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THE PULSATIONS AND THE DYNAMICAL STABILITY OF GASEOUS MASSES IN UNIFORM ROTATION

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ABSTRACT

A variational principle, applicable to axisymmetric oscillations of uniformly rotating axisymmetric configurations, is established On the assumption that the Lagrangian displacement (describing the oscillation) at any point is normal to the level surface (of constant total potential) through that point, it is shown how the variational expression, for the frequencies of oscillation, can be reduced to simple quadratures. The reduction is explicitly carried out for certain stratifications of special interest.

Some new results on the oscillations of slowly rotating configurations are included; and a number of related observations on their stability are also made.

I. INTRODUCTION

A slow uniform rotation affects the radial modes of adiabatic oscillation of an initial spherical distribution of mass in two essentially different ways (cf. Cowling and Newing 1949; see also Chandrasekhar and Lebovitz 1962*d*): *first*, the term in the centrifugal acceleration in the equation governing equilibrium modifies the initial distribution of density and pressure; and *second*, the term in the Coriolis acceleration, in the linearized equations governing small departures from equilibrium, further modifies the characteristic frequencies. In general both these effects are of order Ω^2 (the square of the angular velocity of rotation); and they are found to contribute terms of opposite signs to σ^2 (the square of the characteristic frequency of oscillation). In view particularly of this last circumstance, one cannot be certain whether, in a given situation, rotation will have a stabilizing or a destabilizing effect on the radial oscillations. There is, however, one important exception.

It is known that the condition for the dynamical stability of spherical masses (determined by the stability of the fundamental mode of radial oscillation) is that the ratio of the specific heats γ exceeds $\frac{4}{3}$, in case it is a constant.[1] Now if $\gamma - \frac{4}{3}$ is $O(\Omega^2)$ and $\Omega \rightarrow 0$, then it is an immediate consequence of a formula due to Ledoux (1945) that the effect of rotation on σ^2 can be written down at once, without having to determine its effect on the equilibrium distribution.

Now Ledoux's formula for σ^2, for a slowly rotating configuration, is

$$\sigma^2 = (3\gamma - 4)\frac{|\mathfrak{W}|}{I} + \tfrac{2}{3}(5 - 3\gamma)\Omega^2, \tag{1}$$

where $\mathfrak{W}$ is the gravitational potential energy and

$$I = \int_{\mathfrak{R}} \rho(x) \, |x|^2 dx \tag{2}$$

(where the integration is effected over the domain $\mathfrak{R}$ occupied by the fluid) is the moment of inertia about the center of mass. (Note that in eq. [1] both $\mathfrak{W}$ and I refer to the *rotating* configuration and therefore include, implicitly, terms of order Ω^2.)

[1] In case γ is not a constant, the condition is replaced by $\bar{\gamma} > \frac{4}{3}$, where $\bar{\gamma}$ is a pressure weighted average (cf. § IV*a*, however) In this introductory section, we shall restrict our remarks to the case γ = constant; the restriction will not be made in the later analytical developments.

While Ledoux originally derived his formula from an application of the scalar form of the virial theorem,[2] it was soon shown by Cowling and Newing (1949) that the formula equally follows from a variational expression for σ^2 with the *same* linear substitution for the Lagrangian displacement $\boldsymbol{\xi}$ (in terms of which σ^2 is expressed) that was made by Ledoux. And since in the limit $\gamma \rightarrow \frac{4}{3}$, the correct proper solution for $\boldsymbol{\xi}$ is indeed a linear function of the coordinates (cf. Rosseland 1949, p. 20, remarks following eq. [3.14]), it follows that *the formula*

$$\sigma^2 = (3\gamma - 4)\frac{|\mathfrak{W}_0|}{I_0} + \tfrac{2}{3}\Omega^2 + O(\Omega^4) \quad [\gamma - \tfrac{4}{3} = O(\Omega^2)] \tag{3}$$

(where $\mathfrak{W}_0$ and I_0 now refer to the *non-rotating* configuration) *is an exact one.* The critical value of γ for marginal stability is therefore

$$\gamma_c(\text{Rot.}) = \tfrac{4}{3} - \frac{2\Omega^2 I_0}{9|\mathfrak{W}_0|} + O(\Omega^4). \tag{4}$$

The result (4) is of importance when effects besides rotation alter the value of γ_c from its "classical value" $\frac{4}{3}$. Thus, it is known that the post-Newtonian effects of general relativity have a destabilizing effect on radial pulsations and that (cf. Chandrasekhar 1964*b*)

$$\gamma_c(\text{G.R.}) = \tfrac{4}{3} + K\frac{R_S}{R} + O\left(\frac{R_S{}^2}{R}\right), \tag{5}$$

where K is a certain calculable constant, $R_S = 2GM/c^2$ is the Schwarzschild radius, and R is the radius for the configuration. Now if the effects of general relativity and of rotation are both present, and are both considered as first-order effects, then under their combined influence we must have

$$\gamma_c(\text{Rot.} + \text{G.R.}) = \tfrac{4}{3} + K\frac{R_S}{R} - \frac{2\Omega^2 I_0}{9|\mathfrak{W}_0|} + O\left(\Omega^4, \Omega^2\frac{R_S}{R}, \frac{R_S{}^2}{R^2}\right). \tag{6}$$

This last result is the essential content of some recent papers by Fowler (1966) and Durney and Roxburgh (1967).

While the stabilizing effect of rotation in the limit $\gamma \rightarrow \frac{4}{3}$ is an unambiguous result, it is not clear how small $\gamma - \frac{4}{3}$ must be to be of "order Ω^2" in a given practical situation. For the destabilizing effect of the distortion of the spherical distribution is the more dominating effect for centrally condensed configurations when γ is not too close to the value $\frac{4}{3}$ (see Table 1 in § IV*c* below). And it is pertinent to observe in this connection that the stabilizing contribution $\frac{2}{3}\Omega^2$ in equation (3) represents in turn a somewhat delicate balance between the stabilizing contribution ($+8\Omega^2/3$) derived from the requirement of the conservation of angular momentum and the destabilizing contribution ($-2\Omega^2$) derived from the distortion of the configuration (see § IV*b* below). It is natural to wonder whether this balance might not be upset when the distortion ceases to be of order Ω^2. The question can be stated differently. Consider the variation of γ_c along a sequence of configurations of increasing Ω^2. We know that $\gamma_c(\Omega^2)$, along such a sequence, has a negative slope at $\Omega^2 = 0$. What is the behavior of the curve $\gamma_c(\Omega^2)$ as Ω^2 increases? Does it always remain below[3] $\gamma = \frac{4}{3}$? The present paper is the first of two devoted to a consideration of these questions.

[2] For a somewhat more complete treatment, along the same lines but without any restriction on Ω^2, based on the tensor form of the virial theorem see Chandrasekhar and Lebovitz (1962*a*, § VII); and for the relation of *this* treatment to a variational formulation see Clement (1964, § III).

[3] It is known that for compressible Maclaurin spheroids this is the case (Chandrasekhar and Lebovitz 1962*c*, Fig. 1). But these configurations are, of course, atypical since their mass distribution is highly unrealistic.

The plan of this paper is as follows. In § II, we formulate and discuss the variational principle appropriate for axisymmetric oscillations of a uniformly rotating axisymmetric configuration. In § III we introduce a class of trial functions that appears specially convenient for the present problem. The formulae of § III are applied in § IV to some familiar problems; and it is shown how they lead to results in agreement with those obtained by other methods. In § V we introduce a coordinate system that enables the reduction of the various multiple integrals appearing in the variational expression to simple quadratures; and in §§ VI, VII, and VIII we carry out the necessary reductions for some special choices of equilibrium stratification that we intend to consider in our second paper.

II. THE VARIATIONAL PRINCIPLE

In this section we shall show that the characteristic value problem for axisymmetric perturbations of an axisymmetric uniformly rotating configuration can be put in the standard Rayleigh-Ritz form[4] (see also Lynden-Bell and Ostriker 1967) just as in the absence of rotation (cf. Chandrasekhar 1964*a*).

The equation governing equilibrium is

$$\text{grad}\, p = \rho\, \text{grad}\, U\,, \tag{7}$$

where p is the pressure, ρ is the density, and

$$U = \mathfrak{V} + \tfrac{1}{2}\Omega^2(x_1^2 + x_2^2)\,. \tag{8}$$

In equation (8) $\mathfrak{V}$ denotes the gravitational potential and it has been assumed that the rotation is about the x_3-axis. Equation (7) implies that surfaces of constant p, constant ρ, and constant U all coincide. Accordingly, we may now write

$$p = p(U) \qquad \text{and} \qquad \rho = \rho(U)\,. \tag{9}$$

The linear equations governing small perturbations of such an equilibrium configuration are (cf. Clement 1965; Lebovitz 1967)

$$\rho\left(\frac{\partial^2 \boldsymbol{\xi}}{\partial t^2} + 2\boldsymbol{\Omega}\times\frac{\partial \boldsymbol{\xi}}{\partial t}\right) = \boldsymbol{L}[\boldsymbol{\xi}]\,, \tag{10}$$

where $\boldsymbol{\xi}$ is the Lagrangian displacement and the operator $\boldsymbol{L}$ is defined by

$$\boldsymbol{L}[\boldsymbol{\xi}] = -\text{grad}\,\Delta p + \Delta\rho\,\text{grad}\,U + \rho\,\text{grad}\,\Delta U\,, \tag{11}$$

where Δp, $\Delta\rho$, and ΔU are the Lagrangian changes in the respective variables resulting from the displacement $\boldsymbol{\xi}$. These changes are given by

$$\Delta\rho = -\rho\,\text{div}\,\boldsymbol{\xi}, \qquad \Delta p = -\gamma p\,\text{div}\,\boldsymbol{\xi}\,, \tag{12}$$

and

$$\Delta U = \delta\mathfrak{V} + \boldsymbol{\xi}\cdot\text{grad}\,U\,. \tag{13}$$

In equation (13), $\delta\mathfrak{V}$ represents the *Eulerian change*[5] in $\mathfrak{V}$ and is given by

$$\delta\mathfrak{V} = G\int_{\mathfrak{R}} \rho(\boldsymbol{x}')\,\xi_j(\boldsymbol{x}',t)\frac{\partial}{\partial x_j'}\frac{1}{|\boldsymbol{x}-\boldsymbol{x}'|}\,d\boldsymbol{x}'\,. \tag{14}$$

[4] One of us (N. R. L.) wishes to thank Dr. James Bardeen for a useful conversation on this point.

