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ON THE ELLIPSOIDAL FIGURES OF EQUILIBRIUM OF HOMOGENEOUS MASSES

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1. Introduction. In a series of papers published during the past four years, the classical problems pertaining to the various sequences of ellipsoidal figures of equilibrium of homogeneous masses have been reconsidered with a view to clarifying the many aspects of the sequences which had been only partially or imperfectly examined in the earlier literature. In this paper, we shall attempt to explain in general terms the origin of some of the more striking aspects of the sequences.

2. The different ellipsoidal sequences. We shall first briefly describe the different sequences which have been considered noticing particularly those features which appear to call for some explanation.

a) *The Maclaurin and the Jacobian sequences.* These are the figures of equilibrium of uniformly rotating masses. The Maclaurin sequence is a sequence of oblate spheroids along which the eccentricity (e) of the meridional sections increases from zero to one. A feature of the sequence which caused considerable surprise, at the time it was discovered by D'Alembert, is that along the sequence the square (Ω^2) of the angular velocity of rotation is not a parameter of unrestricted range; and that for each value of Ω^2, less than a certain determinate maximum, there are two permissible spheroidal figures of equilibrium. It was indeed this last circumstance that led Jacobi first to suspect, and then to verify, that a sequence of genuine tri-axial ellipsoids of equilibrium branches off from the Maclaurin sequence. This is the first of several *points of bifurcation* which distinguish the permissible sequences of figures of equilibrium of uniformly rotating homogeneous masses. Besides the point along the Maclaurin sequence where the Jacobian sequence of ellipsoids branches off, greatest interest has been attached to the point of bifurcation, discovered by Poincaré, along the Jacobian sequence, where a new sequence of pear-shaped configurations branches off.

b) *The Jeans sequence.* The Jeans sequence is a sequence of prolate spheroids in equilibrium under the constant tidal action of a fixed rigid spherical mass M'. (This problem is «unphysical» to the extent that it considers the two objects as at rest and ignores the relative accelerations to which they must be subject.) The parameter which measures the intensity of the tidal field and determines the eccentricity (e) of the Jeans spheriod is

$$\mu = GM'/R^3 \tag{1}$$

where R is the distance between the centers of mass of the two objects.

Along the Jeans sequence, the eccentricity of the spheroids varies from zero to one. But equilibrium figures do not exist when R is less than a certain minimum value; and for each value of R, in excess of this minimum, there are two permissible figures of equilibrium.

c) The Roche sequences. Roche's problem is concerned with the equilibrium of a homogeneous mass M rotating about a rigid spherical mass M' in a circular Keplerian orbit of radius R. Under these circumstances, the mass M is subject not only to the centrifugal force appropriate to the angular velocity given by

$$\Omega^2 = G(M + M')/R^3, \tag{2}$$

but also to the tidal action of M'; the magnitude of the latter is measured by (1). By writing

$$\Omega^2 = (1 + p)\,\mu \quad \text{where} \quad p = M/M', \tag{3}$$

it is clear that we obtain different sequences for different initially assigned values of p.

Along each Roche sequence, Ω^2 attains a maximum value (and simultaneously, R attains a minimum value); and there are two figures of equilibrium for each allowed separation.

In the limit $p \to \infty$, the Roche sequence tends to the combined Maclaurin-Jacobi sequence (which consists of the Maclaurin sequence up to the point of bifurcation and of the Jacobian sequence beyond). Also, by letting p take the «unphysical» value –1, we formally obtain the Jeans sequence. The relationships between the Maclaurin, the Jacobi, and the Roche sequences are exhibited in Figure 1.

d) The Darwin sequence. Darwin's problem is concerned with the equilibrium of two homogeneous masses rotating about one another in a manner which maintains their relative dispositions. It differs from Roche's problem in that allowance is made for the centrifugal and the tidal distortions of *both* components.

The results for the case, when the two components are of equal mass and are further congruent, are shown in Figure 2. It will be observed that along the Darwin sequence the maximum angular velocity of orbital rotation does *not* occur at the distance of closest approach.

