J. Astrophys. Astr. (1985) **6,** 247–260

Gravitational Pertubation of Homogeneous Collisionless Dark Matter

T. Padmanabhan & M. M. Vasanthi *Tata Institute of Fundamental Research, Homi Bhabha Road, Bombay 400005*

Received 1985 March 15; accepted 1985 August 12

Abstract. The effect of a perturbing mass on a homogeneous collisionless cloud of dark matter is considered in the linear approximation. It is shown that gravitational potential can have turning points, in sharp contrast with gravitating systems of finite extent. The model offers a reasonable explanation for the observed secondary maxima in the density distribution of rich clusters. The relevance of the model to the flatness of the rotation curves of galaxies is also discussed.

Key words: dark matter, collisionless—cluster of galaxies, dark matter—galaxies, rotation curves

1. Introduction

One of the most intriguing problems in present day astrophysics is related to the fact that estimates of visible matter (at various length scales) fall systematically short of the amount of gravitating matter at the same scale (Rood 1981; Bahcall 1977; Peebles 1979; Faber & Gallager 1979). This problem, usually christened ‘Missing mass problem’, has attracted considerable attention of late in the form of wide variety of explanations. (For a study of the systematics at various length scales, see Cowsik & Vasanthi 1986.) The explanations range from using the relics of the big bang to modifying Newton’s law of gravity (see for *e.g.* Cowsik & McClelland 1973; Pagels & Primack 1982; Olive & Turner 1982; Cabibbo, Farrar & Maiani 1981; Peebles 1982; Sikivie 1982; Milgrom 1983).

Among these explanations there is a sense of naturalness in suggesting that the relics of big bang provide the dark matter. For example, considerable amount of work has been done in recent years to understand the dynamics of the universe dominated by massive neutrinos (Davis *et al.* 1981; Chubb 1983; Doroshkevich *et al.* 1981; Bond, Efstathiou & Silk 1980; Wasserman 1981; Peebles 1982; Melott 1983; Cowsik 1983; Cowsik 1986; Sato & Takahara 1981; Schramm & Steigman 1981; Klinkhamer & Norman 1981).

In the standard big bang scenario, stable massive neutrinos (with mass of the order of ~ 20 eV) would decouple from the rest of the matter at a very early epoch (~ 1 MeV). After this epoch, these neutrinos free-stream in space time as collisionless particles, interacting only through gravity. Such a collisionless species can condense in any potential well and provide the missing mass.

In this paper we shall consider certain mathematical features of such collisionless dark matter which permeates throughout the universe. We shall see that the gravitational effects in such an infinite medium can have certain peculiarities not exhibited by finite bound gravitating systems. Though we shall call the constituents of dark matter as neutrinos, our analysis will be applicable to any other collisionless relic of big bang.

The mathematical formalism is presented in Section 2 wherein we solve the collisionless Boltzmann equation self-consistently in the linear approximation. We show that gravitational potential in an infinite medium can have maxima and minima.

It is possible that these non-trivial turning points in the gravitational potential are responsible for the phenomena of secondary maxima observed in a number of clusters of galaxies (Baier 1983). We show in Section 3 that the observed features are reasonably well described by our model. In addition, the present model may also provide (at least a partial) explanation to the flat rotation curves of spiral galaxies (Rubin 1979; Rubin *et al.* 1980, 1982; Bosma 1978).

2. Mathematical formalism

2.1 *Collisionless Gas in Linear Approximation*

Consider a system of collisionless neutrinos of mass m_ν, in a gravitational potential ϕ (x, t). Let f_{total} (x, v, t) denote the number of neutrinos in the phase space interval x+d^{3}x, v+d^{3}v) at a time t. (The subscript 'total' is added for future notational convenience.) Conservation of phase-space density leads to the collisionless Boltzmann equation

$$\left[\frac{\partial}{\partial t}+\mathbf{v}\cdot\nabla-\nabla\phi\cdot\frac{\partial}{\partial \mathbf{v}}\right]f_{\text{total}}(\mathbf{x},\mathbf{v},t)=0. \tag{1}$$

The gravitational potential ϕ(x, t) satisfies Poisson's equation

$$\nabla^2\phi(\mathrm{x}, t) = 4\pi G m_\nu \int f(\mathrm{x}, \mathrm{v}, t)\mathrm{d}^3\mathrm{v}+4\pi G\rho_{\text{ext}}(\mathrm{x},t). \tag{2}$$

Here ρ_{ext} (x, t) denotes the mass density of gravitating matter other than the neutrinos. (We shall call them, somewhat loosely, as 'galaxies'!) As already emphasized, we expect the gravitational field of a cluster of galaxies to be dominated by the neutrinos rather than by the ρ_{ext}(x, t).

In writing down Equations (1) and (2), we have already neglected the expansion of the universe and other general relativistic effects. Such an approximation is definitely valid during recent epochs. The introduction of the expansion of the universe is necessary to discuss the evolution of perturbation in a collisionless gas and will be taken up in a subsequent paper.