[5] Quite generally the operations Δ and δ leading to the Lagrangian and the Eulerian changes, respectively, are related by

$$\Delta = \delta + \boldsymbol{\xi}\cdot\text{grad}\,.$$

It can be proved in great generality (cf. Clement 1964; also Lynden-Bell and Ostriker 1967) that the operator $\boldsymbol{L}$ is *symmetric*, i.e., for any smooth functions $\boldsymbol{\xi}$ and $\boldsymbol{\eta}$,

$$\int_{\mathfrak{R}} \boldsymbol{\eta} \cdot \boldsymbol{L}[\boldsymbol{\xi}]\, dx = \int_{\mathfrak{R}} \boldsymbol{\xi} \cdot \boldsymbol{L}[\boldsymbol{\eta}]\, dx . \tag{15}$$

(It is this symmetry of $\boldsymbol{L}$ that insures the general variational formulation of the underlying characteristic value problem.)

We now suppose that the perturbation considered is also axisymmetric. In cylindrical polar coordinates ϖ, $z(=x_3)$, and φ, the assumption of axisymmetry means that the components ξ_ϖ, ξ_z, and ξ_φ are all independent of φ. Two immediate consequences of this assumption are (1) that *ξ_φ does not occur in $\boldsymbol{L}[\boldsymbol{\xi}]$ and* (2) *that the φ-component of $\boldsymbol{L}$ vanishes.* The latter consequence implies, according to equation (10), that

$$\frac{\partial \xi_\varphi}{\partial t} + 2\Omega \xi_\varpi = f(\varpi, z) , \tag{16}$$

where we have integrated with respect to t, and $f(\varpi,z)$ is the "constant" of integration. As one can readily verify, equation (16) expresses the conservation of the z-component of the angular momentum per unit mass. In what follows we shall set $f(\varpi,z) = 0$. For the present, we shall justify this assumption by observing that, for a normal mode for which

$$\boldsymbol{\xi}(x,t) = \boldsymbol{\xi}(x) e^{i\sigma t} , \tag{17}$$

where σ denotes a characteristic frequency, $f(\varpi,z)$ necessarily vanishes as long as $\sigma \neq 0$ (for a further justification see below).

Considering next the ϖ-component of either side of equation (10), we have

$$\rho \left(\frac{\partial^2 \xi_\varpi}{\partial t^2} - 2\Omega \frac{\partial \xi_\varphi}{\partial t} \right) = L_\varpi[\boldsymbol{\xi}] , \tag{18}$$

or eliminating ξ_φ with the aid of equation (16) (with $f = 0$), we have

$$\rho \frac{\partial^2 \xi_\varpi}{\partial t^2} = L_\varpi[\boldsymbol{\xi}] - 4\rho\Omega^2 \xi_\varpi . \tag{19}$$

The z-component of either side of equation (10) gives

$$\rho \frac{\partial^2 \xi_z}{\partial t^2} = L_z[\boldsymbol{\xi}] . \tag{20}$$

Since neither $L_\varpi[\boldsymbol{\xi}]$ nor $L_z[\boldsymbol{\xi}]$ involves ξ_φ, equations (19) and (20) contain no reference to ξ_φ: it has effectively been eliminated. For this reason it is convenient to *redefine $\boldsymbol{\xi}$ to be the two-component vector*

$$\boldsymbol{\xi} = \xi_\varpi \mathbf{1}_\varpi + \xi_z \mathbf{1}_z , \tag{21}$$

and this redefinition underlies the rest of this paper. As a consequence $\boldsymbol{\xi}$ no longer defines the complete three-component Lagrangian displacement. To obtain the latter, we must add to $\boldsymbol{\xi}$ the vector $\xi_\varphi \mathbf{1}_\varphi$, where ξ_φ is to be determined in terms of ξ_ϖ with the aid of equation (16) (with f set equal to zero).

With $\boldsymbol{\xi}$ redefined in the manner (21), we may formally combine equations (19) and (20) into the single equation

$$\rho \frac{\partial^2 \boldsymbol{\xi}}{\partial t^2} = \mathfrak{L}[\boldsymbol{\xi}] , \tag{22}$$

where $\mathfrak{L}$ is now an operator acting on vectors in the $(\mathbf{1}_\varpi, \mathbf{1}_z)$-space and whose components are given by

$$\mathfrak{L}_\varpi[\boldsymbol{\xi}] = L_\varpi[\boldsymbol{\xi}] - 4\rho\Omega^2\xi_\varpi$$

and (23)[6]

$$\mathfrak{L}_z[\boldsymbol{\xi}] = L_z[\boldsymbol{\xi}] .$$

The symmetry of this operator $\mathfrak{L}$ follows trivially from that of $\boldsymbol{L}$.

With a dependence of $\boldsymbol{\xi}$ on time of the form (17), equation (22) leads to a characteristic value problem associated with the equation

$$-\sigma^2\rho\boldsymbol{\xi} = \mathfrak{L}[\boldsymbol{\xi}] . \tag{24}$$

From the symmetry of the operator $\mathfrak{L}$ it follows that the formula

$$\sigma^2 = -\frac{\int_{\mathfrak{R}} \boldsymbol{\xi}\cdot\mathfrak{L}[\boldsymbol{\xi}]\,d\boldsymbol{x}}{\int_{\mathfrak{R}} \rho\,|\boldsymbol{\xi}|^2 d\boldsymbol{x}} = -\frac{\int_{\mathfrak{R}} \{\boldsymbol{\xi}\cdot\boldsymbol{L}[\boldsymbol{\xi}] - 4\rho\Omega^2\xi_\varpi^2\}\,d\boldsymbol{x}}{\int_{\mathfrak{R}} \rho\,(\xi_\varpi^2 + \xi_z^2)\,d\boldsymbol{x}} \tag{25}$$

provides a variational basis for the determination of the least characteristic value of σ^2: *it is the minimum value that the ratio appearing in equation (25) can assume for any smooth function* $\boldsymbol{\xi}$. This minimum principle for σ^2 has the important consequence that if for some admissible choice of $\boldsymbol{\xi}$

$$\int_{\mathfrak{R}} \{\boldsymbol{\xi}\cdot\boldsymbol{L}[\boldsymbol{\xi}] - 4\rho\Omega^2\xi_\varpi^2\}\,d\boldsymbol{x} > 0 , \tag{26}$$

the equilibrium configuration is unstable; for the least characteristic value σ^2 is then necessarily negative and leads to an exponentially growing mode.

That the inequality (26) is a sufficient condition for instability has been deduced directly from equation (22) and independently of the theory of characteristic values by Laval, Mercier, and Pellat (1965). Their result is important, from the mathematical point of view, in our present context for two reasons: *first*, the theory in terms of which we have argued has not, to the authors' knowledge, been rigorously established in sufficient generality to apply to the problem being considered; and *second*, since a solution of equation (22) of unstable type leads to a solution of equation (10), also of unstable type, through the simple expedient of choosing an initial value for $\partial\xi_\varphi/\partial t$ so as to make f vanish in equation (16), we are justified in setting f equal to zero as long as we are interested only in establishing sufficient conditions for instability.

III. THE FORM OF THE VARIATIONAL EXPRESSION FOR σ^2 FOR A SPECIAL CHOICE OF $\boldsymbol{\xi}$

In using the variational expression for σ^2 given in equation (25), we must exercise judgment in the choice of suitable trial functions for $\boldsymbol{\xi}$: they must satisfy the requirements which experience and physical considerations suggest as necessary; at the same time they must make the evaluation of the various integrals as simple as possible.

Now the principal obstacle to the requirement of simplicity is the presence of the term in $\delta\mathfrak{B}$ in the operator $\boldsymbol{L}$; for $\delta\mathfrak{B}$ must be determined in terms of the chosen $\boldsymbol{\xi}$, either from equation (14) or, equivalently, from a solution of Poisson's equation,

$$\nabla^2\delta\mathfrak{B} = -4\pi G\delta\rho = 4\pi G \operatorname{div}(\rho\boldsymbol{\xi}) , \tag{27}$$

[6] Note that under the assumption of axisymmetry $\boldsymbol{L}$, like $\mathfrak{L}$, is also an operator in the $(\mathbf{1}_\varpi, \mathbf{1}_z)$-space.

together with appropriate conditions on the boundary $\partial\mathfrak{R}$ of $\mathfrak{R}$. The choice of $\boldsymbol{\xi}$ we shall presently make (eq. [34] below) is motivated, in part, by the desire to avoid this auxiliary calculation.

First, we observe that equation (27) implies, quite generally, that

$$\text{grad}\,\delta\mathfrak{B} = 4\pi G\rho\boldsymbol{\xi} + \text{curl}\,\boldsymbol{A} \tag{28}$$ [7]

for some vector $\boldsymbol{A}$. [Note that since the right-hand side of eq. (27) does not involve ξ_φ, and the left-hand side of eq. (28) has no φ-component, eq. (28) is in fact an equation in the $(\mathbf{1}_\varpi,\mathbf{1}_z)$-space.] The contribution to

$$\int_{\mathfrak{R}} \boldsymbol{\xi}\cdot\mathfrak{L}[\boldsymbol{\xi}]\,d\boldsymbol{x}$$

by the term involving $\delta\mathfrak{B}$ is, therefore,

$$\int_{\mathfrak{R}} \rho\boldsymbol{\xi}\cdot\text{grad}\,\delta\mathfrak{B}\,d\boldsymbol{x} = 4\pi G\int_{\mathfrak{R}} \rho^2|\boldsymbol{\xi}|^2 d\boldsymbol{x} + \int_{\mathfrak{R}} \rho\boldsymbol{\xi}\cdot\text{curl}\boldsymbol{A}\,d\boldsymbol{x}\,. \tag{29}$$

The integral involving $\boldsymbol{A}$ on the right-hand side of equation (29) can be made to vanish if we choose $\boldsymbol{\xi}$ so that

$$\rho\boldsymbol{\xi} = \text{grad}\,\phi \text{ and, on } \partial\mathfrak{R},\ \phi = \phi_0 = \text{constant}; \tag{30}$$

for, then,

$$\begin{aligned}\int_{\mathfrak{R}} \text{grad}\,\phi\cdot\text{curl}\boldsymbol{A}\,d\boldsymbol{x} &= \int_{\mathfrak{R}} \text{div}\,(\phi\,\text{curl}\,\boldsymbol{A})\,d\boldsymbol{x} = \int_{\partial\mathfrak{R}} \phi\,\text{curl}\,\boldsymbol{A}\cdot d\boldsymbol{S} \\ &= \phi_0\int_{\partial\mathfrak{R}} \text{curl}\,\boldsymbol{A}\cdot d\boldsymbol{S} = 0\end{aligned} \tag{31}$$

(where $d\boldsymbol{S}$ is the vector element of area of $\partial\mathfrak{R}$). Hence for $\boldsymbol{\xi}$ of the chosen form

$$\begin{aligned}\int_{\mathfrak{R}} \rho\boldsymbol{\xi}\cdot\text{grad}\,\delta\mathfrak{B}\,d\boldsymbol{x} &= 4\pi G\int_{\mathfrak{R}} \rho^2|\boldsymbol{\xi}|^2 d\boldsymbol{x} \\ &= 4\pi G\int_{\mathfrak{R}} \rho^2(\xi_\varpi^2 + \xi_z^2)\,d\boldsymbol{x}\,;\end{aligned} \tag{32}$$

and this contribution to σ^2 (derived from the term in $\delta\mathfrak{B}$) is formally the same as in the absence of rotation.

