \begin{cases} A & \text{if } \epsilon = 0, \\ A^* & \text{if } \epsilon = 1. \end{cases}$$

For $1 \leq k \leq n$, we define a unitary exchanging map $P_{k,n} : \mathbf{h}^{\otimes n}\otimes\mathcal{H} \to \mathbf{h}^{\otimes n}\otimes\mathcal{H}$ by

$$P_{k,n}(u_1\otimes\cdots\otimes u_n\otimes\xi) := u_{\tau_{k,n}(1)}\otimes\cdots\otimes u_{\tau_{k,n}(n)}\otimes\xi$$

on product vectors, where $\tau_{k,n}$ is the permutation $\{k, k+1, \cdots, n, 1, \ldots, k-1\}$ of $\{1, 2, \cdots, n\}$. Let $\underline{\epsilon} = (\epsilon_1, \epsilon_2, \cdots, \epsilon_n) \in \mathbb{Z}_2^n$. Consider the ampliation of the operator $A^{(\epsilon_k)}$ in $\mathcal{B}(\mathbf{h}^{\otimes n}\otimes\mathcal{H})$ given by

$$A^{(n,\epsilon_k)} := P_{k,n}^*(1_{\mathbf{h}^{\otimes n-1}}\otimes A^{(\epsilon_k)})P_{k,n}.$$

Now we define the operator

$$A^{(\underline{\epsilon})} := \prod_{k=1}^{n} A^{(n,\epsilon_k)} := A^{(n,\epsilon_1)}\cdots A^{(n,\epsilon_n)}$$

as in $\mathcal{B}(\mathbf{h}^{\otimes n}\otimes\mathcal{H})$. Note that as here, through out this article, the product symbol $\prod_{k=1}^{n}$ stands for product with the ordering from 1 to n. For product vectors $\underline{u}, \underline{v} \in \mathbf{h}^{\otimes n}$ one can see that

$$A^{(\underline{\epsilon})}(\underline{u},\underline{v}) = \left(\prod_{i=1}^{n} A^{(n,\epsilon_i)}\right)(\underline{u},\underline{v}) = \prod_{i=1}^{n} A^{(\epsilon_i)}(u_i,v_i) \in \mathcal{B}(\mathcal{H}), \tag{2.2}$$

moreover, for $1 \leq m \leq n$, we see that

$$\left(\prod_{i=1}^{m} A^{(n,\epsilon_i)}\right)(\underline{u},\underline{v}) = \prod_{i=1}^{m} A^{(\epsilon_i)}(u_i,v_i) \prod_{i=m+1}^{n}\langle u_i, v_i\rangle \in \mathcal{B}(\mathcal{H}). \tag{2.3}$$

When $\underline{\epsilon} = \underline{0} \in \mathbb{Z}_2^n$, for simplicity we shall write $A^{(n,k)}$ for $A^{(n,\epsilon_k)}$ and $A^{(n)}$ for $A^{(\underline{\epsilon})}$.

3. Unitary Processes with Independent Increments

Let $\{U_{s,t} : 0 \leq s \leq t < \infty\}$ be a family of unitary operators in $\mathcal{B}(\mathbf{h}\otimes\mathcal{H})$ with $U_{s,s} = 1$ for any $s \geq 0$ and Ω be a fixed unit vector in $\mathcal{H}$. Let us consider the family of unitary operators $\{U_{s,t}^{(\epsilon)}\}$ in $\mathcal{B}(\mathbf{h}\otimes\mathcal{H})$ for $\epsilon \in \mathbb{Z}_2$ given by $U_{s,t}^{(0)} = U_{s,t}$ and $U_{s,t}^{(1)} = U_{s,t}^*$. As in Section 2, for fixed $n \geq 1$, $\underline{\epsilon} \in \mathbb{Z}_2^n$ and each $1 \leq k \leq n$, we define the families of operators $\{U_{s,t}^{(n,\epsilon_k)}\}$ and $\{U_{s,t}^{(\underline{\epsilon})}\}$ in $\mathcal{B}(\mathbf{h}^{\otimes n}\otimes\mathcal{H})$. By identity (2.2), for product vectors $\underline{u}, \underline{v} \in \mathbf{h}^{\otimes n}$ and $\underline{\epsilon} \in \mathbb{Z}_2^n$, we have

$$U_{s,t}^{(\underline{\epsilon})}(\underline{u},\underline{v}) = \prod_{i=1}^{n} U_{s,t}^{(\epsilon_i)}(u_i,v_i).$$

We assume the following on the family of unitary $\{U_{s,t} \in \mathcal{B}(\mathbf{h} \otimes \mathcal{H})\}$.

Assumption A:

(A1) **(Evolution)**[1] For any $0 \leq r \leq s \leq t < \infty$, $U_{s,t}U_{r,s} = U_{r,t}$ and $U_{s,s} = 1$,

(A2) **(Independence of increments)** for any $0 \leq s_i \leq t_i < \infty$ $(i = 1, 2)$ such that $[s_1, t_1) \cap [s_2, t_2) = \emptyset$,

(i) $U_{s_1,t_1}(u_1, v_1)$ commutes with $U_{s_2,t_2}(u_2, v_2)$ and $U^*_{s_2,t_2}(u_2, v_2)$ for any $u_i, v_i \in \mathbf{h}$ $(i = 1, 2)$.

(ii) For pairs (u_i, v_i) and $(p_j, w_j) \in \mathbf{h}(i = 1, 2, \ldots, n; j = 1, 2, \ldots, k)$ and $[a, b)$ and $[r, s)$ disjoint intervals,

$$\left\langle \Omega, \prod_{i=1}^{n} U_{a,b}^{(\epsilon_i)}(u_i, v_i) \prod_{j=1}^{k} U_{r,s}^{(\epsilon'_j)}(p_j, w_j)\Omega \right\rangle = \left\langle \Omega, \prod_{i=1}^{n} U_{a,b}^{(\epsilon_i)}(u_i, v_i)\Omega \right\rangle \left\langle \Omega, \prod_{j=1}^{k} U_{r,s}^{(\epsilon'_j)}(p_j, w_j)\Omega \right\rangle.$$

Assumption B: (Regularity) for any $\infty > t \geq s \geq 0$,

$$\sup\{|\langle \Omega, (U_{s,t} - 1)(u, v)\Omega\rangle| \,:\, \|u\| = \|v\| = 1\} \leq C|t - s|$$

for some positive constant C independent of s, t.

Remark 3.1. Similar sets of assumptions of independence can also be found in the analysis of Levy processes([4]).However here,unlike in [11, 12], the stationarity condition is not assumed.

As in [11, 12], we need further assumptions for Gaussianity and minimality:

Assumption C: (Gaussianity) for each $t \geq s \geq 0$ and any $u_k, v_k \in \mathbf{h}$, $\epsilon_k \in \mathbb{Z}_2$ $(k = 1, 2, 3)$,

$$\lim_{t \downarrow s} \frac{1}{t - s} \left\langle \Omega, \left(\prod_{k=1}^{3} (U_{s,t}^{(\epsilon_k)} - 1)(u_k, v_k) \right) \Omega \right\rangle = 0. \tag{3.1}$$

Assumption D: (Minimality) the set

$$\mathcal{S}_0 = \left\{ U_{\underline{s},\underline{t}}(\underline{u}, \underline{v})\Omega \,:\, \begin{array}{l} \underline{s} = (s_1, s_2, \cdots, s_n),\ \underline{t} = (t_1, t_2, \cdots, t_n);\ 0 \leq \underline{s}, \underline{t} < \infty, \\ s_j \leq t_j;\ \ \underline{u} = \otimes_{k=1}^{n} u_k, \underline{v} = \otimes_{k=1}^{n} v_k \in \mathbf{h},\ n \geq 1 \end{array} \right\}$$

is total in $\mathcal{H}$.

Remark 3.2. The **Assumption D** is not really a restriction, one can as well work by replacing $\mathcal{H}$ by $\mathcal{H}_0$, the closure of the linear span of $\mathcal{S}_0$. In fact, it is easy to see that $U_{s,t}$ leaves $\mathbf{h} \otimes \mathcal{H}_0$ invariant and that it's restriction to $\mathbf{h} \otimes \mathcal{H}_0$ is an isometry. For the unitarity of the restriction, it will be necessary to define $\mathcal{S}_0$ as the span of $\{U^{(\epsilon)}_{\underline{s},\underline{t}}(\underline{u}, \underline{v})\Omega | \underline{s}, \underline{t}; \underline{u}, \underline{v}; \underline{\epsilon}\}$ so that the restriction of $U^*_{s,t}$ to $\mathbf{h} \otimes \mathcal{H}_0$ is an isometry. However, as can be seen in the sequel, we *only* use the isometry of $U_{s,t}$ in this article.

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[1] It may be noted that the evolution equation here is from right to left instead of left to right as was the case in [11], [12]. This is done in order to be in conformity with the notation of [9] enabling us to use the results there (see Appendix) with minimal changes.

3.1. Vacuum Expectation. Let us look at the various evolutions associated with the $\{U_{s,t}\}$. Define a two parameter family of operators $\{T_{s,t}\}$ on $\mathbf{h}$ by

$$\langle u, T_{s,t}v\rangle := \langle \Omega, U_{s,t}(u,v)\Omega\rangle\,, \qquad \forall u,v \in \mathbf{h}.$$

For each $t \geq s \geq 0$, since $U_{s,t}$ is unitary, $T_{s,t}$ is a contractions.

Remark 3.3. The **Assumption B** implies $\|T_{s,t} - 1\| \leq C|t-s|$. In particular, $\lim_{t\downarrow s} T_{s,t} = 1$ uniformly in s.

Lemma 3.4. *Under the* ***Assumptions A*** *and* **B,** *the family* $\{T_{s,t}\}$ *of contractions satisfies*

(i) *for any* $r \leq s \leq t < \infty$, $T_{s,t}T_{r,s} = T_{r,t}$ *and* $T_{s,s} = 1_{\mathbf{h}}$
(ii) *for any* $t' \geq t \geq s \geq 0, \|T_{s,t'} - T_{s,t}\| \leq C|t'-t|$.

Proof. (i) The evolution and independent increment property of $\{U_{s,t}\}$ and the definition of $T_{s,t}$ gives the result.