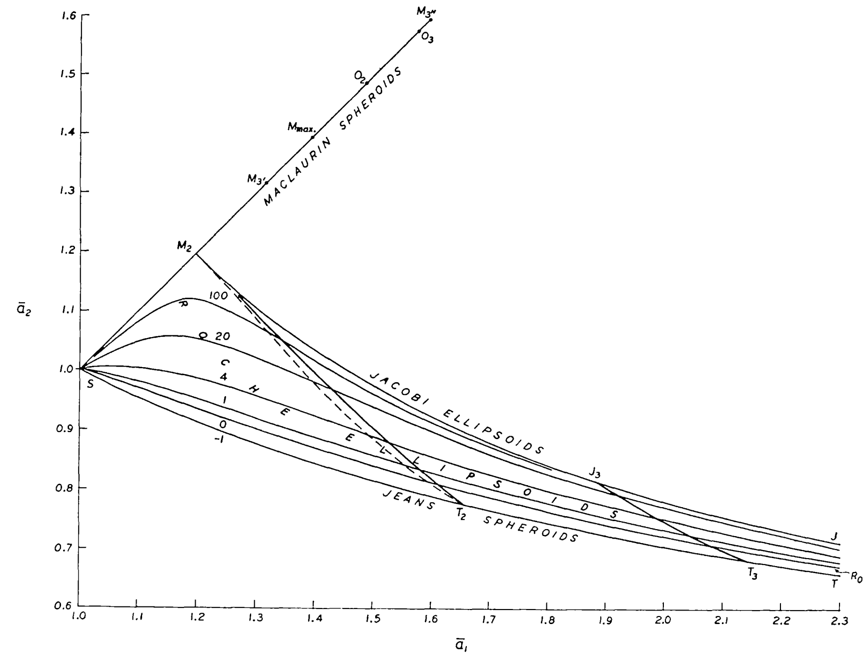

Fig. 1. The relationships among the Maclaurin, the Jacobi, the Jeans, and the Roche sequences. The ordinates and the abscissae are the normalized values, $\bar{a}_1 = a_1/(a_1a_2a_3)^{1/3}$ and $\bar{a}_2 = a_2/(a_1a_2a_3)^{1/3}$, of two of the principal axes of the equilibrium ellipsoid (or spheroid). The undistorted sphere is represented by S, the Maclaurin sequence by SM_3'', the Jacobian sequence by M_2J, the Jeans sequence by ST, and the Roche sequences (labeled by the values of $p = M/M'$ to which they belong) are confined to the domain bounded by the combined Maclaurin-Jacobi sequence, SM_2J, and the Roche sequence, SR_0, for $p = 0$. The first point of bifurcation along the Maclaurin sequences occurs at $M_2\,(e = 0.8127)$; at $M_3'\,(e = 0.8993)$ and $M_3''\,(e = 0.9694)$ occur further neutral points belonging to the third harmonics. At the points $O_2\,(e = 0.9529)$ and $O_3\,(e = 0.9670)$ the Maclaurin spheroid becomes unstable by modes of overstable oscillation belonging to the second and the third harmonics; and at $M_{\max}\,(e = 0.9300)$, Ω^2 attains its maximum values along SM_3''. The Jacobi ellipsoids become unstable by a mode of oscillation belonging to the third harmonics at J_3 ($\bar{a}_1 = 1.8858$; $\bar{a}_2 = 0.8150$) where the pear-shaped sequence branches off; also, along the entire sequence M_2J the Jacobi ellipsoids are characterized by a neutral mode of oscillation belonging to the second harmonics. At the points $T_2\,(e = 0.8830)$ and $T_3\,(e = 0.9477)$ the Jeans spheroids become unstable by modes of oscillation belonging to the second and the third harmonics. The Roche limit where Ω^2 and μ attain their maxima along the different Roche sequences is represented by the dashed curve joining M_2 and T_2; and the locus of points where instability sets in by a mode of oscillation belonging to the second harmonics is shown by the heavy curve joining M_2 and T_2. The locus of the neutral point (belonging to the third harmonics) is shown by the curve joining T_3 and J_3. Note that the Jacobi ellipsoids are to be considered unstable in the limit $p \to \infty$.