The substitution (30) corresponds to the correct solution both when rotation is absent and also when allowance is made for it to $O(\Omega^2)$ (see § IV*b* below); it, therefore, appears as a reasonable choice for the general case even though it cannot correspond to the exact solution.

In this paper, we shall further specialize ϕ by requiring it to be a function of U only (which automatically insures the constancy of ϕ on $\partial\mathfrak{R}$); and since ρ is also a function of U (cf. eq. [9]), the assumption is equivalent to

$$\boldsymbol{\xi} = \chi(U)\ \text{grad}\ U\,. \tag{33}$$

The present assumption (33) implies that, *at every point* $\boldsymbol{x}$, $\boldsymbol{\xi}(\boldsymbol{x})$ *is normal to the level surface through* $\boldsymbol{x}$, while the original assumption (30) required this normality only on the boundary $\partial\mathfrak{R}$ of the configuration. (But as we shall see in § IV*b* even this more specialized choice for $\boldsymbol{\xi}$ is in agreement with the exact solution for the case of slow rotation, and a fortiori for zero rotation.)

[7] In the absence of rotation $\boldsymbol{A} = 0$ (cf. Rosseland 1949).

a) The Reduction of the Variational Expression for σ^2 for the Chosen Form of the Trial Function

It is convenient to define the effective gravity g at any point; it is given by

$$g = |\text{grad } U| . \tag{34}$$

It follows from this definition that for ξ of the form (33)

$$\xi \cdot \text{grad } U = g^2 \chi , \tag{35}$$

and

$$\text{div } \xi = g^2 \chi' + \chi \nabla^2 U , \tag{36}$$

where a prime denotes differentiation with respect to the argument U. By making use of the known relation

$$\nabla^2 U = -4\pi G\rho + 2\Omega^2 , \tag{37}$$

we can rewrite the relation (36) in the form

$$\text{div } \xi = g^2 \chi' - (4\pi G\rho - 2\Omega^2)\chi . \tag{38}$$

We turn now to the reduction of the variational expression for σ^2 for the chosen form of the trial function. We have (cf. eqs. [11] and [23])

$$\int_{\mathfrak{R}} \xi \cdot \mathfrak{L}[\xi] dx = -4\Omega^2 \int_{\mathfrak{R}} \rho \xi_{\varpi}^2 dx - \int_{\mathfrak{R}} \xi \cdot \text{grad } \Delta p dx + \int_{\mathfrak{R}} \xi \cdot (\Delta\rho \text{ grad } U + \rho \text{ grad } \Delta U) dx . \tag{39}$$

With the expression for Δp given in equations (12), the second integral on the right-hand side of equation (39) becomes

$$-\int_{\mathfrak{R}} \xi \cdot \text{grad } \Delta p dx = \int_{\mathfrak{R}} \Delta p \text{ div } \xi dx = -\int_{\mathfrak{R}} \gamma p (\text{div } \xi)^2 dx , \tag{40}$$

where we have integrated by parts and used the condition that Δp vanishes on $\partial\mathfrak{R}$. Similarly, by making use of equations (12), (13), and (32), the third integral on the right-hand side of equation (39) becomes

$$\int_{\mathfrak{R}} \xi \cdot (\Delta\rho \text{ grad } U + \rho \text{ grad } \Delta U) dx = \int_{\mathfrak{R}} \rho [-(\text{div } \xi)(\xi \cdot \text{grad } U) + 4\pi G\rho |\xi|^2 + \xi \cdot \text{grad}(\xi \cdot \text{grad } U)] dx . \tag{41}$$

Now combining equations (39), (40), and (41) and making further use of the relations (34), (35), and (38), we find

$$\sigma^2 \int_{\mathfrak{R}} \rho \chi^2 g^2 dx = \int_{\mathfrak{R}} \gamma p [g^2 \chi' - (4\pi G\rho - 2\Omega^2)\chi]^2 dx - \int_{\mathfrak{R}} \rho \chi^2 \left[2(4\pi G\rho - \Omega^2) g^2 + \text{grad } U \cdot \text{grad } g^2 - 4\Omega^2 \left(\frac{\partial U}{\partial \varpi}\right)^2\right] dx . \tag{42}$$

In this form, the variational expression, besides the known pressure and density distributions in the unperturbed configuration, involves only the single scalar function χ.

There are two alternative forms of equation (42) which we shall find useful. The first of these is obtained with the help of the definition

$$C(x) = \text{div } 1_U = \text{div}\left(\frac{1}{g}\text{ grad } U\right) = \frac{1}{g^2}(g\nabla^2 U - \text{grad } U \cdot \text{grad } g). \tag{43}$$

Apart from a factor -2, C represents the mean curvature of the level surface passing through x. With the aid of equation (37) and the definition (43), we find (cf. the integrand of the second integral on the right-hand side of eq. [42])

$$2g^2(4\pi G\rho - \Omega^2) + \text{grad } U \cdot \text{grad } g^2 = 2g^2(\Omega^2 - \nabla^2 U) + \text{grad } U \cdot \text{grad } g^2 = 2g^2\Omega^2 - 2g^3 C = 2\Omega^2\left[\left(\frac{\partial U}{\partial \varpi}\right)^2 + \left(\frac{\partial U}{\partial z}\right)^2\right] - 2g^3 C. \tag{44}$$

Inserting this last relation in equation (42), we obtain

$$\sigma^2\int_{\mathfrak{R}}\rho\chi^2 g^2 dx = \int_{\mathfrak{R}}\gamma p[g^2\chi' - (4\pi G\rho - 2\Omega^2)\chi]^2 dx + 2\int_{\mathfrak{R}}\rho\chi^2 g^3 C dx + 2\Omega^2\int_{\mathfrak{R}}\rho\chi^2\left[\left(\frac{\partial U}{\partial \varpi}\right)^2 - \left(\frac{\partial U}{\partial z}\right)^2\right]dx. \tag{45}$$

The second form of equation (42) which we shall need is obtained by noting that if ρ vanishes on $\partial\mathfrak{R}$,

$$\int_{\mathfrak{R}}\rho\chi^2 \text{ grad } U \cdot \text{grad } g^2 dx = -\int_{\mathfrak{R}}[\rho\chi^2 g^2\nabla^2 U + g^2 \text{ grad } U \cdot \text{grad}(\rho\chi^2)]dx = \int_{\mathfrak{R}}\left[\rho\chi^2 g^2(4\pi G\rho - 2\Omega^2) - g^4\frac{d}{dU}(\rho\chi^2)\right]dx \tag{46}$$

and hence

$$\sigma^2\int_{\mathfrak{R}}\rho\chi^2 g^2 dx = \int_{\mathfrak{R}}\gamma p[\chi' g^2 - (4\pi G\rho - 2\Omega^2)\chi]^2 dx + \int_{\mathfrak{R}} g^4\frac{d}{dU}(\rho\chi^2)dx - 4\int_{\mathfrak{R}}\rho\chi^2\left[(3\pi G\rho - \Omega^2)g^2 - \Omega^2\left(\frac{\partial U}{\partial \varpi}\right)^2\right]dx. \tag{47}$$

IV. APPLICATIONS TO PREVIOUSLY CONSIDERED STRATIFICATIONS

In this section we shall apply the formulae of § III to the non-rotating and the slowly rotating cases, and also to spheroidally stratified configurations.

a) The Spherical Case: $\Omega = 0$

In this case the following formulae hold:

$$g = -\frac{d\mathfrak{V}}{dr}, \quad \frac{dp}{dr} = -g\rho, \quad \frac{1}{r^2}\frac{d}{dr}(r^2 g) = 4\pi G\rho, \quad \chi' = -\frac{1}{g}\frac{d\chi}{dr}, \quad \text{and} \quad C = -\frac{2}{r}. \tag{48}$$

Also the radial Lagrangian displacement, ξ_r, is given by

$$\xi_r = -g\chi \,. \tag{49}$$

Inserting these relations in equation (45), letting

$$\psi = r^2\xi_r \,, \tag{50}$$

and simplifying, we readily obtain the formula

$$\sigma^2\int_0^R \rho\psi^2 \frac{dr}{r^2} = \int_0^R \left[\gamma p \left(\frac{d\psi}{dr}\right)^2 + \frac{4}{r}\frac{dp}{dr}\psi^2\right]\frac{dr}{r^2}, \tag{51}$$

where R denotes the radius of the configuration. Equation (51) is one of the standard forms in which the variational principle for the radial oscillations of a spherical mass is generally expressed (cf. Chandrasekhar 1964*a*, eq. [49]; also Ledoux and Walraven 1958).

An alternative form of equation (51), which we obtain by integrating by parts the term in dp/dr in equation (51), is

$$\sigma^2\int_0^R \rho\psi^2 \frac{dr}{r^2} = \int_0^R (\gamma - \tfrac{4}{3}) p \left(\frac{d\psi}{dr}\right)^2 \frac{dr}{r^2} + 12\int_0^R p \left(\psi - \tfrac{1}{3} r \frac{d\psi}{dr}\right)^2 \frac{dr}{r^4}. \tag{52}$$

In view of the minimum character of the underlying variational principle (see remarks following eq. [25]), the least proper value of σ^2 must be less than *any* value given by equation (52) so long as the chosen ξ_r (i.e., ψ) is smooth. And with the particular choice

$$\psi = r^3 \qquad \text{or} \qquad \xi_r = r \,, \tag{53}$$

we obtain the inequality

$$\sigma^2 \le \frac{9\int_0^R (\gamma - \frac{4}{3}) p r^2 dr}{\int_0^R \rho r^4 dr} = 9(\bar\gamma - \tfrac{4}{3})\frac{\int_0^R p r^2 dr}{\int_0^R \rho r^4 dr} = 3(\bar\gamma - \tfrac{4}{3})\frac{|\mathfrak{W}|}{I}, \tag{54}$$

where, in the latter two forms, $\bar\gamma$ denotes the pressure weighted average of γ. From the inequality (54) it follows that *a sufficient condition for instability is that* $\bar\gamma < \frac{4}{3}$. This result is of course well known: the basic formula, in precisely the form given in equation (54), occurs in one of Ledoux's early papers (1946, eq. [6]).[8]

A slightly sharper result (than the one we have stated) can be deduced from equation (52) in case γ is not constant. Writing

$$\psi = r^3 + \epsilon\phi(r) \tag{55}$$

(where ϵ will be assumed small), we find from equation (52) that

$$\sigma^2\int_0^R \rho\psi^2 \frac{dr}{r^2} = 9(\bar\gamma - \tfrac{4}{3})\int_0^R p r^2 dr + 6\epsilon\int_0^R (\gamma - \tfrac{4}{3}) p \frac{d\phi}{dr} dr + O(\epsilon^2). \tag{56}$$