(ii) By (i), for a fixed $s \geq 0$ and any $t' \geq t \geq s$, we have

$$\|T_{s,t'} - T_{s,t}\| = \|\left(T_{t,t'} - 1\right) T_{s,t}\| \leq \|T_{s,t}\|\|T_{t,t'} - 1\| \leq C|t'-t|.$$

□

Then we have the following result about the evolutions of the type $T_{s,t}$ by corollary 6.2 in the Appendix:

There exists $G \in L^\infty_{\mathrm{loc}}(\mathbb{R}_+, \mathcal{B}_s(\mathbf{h}))$ (definition is given in Appendix) such that

$$T_{s,t} - 1 = \int_s^t G(\tau) T_{s,\tau} d\tau \tag{3.2}$$

and $\lim_{h\downarrow 0} \frac{T_{t,t+h}-I}{h} = G(t)$ in the strong operator topology for almost every t.

We shall need the following observation (see Equation (6.2) in [11]):

$$\sum_{k\geq 1} \|\left(U_{s,t} - 1\right)(\phi_k, w)\Omega\|^2 = \langle w, \left(1 - T_{s,t}\right) w\rangle + \langle \left(1 - T_{s,t}\right) w, w\rangle \tag{3.3}$$

for any $w \in \mathbf{h}$, where $\{\phi_k\}$ is an complete orthonormal basis of $\mathbf{h}$.

Lemma 3.5. (i) *Under the* ***Assumption C****, for any* $s \geq 0$ *and* $n \geq 3$, $\underline{u}, \underline{v} \in \mathbf{h}^{\otimes n}$ *and* $\underline{\epsilon} \in \mathbb{Z}_2^n$*, we have*

$$\lim_{t\downarrow s} \frac{1}{t-s} \left\langle \Omega, \left(\prod_{k=1}^n \left[\left(U_{s,t}^{(\epsilon_k)} - 1\right)(u_k, v_k)\right]\right)\Omega \right\rangle = 0, \tag{3.4}$$

(ii) *assume* ***B*** *and* ***C****. Then for* $u, v \in \mathbf{h}$, *product vectors* $\underline{p}, \underline{w} \in \mathbf{h}^{\otimes n}$ *and* $\epsilon \in \mathbb{Z}_2$, $\underline{\epsilon'} \in \mathbb{Z}_2^n$*, we have*

$$\begin{aligned} &\lim_{t\downarrow s} \frac{1}{t-s} \left\langle \left(U_{s,t} - 1\right)^{(\epsilon)}(u,v)\Omega, \left(U_{s,t}^{(\underline{\epsilon'})} - 1\right)(\underline{p}, \underline{w})\Omega \right\rangle \qquad (3.5)\\ &\quad = (-1)^{\epsilon} \lim_{t\downarrow s} \frac{1}{t-s} \left\langle \left(U_{s,t} - 1\right)(u,v)\Omega, \left(U_{s,t}^{(\underline{\epsilon'})} - 1\right)(\underline{p}, \underline{w})\Omega \right\rangle. \end{aligned}$$

Proof. (i) The proof is a simple modification of the proof of Lemma 6.6 in [11].

(ii) The idea here is similar to that in the proof of Lemma 6.7 in [11]. For $\epsilon = 0$, it is obvious. To see this for $\epsilon = 1$, put

$$\Phi = \left(U_{s,t}^{(\epsilon')} - 1\right)(\underline{\mathrm{p}}, \underline{\mathrm{w}})\Omega$$

and consider the following

$$\begin{aligned} &\lim_{t\downarrow s}\frac{1}{t-s}\left\langle\left(U_{s,t} + U_{s,t}^* - 2\right)(u,v)\Omega, \Phi\Omega\right\rangle \qquad (3.6)\\ &= -\lim_{t\downarrow s}\frac{1}{t-s}\left\langle\left[\left(U_{s,t}^* - 1\right)\left(U_{s,t} - 1\right)\right](u,v)\Omega, \Phi\Omega\right\rangle\\ &= -\lim_{t\downarrow s}\frac{1}{t-s}\sum_{k\geq 1}\left\langle\left(U_{s,t} - 1\right)(e_k, v)\Omega, \left(U_{s,t} - 1\right)(e_k, u)\Phi\Omega\right\rangle. \end{aligned}$$

On the other hand, we have

$$\begin{aligned} &\left|\frac{1}{t-s}\sum_{k\geq 1}\left\langle\left(U_{s,t} - 1\right)(e_k, v)\Omega, \left(U_{s,t} - 1\right)(e_k, u)\Phi\Omega\right\rangle\right|^2\\ &\leq \left(\sum_{k\geq 1}\frac{1}{t-s}\left\|\left(U_{s,t} - 1\right)(e_k, v)\Omega\right\|^2\right)\left(\sum_{k\geq 1}\frac{1}{t-s}\left\|\left(U_{s,t} - 1\right)(e_k, u)\Phi\Omega\right\|^2\right). \end{aligned}$$

By (3.3) and (iv) in Lemma, the above quantity is equal to

$$\begin{aligned} &2Re\left\langle v, \frac{1-T_{s,t}}{t-s}v\right\rangle\frac{1}{t-s}\left\langle\Phi\Omega, \left[\left(U_{s,t}^* - 1\right)\left(U_{s,t} - 1\right)\right](u,u)\Phi\Omega\right\rangle\\ &= 2Re\left\langle v, \frac{1-T_{s,t}}{t-s}v\right\rangle\frac{1}{t-s}\left\langle\Phi\Omega, \left(2 - U_{s,t}^* - U_{s,t}\right)(u,u)\Phi\Omega\right\rangle. \end{aligned}$$

Since by **Assumption B**, $|\left\langle v, \frac{1-T_{s,t}}{t-s}v\right\rangle| \leq C\|v\|^2$ for any $v \in \mathbf{h}$ and since by the part(i) of this lemma,

$$\lim_{t\downarrow s}\frac{1}{t-s}\left\langle\Phi\Omega, \left(2 - U_{s,t}^* - U_{s,t}\right)(u,u)\Phi\Omega\right\rangle = 0,$$

we obtain by (3.7) that $\lim_{t\downarrow s}\frac{1}{t-s}\left\langle\left(U_{s,t} + U_{s,t}^* - 2\right)(u,v)\Omega, \Phi\Omega\right\rangle =$

$$\lim_{t\downarrow s}\frac{1}{t-s}\sum_{k\geq 1}\left\langle\left(U_{s,t} - 1\right)(e_k, u)\Omega, \left(U_{s,t} - 1\right)(e_k, v)\left(U_{s,t}^{(\epsilon')} - 1\right)(\underline{\mathrm{p}}, \underline{\mathrm{w}})\Omega\right\rangle = 0,$$

which implies (3.6). □

For each $s \geq 0$ and for vectors $u, v, p, w \in \mathbf{h}$ the identity (3.5) gives

$$\begin{aligned} &\lim_{t\downarrow s}\frac{1}{t-s}\left\langle\left(U_{s,t} - 1\right)^{(\epsilon)}(u,v)\Omega, \left(U_{s,t} - 1\right)^{(\epsilon')}(p,w)\Omega\right\rangle \qquad (3.7)\\ &= (-1)^{\epsilon+\epsilon'}\lim_{t\downarrow s}\frac{1}{t-s}\left\langle\left(U_{s,t} - 1\right)(u,v)\Omega, \left(U_{s,t} - 1\right)(p,w)\Omega\right\rangle. \end{aligned}$$

We now introduce the partial trace $\mathrm{Tr}_{\mathcal{H}}$ which is a linear map from $\mathcal{B}_1(\mathbf{h}\otimes\mathcal{H})$ to $\mathcal{B}_1(\mathbf{h})$ defined by

$$\langle u, \mathrm{Tr}_{\mathcal{H}}(B)v\rangle := \sum_{j\geq 1}\langle u\otimes\zeta_j, Bv\otimes\zeta_j\rangle, \qquad \forall u,v\in\mathbf{h}$$

for $B\in\mathcal{B}_1(\mathbf{h}\otimes\mathcal{H})$. In particular, $\mathrm{Tr}_{\mathcal{H}}(B)=\mathrm{Tr}(B_2)\,B_1$ for $B=B_1\otimes B_2$. Then we define a family of operators $\{Z_{s,t}\}_{0\leq s\leq t}$ on the Banach space $\mathcal{B}_1(\mathbf{h})$ by

$$Z_{s,t}(\rho)=\mathrm{Tr}_{\mathcal{H}}\left[U_{s,t}\left(\rho\otimes|\Omega><\Omega|\right)U_{s,t}^*\right], \qquad \rho\in\mathcal{B}_1(\mathbf{h}). \tag{3.8}$$

Thus, for any $u,v,p,w\in\mathbf{h}$, we have

$$\langle p, Z_{s,t}(|w><v|)u\rangle := \langle U_{s,t}(u,v)\Omega, U_{s,t}(p,w)\Omega\rangle\,. \tag{3.9}$$

For $\rho\in\mathcal{B}_1(\mathbf{h})$, by the definition of $Z_{s,t}$ and trace norm (see page no. 47 in [5]), we have

$$\begin{aligned}\|Z_{s,t}(\rho)\|_1 &= \left\|\mathrm{Tr}_{\mathcal{H}}[U_{s,t}\left(\rho\otimes|\Omega><\Omega|\right)U_{s,t}^*]\right\|_1\\ &= \sup_{\phi,\psi:\text{ ons of }\mathbf{h}}\sum_{k\geq 1}\left|\left\langle\phi_k, \mathrm{Tr}_{\mathcal{H}}\left[U_{s,t}\left(\rho\otimes|\Omega><\Omega|\right)U_{s,t}^*\right]\psi_k\right\rangle\right|\\ &\leq \sup_{\phi,\psi:\text{ ons of }\mathbf{h}}\sum_{j,k\geq 1}\left|\left\langle\phi_k\otimes\zeta_j, U_{s,t}\left(\rho\otimes|\Omega><\Omega|\right)U_{s,t}^*\psi_k\otimes\zeta_j\right\rangle\right|\\ &\leq \left\|U_{s,t}\left(\rho\otimes|\Omega><\Omega|\right)U_{s,t}^*\right\|_1\leq\|\rho\|_1.\end{aligned}$$

Thus $Z_{s,t}$ is contractive. For any $u,v\in\mathbf{h}$,

$$\|U_{s,t}(u,v)\Omega\|^2=\langle u, Z_{s,t}(|v><v|)u\rangle$$

and positivity of $Z_{s,t}$ is clear.

Lemma 3.6. *Under the* ***Assumptions A*** *and* ***B****,* $\{Z_{s,t}\}$ *is a family of positive contractive map on* $\mathcal{B}_1(\mathbf{h})$ *satisfying*

(i) *for any* $0\leq r\leq s\leq t<\infty$*,* $Z_{s,t}Z_{r,s}=Z_{r,t}, Z_{s,s}=1$
(ii) *for any* $t'\geq t\geq s\geq 0, \|Z_{s,t'}-Z_{s,t}\|_1\leq 4C|t'-t|$,
(iii) *For any* $\rho\in\mathcal{B}_1(\mathbf{h}), Tr(Z_{s,t}\rho)=Tr(\rho)$.

