Quantum Ratchet motion

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Abstract | Ratchet models provide an useful mechanism for directed transport using nonequilibrium fluctuations of the surrounding. In the dynamics of micro-particles quantum effect is likely to make its presence felt in appropriate situations, particularly at low temperature. In this review we have discussed the aspects of ratchet motion in quantum domain. Making use of a Wigner canonical thermal distribution for description of the statistical properties of the noise, we explore the generic effects of quantization like vacuum field fluctuation and tunneling in fluctuation-induced quantum transport both in the overdamped and the underdamped limits and the energetics of quantum ratchet.

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Vacuum field fluctuation: is the fluctuation arising out of zero point energy of a quantum system which has no classical analog.

Ratchet effect: is the appearance of a systematic directional motion generating useful work from random motion in absence of any macroscopic static forces, gradient or biased time dependent perturbations.

I. Introduction

A. *Preliminary remarks*

Efficient extraction of useful work from heat has remained a key theme over the last couple of centuries. Guided by two laws of thermodynamics we know how an engine works between two temperatures and how and to what extent conversion of heat to work is possible. Recent developments in biology and in other areas which involve transport problems, however, have shown that an engine or motor at a molecular level can act at a single temperature. The working of such a motor/device is based on rectification of non-equilibrium fluctuations in an open system when certain physical conditions are appropriately maintained. Loosely speaking, such devices currently constitute a field — molecular motors or ratchets. Since the characteristic length scale over which a molecular motor or in general a microscopic ratchet works is much smaller compared to that for macroscopic devices, it is imperative that quantum effects are important for appropriate description of situations, particularly at low temperature. The relevant length scale important for the present problem is determined by thermal wave length $\lambda = h/\sqrt{2\pi mkT}$, which takes care of quantum effect (h, Planck constant) as well as thermal fluctuations (of the order kT, k and T being Boltzmann constant and absolute temperature, respectively) and mass (m) of the system. This article is a brief account of some general features of the ratchet problem with a special emphasis on its quantum aspects. The areas of application of the concept of quantum ratchet motion cover Josephson-junction arrays, optical lattices, superconductor among others.

B. *Molecular motors and ratchet devices; an historical background*

The progress in the field of *ratchet effect* and Brownian motors owes its origin in rediscoveries of some of the basic principles in different contexts. Although certain aspects of ratchet effect are contained in early works of Seebeck, Maxwell, Curie and others[1], an important step influencing the field came from Feynman's focus[2] on the ratchet and pawl machine operating between two thermal baths. After the discovery of Brillouin paradox[3], an explanation of the *Seebeck effect* based on the Feynman ratchet model becomes straight-forward (Seebeck's discovery in 1822, of course was without the idea about ratchet mechanism).

The most important direction of the theory of Brownian motor that leads us to the realm of intercellar transport reserach mainly concerns the biochemistry of and underlying physics of the movement of the molecular motors in living cells and molecular ion pumps. In many cases

Detailed balance: is a balance between backward and forward transition from a local minima. It is a rule to be obeyed strictly at equilibrium.

Brownian motion: is an irregular and aminated movement of small microparticles in a fluid medium.

Overdamped limit: is the limit in which the damping constant far exceeds the characteristic frequencies of the system.

of biomolecular transport the concept of ratchet mechanism has been explored. In biomolecular transport Huxley made a seminal contribution to muscle contraction[4], and further research continued in the late 1980s due to Braxton and Yount[5,6] and in the 1990s due to Vale and Oosawa[7], Leibler and Huse[8,9], Cordova, *et al.*[10], Magnasco[11,12], Prost *et al.*[13,14], Astumian and Bier[15,16], Peskin *et al.*[17,18] and many others. In the case of molecular pumps, important development started with the theoretical interpretation of previously known experimental finding[19,20] as a ratchet effect in 1986 by Tsong, Astumian and coworkers[21,22].

From the physical side, a ratchet effect in voltage rectification by DC-SQUID (super-conducting quantum interference device) in presence of a magnetic field and an unbiased AC-current has been experimentally verified along with theoretical interpretation by Waele et al.[23,24]. Since mid-1970s, directed transport induced by time-periodic forces in a spatially periodic structures with broken symmetry has been the subject of several hundred experimental and theoretical works. A seminal work in the experimental investigation of so-called photovoltaic and photorefractive effect was done by Glass et al.[25] in 1974. A few years later Belinicher, Sturman and coworkers[26,27] suggested a general theoretical framework to identify the two main ingredients of ratchet effect in periodic systems as the breaking of thermal equilibrium (detailed balance symmetry) and the spatial symmetry. The much more general validity of a tilting ratchet scheme was also explored beyond the specific experimental system.

Generation of a DC-output by two superimposed sinusoidal AC-inputs in a spatially symmetric and periodic system was experimentally verified by Seeger and Maurer[28]. This phenomenon was theoretically analyzed by Wonneberger[29]. The occurrence of a ratchet effect had been predicted by Bug and Berne[30] for the simplest variant of a pulsating ratchet scheme. Around this time a ratchet model with a symmetric periodic potential and state dependent temperature with same periodicity (but out of phase) had been studied theoretically by B*ü*ttiker[31] and van Kampen[32].

The independent investigation of the on-off ratchet scheme by Ajdari and Prost[33] and tilting ratchet scheme by Magnasco[11] provided the a new impetus to theoretical and experimental activities in statistical physics and bio-physics community. Although initially the modeling of molecular motors was one of the main motivations, the scope of the ratchet effect has subsequently been broadened to cover an increasing number of physical and technological applications.

In order to understand the underlying mechanism of the generation of unidirectional motion from non-equilibrium fluctuations, several models have been proposed[34–37,39–56]. A recent ratchet effect for fluxons in Josephson-junction array[61], a ratchet effect in cold atoms using an asymmetric optical lattice[62], and the lowering of vortex density in superconductors using the ratchet effect[63] are worthy of special attention. The efficiency of ratchet devices[64–67] and comparison with different types of bio-molecular motors comprise a recently developing field.

The study of the ratchet system is motivated in recent years by the recognition of the effect in the quantum domain. Reimann *et al*[68] investigated an adiabatically rocked ratchet system to show that quantum corrections enhance classical transport at low temperature. Two models of quantum ratchet have also been proposed by Yukawa *et al*[69]. Based on the perturbative approach Scheidl and Vinokur[70] have investigated quantum Brownian motor in ratchet potentials to identify the characteristic scales of response functions of the system. Carlo *et al*[71] have studied a typical model quantum chaotic dissipative ratchet to analyze the directed transport from a quantum strange attractor. The latter research analyzes a quantum ratchet based on an asymmetric (triangular) quantum dot[72], an asymmetric antidot array[73], and surface electro-migration[74].

C. *Scope and plan of the review*

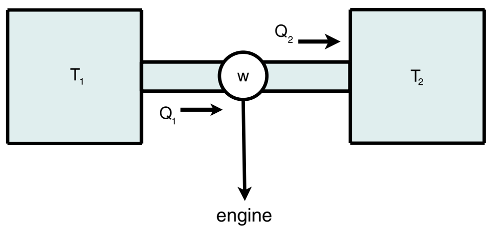
Our object in this review is to discuss some basic aspects of quantum ratchet and the related problems of quantum transport. Since the underlying issues are based on the theories of Brownian motion in thermodynamically open systems, our aim here is to develop systematically a scheme for quantum Brownian motion followed by an extension to transport problem, keeping in view of the inherent thermodynamic consistencies. The outlay of the review is as follows:

(1) In the next section (Sec. II) we have systematically developed the basic principle of ratchet motion and directed transport in classical domain.

(2) In Sec. III we have derived the basic equations describing quantum stochastic dynamics on a general footing followed by an overdamped description. Thermodynamic consistency has been stressed to avoid the pitfalls of fictitious current generation.

(3) In Sec. IV we show how an external non-equilibrium fluctuation can break the condition of detailed balance inducing a directed quantum

Figure 1: A schematic illustration for a typical heat engine.



transport. An analysis of the essential features of quantum ratchet devices and their efficiencies in overdamped limit is made.
(4) In Sec. V we discuss the quantum ratchet problem in the underdamped regime and its utility in the separation of quantum particles. The rectification efficiency in weak friction regime is also formulated.
(5) The review is concluded in Sec. VI.