If, for the sake of simplicity, we suppose that $\bar\gamma = \frac{4}{3}$, we easily see that the right-hand side of equation (56) can be made negative by appropriate choices of $\phi(r)$ and of ϵ, the "appropriateness" depending on the precise nature of the variability of γ. In any event, it follows that *when γ is not a constant and $\bar\gamma = \frac{4}{3}$, the configuration is already dynamically unstable.*

[8] It should also be noted that Ledoux made use of the relation (54) (with the equality sign) to estimate the extent of the regions in which γ may take different constant values ($\gamma < \frac{4}{3}$ in some regions and $\gamma > \frac{4}{3}$ in other regions) before instability can set in.

b) The Case When Ω^2 Is Small and Effects of Order Ω^2, Only, Are Taken into Account

In this case, we write

$$f(r,\mu) = f_0(r) + \Omega^2 f_1(r,\mu) + O(\Omega^4) , \tag{57}$$

where μ is the cosine of the colatitude and f stands for any of the variables ρ, p, U, etc. The variables distinguished with the subscript "0" represent some appropriately chosen undistorted configuration (cf. Chandrasekhar 1933, or Chandrasekhar and Lebovitz 1962*d*, § II). In the present approximation, in which all terms of order Ω^4 and higher are neglected, it will suffice to extend the range of integration, in evaluating an integral, over the volume of the undistorted spherical configuration provided only the integrand vanishes over the boundary (for an explicit proof see, e.g., Clement 1965, Appendix I). For the integrals that appear in equation (45), the required condition is met if we restrict our consideration to equilibrium configurations for which the density ρ vanishes on the boundary; and this we shall do in the present subsection.

A conclusion of some interest may be drawn immediately from equation (45) if we suppose that the trial function χ may also be decomposed in the same manner (57). In that case the last term of equation (45) makes a contribution to σ^2 of the amount

$$2\Omega^2 \frac{\int_{\mathfrak{R}} \rho\chi^2[(\partial U/\partial \varpi)^2 - (\partial U/\partial z)^2]dx}{\int_{\mathfrak{R}} \rho\chi^2[(\partial U/\partial \varpi)^2 + (\partial U/\partial z)^2]dx} . \tag{58}$$

Consistently with neglecting terms of $O(\Omega^2)$, we must suppose that in the foregoing expression ρ, U, and χ are functions of r only. Accordingly, we may write

$$\frac{\partial U}{\partial \varpi} = \frac{dU}{dr} \sin \vartheta \quad \text{and} \quad \frac{\partial U}{\partial z} = \frac{dU}{dr} \cos \vartheta . \tag{59}$$

On evaluating the expression (58) under these conditions, we obtain the value $\frac{2}{3}\Omega^2$. This stabilizing contribution, already discussed in § I, is the *only* contribution of $O(\Omega^2)$ to σ^2 if $\gamma - \frac{4}{3}$ is small enough. It is therefore of interest to note that this stabilizing contribution is made up of two parts. Thus, tracing the effect of the conservation of angular momentum through the equations, we find that its contribution is (cf. eq. [42])

$$\frac{4\Omega^2 \int_{\mathfrak{R}} \rho\chi^2(\partial U/\partial \varpi)^2 dx}{\int_{\mathfrak{R}} \rho\chi^2[(\partial U/\partial \varpi)^2 + (\partial U/\partial z)^2]dx} = \tfrac{8}{3}\Omega^2 ; \tag{60}$$

and the remaining contribution $-2\Omega^2$ may be traced to the distortion of the equilibrium configuration and the perturbation of the gravitational potential. The particular coefficients $\frac{8}{3}$ and -2 that we find here are due to the replacement of ρ, U, and χ by functions of r only. If the configuration is strongly distorted, and these replacements become invalid, it may be that this stabilizing influence is strongly modified or even destroyed. Reference to this possibility has already been made in § I.

Returning to equation (57), we next observe that $f_1(r,\mu)$ will normally have the further decomposition

$$f_1(r,\mu) = f_{10}(r) + f_{12}(r)P_2(\mu) , \tag{61}$$

corresponding to the fact that, in the present approximation, the effect of the rotation is a P_2-deformation superposed on a uniform expansion (cf. Milne 1923). We shall adopt the same decomposition (61) for the trial function χ, as well.

We have already remarked that, under the assumption that ρ vanishes on the boundary, it is legitimate to extend all integrations only over the undistorted sphere. Under these circumstances, the coefficients ρ_{12}, p_{12}, χ_{12}, etc., do not contribute to any of the integrals. And since we shall prescribe χ_0 (for example, identify it with the proper solution belonging to σ_0^2, the characteristic value in the absence of rotation), only χ_{10} is left at our disposal in the selection of a trial function. In fact, it may appear that we do not have a choice even in the selection of χ_{10}, since we have also required χ to be a function of U only. We shall show that it is nevertheless legitimate to regard χ_{10} as at our disposal.

The condition, that χ be a function of U only, is

$$\frac{\partial \chi}{\partial r}\frac{\partial U}{\partial \mu}-\frac{\partial \chi}{\partial \mu}\frac{\partial U}{\partial r}=0\,. \tag{62}$$

On substituting for χ and U in accordance with equations (57) and (61), we obtain

$$\Omega^2\left(\frac{d\chi_0}{dr}+\Omega^2\frac{\partial \chi_1}{\partial r}\right)\frac{\partial U_1}{\partial \mu}-\Omega^2\left(\frac{dU_0}{dr}+\Omega^2\frac{\partial U_1}{\partial r}\right)\frac{\partial \chi_1}{\partial \mu}=0\,. \tag{63}$$

Neglecting terms of $O(\Omega^4)$, as we must, and substituting for χ_1 and U_1 in accordance with equation (61), we are left with

$$\frac{d\chi_0}{dr}U_{12}=\frac{dU_0}{dr}\chi_{12}\,. \tag{64}$$

Equation (64) uniquely specifies χ_{12} if χ_0 is prescribed; but χ_{10} is left entirely arbitrary.

We now return to equation (42) and simplify the right-hand side of this equation correctly to $O(\Omega^2)$ when the functions representing the various variables have the decompositions assumed.

To facilitate comparison with the non-rotating case, we shall write (cf. eqs. [49] and [50])

$$\chi_0=\frac{\psi_0(r)}{r^2 dU_0/dr}\,. \tag{65}$$

Since the coefficients of $P_2(\mu)$ in the expansion of the various quantities do not contribute to any of the integrals, it will suffice to include only the f_{10}-terms in writing out the expansions of the various quantities to $O(\Omega^2)$. Thus, we find

$$C=-\frac{2}{r}+\ldots\,, \tag{66}$$

and

$$\chi^2 g^2=\frac{\psi_0^2}{r^4}+2\Omega^2\frac{\psi_0 q}{r^4}+\ldots\,, \tag{67}$$

where

$$q=\frac{dU_{10}/dr}{dU_0/dr}\psi_0+r^2\frac{dU_0}{dr}\chi_{10}\,. \tag{68}$$

In equations (66) and (67) the dots stand for those terms of order Ω^2 that occur with the factor $P_2(\mu)$; and for the reasons we have stated, we have not written out these terms.

Turning now to the reduction of equation (45) and considering the first integral on the right-hand side, we find after a straightforward calculation,

$$\int_{\mathfrak{R}}\gamma p\left[g^2\frac{d\chi}{dU}-(4\pi G\rho-2\Omega^2)\chi\right]^2 dx=\int_{\mathfrak{R}}\gamma p\left(\frac{d\psi_0}{dr}\right)^2\frac{dx}{r^4}+2\Omega^2\int_{\mathfrak{R}}\gamma p_0\frac{d\psi_0}{dr}\frac{dq}{dr}\frac{dx}{r^4}\,. \tag{69}$$

In considering the second integral on the right-hand side of equation (45), we observe that

$$\rho g = |\operatorname{grad} p| = \left[\left(\frac{\partial p}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial p}{\partial \mu}\right)^2\right]^{1/2} = -\frac{\partial p}{\partial r} + O(\Omega^4). \tag{70}$$

Consequently, by virtue of equations (66) and (67), we have

$$\begin{aligned} 2\int_{\mathfrak{R}} \rho g^3 C \chi^2 dx &= 4\int_{\mathfrak{R}} \frac{\partial p}{\partial r}(\psi_0^2 + 2\Omega^2\psi_0 q)\frac{dx}{r^5} \\ &= -4\int_{\mathfrak{R}} p \frac{d}{dr}\left(\frac{\psi_0^2}{r^3}\right)\frac{dx}{r^2} + 8\Omega^2\int_{\mathfrak{R}} q\psi_0 \frac{dp_0}{dr}\frac{dx}{r^5}, \end{aligned} \tag{71}$$

where we have performed an integration by parts. With the foregoing reductions, equation (45) becomes

$$\begin{aligned} \sigma^2\int_{\mathfrak{R}} \rho \chi^2 g^2 dx = &\int_{\mathfrak{R}} p\left[\gamma\left(\frac{d\psi_0}{dr}\right)^2 - 4r^2\frac{d}{dr}\left(\frac{\psi_0^2}{r^3}\right)\right]\frac{dx}{r^4} + \tfrac{2}{3}\Omega^2\int_{\mathfrak{R}} \rho_0\psi_0^2\frac{dx}{r^4} \\ &+ 2\Omega^2\int_{\mathfrak{R}}\left(\gamma p_0\frac{d\psi_0}{dr}\frac{dq}{dr} + 4q\frac{\psi_0}{r}\frac{dp_0}{dr}\right)\frac{dx}{r^4}. \end{aligned} \tag{72}$$

We can recover Ledoux's formula from equation (72). (Note that the variables ρ and p still contain terms of order Ω^2.) Putting $\psi_0 = r^3$ we obtain

$$\sigma^2\int_{\mathfrak{R}} \rho\chi^2 g^2 dx = 9\int_{\mathfrak{R}}(\gamma - \tfrac{4}{3})p\,dx + \tfrac{2}{3}\Omega^2\int_{\mathfrak{R}}\rho_0 r^2 dx + 2\Omega^2\int_{\mathfrak{R}}(3\gamma - 4)p_0\frac{dq}{dr}\frac{dx}{r^2}, \tag{73}$$

after an integration by parts in the third integral on the right-hand side. Since χ_{10} is still at our disposal, we may choose it in such a way that q vanishes. By equation (68), we accomplish this with the choice

$$\chi_{10} = -r\frac{dU_{10}/dr}{(dU_0/dr)^2}. \tag{74}$$

With χ_{10} so chosen, equation (73) becomes (cf. also eq. [67])

$$\sigma^2\int_{\mathfrak{R}}\rho r^2 dx = 9\int_{\mathfrak{R}}(\gamma - \tfrac{4}{3})p\,dx + \tfrac{2}{3}\Omega^2\int_{\mathfrak{R}}\rho_0 r^2 dx. \tag{75}$$

And equation (75) reduces to Ledoux's formula (1) if we make use of the relation

$$3\int_{\mathfrak{R}} p\,dx = |\mathfrak{W}| - \tfrac{2}{3}I\Omega^2, \tag{76}$$

provided by the virial theorem in the case of slow rotation.