Proof. (i)To prove evolution property of $Z_{s,t}$ it is enough to show that

$$\langle U_{r,t}(u,v)\Omega, U_{r,t}(p,w)\Omega\rangle=\langle p, Z_{r,t}(|w><v|)u\rangle=\langle p, Z_{s,t}Z_{r,s}(|w><v|)u\rangle$$

for any $u,v,p,w\in\mathbf{h}$. This can be checked by using the evolution and independent increment properties of the unitary family $U_{s,t}$.

(ii) For any rank one operator $\rho=|w><v|$, $w,v\in\mathbf{h}$, we have

$$
\begin{aligned}
&\|(Z_{s,t}-1)(|w><v|)\|_1\\
&= \sup_{\{\phi\},\{\psi\}\ ONB\ of\ \mathbf{h}} \sum_{k\geq 1} |\langle \phi_k, (Z_{s,t}-1)(|w><v|)\psi_k\rangle|\\
&= \sup_{\phi,\psi} \sum_{k\geq 1} |\langle U_{s,t}(\psi_k,v)\Omega, U_{s,t}(\phi_k,w)\Omega\rangle - \overline{\langle \psi_k, v\rangle}\langle \phi_k, w\rangle|\\
&\leq \sup_{\phi,\psi} \sum_{k\geq 1} |\langle (U_{s,t}-1)(\psi_k,v)\Omega, (U_{s,t}-1)(\phi_k,w)\Omega\rangle|\\
&\quad + \sup_{\phi,\psi} \sum_{k\geq 1} |\overline{\langle \psi_k, v\rangle}\langle \Omega, (U_{s,t}-1)(\phi_k,w)\Omega|\\
&\quad + \sup_{\phi,\psi} \sum_{k\geq 1} |\overline{\langle \Omega, (U_{s,t}-1)(\psi_k,v)\Omega\rangle}\langle \phi_k, w\rangle|\\
&\leq \sup_{\phi,\psi} \left[\sum_{k\geq 1} \|(U_{s,t}-1)(\psi_k,v)\Omega\|^2\right]^{1/2} \left[\sum_{k\geq 1} \|(U_{s,t}-1)(\psi_k,w)\Omega\|^2\right]^{1/2}\\
&\quad + \sup_{\phi,\psi} \left[\sum_{k\geq 1} |\langle \psi_k, v\rangle|^2\right]^{1/2} \left[\sum_{k\geq 1} |\langle \phi_k, (T_{s,t}-1)w\rangle|^2\right]^{1/2}\\
&\quad + \sup_{\phi,\psi} \left[\sum_{k\geq 1} |\langle \phi_k, w\rangle|^2\right]^{1/2} \left[\sum_{k\geq 1} |\langle \psi_k, (T_{s,t}-1)v\rangle|^2\right]^{1/2}.
\end{aligned}
$$

Hence by identity (3.3) and **Assumption B** we obtain

$$
\begin{aligned}
&\|(Z_{s,t}-1)(|w><v|)\|_1\\
&\leq 2\|(T_{s,t}-1)\|\|w\|\|v\| + \|(T_{s,t}-1)w\|\|v\| + \|(T_{s,t}-1)v\|\|w\|\\
&\leq 4C|t-s|\|w\|\ \|v\|.
\end{aligned}
$$

Now any for $\rho = \sum_k \lambda_k |\phi_k >< \psi_k| \in \mathcal{B}_1(\mathbf{h})$, where $\{\phi_k\}$ and $\{\psi_k\}$ are two orthonormal bases of $\mathbf{h}$ and we have

$$\|Z_{s,t}(\rho) - \rho\|_1 \leq 4C\left(\sum_k |\lambda_k|\right)|t-s| \leq 4C\|\rho\|_1|t-s|$$

and hence

$$\|Z_{s,t}-1\| \leq 4C|t-s|. \tag{3.10}$$

By evolution property and contractivity of $\{Z_{s,t}\}$

$$\|Z_{s,t'} - Z_{s,t}\| = \|(Z_{t,t'}-1)\, Z_{s,t}\| \leq \|Z_{s,t}\|\|Z_{t,t'}-1\| \leq 4C|t'-t|.$$

(iii) It can be proved as in lemma 6.5 in [11] □

The Corollary 6.2 in the Appendix leads to following result for the evolution $Z_{s,t}$.: Under the **Assumptions A** and **B** there exists $\mathcal{L} \in L^\infty_{\mathrm{loc}}(\mathbb{R}_+, \mathcal{B}_s(\mathcal{B}_1(\mathbf{h})))$

(see Appendix for definition) such that

$$Z_{s,t} - 1 = \int_s^t \mathcal{L}(\tau) Z_{s,\tau} d\tau, \quad \lim_{h\downarrow 0} \frac{Z_{t,t+h} - I}{h} = \mathcal{L}(t). \tag{3.11}$$

4. Construction of Noise Space

Consider the algebra M generated by the tuples $(\underline{u}, \underline{v}, \underline{\epsilon})$ with multiplication structure given by $(\underline{u}, \underline{v}, \underline{\epsilon}) \cdot (\underline{p}, \underline{w}, \underline{\epsilon}') = (\underline{u} \otimes \underline{w}, \underline{v} \otimes \underline{z}, \underline{\epsilon} \oplus \underline{\epsilon}')$. For each $s \geq 0$ we define a scalar valued map K_s on $M \times M$ by setting, for $(\underline{u}, \underline{v}, \underline{\epsilon}), (\underline{p}, \underline{w}, \underline{\epsilon}') \in M$,

$$K_s\left((\underline{u}, \underline{v}, \underline{\epsilon}), (\underline{w}, \underline{z}, \underline{\epsilon}')\right) := \lim_{t\downarrow s} \frac{1}{t-s} \left\langle \left(U_{s,t}^{(\underline{\epsilon})} - 1\right)(\underline{u}, \underline{v})\Omega, \left(U_{s,t}^{\underline{\epsilon}'} - 1\right)(\underline{p}, \underline{w})\Omega \right\rangle$$

if the limit exists.

Theorem 4.1. *For almost every $s \geq 0$*

(i) *the map K_s is a positive definite kernel on M,*

(ii) *there exists a unique (up to unitary equivalence) separable Hilbert space $\mathbf{k}_s$, an embedding $\eta_s : M \to \mathbf{k}_s$ such that*

$$\{\eta_s(\underline{u}, \underline{v}, \underline{\epsilon}) : (\underline{u}, \underline{v}, \underline{\epsilon}) \in M\} \quad \textit{is total in } \mathbf{k}_s, \tag{4.1}$$

$$\langle \eta_s(\underline{u}, \underline{v}, \underline{\epsilon}), \eta_s(\underline{p}, \underline{w}, \underline{\epsilon}')\rangle = K_s\left((\underline{u}, \underline{v}, \underline{\epsilon}), (\underline{p}, \underline{w}, \underline{\epsilon}')\right), \tag{4.2}$$

(iii) *for any $(\underline{u}, \underline{v}, \underline{\epsilon}) \in M$, $\underline{u} = \otimes_{i=1}^n u_i$, $\underline{v} = \otimes_{i=1}^n v_i$ and $\underline{\epsilon} = (\epsilon_1, \cdots, \epsilon_n)$*

$$\eta_s(\underline{u}, \underline{v}, \underline{\epsilon}) = \sum_{i=1}^n \prod_{k \neq i} \langle u_k, v_k \rangle \, \eta_s(u_i, v_i, \epsilon_i), \tag{4.3}$$

(iv) *$\eta_s(u, v, 1) = -\eta_s(u, v, 0)$ for any $u, v \in \mathbf{h}$,*

(v) *for fixed $u, v, p, w \in \mathbf{h}$, the map $s \mapsto K_s((u, v), (p, w)) = \langle \eta_s(u, v), \eta_s(p, w)\rangle$ is Lebesgue measurable and locally bounded in $\mathbb{R}_+$.*

Proof. (i) The proof is exactly same as the proof of Lemma 7.1 in [11]. By Lemma 3.5, for elements $(\underline{u}, \underline{v}, \underline{\epsilon}), (\underline{p}, \underline{w}, \underline{\epsilon}') \in M$, $\underline{\epsilon} \in \mathbb{Z}_2^m$ and $\underline{\epsilon}' \in \mathbb{Z}_2^n$, we have

$$\begin{aligned} &K_s\left((\underline{u}, \underline{v}, \underline{\epsilon}), (\underline{p}, \underline{w}, \underline{\epsilon}')\right) \qquad (4.4)\\ &= \lim_{t\downarrow s} \frac{1}{t-s} \left\langle \left(U_{s,t}^{(\underline{\epsilon})} - 1\right)(\underline{u}, \underline{v})\Omega, \left(U_{s,t}^{(\underline{\epsilon}')} - 1\right)(\underline{p}, \underline{w})\Omega \right\rangle \\ &= \sum_{1\leq i\leq m,\ 1\leq j\leq n} \prod_{k\neq i} \overline{\langle u_k, v_k\rangle} \prod_{l\neq j} \langle p_l, w_l\rangle \\ &\quad \times \lim_{t\downarrow s} \frac{1}{t-s} \left\langle (U_{s,t} - 1)^{(\epsilon_i)}(u_i, v_i)\Omega, (U_{s,t} - 1)^{(\epsilon_j')}(p_j, w_j)\Omega \right\rangle. \end{aligned}$$

Since

$$\begin{aligned} &\langle (U_{s,t} - 1)(u, v)\Omega, (U_{s,t} - 1)(p, w)\Omega\rangle \\ &\quad = \langle U_{s,t}(u, v)\Omega, U_{s,t}(p, w)\Omega\rangle - \overline{\langle u, v\rangle}\langle p, w\rangle \\ &\qquad - \overline{\langle u, v\rangle}\,\langle \Omega, (U_{s,t} - 1)(p, w)\Omega\rangle - \overline{\langle \Omega, (U_{s,t} - 1)(u, v)\Omega\rangle}\langle p, w\rangle \\ &\quad = \langle p, (Z_{s,t} - 1)(|w\rangle\langle v|)u\rangle - \overline{\langle u, v\rangle}\,\langle p, (T_{s,t} - 1)\,w\rangle - \overline{\langle u, (T_{s,t} - 1)\,v\rangle}\langle p, w\rangle, \end{aligned}$$

the existence of the limits on the right hand side of (4.4) follows from the identity (3.5) and by the equations (3.2) and (3.11), K_s is given as

$$\begin{aligned} &K_s((u,v,\epsilon),(p,w,\epsilon')) \qquad (4.5)\\ &= (-1)^{\epsilon+\epsilon'} \lim_{t\downarrow s} \left\{ \left\langle p, \frac{Z_{s,t}-1}{t-s}(|w><v|)u \right\rangle - \overline{\langle u,v\rangle} \left\langle p, \frac{T_{s,t}-1}{t-s} w \right\rangle \right\} \\ &\quad - (-1)^{\epsilon+\epsilon'} \lim_{t\downarrow s} \overline{\left\langle u, \frac{T_{s,t}-1}{t-s} v \right\rangle} \langle p,w\rangle \\ &= (-1)^{\epsilon+\epsilon'} \left\{ \langle p, \mathcal{L}(s)(|w><v|)u\rangle - \overline{\langle u,v\rangle}\, \langle p, G(s)w\rangle - \overline{\langle u, G(s)v\rangle} \langle p,w\rangle \right\}. \end{aligned}$$

(ii) For each $s \geq 0$, the Kolmogorov's construction [10] to the pair (M, K_s) provides a Hilbert space $\mathbf{k}_s$ as the closure of the span of $\{\eta_s(\underline{u},\underline{v},\underline{\epsilon}) : (\underline{u},\underline{v},\underline{\epsilon}) \in M\}$.

