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Abstract

Consider an irreducible algebraic space curve C which is implicitly defined as the intersection of two algebraic surfaces. There always exists a birational correspondence between the points of C and the points of an irreducible plane curve P whose genus is the same as that of C. Thus C is rational iff the genus of P is zero. When f and g are not tangent along C we present a method of obtaining a projected plane curve P together with a birational mapping betweeen the points of P and C. Together with [4], this method yields an algorithm to compute the genus of C and if the genus is zero, the rational parametric equations for implicitly defined rational space curves C. As a biproduct, this method also yields the implicit and parametric equations of a rational surface containing the space curve.

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1 Introduction

Consider an irreducible algebraic space curve C which is implicitly defined as the intersection of two algebraic surfaces $f(x,y,z) = 0$ and $g(x,y,z) = 0$. We concern ourselves with space curves defined by two surfaces since they are of direct interest to applications in computer-aided design and computer graphics, see Boehm, et. al [7]. Irreducible space curves in general, defined by more than two surfaces are difficult to handle equationally and one needs to resort to computationally less efficient ideal-theoretic methods, Buchberger [9]. However general space curves is a topic with various unresolved issues of mathematical and computational interest and an area of important future research, Abhyankar [1].

Now for an irreducible algebraic space curve C as above, there always exists a birational correspondence between the points of C and the points of an irreducible plane curve P whose genus is the same as that of C, see Walker [18]. Birational correspondence between C and P means that the points of C can be given by rational functions of points of P and vice versa (i.e a 1-1 mapping, except for a finite number of exceptional points, between points of C and P). Together, (i) the method of computing the genus and rational parameterization of algebraic plane curves, Abhyankar and Bajaj [4], and (ii) the method of this paper of constructing a plane curve P along with a birational mapping between the points of P and the given space curve C, gives an algorithm to compute the genus of C and if genus $= 0$ the rational parametric equations of C.

In this paper we now show how, given an irreducible space curve C, defined implicitly as the intersection of two algebraic surfaces, one is able to construct the equation of a plane curve P and a birational mapping between the points of P and C. As a first attempt in constructing P, we may consider the projection of the space curve C along any of the coordinate axis yielding a plane curve whose points are in correspondence with the points of C. Projecting C along, say the z axis, can be achieved by computing the Sylvester resultant of f and g, treating them as polynomials in z, yielding a single polynomial in x and y the coefficients of f and g. The Sylvester resultant eliminates one variable, in this case z, from two equations, see Salmon [14]. Efficient methods are known for computing this resultant for polynomials in any number of variables, see Collins [11], Bajaj and

Royappa [5]. The Sylvester resultant of f and g thus defines a plane algebraic curve P. However this projected plane curve P in general, is not in birational correspondence with the space curve C. For a chosen projection direction it is quite possible that most points of P may correspond to more than one point of C (i.e. a multiple covering of P by C) and hence the two curves are not birationally related. However this approach may be rectified, as explained in this paper, by choosing a valid projection direction which yields a birationally related, projected plane curve P.

There remains the problem of constructing the birational mapping between points on P and C. Let the projected plane curve P be defined by the polynomial $h(\tilde{x}, \tilde{y})$. The map one way is linear and is given trivially by $\tilde{x} = x$ and $\tilde{y} = y$. To construct the reverse rational map one only needs to compute $z = I(\tilde{x}, \tilde{y})$ where I is a rational function. We show in this paper how it is always possible to construct this rational function by use of a polynomial remainder sequence along a valid direction. In fact the resultant is no more than the end result of a polynomial remainder sequence, see Bocher [6], Collins [10].

Note additionally that the reverse rational map, $z = I(\tilde{x}, \tilde{y})$ where I is a rational function is also the rational parametric equation of a rational surface containing the space curve C. Hence constructing a birational mapping between space and plane curves which always exists, also yields an explicit rational surface containing the space curve. By an explicit rational surface we mean one with a known or trivially derivable rational parameterization. For irreducible space curves C, a method of obtaining an explicit rational surface containing C, is given (without proof) in Snyder and Sisam [17]. The technique presented here is similar, but uses a subresultant polynomial remainder sequence, which for an appropriately chosen coordinate direction, provides an efficient way of obtaining the reverse rational map as well as an explicit rational surface containing C.