We now return to equation (72) to complete the reduction. Substituting for p and ρ their expansions according to equations (57) and (61) and writing $\sigma^2 = \sigma_0^2 + \Omega^2\sigma_2^2$, we obtain by equating the terms independent of Ω^2 and proportional to Ω^2,

$$\sigma_0^2\int_{\mathfrak{R}}\rho_0\psi_0^2\frac{dx}{r^4} = \int_{\mathfrak{R}} p_0\left[\gamma\left(\frac{d\psi_0}{dr}\right)^2 - 4r^2\frac{d}{dr}\left(\frac{\psi_0^2}{r^3}\right)\right]\frac{dx}{r^4}, \tag{77}$$

and

$$\begin{aligned} \sigma_2^2\int_{\mathfrak{R}}\rho_0\psi_0^2\frac{dx}{r^4} = &\, 2\int_{\mathfrak{R}}\left[\gamma p_0\frac{d\psi_0}{dr}\frac{dq}{dr} + \left(\frac{4}{r}\frac{dp_0}{dr} - \sigma_0^2\rho_0\right)\psi_0 q\right]\frac{dx}{r^4} \\ &+ \int_{\mathfrak{R}} p_{10}\left[\gamma\left(\frac{d\psi_0}{dr}\right)^2 - 4r^2\frac{d}{dr}\left(\frac{\psi_0^2}{r^3}\right)\right]\frac{dx}{r^4} - \sigma_0^2\int_{\mathfrak{R}}\rho_{10}\psi_0^2\frac{dx}{r^4} + \tfrac{2}{3}\int_{\mathfrak{R}}\rho_0\psi_0^2\frac{dx}{r^4}. \end{aligned} \tag{78}$$

If we now assume that ψ_0 is the proper solution belonging to the characteristic value σ_0^2, then ψ_0 satisfies the Euler equation[9]

$$\frac{d}{dr}\left(\frac{\gamma p_0}{r^2}\frac{d\psi_0}{dr}\right)-\frac{4}{r^3}\frac{dp_0}{dr}\psi_0=-\sigma_0^2\frac{\rho_0\psi_0}{r^2}. \tag{79}$$

Equation (79) not only implies equation (77) but also implies (as one can readily verify) that the first term on the right-hand side of equation (78) vanishes for arbitrary q.[10] We therefore obtain

$$\sigma_2^2\int_0^R \rho_0\psi_0^2\frac{dr}{r^2}=\int_0^R p_{10}\left[\gamma\left(\frac{d\psi_0}{dr}\right)^2-4r^2\frac{d}{dr}\left(\frac{\psi_0^2}{r^3}\right)\right]\frac{dr}{r^2} -\sigma_0^2\int_0^R \rho_{10}\psi_0^2\frac{dr}{r^2}+\tfrac{2}{3}\int_0^R \rho_0\psi_0^2\frac{dr}{r^2}. \tag{80}$$

We notice the remarkable fact that this formula does not contain χ_{10}. Consequently to find the coefficient σ_2^2, one needs to know only the distortion of the equilibrium configuration (i.e., p_{10} and ρ_{10}) and the proper solution ψ_0 appropriate to the undistorted configuration. For the special case of polytropes, a formula equivalent to (80) was derived by Clement (1965, eq. [65]).

The present formulation of the variational principle with the trial function selected according to equation (33) is now seen to imply no loss of generality, since it leads to equation (80) which, in view of Clement's work, is exact.

c) The Evaluation of σ_2^2 for Distorted Polytropes

As we have already stated, the form which equation (80) takes for polytropes was derived by Clement. Using his formula, we have evaluated σ_2^2 for polytropic indices $n = 1.5$, 2.0, and 3.0 and for various values of γ. The results are listed in Table 1. We see that σ_2^2 becomes negative when γ is sufficiently different from $\frac{4}{3}$; and also that the value of γ at which σ_2^2 changes sign decreases with increasing n. We have referred to these facts in § I. However, it should be noted that there is some ambiguity in the comparison implied by the listings in Table 1, namely, that the characteristic frequencies of oscillation of a non-rotating and a rotating polytrope having the same parameters, n, ρ_c (central density), and K (the constant of proportionality in the pressure-density relation), are strictly comparable. It is by no means obvious that this is the case: the two configurations that are "compared" have, for example, different masses and different volumes (cf. Chandrasekhar and Lebovitz 1962*d*, § IX).

d) The Compressible Maclaurin Spheroid

The pulsation frequency σ^2 and the critical value γ_c of the ratio of specific heats for marginal stability have been worked out exactly for this model (Chandrasekhar and Lebovitz 1962*c*). This model therefore provides a useful test case for the application of equation (45).

The equilibrium configuration is described by the solution

$$p=\rho U \quad \text{and} \quad U=\kappa\left(1-\frac{\varpi^2}{a_1^2}-\frac{z^2}{a_3^2}\right), \tag{81}$$

where

$$\kappa=\pi G\rho a_1^2 a_3^3 A_3 \tag{82}$$

[9] Eddington's pulsation equation in this context.

[10] In fact the integral in question is proportional to the first variation of σ_0^2 as defined by eq. (77).

and the notation is that of Chandrasekhar and Lebovitz (1962*b*). With this solution, we readily find that

$$\operatorname{grad} U = -2\kappa\left(\frac{\varpi}{a_1^2}\,1_\varpi + \frac{z}{a_3^2}\,1_z\right), \qquad g = 2\kappa\left(\frac{\varpi^2}{a_1^4}+\frac{z^2}{a_3^4}\right)^{1/2},$$

and

$$C = -\left(\frac{\varpi^2}{a_1^4}+\frac{z^2}{a_3^4}\right)^{-3/2}\left[\left(\frac{1}{a_1^2}+\frac{1}{a_3^2}\right)\frac{\varpi^2}{a_1^4}+\frac{2}{a_1^2}\frac{z^2}{a_3^4}\right]. \tag{83}$$

And as a suitable trial function we shall choose

$$\chi = \text{constant} \tag{84}$$

since, in the absence of rotation, this choice leads to the exact solution. With the foregoing substitutions, we find from equation (45), after elementary integrations,

$$\sigma^2 = \frac{2\kappa}{a_1^2}\left[\left(2+\frac{a_1^2}{a_3^2}\right)\gamma - 4\,\frac{2a_1^2+a_3^2}{a_1^2+2a_3^2}\right] + 2\Omega^2\,\frac{2a_3^2-a_1^2}{2a_3^2+a_1^2}, \tag{85}$$

or, in terms of the eccentricity e of the meridional section, we have

$$\frac{\sigma^2}{\pi G\rho} = \left[2(3-2e^2)\gamma - 8\,\frac{(3-e^2)(1-e^2)}{3-2e^2}\right]A_3 + 2\,\frac{\Omega^2}{\pi G\rho}\,\frac{1-2e^2}{3-2e^2}. \tag{86}$$

For small values of e, equation (86) gives

$$\frac{\sigma^2}{\pi G\rho} = 4(\gamma - \tfrac{4}{3}) + \tfrac{16}{45}(5-3\gamma)\,e^2 + O(e^4); \tag{87}$$

this agrees with the exact formula (and with eq. [1]) through terms of $O(\Omega^2)$.

TABLE 1

VALUES OF σ_0^2 AND σ_2^2 FOR POLYTROPES*

γ	σ_0^2	σ_2^2
	$n=1\ 5$	
1 35.	0 007755	+0 30985
1 375	019334	+0 27430
1 40	030850	+0 23837
1 45	053710	+0 16546
1 55..	098812	+0 01594
1 5605	......	0
1.60	121094	−0 06051
1 6667	0 150560	−0 16394
	$n=2\ 0$	
1 40..	0 024752	+0 21248
1 45.	.042805	+0 11626
1 5076	....	0
1 55	.077773	−0 08913
1 60	094757	−0 19761
1 6667	0 116956	−0 34747
	$n=3\ 0$	
1 35.	0 004010	+0 27697
1 375	.009755	+0 18085
1 40	015182	+0 06946
1 4140	....	0
1 45	.025110	−0 20294
1 50	033894	−0 54011
1 55	041680	−0 92946
1 60	048642	−1 35145
1 65	054951	−1 78510
1 6667	056935	−1 93002
1 70	0 060755	−2 21784

* The values of σ_0^2 listed are those derived by Roberts (1963) by a direct numerical integration of the pulsation equation; and his "exact" proper solutions were used in the evaluation of σ_2^2.

Values of σ^2 and γ_c, calculated with the aid of equation (86), are listed in Tables 2 and 3 and are further compared with the exact values given in Chandrasekhar and Lebovitz (1962*c*, Tables 1 and 2A). The agreement is satisfactory and encourages the hope that the chosen formulation of the variational principle will give equally satisfactory results in other cases as well.

TABLE 2

THE EXACT AND THE APPROXIMATE VALUES OF σ^2 FOR THE COMPRESSIBLE MACLAURIN MODEL

(σ^2 Is Measured in the Unit $\pi G\rho$)

e	σ^2 (exact)	σ^2 (approx)	σ^2 (exact)	σ^2 (approx)	σ^2 (exact)	σ^2 (approx.)
	$\gamma=1$ 3		$\gamma=1$ 33 . . .		$\gamma=1.4$	
0 0.	−0 13322	−0 1332	0 00	0 00	0 26688	0 2669
05	− .13237	− 1324	00088	.0009	.26737	.2674
.10	− .12948	− .1294	00350	.0036	26945	.2695
.15	− .12447	− .1244	.00806	.0081	.27311	.2732
.20	− 11749	− .1174	.01440	.0145	.27817	.2783
.25	− .10839	− .1081	.02267	0230	.28477	.2851
.30	− 09708	− .0965	03295	0336	.29294	.2938
.35	− .08341	− 0823	04536	.0466	.30277	.3044
.40	− .06723	− 0651	.06004	.0625	31436	.3175
.45	−0 04829	−0 0443	.07720	.0816	.32780	.3335
.50			0 09708	0 1049	.34324	.3531
0.60		..	.	.	0 38059	0 4089
	$\gamma=1.5$		$\gamma=1.6$		$\gamma=1$ 666 . . .	
0 0.	0 66684	0 6668	1 06668	1 0667	1 33310	1 3331
.05	.66711	6671	1 06684	1 0669	1 33334	1 3334
.10	.66838	.6685	1 06654	1 0674	1 33330	1 3333
.15	67066	.6707	1 06616	1 0683	1 33353	1 3334
.20	67375	.6740	1 06561	1 0697	1 33402	1 3335
.25	.67771	.6784	1 06482	1 0717	1 33520	1 3339
.30	.68250	6840	1 06369	1 0743	1 33775	1 3345
.35	.68810	.6911	1 06204	1 0778	1 34298	1 3356
.40	.69444	.7000	1 05972	1 0826	1 35314	1 3376
.45	.70142	.7113	1 05647	1 0890	1 37178	1 3409
.50	.70891	.7255	1 05197	1 0979	1.40306	1 3462
0.60	0.72434	0.7686	1 03721	1 1283	1 51536	1.3681