(iii) Again as in [11], for any $(\underline{p},\underline{w},\underline{\epsilon}') \in M$, by Lemma 3.5, we have

$$\begin{aligned} \langle \eta_s(\underline{u},\underline{v},\underline{\epsilon}), \eta_s(\underline{p},\underline{w},\underline{\epsilon}')\rangle &= K_s\left((\underline{u},\underline{v},\underline{\epsilon}),(\underline{p},\underline{w},\underline{\epsilon}')\right) \\ &= \sum_{i=1}^{n} \prod_{k\neq i} \overline{\langle u_k, v_k\rangle}\, \langle \eta_s(u_i,v_i,\epsilon_i), \eta_s(\underline{p},\underline{w},\underline{\epsilon}')\rangle . \end{aligned}$$

Since $\{\eta_s(\underline{p},\underline{w},\underline{\epsilon}') : (\underline{p},\underline{w},\underline{\epsilon}') \in M\}$ is a total subset of $\mathbf{k}_s$, (4.3) follows.

(iv) By (3.5), we have

$$\langle \eta_s(u,v,1), \eta_s(\underline{p},\underline{w},\underline{\epsilon}')\rangle = \langle -\eta_s(u,v,0), \eta_s(\underline{p},\underline{w},\underline{\epsilon}')\rangle$$

and hence $\eta_s(u,v,1) = -\eta_s(u,v,0)$.

By parts (iii) and (iv) of this theorem, it is clear that $\mathbf{k}_s$ is spanned by the family $\{\eta_s(u,v) : u,v \in \mathbf{h}\}$, where we have written $\eta_t(u,v)$ for $\eta_t(u,v,0)$.

Since $G \in L^\infty_{\rm loc}(\mathbb{R}_+, \mathcal{B}_s(\mathbf{h}))$ and $\mathcal{L} \in L^\infty_{\rm loc}(\mathbb{R}_+, \mathcal{B}_s(\mathcal{B}_1(\mathbf{h})))$ it follows from (4.5) that $\eta_s(.,.) : \mathbf{h}\times\mathbf{h} \to \mathbf{k}_s$ is sesquilinear and continuous and thus separability of $\mathbf{k}_s$ follows from that of $\mathbf{h}$.

(v) This follows similarly as for (iv). □

For any two orthonormal bases $\{\phi_k\}, \{\psi_k\}$ of $\mathbf{h}$, the collection of vectors

$$\{\eta_s(\phi_k, \psi_l) : k,l \geq 1\}$$

is a countable total family in $\mathbf{k}_s$ and

$$s \mapsto \langle \eta_s(u,v), \eta_s(p,w)\rangle = K_s((u,v),(p,w))$$

is a Lebesgue measurable function. Thus $s \mapsto \langle \eta_s(u,v), \eta_s(\phi_k,\psi_l)\rangle$ is measurable and therefore the family $\{\mathbf{k}_s : s \geq 0\}$ spanned by $\{\eta_s(u,v) : s\geq 0, u,v \in \mathbf{h}\}$, is a measurable field of Hilbert spaces (Chapter 8, [3]).

For any $T \geq 0$, define $K^T((u,v),(p,w)) = \int_0^T K_s((u,v),(p,w))ds$

$$= \int_0^T \{\langle p, \mathcal{L}(s)(|w><v|)u\rangle - \overline{\langle u,v\rangle}\, \langle p, G(s)w\rangle - \overline{\langle u, G(s)v\rangle}\langle p,w\rangle\} ds.$$

Since each K_s is positive definite it can be seen that K^T is a positive definite kernel. Let the associated Hilbert space $\mathbf{k}^T$. There exists a family of vectors $\eta^T(u,v)$ which spans the Hilbert space $\mathbf{k}^T$ such that

$$\begin{aligned}&\langle \eta^T(u,v), \eta^T(p,w)\rangle = K^T((u,v),(p,w))\\ &= \int_0^T K_s((u,v),(p,w))ds = \int_0^T \langle \eta_s(u,v), \eta_s(p,w)\rangle ds.\end{aligned}$$

Comparing the two expressions for K^T, it follows that

$$\langle \eta_t(u,v), \eta_t(p,w)\rangle = \langle p, \{\mathcal{L}(t)(|w><v|) - |G(t)w><v| - |w><G(t)v|\}u\rangle. \quad (4.6)$$

In $\mathbf{k}^T$ there exists a bounded self adjoint operator A with absolutely continuous simple spectrum such that $A\eta^T(u,v)(s) = s\eta_s(u,v)$ for almost every $s \in [0,T]$ and $\mathbf{k}^T$ is the direct integral $\int_{[0,T]}^{\oplus} \mathbf{k}_s ds$ ([3]). There is natural isometric embedding of $\mathbf{k}^T$ in $\mathbf{k}^{T'}$ for $T \leq T'$ by setting $\eta_s^{T,T'}(u,v) = \eta_s^T(u,v)$ for all $0 \leq s \leq T$ and 0 for $s \in (T,T']$.

Remark 4.2. The integral $\int_{\mathbb{R}_+} K_s((u,v),(u,v))ds = \int_{\mathbb{R}_+} \|\eta_s(u,v)\|^2 ds$ need not exist and therefore $\int_{\mathbb{R}_+}^{\oplus} \mathbf{k}_s ds$ may not be defined.

Lemma 4.3. *Under the hypothesis of Theorem 4.1, we have the following:*

(i) *There exists a unique strong measurable family of bounded operators* $L(t)$: $\mathbf{h} \to \mathbf{h} \otimes \mathbf{k}_t$ *such that*

$$\|L(t)v\|^2 = -2\mathrm{Re}\,\langle v, G(t)v\rangle\,, \qquad \forall v \in \mathbf{h}.$$

(ii) *The map* $t \mapsto L(t)$ *is locally norm bounded.*

Proof. (i) By the identity (4.5), for any $u, v \in \mathbf{h}$, we have for almost every $t \geq 0$

$$\|\eta_t(u,v)\|^2 = \langle u, \mathcal{L}(t)(|v><v|)u\rangle - \overline{\langle u,v\rangle}\,\langle u, G(t)v\rangle - \overline{\langle u, G(t)v\rangle}\langle u,v\rangle.$$

and thus

$$\begin{aligned}&\sum_k \|e_k \otimes \eta_t(e_k,v)\|^2 = \sum_k \|\eta_t(e_k,v)\|^2\\ &= \sum_k \Big[\langle e_k, \mathcal{L}(t)(|v><v|)e_k\rangle - \overline{\langle e_k, v\rangle}\,\langle e_k, G(t)v\rangle - \overline{\langle e_k, G(t)v\rangle}\,\langle e_k, v\rangle\Big]\\ &= \mathrm{Tr}\,(\mathcal{L}(t)(|v><v|)) - \langle v, G(t)v\rangle - \overline{\langle v, G(t)v\rangle}.\end{aligned}$$

Moreover, since $Z_{s,t}$ is trace preserving it follows that $\mathrm{Tr}\,(\mathcal{L}(t)(|v><v|)) = 0$. Therefore $\sum_k \|e_k \otimes \eta_t(e_k,v)\|^2 = -2Re\,\langle v, G(t)v\rangle$. This implies that $\sum_k e_k \otimes \eta_t(e_k,v)$ is convergent in norm and in fact for almost every t it defines a bounded operator $L(t) : \mathbf{h} \to \mathbf{h} \otimes \mathbf{k}_t$ given by $L(t)v = \sum_k e_k \otimes \eta_t(e_k,v)$ with

$$\|L(t)v\|^2 = -2Re\,\langle v, G(t)v\rangle\,. \quad (4.7)$$

The strong measurability of $t \mapsto L(t)$ follows from the definition.

The part (ii) follows from the local norm boundedness of $G(.)$. □

5. Hudson-Parthasarathy (HP) Evolution Systems and Equivalence

5.1. HP Evolution Systems. In order to simplify the discussion of the existence and uniqueness of the solution of HP type quantum stochastic differential equation in $\Gamma_{sym}(\int_{\mathbb{R}_+}^{\oplus} \mathbf{k}_s ds)$ and to be able to refer to the existing literature, it is convenient to introduce the following point of view which allow us to embed the process in the standard Fock space $\Gamma = \Gamma_{sym}(L^2(\mathbb{R}_+, \mathbf{k}))$ where $\mathbf{k} = l^2(\mathbb{N})$.

Note that for almost every $t \geq 0$, $\mathbf{k}_t$ is a complex separable Hilbert space. Setting $d(t) =$ the dimension of $\mathbf{k}_t$, $d : \mathbb{R}_+ \to \mathbb{N} \cup \{\infty\}$ is measurable and defining $\Lambda_n = \{t : d(t) = n\}$, $\mathbb{R}_+$ can be written as disjoint union $\bigcup_{n=1}^{\infty} \Lambda_n$ of measurable sets. Let us consider the Hilbert space $l^2(\mathbb{N})$ with a fixed orthonormal basis $\{E_j : j \geq 0\}$. Now for $t \in \Lambda_n$, $n < \infty$ we embed $\mathbf{k}_t$ as the n dimensional subspace $Span\{E_j : 1 \leq j \leq n\}$ of $\mathbf{k}$ and for $t \in \Lambda_\infty$, $\mathbf{k}_t$ identified with $\mathbf{k}$. Then the direct integral $\int_{\mathbb{R}_+}^{\oplus} \mathbf{k}_t dt = \bigoplus_{n \geq 1} L^2(\Lambda_n, \mathbb{C}^n) \bigoplus L^2(\Lambda_\infty, \mathbf{k})$. If $\Lambda_\infty = \emptyset$, then $\int_{\mathbb{R}_+}^{\oplus} \mathbf{k}_t dt$ is isometrically embedded in $L^2(\mathbb{R}_+, \mathbf{k})$.