The general solution to Equations (1) and (2) is unknown. To make any progress we have to make reasonable approximations. To be specific, let us consider a cluster of galaxies with a neutrino halo, and treat galaxies as a perturbation in the neutrino background. In the (trivial) zeroth order approximation, we shall entirely neglect the galaxies and assume neutrinos to be distributed homogeneously all over the universe. Such a homogeneous distribution of matter does not produce any gravitational potential. Thus, in the zeroth order, we can take

$$f_{\text{total}}(\mathrm{x}, \mathrm{v}, t) = f_0(\mathrm{v});\ \ \rho_{\text{ext}}(\mathrm{x}, t) = 0 \tag{3}$$

$$\phi(\mathrm{x}, \mathrm{t}) = 0. \tag{4}$$

Let us now 'switch on' the galaxies in the form of ρ_{ext} (x, t). These galactic perturbations will induce clustering in the neutrino background and will make the distribution function space-dependent. Such a space-dependent distribution function will in turn produce a gravitational potential of its own. Thus, in the first order, the potential ϕ (x, t) arises from two sources: (a) A part ϕ_{ext} (x, t) which comes from ρ_{ext}(x, t) and satisfies the equation,

$$\nabla^2 \phi_{\text{ext}}(\text{x}, t) = 4\pi G \rho_{\text{ext}}(\text{x}, t) \tag{5}$$

and (b) a part ($\phi - \phi_{\text{ext}}$) which is due to perturbed neutrino distribution, which we shall call f (x, v,t). That is,

$$f_{\text{total}}(\text{x, v}, t) = f_0(\text{v}) + f(\text{x, v}, t), \tag{6}$$

$$\nabla^2[\phi(\text{x, t}) - \phi_{\text{ext}}(\text{x}, t)] = 4\pi G m_\nu \int f(\text{x, v } t) \, \text{d}^3\text{v}. \tag{7}$$

We shall assume that $f \ll f_0$ and linearize (1) in f. This gives,

$$\left[\frac{\partial}{\partial t} + \mathbf{v} \cdot \frac{\partial}{\partial \mathbf{x}}\right] f(\mathbf{x}, \mathbf{v}, t) = \frac{\partial \phi}{\partial \mathbf{x}} \cdot \frac{\partial f_0}{\partial \mathbf{v}}. \tag{8}$$

Once the form of the perturbation, ρ_{ext} (or ϕ_{ext}(x, t)) is specified, Equations (7) and (8) determine the perturbed distribution of dark matter in a cluster.

These equations can be solved in a straightforward manner using Fourier transforms. It is convenient to define the 'one-sided' Fourier transform (Lifshitz & Pitaevskii 1981) of f (x, v, t) (and similarly for ϕ (x ,t)) by,

$$f_{\omega\mathbf{k}}(\mathbf{v}) = \int_0^\infty \text{d}t \int \text{d}^3\mathbf{x} \, f(\mathbf{x}, \mathbf{v}, t) \, \text{e}^{-i\mathbf{k}\cdot\mathbf{x}} \, \text{e}^{i\omega t}. \tag{9}$$

The inverse transform is given by

$$f(\mathbf{x}, \mathbf{v}, t) = \int_{-\infty + i\sigma}^{+\infty + i\sigma} \frac{\text{d}\omega}{2\pi} \frac{\text{d}^3\mathbf{k}}{(2\pi)^3} f_{\mathbf{k}\omega}(\mathbf{v}) \, \text{e}^{i(\mathbf{k}\cdot\mathbf{x} - \omega t)}. \tag{10}$$

Here the ω integral is taken along a straight line in the complex ω-plane parallel to and above the real axis passing above all the singularities of $f_{\text{k}\omega}$. We multiply both sides of (8)by $\text{e}^{i\omega t}$ and integrate with respect to t. Defining f_{k}(v, t) and g_{k}(v) by,

$$f_{\text{k}}(\text{v}, t) = \int \text{d}^3 \text{x} \, f(\text{x, v}, t) \text{e}^{-i\text{k}\cdot\text{x}}, \tag{11}$$

$$g_{\text{k}}(\text{v}) = f_{\text{k}}(\text{v}, t)|_{t=0} = f_{\text{k}}(\text{v}, 0). \tag{12}$$

We can write (8) in Fourier space as,

$$i(\mathbf{k}\cdot\mathbf{v} - \omega) f_{\mathbf{k}\omega}(\mathbf{v}) - g_{\mathbf{k}} = i\mathbf{k} \cdot \frac{\partial f_0}{\partial \mathbf{v}} \phi_{\mathbf{k}\omega}. \tag{13}$$

Equation (7) in the Fourier space reads as

$$-\text{k}^2 (\phi_{\text{k}\omega} - \phi_{\text{k}\omega}(\text{ext})) = 4\pi G m_\nu \int f_{\text{k}\omega}(\text{v}) \text{d}^3\text{v}. \tag{14}$$

Equation (13) can be solved to give,

$$f_{\mathbf{k}\omega}(\mathbf{v}) = \frac{i\mathbf{k} \cdot \dfrac{\text{d}f_0}{\text{d}\mathbf{v}} \phi_{\mathbf{k}\omega} + g_{\mathbf{k}\omega}}{i(\mathbf{k}\cdot\mathbf{v} - \omega)} \tag{15}$$

Using (14) and (15) we can determine $\phi_{k\omega}$ to be,

$$\phi_{\mathbf{k}\omega} = \frac{1}{\varepsilon(\mathbf{k},\omega)}\,\phi^{\text{ext}}_{\mathbf{k}\omega} - \frac{4\pi G m_\nu}{\mathbf{k}^2}\int \frac{g_{\mathbf{k}}(\mathbf{v})\,\mathrm{d}^3\mathbf{v}}{i(\mathbf{k}\cdot\mathbf{v}-\omega)} \tag{16}$$

where

$$\varepsilon_{(\mathbf{k},\omega)} = 1 + \frac{4\pi G m_\nu}{\mathbf{k}^2}\int \frac{\left(\mathbf{k}\cdot\dfrac{\mathrm{d}f_0}{\mathrm{d}\mathbf{u}}\right)}{(\mathbf{k}\cdot\mathbf{u}-\omega)}\,\mathrm{d}^3\,\mathbf{u}. \tag{17}$$

Given the ϕ^{ext} and the initial condition g_k (v), Equation (16) determines $\phi_{k\omega}$. Using $\phi_{k\omega}$ in (15) gives us the perturbed distribution of dark matter. Thus (15) and (16) solve the problem completely in the linear approximation. The potential $\phi(\text{x}, t)$ and the distribution function $f(\text{x}, \text{v}, t)$ can be obtained by the inverse transformations (10) of Equations (15) and (16).