TABLE 3

COMPARISON OF THE EXACT AND THE APPROXIMATE VALUES OF γ_c

e	γ_c (exact)	γ_c (approx)	e	γ_c (exact)	γ_c (approx.)
0 0	1 3333	1 3333	0 5	1.3071	1 3052
.1	1 3324	1 3324	.6	1.2920	1.2863
.2	1 3297	1 3297	.7	1.2693	1 2533
3	1 3249	1 3247	.8	1 2318	1 1848
0 4	1 3176	1 3170	0 9	1 1535	1.0009

V. THE REDUCTION TO QUADRATURES OF THE INTEGRALS IN THE VARIATIONAL EXPRESSIONS FOR σ^2

In order to facilitate the evaluation of the integrals appearing in equation (42), and in its alternative forms (45) and (47), we shall introduce a system of coordinates s, ϑ, and φ (say) such that s is constant on the level surfaces, U = constant. Since the trial function $\chi(U)$, as well as $\rho(U)$ and $p(U)$, are then independent of ϑ and φ, the integrations with respect to these variables may be performed, independently of the choice of χ, for any given (or assumed) stratification. In this manner the triple integrals, which occur in the variational expression for σ^2, can be reduced to quadratures over s and put into a form analogous to that of equation (51).

The derivation of the basic formulae (eqs. [98] and [99] below) does not require that the surfaces in question be the level surfaces of a rotating mass; so we shall avoid this terminology. Let the surface $\mathfrak{S}$, inclosing the simply connected domain $\mathfrak{R}(s)$, be specified by a smooth function $S(x,s)$ as follows:

$$\mathfrak{S}(s) = \{x: S(x,s) = 0\} ,$$

with (88)

$$\mathfrak{R}(s) = \{x: S(x,s) \leq 0\} .$$

For definiteness we shall suppose that

$$\frac{\partial S}{\partial s} < 0 \qquad \text{in } \mathfrak{R} , \tag{89}$$

where $\mathfrak{R}$ is defined by equation (88) with s set equal to its maximum value s_1 (say), i.e.,

$$\mathfrak{R} = \mathfrak{R}(s_1) . \tag{90}$$

The condition (89) implies in a natural way that $\mathfrak{S}(t_1)$ is interior to $\mathfrak{S}(t_2)$ if $t_1 < t_2$. It further implies that there is a unique surface $\mathfrak{S}$ passing through a given point x. Finally, we shall also require that there is a minimum value of s, s_0 (say), such that $S(x,s_0)$ is positive unless $x = 0$; this insures that s_0 corresponds to the origin.

Now let the position vector of a point lying on the surface $\mathfrak{S}(s)$ be given by

$$x = X(s,\vartheta,\varphi) , \tag{91}$$

where the surface parameters ϑ and φ have a domain $\mathfrak{D}(s)$ (say); as ϑ and φ range over $\mathfrak{D}(s)$, they should provide all points of $\mathfrak{S}(s)$ in a one-one fashion. With these definitions, the element of volume in $\mathfrak{R}$ is

$$|J|\, ds\, d\vartheta\, d\varphi , \tag{92}$$

where

$$J = \frac{\partial X}{\partial s} \cdot \frac{\partial X}{\partial \vartheta} \times \frac{\partial X}{\partial \varphi} \tag{93}$$

is the Jacobian of the transformation.

If we substitute from equation (91) into the equation $S(x,s) = 0$ defining $\mathfrak{S}(s)$, we obtain an identity in s, ϑ, and φ; and differentiating this identity, we find

$$\frac{\partial X}{\partial s} \cdot \text{grad}\, S = -\frac{\partial S}{\partial s} , \qquad \frac{\partial X}{\partial \vartheta} \cdot \text{grad}\, S = 0 , \qquad \text{and} \qquad \frac{\partial X}{\partial \varphi} \cdot \text{grad}\, S = 0 . \tag{94}$$

The last two of these equations imply that there exists a scalar function $\alpha(x)$ such that

$$\frac{\partial X}{\partial \vartheta} \times \frac{\partial X}{\partial \varphi} = \alpha(x)\, \text{grad}\, S . \tag{95}$$

it follows that

$$\frac{\partial S}{\partial s} = -s + \frac{e\,e'}{(1-e^2)^2}\,z^2 \qquad \left(e' = \frac{d\,e}{d\,s}\right), \tag{101}$$

$$|J| = -\frac{|(\partial \boldsymbol{X}/\partial\vartheta) \times (\partial \boldsymbol{X}/\partial\varphi)|}{|\text{grad}\, S|}\,\frac{\partial S}{\partial s}, \tag{97}$$

where the minus sign has been inserted to accord with our convention that $\partial S/\partial s$ is negative.

Returning to our principal problem, we are primarily interested in evaluating integrals whose integrands have the form $G(s)H(\boldsymbol{x})$, where H, in general, depends on ϑ and φ, as well as s (cf. eq. [42]). We now see that

$$\int_{\mathfrak{R}} G(s)\,H(\boldsymbol{x})\,d\boldsymbol{x} = \int_{s_0}^{s_1} G(s)\,W(s)\,ds\,, \tag{98}$$

where

$$W(s) = \iint_{\mathfrak{D}(s)} |J|\,H(\boldsymbol{X})\,d\vartheta d\varphi\,, \tag{99}$$

where, as the notation implies, the dependence of H on $\boldsymbol{x}$ must be replaced, by means of the transformation equation (91), by its dependence on s, ϑ, and φ.

In our present problem we may choose for φ the azimuthal angle. All the integrands are then independent of φ, and the formula (99) is accordingly simplified. We shall find in §§ VI–VIII that the evaluation of four integrals like the one on the right-hand side of equation (99) will reduce all the integrals appearing in equation (42) to single integrals over s.

VI. SPHEROIDAL STRATIFICATIONS

It is possible to show that, in a rigorous theory, a uniformly rotating configuration cannot be stratified on spheroids unless the density is everywhere the same (cf. Dive 1952). Since, however, on an approximate theory, slowly rotating configurations *are* stratified on spheroids, it is reasonable, when seeking approximate equilibrium configurations, to assume spheroidal stratification and try to find that set of spheroidal surfaces which is, in some sense, the best. This is the approach taken by Roberts (1962, 1963) and by Hurley and Roberts (1964*a*, *b*), who have constructed rapidly rotating polytropes selecting the "best" stratification on the basis of a variational principle. We shall adapt equation (47), appropriately, for a study of the oscillations of these models of Hurley and Roberts.

The function S of equations (88) may, in the present case, be written as

$$S(\varpi, z, s) = \tfrac{1}{2}\left(-s^2 + \varpi^2 + \frac{z^2}{1-e^2}\right), \tag{100}$$

where e is, in general, a function of s. The coordinate s is the equatorial radius of a given level surface and $e(s)$ is the eccentricity of its meridional section. From equation (100)

It is convenient to introduce the coordinates s, ϑ, and φ through the transformation formulae

$$x_1 = s \sin \vartheta \cos \varphi , \quad x_2 = s \sin \vartheta \sin \varphi , \quad \text{and} \quad x_3 = (1 - e^2)^{1/2} s \cos \vartheta \; ; \tag{103}$$

ϑ is then the "mean anomaly" and φ is the usual azimuthal angle.

We now calculate $|J|$ as given by equation (97). Since $e'(s) \geq 0$ in all cases of interest (cf. Hurley and Roberts 1964*b;* in the slowly rotating case, this inequality follows from Clairaut's equation) we may define, without ambiguity,

$$\beta(s) = \left(\frac{s e e'}{1 - e^2}\right)^{1/2} \qquad (e' \geq 0) . \tag{104}$$

With this definition, equation (101) takes the alternative form

$$\frac{\partial S}{\partial s} = -s(1 - \beta^2 \cos^2 \vartheta) . \tag{105}$$

Our earlier requirement (89) that $\partial S/\partial s \leq 0$ now implies that $\beta < 1$; we shall assume that this is the case. [It can be verified that the difficulty encountered by Hurley and Roberts (1964*a*) with the crossing of the level surfaces is related to a violation of this inequality.]

A straightforward calculation now gives

$$J = (1 - e^2)^{1/2} s^2 (1 - \beta^2 \cos^2 \vartheta) \sin \vartheta \; ; \tag{106}$$

and equation (99), for the stratification we are presently considering, becomes

$$W(s) = 2\pi (1 - e^2)^{1/2} s^2 \int_{-1}^{+1} (1 - \beta^2 \mu^2) H(s, \mu) d\mu , \tag{107}$$

where we have used $\mu = \cos \vartheta$ as the variable of integration.

Inspection of equation (47) now shows that to reduce the various integrals occurring in this equation to quadratures over s, it is necessary to evaluate four functions $W(s)$ with the choices

$$H_0 = 1 , \quad H_1 = \left(\frac{\partial U}{\partial \varpi}\right)^2 , \quad H_2 = g^2 = \left(\frac{\partial U}{\partial \varpi}\right)^2 + \left(\frac{\partial U}{\partial z}\right)^2 , \quad \text{and } H_3 = g^4 . \tag{108}$$

We shall denote the corresponding functions by $W_i(s)$ ($i = 0, 1, 2,$ and 3). It will appear that all these functions can be expressed in terms of elementary functions.