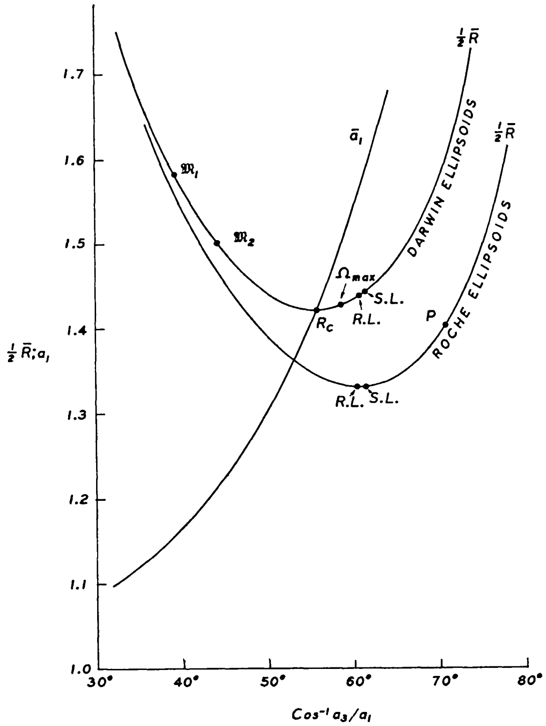

Fig. 2. The variation of the distance ($\bar{R}$) between the centers of mass of the two components and of the semi-major axis ($\bar{a}_1$) along the Darwin sequence of congruent components of equal mass. Along the sequence we distinguish the points $\mathfrak{M}_1$, $\mathfrak{M}_2$, and Ω_{max} where the total angular momentum of the system ($\mathfrak{M}_1$), the orbital angular momentum and the angular momentum of only one of the components ($\mathfrak{M}_2$), and the angular velocity (Ω) attain their extreme values. At R_c, where the curves $\frac{1}{2}\bar{R}$ and $\bar{a}_1$ cross, the two components are in contact; contact occurs very nearly at the distance of closest approach. The Roche limit (where the equilibrium ellipsoid can be deformed into a neighboring equilibrium ellipsoid by an infinitesimal solenoidal displacement of the form (21)) occurs at *R. L.*; and instability by a mode of oscillation belonging to the second harmonics sets in at *S. L.* The results for the Roche sequence for the case $M/M' = 1$ are included for comparison; along this sequence, besides the Roche limit (*R. L.*) and the stability limit (*S. L.*), we have also the point (*P*) where instability by a mode of oscillation belonging to the third harmonics sets in.

3. Integral properties derived from the virial theorem and its extensions. The principal features of the different sequences described in the preceding section are most easily understood in terms of the integral properties one obtains by taking the first and the second moments,* with respect to the space co-ordinates x_j, of the equation of motion governing the fluid.

For the case of a mass rotating uniformly with an angular velocity Ω about the x_3-axis, the moment equations are:

* The zero-order moment is related to the motion of the center of mass and is not of interest in the present connection.

$$\mathfrak{W}_{ij} + \Omega^2 (I_{ij} - \delta_{i3} I_{3j}) + \Pi \delta_{ij} = 0 \tag{4}$$

and

$$\mathfrak{W}_{ij;k} + \mathfrak{W}_{ik;j} + \Omega^2 (I_{ijk} - \delta_{i3} I_{3jk}) + \Pi_j \delta_{ki} + \Pi_k \delta_{ij} = 0 , \tag{5}$$

where

$$\Pi = \int_V p d\mathbf{x} , \quad \Pi_i = \int_V p x_i \, d\mathbf{x} \tag{6}$$