For any subset $\mathbf{D} \subseteq L^2(\mathbb{R}_+, \mathbf{k})$, let $\mathcal{E}(\mathbf{D})$ be the subspace of Γ which is spanned by the set $\{\mathbf{e}(f) : f \in \mathbf{D}\}$ of exponential vectors defined as:

$$\mathbf{e}(f) := \oplus_{n \geq 0} \frac{f^{\otimes n}}{\sqrt{n!}}.$$

For $0 \leq s < t < \infty$ and $f \in \mathcal{K} = L^2(\mathbb{R}_+, \mathbf{k})$, we denote the functions $1_{[0,s]} f$, $1_{(s,t]} f$ and $1_{[t,\infty)} f$ by $f_{s]}$, $f_{(s,t]}$ and $f_{[t}$, where 1_A is the indicator function of $A \subset [0, \infty)$. Then the Hilbert spaces $\mathcal{K}$ and Γ can be decomposed as $\mathcal{K} = \mathcal{K}_{s]} \oplus \mathcal{K}_{[s,t)} \oplus \mathcal{K}_{[t}$ and $\Gamma = \Gamma_{s]} \otimes \Gamma_{[s,t)} \otimes \Gamma_{[t}$ via the unitary isomorphism given by:

$$\Gamma \ni \mathbf{e}(f) \quad \longleftrightarrow \quad \mathbf{e}(f_{s]}) \otimes \mathbf{e}(f_{(s,t]}) \otimes \mathbf{e}(f_{[t}) \in \Gamma_{s]} \otimes \Gamma_{[s,t)} \otimes \Gamma_{[t},$$

where $\mathcal{K}_{s]} = L^2([0, s), \mathbf{k})$, $\mathcal{K}_{[s,t)} = L^2([s, t), \mathbf{k})$, $\mathcal{K}_{[t} = L^2([t, \infty), \mathbf{k})$ and $\Gamma_{s]} = \Gamma(\mathcal{K}_{s]})$, $\Gamma_{[s,t)} = \Gamma(\mathcal{K}_{[s,t)})$, $\Gamma_{[t} = \Gamma(\mathcal{K}_{[t})$.

Let us consider the Hudson-Parthasarathy (HP) type equation on $\mathbf{h} \otimes \Gamma$:

$$V_{s,t} = 1_{\mathbf{h}\otimes\Gamma} + \sum_{\mu,\nu \geq 0} \int_s^t L_\nu^\mu(\tau) V_{s,\tau} \Lambda_\mu^\nu(d\tau). \tag{5.1}$$

Here the coefficients $L_\nu^\mu(\tau)$ $(\mu, \nu \geq 0)$ are operators in $\mathbf{h}$ and $\Lambda_\mu^\nu(t)$ are fundamental processes define by

$$\Lambda_\nu^\mu(t) = \begin{cases} t 1_{\mathbf{h}\otimes\Gamma} & \text{for } (\mu, \nu) = (0, 0), \\ a\left(1_{[0,t]} \otimes E_j(t)\right) & \text{for } (\mu, \nu) = (j, 0), \\ a^\dagger\left(1_{[0,t]} \otimes E_k(t)\right) & \text{for } (\mu, \nu) = (0, k), \\ \Lambda\left(1_{[0,t]} \otimes |E_k(t) >< E_j(t)|\right) & \text{for } (\mu, \nu) = (j, k), \end{cases} \tag{5.2}$$

where $E_j(t) = E_j$ for $j \in \{1, 2, \cdots d(t)\}$ and $E_j(t) = 0$ otherwise. With respect to the orthonormal basis $E_j(t)$ we have bounded operators $\{L_j(t) : t \geq 0, j \geq 1\}$ in $\mathbf{h}$ such that

$$\langle u, L_j(t) v\rangle = \langle E_j, \eta_t(u, v)\rangle = \langle u \otimes E_j, L(t) v\rangle, \forall u, v \in \mathbf{h}. \tag{5.3}$$

For the details about quantum stochastic calculus see [10, 6]).

Now, let us state the main result of this article.

Theorem 5.1. (i) *The HP equation*

$$V_{s,t} = 1_{\mathbf{h}\otimes\Gamma} + \sum_{\mu,\nu\geq 0}\int_s^t L^\mu_\nu(\tau)V_{s,\tau}\Lambda^\nu_\mu(d\tau) \tag{5.4}$$

on $\mathbf{h}\otimes\Gamma_{\mathrm{sym}}(\mathcal{K})$ *with coefficients* $L^\mu_\nu(t)$ *given by*

$$L^\mu_\nu(t) = \begin{cases} G(t) & \text{for } (\mu,\nu)=(0,0),\\ L_j(t) & \text{for } (\mu,\nu)=(j,0),\\ -L_k(t)^* & \text{for } (\mu,\nu)=(0,k),\\ 0 & \text{for } (\mu,\nu)=(j,k),\end{cases} \tag{5.5}$$

with the unitarity condition (4.7) admit a unique unitary solution $V_{s,t}$.

(ii) *Assume* ***A, B, C*** *and* ***D***. *Then there exists a unitary isomorphism* $\tilde{\tilde{\Xi}}$: $\mathbf{h}\otimes\mathcal{H}\to\mathbf{h}\otimes\Gamma$ *such that*

$$U_{s,t} = \tilde{\tilde{\Xi}}^* V_{s,t}\tilde{\tilde{\Xi}}, \qquad \forall\, 0\leq s\leq t<\infty. \tag{5.6}$$

Proof. (i) The existence of the strong solution $V_{s,t}$ of the equation (5.4) follows exactly as in Proposition 27.5 of ([10]) since for any $\Psi\in\mathbf{h}\otimes\Gamma$, we have

$$\begin{aligned}\sum_{j=1}^\infty \|(L_j(\tau)\otimes I)\Psi\|^2 &= \sum_j\sum_i \|(L_j(\tau)v_i)\otimes E_i\|^2 = \sum_i\sum_j\|L_j(\tau)v_i\|^2\\ &= \sum_i \|L(\tau)v_i\|^2 \ \leq sup_{0\leq\tau\leq T}(\|L(\tau)\|^2)\sum_i\|v_i\|^2\\ &= \mathrm{sup}_{0\leq\tau\leq\mathrm{T}}(\|\mathrm{L}(\tau)\|^2)\|\Psi\|^2,\end{aligned}$$

where we have written $\Psi=\sum v_i\otimes E_i$,$\{E_i\}$ an ONB in Γ.

The isometry of $V_{s,t}$ follows easily as in the proof of the theorem 27.8 of ([10]). On the other hand for the proof of co-isometry of $V_{s,t}$ we proceed as in Theorem 5.3.3 of ([6]) and for $f,g\in L^2\bigcap L^\infty(\mathbb{R}_+,\mathbf{k})$ define $Y_{g,f}(t):\mathcal{B}(\mathbf{h})\to\mathcal{B}(\mathbf{h})$ by $Y_{g,f}(t)X=[\sum_j\overline{g_j(t)}L_j(t),X]-[\sum_j f_j(t)L_j(t)^*,X]+\{\sum_j L_j(t)^*XL_j(t)+XG(t)+G(t)^*X\}$, so that if we set$X_{g,f}(s,t)=\langle .\otimes e(g),(V_{s,t}V^*_{s,t}).\otimes e(f)\rangle$, then $X_{g,f}(s,.)$ satisfy the equation

$$X_{g,f}(s,t) = \langle e(g),e(f)\rangle I_{\mathbf{h}} + \int_s^t Y_{g,f}(\tau)X_{g,f}(s,\tau)d\tau. \tag{5.7}$$

By the equation (4.7) ,we note that $\langle e(g),e(f)\rangle I_{\mathbf{h}}$ is a solution of the linear equation (5.7) and hence by the uniqueness of the solution of the $\mathcal{B}(\mathbf{h})$-valued initial value problem we have that $X_{g,f}(s,t)=\langle e(g),e(f)\rangle I_{\mathbf{h}}$ or $V_{s,t}$ is a co-isometry, leading to the unitarity of the same. We postpone the proof of part (ii) to the next two subsections. □

For $\underline{\epsilon}=(\epsilon_1,\epsilon_2,\cdots,\epsilon_n)\in\mathbb{Z}_2^n$, we define $V^{(\underline{\epsilon})}_{s,t}\in\mathcal{B}(\mathbf{h}^{\otimes n}\otimes\Gamma)$ by setting $V^{(\epsilon)}_{s,t}\in\mathcal{B}(\mathbf{h}\otimes\Gamma)$ by

$$V^{(\epsilon)}_{s,t} = \begin{cases} V_{s,t} & \text{for } \epsilon=0,\\ V^*_{s,t} & \text{for } \epsilon=1.\end{cases}$$

The next result verifies the properties of **Assumption A** for the family $V_{s,t}$ with $\Omega=\mathbf{e}(0)\in\Gamma$.

Lemma 5.2. *The unitary solution* $\{V_{s,t}\}$ *of HP equation* (5.4) *satisfies*

(i) *for any* $0 \le r \le s \le t < \infty$, $V_{r,t} = V_{s,t}V_{r,s}$,

(ii) **Assumption A** *holds for the family* $\{V_{s,t}\}$ *with the distinguished vector* $e(0)$ *in* Γ,

(iii) *for any* $0 \le s \le t < \infty$,

$$\langle \boldsymbol{e}(0), V_{s,t}(u,v)\boldsymbol{e}(0)\rangle = \langle u, T_{s,t}v\rangle, \qquad \forall\, u,v \in \mathbf{h}.$$

Proof. (i) For fixed $0 \le r \le s \le t < \infty$, we set $W_{r,t} = V_{s,t}V_{r,s}$. Then by (5.4), we have

$$\begin{aligned} W_{r,t} &= V_{r,s} + \sum_{\mu,\nu\ge 0}\int_s^t L^\mu_\nu(\tau)V_{s,\tau}V_{r,s}\Lambda^\nu_\mu(d\tau) \\ &= W_{r,s} + \sum_{\mu,\nu\ge 0}\int_s^t L^\mu_\nu(\tau)W_{r,\tau}\Lambda^\nu_\mu(d\tau), \end{aligned}$$

since $W_{r,s} = V_{r,s}V_{s,s} = V_{r,s}$. Thus the family $\{W_{r,t}\}$ of unitary operators also satisfies the HP equation (5.4) for $V_{r,t}$. Hence by the uniqueness of the solution of this quantum stochastic differential equation, $W_{r,t} = V_{r,t}$ for any $0 \le r \le s \le t < \infty$, and the result follows.

