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THE DEFORMED FIGURES OF THE DEDEKIND ELLIPSOIDS IN THE POST-NEWTONIAN APPROXIMATION TO GENERAL RELATIVITY: CORRECTIONS AND AMPLIFICATIONS

S. CHANDRASEKHAR AND DONNA D. ELBERT
University of Chicago
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ABSTRACT

Two errors in the analysis of an earlier paper (*Ap. J.*, **192**, 731, 1974) on the same subject are corrected. It is found that, as a consequence of the corrections, the solution to the post-Newtonian equations (appropriate to determining the deformed figures of the Dedekind ellipsoid) now diverges at a point where the axes of the ellipsoid are in the ratios 1:0.3370:0.2850. In addition, the fourth-harmonic oscillations of the Dedekind ellipsoid are considered; and it is found that it becomes dynamically unstable when its axes are in the ratios 1:0.3121:0.2680.

Subject headings: relativity — rotation

I. INTRODUCTION

In an earlier paper (Chandrasekhar and Elbert 1974; this paper will be referred to hereafter as *loc. cit.*), we considered the deformed figures of the Dedekind ellipsoid in the first post-Newtonian approximation to general relativity. In view of the complexity of the analysis (and the fact that the work had been carried out intermittently over several years), we have always felt uneasy that the work had not been independently checked by someone else. About a year ago, at our request, Dr. Monique Tassoul undertook to check the analysis of the paper; and she promptly discovered two errors: *first*, that the assumed post-Newtonian velocity-field was not general enough to carry out the analysis consistently; and *second*, that one of the boundary conditions, namely, that the normal component of the streaming velocity must vanish on the free boundary, had not been properly applied. We are immensely grateful to Dr. Tassoul for her patience in scrutinizing the analysis and discovering the errors. While the modifications necessary to amend the errors are readily enough made, the corrections, both substantive and otherwise, are too numerous to make for an easy understanding. On that account, we have preferred to replace entire sections of the paper, so that with the deletions and substitutions the paper can be read coherently.

In addition to the corrections (included in Part I), we have made some amplifications (included in Part II) relating to the fourth-harmonic oscillations of the Dedekind ellipsoid (for which the necessary equations were set out in full in an Appendix to the earlier paper and which is, of course, unaffected by the errors noted).

PART I: CORRECTIONS

Delete the entire text between the beginning of the paragraph following equation (16) on page 733 and the end of § V *on page 738 and replace by the following:*

Turning next to the post-Newtonian terms on the right-hand side of equation (11) which are not expressed as gradients and inserting for the various quantities, we find apart from a factor $(\pi G\rho)^2/c^2$,

$$-x_1[4Q_2(Q_1 + Q_2)I + 8a_2^2B_{12}(A_1 + A_2)]$$

$$+x_1^3[4A_1Q_2(Q_1 + Q_2) - Q_1Q_2^3 + 8a_2^2B_{12}(3A_{11} + A_{12})]$$

$$+x_1x_2^2[4A_2Q_2(Q_1 + Q_2) - Q_1^3Q_2 + 8a_2^2B_{12}(3A_{22} + A_{12}) + 8Q_1(A_1Q_1 + A_2Q_2)]$$

$$+x_1x_3^2[4A_3Q_2(Q_1 + Q_2) + 8a_2^2B_{12}(A_{13} + A_{23})] \qquad (\alpha = 1), \quad (17)$$

and

$$-x_2[4Q_1(Q_1+Q_2)I + 8a_1^2B_{12}(A_1+A_2)]$$
$$+x_2^3[4A_2Q_1(Q_1+Q_2) - Q_2Q_1^3 + 8a_1^2B_{12}(3A_{22}+A_{12})]$$
$$+x_2x_1^2[4A_1Q_1(Q_1+Q_2) - Q_2^3Q_1 + 8a_1^2B_{12}(3A_{11}+A_{12}) + 8Q_2(A_1Q_1+A_2Q_2)]$$
$$+x_2x_3^2[4A_3Q_1(Q_1+Q_2) + 8a_1^2B_{12}(A_{23}+A_{13})] \qquad (\alpha=2), \quad (18)$$

and

$$+x_3x_1^2(8a_1^2Q_2^2A_{13}) + x_3x_2^2(8a_2^2Q_1^2A_{23}) \qquad (\alpha=3). \quad (19)$$

We observe that the coefficient of $x_1x_2^2$ in the expression (17) is not equal to the coefficient of $x_2x_1^2$ in the expression (18). Also, the coefficients of $x_1x_3^2$ and $x_2x_3^2$ in the expressions (17) and (18) are not, respectively, equal to the coefficients of $x_3x_1^2$ and $x_3x_2^2$ in the expression (19). Accordingly, these terms cannot be expressed as the gradient of a scalar function. But equation (11) requires that when these terms are combined with those derived from $v_\beta\partial v_\alpha/\partial x_\beta$ they must be so expressible.

b) *The Post-Newtonian Velocity Field*

As we have already remarked, the Newtonian velocity field, specified in equations (1) and (2), is not consistent with the post-Newtonian equation of continuity (9) to the requisite order. To rectify this situation, we let

$$\frac{v_1}{(\pi G\rho)^{1/2}} = Q_1x_2 + \frac{\pi G\rho}{c^2}\delta v_1, \quad \frac{v_2}{(\pi G\rho)^{1/2}} = Q_2x_1 + \frac{\pi G\rho}{c^2}\delta v_2, \quad \text{and} \quad \frac{v_3}{(\pi G\rho)^{1/2}} = 0 + \frac{\pi G\rho}{c^2}\delta v_3, \quad (20)$$

where δv_1, δv_2, and δv_3 are quantities that are to be determined consistently with equations (5) and (11). With Q_1 and Q_2 defined as in equation (2), equation (5) is satisfied to zero-order (as we should indeed expect); in the next higher order we obtain