$$I_{ij} = \int_V \varrho x_i x_j \, d\mathbf{x} , \quad I_{ijk} = \int_V \varrho x_i x_j x_k \, d\mathbf{x} , \tag{7}$$

$$\mathfrak{W}_{ij} = -\frac{1}{2} \int_V \varrho \, \mathfrak{B}_{ij} \, d\mathbf{x} , \quad \mathfrak{W}_{ij;k} = -\frac{1}{2} \int_V \varrho \, \mathfrak{B}_{ij} \, x_k \, d\mathbf{x} , \tag{8}$$

and $\mathfrak{B}_{ij}$ is the tensor potential

$$\mathfrak{B}_{ij}(\mathbf{x}) = G \int_V \varrho(\mathbf{x}') \frac{(x_i - x_i')(x_j - x_j')}{|\mathbf{x} - \mathbf{x}'|^3} \, d\mathbf{x}'. \tag{9}$$

In the foregoing definitions, the integrations are effected over the entire volume V occupied by the fluid; also, p denotes the pressure and ϱ the density.

Equations (4) and (5) are the second and the third order virial equations governing the equilibrium of uniformly rotating bodies. They are entirely general: they are in no way dependent on any constitutive relations that may exist; and, equations similar to them can be readily written down for bodies subject to centrifugal and tidal forces, simultaneously.

Writing out equation (4) explicitly for the different components and eliminating Π between them, we obtain the set of equations

$$I_{13} = I_{23} = 0 , \ \mathfrak{W}_{13} = \mathfrak{W}_{23} = 0 , \tag{10}$$

$$\mathfrak{W}_{12} + \Omega^2 I_{12} = 0 , \tag{11}$$

$$\mathfrak{W}_{11} - \mathfrak{W}_{22} + \Omega^2 (I_{11} - I_{22}) = 0 , \tag{12}$$

and

$$\mathfrak{W}_{11} + \mathfrak{W}_{22} - 2 \mathfrak{W}_{33} + \Omega^2 (I_{11} + I_{22}) = 0 . \tag{13}$$

We can write down a similar set of «canonical» equations for the third-order tensors $\mathfrak{W}_{ij;k}$ and I_{ijk}. But for the illustrative purposes of this paper, it will suffice to note the following particular case of equation (5):

$$2 \mathfrak{W}_{12;2} + \Omega^2 I_{122} = 0 . \tag{14}$$

4. The points of bifurcation along the Maclaurin and the Jacobian sequences. With the integral properties provided by the virial equations (4) and (5), it is a particularly simple matter to exhibit and isolate the points of bifurcation along the Maclaurin and the Jacobian sequences.

Considering first the point of bifurcation along the Maclaurin sequence where the Jacobian sequence branches off, we observe that at this point we should be able to deform the Maclaurin spheroid into a tri-axial ellipsoid without, in any way, affecting the equilibrium. And this fact alone suffices, as we shall presently see, to isolate the point of bifurcation in question.

Now an infinitesimal displacement ξ that deforms a spheroid into a tri-axial ellipsoid and does not, at the same time, affect its angular momentum is

$$\xi_1 = \alpha x_2 \,, \; \xi_2 = \beta x_1 \,, \text{ and } \xi_3 = 0 \,, \tag{15}$$

where α and β are two infinitesimal constants. A necessary condition that the displacement does not violate the equilibrium of the spheroid is that the first variations of equations (10) – (13) vanish.

It is evident that

$$\delta \mathfrak{W}_{11} = \delta \mathfrak{W}_{22} = \delta \mathfrak{W}_{33} = 0 \text{ and } \delta I_{11} = \delta I_{22} = \delta I_{13} = \delta I_{23} = 0 \,, \tag{16}$$

where $\delta \mathfrak{W}_{11}$, $\delta \mathfrak{W}_{22}$, etc., are the variations in the respective quantities induced by the displacement (15). Equations (10), (12), and (13) are, therefore, invariant to the displacement (15). But it is not true that the corresponding first variation of equation (11) vanishes at an arbitrarily selected point along the Maclaurin sequence. A necessary condition for the occurrence of a point of bifurcation where a sequence of ellipsoids branches off is, therefore,

$$\delta \mathfrak{W}_{12} + \Omega^2 \, \delta I_{12} = 0 \,. \tag{17}$$

It can be readily verified that the condition which follows from equation (17) as a requirement for not violating equilibrium is exactly the same as the one which determines the point where the Jacobian sequence branches off from the Maclaurin sequence.