From Equation (16), it is clear that $\phi_{k\omega}$ arises from two different sources. Initial inhomogenities in the medium, characterized by g_k, propagate in time and contribute as the second term in (16). As we are more interested in the effect of external galactic perturbations, we shall take the initial condition,

$$g_k = 0 \tag{18}$$

leading to,

$$\phi_{\mathbf{k}\omega} = \frac{1}{\varepsilon(\mathbf{k},\omega)}\,\phi^{\text{ext}}_{\mathbf{k}}. \tag{19}$$

This equation shows that the neutrino background acts as a polarizable medium. The potential in the medium is scaled by a factor ε ($\mathbf{k}$, ω) which may be called the 'gravitational permittivity' of the medium. As one can see from (17), $\varepsilon(\mathbf{k}, \omega)$ is completely determined by the background distribution function f_0 (v).

There is a minor mathematical point which is worth taking note of at this stage. Expressions like (16), (17) *etc.* contain integrals with integrands that have poles in the real axis (for example, the integral in (17) has a pole at $\omega = \mathbf{k}\cdot\mathbf{u}$). Thus, one has to specify the contour of integration for these expressions. This is a well-known feature in the theory of collisionless plasmas (Lifshitz & Pitaevskii 1981). We assume that the potential was zero at $t = -\infty$ and was switched on adiabatically:

$$(\text{t}) \sim \phi_\omega e^{-i\omega t} e^{Pt} \qquad (P \geqslant 0) \tag{20}$$

The exp (pt) factor makes ϕ vanish in the past infinity. The limit of ($p \to 0$) is taken at the end of the calculations. (As it stands (20) diverges at the future infinity; however, because of causality, the behaviour of $\phi(t)$ at future infinity cannot affect physics at any finite time.) Clearly this operation is equivalent to replacing ω by ($\omega + ip$) and taking ($p \to 0$) limit in the end. Such a procedure makes the integral in (16), (17) *etc.* well-defined. Whenever a pole in the real axis is encountered we shall assume that ω has an infinitestimal positive imaginary part.

To understand the effects of $\varepsilon(\mathbf{k}, \omega)$ and the polarization of the medium, we have to consider suitable 'test perturbations' in the form of $\rho^{\text{ext}}(\text{x}, t)$. Let us assume that $\rho^{\text{ext}}(\text{x}, t)$ arises due to a set of N galaxies. We shall denote the trajectory of the nth galaxy (n =1, 2, . . . , N) by $R_n(t)$. Then if the mass of the nth galaxy is M_n, we have,

$$\rho^{\text{ext}}(\mathbf{x}, t) = \sum_{n=1}^{N} M_n\,\delta(\mathbf{x} - \mathbf{R}_n(t)). \tag{21}$$

Since

$$\nabla^2 \phi^{\text{ext}}(\text{x}, t) = 4\pi G \rho^{\text{ext}}(\text{x}, \text{t}), \tag{22}$$

we get,

$$\phi_{\mathbf{k}}^{\text{ext}} = -\frac{4\pi G}{\mathbf{k}^2} \sum_n M_n \int_0^\infty \mathrm{d}t \exp i[\omega t - \mathbf{k} \cdot \mathbf{R}_n(t)]. \tag{23}$$

Two simple cases are of physical importance: (i) The galaxy is at rest; $R(t) = R$ = constant, and (ii) The galaxy is moving with a uniform velocity; $\mathbf{R}(t) = \mathbf{u}t$. The effect of N galaxies in similar state of motion can be found by superposing individual galaxies. In these two cases $\phi_{\text{k}}^{\text{ext}}$ are given by

$$\phi_{\mathbf{k}}^{\text{ext}} = -\frac{4\pi GM}{\mathbf{k}^2} i\mathbf{k}\mathrm{e}^{-i\mathbf{k}\cdot\mathbf{R}}, \qquad \mathbf{R}(t) = \mathbf{R}, \quad \text{case i}; \tag{24}$$

$$\phi_{\mathbf{k}}^{\text{ext}} = \frac{4\pi G M i}{\mathbf{k}^2(\mathbf{k}\cdot\mathbf{u} - \omega)}, \qquad \mathbf{R}(t) = \mathbf{u}t, \quad \text{case ii}. \tag{25}$$

Case (i) leads to an interesting steady-state distribution of dark matter which we shall discuss in the remaining sections of this paper. The situation described in case (ii) is of relevance in the discussion of dynamical friction and galactic segregation. This will be discussed in a future publication.

2.2 *The Steady-State Distribution*

Rigorously speaking, galaxies in a cluster cannot be considered to be stationary. However, once the cluster has reached a steady-state configuration, one can meaningfully discuss a time-independent distribution function. This time-independent distribution of galactic matter will induce clustering in the neutrino gas. We are interested in the form of this distribution function.