II. The Ratchet motion

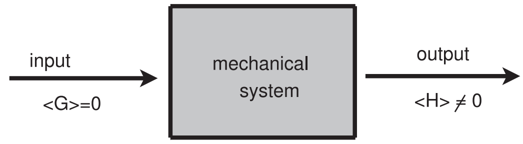
A. *The basic principle of ratchet motion*

Let us start from a statement of second law of thermodynamics "*heat can not be converted into work in a cyclic process at a single temperature*". To convert heat into work we need at least two heat reservoirs at temperature T_1 and T_2 ($T_1 > T_2$). If Q_1 amount heat energy flows out from the hot reservoir, a part of the heat energy is partially converted to work by the heat engine and rest of the heat is gained by the cold reservoir (schematically presented in Fig. 1), the work done W and efficiency of the heat engine η are given by

$$Q_1 - Q_2 = W; \qquad \eta = \frac{T_1 - T_2}{T_1}. \tag{2.1}$$

Now our question is "*can an engine work at a single temperature* ?". Answer to this question concerns underlying mechanism of the movement of the molecular motors in the living cells. Molecular motors are amazing biological machines that are responsible for most forms of movement, e, g, myosin is a cytoplasmic motor, which moves on acting filaments and carries the essential materials into the different parts of the cell and outside the cells as required for a living organism. All biological motors work at a single temperature. To explain this type of movement and the underlying mechanism we would like to look for a typical machine which can work at a single temperature.

Figure 2: A schematic illustration of input output device.



We now proceed from a statistical point of view. A system interacting weakly with surrounding can be described by a Gibbs state. An alternate approach to describe thermodynamic equilibrium state (may be stationary state of the system) is the Langevin approach. Let us consider a Brownian particle driven by a thermal or/and non-thermal stochastic forces. In absence of deterministic potential force, the particle diffuses and the motion remain unbiased. If the Brownian particle is additionally driven by a nonzero bias force one can obtain the directed motion determined by this force. This is a trivial example of transport. The nontrivial case, the focal theme of this section (and Sec. IV and V in quantum mechanical context) is the transport when a bias force is zero and all random forces are of zero-mean values. The fundamental question is: can transport be induced by a stochastic force, random perturbation or by noise. The problem is schematically illustrated as an input–output device in Fig. 2. As input we may consider thermal and non-thermal fluctuations, noise or a periodic external force; all of zero averages over many realizations, $\langle G \rangle = 0$ and as an output we want to retrieve something with non-zero average $\langle H \rangle \neq 0$.

In our chosen system we consider a Brownian particle moving in a spatially periodic potential, $V(x) = V(x + L)$, so that the average potential force is zero over a period L. In output we want to obtain a non-zero average valued quantity which characterizes transport or directed motion of the Brownian particles. A simple quantifier is the average velocity of the particles. If the velocity is zero then the particles are not macroscopically transported. Or in other words macroscopic directed motion is observed only when $\langle v \rangle \neq 0$.

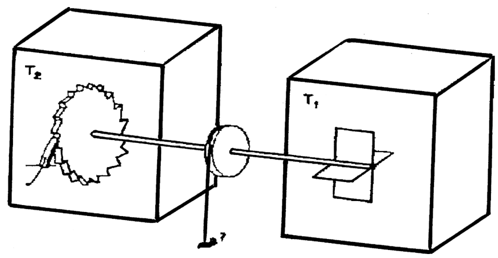
We may summarize the problem in the following form: how to convert diffusive random motion of the Brownian particles into a directed motion without any biasing force or gradient.

B. *Feynman ratchet and directed motion*

Let us consider briefly the Feynman ratchet and pawl device (illustrated in Fig. 3). For a detailed discussion we refer to[2]. The machine consists of an

: Fokker–Planck equation describes the time evolution of probability distribution function of one or more stochastic variables.

Figure 3: A schematic illustration of ratchet and pawl mechanism.



axle with vanes in one side and a ratchet (a wheel with asymmetric teeth) in the other end. The two ends of the device are inserted into two gas reservoirs at different temperatures T_1 and T_2, respectively. Due to pawl mechanism the axle of the device can rotate only in one direction. Because of the random collision with gas molecules of the reservoirs, vanes as well as the ratchet undergo random motion. So one can expect a net directed motion of the ratchet even at same temperature which means an average directed motion is generated out of thermal fluctuations. But, this is in contradiction with the second law of thermodynamics. The paradox is resolved in the following way[2]. The pawl is first accidentally raised up by gaining the required energy at a temperature T and the jumps back to its original position and executes jittering motion and undergoes friction with ratchet and eventually gets heated. The friction as a consequence leads to heating of the gas molecules. Thus even if one starts with two different temperatures for the two reservoirs, dissipation through the pawl leads to equalization of temperature. If the temperatures T_1 and T_2 are not same, then the case is trivial and an average directed motion is observed as a consequence of temperature gradient. Based on this analysis of the Feynman ratchet we shall construct a mathematical model with specific minimal desired features. We therefore note the following requirements.

(a) Ratchet is a spatially periodic system. It corresponds to a spatially periodic potential $V(x) = V(x+L)$.

(b) The symmetry of the ratchet is broken with respect to pawl mechanism (teeth are asymmetric). It corresponds to breaking of the reflection symmetry of the potential, $V(x-c) \neq V(x+c)$ where in c is any real number except zero.

(c) The average force acting on vanes caused by collisions of gas molecules is zero. It corresponds to zero-mean thermal fluctuations.

(d) The directed motion is induced due to a temperature gradient. It is trivial and instead of this we assume that a driving non-thermal force of zero mean acts on the system.

C. *Formulation of the ratchet model*

We now describe the ratchet model in the following way: consider a Brownian particle under the action of a periodic potential $(V(x) = V(x+L))$. The length and barrier height of the potential are L and $\Delta V = V_{max} - V_{min}$, respectively. The equation of motion for the particle is given by

$$m\ddot{x} + \gamma\dot{x} = -V'(x) + \Gamma(t) + \varepsilon(t). \qquad (2.2)$$

The first term of left hand side of the above equation represents the inertial term and the second term describes the force due to friction, which is directly proportional to the velocity of the particle, γ is the friction coefficient, $-V'(x)$ is the potential force, which has a zero average value over a period L.

$$\langle(-V'(x))\rangle_L = -\frac{1}{L}\int_x^{x+L} V'(x)\,dx$$
$$= \frac{1}{L}[V(x) - V(x+L)] = 0. \qquad (2.3)$$

The stochastic term $\Gamma(t)$ describes thermal fluctuation and is modeled by a $\delta-$ correlated Gaussian white noise with the properties.

$$\langle\Gamma(t)\rangle = 0;\ \langle\Gamma(t)\Gamma(t')\rangle = 2\gamma kT\delta(t-t'). \qquad (2.4)$$

where k stands for Boltzmann constant and T is the temperature of the system. Another stochastic term $\varepsilon(t)$ describes non-thermal and non-equilibrium fluctuation. We shall select specific prescription for $\varepsilon(t)$ in the next section. We begin by examining the equilibrium situation by dropping the second stochastic term from Eq. (2.2). The equation of motion is then given by

$$m\ddot{x} + \gamma\dot{x} = -V'(x) + \Gamma(t). \qquad (2.5)$$

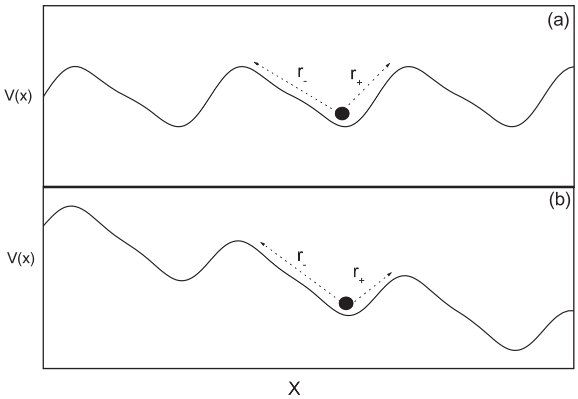
The Fokker-Planck equation corresponding to Eq. (2.5) is given by

$$\frac{\partial P(x,\dot{x},t)}{\partial t} = \left[-\frac{\partial}{\partial x}\dot{x} + \frac{\partial}{\partial \dot{x}}\left(\frac{\gamma}{m}\dot{x} - \frac{f(x)}{m}\right)\right.$$
$$\left. + \frac{\gamma kT}{m^2}\frac{\partial^2}{\partial x^2}\right]P(x,\dot{x},t), \qquad (2.6)$$

where $f(x) = -V'(x)$ presents the potential force. The above equation can be solved in the stationary state. The stationary probability density is

$$P_{st}(x,\dot{x}) = N\exp\left[-\frac{m\dot{x}^2}{2} + \int_0^x \frac{f(y)}{kT}dy\right], \qquad (2.7)$$

Figure 4: Schematic illustration of the generalized potential. (a) A periodic potential with zero slope; in this case transition probability from a local minimum of the generalized potential to the left valley and right valley are equal ($r_+ = r_-$) and so net directed movement of the particle is not possible ($v \sim r_+ - r_- = 0$). (b) A periodic potential with non-zero slope; so in that case $r_+ \neq r_-$ and $v \sim r_+ - r_- \neq 0$.



where N is the normalization constant which can obtained as

$$\int_{-\infty}^{+\infty} d\dot{x} \int_0^L dx\, P_{st}(x, \dot{x}) = 1. \qquad (2.8)$$

The mean velocity is

$$\langle \dot{x} \rangle_s = \int_{-\infty}^{+\infty} \dot{x}\, d\dot{x} \int_0^L dx\, P_{st}(x, \dot{x}). \qquad (2.9)$$

It is easy to check that in the stationary state the mean velocity is equal to zero. So there is no directed motion of the Brownian particle for the potential $V(x)$, which is a consequence of the principle of detailed balance[38]. To induce transport we require intervention of some non-thermal force.