$$\frac{\partial}{\partial x_1}\delta v_1 + \frac{\partial}{\partial x_2}\delta v_2 + \frac{\partial}{\partial x_3}dv_3 = -\frac{1}{\pi G\rho}\left(Q_1x_2\frac{\partial}{\partial x_1} + Q_2x_1\frac{\partial}{\partial x_2}\right)\left(v^2 + 2U + \frac{p}{\rho}\right), \quad (21)$$

or, inserting for v^2, U, and p/ρ their Newtonian values (as we may for determining the post-Newtonian terms δv_α), we obtain

$$\frac{\partial}{\partial x_1}\delta v_1 + \frac{\partial}{\partial x_2}\delta v_2 + \frac{\partial}{\partial x_3}\delta v_3 = -2\left[Q_1\left(Q_2^2 - 2A_1 - \frac{a_3^2}{a_1^2}A_3\right) + Q_2\left(Q_1^2 - 2A_2 - \frac{a_3^2}{a_2^2}A_3\right)\right]x_1x_2. \quad (22)$$

A *particular solution* of equation (22) is given by

$$\delta v_1 = q_1x_1^2x_2, \quad \delta v_2 = q_2x_2^2x_1, \quad \text{and} \quad \delta v_3 = q_3x_3x_1x_2, \quad (23)$$

provided

$$q_1 + q_2 + \tfrac{1}{2}q_3 = -\left[Q_1\left(Q_2^2 - 2A_1 - \frac{a_3^2}{a_1^2}A_3\right) + Q_2\left(Q_1^2 - 2A_2 - \frac{a_3^2}{a_2^2}A_3\right)\right]. \quad (24)$$

For the velocity field (20) with δv_α given by equations (23), we find that

$$\frac{v_\beta}{\pi G\rho}\frac{\partial v_1}{\partial x_\beta} = Q_1Q_2x_1 + \frac{\pi G\rho}{c^2}[(Q_1q_2 + 2q_1Q_1)x_1x_2^2 + Q_2q_1x_1^3], \quad (25)$$

$$\frac{v_\beta}{\pi G\rho}\frac{\partial v_2}{\partial x_\beta} = Q_1Q_2x_2 + \frac{\pi G\rho}{c^2}[(Q_2q_1 + 2q_2Q_2)x_2x_1^2 + Q_1q_2x_2^3], \quad (26)$$

$$\frac{v_\beta}{\pi G\rho}\frac{\partial v_3}{\partial x_\beta} = 0 + \frac{\pi G\rho}{c^2}(Q_2q_3x_3x_1^2 + Q_1q_3x_3x_2^2). \quad (27)$$

The Newtonian terms on the left-hand side of equations (25) and (26) are clearly expressible as the gradient of $\frac{1}{2}Q_1Q_2(x_1^2 + x_2^2)$. We now require that when the terms (17) and (18) are combined with the terms in (25) and (26) (in accordance with eq. [11]) they are also expressible as the gradient of a scalar function. This latter requirement can be met if

$$4A_2Q_2(Q_1+Q_2) - Q_1^3Q_2 + 8a_2^2B_{12}(3A_{22}+A_{12}) + 8Q_1(A_1Q_1+A_2Q_2)$$
$$- [4A_1Q_1(Q_1+Q_2) - Q_2^3Q_1 + 8a_1^2B_{12}(3A_{11}+A_{12}) + 8Q_2(A_1Q_1+A_2Q_2)]$$
$$= Q_1(q_2+2q_1) - Q_2(q_1+2q_2)\,. \quad (28)$$

Equations (24) and (28) provide two equations for the three "unknowns" q_1, q_2, and q_3.

We now proceed to write down the general solution of equation (22). It will appear that in the post-Newtonian Dedekind configuration, the velocity field can, at most, be a cubic polynomial in the coordinates. Consistent with this requirement and compatible with the expression for δU (eq. [44] below) we shall write the general solution in the form

$$\delta v_1 = (q_1+q)x_2x_1^2 + r_1x_2^3 + t_1x_2x_3^2$$

and

$$\delta v_2 = (q_2-q)x_1x_2^2 + r_2x_1^3 + t_2x_1x_3^2\,, \quad (29)$$

where q, r_1, r_2, t_1, and t_2 are constants, unspecified for the present. By virtue of equation (24), equation (22) is identically satisfied by the solution (29); the constants in the solution (29) are, therefore, not restricted in any manner by equation (22).

The additional terms in the solution (29) contribute to the right-hand sides of equations (25) and (26) the further post-Newtonian terms

$$(Q_1r_2+Q_2q)x_1^3 + (Q_1t_2+Q_2t_1)x_1x_3^2 + (3r_1Q_2+qQ_1)x_1x_2^2 \quad (30)$$

and

$$(Q_2r_1-Q_1q)x_2^3 + (Q_1t_2+Q_2t_1)x_2x_3^2 + (3r_2Q_1-qQ_2)x_2x_1^2\,, \quad (31)$$

respectively. To satisfy the requirement that the right-hand side of equation (11) continues to be the gradient of a scalar function, we impose the condition

$$3r_1Q_2 + qQ_1 = 3r_2Q_1 - qQ_2$$

or

$$q(Q_1+Q_2) = 3(r_2Q_1 - r_1Q_2)\,, \quad (32)$$

which maintains the equality of the coefficients of $x_1x_2^2$ and $x_2x_1^2$.