It is obvious that the time independent solution is determined by the static part of the permittivity *viz.* $\varepsilon(\text{k}, 0)$. (To see this, note that when the inverse transform of ϕ_{kw} is taken, using (20) and (25) it is the pole at $\omega = 0$ that produces the static part of the potential.) Considering the importance of the result, we shall derive it more directly. When f, ϕ and ρ^{ext} are independent of time, it is easy to show that

$$\phi_{\mathbf{k}} = -\frac{4\pi G}{\mathbf{k}^2} \frac{\rho_{\mathbf{k}}^{\text{ext}}}{\varepsilon(\mathbf{k}, 0)} \tag{26}$$

with

$$\varepsilon(\mathbf{k}, 0) = 1 + \frac{4\pi G m_\nu}{\mathbf{k}^2} \int \frac{\mathbf{k} \cdot \frac{\mathrm{d}f_0}{\mathrm{d}\mathbf{v}}}{\mathbf{k}\cdot\mathbf{v}} \mathrm{d}^3\mathbf{v}. \tag{27}$$

Assuming f_0 (v) depends only on |v| (which is a reasonable assumption because there are no preferred directions in the velocity space), we get

$$\varepsilon(\mathbf{k}, 0) = 1 + \frac{8\pi G m_\nu}{k^2} \int_0^\infty \left(\frac{\mathrm{d}f_0}{\mathrm{d}v}\right) 2\pi v \, \mathrm{d}v \tag{28}$$

$$= (1 - k_J^2/k^2)$$

where,

$$\frac{1}{\sigma^2} = \frac{4\pi}{n}\int_0^\infty f_0\,\mathrm{d}v, \quad k_J^2 = \frac{4\pi Gnm_\nu}{\sigma^2}.$$

Thus the static part of the permittivity is essentially determined by the Jean's length k_J^{-1} of the background medium. In this case it is instructive to compare (28) with the static part of the permittivity of an electromagnetically active medium. In the electromagnetic case, (28) is replaced by,

$$\varepsilon(\mathbf{k}, 0) = 1 + \frac{k_D^2}{k^2}; \quad k_D^2 = \frac{4\pi\rho_0}{\sigma^2}\left(\frac{e}{m}\right)^2 \tag{29}$$

where k_D^{-1} is the Debye length for the medium. Note that the sign of k^2 is different in (28) and (29).

Substituting (28) into (26), we get the potential to be,

$$\phi_{\mathbf{k}} = -\frac{4\pi G\rho_{\mathbf{k}}^{\text{ext}}}{(k^2 - k_J^2)}. \tag{30}$$

Let us calculate the potential at any point x, due to a galaxy of mass M kept at the origin. Taking $\rho^{\text{ext}}(x) = M\delta(x)$ and using the inverse transformations we get,

$$\phi = -4\pi GM\int\frac{\mathrm{d}^3k}{(2\pi)^3}\frac{e^{ik\cdot x}}{(k^2 - k_J^2)}. \tag{31}$$

In this section we have not bothered to show the $i\varepsilon$ explicitly. It is easy to see (using reasonings similar to that of Equation 20) that k_J should be treated as having an infinitesimal positive part. With this prescription, (31) gives

$$\phi = -\frac{GM}{|\mathbf{x}|}\cos k_J|\mathbf{x}|. \tag{32}$$

Thus the polarization of the medium introduces an extra sinusoidal dependence in $\phi(x)$. This is to be contrasted with the electromagnetic case in which one would have used (29) rather than (28). The change in the sign of k^2 term has the effect of replacing (32) by,

$$\phi = -\frac{Q}{|\mathbf{x}|}\exp(-k_D|\mathbf{x}|) \tag{33}$$

leading to the well-known Debye shielding. Equation (32) emphasizes the fact that gravitational effects cannot be shielded. This oscillatory behaviour of the gravitational field of a test mass in a collisionless gas was derived earlier by Marochnik (1968) in the context of a star in a star cluster.

Let us look at the physics described by Equation (32). In the absence of neutrino background, a galaxy of mass M kept at the origin will produce the Newtonian $|x|^{-1}$ potential. Any test particle, say, a galaxy, will feel this force and will be attracted towards the origin. When the neutrino cloud is present, the situation can be very different. The galaxy at the origin perturbs the background and leads to inhomogenities. Any test particle (at a point x) will now feel the combined effects of the perturbed medium as well as the galaxy at the origin. Depending on the relative distribution of inhomogenities, the test particle may feel a force either towards the origin or away from the origin. Thus—as is clear from (32)—the sign and slope of $\phi(x)$ can be positive or negative, implying either attraction or repulsion.

This peculiar feature, which has no analogue in finite, bound, gravitating systems, can also be understood from another angle. The gravitational potential produced by any spherically symmetric distribution of matter with density $\rho(r)$ satisfies the equation,

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\phi}{\partial r}\right) = 4\pi G\rho(r). \tag{34}$$

Therefore,

$$\frac{\partial\phi}{\partial r} = \frac{G}{r^2}\int_0^r \rho(x)4\pi x^2\,\mathrm{d}x. \tag{35}$$

A turning point for ϕ at, say, $r = R$ would imply the vanishing of the $\partial\phi/\partial r$ at $r = R$, and hence the vanishing of the integral in (35). In normal circumstances $\rho(r)$ is always greater than zero and thus the integral cannot vanish. This argument however assumes that $\rho(r)$ has only finite extent in space and falls faster than r^{-3} at large r. In an infinite homogeneous medium, for example, the integrals like the one in (35) do not exist. We have to do a more careful job. Suppose that the gravitating matter consists of two components $\rho_1(r)$ and $\rho_2(r)$. Let $\rho_1(r)$ fall faster than r^{-3} at large distances. However, suppose $\rho_2(r)$ is equal to $\bar{\rho}_2 + f(r)$ where $\bar{\rho}_2$ is a constant homogeneous distribution throughout space and $f(r)$ is the deviation from the homogeneity which may be *positive* or *negative* (*i.e.* $\rho_2(r)$ may be enhanced or depreciated from the mean value $\bar{\rho}_2$). Of course f (r) $<|\bar{\rho}_2(r)|$ so that ρ_2 is always greater than zero. The gravitational potential ϕ due to this distribution satisfies the equation