An overdamped situation is a condition of interest in many situations. The overdamped equation of motion for the Brownian particle is given by

$$\gamma \dot{x} = -V'(x) + \Gamma(t). \qquad (2.10)$$

The equation for probability density function $P(x, t)$ corresponding to Eq. (2.10) is given by

$$\frac{\partial P(x,t)}{\partial t} = -\frac{\partial J(x,t)}{\partial x}. \qquad (2.11)$$

where the probability current is

$$J(x,t) = f(x)P(x,t) - D\frac{\partial P(x,t)}{\partial x}. \qquad (2.12)$$

$(D = kT)$. In the stationary state $P(x) = Lt_{t\to\infty} P(x,t)$, J is a constant as given by

$$J = f(x)P(x) - D\frac{\partial P(x)}{\partial x}. \qquad (2.13)$$

The solution of above equation for $P(x)$ reads as

$$P(x) = -\frac{J}{D}\exp[-\psi(x)] \int_0^x \exp[\psi(y)]\, dy + N\exp[-\psi(x)], \qquad (2.14)$$

where

$$\psi(x) = -\int_0^x \frac{f(y)}{D}\, dy, \qquad (2.15)$$

or $\psi(x) = \frac{V(x)}{D}$ and N is a constant. Periodic boundary condition implies that

$$\psi(x) = \psi(x+1). \qquad (2.16)$$

For periodic boundary condition on (2.14) and from (2.16) it follows that,

$$\frac{J}{D} \int_x^{x+1} \exp[\psi(y)]\, dy = 0. \qquad (2.17)$$

Since the above integral is non-zero an overdamped Langevin equation with periodic boundary condition shows $J = 0$. Therefore we have

$$P(x) = N\exp[-\psi(x)]. \qquad (2.18)$$

Normalization constant N is $[\int_0^1 \exp[\int_0^x \frac{f(y)}{D} dy]]^{-1}$. The zero current situation can be physically explained in the following way: the generalized potential $\psi(x)$ has no slope (Fig. 4(a)). It means that the transition rates from a state of local minimum of generalized potential $\psi(x)$ to the left valley and right valley are same. The stationary mean velocity depends on the difference between transition rates in the positive and negative directions. Hence $J = 0$, implies detailed balance. Now the question is *how to break this detailed balance*? The hint comes from the expression for generalized potential (2.15). It is easy to check that a non-zero slope for generalized potential $\Delta\psi \neq 0$ (Fig. 4(b)) may result if D becomes space-dependent, i. e., $D(x)$. *So for a directed motion we must have to break the condition of detailed balance by introducing non-equilibrium fluctuation or noise or an external periodic force. Addition of any external non-equilibrium fluctuation to the system is not sufficient to break the detailed balance for generation of directed transport but also the external non-equilibrium fluctuating force must be* *sufficiently correlated to make the diffusion coefficient space dependent which makes generalized potential asymmetric.*

While in the recent context of ratchet a major emphasis is laid on molecular pumps and motors in the realm of biophysics and chemistry, it would seem that a Brownian particle being a microscopic object[80], quantum effect is likely to make its presence felt in appropriate situation, particularly at low temperature. One thus expects the directed current or rectification of noise to be important in transport of quantum particles in quantum dots, wires related nanodevices[51,68] and also in the context of superionic conductors[81–83]. Furthermore such studies are important also from the point of view of quantum-classical correspondence. To incorporate the elements of quantum theory in a ratchet device it is necessary to satisfy two basic requirements. First, any approximation in the treatment of quantization of classical motion must not break the symmetry of the ratchet device, or in other words, more specifically, quantization should not bring in any additional tilt to the potential or break its inversion symmetry or symmetry of the detailed balance. Second, the forcing must be unbiased, so that after appropriate averaging over ensemble or over the period of space or time no directional component should remain. Any approximation pertaining to the problem must conform to these requirements in any correct quantum formalism.

Keeping in view of the above considerations we first formulate in the next section the quantum stochastic dynamics and its approach to equilibrium. The introduction of the non-equilibrium fluctuating force with a sufficient correlation breaks the symmetry of the detailed balance to produce directed quantum transport. The aim of the present review is to explore the quantum effect arising out in a quantum ratchet device[76,78,79].

III. A quantum system in a periodic potential at equilibrium

A. *General aspects*

We consider a particle of mass m moving in a periodic classical potential $V(x)$. The particle is coupled to a set of harmonic oscillators of unit mass acting as a bath. This is represented by the following system-reservoir Hamiltonian[84,85]

$$\hat{H} = \frac{\hat{p}^2}{2m} + V(\hat{x}) + \sum_{j=1}^{N} \left\{ \frac{\hat{p}_j^2}{2} + \frac{1}{2}\kappa_j(\hat{q}_j - \hat{x})^2 \right\} \quad (3.1)$$

Here $\hat{x}$ and $\hat{p}$ are the coordinate and momentum operators of the particle and $\{\hat{q}_j, \hat{p}_j\}$ are the set of coordinate and momentum operators for the reservoir oscillators coupled linearly through the coupling constants $\kappa_j (j = 1, 2, \ldots)$. For the spatially periodic potential, we have $V(x) = V(x+L)$, where L is the length of the period. The coordinate and momentum operators follow the usual commutation rules $\{\hat{x}, \hat{p}\} = i\hbar$ and $\{\hat{q}_i, \hat{p}_j\} = i\hbar\delta_{ij}$. Eliminating the bath degrees of freedom in the usual way we obtain the operator Langevin equation for the particle

$$m\ddot{\hat{x}} + \int_0^t dt'\gamma(t-t')\dot{\hat{x}}(t') + V'(\hat{x}) = \hat{\Gamma}(t), \quad (3.2)$$

(Overdots refers to differentiation with respect to time t) where noise operator $\hat{\Gamma}(t)$ and the memory kernel are given by

$$\hat{\Gamma}(t) = \sum_j \left[\{\hat{q}_j(0) - \hat{x}(0)\}\kappa_j \cos\omega_j t + \kappa_j^{1/2}\hat{p}_j(0)\sin\omega_j t \right], \quad (3.3)$$

and

$$\gamma(t) = \sum_j \kappa_j \cos\omega_j t, \quad (3.4)$$

respectively, with $\kappa_j = \omega_j^2$. Following Ref[85–88] we then carry out a quantum mechanical average $\langle \ldots \rangle$ over the product separable bath modes with coherent states and the system mode with an arbitrary state at $t = 0$ in Eq. (3.2) to obtain a generalized quantum Langevin equation as

$$m\ddot{x} + \int_0^t dt'\gamma(t-t')\dot{x}(t') + V'(x) = \Gamma(t) + Q(x, \langle\delta\hat{x}^n\rangle), \quad (3.5)$$

where the quantum mechanical mean value of the position operator $\langle\hat{x}\rangle = x$ and

$$Q(x, \langle\delta\hat{x}^n\rangle) = V'(x) - \langle V'(\hat{x})\rangle, \quad (3.6)$$

which by expressing $\hat{x}(t) = x(t) + \delta\hat{x}(t)$ in $V(\hat{x})$ and using a Taylor series expansion around x may be rewritten as

$$Q(x, \langle\delta\hat{x}^n\rangle) = -\sum_{n\geq 2} \frac{1}{n!} V^{n+1}(x)\langle\delta\hat{x}^n\rangle, \quad (3.7)$$