Turning next to the consideration of the terms in $x_1x_3^2$ and $x_3x_1^2$, and $x_2x_3^2$ and $x_3x_2^2$, we first observe that the terms on the right-hand side of equation (27) combine with the terms (19) to give

$$x_3x_1^2(8a_1^2Q_2^2A_{13} - Q_2q_3) \qquad \text{and} \qquad x_3x_2^2(8a_2^2Q_1^2A_{23} - Q_1q_3)\,. \quad (33)$$

The corresponding terms in $x_1x_3^2$ and $x_2x_3^2$ are obtained by combining the terms in the expressions (17) and (30) and in the expressions (18) and (31); we thus obtain

$$x_1x_3^2[4A_3Q_2(Q_1+Q_2) + 8a_2^2B_{12}(A_{13}+A_{23}) - (Q_1t_2+Q_2t_1)]$$

and

$$x_2x_3^2[4A_3Q_1(Q_1+Q_2) + 8a_1^2B_{12}(A_{23}+A_{13}) - (Q_1t_2+Q_2t_1)]\,. \quad (34)$$

The integrability of equation (11) requires that the coefficients of $x_3x_1^2$ and $x_3x_2^2$ in the expressions (33) agree, respectively, with those of $x_1x_3^2$ and $x_2x_3^2$ in the expressions (34); and these requirements give

$$4A_3Q_2(Q_1+Q_2) + 8a_2^2B_{12}(A_{13}+A_{23}) - (Q_1t_2+Q_2t_1) = 8a_1^2Q_2^2A_{13} - Q_2q_3 \quad (35)$$

and

$$4A_3Q_1(Q_1+Q_2) + 8a_1^2B_{12}(A_{23}+A_{13}) - (Q_1t_2+Q_2t_1) = 8a_2^2Q_1^2A_{23} - Q_1q_3\,. \quad (36)$$

Multiplying equation (35) by Q_1 and subtracting from it equation (36) multiplied by Q_2, we find (on using the relation $Q_1Q_2 = -2B_{12}$) that

$$Q_1t_2 + Q_2t_1 = 0\,. \quad (37)$$

It now readily follows from equation (35) (or [36]) that

$$q_3 = -4A_3(Q_1+Q_2) + 4a_1^2Q_2A_{13} + 4a_2^2Q_1A_{23}\,. \quad (38)$$

Equations (24), (28), and (38) suffice to determine q_1, q_2, and q_3 uniquely; and the particular solution sought becomes determinate. It remains to determine q, r_1, r_2, t_1, and t_2; and so far we have obtained two equations, namely, equations (32) and (37), between them.

With the reductions that have been effected until now, we can write the integral of equation (11) in the form

$$\begin{aligned}\frac{1}{\pi G\rho}\frac{p}{\rho} = {} & a_3^2 A_3\left(1-\sum_{\mu=1}^{3}\frac{x_\mu^2}{a_\mu^2}\right)+\delta U \\ & +\frac{\pi G\rho}{c^2}\Bigg\{2\Phi+2v^2U+\frac{1}{2}\left(\frac{p}{\rho}\right)^2 \\ & +\tfrac{1}{4}x_1^4[4A_1Q_2(Q_1+Q_2)-Q_1Q_2^3+8a_2^2B_{12}(3A_{11}+A_{12})-Q_2q_1-(Q_1r_2+Q_2q)] \\ & +\tfrac{1}{4}x_2^4[4A_2Q_1(Q_1+Q_2)-Q_2Q_1^3+8a_1^2B_{12}(3A_{22}+A_{12})-Q_1q_2-(Q_2r_1-Q_1q)] \\ & +\tfrac{1}{2}x_1^2x_2^2\left[\begin{array}{l}4A_2Q_2(Q_1+Q_2)-Q_1^3Q_2+8a_2^2B_{12}(3A_{22}+A_{12})+8Q_1(A_1Q_1+A_2Q_2) \\ \qquad -Q_1q_2-2q_1Q_1-(Q_1q+3r_1Q_2) \\ \textit{or} \\ 4A_1Q_1(Q_1+Q_2)-Q_2^3Q_1+8a_1^2B_{12}(3A_{11}+A_{12})+8Q_2(A_1Q_1+A_2Q_2) \\ \qquad -Q_2q_1-2q_2Q_2+(Q_2q-3r_2Q_1)\end{array}\right] \\ & +\tfrac{1}{2}x_1^2x_3^2[4A_3Q_2(Q_1+Q_2)+8a_2^2B_{12}(A_{13}+A_{23})] \\ & +\tfrac{1}{2}x_2^2x_3^2[4A_3Q_1(Q_1+Q_2)+8a_1^2B_{12}(A_{13}+A_{23})] \\ & -\tfrac{1}{2}x_1^2[4Q_2(Q_1+Q_2)I+8a_2^2B_{12}(A_1+A_2)] \\ & -\tfrac{1}{2}x_2^2[4Q_1(Q_1+Q_2)I+8a_1^2B_{12}(A_1+A_2)]\Bigg\}, \end{aligned} \tag{39}$$

where δU is the change in the Newtonian gravitational potential of the deformed post-Newtonian configuration. For the sake of brevity, we shall rewrite equation (31) in the form

$$\begin{aligned}\frac{1}{\pi G\rho}\frac{p}{\rho} = {} & a_3^2 A_3\left(1-\sum_{\mu=1}^{3}\frac{x_\mu^2}{a_\mu^2}\right)+\delta U \\ & +\frac{\pi G\rho}{c^2}\{\alpha_1x_1^2+\alpha_2x_2^2+\alpha_3x_3^2+\alpha_{33}x_3^4+\alpha_{13}x_1^2x_3^2+\alpha_{23}x_2^2x_3^2 \\ & +[\alpha_{11}-\tfrac{1}{4}(Q_1r_2+Q_2q)]x_1^4+[\alpha_{22}-\tfrac{1}{4}(Q_2r_1-Q_1q)]x_2^4 \\ & +[\alpha_{12}-\tfrac{1}{2}(Q_1q+3r_1Q_2)\ \textit{or}\ +\tfrac{1}{2}(Q_2q-3r_2Q_1)]x_1^2x_2^2\}, \end{aligned} \tag{40}$$

where α_1, α_{11}, α_{12}, etc., are quantities which can be read off by comparison with equation (39).