$$\nabla^2\phi = 4\pi G(\rho_1+\rho_2+f). \tag{36}$$

Because of linearity, we can write $\phi = \psi+n$ where ψ and η satisfy the equations

$$\nabla^2\psi = 4\pi G(\rho_2, \tag{37}$$

$$\nabla^2\eta = 4\pi G(\rho_1+\mathrm{f}). \tag{38}$$

We realise that the potential ψ produced by a distribution $\bar{\rho}_2$ (constant throughout the universe) is formally infinite. But this is of no concern because such a homogeneous distribution of matter does not provide any gravitational force. In other words, the dynamics is completely determined by $\eta(r)$. This $\eta(r)$ can have nontrivial maxima and minima because the source for $\eta(r)$ (which is $\rho_1(r) + f(r)$) need not be positive definite. This is precisely what happens when a homogeneous background of neutrinos is present in the universe. Deviations from homogeneity, which are responsible for gravitational force may be positive or negative. Since such an homogeneous background distribution is of relevance in many a cosmological context (see for example: Peebles 1980 and Weinberg 1972), the above mentioned feature should be kept in mind.

Using (26) and (28) one can immediately obtain the distribution of dark matter:

$$\rho_{\mathbf{k}}^{\nu} = m_\nu \int f_k(\mathbf{v})\,\mathrm{d}^3\mathbf{v} \tag{39}$$

$$= \rho_{\mathbf{k}}^{\text{ext}}\frac{k_J^2}{(k^2-k_J^2)} = -\frac{k_J^2}{4\pi G}\phi_{\mathbf{k}}. \tag{40}$$

In other words,

$$\rho^{\nu}(\mathbf{x}) = -\frac{k_J^2}{4\pi G}\phi(\mathbf{x}). \tag{41}$$

This equation relates the spatial distribution of dark matter to the overall gravitational potential. Clearly, when $\phi(x)$ is negative [regions of attraction] ρ_v (x) is positive (density enhancement) and vice versa. In the case of a test galaxy at origin, we can use (32) in (41) to obtain,

$$\rho_v(\mathbf{x}) = \frac{Mk_J^3}{(4\pi)} \cdot \frac{\cos k_J|\mathbf{x}|}{k_J|\mathbf{x}|}. \tag{42}$$

The validity of linear perturbation theory requires the condition

$$|\rho_v(\mathbf{x})| \ll \rho_0 \tag{43}$$

which must be kept in mind in using expressions like (42).

The potential due to a distribution of galaxies can be found by superposing potentials of the form (32). For a distribution of galaxies represented by ρ^{ext} (x), this is given by the integral,

$$\phi(\mathbf{x}) = -G \int \frac{\rho^{\text{ext}}(\mathbf{r})}{|\mathbf{r}-\mathbf{x}|} \cos k_J|\mathbf{r}-\mathbf{x}|\, \mathrm{d}^3\mathbf{r}. \tag{44}$$

If $\rho^{\text{ext}}(r)$ is assumed to be spherically symmetric (which is a reasonable assumption for most large clusters of galaxies), then the angular integrations can be performed to give,

$$\phi(\mathbf{x}) = -4\pi G \left[\frac{\sin k_J x}{k_J x} \int_x^\infty \mathrm{d}r\, r\rho(r) \cos k_J r + \frac{\cos k_J x}{k_J x} \int_0^x \mathrm{d}r\, r\rho(r) \sin k_J r\right]. \tag{45}$$

Even though (45) will lead to a much more complicated form than (32) the qualitative features will be the same. In particular one does expect the sinusoidal behaviour of the potential, at least at large x. As a prototype, consider two simple forms for $\rho(r)$: (i) exponential fall with

$$\rho(r) = \bar{\rho} \frac{\mathrm{e}^{-r/R}}{(r/R)}, \tag{46}$$

and (ii) a box type fall off with

$$\rho(r) = \begin{cases} \bar{\rho} & \text{for } r \leq R, \\ 0 & \text{for } r > R. \end{cases} \tag{47}$$

For these two cases (45) can be evaluated analytically. With (46) we get, with $M_c = \frac{4\pi}{3} \bar{\rho} R^3$,

$$\phi(x) = -\frac{3GM_c}{|\mathbf{x}|} - \frac{1}{(1+k_J^2R^2)} [\cos k_J x - \mathrm{e}^{-x/R}] \tag{48}$$

and for (47) we get

$$\phi(x) = \frac{3GM_c}{R} \frac{1}{(k_J R)^2} \left[1 - \frac{\sin k_J x}{k_J x} (k_J R \sin k_J R + \cos k_J R)\right], \quad x < R \tag{49}$$

$$= \frac{3GM_c}{R} \frac{1}{(k_J R)^2} \frac{\cos k_J x}{k_J x} [k_J R \cos k_J R - \sin k_J R], \qquad x > R. \tag{50}$$

Both (49) and (50) exhibit the 'cosine' dependence at large distances. It is also clear from (45) that for large x, it is the second integral in the right-hand side that contributes most, leading to the cosine dependence.

In the above analysis we have specified $\rho^{\text{ext}}(x)$ in an *ad hoc* manner. In reality, $\rho^{\text{ext}}(x)$ will be determined in a self-consistent manner by the response of the galaxies to the gravitational potential. This leads to some interesting tests of the above model which we shall indicate in the next section.