The above expansion implies that the nonzero anharmonic terms beyond $n \geq 2$ contain quantum dispersions $\langle\delta\hat{x}^n\rangle$. Although we develop this section in general terms, we are specifically concerned here typically with periodic nonlinear potentials of the type $\sin\frac{2\pi x}{L}$ or $\cos\frac{2\pi x}{L}$ or their linear combinations and the like which have been used earlier in several contexts. The nonlinearity of the potential is an important source of quantum correction in addition to the quantum noise of the heat bath. The calculation of Q rests on the quantum correction terms $\langle \delta \hat{x}^n \rangle$ which one determines by solving a set of quantum correction equations as given in Ref.[76,85]. Furthermore the c-number Langevin force is given by

$$\Gamma(t) = \sum_j \Big[\big\{ \langle \hat{q}_j(0) \rangle - \langle \hat{x}(0) \rangle \big\} \omega_j^2 \cos\omega_j t + \omega_j \hat{p}_j(0) \sin\omega_j t \Big], \quad (3.8)$$

which must satisfy noise characteristics of the bath at equilibrium,

$$\langle \Gamma(t) \rangle_S = 0, \quad (3.9)$$

$$\langle \Gamma(t)\Gamma(t') \rangle_S = \frac{1}{2} \sum_j \omega_j^2 \, \hbar\omega_j \left(\coth \frac{\hbar\omega_j}{2kT} \right) \cos\omega_j(t-t'), \quad (3.10)$$

Eq. (3.10) expresses the quantum fluctuation-dissipation relation. The above conditions (3.9)–(3.10) can be fulfilled provided the initial shifted coordinates $\{\langle \hat{q}_j(0) \rangle - \langle \hat{x}(0) \rangle\}$ and momenta $\langle \hat{p}_j(0) \rangle$ of the bath oscillators are distributed according to the canonical thermal Wigner distribution[94,95] of the form

$$P_j([\langle \hat{q}_j(0) \rangle - \langle \hat{x}(0) \rangle], \langle \hat{p}_j(0) \rangle) = N \exp\left\{ - \frac{\frac{1}{2}\langle \hat{p}_j(0) \rangle^2 + \frac{1}{2}\omega_j^2 [\langle \hat{q}_j(0) \rangle - \langle \hat{x}(0) \rangle]^2}{\hbar\omega_j [n(\omega_j) + \frac{1}{2}]} \right\}, \quad (3.11)$$

so that the statistical averages $\langle \ldots \rangle_s$ over the quantum mechanical mean value O of the bath variables are defined as

$$\langle O_j \rangle_s = \int O_j \, P_j \, d\langle \hat{p}_j(0) \rangle \, d\{\langle \hat{q}_j(0) \rangle - \langle \hat{x}(0) \rangle\}. \quad (3.12)$$

Here $n(\omega)$ is given by Bose–Einstein distributions $(e^{\frac{\hbar\omega}{kT}} - 1)^{-1}$. P_j is the exact solution of Wigner equation for harmonic oscillator[94,95] and forms the basis for description of the quantum noise characteristics of the bath kept in thermal equilibrium at temperature T. In the continuum limit the fluctuation-dissipation relation (3.10) can be written as

$$\langle \Gamma(t)\Gamma(t') \rangle = \frac{1}{2} \int_0^\infty d\omega \, \kappa(\omega) \, \rho(\omega) \, \hbar\omega \times \coth\left(\frac{\hbar\omega}{2kT} \right) \cos\omega(t-t'), \quad (3.13)$$

where we have introduced the density of the modes $\rho(\omega)$. Since we are interested in the Markovian limit in the present context, we assume $\kappa(\omega)\rho(\omega) = \frac{2}{\pi}\gamma$ and the variation of coth function with frequency to be very slow, Eq. (3.13) then yields[89,90,99]

$$\langle \Gamma(t)\Gamma(t') \rangle = 2D_q \delta(t-t'), \quad (3.14)$$

with

$$D_q = \frac{1}{2}\gamma\hbar\omega_0 \coth\frac{\hbar\omega_0}{2kT}, \quad (3.15)$$

ω_0 refers to static frequency limit. Furthermore from Eq. (3.4) in the continuum limit we have

$$\gamma(t-t') = \gamma\,\delta(t-t'), \quad (3.16)$$

γ is the dissipation constant in the Markovian limit. In this limit Eq. (3.5) therefore reduces to

$$m\ddot{x} + \gamma\dot{x} + V'(x) = \Gamma(t) + Q(x, \langle \delta\hat{x}^n \rangle). \quad (3.17)$$

It is useful to work with dimensionless variables for the present problem to keep track of the relations between the scales of energy, length and time. The period L of the periodic potential $V(x)$ determines in a natural way the characteristic length scale of the system. Therefore the position of the Brownian particle is scaled as

$$\overline{x} = x/L.$$

Next we consider the timescales of the system. In absence of the potential and the noise term the velocity of the particle $\dot{x}(t) \sim \exp(-t/\tau_L)$ with $\tau_L = m/\gamma$, which represents the correlation time scale of the velocity the Brownian particle. To identify the next characteristic time τ_0 we consider the deterministic overdamped motion due to the potential as $\gamma\frac{dx}{dt} = -\frac{dV(x)}{dx}$. Then τ_0 is determined from $\gamma\frac{L}{\tau_0} = -\frac{\Delta V}{L}$ as $\tau_0 = \frac{\gamma L^2}{\Delta V}$ where ΔV is the barrier height of the original potential. Hence time is scaled as $\overline{t} = \frac{t}{\tau_0}$. Furthermore the potential, the noise and the quantum correction terms are re-scaled as $\overline{V}(\overline{x}) = V(x)/\Delta V$, $\overline{\Gamma}(\overline{t}) = \Gamma(t)/(\Delta V/L)$ and $Q/(\Delta V/L)$, respectively. Hence dimensionless quantum Langevin equation reads as

$$\mu^*\ddot{x} + \dot{x} = f(x) + \Gamma(t). \quad (3.18)$$

Here x is $\overline{x}$ and the t is $\overline{t}$ and the over-dot(.) refers to differentiation with respect to scaled time $\overline{t}$. Dimensionless mass $\mu^* = \frac{m}{\gamma\tau_0} = \frac{\tau_L}{\tau_0}$ and

$$f(x) = -V'(x) + Q(x, \langle \delta\hat{x}^n \rangle). \quad (3.19)$$

The noise properties of the quantum bath are then rewritten as

$$\langle\Gamma(t)\rangle_s = 0,$$
$$\langle\Gamma(t)\Gamma(t')\rangle_s = 2D_q\delta(t-t'),$$

where

$$D_q = \frac{\frac{1}{2}\hbar\omega_0\coth\frac{\hbar\omega_0}{2kT}}{\Delta V}.$$

B. *Quantum ratchet under equilibrium condition*

The Fokker–Planck equation corresponding to Eq. (3.18) is given by

$$\frac{\partial P(x,\dot{x},t)}{\partial t} = \left[-\frac{\partial}{\partial x}\dot{x}+\frac{\partial}{\partial \dot{x}}\left(\frac{\dot{x}}{\mu^*}-\frac{f(x)}{\mu^*}\right) + \frac{D_q}{\mu^{*2}}\frac{\partial^2}{\partial x^2}\right]P(x,\dot{x},t). \quad (3.20)$$

The above equation can be solved in the stationary state. The stationary probability density is

$$P_{st}(x,\dot{x}) = N\exp\left[-\frac{\mu^*\dot{x}^2}{2}+\int_0^x\frac{f(y)}{D_q}dy\right], (3.21)$$

where N is the normalization constant which can obtained as

$$\int_{-\infty}^{+\infty}d\dot{x}\int_0^L dx\,P_{st}(x,\dot{x}) = 1. \quad (3.22)$$

It is easy to check that in the stationary state the mean velocity is equal to zero;

$$\langle\dot{x}\rangle_s = \int_{-\infty}^{+\infty}\dot{x}\,d\dot{x}\int_0^L dx\,P_{st}(x,\dot{x}). \quad (3.23)$$

Several points are now in order: (i) Eq. (3.23) suggests that the stationary distribution (3.21) is an equilibrium distribution because of the zero current condition. (ii) The equilibrium distribution Eq. (3.21) formally contains quantum corrections to all orders in $Q(x,\langle\delta\hat{x}^n\rangle)$. (iii) Since $Q(x,\langle\delta\hat{x}^n\rangle)$ essentially arises due to nonlinear part of the potential the nonlinearity and the quantum effects are entangled in this quantity modifying the classical part of the potential. Thus the classical force $-V'(x)$ is modified by the quantum contribution. (iv) Since in the present scheme one may express the quantum mechanical operator $\hat{x} = x+\delta\hat{x}$ or $\dot{\hat{x}} = \dot{x}+\delta\dot{\hat{x}}$ where x and $\dot{x}$ are quantum mechanical mean values and $\langle\delta\hat{x}\rangle = \langle\delta\dot{\hat{x}}\rangle = 0$ by construction and $[\delta\hat{x},\delta\dot{\hat{x}}] = i\hbar$ as noted earlier, it follows that

$$\langle\dot{\hat{x}}\rangle_{qs} = \langle\dot{x}+\delta\dot{\hat{x}}\rangle_{qs} = \langle\dot{x}\rangle_s+\langle\langle\delta\dot{\hat{x}}\rangle\rangle_s = \langle\dot{x}\rangle_s. (3.24)$$