In a similar way, the point along the Jacobian sequence where the sequence of the pear-shaped configurations branches off can be isolated. Thus, an infinitesimal displacement which will deform a tri-axial ellipsoid (with semi-axes a_1, a_2, and a_3) into a pear-shaped body, preserving its homogeneity, is given by

$$\xi_j = \text{constant} \, \frac{\partial}{\partial x_j} \, x_1 \left(\sum_{i=1}^{3} \frac{x_i^2}{a_i^2 + \lambda} - 1 \right) , \tag{18}$$

where λ is the larger of the roots of the equation

$$\frac{3}{a_1^2+\lambda}+\frac{1}{a_2^2+\lambda}+\frac{1}{a_3^2+\lambda}=0\,. \tag{19}$$

The application of the displacement (18) to an arbitrary member of the Jacobian sequence will affect the equilibrium of the body; a necessary condition that it does not is that the first variations, of the entire set of the «canonical equations» which follows from equation (5) vanish. In particular, we must certainly require that (cf. eq. [14])

$$2\,\delta\,\mathfrak{W}_{12;2}+\Omega^2\,\delta I_{122}=0 \tag{20}$$

where $\delta\mathfrak{W}_{12;2}$ and δI_{122} are the variations in $\mathfrak{W}_{12;2}$ and I_{122} due to the deformation of the ellipsoid caused by the displacement (18). And it can be verified that the condition which follows from equation (20), as a requirement for not violating equilibrium, is equivalent to the one which determines the point where the sequence of the pear-shaped configurations branches off from the Jacobian sequence.

It is clear that at points of bifurcation, like the two we have considered, there must occur neutral modes of oscillation belonging to zero proper frequencies. But one cannot be certain, on this ground alone, that instability occurs in the sense that on one or the other side of the point of bifurcation the object is characterized by an imaginary (or a complex) frequency of oscillation in an analysis of its normal modes. Thus, along the Maclaurin sequence, in the absence of any viscous or dissipative mechanism, the normal mode that becomes neutral at the point of bifurcation has real proper frequencies on both sides of the point; but if viscosity is present* instability (with an e-folding time dependent on the magnitude of the viscosity) does occur on the side of the Jacobian sequence. On the other hand, along the Jacobian sequence, where the sequence of the pear-shaped configurations branches off, true instability sets in: beyond the point of bifurcation, there exists, for the Jacobi ellipsoid, a proper mode of oscillation belonging to an imaginary characteristic frequency. That such an unstable mode (belonging to the third harmonics) exists for the Jacobi ellipsoid was established by Cartan in 1924. But the underlying characteristic value problem was solved only recently; and Figure 3 exhibits the behavior along the sequence of the characteristic frequency which leads to the instability of the Jacobi ellipsoid.

We may refer to two related matters. *First*, along the entire Jacobian sequence, there exists a non-trivial neutral mode of oscillation belonging to the second harmonics. A Jacobi ellipsoid has thus only four modes of oscillation, with finite frequencies, which belong to the second harmonics; this is in contrast to the Maclaurin spheroid which has five such modes. This curious circumstance is due to the fact that the equation which determines

* To avoid possible misunderstandings, it may be stated that allowance for viscosity has no effect either on the structure of equilibrium configuration or on the location of the point of bifurcation.