It is important to note that conversely knowing the rational parametric equations of a rational surface containing a space curve, yields a birational mapping between points on the space curve and a plane curve. Namely, if one of the two surfaces f or g defining the space curve C, or actually any known surface in $I(f, g)$, the Ideal[1] of the curve generated

[1] $I(f, g) = \{h(x, y, z) \mid h = \alpha f + \beta g$ for any polynomials $\alpha(x, y, z)$ and $\beta(x, y, z)\}$.

by f and g is rational with a known rational parameterization, then points on C are easily mapped to a single polynomial equation $h(s,t) = 0$ describing a plane curve P in the parametric plane $s-t$ of the rational surface. This mapping between the (x,y,z) points of C and the (s,t) points of P is birational with the reverse rational map, from the points on P to points on C being given by the parametric equations of the rational surface. For space curves C which have a quadric or a rational cubic surface in its Ideal, the plane curve P and the rational mapping from the points on P to C are easily constructed by using known techniques for parameterizing these rational surface, see Abhyankar and Bajaj [2,3], Sederberg and Snively [16].

The rest of this paper is structured as follows. Section 2 describes a method of choosing a valid direction of projection for the space curve C. This then also yields a projected plane curve P in birational correspondence to C. Using these results, Section 3 describes a method of constructing the reverse rational map between points on the plane curve P and points on C.

2 Valid Projection Direction

To find an appropriate axis of projection, the following general procedure may be adopted. Consider the linear transformation $x = a_1x_1 + b_1y_1 + c_1z_1$, $y = a_2x_1 + b_2y_1 + c_2z_1$ and $z = a_3x_1 + b_3y_1 + c_3z_1$. On substituting into the equations of the two surfaces defining the space curve we obtain the transformed equations $f_1(x_1,y_1,z_1) = 0$ and $g_1(x_1,y_1,z_1) = 0$. Next compute the $Res_{z_1}(f_1,g_1)$ which is a polynomial $h(x_1,y_1)$ describing the projection along the Z axis of the space curve C onto the $z=0$ plane.

Since C is irreducible and f and g are not tangent along C, the order of $h(x_1,y_1)$ is exactly equal to the projection degree, see [1]. By order of $h(x_1,y_1)$ we mean k, if $h(x_1,y_1) = (g(x_1,y_1))^k$. For a birational mapping we desire a projection degree equal to one. Hence, we choose the coefficients of the linear transformation, a_i, b_i and c_i such that (i) the determinant of a_i, b_i and c_i is non zero and (ii) the equation of the projected plane curve $h(x_1,y_1)$ is not a power of an irreducible polynomial. The latter can be

achieved by making the discriminant $Res_{x_1}(h, h_{x_1})$ to be non zero. Note, a random choice of coefficients would also work with probability 1, since the set of coefficients which make the determinant and $Res_{x_1}(h, h_{x_1})$ equal to zero, are restricted to the points of a lower dimensional hypersurface. See [15] where the notion of randomized computations with algebraic varieties is made precise. A suitable choice of coefficients thus ensures that the projected irreducible plane curve given by $h(x_1, y_1)$ is in birational correspondence with the irreducible space curve and thus of the same genus. The parameterization methods of Abhyankar and Bajaj [4] for algebraic plane curves are now applicable and thereby yield a genus computation as well as an algorithm for rationally parameterizing the space curve.

3 Constructing the Birational Map

We choose a valid projection direction by the method described in the earlier section. Without loss of generality let this direction be the Z axis. Let the surfaces $f(x, y, z) = 0$ and $g(x, y, z) = 0$ be of degrees m_1 and m_2 in z, respectively. Again, without loss of generality, assume $m_1 \geq m_2$. Let $F_1 = f(x, y, z)$ and $F_2 = g(x, y, z)$ be given by

$$\begin{aligned} F_1 &= f_0\, z^{m_1} + f_1\, z^{m_1-1} + \ldots + f_{m_1-1}\, z + f_{m_1} \\ F_2 &= g_0\, z^{m_2} + g_1\, z^{m_2-1} + \ldots + g_{m_2-1}\, z + g_{m_2} \end{aligned} \tag{1}$$

with f_j, $(j = 0 \ldots m_1)$ and g_k, $(k = 0 \ldots m_2)$, denoting polynomials in x, y. Then, there exist polynomials $F_{i+2}(x, y, z)$, for $i = 1 \ldots k$, such that $A_i\, F_i = Q_i\, F_{i+1} + B_i\, F_{i+2}$ with m_{i+2}, the degree of z in F_{i+2}, less than m_{i+1}, the degree of z in F_{i+1} and certain polynomials $A_i(x, y)$, $Q_i(x, y, z)$ and $B_i(x, y)$. The polynomials F_{i+2}, $i = 1, 2, \ldots$ form, what is known as a polynomial remainder sequence and can be computed in various different ways, as we now describe.