First, we note that since the potential U is a function of s only, we may write (to avoid ambiguity)

$$U = u(s) \; ; \tag{109}$$

and we have

$$\left(\frac{\partial U}{\partial \varpi}\right)^2 = \left(\frac{du}{ds}\right)^2 \left(\frac{\partial S}{\partial \varpi}\right)^2 \quad \text{and} \quad \left(\frac{\partial U}{\partial z}\right)^2 = \left(\frac{du}{ds}\right)^2 \left(\frac{\partial S}{\partial z}\right)^2 . \tag{110}$$

On the other hand, the derivatives of $S(\varpi, z)$ may be obtained by implicit differentiation of equation (100); we find

$$\frac{\partial S}{\partial \varpi} = \frac{(1 - \mu^2)^{1/2}}{1 - \beta^2 \mu^2} \quad \text{and} \quad \frac{\partial S}{\partial z} = \frac{\mu}{(1 - e^2)^{1/2} (1 - \beta^2 \mu^2)} . \tag{111}$$

Writing out the functions $W_i(s)$, and simultaneously defining the new functions $V_i(s)$ (for later convenience), we have

$$4\pi V_0(s) \equiv W_0(s) = 4\pi(1-e^2)^{1/2}s^2\int_0^1(1-\beta^2\mu^2)\,d\mu = 4\pi(1-e^2)^{1/2}(1-\tfrac{1}{3}\beta^2)\,s^2\,, \tag{112}$$

$$4\pi\left(\frac{du}{ds}\right)^2 V_1(s) \equiv W_1(s) = 4\pi(1-e^2)^{1/2}s^2\left(\frac{du}{ds}\right)^2\int_0^1\frac{1-\mu^2}{1-\beta^2\mu^2}\,d\mu = 4\pi\,\frac{(1-e^2)^{1/2}}{\beta^2}\left[1-\frac{1-\beta^2}{2\beta}\log\left(\frac{1+\beta}{1-\beta}\right)\right]s^2\left(\frac{du}{ds}\right)^2, \tag{113}$$

and

$$4\pi\left(\frac{du}{ds}\right)^{2(i-1)} V_i(s) \equiv W_i(s) = 4\pi(1-e^2)^{1/2}s^2\left(\frac{du}{ds}\right)^{2(i-1)} \times\int_0^1\left(1+\frac{e^2\mu^2}{1-e^2}\right)^{i-1}\frac{d\mu}{(1-\beta^2\mu^2)^{2i-3}} \qquad (i=2,3)\,. \tag{114}$$

The integrals over μ appearing in the definitions of the functions $V_2(s)$ and $V_3(s)$ are also readily evaluated, and we find

$$\int_0^1\left(1+\frac{e^2\mu^2}{1-e^2}\right)\frac{d\mu}{1-\beta^2\mu^2} = \frac{1}{\beta^2}\left[\left(\beta^2+\frac{e^2}{1-e^2}\right)f(\beta)-\frac{e^2}{1-e^2}\right],$$

and

$$\int_0^1\left(1+\frac{e^2\mu^2}{1-e^2}\right)^2\frac{d\mu}{(1-\beta^2\mu^2)^3} = \frac{1}{8\beta^4}\left\{\left[3\beta^4-\frac{2e^2}{1-e^2}\beta^2+\frac{3e^4}{(1-e^2)^2}\right]f(\beta) + \frac{1}{(1-\beta^2)^2}\left[(5-3\beta^2)\beta^4+\frac{2e^2}{1-e^2}(1+\beta^2)\beta^2+\frac{e^4}{(1-e^2)^2}(5\beta^2-3)\right]\right\}, \tag{115}$$

where

$$f(\beta) = \frac{1}{2\beta}\log\left(\frac{1+\beta}{1-\beta}\right) = \sum_{k=0}^{\infty}\frac{\beta^{2k}}{2k+1}\,. \tag{116}$$

With the foregoing reductions equation (47) finally takes the form

$$\sigma^2\int_{s_0}^{s_1}\rho\chi^2\left(\frac{du}{ds}\right)^2 V_2\,ds = \int_{s_0}^{s_1}\left(\frac{du}{ds}\right)^3\frac{d}{ds}(\rho\chi^2)V_3\,ds + \int_{s_0}^{s_1}\gamma p\left[\left(\frac{d\chi}{ds}\right)^2\left(\frac{du}{ds}\right)^2 V_3 - 2(4\pi G\rho-2\Omega^2)\frac{d\chi}{ds}\frac{du}{ds}\chi V_2 + (4\pi G\rho-2\Omega^2)^2\chi^2 V_0\right]ds - 4\int_{s_0}^{s_1}\rho\chi^2\left(\frac{du}{ds}\right)^2[(3\pi G\rho-\Omega^2)V_2-\Omega^2 V_1]\,ds\,. \tag{117}$$

This equation represents a generalization, to the case when rotation is present, of equation (51) valid in the absence of rotation. The Euler equation based on equation (117) will provide a generalization of Eddington's pulsation equation analogous to Roberts' generalization of the Lane-Emden equation (Roberts 1962, eq. [2.31]). However, we shall have no reason to write down this generalization.

VII. THE ROCHE STRATIFICATION

The spheroidal stratification considered in § VI is an extrapolation to larger rotations of the stratification that is known to be correct for slow rotation. In contrast, a stratification that will apply to all rotations, with sufficient approximation under most circumstances of practical interest, is that of Roche which one obtains when all the mass (M) is concentrated at the center. The level surfaces in this limit are given by (cf. Jeans 1929, p. 252)

$$U_{\text{Roche}} = \frac{GM}{r} + \tfrac{1}{2}\Omega^2 r^2 \sin^2 \vartheta . \tag{118}$$

While these level surfaces are strictly applicable only in the limit of infinite central condensation, they do provide a very good approximation to the true level surfaces of rotating masses of even moderate central condensations such as a polytrope of index $n = 3$. For example, the ellipticity ϵ of a polytrope $n = 3$ for slow rotation is given by (cf. Chandrasekhar 1933)

$$\epsilon = 0.7717 \frac{\Omega^2}{2\pi G \rho_m} \tag{119}$$

(where ρ_m denotes the mean density), while for the Roche surfaces

$$\epsilon = 0.75 \frac{\Omega^2}{2\pi G \rho_m} \qquad \left(\rho_m = \frac{M}{\frac{4}{3}\pi R^3}\right). \tag{120}$$

The discrepancy, for slow rotation, is thus less than 3 per cent even for the outermost boundary. The assumption that even for rapid rotation the Roche surfaces provide a very good approximation to the true level surfaces of rotating masses, with central condensations comparable to the polytrope $n = 3$, has been confirmed by Monaghan and Roxburgh (1965*a*, *b*) and by Ostriker and Mark.[11] Indeed Roxburgh, Griffith, and Sweet (1965) have made the approximation provided by the Roche surfaces the first step in a rapidly converging iteration scheme for obtaining the exact solutions under a wide variety of circumstances. It would therefore appear that, for a discussion of the oscillations and the stability of rapidly rotating masses, the reduction of equation (45) for the Roche stratification will provide a good starting point.

It is important to observe that the foregoing remarks on the good agreement of the level surfaces in rotating configurations with the *Roche surfaces* are not meant to suggest that the march of U in these configurations agrees well with the march of U in the *Roche model.* They are meant rather to point out that the level surfaces in rotating configurations (of even moderate central condensations) can be *geometrically* well approximated by the Roche surfaces so that in the same approximation we may consider U as a function of U_{Roche}, regarding the latter only as a labeling parameter. In other words, the assumption that we propose to make is

$$U = f(U_{\text{Roche}}) , \tag{121}$$

where f is some function (to be determined empirically from the integrations such as those of Ostriker and Mark).

Returning to Roche's equation (118), we first observe that the condition, that for equilibrium the effective gravity should be directed inward, requires that the equatorial radius of the outermost level surface does not exceed the critical value

$$\varpi_c = (GM/\Omega^2)^{1/3} . \tag{122}$$

[11] We are greatly indebted to Dr. J. Ostriker and Mr. J. Mark for providing us with the results of their exact numerical integrations for rapidly rotating polytropes.

By choosing ϖ_c as the unit of length and Ω as the unit of frequency, we can rewrite equation (118) in the form

$$\nu = \frac{1}{r} + \tfrac{1}{2} r^2 \sin^2 \vartheta\,, \tag{123}$$

if U_{Roche} is measured in the unit $(GM\Omega)^{2/3}$; and we suppose that U (also measured in the same unit) is some determinate function only of ν. Accordingly, the corresponding form of the function S of equations (88) is

$$S \equiv S(r, \mu, s) = \frac{1}{r} + \tfrac{1}{2} r^2 (1-\mu^2) - \nu[U(s)]. \tag{124}$$

We now write the variational expression (42) for σ^2 in the form

$$\begin{aligned}\left(\frac{\sigma}{\Omega}\right)^2 \int_{\mathfrak{R}} \rho \chi^2 g^2 dx = \int_{\mathfrak{R}} \gamma p \left[\left(\frac{d\chi}{dU}\right)^2 g^4 - 4(2Q-1)\chi \frac{d\chi}{dU} g^2 + 4(2Q-1)^2 \chi^2\right] dx \\ - \int_{\mathfrak{R}} \rho \chi^2 \left[2(4Q-1) g^2 + \text{grad}\, U \cdot \text{grad}\, g^2 - 4\left(\frac{\partial U}{\partial \varpi}\right)^2\right] dx\,,\end{aligned} \tag{125}$$

where g is measured in the unit $(GM\Omega^4)^{1/3}$, and

$$Q = \pi G \rho / \Omega^2\,; \tag{126}$$

also the unit in which p is measured is $(GM\Omega)^{2/3}$ times the unit of ρ.

Since we have supposed that U is some function of ν only,

$$\text{grad}\, U = \frac{dU}{d\nu} \text{grad}\, \nu\,, \qquad g^2 = \left(\frac{dU}{d\nu}\right)^2 |\text{grad}\, \nu|^2\,,$$

and

$$\text{grad}\, U \cdot \text{grad}\, g^2 = \left(\frac{dU}{d\nu}\right)^3 [\text{grad}\, \nu \cdot \text{grad}(|\text{grad}\, \nu|^2)] + 2\left(\frac{dU}{d\nu}\right)^2 \frac{d^2 U}{d\nu^2} |\text{grad}\, \nu|^4\,; \tag{127}$$

whereas from equation (123) we find

$$\left(\frac{\partial \nu}{\partial \varpi}\right)^2 = \frac{(1-r^3)^2}{r^4}(1-\mu^2)\,, \qquad |\text{grad}\, \nu|^2 = \frac{1}{r^4} - \left(\frac{2}{r} - r^2\right)(1-\mu^2)\,,$$

and

$$\text{grad}\, \nu \cdot \text{grad}(|\text{grad}\, \nu|^2) = \frac{4}{r^7} - 2\left(\frac{3}{r^4} + \frac{3\mu^2}{r} - r^2\right)(1-\mu^2)\,. \tag{128}$$

Further, taking $s = -\nu$ (the minus sign is needed because ν decreases outward, whereas s, by assumption, must increase outward), we find that the required Jacobian is

$$|J| = \frac{r^4(\nu, \mu)}{1 - r^3(\nu, \mu)(1-\mu^2)} = \frac{r^4(\nu, \mu)}{3 - 2\nu r(\nu, \mu)}\,, \tag{129}$$ [12]

where the latter form is obtained by making use of equation (123).