3. Comparison with observation

The discussion in the previous sections has been purely kinematic. In order to apply these results to any realistic astrophysical system, one has to consider the dynamics of the model as well. In particular, is it possible for a homogeneous distribution of collisionless relic to arise in standard big bang model? In the conventional picture of ν-dominated universe with adiabatic fluctuations the first structures to form and collapse are the super clusters (Bond, Efstathiou & Silk 1980; Bond & Sazlay 1983). In such a picture, it will be very difficult to obtain a homogeneous distribution. On the contrary, there are other scenarios in which such a situation can arise. One simple possibility would be isothermal ν-fluctuations. A more interesting situation, however can arise if there existed an unstable heavy neutrino which decays to the stable light neutrino (Simpson 1985; Fukugita & Yanagida 1984; Padmanabhan & Vasanthi 1985). The decay products would be relativistic at the time of decay and can provide a homogeneous background at reasonably low redshifts. While the dynamics of such a model is yet to be investigated fully, it is quite likely to be very different from the standard scenario. In general, dark matter observations can be explained with relative ease, if there are two components to dark matter: one of which is distributed reasonably homogeneously and the other clustered at smaller scales.

The above comments as well as the discussion in the following sections are somewhat tentative, and are intended only to point out certain possibilities. Whether these possibilities can be implemented realistically in a consistent astrophysical scenario is a question of dynamics and is beyond the scope of the present paper.

3.1 *Secondary Maxima in Clusters of Galaxies*

Given a reliable functional form of visible matter density in a cluster, one may attempt a self-consistent model-building based on the above discussion. However, as discussed in the last section, the theory predicts a sinusoidal dependence of the potential on the radial distance, as shown in Fig. 1. Qualitatively, we expect test galaxies to cluster around the minima of the potential. In other words, at least in some clusters, one expects a secondary maximum in the density distribution of visible matter. (Of course, such a maximum is observable only when it occurs well within the size of the cluster.)

From Equation (32) it can be seen that the second minimum of the potential occurs around $k_J x_m \simeq 2\pi$. Solving the equation for the turning points,

$$\cos k_J x + k_J x \sin k_J x = 0 \tag{51}$$

one finds the value to be,

$$x_m \simeq 6.1\, k_J^{-1}. \tag{52}$$

Using the definition of k_J we estimate

$$x_m = 0.6 \left(\frac{\rho_0}{1.5 \times 10^{-25}\ \text{g cm}^{-3}}\right)^{-1/2} \left(\frac{\sigma}{10^3\ \text{km s}^{-1}}\right) \text{Mpc}. \tag{53}$$

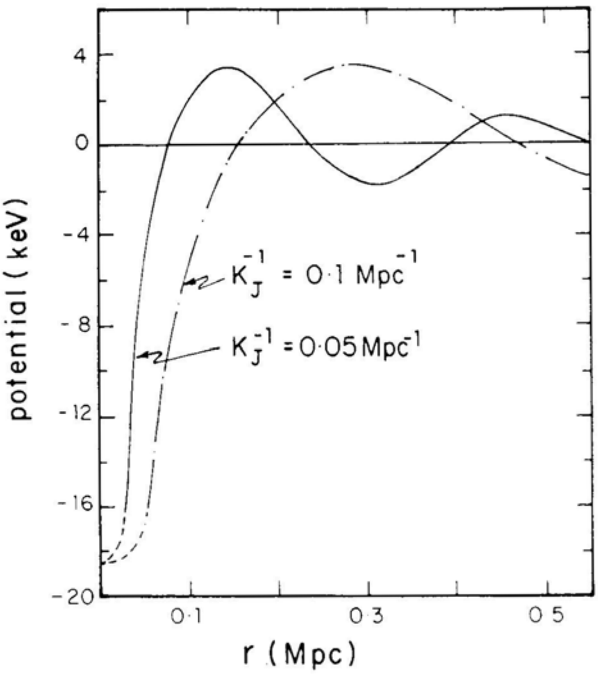

Figure 1. The perturbations of the cluster potential by the galaxies in the core.

We have scaled the expression with the value of ρ_0 usually quoted for the Coma cluster of galaxies. It is believed that ρ_0 for other clusters are somewhat lower. We have scaled the neutrino velocity dispersion σ by a typical galactic velocity dispersion in Coma cluster (900–1200 km s^{-1}). There is *no* deep theoretical reason to expect σ to have the same value, though some models suggest this possibility. Of course, Pauli principle sets the lower-limit

$$\sigma > v_F \simeq (30\,\mathrm{km\,s^{-1}})\left(\frac{\rho}{10^{-25}\,\mathrm{g\,cm^{-3}}}\right)^{1/3}\left(\frac{m_\nu}{50\,\mathrm{eV}}\right)^{-4/3}. \qquad (54)$$

Because of these uncertainties the value of x_m will definitely vary from cluster to cluster. We may naively expect a secondary maximum within one order of (53) (*i.e.* in the range of 0.3 to 3 Mpc).

In the Coma cluster of galaxies, the velocity dispersion curves suggest a secondary maximum at about 20 arcmin from the centre which corresponds to a distance scale of about 0.7 Mpc.

A host of other clusters show evidence for a secondary maximum in the density distribution. It is worth noting that no other simple explanation exists for this feature (Baier 1983). We give in Table 1 a list of clusters (which exhibit the secondary maximum), using the values of the redshift to these clusters (Hoessel, Gunn & Thuan 1980).

We have estimated the distances x_m. It may be noted that all these values fall between 0.49 Mpc and 1.44 Mpc, giving excellent qualitative agreement with the value in (53). Assuming that the missing mass density in all these clusters is of the order of

Table 1. List of clusters exhibiting secondary maximum ($H = 50\ h$ km s^{-1} Mpc^{-1}.)