IV. THE NATURE OF THE POST-NEWTONIAN DEFORMATION AND THE CHANGE IN THE GRAVITATIONAL POTENTIAL CAUSED BY IT

We shall suppose that the post-Newtonian figure is obtained by a deformation of the Newtonian figure by the application of a suitable Lagrangian displacement at each point of its interior and the boundary. It is clear that the nature of the deformation considered in Paper III, § IV, in the context of the Jacobian figures will suffice equally in the present context of the Dedekind figures. We shall suppose, then, that (cf. Paper III, eqs. [47], [48], and [56])

$$\xi = \frac{\pi G\rho a_1^2}{c^2}\sum_{i=1}^{5}S_i\xi^{(i)} \tag{41}$$

where

$$\xi^{(1)} = (x_1, 0, -x_3), \qquad \xi^{(2)} = (0, x_2, -x_3), \qquad \xi^{(3)} = \frac{1}{a_1^2}(\tfrac{1}{3}x_1^3, -x_1^2x_2, 0),$$

$$\xi^{(4)} = \frac{1}{a_1^2}(0, \tfrac{1}{3}x_2^3, -x_2^2x_3), \qquad \text{and} \qquad \xi^{(5)} = \frac{1}{a_1^2}(-x_3^2x_1, 0, \tfrac{1}{3}x_3^3). \tag{42}$$

The deformation of the Dedekind ellipsoid by the displacement (33) will change the gravitational potential U by the amount

$$\delta U = \frac{\pi G \rho a_1^2}{c^2} \sum_{i=1}^{5} S_i \delta U^{(i)} , \tag{43}$$

where expressions for $\delta U^{(i)}$ are given in Paper III, equations (70)–(73); and as in Paper III, we can write

$$\delta U = \frac{(\pi G \rho)^2}{c^2} \left\{ a_1^2 \sum_{i=1}^{2} S_i \left[u_0^{(i)} + \sum_{\mu=1}^{3} u_\mu^{(i)} x_\mu^2 \right] + \sum_{i=3}^{5} S_i \left[u_0^{(i)} + \sum_{\mu=1}^{3} u_\mu^{(i)} x_\mu^2 + \sum_{\mu=1}^{3} u_{\mu\mu}^{(i)} x_\mu^4 + \sum_{\mu,\nu}^{12,23,31} u_{\mu\nu}^{(i)} x_\mu^2 x_\nu^2 \right] \right\} . \tag{44}$$

V. THE DETERMINATION OF THE POST-NEWTONIAN FIGURE

Returning to equation (40), we shall now rewrite it in the form

$$\frac{1}{\pi G \rho} \frac{p}{\rho} = a_3^2 A_3 \left(1 - \sum_{\mu=1}^{3} \frac{x_\mu^2}{a_\mu^2} \right) + \frac{\pi G \rho}{c^2} \left[\sum_{\mu=1}^{3} P_\mu x_\mu^2 + \sum_{\mu=1}^{3} P_{\mu\mu} x_\mu^4 + \sum_{\mu,\nu}^{12,23,31} P_{\mu\nu} x_\mu^2 x_\nu^2 \right] , \tag{45}$$

where

$$P_\mu = \alpha_\mu + \sum_{i=1}^{2} S_i a_1^2 u_\mu^{(i)} + \sum_{i=3}^{5} S_i u_\mu^{(i)} ,$$

$$P_{11} = \alpha_{11} - \tfrac{1}{4}(Q_1 r_2 + Q_2 q) + \sum_{i=3}^{5} S_i u_{11}^{(i)} ,$$

$$P_{22} = \alpha_{22} - \tfrac{1}{4}(Q_2 r_1 - Q_1 q) + \sum_{i=3}^{5} S_i u_{22}^{(i)} ,$$

$$P_{33} = \alpha_{33} + \sum_{i=3}^{5} S_i u_{33}^{(i)} ,$$

$$P_{12} = \alpha_{12} - \tfrac{1}{2}(Q_1 q + 3 r_1 Q_2) + \sum_{i=3}^{5} S_i u_{12}^{(i)} ,$$

$$P_{13} = \alpha_{13} + \sum_{i=3}^{5} S_i u_{13}^{(i)} , \quad \text{and} \quad P_{23} = \alpha_{23} + \sum_{i=3}^{5} S_i u_{23}^{(i)} . \tag{46}$$

It remains to apply the proper boundary conditions to the solutions which we have found for the velocity field (eqs. [20] and [29]) and the pressure distribution (eq. [46]) and determine the 10 constants q, r_1, r_2, t_1, t_2, S_1, S_2, S_3, S_4, and S_5.

The boundary conditions that have to be applied on the bounding surface,

$$S(x) = \sum_{\mu=1}^{3} \frac{x_\mu^2}{a_\mu^2} - 1 - \frac{2\pi G \rho}{c^2} \left[S_1 a_1^2 \left(\frac{x_1^2}{a_1^2} - \frac{x_3^2}{a_3^2} \right) + S_2 a_1^2 \left(\frac{x_2^2}{a_2^2} - \frac{x_3^2}{a_3^2} \right) + S_3 \left(\frac{x_1^4}{3a_1^2} - \frac{x_1^2 x_2^2}{a_2^2} \right) + S_4 \left(\frac{x_2^4}{3a_2^2} - \frac{x_2^2 x_3^2}{a_3^2} \right) + S_5 \left(\frac{x_3^4}{3a_3^2} - \frac{x_3^2 x_1^2}{a_1^2} \right) \right] = 0 , \tag{47}$$

of the deformed ellipsoid, are that *the normal component of the velocity and the pressure vanish on it identically.*

The requirement that the normal component of the velocity vanishes on the surface defined by equation (47) is

$$v_\mu \frac{\partial S}{\partial x_\mu} = 0 \qquad on \qquad S(x) = 0 \tag{48}$$