The relation between three types of averages e.g., $\langle\ldots\rangle_{qs}$, quantum statistical; $\langle\ldots\rangle_s$ statistical average over quantum mechanical mean and $\langle\ldots\rangle$, quantum mechanical mean must be clearly distinguished. The relation Eq. (3.24) expresses the usual quantum current as a simple statistical average of the quantum mechanical mean value in the present c-number scheme and the decisive advantage of using this formalism is quite apparent. (v) In absence of quantum correction term $Q(x,\langle\delta\hat{x}^n\rangle)$ and $D_q \to \frac{\gamma kT}{\Delta V}$ as one approaches the classical limit ($kT \gg \hbar\omega$), the quantum Langevin equation (3.18) reduces to classical Langevin equation. (vi) The zero current situation or equivalently the equilibrium distribution function (3.21) ensures the condition of detailed balance in absence of any external driving. This condition is a necessity in the present context and the formalism since it guarantees that the quantum correction term does not give any tilt or bring any asymmetry on the classical periodic potential generating any unphysical current.

Under overdamped condition the inertial term may be neglected and one obtains from Eq. (3.18)

$$\dot{x} = f(x)+\Gamma(t), \quad (3.25)$$

where over-dot (.) refers to differentiation with respect to dimensionless time t defined as $\overline{t} = \frac{t}{\tau_0}$ and $\overline{x} = \frac{x}{L}$. Therefore Eq. (3.25) gives the relation $\langle\frac{dx}{dt}\rangle_s = \frac{L}{\tau_0}\langle\dot{x}\rangle_s = v_0\langle f(x)\rangle_s$ or

$$\left\langle\frac{dx}{dt}\right\rangle_s = v_0\int_0^1 f(x)P_{st}(x)dx. \quad (3.26)$$

Here we have denoted the characteristic velocity $v_0 = L/\tau_0$. The equation for probability density function $P(x,t)$ corresponding to Eq. (3.25) is given by

$$\frac{\partial P(x,t)}{\partial t} = -\frac{\partial J(x,t)}{\partial x}, \quad (3.27)$$

where the probability current

$$J(x,t) = f(x)P(x,t)-D_q\frac{\partial P(x,t)}{\partial x}. \quad (3.28)$$

In the stationary state $P(x) = Lt_{t\to\infty}P(x,t)$, J is constant as

$$J = f(x)P(x)-D_q\frac{\partial P(x)}{\partial x}. \quad (3.29)$$

The solution of above equation for $P(x)$ reads as

$$P(x) = -\frac{J}{D_q}\exp[-\psi(x)]\int_0^x\exp[\psi(y)]\,dy + N\exp[-\psi(x)], \quad (3.30)$$

Telegraphic noise: is a stochastic process in which a random variable can assume two values.

where

$$\psi(x) = -\int_0^x \frac{f(y)}{D_q} dy,$$

or $\psi(x) = \frac{V(x) - V(x) + \langle V(\hat{x}) \rangle}{D_q}$ and N is constant. The last relation follows from (3.6) and (3.19). Since $V(x)$ is periodic, *i.e.* $V(x) = V(x+1)$ we must have

$$\psi(x) = \psi(x+1). \tag{3.31}$$

For periodic boundary condition on (3.30) and from (3.31) it follows that,

$$\frac{J}{D_q} \int_x^{x+1} \exp[\psi(y)]\, dy = 0. \tag{3.32}$$

Since the above integral is non-zero an overdamped Langevin equation with periodic boundary condition shows $J = 0$. This corresponds to an equilibrium situation with probability density function from (3.30),

$$P(x) = N \exp[-\psi(x)]. \tag{3.33}$$

Normalization constant N is $[\int_0^1 \exp[-\psi(x)]]^{-1}$. Therefore the quantum correction $Q(x, \langle \delta\hat{x}^n \rangle)$ in $\psi(x)$, as expected, can not break the detailed balance in the quantum system, nor the symmetry of the potential. This conclusion is an important check of the present formalism for a correct description of the equilibrium.

IV. Breaking of detailed balance by non-equilibrium external fluctuation and noise-induced quantum transport under overdamped condition

Since at equilibrium detailed balance in the quantum stochastic system under overdamped condition forbids any transport we introduce an external forcing agency on the system. The dynamics of the particle is described by the equation

$$\dot{x} = f(x) + \Gamma(t) + \epsilon(t), \tag{4.1}$$

where $\Gamma(t)$ is the quantum internal noise of the bath with the properties as noted earlier. $\epsilon(t)$ is the an external non-equilibrium fluctuation. In our present analysis of quantum ratchet problem we have chosen two specific forms of $\epsilon(t)$, (A) an asymmetric exponentially correlated telegraphic noise (B) a periodic force, e.g., $\sin\omega t$.

A. *Generation of directed motion by a random telegraphic noise*

In our first analysis, $\epsilon(t)$ is a random telegraphic noise also known as dichotomous noise, which takes two possible values $\epsilon(t) = \{-a, b\}$. If the probability of jumps per unit time from one state are given by $P(-a \to b) = \mu_a$ and $P(b \to -a) = \mu_b$ and if we assume $a\mu_b = b\mu_a$, then this external stochastic process can be described by the first two moments as

$$\langle \epsilon(t) \rangle = 0, \tag{4.2}$$

$$\langle \epsilon(t)\epsilon(s) \rangle = \frac{Q_I}{\tau} \exp\left[-\frac{|t-s|}{\tau}\right], \tag{4.3}$$

where the correlation time of the noise $\tau = \frac{1}{\mu_a + \mu_b}$ and the noise intensity $Q_I = \tau ab$, τ_a and τ_b are mean waiting times in the states a and b ($\mu_a = \frac{1}{\tau_a}, \mu_b = \frac{1}{\tau_b}$) respectively. Therefore the three parameters intensity Q_I, correlation time τ, and asymmetry $\theta = b - a$ are the characteristics of the noise. For symmetrical noise $a = b$. The quantum corrections in $f(x)$ which we consider in principle are in all orders. The quantum equation of motion for joint probability densities can be mapped into a classical setting by defining

$$P_+(x,t) = P(x,b,t); \quad P_-(x,t) = P(x,-a,t),$$

so that Fokker–Planck equation with jump processes are given by

$$\begin{aligned} \frac{\partial P_+(x,t)}{\partial t} &= -\frac{\partial}{\partial x}[f(x) + b]P_+(x,t) \\ &\quad + D_q \frac{\partial^2}{\partial x^2} P_+(x,t) - \mu_b P_+(x,t) \\ &\quad + \mu_a P_-(x,t), \end{aligned} \tag{4.4}$$

$$\begin{aligned} \frac{\partial P_-(x,t)}{\delta t} &= -\frac{\partial}{\partial x}[f(x) - a]P_-(x,t) \\ &\quad + D_q \frac{\partial^2}{\partial x^2} P_-(x,t) + \mu_b P_+(x,t) \\ &\quad - \mu_a P_-(x,t). \end{aligned} \tag{4.5}$$

The total probability density $P(x,t)$ at any time is given by

$$P(x,t) = P_+(x,t) + P_-(x,t). \tag{4.6}$$

Eqs. (4.4) and (4.5) yield the equation of the motion for $P(x,t)$ as

$$\frac{\partial P(x,t)}{\partial t} = -\frac{\partial}{\partial x}[f(x)]P(x,t) - \frac{\partial}{\partial x} W(x,t) + D_q \frac{\partial^2}{\partial x^2} P(x,t), \tag{4.7}$$

where $W(x,t)$ is an auxiliary distribution function

$$W(x,t) = bP_+(x,t) - aP_-(x,t), \quad (4.8)$$

which follows the equation

$$\frac{\partial W(x,t)}{\partial t} = -\frac{\partial}{\partial x}[f(x)+\theta]W(x,t) + D_q\frac{\partial^2}{\partial x^2}W(x,t) - \frac{1}{\tau}W(x,t) - ab\frac{\partial}{\partial x}P(x,t). \quad (4.9)$$

The normalization conditions are

$$\int_c^{c+1} P(x,t)dx = 1;$$
$$\int_c^{c+1} W(x,t)dx = 0. \quad (4.10)$$

In the stationary state we obtain an expression for the constant current J as

$$-D_qP'(x) + f(x)P(x) + W(x) = J, \quad (4.11)$$

and also we have

$$\tau D_q W''(x) - \tau[\theta + f(x)]W'(x) - [1+\tau f'(x)]W(x) = Q_I P'(x), (4.12)$$

$P(x)$ and $W(x)$ are the stationary solutions of the coupled equations (4.7) and (4.9). It is difficult to solve analytically the above two equations for arbitrary potential. In what follows we consider the solutions under two specific cases (a) large correlation time and (b) small correlation time.