(ii) For any $0 \le s \le t < \infty$, the solution $V_{s,t} \in \mathcal{B}(\mathbf{h}\otimes\Gamma_{[s,t]})$. Therefore, for $p,w \in \mathbf{h}$, $V_{s,t}(p,w) \in \mathcal{B}(\Gamma_{[s,t]})$ and the Assumptions $\mathbf{A}2$(i) and $\mathbf{A}2$(ii) are verified by the property of the continuous tensor-factorization of the Fock space.

(iii) Let us define

$$\left\langle u, \widetilde{T}_{s,t}v\right\rangle := \langle \mathbf{e}(0), V_{s,t}(u,v)\mathbf{e}(0)\rangle, \qquad \forall\, u,v\in\mathbf{h}.$$

Then $\widetilde{T}_{s,t}$ is a contractive family of operators and by (5.4), we have that

$$\widetilde{T}_{s,t} = 1 + \int_s^t G(\tau)\widetilde{T}_{s,\tau}d\tau. \tag{5.8}$$

Thus $\widetilde{T}_{s,t} - T_{s,t}$ satisfies the differential equation

$$\widetilde{T}_{s,t} - T_{s,t} = \int_s^t G(\tau)(\widetilde{T}_{s,\tau} - T_{s,\tau})d\tau.$$

Since $G(\tau)$ is locally norm bounded, an iteration of this equation will lead to $\widetilde{T}_{s,t} = T_{s,t}$ for almost all s,t and therefore by continuity also for all s,t. □

Consider the family of operators $\widetilde{Z}_{s,t}$ defined by

$$\widetilde{Z}_{s,t}(\rho) = \mathrm{Tr}_\Gamma\left[V_{s,t}(\rho\otimes|\mathbf{e}(0)><\mathbf{e}(0)|)V^*_{s,t}\right], \qquad \forall\,\rho\in\mathcal{B}_1(\mathbf{h}).$$

As for $Z_{s,t}$, it can be seen that $\widetilde{Z}_{s,t}$ is a contractive family of maps on $\mathcal{B}_1(\mathbf{h})$ and, in particular, for any $u,v,p,w\in\mathbf{h}$,

$$\left\langle p, \widetilde{Z}_{s,t}(|w><v|)u\right\rangle = \langle V_{s,t}(u,v)\mathbf{e}(0), V_{s,t}(p,w)\mathbf{e}(0)\rangle\,.$$

Lemma 5.3. *The family* $\{\widetilde{Z}_{s,t}\}$ *is a uniformly continuous evolution of contraction on* $\mathcal{B}_1(\mathbf{h})$ *and* $\widetilde{Z}_{s,t} = Z_{s,t}$, *where* $Z_{s,t}$ *is given as in* (3.8).

Proof. By (5.4) and Itô's formula, for $u, v, p, w \in \mathbf{h}$

$$
\begin{aligned}
&\left\langle p, \left[\widetilde{Z}_{s,t} - 1\right] (|w >< v|)u \right\rangle \\
= \;& \langle V_{s,t}(u,v)\mathbf{e}(0), V_{s,t}(p,w)\mathbf{e}(0)\rangle - \overline{\langle u, v\rangle}\, \langle p, w\rangle \\
= \;& \int_s^t \langle V_{s,\tau}(u,v)\mathbf{e}(0), V_{s,\tau}(G(\tau)^* p, w)\mathbf{e}(0)\rangle \, d\tau \\
&+ \int_s^t \langle V_{s,\tau}(G(\tau)^* u, v)\mathbf{e}(0), V_{s,\tau}(p,w)\mathbf{e}(0)\rangle \, d\tau \\
&+ \sum_j \int_s^t \langle V_{s,\tau}(L_j(\tau)^* u, v)\mathbf{e}(0), V_{s,\tau}(L_j(\tau)^* p, w)\mathbf{e}(0)\rangle \, d\tau \\
= \;& \int_s^t \Big\langle p, \{G(\tau)\tilde{Z}_{s,\tau}(|w >< v|) + \tilde{Z}_{s,\tau}(|w >< v|)G(\tau)^* \\
&+ \sum_{j\ge 1} L_j(\tau)\tilde{Z}_{s,\tau}(|w >< v| L_j(\tau)^*\}u \Big\rangle \, d\tau.
\end{aligned}
$$

Thus by identity (5.3) for $\{L_j(t)\}$ and (4.6), we have that

$$
\left\langle p, \left[\widetilde{Z}_{s,t} - 1\right] (\rho)u \right\rangle = \int_s^t \left\langle p, \mathcal{L}(\tau)\widetilde{Z}_{s,\tau}(\rho)u \right\rangle d\tau, \tag{5.9}
$$

where $\rho = |w >< v|$. Thus the family $\{\widetilde{Z}_{s,t}\}$ satisfies the equation

$$
\widetilde{Z}_{s,t}(\rho) = \rho + \int_s^t \mathcal{L}(\tau)\widetilde{Z}_{s,\tau}(\rho) d\tau, \qquad \rho \in \mathcal{B}_1(\mathbf{h}).
$$

Therefore, proceeding as in the proof of Lemma 5.2 (iii) we can conclude that $\widetilde{Z}_{s,t} = Z_{s,t}$. □

5.2. Minimality of HP Evolution Systems. In this section we shall show the minimality of the HP evolution system $\{V_{s,t}\}$ discussed in Section 5.1 which will be needed to prove (ii) in Theorem 5.1, i.e., to establish unitary equivalence of $U_{s,t}$ and $V_{s,t}$. We shall prove here that the subset

$$
\mathcal{S}' = \left\{ V_{\underline{s},\underline{t}}(\underline{u},\underline{v})\mathbf{e}(0) \;:\; \begin{array}{l} \underline{s} = (s_1, s_2, \cdots, s_n),\ \underline{t} = (t_1, t_2, \cdots, t_n);\ 0 \le \underline{s}, \underline{t} < \infty, \\ s_j \le t_j;\ \ \underline{u} = \otimes_{i=1}^n u_i, \underline{v} = \otimes_{i=1}^n v_i \in \mathbf{h},\ n \ge 1 \end{array} \right\}
$$

is total in the symmetric Fock space $\Gamma(\mathcal{K}) \subseteq \Gamma(L^2(\mathbb{R}_+, \mathbf{k}))$, where

$$
V_{\underline{s},\underline{t}}(\underline{u},\underline{v})\mathbf{e}(0) := V_{s_1,t_1}(u_1,v_1)\cdots V_{s_n,t_n}(u_n,v_n)\mathbf{e}(0).
$$

Let $T \ge 0$ be fixed and as in ([11]), we note that for any $0 \le s < t \le T$, $u, v \in \mathbf{h}$,

$$
\frac{1}{t-s}\left[V_{s,t} - 1\right](u,v)\mathbf{e}(0) = \gamma(s,t,u,v) + \rho(s,t,u,v) + \zeta(s,t,u,v) + \varsigma(s,t,u,v), \tag{5.10}
$$

where these vectors in the Fock space Γ are given by

$$\gamma(s,t,u,v) := \frac{1}{t-s}\sum_{j\geq 1}\int_s^t \langle u, L_j(\lambda)v\rangle\, a_j^\dagger(d\lambda)\,\mathbf{e}(0),$$
$$\rho(s,t,u,v) := \frac{1}{t-s}\int_s^t \langle u, G(\lambda)v\rangle\, d\lambda\ \mathbf{e}(0),$$
$$\zeta(s,t,u,v) := \frac{1}{t-s}\sum_{j\geq 1}\int_s^t (V_{s,\lambda}-1)\,(L_j(\lambda)^*u, v)\, a_j^\dagger(d\lambda)\,\mathbf{e}(0),$$
$$\varsigma(s,t,u,v) := \frac{1}{t-s}\int_s^t (V_{s,\lambda}-1)\,(G(\lambda)^*u, v)\, d\lambda\,\mathbf{e}(0).$$

Note that any $\phi\in\Gamma$ can be written as $\phi=\phi^{(0)}\oplus\phi^{(1)}\oplus\cdots$, where $\phi^{(n)}$ is in the n-fold symmetric tensor product $L^2(\mathbb{R}_+,\mathbf{k})^{\otimes_s n}\equiv L^2(\Sigma_n)\otimes\mathbf{k}^{\otimes n}$. Here Σ_n is the n-simplex $\{\underline{t}=(t_1,t_2,\cdots,t_n): 0\leq t_1<t_2\cdots<t_n<\infty\}$.

Lemma 5.4. *Let $u,v\in\mathbf{h}$ and let $C_T=4e^T\sup\{\|L(\lambda)\|^2+\|G(\lambda)\|^2 : 0\leq\lambda\leq t\}$. Then for any $0\leq s\leq t\leq T$,*
(i)

$$\|(V_{s,t}-1)v\boldsymbol{e}(0)\|^2\leq C_T|t-s|\|v\|^2. \tag{5.11}$$

(ii) $\|(V_{s,t}-1)(u,v)\ \boldsymbol{e}(0)\|^2\leq C_T\|u\|^2\|v\|^2|t-s|$.
(iii) For any $u\in\mathbf{h}$

$$\|\sum_{j\geq 1}\int_s^t V_{s,\lambda}(u,L_j(\lambda)v)a_j^\dagger(d\lambda)\boldsymbol{e}(0)\|^2$$
$$\leq C_T\|u\|^2\|v\|^2|t-s|.$$

Proof. (i) By estimates of quantum stochastic integration (Proposition 27.1, [10])

$$\begin{aligned}
&\|(V_{s,t}-1)v\mathbf{e}(0)\|^2\\
&=\|\sum_{j\geq 1}\int_s^t V_{s,\lambda}L_j(\lambda)a_j^\dagger(d\lambda)\ v\mathbf{e}(0)+\int_s^t V_{s,\lambda}G(\lambda)d\lambda\ v\mathbf{e}(0)\|^2\\
&\leq 2e^T\int_s^t\{\sum_{j\geq 1}\|L_j(\lambda)v\|^2+\|G(\lambda)v\|^2\}d\lambda\\
&\leq 2e^T\|v\|^2\int_s^t\{\|L(\lambda)\|^2+\|G(\lambda)\|^2\}d\lambda\\
&=\|v\|^2C_T|t-s|.
\end{aligned}$$

(ii) By Lemma 2.1 (ii), we have $(V_{s,t}-1)(u,v)\mathbf{e}(0)=F_u^*(V_{s,t}-1)v\mathbf{e}(0)$ and therefore the result follows from (i).