For the velocity field specified by equation (20), equation (48) gives

$$\left(Q_1 x_2 + \frac{\pi G \rho}{c^2} \delta v_1 \right) \frac{\partial S}{\partial x_1} + \left(Q_2 x_1 + \frac{\pi G \rho}{c^2} \delta v_2 \right) \frac{\partial S}{\partial x_2} + \frac{\pi G \rho}{c^2} \delta v_3 \frac{\partial S}{\partial x_3} = 0 \qquad on \qquad S(x) = 0 \tag{49}$$

For Q_1 and Q_2 related as in equation (2), equation (49) is automatically satisfied in the Newtonian approximation; in the post-Newtonian approximation, equation (49) gives

$$-2(S_1 - S_2)Q_1 + x_1^2\left[\frac{1}{a_1^2}(q_1 + q) + \frac{r_2}{a_2^2} - \frac{4S_3}{3a_1^2}Q_1 + 2\frac{S_3}{a_2^2}Q_2\right] + x_2^2\left[\frac{1}{a_2^2}(q_2 - q) + \frac{r_1}{a_1^2} - \frac{4S_4}{3a_2^2}Q_2 + 2\frac{S_3}{a_2^2}Q_1\right] + x_3^2\left[\left(\frac{t_1}{a_1^2} + \frac{t_2}{a_2^2} + \frac{q_3}{a_3^2}\right) + 2\frac{S_5}{a_1^2}Q_1 + 2\frac{S_4}{a_3^2}Q_2\right] = 0 \tag{50}$$

on

$$\frac{x_1^2}{a_1^2} + \frac{x_2^2}{a_2^2} + \frac{x_3^2}{a_3^2} = 1\,. \tag{51}$$

The condition, that the expression on the left-hand side of equation (50) vanishes on the surface of the undistorted ellipsoid, is that the coefficients of x_1^2, x_2^2, and x_3^2 in the expression are, respectively, equal to

$$2(S_1 - S_2)\frac{Q_1}{a_1^2}, \quad 2(S_1 - S_2)\frac{Q_1}{a_2^2}, \quad \text{and} \quad 2(S_1 - S_2)\frac{Q_1}{a_3^2}. \tag{52}$$

These conditions lead to the equations

$$\left[a_1^2\frac{Q_1 + Q_2}{3Q_1} - (a_1^2 - a_2^2)\right]q = 2(a_1^2 + a_2^2)(S_1 - S_2) - (a_1^2q_2 + a_2^2q_1) + [\tfrac{4}{3}a_2^2Q_1 - 2a_1^2(Q_1 + Q_2)]S_3 + \tfrac{4}{3}a_1^2Q_2S_4\,, \tag{53}$$

$$r_1 = \frac{a_1^2}{a_2^2}[2Q_1(S_1 - S_2) + \tfrac{4}{3}Q_2S_4 - 2Q_1S_3 + (q - q_2)]\,, \tag{54}$$

$$r_2 = \frac{1}{a_1^2}[2a_2^2Q_1(S_1 - S_2) + (\tfrac{4}{3}a_2^2Q_1 - 2a_1^2Q_2)S_3 - a_2^2(q + q_1)] \tag{55}$$

and

$$\frac{t_1}{a_1^2} + \frac{t_2}{a_2^2} + \frac{q_3}{a_3^2} = \frac{2}{a_3^2}Q_1(S_1 - S_2) - 2\frac{S_5}{a_1^2}Q_1 - 2\frac{S_4}{a_3^2}Q_2\,. \tag{56}$$

In obtaining the solutions (53), (54), and (55) for q, r_1, and r_2, we have made use of the relations (32).

We observe that the constants q, r_1, and r_2 are expressed in terms of the S's. Accordingly, all of the expressions listed in equations (46) are expressible in terms of the S's.

Turning next to the boundary condition which requires the vanishing of the pressure p on the bounding surface defined by equation (47), we observe that in view of the formal identity of equation (45) and equation (76) of Paper III, the discussion of the boundary condition in Paper III, § VII (eqs. [76]–[87]), applies unchanged. Therefore, with the definitions (cf. Paper III, eq. [81])

$$\bar{Q}_1 = P_1 - 2a_3^2A_3S_1\,, \quad \bar{Q}_2 = P_2 - 2a_3^2\frac{a_1^2}{a_2^2}A_3S_2\,, \quad \bar{Q}_3 = P_3 + 2a_1^2A_3(S_1 + S_2)\,,$$

$$Q_{11} = P_{11} - \frac{2a_3^2A_3}{3a_1^2}S_3\,, \quad Q_{22} = P_{22} - \frac{2a_3^2A_3}{3a_2^2}S_4\,, \quad Q_{33} = P_{33} - \tfrac{2}{3}A_3S_5\,,$$

$$Q_{12} = P_{12} + \frac{2a_3^2A_3}{a_2^2}S_3\,, \quad Q_{23} = P_{23} + 2A_3S_4\,, \quad Q_{31} = P_{31} + \frac{2a_3^2A_3}{a_1^2}S_5\,, \tag{57}$$

the boundary condition yields the equations

$$a_1^4Q_{11} + a_2^4Q_{22} - a_1^2a_2^2Q_{12} = 0\,, \quad a_2^4Q_{22} + a_3^4Q_{33} - a_2^2a_3^2Q_{23} = 0\,,$$

$$a_3^4Q_{33} + a_1^4Q_{11} - a_3^2a_1^2Q_{31} = 0\,,$$

$$a_1^4Q_{11} - a_2^4Q_{22} + a_1^2\bar{Q}_1 - a_2^2\bar{Q}_2 = 0\,, \quad \text{and} \quad a_3^4Q_{33} - a_1^4Q_{11} + a_3^2\bar{Q}_3 - a_1^2\bar{Q}_1 = 0\,. \tag{58}$$

By virtue of equations (53)–(56), the foregoing equations provide five linear equations for S_1, S_2, S_3, S_4, and S_5; and they will, accordingly, suffice to determine them. With the S's thus determined, equations (53)–(56) will determine q, r_1, and r_2; and equations (37) and (56) will determine t_1 and t_2; and the solution of the problem will be completed.