1 Large correlation time

We return to Eqs. (4.11) and (4.12) and rewrite (4.12) as

$$D_q\frac{d^2W(x)}{dx^2} - \frac{d}{dx}\{\theta + f(x)\}W(x) - \frac{1}{\tau}W(x) = ab\frac{dP(x)}{dx}. \quad (4.13)$$

For $\tau \gg 1$ we neglect the term with $\frac{1}{\tau}$. Integration over Eq. (4.13) then leads to

$$D_qW'(x) - [\theta + f(x)]W(x) = abP(x) + D. \quad (4.14)$$

D is constant. We now put the equilibrium solution for $P(x)$, Eq. (3.33) in (4.14) and solve it for $W(x)$ to obtain (we put $D=0$ to make the system free from bias due to external fluctuating force averaged over a period)

$$W(x) = \exp[-\psi_1(x)] \times \left\{\frac{abN}{D_q}\int_0^x \exp\psi_2(y)dy + C_m\right\}, \quad (4.15)$$

$$C_m = -\frac{abN}{D_q} \times \frac{\int_{c_0}^{c_0+1}\exp[-\psi_1(x)]\int_0^x \exp[\psi_2(y)]\,dy\,dx}{\int_{c_0}^{c_0+1}\exp[-\psi_1(x)]\,dx}, \quad (4.16)$$

$$\psi_1(x) = -\int_0^x \frac{\theta + f(y)}{D_q}dy;$$
$$\psi_2(x) = -\int_0^x \frac{\theta}{D_q}dy;$$
$$\psi(x) = -\int_0^x \frac{f(y)}{D_q}dy. \quad (4.17)$$

Here N is given by normalization constant in (3.33). Putting the solutions for $W(x)$ and $P(x)$ (4.15) and (3.33) in (4.11) as a first approximation we obtain the expression for current as the lowest order iterative solution which is given by

$$J = \frac{\int_x^{x+1}\exp[-\psi_2(y)]\{\frac{abN}{D_q}\int_0^y \exp[\psi_2(z)]\,dz + C_m\}\,dy}{\int_x^{x+1}\exp[-\psi(y)]\,dy}. \quad (4.18)$$

The expression for current is valid for large correlation time of the dichotomous noise but formally takes into consideration of quantum effects to all orders. In order to check the consistency of the above expression we now examine the following limiting situations. First, we consider the dichotomous noise to be symmetric, *i.e.*, $\theta = 0$. J then reduces to

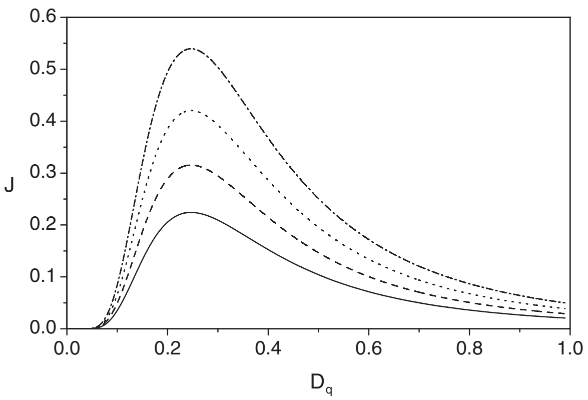
$$J = \frac{\int_x^{x+1}\{\frac{abN}{D_q}y + C_m\}\,dy}{\int_x^{x+1}\exp[-\psi(y)]\,dy}, \quad (4.19)$$

and

$$C_m = -\frac{abN}{D_q}\frac{\int_{c_0}^{c_0+1}x\exp[-\psi(x)]\,dx}{\int_{c_0}^{c_0+1}\exp[-\psi(x)]\,dx}. \quad (4.20)$$

If now $V(x)$ is assumed to be of inversion symmetric, then $\langle V(\hat{x})\rangle$ is also symmetric and C_m

Figure 5: Variation of current (J) with quantum diffusion coefficient D_q in the large correlation time limit for $\theta = 1.0$ and (i) $a = 1.0$ and $b = 2.0$ (solid line), (ii) $a = 1.25$ and $b = 2.25$ (dashed line), (iii) $a = 1.5$ and $b = 2.5$ (dot line) and (iv) $a = 1.75$ and $b = 2.75$ (dash-dot line).



would be zero and $J = 0$ in such situation since

$$\int_{-1/2}^{+1/2} x \exp[-\psi(x)]\, dx = 0\,;$$

$$\int_{-1/2}^{+1/2} \frac{abN}{D_q}\, x\, dx = 0. \qquad (4.21)$$

Therefore with symmetric potential and symmetric dichotomous noise, the current is zero even in the presence of quantum corrections. To obtain a quantum current it is necessary that either the periodic potential should be asymmetric and/or noise $\epsilon(t)$ should be asymmetric and vice-versa.

We now proceed to analyze the current under non-equilibrium condition and the related quantum effects. One of the prime quantities for this analysis is the potential $V(x)$ or the corresponding force term $f(x)$ given by

$$f(x) = -[V'(x) - Q(x, \langle \delta\hat{x}^n \rangle)] = -\frac{\partial}{\partial x}\left[V(x) + \sum_{n \geq 2} \frac{1}{n!} V^n(x) \langle \delta\hat{x}^n \rangle \right]. \qquad (4.22)$$

Taking up to leading order quantum correction the effective potential force $f(x)$ is given[76,89,90,78],

$$f(x) = -[V'(x) + \Delta_q V'''(x)[V'(x)]^2]. \qquad (4.23)$$

where $\Delta_q = \frac{\hbar/\omega_0}{[V'(q_c)]^2}$, q_c is a reference point, where quantum correction term $\langle \delta\hat{x}^2 \rangle$ has a minimum value $(\hbar/\omega_0)$. We now emphasize an important point. If the potential is symmetric, then the quantum correction in Eq. (4.23) is an odd function just as $V'(x)$. This implies that quantum correction to classical potential has not destroyed the inversion symmetry of $V(x)$. Thus the approximation in deriving the leading order quantum effect is consistent with symmetry requirement of the problem. For a symmetric cosine potential with period 2π

$$V(x) = \frac{1}{2}(\cos x + 1). \qquad (4.24)$$

The $f(x)[= \pi\{\sin 2\pi x + \Delta_q \sin^3 2\pi x\}]$ and other related quantities $\psi_1(x)$, $\psi_2(x)$ and $\psi(x)$ can be calculated explicitly in the usual way. In Fig. 5 we illustrate the variation of current for a fixed value of the system non-linearity $\Delta_q (= 0.04)$ and asymmetry parameter $\theta (= 1.0)$ as a function of quantum diffusion coefficient D_q. One observes that with increase of D_q the magnitude of current increases to a maximum followed by a decrease. For a fixed D_q with increase of the strength of the external dichotomous noise (proportional to the product ab) the current increases.The effect of quantization of a classical ratchet is shown in Fig. 6, where we make a comparison of the current vs temperature profile for the classical and the quantum cases for $a = 1.75$, $b = 2.75$. One observes that in the low temperature region the classical current is significantly lower in magnitude than the quantum current, and at high temperature the effect of quantization become insignificant.

2 Short correlation time

In the regime of short correlation time $\tau \ll 1$ of dichotomous noise we follow Kula *et al*[93] to expand $P(x)$, $W(x)$ and J in power series with τ as a smallness parameter;

$$P(x) = \sum_{n=0}^{\infty} \tau^n P_n(x); \quad W(x) = \sum_{n=0}^{\infty} \tau^n W_n(x) \text{ and}$$

$$J = \sum_{n=0}^{\infty} \tau^n J_n. \qquad (4.25)$$

Making use of the above expressions in (4.11) and (4.12) we obtain the following set of equations,

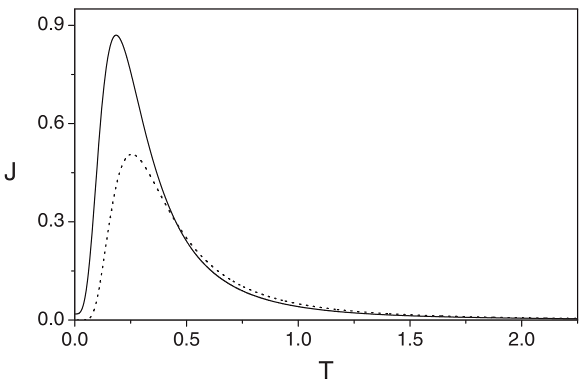
$$J_n = -D_q P'(x) + f(x) P_n(x) + W_n(x), \qquad (4.26)$$

$$W_n(x) = D_q W''_{n-1}(x) - [\theta + f(x)] W'_{n-1}(x) - f'(x) W_{n-1}(x) - Q_I P'_n(x), \qquad (4.27)$$

with $n = 1, 2, 3 \ldots$

$$W_0(x) = -Q_I P'_0(x).$$

Figure 6: Comparison of quantum (solid line) and classical (dotted line) current (J) vs temperature (T) profile for the parameter set $a=1.75$, $b=2.75$, in the large correlation time limit.