(iii) By lemma 2.1,

$$\begin{aligned}
&\|\sum_{j\geq 1}\int_s^t V_{s,\lambda}(L_j(\lambda)^*u,v)a_j^+(d\lambda)e(0)\|^2 \\
&= \|\sum_j \int_s^t F_u^*(L_j(\lambda)\otimes I_\Gamma)V_{s,\lambda}a_j^+(d\lambda)ve(0)\|^2 \\
&\leq 2e^T\|u\|^2\|v\|^2 sup_{0\leq\lambda\leq T}\|L(\lambda)\|^2|t-s| \\
&\leq C_T\|u\|^2\|v\|^2|t-s|,
\end{aligned}$$

where we have used the standard estimate of a quantum stochastic integral. □

Lemma 5.5. *For any* $u, v \in \mathbf{h}, 0 \leq s \leq t \leq T$,

(i) $\sup\{\|\zeta(s,t,u,v)\|^2 : 0 \leq s \leq t \leq T\} \leq C_T^2\|u\|^2\|v\|^2$ *and* $\|\varsigma(s,t,u,v)\| \leq C_T|t-s|^{1/2}\|u\|\|v\|$.

(ii) *For any* $\phi \in \Gamma(L^2(\mathbb{R}_+,\mathbf{k}))$, $\lim_{t\downarrow s}\langle\phi,\zeta(s,t,u,v)\rangle = 0$ *and*

$$\lim_{t\downarrow s}\langle\phi,\gamma(s,t,u,v)\rangle = \sum_{j\geq 1}\langle u, L_j(s)v\rangle\overline{\phi_j^{(1)}}(s) = \langle\phi^{(1)}(s),\eta_s(u,v)\rangle,\ \ a.e.\ s\geq 0.$$

Proof. (i) By Lemma 5.4, part (iii), we have

$$\begin{aligned}
&\|\zeta(s,t,u,v)\|^2 = \frac{1}{|t-s|^2}\|\sum_{j\geq 1}\int_s^t (V_{s,\lambda}-1)(L_j(\lambda)^*u,v)a_j^\dagger(d\lambda)\ \mathbf{e}(0)\|^2 \\
&= |t-s|^{-2}\|\sum_j\int_s^t F_u^*(L_j(\lambda)\otimes I_\Gamma)(V_{s,\lambda}-I)a_j^+(d\lambda)ve(0)\|^2 \\
&\leq 2e^T\|u\|^2|t-s|^{-2}sup_\lambda\|L(\lambda)\|^2\int_s^t\|(V_{s,\lambda}-I)ve(0)\|^2d\lambda \\
&\leq C_T^2\|u\|^2\|v\|^2,
\end{aligned}$$

where we have used the estimate (5.10). Similarly,

$$\begin{aligned}
\|\varsigma(s,t,u,v)\| &= \frac{1}{|t-s|}\|\int_s^t(V_{s,\lambda}-1)(G(\lambda)^*u,v)d\lambda\ \mathbf{e}(0)\| \\
&= |t-s|^{-1}\|\int_s^t F_u^*(G(\lambda)\otimes I_\Gamma)(V_{s,\lambda}-I)ve(0)d\lambda\| \\
&\leq \|u\||t-s|^{-1}sup_\lambda(\|G(\lambda)\|)\int_s^t\|(V_{s,\lambda}-I)ve(0)\|d\lambda \\
&\leq C_T|t-s|^{1/2}\|u\|\|v\|.
\end{aligned}$$

(ii) For any $f \in L^2(\mathbb{R}_+,\mathbf{k})$,

$$\langle \mathbf{e}(f), \zeta(s,t,u,v)\rangle = \langle \mathbf{e}(f), \frac{1}{t-s}\sum_{j\geq 1}\int_s^t (V_{s,\lambda}-1)(L_j(\lambda)^* u, v) a_j^\dagger(d\lambda)\ \mathbf{e}(0)\rangle$$
$$= \frac{1}{t-s}\sum_{j\geq 1}\int_s^t \overline{f_j(\lambda)}\langle \mathbf{e}(f), (V_{s,\lambda}-1)(L_j(\lambda)^* u, v)\ \mathbf{e}(0)\rangle d\lambda$$
$$= \frac{1}{t-s}\int_s^t R(s,\lambda) d\lambda,$$

where $R(s,\lambda) = \sum_{j\geq 1}\overline{f_j(\lambda)}\langle \mathbf{e}(f), (V_{s,\lambda}-1)(L_j(\lambda)^* u, v)\ \mathbf{e}(0)\rangle$. Note that the complex valued function $R(s,\lambda)$ is locally integrable in λ and continuous in s and and therefore it makes sense to talk about $R(s,s)$ which is 0. So we get

$$\lim_{t\downarrow s}\langle \mathbf{e}(f), \zeta(s,t,u,v)\rangle = 0.$$

Since $\zeta(s,t,u,v)$ is uniformly bounded in s,t

$$\lim_{t\downarrow s}\langle \phi, \zeta(s,t,u,v)\rangle = 0, \forall \phi \in \Gamma.$$

We also have

$$\langle \phi, \gamma(s,t,u,v)\rangle = \frac{1}{t-s}\sum_{j\geq 1}\int_s^t \langle u, L_j(\lambda) v\rangle \overline{\phi_j^{(1)}}(\lambda) d\lambda. \tag{5.12}$$

Since

$$|\sum_{j\geq 1}\langle u, L_j(\lambda)v\rangle\overline{\phi_j^{(1)}}(\lambda)|^2 \leq \|u\|^2\sum_{j\geq 1}\|L_j(\lambda)v\|^2|\phi_j^{(1)}(\lambda)|^2 \leq C_\tau\|v\|^2\|\phi^{(1)}(\lambda)\|^2,$$

the function $\sum_{j\geq 1}\langle u, L_j(\lambda)v\rangle\overline{\phi_j^{(1)}}(\lambda)$ is in L^2 and hence locally integrable. Thus we get

$$\lim_{t\downarrow s}\langle \phi, \gamma(s,t,u,v)\rangle = \sum_{j\geq 1}\langle u, L_j(s)v\rangle\overline{\phi_j^{(1)}}(s) = \langle \phi^{(1)}(s), \eta_s(u,v)\rangle \text{ a.e. } s\geq 0.$$

□

Lemma 5.6. *For* $n \geq 1$, $\underline{s} \in \Sigma_n$ *and* $u_k, v_k \in \mathbf{h} : k = 1, 2, \cdots, n, \phi \in \Gamma(L^2(\mathbb{R}_+, \mathbf{k}))$ *and disjoint* $[s_k, t_k)$,

(i) $\lim_{\underline{t}\downarrow\underline{s}}\langle \phi, \prod_{k=1}^n M(s_k,t_k,u_k,v_k)\ \mathbf{e}(0)\rangle = 0$, *where*

$$M(s_k,t_k,u_k,v_k) = \frac{(V_{s_k,t_k}-1)}{t_k-s_k}(u_k,v_k) - \rho(s_k,t_k,u_k,v_k) - \gamma(s_k,t_k,u_k,v_k)$$

and $\lim_{\underline{t}\downarrow\underline{s}}$ *means* $t_k \downarrow s_k$ *for each* k.

(ii) $\lim_{\underline{t}\downarrow\underline{s}}\langle \phi, \otimes_{k=1}^n\gamma(s_k,t_k,u_k,v_k)\rangle = \langle \phi^{(n)}(s_1,s_2,\cdots,s_n), \eta_{s_1}(u_1,v_1)\otimes\cdots\otimes \eta_{s_n}(u_n,v_n)\rangle$.

Proof. (i) First note that $M(s,t,u,v)\mathbf{e}(0) = \zeta(s,t,u,v) + \varsigma(s,t,u,v)$. So by the above observations in Lemma 5.5, $\{M(s,t,u,v)\mathbf{e}(0)\}$ is uniformly bounded in

$s, t \leq \tau$ and $\lim_{t\downarrow s}\langle \mathbf{e}(f), M(s,t,u,v)\mathbf{e}(0)\rangle = 0, \forall f \in L^2(\mathbb{R}_+, \mathbf{k})$. Since the intervals $[s_k, t_k)$'s are disjoint for different k's,

$$\langle \mathbf{e}(f), \prod_{k=1}^{n} M(s_k, t_k, u_k, v_k)\ \mathbf{e}(0)\rangle = \prod_{k=1}^{n}\langle \mathbf{e}(f_{[s_k,t_k)}), M(s_k, t_k, u_k, v_k)\ \mathbf{e}(0)\rangle$$

and thus $\lim_{\underline{t}\downarrow\underline{s}}\langle \mathbf{e}(f), \prod_{k=1}^{n} M(s_k, t_k, u_k, v_k)\ \mathbf{e}(0)\rangle = 0$.

Since $\prod_{k=1}^{n} M(s_k, t_k, u_k, v_k)\ \mathbf{e}(0)$ is uniformly bounded in s_k, t_k requirement follows for $\phi \in \Gamma$.

(ii) It can be proved similarly as part (iii) of the previous Lemma. □

Lemma 5.7. *Let $\phi \in \Gamma$ be such that*

$$\langle \phi, \psi\rangle = 0, \qquad \forall\ \psi \in \mathcal{S}'. \tag{5.13}$$

Then we have

(i) $\phi^{(0)} = 0$ *and* $\phi^{(1)} = 0$,
(ii) *for any* $n \geq 0$, $\phi^{(n)} = 0$,
(iii) *the set* $\mathcal{S}'$ *is total in the Fock space* Γ.

Proof. (i) For any $s \geq 0$, $V_{s,s} = 1_{\mathbf{h}\otimes\Gamma}$ and so, in particular, (5.13) gives, for any $u, v \in \mathbf{h}$,

$$0 = \langle \phi, V_{s,s}(u,v)\mathbf{e}(0)\rangle = \langle u, v\rangle\ \overline{\phi^{(0)}}$$

and hence $\phi^{(0)} = 0$.