Section VI (*The Binding Energy*) *has no corrections. Delete* § VII (*including Table 1*) *and replace by the following new* § VII:

VII. NUMERICAL RESULTS

In Table 1 the various constants which determine the deformed figure of the Dedekind ellipsoid are listed. The table also includes $(\Delta E)_{\text{proper vol}}$.

It will be observed that the solution of the post-Newtonian equations diverges at

$$a_2/a_1 = 0.3370 \qquad \text{and} \qquad a_3/a_1 = 0.2850\,;$$

this point is much further along the sequence than the erroneous calculations had indicated. Some further remarks concerning this point of divergence are made in Part II.

1978ApJ...220..303C

TABLE 1

THE CONSTANTS OF THE POST-NEWTONIAN DEDEKIND CONFIGURATIONS

a_2	q_1	q_2	q_3	S_1	S_2	S_3	S_4	S_5	q	r_1	r_2	t_1	t_2	$(\Delta E)_{\text{prop. vo}}$
0.85	2.666	−2.984	0.3414	−1.120	−0.744	+0.3850	−1.917	−11.91	−2.978	−0.313	−0.049	−4.718	−3.409	−1.127
0.80	2.607	−3.041	0.4691	−0.9854	−0.7990	+0.3374	−1.670	−12.56	−3.010	−0.399	−0.106	−6.835	−4.375	−0.9577
0.75	2.544	−3.101	0.6052	−0.8566	−0.8451	+0.3095	−1.392	−13.38	−3.058	−0.4982	−0.1658	−9.386	−5.280	−0.7942
0.70	2.478	−3.163	0.7511	−0.7315	−0.8833	+0.3044	−1.068	−14.41	−3.129	−0.6132	−0.2314	−12.54	−6.144	−0.6384
0.66	2.424	−3.215	0.8758	−0.6330	−0.9099	+0.3213	−0.7614	−15.45	−3.212	−0.7210	−0.2903	−15.67	−6.824	−0.5187
0.65	2.410	−3.228	0.9082	−0.6085	−0.9163	+0.3292	−0.6763	−15.75	−3.238	−0.7508	−0.3062	−16.56	−6.995	−0.4897
0.64	2.396	−3.241	0.9412	−0.5838	−0.9225	+0.3387	−0.5872	−16.06	−3.267	−0.7819	−0.3227	−17.50	−7.168	−0.4612
0.63	2.382	−3.255	0.9747	−0.5592	−0.9287	+0.3500	−0.4937	−16.40	−3.299	−0.8146	−0.3399	−18.50	−7.342	−0.4328
0.62	2.368	−3.269	1.0087	−0.5343	−0.9349	+0.3634	−0.3951	−16.75	−3.334	−0.8490	−0.3578	−19.56	−7.518	−0.4047
0.61	2.354	−3.283	1.0434	−0.5093	−0.9411	+0.3789	−0.2914	−17.13	−3.373	−0.8850	−0.3765	−20.69	−7.697	−0.3768
0.60	2.339	−3.297	1.0786	−0.4841	−0.9474	+0.3970	−0.1817	−17.54	−3.416	−0.9235	−0.3963	−21.89	−7.880	−0.3492
0.59	2.325	−3.311	1.114	−0.4586	−0.9539	+0.4177	−0.0652	−17.97	−3.465	−0.9643	−0.4172	−23.17	−8.067	−0.3217
0.58	2.311	−3.325	1.151	−0.4327	−0.9606	+0.4415	+0.0585	−18.43	−3.518	−1.0079	−0.4392	−24.55	−8.259	−0.2944
0.57	2.297	−3.340	1.188	−0.4063	−0.9677	+0.4686	+0.1906	−18.93	−3.579	−1.0545	−0.4627	−26.03	−8.457	−0.2671
0.56	2.282	−3.355	1.226	−0.3793	−0.9751	+0.4997	+0.3321	−19.46	−3.647	−1.105	−0.4878	−27.62	−8.662	−0.2401
0.55	2.268	−3.370	1.264	−0.3516	−0.9832	+0.5352	+0.4842	−20.04	−3.724	−1.160	−0.5148	−29.35	−8.877	−0.2129
0.50	2.198	−3.449	1.469	−0.1958	−1.0371	+0.8074	+1.466	−23.80	−4.302	−1.539	−0.6907	−40.63	−10.159	−0.0737
0.45	2.130	−3.535	1.696	+0.0272	−1.146	+1.394	+3.149	−30.26	−5.556	−2.330	−1.0051	−60.20	−12.19	+0.0901
0.40	2.069	−3.633	1.950	+0.5029	−1.467	+3.061	+7.072	−44.97	−9.206	−4.890	−1.796	−105.55	−16.89	+0.3717
0.35	2.021	−3.746	2.239	+4.310	−4.548	+18.42	+37.78	−155.6	−43.64	−34.11	−8.588	−453.5	−55.55	+2.303
0.34	2.013	−3.771	2.301	+20.2	−17.7	+83.6	+164.	−606.	−190.	−165.	−37.1	−1876.	−217.0	+10.20
0.33	2.007	−3.797	2.365	−9.328	+6.748	−37.47	−69.94	+227.8	+82.24	+78.80	+15.84	+759.5	+82.71	−4.458
0.32	2.001	−3.824	2.432	−4.112	+2.451	−16.16	−28.40	+79.40	+34.30	+36.75	+6.500	+290.6	+29.76	−1.858
0.30	1.992	−3.882	2.570	−2.152	+0.8497	−8.186	−12.28	+21.05	+16.43	+22.35	+2.972	+107.71	+9.694	−0.8683
0.28	1.989	−3.946	2.718	−1.567	+0.387	−5.832	−6.948	+1.100	+11.21	+19.64	+1.903	+46.1	+3.61	−0.563
0.25	1.996	−4.054	2.958	−1.221	+0.1250	−4.447	−2.644	−16.19	+8.232	+21.56	+1.225	−5.537	−0.3461	−0.3618