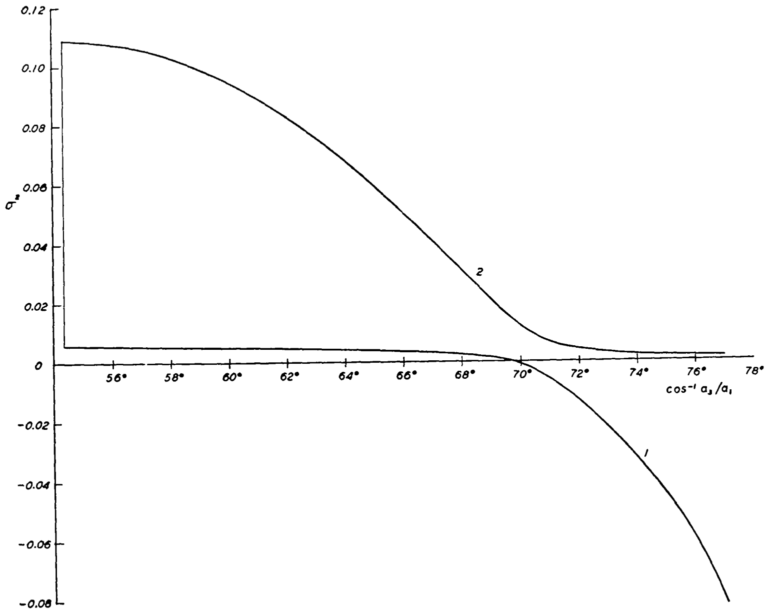

Fig. 3. The squares of the characteristic frequencies σ^2 (in the unit $\pi G\varrho$) of the two lowest modes of oscillation (belonging to the third harmonics) of the Jacobi ellipsoid. The instability of the Jacobi ellipsoid sets in via the mode labeled by 1.

Ω^2 along the Jacobian sequence coincides with the condition (17) without any prior requirements*.

Second, even as the condition (20) enables us to determine the onset of instability along the Jacobian sequence by a mode of oscillation belonging to the third harmonics, analogous conditions (derived from appropriately generalized virial equations of the third order) enable us to determine the corresponding points of instability along the Jeans, the Roche, and the Darwin sequences.

5. The occurrence of a maximum Ω^2 along the Maclaurin and the Roche sequences. We have already remarked that along the Maclaurin and the Roche sequences, Ω^2 attains a maximum. We shall now show that this attainment of a maximum Ω^2, in the two cases, has a common origin.

Quite generally, we ask: *can we deform an equilibrium ellipsoid into a neighboring equilibrium ellipsoid which belongs to the same sequence?*

* From another point of view, the real «curiosity» (as Kelvin has expressed) is the *existence* of the Jacobi ellipsoids: for, if ellipsoidal figures of equilibrium exist, then their equilibrium cannot be affected by a simple rotation about the x_3 -axis; and this is exactly what the displacement (15) accomplishes: it rotates the ellipsoid about the x_3 -axis by the infinitesimal angle $\delta\varphi = (\alpha a_2^2 + \beta a_1^2)/(a_2^2 - a_1^2)$. In other words, the invariance of the equilibrium to rotations about the x_3 -axis requires that the angular velocity be determined consistently with equation (17)!

Now a tri-axial ellipsoid can be deformed into a neighboring ellipsoid by the displacement

(21) $$\xi_i = \alpha_i x_i \text{ (no summation over the repeated indices)},$$

where the α_i 's are infinitesimal constants satisfying the condition,

(22) $$\sum_{i=1}^{3} \alpha_i = 0 ,$$

required to preserve the homogeneity of the ellipsoid.

In general, the deformation of an ellipsoid by the displacement (21) will affect its equilibrium. Thus, in the case of uniformly rotating masses, the condition that equilibrium be unaffected, will require, in particular, that (cf. eqs. [12] and [13])

(23) $$\delta \mathfrak{W}_{11} - \delta \mathfrak{W}_{22} + \Omega^2 (\delta I_{11} - \delta I_{22}) = 0$$

and

(24) $$\delta \mathfrak{W}_{11} + \delta \mathfrak{W}_{22} - 2\, \delta \mathfrak{W}_{33} + \Omega^2 (\delta I_{11} + \delta I_{22}) = 0 ,$$

for the deformation caused by the displacement (21); the remaining equations (10) and (11) are clearly invariant to this displacement.