Serial No.	Cluster name	Redshift Z	Position of secondary max arcmin	Position of secondary max h^{-1} Mpc	k_J^{-1} Mpc
1	A147	0.044	7.4	0.57	0.09
2	A576	0.0392	7.3	0.50	0.08
3	A671	0.0497	8.0	0.69	0.11
4	A1225	0.1033	8.0	1.44	0.24
5	A1227	0.0339	8.0	0.47	0.08
6	A1383	0.0598	13.5	1.41	0.23
7	A1775	0.0718	11.5	1.44	0.24
8	A2029	0.0777	7.5	1.02	0.17
9	A2666	0.0273	8.0	0.38	0.06
10	A1656 (Coma)	0.0224	20.0	0.78	0.13

10^{-25} g cm^{-3} (which actually is somewhat large), one may conclude that the dark matter has a velocity dispersion of a few thousands of kilometres per second.

3.2 *Rotation Curves of Galaxies*

Let us consider a galaxy situated somewhere near the first minimum of the potential (as we saw in Section 3.1, this occurs at $r = R$ with $R = 6.1\ k_J^{-1}$). We assume that the centre of the galaxy is located at R and the linear extent of the galaxy is small compared to cluster scale. A star moving in the galaxy at a distance ξ from the centre of galaxy will be subject to the combined gravitational force of the galaxy and the cluster. The potential due to the galaxy at ξ is of the order of

$$\phi_G \simeq -\frac{GM_G}{\xi} \tag{55}$$

where M_G is the mass of the galaxy. The cluster potential at a point r is of the order of

$$\phi_c \simeq -\frac{GM_c}{r}\cos k_J r \tag{56}$$

where M_c is the core mass of the cluster. In our case $r = R + \xi$ with $\xi \ll R$. Expanding in a Taylor series and noticing that $\phi'_c(r)$ is zero, we get

$$\phi_c(\xi) = \left[\frac{GM_c k_J^2}{2R}\cos k_J R\right]\xi^2 + \text{constant} \tag{57}$$

$$\simeq \frac{GM_c k_J^2}{2R}\xi^2, \tag{58}$$

where we have approximated cos $k_J r$ by unity and dropped an unimportant constant. Thus the total potential felt by the star at ξ is given by

$$\phi_{\text{total}} = \frac{GM_c k_J^2}{2R}\xi^2 - \frac{GM_G}{\xi}. \tag{59}$$

The rotational velocity of a star at ξ is given by

$$v^2(\xi) = \xi \frac{\partial \phi_{\text{total}}}{\partial \xi} = \frac{GM_c k^2}{2R} \xi^2 + \frac{GM_G}{\xi}. \tag{60}$$

Using $kR = 6.1$, v^2 can be written as

$$\left(\frac{v}{v_m}\right)^2 = 5.89 \left(\frac{M_c}{M_G}\right)^{2/3} \left[x^2 + 0.03 \frac{M_G}{M_c} \frac{1}{x}\right] \tag{61}$$

where

$$v_m^2 = 6.3 \frac{GM_c}{R} \left(\frac{M_G}{M_c}\right)^{2/3}$$

and

$$x = (\xi/R).$$

Since v^2 (ξ) contains two terms, one increasing with and the other decreasing with ξ, it is easy to see that v^2 can be flat for a range of ξ.

We show in Fig. 2 the result of a numerical computation of v; for a set of fiducial values of parameters (M_c = 1.5 × 1014 $M_\odot$, M_G = 4 ×1011 $M_\odot$). The density distribution of galaxies was smoothened out near $\xi = 0$ to avoid the singularity at the origin. As can be seen from Fig. 2, the rotation curve is reasonably flat for a large range in ξ. The flatness of rotation curves is probably the most convincing evidence for the dark matter. However, one should *not* consider rotation curves as a "crucial" test for dark matter modelling. It is fairly straightforward to explain flat rotation curves once some form of dark matter distribution is invoked. All these models require some special

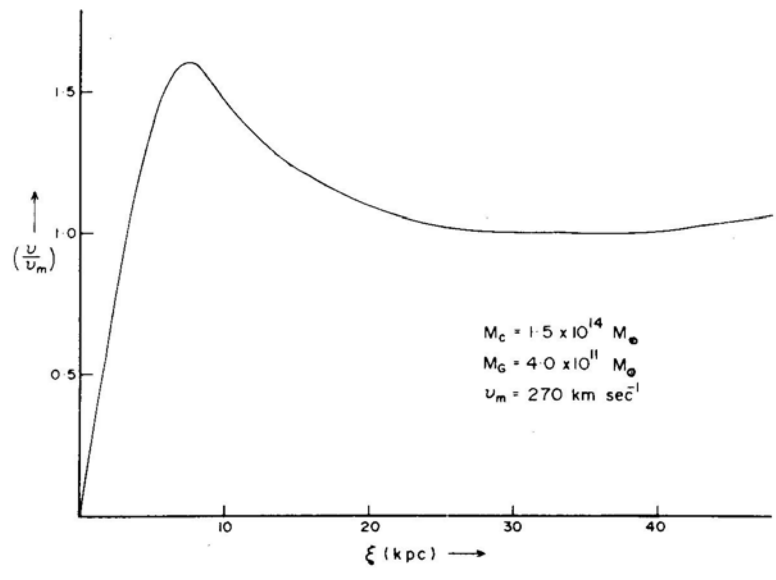

Figure 2. Rotation curve of a fiducial galaxy. The rotational velocity v, is plotted (in units of the flat value v_m) against the distance ξ from the centre of the galaxy. Note that the curve is reasonably flat from $\simeq$ 20 kpc to 50 kpc. M_c is the cluster core mass, and M_G is the galaxy mass. The flatness of the curve does not depend sensitively on the value of these parameters.

alignment of the galaxy with respect to the dark matter distribution (they are usually taken to be concentric). In our case we have assumed that the galaxy is near the mininum of the potential. Thus we do not feel that explanation of rotation curve is a sufficiently critical test of the dark matter distribution. The picture presented here is not complete to every detail (Explanation for flat rotation curves using a neutrino background is also attempted in Cowsik & Ghosh (1986) & Basdevant (1984)). In particular, the present model is incapable of explaining the rotation curves of field galaxies.