PART II: AMPLIFICATIONS

VIII. THE EVEN MODES OF OSCILLATION OF THE DEDEKIND ELLIPSOID BELONGING TO THE FOURTH HARMONICS

We have emphasized, in the context of the deformed post-Newtonian figures of the Maclaurin spheroids and the Jacobi ellipsoids, the crucial importance of the relationship between the points where the solution to the post-Newtonian equations diverge and where the Newtonian configuration becomes unstable (secularly or dynamically) by a mode of oscillation belonging to an appropriate harmonic. Along the Maclaurin and the Jacobi sequences the two points coincide. Thus, the Jacobi ellipsoid allows a nontrivial neutral mode of deformation belonging to the fourth harmonic at precisely the same point where the solution to the post-Newtonian equations diverge (cf. Paper III). In contrast, as we have shown in *loc. cit.* (Appendix), *the Dedekind ellipsoid allows, nowhere along its sequence, a nontrivial neutral mode of deformation belonging to the fourth harmonics* (though it does allow a "trivial" neutral deformation along its entire sequence). But the question whether the Dedekind ellipsoid allows a nontrivial *dynamical* neutral mode of deformation was left open. To answer this question, one must treat the full set of dynamical equations which govern small time-dependent perturbations about equilibrium. We have now carried out such a treatment with results that we shall now describe.

In the Appendix to the earlier paper (*loc. cit.*) we have written out in full the complete set of the fourth-order virial-equations which will enable us to determine the characteristic frequencies of the relevant fourth-harmonic oscillations of the Dedekind ellipsoid. Thus, by setting the dependence on time of all the quantities governing the perturbation as $e^{\lambda t}$, we shall obtain from the 16 equations included in *loc. cit.*, equation (A21), a set of 12 equations after the elimination of $\delta\Pi_{11}$, $\delta\Pi_{22}$, $\delta\Pi_{33}$, and $\delta\Pi_{12}$. These 12 equations supplemented by the four divergence-conditions which follow from *loc. cit.* (eqs. [A17] and [A23] with V_{12} set equal to zero) will provide us with a homogeneous system of 16 linear equations for the 16 fourth-order virials which are even in the index 3. The vanishing of the determinant of this system will give us the required characteristic equation for λ. By suitably combining the rows and columns of the characteristic determinant of order 16, we can reduce it to one of order 6 in which λ occurs either as λ^2 or λ^4. A sample of the determinant obtained for the case $a_2/a_1 = 0.31$ is given below.

We supplied several such reduced determinants to P. S. Marcus and W. H. Press, who kindly programmed them for evaluating the roots. In Table 2 we list the squares of the characteristic frequencies (σ) for some values of a_2/a_1 along the Dedekind sequence.

We observe that the characteristic equation allows 11 roots for σ^2 including one double root and one zero root; and further that it becomes dynamically unstable along the sequence by a mode of oscillation belonging to this group. In Table 3 we compare the points of onset of the third and the fourth harmonic instabilities along the Jacobi and the Dedekind sequences.

$$
\begin{vmatrix}
\begin{array}{l}+0.5716879\lambda^4\\+3.6982788\lambda^2\\+2.6558027\end{array} & \begin{array}{l}\\+2.7322378\lambda^2\\+2.4164324\end{array} & \begin{array}{l}\\-9.0661380\lambda^2\\-41.4159200\end{array} & \begin{array}{l}\\+3.9825714\lambda^2\\-1.5803540\end{array} & \begin{array}{l}+1.1032985\lambda^4\\+2.0617310\lambda^2\\-0.2082314\end{array} & \begin{array}{l}+0.3296352\lambda^4\\-0.8173564\lambda^2\\ \end{array} \\
\begin{array}{l}-0.0549392\lambda^4\\-0.5938072\lambda^2\\-0.2137989\end{array} & \begin{array}{l}\\+0.0047560\lambda^2\\+0.0042063\end{array} & \begin{array}{l}-0.7736632\lambda^4\\-25.9573668\lambda^2\\-49.7534386\end{array} & \begin{array}{l}+3.4301273\lambda^4\\+2.6356764\lambda^2\\-2.5888895\end{array} & \begin{array}{l}\\+0.3257403\lambda^2\\-0.9657067\end{array} & \begin{array}{l}-0.3296352\lambda^4\\-0.7776569\lambda^2\\ \end{array} \\
\begin{array}{l}-0.2142358\lambda^4\\-0.5840680\lambda^2\\-0.2980414\end{array} & \begin{array}{l}-0.3185932\lambda^4\\-0.5684864\lambda^2\\-0.2535775\end{array} & \begin{array}{l}-0.0216130\lambda^4\\+1.4000537\lambda^2\\-1.0740158\end{array} & \begin{array}{l}-0.6891162\lambda^4\\-0.2659677\lambda^2\\-0.1447172\end{array} & \begin{array}{l}+0.2078641\lambda^4\\+0.1997145\lambda^2\\-0.0649373\end{array} & \begin{array}{l}+0.1249765\lambda^4\\+0.0130613\lambda^2\\ \end{array} \\
\begin{array}{l}-0.5455527\lambda^4\\+2.7911903\lambda^2\\-2.4117127\end{array} & \begin{array}{l}-2.2344812\lambda^4\\-4.6406562\lambda^2\\-2.3564776\end{array} & \begin{array}{l}-18.4414895\lambda^4\\-191.8010840\lambda^2\\-29.5369760\end{array} & \begin{array}{l}+32.4615837\lambda^4\\-14.7837560\lambda^2\\-5.1755040\end{array} & \begin{array}{l}-3.8838134\lambda^4\\-6.4421870\lambda^2\\+0.8506658\end{array} & \begin{array}{l}-7.0481481\lambda^4\\+1.0011837\lambda^2\\ \end{array} \\
\begin{array}{l}\\+0.0250508\lambda^2\\+0.0135801\end{array} & \begin{array}{l}\\+0.0083538\lambda^2\\+0.0073882\end{array} & \begin{array}{l}-0.0405966\lambda^4\\+0.0699097\lambda^2\\+1.4260702\end{array} & \begin{array}{l}\\-0.0078246\lambda^2\\+0.0973182\end{array} & \begin{array}{l}-0.0039013\lambda^4\\-0.0386890\lambda^2\\+0.0251057\end{array} & \begin{array}{l}+0.3296352\lambda^4\\+0.3141873\lambda^2\\ \end{array} \\
\begin{array}{l}\\+0.0924928\lambda^2\\+0.1413118\end{array} & \begin{array}{l}\\+0.0869280\lambda^2\\+0.0768804\end{array} & \begin{array}{l}+3.8525682\lambda^4\\+15.5445262\lambda^2\\+14.8394397\end{array} & \begin{array}{l}+3.4301273\lambda^4\\+2.9522350\lambda^2\\+1.0126760\end{array} & \begin{array}{l}+0.0405966\lambda^4\\+0.1837705\lambda^2\\+0.2612455\end{array} & \begin{array}{l}-0.3296352\lambda^4\\-0.0558117\lambda^2\\ \end{array}
\end{vmatrix} = 0
$$