The probability functions $P_n(x)$ obey the periodicity conditions and they are normalized over dimensionless period $(L=1)$. We thus obtain the zero order contributions as

$$J_0 = 0, \tag{4.28}$$

$$P_0(x) = N\exp\left[\int_0^x \frac{f(y)}{D_q+Q_I}\,dy\right], \tag{4.29}$$

with normalization constant

$$N^{-1} = \int_0^1 \exp\left[\int_0^x \frac{f(y)}{D_q+Q_I}\,dy\right]dx. \tag{4.30}$$

The higher order contributions can be obtained following Kula *et al.* For the present purpose the leading order current is given by

$$\begin{aligned} J_1 = & \left[\int_0^1 N^{-1}P_0(x)dx\int_0^1 NP_0^{-1}(x)dx\right]^{-1} \\ & \times\left[\frac{\theta Q_I}{(D_q+Q_I)^2}\int_0^1 f^2(x)dx \right.\\ & \left. + \frac{Q_I^2}{(D_q+Q_I)^3}\int_0^1 f^3(x)dx\right]. \end{aligned} \tag{4.31}$$

The key quantity for the above equation is the force term $f(x)$ with leading order nonlinear correction (4.23). For the symmetric smooth cosine potential of the form $V(x)=\frac{1}{2}[\cos 2\pi x+1]$, $f(x)$ is an odd function. For asymmetric dichotomous fluctuations $(\theta\neq 0)$ and symmetric potential the leading order current is proportional to the integral $f^2(x)$; the integral over $f^3(x)$ being zero. On the other hand the current is proportional to the integral over $f^3(x)$ for symmetric$(\theta=0)$ dichotomous noise. Therefore it is apparent that in the short correlation time limit it is not possible to obtain any noise induced transport with symmetric noise and symmetric potential.

For $f(x)$ given earlier one obtains explicitly the quantum current

$$J_1 = \frac{\pi^2}{2I_1\,I_2}\,\frac{\theta Q_I}{(D_q+Q_I)^2}\left[\frac{5}{8}\Delta_q^2-\frac{3}{2}\Delta_q+1\right] \tag{4.32}$$

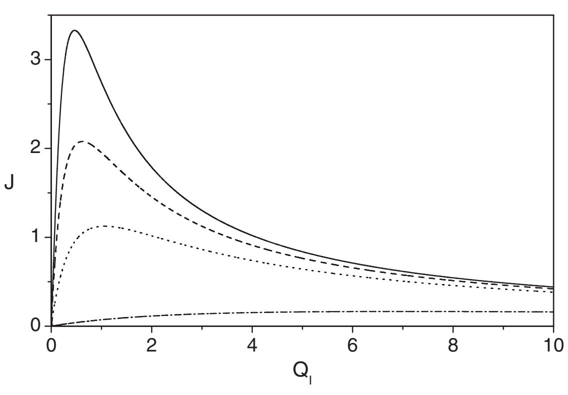
I_1 and I_2 are given by

$$I_1 = \int_0^1 \exp\left[-\int_0^x \frac{f(y)}{(D_q+Q_I)}\,dy\right]dx;$$

$$I_2 = \int_0^1 \exp\left[\int_0^x \frac{f(y)}{(D_q+Q_I)}\,dy\right]dx.$$

We now numerically illustrate the behavior of quantum current given by Eq. (4.32). The effect of quantization of the reservoir is apparent in Fig. 7 in the variation of current with Q_I for several values of quantum diffusion coefficient D_q of the heat bath for fixed $\theta(=1.0)$ and $\tau(=0.1)$. For small D_q, the current falls off monotonically after reaching maxima. The maxima and the current drops for higher values of quantum diffusion coefficient since thermalization prevails over the dynamics, in general. In Fig. 8 we compare the current vs temperature profile for the classical and the quantum $(\Delta_q=0.3)$ cases for fixed $a(=1.0)$, $b(2.0)$ and $\tau(=0.1)$. We observe again that in the low temperature range the current is significantly higher for the quantum case. In order to examine the influence of the correlation time τ of the nonthermal noise on current we plot

Figure 7: Variation of current (J) with nonthermal noise strength (Q_I) for different values of quantum diffusion coefficients $D_q=0.25$ (solid line), $D_q=1.5$ (dashed line), $D_q=3.0$ (dotted line) and $D_q=7.0$ (dash-dot line) for the parameter set $\theta=1.0$ and $\tau=0.01$ in the short correlation time limit.



Superionic conductor: is a typical crystal lattice, where the mobile ions are constrained to flow through a confining network of voids and narrow channels as defined by the interaction potential.

Square wave: is a basic kind of non-sinusoidal wave form encountered in electronics and signal processing. In the present problem we have used zero mean square wave force for generation of directed motion.

in Fig. 9 the variation of current as a function of D_q for several values of τ for fixed values of $a(=1.0)$, $b(=2.0)$, $\theta(=1.0)$ and $\Delta_q(=0.04)$. All the bell-shaped curves exhibit maxima at optimal D_q values. Increase in correlation time τ results in enhancement of directed motion, and shift of the maxima, towards the origin. Physically this implies that departure from equilibrium is increasingly favored for larger correlation time of the external noise in this region. A possible experimental realization of the distinctive behavior of this quantum ratchet in contrast to its classical counterpart in a superionic conductor driven by a dichotomous noisy electric field at low temperature may be suggested. A schematic diagram for the superionic conductor in a dichotomous noisy electric field is given in Fig. 10. A typical superionic conductor may be AgI. This system has been traditionally used[81–83] for measurement of current in presence of an external electric field directly or in terms of frequency dependent mobility. Typically in a superionic conductor like AgI, I^- ions form the lattice allowing the Ag^+ ions to move in a periodic potential of the form $\cos\frac{2\pi x}{L}$, L being the lattice spacing. The lattice vibrations contribute to both the Langevin force as well as the frictional force on the Ag^+ ions maintaining the detailed balance at the thermal equilibrium. For slowly moving Ag^+ ions compared to lattice vibrations, white noise approximation is sufficient. At very low temperature the thermal fluctuation gets suppressed making thermal wave length much longer. At the same time, the average thermal photon number $\bar{n}$ leads to zero and vacuum fluctuations come into play. In this limit the current is significantly enhanced in the quantum domain. An application of an external electric field at both ends of the conductor which fluctuates randomly between two values in an asymmetric way obeying the prescribed noise statistics, is expected to result in an observable current.

Figure 10: Highly mobile Ag^+ ions (solid circles) in the nearly fixed iodide lattice (hollow circle).

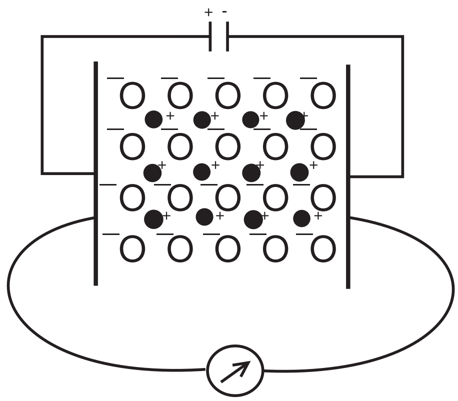


Figure 8: Comparison of quantum (solid line)and classical (dotted line) current (J) vs temperature (T) profile for the parameter set $a=1.0, b=2.0$ and $\tau=0.1$ in the short correlation time limit.