(ii) By (5.13), $\langle \phi, [V_{s,t} - 1]\,(u,v)\mathbf{e}(0)\rangle = 0$ for any $0 \leq s < t \leq \tau < \infty$ and $u, v \in \mathbf{h}$. By HP equation (5.4) and part (iii) of Lemma 5.5 , we have

$$\begin{aligned} 0 &= \lim_{t\downarrow s}\frac{1}{t-s}\ \langle \phi, [V_{s,t} - 1](u,v)\mathbf{e}(0)\rangle \\ &= \sum_{j\geq 1}\langle u, L_j(s)v\rangle\ \overline{\phi_j^{(1)}(s)} \\ &= \langle \phi^{(1)}(s), \eta_s(u,v)\rangle. \end{aligned}$$

So $\left\langle \phi^{(1)}(s), \eta_s(u,v)\right\rangle = 0$ for any $u, v \in \mathbf{h}$ for almost every s. Since $\{\eta_s(u,v) : u, v \in \mathbf{h}\}$ is total in $\mathbf{k}_s$, it follows that $\phi^{(1)}(s) = 0 \in \mathbf{k}_s$ for almost every $0 \leq s \leq \tau$, i.e, $\phi^{(1)} = 0$.

(iii) We prove this by induction. The result is already proved for $n = 0, 1$. For $n \geq 2$, assume as induction hypothesis that for all $m \leq n-1$, $\phi^{(m)}(\underline{s}) = 0$, for almost every $\underline{s} \in \Sigma_m$ ($s_i \leq \tau$ for $i = 1, 2, \cdots, m$). To show that $\phi^{(n)} = 0$, we note that by a similar argument as in [11],

$$\left\langle \phi^{(n)}(s_1, s_2, \cdots, s_n), \eta_{s_1}(u_1, v_1) \otimes \cdots \otimes \eta_{s_n}(u_n, v_n)\right\rangle = 0.$$

for almost every $\underline{s} \in \Sigma_n$ ($s_i \leq \tau$). Since $\{\eta_s(u,v) : u, v \in \mathbf{h}\}$ is total in $\mathbf{k}_s$, it follows that $\phi^{(n)}(s_1, s_2, \cdots, s_n) = 0 \in \mathbf{k}_{s_1} \otimes \cdots \otimes \mathbf{k}_{s_n}$ for almost every $(s_1, s_2, \cdots, s_n) \in \Sigma_n$. □

5.3. Unitary Equivalence. We shall now prove (ii) in Theorem 5.1 that the unitary evolution $\{U_{s,t}\}$ on $\mathbf{h}\otimes\mathcal{H}$ is unitarily equivalent to the unitary solution $\{V_{s,t}\}$ of HP equation (5.4). To prove this we need the following two results.

Lemma 5.8. *Let $U_{\underline{s},\underline{t}}(\underline{u},\underline{v})\Omega$ and $U_{\underline{s}',\underline{t}'}(\underline{p},\underline{w})\Omega$ be in $\mathcal{S}$, where $\underline{v},\underline{z}\in\mathbf{h}^{\otimes n}$. Then there exist an integer $m\geq 1$, $\underline{a}=(a_1,a_2,\cdots,a_m)$, $\underline{b}=(b_1,b_2,\cdots,b_m)$ with $0\leq a_1\leq b_1\leq\cdots\leq a_m\leq b_m<\infty$, partition $R_1\cup R_2\cup R_3=\{1,\cdots,m\}$ with $|R_i|=m_i$, family of vectors $x_{k_l},g_{k_i}\in\mathbf{h}$ and $y_{k_l},h_{k_i}\in\mathbf{h}$ for $l\in R_1\cup R_2$ and $i\in R_2\cup R_3$ such that*

$$U_{\underline{s},\underline{t}}(\underline{u},\underline{v})=\sum_{\underline{k}}\prod_{l\in R_1\cup R_2}U_{a_l,b_l}(x_{k_l},y_{k_l}),$$

$$U_{\underline{s}',\underline{t}'}(\underline{p},\underline{w})=\sum_{\underline{k}}\prod_{l\in R_2\cup R_3}U_{a_l,b_l}(g_{k_l},h_{k_l}).$$

Proof. It follows from the evolution hypothesis of $\{U_{s,t}\}$ that for $r\in[s,t]$ and a complete orthonormal basis $\{f_j\}\in\mathbf{h}$ we can write

$$U_{s,t}(u,v)=\sum_{j\geq 1}U_{s,r}(u,f_j)U_{r,t}(f_j,v).$$

□

Remark 5.9. Since the family of unitary operators $\{V_{s,t}\}$ on $\mathbf{h}\otimes\Gamma$ enjoy all the properties satisfy by family of unitary operators $\{U_{s,t}\}$ on $\mathbf{h}\otimes\mathcal{H}$, the above lemma also hold if we replace $U_{s,t}$ by $V_{s,t}$.

Lemma 5.10. *For $U_{\underline{s},\underline{t}}(\underline{u},\underline{v})\Omega,U_{\underline{s}',\underline{t}'}(\underline{p},\underline{w})\Omega\in\mathcal{S}$, we have*

$$\left\langle U_{\underline{s},\underline{t}}(\underline{u},\underline{v})\Omega,U_{\underline{s}',\underline{t}'}(\underline{p},\underline{w})\Omega\right\rangle=\left\langle V_{\underline{s},\underline{t}}(\underline{u},\underline{v})\boldsymbol{e}(0),V_{\underline{s}',\underline{t}'}(\underline{p},\underline{w})\boldsymbol{e}(0)\right\rangle. \qquad (5.14)$$

Proof. The proof of (5.14) is very similar to that in [11]. In fact, for

$$0\leq s\leq t<\infty,\langle U_{s,t}(u,v)\Omega,U_{s,t}(p,w)\Omega\rangle=\langle p,Z_{s,t}(|w><v|)u\rangle$$

while

$$\langle V_{s,t}(u,v)\mathbf{e}(0),V_{s,t}(p,w)\mathbf{e}(0)\rangle=\left\langle p,\widetilde{Z}_{s,t}(|w><v|)u\right\rangle$$

but $\widetilde{Z}_{s,t}=Z_{s,t}$. □

Now defining a map $\Xi:\mathcal{H}\to\Gamma$ by sending $U_{\underline{s},\underline{t}}(\underline{u},\underline{v})\Omega\in\mathcal{S}$ to $V_{\underline{s},\underline{t}}(\underline{u},\underline{v})\mathbf{e}(0)\in\mathcal{S}'$, as in [11], we can establish unitary equivalence of HP evolution $V_{s,t}$ with the evolution $U_{s,t}$ we started with.

6. Appendix

Let X be a complex separable Banach space with the *$Radon-Nikodym$ property*, i.e., every $f\in\mathrm{Lip}(\mathbb{R},X)\equiv\{f:\mathbb{R}\to X|\|f(t)-f(s)\|\leq C|t-s|$ for some $0<C<\infty\}$ is differentiable almost everywhere. In such a case, $f'\in L^\infty_{\mathrm{loc}}(\mathbb{R},X)$ and

$$f(t)-f(s)=\int_s^t f'(\tau)d\tau. \qquad (6.1)$$

It is known [1] that separable reflexive Banach spaces and separable dual Banach spaces have the Radon-Nikodym property. Thus the cases relevant to our problem in which $X = \mathbf{h}$ and $X = \mathcal{B}_1(\mathbf{h})$ qualify as spaces with Radon-Nikodym property. We shall denote by $\mathcal{B}_s(X)$ the linear space $\mathcal{B}(X)$ equipped with strong operator topology.

Let $\{\tilde{S}_{s,t} | s, t \in \mathbb{R}, s \leq t\}$ be a contractive evolution acting on a complex separable Banach space X, i.e., $\|\tilde{S}_{s,t}\| \leq 1$ and $\tilde{S}_{r,t} = \tilde{S}_{s,t}\tilde{S}_{r,s}, \tilde{S}_{s,s} = 1$ for $r \leq s \leq t$. Then we have the following theorem [9] characterizing such evolution.

Theorem 6.1. *Let the Banach space X have the Radon-Nikodym property and let the evolution $\tilde{S}_{s,t}$ satisfy uniform Lipshitz condition: $\|\tilde{S}_{s,t} - 1\| \leq C|t-s|$ for $s, t \in \mathbb{R}$ and $s \leq t$. Then there exists an operator valued function $\tilde{G} \in L^\infty_{\text{loc}}(\mathbb{R}, \mathcal{B}_s(X))$ such that $\tilde{S}_{s,t} = 1 + \int_s^t \tilde{g}(\tau)\tilde{S}_{s,\tau} d\tau$.*

This theorem is proven in [9]. We need to adapt this for the evolutions (viz., $T_{s,t}$ and $Z_{s,t}$) that we have constructed earlier where $s, t \in \mathbb{R}_+$.

Given a contractive evolution $S_{s,t}$ on $\mathbb{R}_+$, we can extend it to define a contractive evolution $\tilde{S}_{s,t}$ on $\mathbb{R}$ as follows:

$$\tilde{S}_{s,t} = \begin{cases} S_{s,t} & \text{if} \quad 0 \leq s \leq t \\ 1 & \text{if} \quad s \leq t \leq 0 \\ S_{0,t} & \text{if} \quad s \leq 0 \leq t. \end{cases}$$

It is easy to check that this $\tilde{S}_{s,t}$ is a contractive evolution on $\mathbb{R}$. Furthermore, it is clear that $\tilde{S}_{s,t}$ satisfies Lipshitz condition on $\mathbb{R}$ if $S_{s,t}$ does the same on $\mathbb{R}_+$.

Corollary 6.2. *Let X be either $\mathbf{h}$ or $\mathcal{B}_1(\mathbf{h})$ and let $T_{s,t}$ and $Z_{s,t}$ be contractive evolutions on $\mathbb{R}_+$ respectively. Then there exist operator valued functions $G \in L^\infty_{\text{loc}}(\mathbb{R}_+, \mathcal{B}_s(\mathbf{h}))$ and $\mathcal{L} \in L^\infty_{\text{loc}}(\mathbb{R}_+, \mathcal{B}_s(\mathcal{B}_1(\mathbf{h})))$ respectively such that*

$$T_{s,t} = 1 + \int_s^t G(\tau)T_{s,\tau} d\tau$$

and

$$Z_{s,t} = 1 + \int_s^t \mathcal{L}(\tau)Z_{s,\tau} d\tau.$$

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Un Cig Ji: Department of Mathematics, Research Institute of Mathematical Finance, Chungbuk National University, Cheongju 361-763, Korea
E-mail address: uncigji@chungbuk.ac.kr

Lingaraj Sahu: Indian Institute of Science Education and Research (IISER) Mohali, MGSIPAP Complex Sector 26, Chandigarh -16, India
E-mail address: lingaraj@iisermohali.ac.in

Kalyan B. Sinha: Jawaharlal Nehru Centre for Advanced Scientific Research, Jakkur, Bangalore-64, India; Department of Mathematics, Indian Institute of Science, Bangalore-12, India
E-mail address: kbs_jaya@yahoo.co.in