TABLE 2

SQUARES OF THE CHARACTERISTIC FREQUENCIES σ^2 (in the unit $\pi G\rho$) OF THE EVEN FOURTH-HARMONIC OSCILLATIONS OF THE DEDEKIND ELLIPSOID

$a_2/a_1 = 0.615$	$a_2/a_1 = 0.55$	$a_2/a_1 = 0.32$	$a_2/a_1 = 0.31$	$a_2/a_1 = 0.30$
7.15328	6.75468	4.76401	+4.68674	+4.61511
8.99693	8.88237	8.15461	+8.10758	+8.05896
3.70620	3.70794	3.38475	+3.32826	+3.26378
1.60133	1.51047	1.14494	+1.14970	+1.15778
1.36249	1.29913	0.90860	+0.88441	+0.85954
1.36249	1.29912	0.90861	+0.88442	+0.85955
0.87897	0.85261	0.88275	+0.86562	+0.84515
0.66716	0.69244	0.53503	+0.52015	+0.50484
0.28510	0.23780	0.061462	+0.058864	+0.056562
0.063227	0.060673	0.0058232	−0.0015608	−0.0089902
0	0	0	0	0

IX. CONCLUDING REMARKS

We return to the question of the relationship between the point where the solution to the post-Newtonian figures of the Dedekind ellipsoid diverges and where the Dedekind ellipsoid becomes secularly or dynamically unstable. We have already seen in *loc. cit.* that the Dedekind ellipsoid, nowhere along its sequence, allows a nontrivial neutral mode of secular instability; and we have now found that the point of onset of dynamical instability is different from the point where the post-Newtonian solution diverges: $a_2/a_1 = 0.3121$ as against $a_2/a_1 = 0.3370$. We now ask whether we should indeed have expected any relationship between the two points.

Let us recall that we replaced the assumption of constant energy-density $\epsilon(=\rho c^2 + \Pi)$ by the equivalent assumption $\rho =$ constant and $\Pi = 0$. On the latter assumption, we can consider the post-Newtonian terms in the equations of hydrodynamic equilibrium as inducing a deformation of the Newtonian ellipsoid by divergence-free Lagrangian displacements; and indeed by displacements which are "congruent modulo the ellipsoid" (in the sense defined in Chandrasekhar 1969, p. 108).[1] At the same time, the post-Newtonian equation of continuity (*loc. cit.*, eq. [5]) requires that the increment in the Newtonian velocity field is not divergence-free as is manifest from equation (22). Indeed, it is this non-solenoidal character of the velocity increments that required us to introduce the terms in q, r_1, r_2, t_1, and t_2 in the post-Newtonian velocity field given by equations (29). In contrast, the solenoidal character of the velocity field (as necessarily required by the assumption of uniform density) underlies *all* Newtonian perturbations. This fact is reflected in the circumstance that the operator which is inverted in the solution of the post-Newtonian equations is in a space of 10 dimensions in contrast to the corresponding Newtonian operator which is in a space of five dimensions. On these accounts, one should perhaps not be surprised that the solution for the post-Newtonian figures of the Dedekind ellipsoid diverges at a point not related in any way with Newtonian instabilities. And these facts once again emphasize the importance of constructing Dedekind-like figures in the exact framework of general relativity on the basis of more physically reasonable equations of state. We should then discover how fully relativistic Dedekind-sequences may terminate.

We wish to express our profound gratitude to Dr. Monique Tassoul for discovering the errors in the earlier paper. We are also indebted to P. S. Marcus and W. H. Press for their assistance in solving for the characteristic frequencies of oscillation of the Dedekind ellipsoids listed in Table 2.

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[1] It should perhaps be emphasized that the fact that the deformed figure of the post-Newtonian ellipsoid can be derived by a Lagrangian displacement, modulo the ellipsoid, and of $O(c^{-2})$, is used *only* in the context of determining the change δU, of $O(c^{-2})$, in the Newtonian gravitational potential.

TABLE 3

POINTS OF ONSET OF THIRD AND FOURTH HARMONIC INSTABILITIES ALONG THE JACOBI AND THE DEDEKIND SEQUENCES

HARMONIC	JACOBI a_2/a_1	JACOBI a_3/a_1	DEDEKIND a_2/a_1	DEDEKIND a_3/a_1
3rd	0.4322	0.3451	0.4413	0.3504
4th	0.2972	0.2575	0.3121	0.2680

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S. Chandrasekhar and Donna D. Elbert: Laboratory for Astrophysics and Space Research, University of Chicago, 933 E. 56th Street, Chicago, IL 60637