For the displacement (21), the first variations of the diagonal elements of $\mathfrak{W}_{ij}$ and I_{ij} do not vanish. Equations (22) – (24) provide, in fact, a set of three linear homogeneous equations for the α_i's. The existence of a non-vanishing solenoidal displacement of the form (21) requires, as a necessary condition, the vanishing of the determinant of the equations for the α_i's which follow from equations (22) – (24). It can be readily shown that, when applied to the Maclaurin sequence, this condition is the same as the one which determines the maximum of Ω^2 along the sequence.

Similarly, by using the appropriate generalizations of equations (23) and (24), we find that the condition, that a Roche ellipsoid allows an infinitesimal solenoidal displacement which will deform it into a neighboring equilibrium ellipsoid, is met exactly where Ω^2 attains its maximum value.

In the context of the foregoing remarks relative to the Roche ellipsoids, it should be noted that, along the Darwin sequence, the point at which a Darwin ellipsoid allows deformation by a solenoidal displacement of the form (21), without violation of its equilibrium, is quite different from the points where Ω^2 and R attain their respective extremes (see Fig. 2).

6. The point where instability sets in along a Roche sequence. It is clear from the remarks in Section 5 that we have no basis for associating with our ability to deform a Roche ellipsoid into a neighboring equilibrium ellipsoid any indication relative to its stability or instability. Indeed, a rigorous theory of small oscillations shows that instability occurs, by a mode of oscillation belonging to the second harmonics, subsequently to the attainment of the maximum of Ω^2 at the so-called Roche limit. Moreover, at the point where instability occurs, one of the characteristic roots σ^2 (belonging to modes with a time-dependence of the form $e^{i\sigma t}$) vanishes; and Ω^2 does not attain its maximum at this point. It is important to note that, in this context, no simple consideration, based on the equations governing equilibrium, can correctly predict the onset of instability since at that point the corresponding normal mode is *not* stationary: its dependence on time is that of a polynomial in t (which is not inconsistent with $\sigma^2 = 0$ with an assumed time dependence of the form $e^{i\sigma t}$); and this particular dependence arises from two characteristic roots vanishing simultaneously at the point. Thus, we have here an example in which $\sigma^2 = 0$ at the onset of instability; but the corresponding normal mode is not stationary. It is on this last account that there is no point of bifurcation along a Roche sequence.

And finally, it is worth noting that according to the theory of the small oscillations (referred to in the preceding paragraph), in the limit $p \to \infty$, the entire Jacobian part of the combined Maclaurin-Jacobi sequence must be considered as unstable.

7. Concluding remarks. In the earlier discussions relative to the stability of the various ellipsoidal figures of equilibrium, criteria that were mostly used were based on the occurrence of points of bifurcations (as in the case of the Maclaurin and the Jacobian sequences) or on the occurrence of a maximum or a minimum of some particular parameter which labels the configurations along a sequence (as in the case of the Jeans, the Roche, and the Darwin sequences*). In the case of the Roche sequences, such criteria have been shown to have no real relevance to the question of the onset of instability. But in the case of the Darwin sequence, the question has been only partially settled. In this latter case, Darwin and Jeans have used as criteria for distinguishing stability, the occurrence of minima in the total angular momentum of the system or in «that part of the angular momentum which is liable to variation when tides cannot be raised in the secondary» (Jeans). The points where the minima of these quantities occur are denoted by $\mathfrak{M}_1$ and $\mathfrak{M}_2$ in Figure 2. But instability, in the strict sense we are using that term, certainly does not arise at either point by any natural mode of oscillation of either component by itself. The question still remains whether the tidal coupling between the two components can induce a further instability or at least a neutral mode of oscillation. No treatment of such coupled oscillations exists at the present time; and it would appear that only by such a treatment can criteria similar to those of Darwin and Jeans emerge.

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* And sometimes, even the Maclaurin sequence!