Let $lc(A)$ denote the leading coefficient of polynomial A, viewed as a polynomial in z, (i.e. coefficient of term with highest z degree). Further let c_i denote $lc(F_i)$. To compute F_{i+2} from F_i and F_{i+1} we first begin with $R_i^0 = F_i$ and then,

$$for \quad k = 1, \ldots, m_i - m_i + 1 + 1$$

$$
\begin{aligned}
if \quad & lc(R_i^k - 1) \;=\; 0 \\
then \quad & R_i^k \;=\; R_i^k - 1 \\
else \quad & R_i^k \;=\; c_{i+1}\, R_i^{k-1} \;-\; z^{m_i - m_{i+1}+1-k}\, lc(R_i^{k-1})\, F_{i+1}
\end{aligned} \tag{2}
$$

The polynomial $R_i^{m_i - m_{i+1}+1}$ is known as the psuedo-remainder of F_i and F_{i+1}. Using Collin's reduced PRS method [10], one constructs the polynomial $F_{i+2} = \frac{R_i^{m_i - m_{i+1}+1}}{d_{i-1}}$ where $d_0 = 1$ and $d_i = c_{i+1}^{m_i - m_{i+1}+1}$. Using Brown's subresultant PRS scheme [8], one constructs the polynomial $F_{i+2} = (-1)^{m_i - m_{i+1}+1} \frac{R_i^{m_i - m_{i+1}+1}}{c_i\, E_{m_i}}$ where $E_{m_1} = 1$ and $E_{m_{i+1}} = \frac{c_{i+1}^{m_i - m_{i+1}}}{E_{m_i}^{m_i - m_{i+1}-1}}$. As shown by Loos [13], both the above methods, as well as others, follow naturally from the subresultant theorem of Habicht [12].

Thus starting with polynomials F_1 and F_2 one constructs the polynomial remainder sequence, $F_1, F_2, F_3, \ldots F_i, \ldots F_r$ with m_i, the z degree of F_i less than m_{i-1}, the z degree of F_{i-1} and $m_r = 0$ (i.e. F_r being independent of z). We choose the subresultant PRS scheme for its computational superiority and also because each $F_i = S_{m_{i-1}-1}, 1 \leq i \leq r$, where S_k is the k^{th} subresultant of F_1 and F_2, see [6, 8, 10, 12].

Now for any i, if F_{i-1} and F_i are of degree greater than two and F_{i+1} is independent of z then the Z axis is not a valid projection direction. This may be seen as follows. Since the Z axis was chosen as a valid projection direction, the $Res_z[f(x,y,z), g(x,y,z)] = Res_z[F_1, F_2]$ is non-zero and not a multiple of some irreducible polynomial. This holds for any two surfaces $f = F_{i-1}$ and F_i in the polynomial remainder sequence where each of the subresultants is also not a multiple of some irreducible polynomial. To complete the argument, it remains to see that by induction if F_{i-1} and F_i are of say degree three and two respectively and F_{i+1} is independent of z then the $Res_z(F_{i-1}, F_i)$ is equal to some $h^3(x,y)$, which is impossible.

Hence in the polynomial remainder sequence there exists a polynomial remainder which is linear in z, i.e., $F_{r-1} = z\Phi_1(x,\, y) - \Phi_2(x,\, y) = 0$. Thus on computing the polynomial remainder sequence and obtaining F_{r-1}, one is able to construct the required inverse map, $z = \frac{\Phi_2(x,\, y)}{\Phi_1(x,\, y)}$, which also is a rational surface containing the space curve. The rational parameterization of this rational surface is trivially given by $x = s, y = t$ and $z = \frac{\Phi_2(s,\, t)}{\Phi_1(s,\, t)}$.

4 Conclusion

The method of the earlier sections of constructing the inverse rational map as well as a rational surface containing the space curve can be applied for reducible as well as irreducible curves, defined implicitly as the intersection of two surfaces. The one limitation however is the assumption of non-tangency of the surfaces meeting along the space curve. It remains open to construct a birational map as well as a rational surface containing a space curve when the two surfaces defining the space curve are also tangent along the entire curve.

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