With the definitions (cf. eq. [107])

[12] Eq. (129) follows directly from a formula given in § VIII (eq. [133]) which is applicable to entirely general axisymmetric level surfaces: we have only to identify S in that formula with the surface S defined by equation (123) and let $s = -\nu$.

$$V_0(\nu)=\nu^4\int_0^1|J|\,d\mu\,,\qquad V_1(\nu)=\int_0^1|J|\,|\text{grad}\,\nu|^2 d\mu\,,$$

$$V_2(\nu)=\frac{1}{\nu^4}\int_0^1|J|\,|\text{grad}\,\nu|^4 d\mu\,,\qquad V_4(\nu)=\int_0^1|J|\left(\frac{\partial\nu}{\partial\varpi}\right)^2 d\mu\,,\tag{130}$$

[13]

$$V_3(\nu)=\frac{1}{\nu^4}\int_0^1|J|\,\text{grad}\,\nu\cdot\text{grad}(|\text{grad}\,\nu|^2)\,d\mu\,,$$

equation (24) may now be written in the form

$$\left(\frac{\sigma}{\Omega}\right)^2\int\rho\chi^2\left(\frac{dU}{d\nu}\right)^2 V_1 d\nu$$

$$=-\int\rho\chi^2\left[2(4Q-1)\left(\frac{dU}{d\nu}\right)^2 V_1+\nu^4\left(\frac{dU}{d\nu}V_3+2\,\frac{d^2U}{d\nu^2}V_2\right)\left(\frac{dU}{d\nu}\right)^2-4V_4\right]d\nu\tag{131}$$

$$+\int\gamma p\left[\left(\frac{d\chi}{d\nu}\right)^2\left(\frac{dU}{d\nu}\right)^2\nu^4V_2-4(2Q-1)\chi\,\frac{d\chi}{d\nu}\frac{dU}{d\nu}V_1+4(2Q-1)^2\chi^2\frac{V_0}{\nu^4}\right]d\nu\,,$$

where the integrations are over the appropriate range of ν. Table 4 gives V_0, V_1, V_2, V_3, and V_4 as functions of $1/\nu$.

In applying equation (129), we may substitute for $p(\nu)$ and $\rho(\nu)$ the distributions derived for configurations of at least moderate central condensations: as we have stated, the Roche stratification may be expected to provide a sufficiently good approximation for them.

VIII. GENERAL AXISYMMETRIC CONFIGURATIONS

A general axisymmetric configuration may be specified by giving the distance $R(s,\mu)$ from the center to a point on the level surface at colatitude $\vartheta=\cos^{-1}\mu$. The associated coordinate transformation (91) has the form

$$x_1=R(s,\mu)(1-\mu^2)^{1/2}\cos\varphi,\quad x_2=R(s,\mu)(1-\mu^2)^{1/2}\sin\varphi,\quad\text{and}\quad x_3=R(s,\mu)\mu\,;\tag{132}$$

and a straightforward calculation gives (cf. eq. [97])

$$J=-R^2(s,\mu)\frac{\partial S/\partial s}{|\partial S/\partial r|}\,.\tag{133}$$

We may take

$$S(r,s,\mu)=r-R(s,\mu)\,,\tag{134}$$

as the equation defining the level surfaces; but to be in agreement with our convention (89) regarding s we must assume $\partial R/\partial s>0$. With this definition of S equation (131) gives

$$|J|=R^2\frac{\partial R}{\partial s}=-R^2\frac{\partial R}{\partial U}\,,\tag{135}$$

where, as in the preceding section (see n. 12), we have taken $s=-U$. It also follows from equation (132) (with $s=-U$) that

$$\left(\frac{\partial U}{\partial\varpi}\right)^2=\frac{(R+\mu\partial R/\partial\mu)^2}{R^2(\partial R/\partial U)^2}(1-\mu^2)\qquad\text{and}\qquad g^2=\frac{R^2+(1-\mu^2)(\partial R/\partial\mu)^2}{R^2(\partial R/\partial U)^2}\,.\tag{136}$$

[13] The factor ν^4 in the definition of $V_0(\nu)$ and the factor ν^{-4} in the definitions of $V_2(\nu)$ and $V_3(\nu)$ have been inserted in order that these functions may not vary over too wide a range of values.

TABLE 4

THE WEIGHT FUNCTIONS FOR THE ROCHE STRATIFICATION

ν^{-1}	V_0	V_1	V_2	V_3	V_4
0 0.	1 000000	1 000000	1 000000	0	0 666667
.01	1 000000	0 999988	1 000008	0 040000	666667
.02	1 000001	0 999990	0 999978	0 079999	666662
.03	1 000048	0 999980	0 999912	0 119996	666646
.04	1 000125	0 999956	0 999788	0 159986	666616
05	1 000248	0 999916	0 999584	0 199967	666567
06	1 000430	0 999855	0 999281	0 239931	666494
07	1 000685	0 999771	0 998857	0 279872	666392
08	1 001024	0 999658	0 998294	0 319781	.666257
.09	1 001460	0 999513	0 997572	0 359650	666083
10	1 002004	0 999332	0 996669	0 399466	.665866
11	1 002670	0 999111	0 995568	0 439218	665600
.12	1 003470	0 998846	0 994247	0 478893	665282
13	1 004417	0 998532	0 992688	0 518474	664905
14	1 005524	0 998165	0 990872	0 557947	664465
15	1 006805	0 997742	0 988778	0 597294	.663957
16	1 008273	0 997258	0 986387	0 636495	663375
17	1 009943	0 996708	0 983682	0 675531	662715
18	1 011829	0 996089	0 980642	0 714380	661972
19	1 013948	0 995396	0 977250	0 753018	661139
20	1 016313	0 994623	0 973488	0 791422	.660211
.21	1 018944	0 993768	0 969337	0 829564	659183
22	1 021856	0 992824	0 964780	0 867419	658049
23	1 025068	0 991788	0 959801	0 904955	656804
24	1 028600	0 990654	0 954382	0 942143	655440
25	1 032472	0 989416	0 948507	0 978951	653952
26	1 036706	0 988071	0 942161	1 015343	652333
27	1 041327	0 986611	0 935329	1 051284	65057/
.28	1 046358	0 985032	0 927996	1 086737	648676
29	1 051828	0 983328	0 920148	1 121661	.646624
30	1 057766	0 981492	0 911773	1 156015	644412
31	1 064204	0 979518	0 902858	1 189754	642033
32	1 071176	0 977399	0.893392	1 222832	639478
33	1 078721	0 975128	0 883365	1 255202	636739
34	1 086879	0 972697	0 872767	1 286810	633806
35	1 095698	0 970099	0 861590	1 317605	630668
36	1 105227	0 967325	0 849827	1 347529	627316
37	1 115522	0 964367	0 837472	1 376522	.623739
38	1 126646	0 961214	0 824521	1 404521	619924
39	1 138669	0 957856	0 810971	1 431459	615858
40	1 151669	0 954282	0 796820	1 457267	611527
.41	1 165733	0 950481	0 782069	1 481868	606916
42	1 180962	0 946439	0 766721	1 505185	602010
43	1 197470	0 942143	0 750779	1 527132	596789
44	1 215386	0 937577	0 734250	1 547620	.591234
45	1 234860	0 932725	0 717144	1 566553	585324
46	1 256065	0 927569	0 699471	1 583827	579036
47	1 279202	0 922087	0 681245	1 599333	572342
48	1 304508	0 916258	0 662484	1 612951	565213
49	1 332263	0 910056	0 643209	1 624552	557617
50	1 362800	0 903452	0 623444	1 633996	549517
51	1 396521	0 896416	0 603216	1 641129	540869
52	1 433917	0 888910	0 582559	1 645782	.531625
53	1 475589	0 880892	0 561510	1 647/70	521730
54	1 522291	0 872313	0 540112	1 646884	511118
55	1 574979	0 863115	0 518414	1 642890	499712
56	1 634889	0 853231	0 496473	1 635520	487419
57	1 703651	0 842577	0 474353	1 624470	474128
58	1 783476	0 831052	0 452127	1 609379	459702
59	1 877447	0 818531	0 429880	1 589821	443965
60	1 990038	0 804849	0 407711	1.565273	426694
61	2 128060	0 789791	0 385734	1 535080	407587
62	2 302565	0 773057	0 364084	1 498384	386227
63	2 533135	0 754208	0 342927	1 453995	.361994
64	2 859343	0 732543	0 322466	1 400126	.333888
65	3 380898	0 706770	0.302972	1 333695	.300056
0.66	4.506312	0 673723	0 284843	1.247630	0.255881

It is now clear from equations (98) and (99) that to reduce equation (47) to simple quadratures over s, we need the following functions:

$$V_0(U)=\int_{-1}^{+1}|J|\,d\mu\,,\quad V_1(U)=\int_{-1}^{+1}g^2|J|\,d\mu\,,\quad V_2(U)=\int_{-1}^{+1}g^4|J|\,d\mu\,,$$

and

$$V_3(U)=\int_{-1}^{+1}\left(\frac{\partial U}{\partial\varpi}\right)^2|J|\,d\mu\,. \tag{137}$$

In terms of these functions equation (47) becomes

$$\sigma^2\int_{U_0}^{U_1}\rho\chi^2V_1dU=\int_{U_1}^{U_0}\frac{d}{dU}(\rho\chi^2)V_2dU-4\int_{U_0}^{U_1}\rho\chi^2[(3\pi G\rho-\Omega^2)V_1-\Omega^2V_3]dU$$
$$+\int_{U_0}^{U_1}\gamma p\left[\left(\frac{d\chi}{dU}\right)^2V_2-2(4\pi G\rho-2\Omega^2)\chi\frac{d\chi}{dU}V_1+(4\pi G\rho-2\Omega^2)^2\chi^2V_0\right]dU\,. \tag{138}$$

Equation (136) should be especially suitable for the numerical evaluation of σ^2 for the rotating polytropes of James (1964, 1967).

IX. CONCLUDING REMARKS

That slow rotation has a stabilizing effect, in lowering the critical values of γ (for marginal stability) below the value $\frac{4}{3}$, is an unambiguous theoretical result. When one seeks the origin of the stabilizing effect in the analytical treatment, one finds that it is directly traceable to the Coriolis term in the equations of motion which leads to the relation $\xi_\varphi=(2i\Omega/\sigma)\xi_\varpi$ (cf. eq. [16]) between the φ- and the ϖ-components of the Lagrangian displacement. But this is not the only effect of rotation: its effect on the equilibrium distribution of the pressure and the density, resulting particularly in a general expansion of the configuration, may have a destabilizing, and perhaps even a decisive, influence. The principal reason for adapting the general variational principle so as to be applicable to a wide variety of initial conditions is the hope that it will resolve these basic ambiguities. We shall return to these questions in a second paper which will include the results of the application of the different formulae developed in this paper.

On a somewhat different aspect of the problem, the present paper has been restricted to a consideration of the effects of uniform rotation. The extension of the analysis to allow for non-uniform rotation should be of considerable interest in view of the greater amount of angular momentum that can be stored in the configuration. Such an extension should be particularly simple when Ω is a function of ϖ only; for then not only are the considerations of § II almost unaltered (the only difference is that $4\Omega^2$ in eq. [19] is replaced by Rayleigh's discriminant $d[\varpi^4\Omega^2]/\varpi^3d\varpi$) but also the form (33) of the trial function would be applicable since equations (9) continue to hold. We hope to return to a consideration of these extensions in later papers.

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