4. Conclusions

It is interesting to observe that grativational perturbations in an infinite medium may actually lead to some observable consequences. Various aspects of this work requires further study. It is necessary to develop a model for dark matter distribution by evolving the collisionless Boltzmann equation from the past taking the expansion of the universe into consideration. Such an investigation is especially important in deciding whether dark matter condensates are truly isolated finite gravitating systems (like, say, galaxies or clusters of galaxies) or whether they extend throughout the universe with an increased density contrast near gravitating objects (Padmanabhan & Vasanthi 1985). It would be also interesting to see how sensitively our results depend on various approximations made in this paper (for example, homogeneity of background, linearization of equations *etc.*). Even at this stage the idea that neutrinos or some other 'inos' of finite mass play an essential role in the dynamics of the universe and are responsible for a wide variety of phenomena otherwise not understandable seems to be quite attractive.

Acknowledgement

We are thankful to Professor R. Cowsik who has made significant contribution to the evolution of ideas presented in this paper.

References

Bahcall, N. A. 1977, *A. Rev. Astr. Astrophys.,* **15,** 505.
Baier, F.W.1983, *Astr. Nachr.,* **304,** 211.
Basdevant, J. L.1984, *Astr. Astrophys.,* (Submitted).
Bond, J. R., Efstathiou, G, Silk, J. 1980, *Phys. Rev. Lett.,* **45,** 1980.
Bond, J. R., Szalay, A. S. 1983, *Astrophys. J.*, **274**, 443.
Bosma, A. 1978, *Ph.D. Thesis,* Gröningen University.
Cabbibo, N., Farrar, G R., Maiani, L. 1981, *Phys. Letts.,* **105B,** 155.
Chubb, T. A. 1983, Preprint (E.O. Hulburt center for space research).
Cowsik, R.1983, *Current Sci*., **52,** 1104.
Cowsik, R. 1985, *J. Astrophys. Astr.,* 7, (inpress).
Cowsik, R., Ghosh, P. 1985, J. *Astrophys. Astr.,* **7**, (in press).
Cowsik, R., McClelland, J.1973, *Astrophys. J.,***180,** 7.
Cowsik, R.,Vasanthi, M.M.1985, *J. Astrophys. Astr.,* **7**, (in press).
Davis, M., Lecar, M., Pryor, C, Witten, E. 1981, *Astrophys. J.,* **250,** 423.

Doroshkevich, A. G., Khlopov, M. Yu., Sunyaev, R. A., Szalay, A. S., Zeldovich, Ya. B. 1981, *Ann. N.Y. Acad. Sci.,* **375,** 32.
Faber, S. M., Gallagher, J. S. 1979, *A. Rev. Astr. Astrophys.,* **17,** 135.
Fukugita, M., Yanagida, 1984, *Phys. Letts.,* **144B,** 386.
Hoessel, J. G., Gunn, J. E., Thuan, T. X. 1980, *Astrophys. J.,* **241,** 486.
Klinkhamer, F. R., Norman, C. A. 1981, *Astrophys. J.,* **243,** L1.
Lifshitz, E. M., Pitaevskii, L. P. 1981, *Physical Kinetics—Landau and Lifshitz course of Theoretical Physics,* Vol. 10 (Pergamon Press).
Marochnik, L. S. 1968, *Sov. Astr.,* **11,** 873.
Melott, A. L. 1983, *Astrophys. J.,* **264,** 59.
Milgrom, M. 1983, *Astrophys. J.,* **270,** 365.
Olive, K. A., Turner, M. S. 1982, *Phys. Rev.,* **D25**, 213.
Padmanabhan, T., Vasanthi, M. M. 1985, *J. Astrophys. Astr.* (Submitted).
Padmanabhan, T., Vasanthi, M. M. 1985, *Nature,* **317,** 335.
Pagels, H., Primack, J. R. 1982, *Phys. Rev. Lett,* **48,** 223.
Peebles, P. J. E. 1979, *Les Houches Summer School Lecture Notes.*
Peebles, P. J. E., 1980, *Large Scale Structure of the Universe,* Princeton Univ. Press, New Jersey.
Peebles, P. J. E. 1982, *Astrophys. J.,* **258,** 415.
Rood, J. H. 1981, *Rep. Prog. Phys.,* **44,** 1077.
Rubin, V. C. 1979, *Comm. Astrophys.,* **8,** 79.
Rubin, V. C, Ford, W. K. J., Thonnard, N., Burstein, D. 1982; *Astrophys. J.,* **261,** 439.
Rubin, V. C, Thonnard, N., Ford, W. K. J. 1982, *Astr. J.,* **87,** 477.
Rubin, V. C, Ford, W. K., Thonnard, N. 1980, *Astrophys. J.,* **238,** 471.
Sato, H., Takahara, F. 1981, *Progress Theor. Phys.,* **66,** 508.
Schramm, D. N., Steigman, G. 1981, *Astrophys. J.,* **243,** 1.
Sikivie, P. 1982, *Phys. Rev. Lett.,* **48,** 1156.
Simpson, J. J. 1985, *Phys. Rev. Lett.,* **54,** 1891.
Wasserman, J. 1981, *Astrophys. J.,* **248,** 1.
Weinberg, S. 1972, *Gravitation and Cosmology,* Wiley, New York.