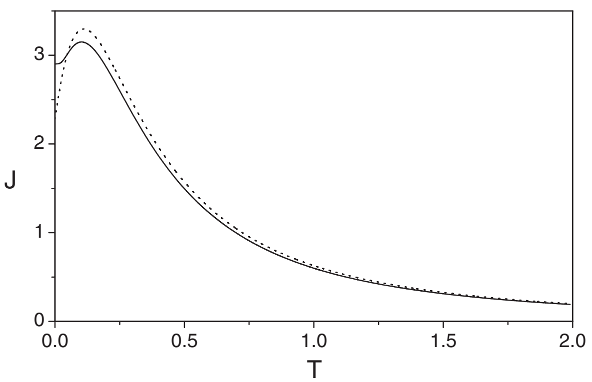
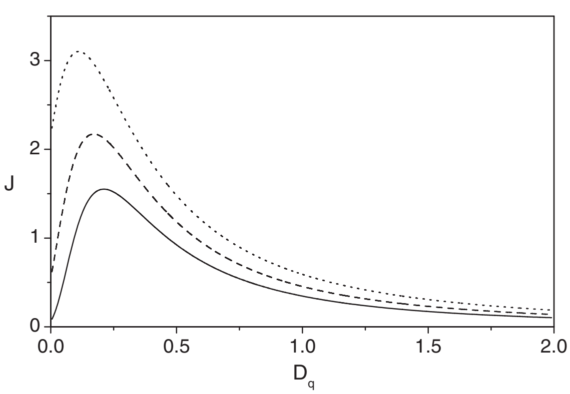


Figure 9: Variation of current (J) with quantum diffusion coefficient for different values of correlation time τ of nonthermal noise $\tau=0.05$ (solid line), $\tau=0.07$ (dashed line) and $\tau=0.1$ (dotted line) for the parameter set $a=1.0$, $b=2.0$ and $\theta=1.0$ in the short correlation time limit.



B. *Generation of directed motion by an external periodic force*

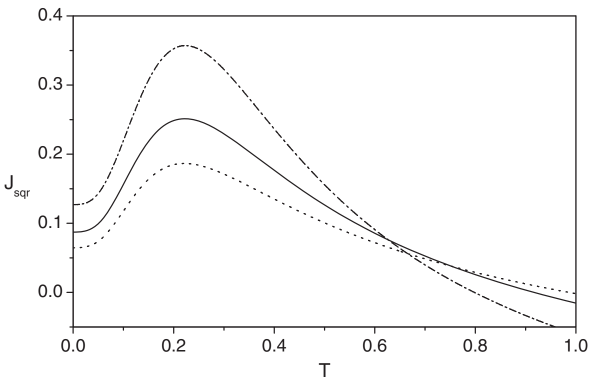
In this section we analyze the essential features of a quantum ratchet device where we use an external periodic signal,

$$\epsilon(t) = A(t) = A_0 \sin\omega t, \tag{4.33}$$

to break the detailed balance of the system. The stochastic dynamics of the particle is given by Eq. (4.1). We replace the external fluctuating force term by $A(t)$)

$$\dot{x} = f(x) + \Gamma(t) + A(t). \tag{4.34}$$

Figure 11: A plot of quantum current vs T for different strength of external periodic force (i) $A_0 = 1.0$ (dotted line), (ii) $A_0 = 1.2$ (solid line), (iii) $A_0 = 1.5$ (dash-dot line) and $l = 0.01$. (All the quantities are dimensionless).



$f(x)$ and $\Gamma(t)$ are same as discussed in the earlier section. The Fokker-Planck equation corresponding to Eq. (4.34) is given by

$$\frac{\partial P(x,t)}{\partial t} = -\frac{\partial J(x,t)}{\partial x}, \tag{4.35}$$

where

$$J(x,t) = -D_q \frac{\partial P(x,t)}{\partial x} + [f(x) + A(t)]P(x,t). \tag{4.36}$$

If the forcing frequency is very low, there is enough time for the system to reach the steady state during the period τ and the above equation can be solved analytically for J as a function of A, using periodic boundary and normalization conditions

$$P(x+1) = P(x); \quad \int_c^{c+1} P(x)dx = 1. \tag{4.37}$$

We then obtain

$$\begin{aligned} J(A) &= [\exp[\psi(1)] - \exp[\psi(0)]] \\ &\quad [N\left[\{\exp[\psi(1)]\}[\int_0^1 \exp[\psi(x)]dx - C_2]\right. \\ &\quad + C_2 \exp[\psi(0)]]]^{-1}, \end{aligned} \tag{4.38}$$

where

$$\begin{aligned} N &= \frac{1}{D_q}\int_0^1 \exp[\psi(x)]dx; \\ C_2 &= \frac{\int_0^1 \exp[\psi(x)]dx \int_0^x dy \exp[\psi(y)]}{\int_0^1 \exp[\psi(x)]dx}, \end{aligned} \tag{4.39}$$

$$\begin{aligned} \psi(x) &= \int_c^x \frac{f(y)+A}{D_q} dy; \\ \psi(1) &= \int_c^1 \frac{f(x)+A}{D_q} dx. \end{aligned} \tag{4.40}$$

The average current over a forcing period is given by

$$J_{av} = \frac{1}{\tau}\int_0^\tau J(A(t))dt. \tag{4.41}$$

Average square wave current of amplitude A_0 is given by

$$J_{sqr} = \frac{1}{2}[J(A_0) + J(-A_0)]. \tag{4.42}$$

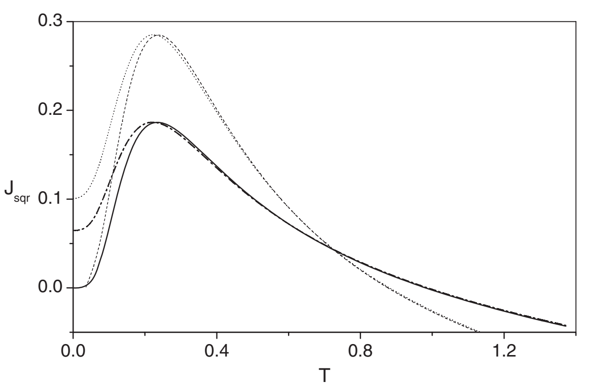
We now proceed to analyze the current under non-equilibrium condition and the related quantum effects. One of the prime quantities for this analysis is the potential $V(x)$ or the corresponding force term $f(x)$. Taking quantum correction up to a leading order, the effective potential force is given by

$$\begin{aligned} f(x) &= -[V'(x) - Q(x, \langle\delta\hat{x}^n\rangle)] = -[V'(x) \\ &\quad + \Delta_q V'''(x)[V'(x)]^2]. \end{aligned} \tag{4.43}$$

In present problem we consider an asymmetric potential of period 2π,

$$V(x) = -\sin x - 0.25\sin 2x. \tag{4.44}$$

Figure 12: A currentJ_{sqr} vs temperature (T) plot comparing classical (dashed line and solid line) and quantum (dotted line and dashed dot line) limit for different strength of external periodic force (i) $A_0 = 1.3$ (dotted line and dashed line), (ii) $A_0 = 1.0$ (dashed dot line and solid line) and $l = 0.0$. (All the quantities are dimensionless).



Efficiency: of ratchet device is defined as the fraction of the total consumption energy which is accessible to do useful task.

If the potential is symmetric, then the quantum correction is an odd function just as $V'(x)$. This implies that quantum correction to classical potential has not destroyed the inversion symmetry of $V(x)$ as in the previous case. It also clear that if the potential is periodic then the contribution due to quantum correction to the classical potential i.e., $\int_0^x Q(x, \langle\delta\hat{x}^n\rangle)dx$ is a periodic function of x. Assuming the form of potential of Eq. (4.44), the expression for quantum correction can be calculated in the usual way[76,85,89,90]. Physically the correction terms account for the quantum fluctuation or dispersion around the classical path of a dynamical system. In presence of strong dissipation these fluctuations are small since it is well-known that dissipation enhances classicality[96]. The role of effective potential of the similar nature which gives rise to leading order quantum correction to classical Langevin force had also been noted earlier, e. g., in the analysis of strong friction limit of quantum stochastic processes etc.[91,92,100]. We emphasize that the approximate forms of quantum correction must satisfy the basic symmetry requirement, appropriate equilibrium distribution and other thermodynamic consistency condition as pointed out in the earlier section. In Fig. 11 we illustrate the variation of current as a function of temperature (T) for different values of the amplitude of external derive (A_0). One observes that with increase of D_q (proportional to temperature) the magnitude of current increases to a maximum followed by decrease and a current reversal at high temperature. At higher temperature the system is thermalized as a result of which organized motion in a preferential direction decreases and the motion towards the load dominates. For fixed D_q with increase of the value of A_0 the magnitude of the current increases. The effect of quantization of a classical ratchet is shown in Fig. 12, where we present a comparison of the current vs temperature profile for the classical and the quantum cases. One observes that at the low temperature region the classical current is significantly lower in magnitude than quantum current and at the higher temperature the effect of quantization becomes insignificant. This may be interpreted in terms of an interplay between quantum diffusion coefficient D_q and the potential force term $f(x)$. $f(x)$ contains quantum correction arising due to nonlinearity of the potential. As temperature $T \to 0$, D_q approaches to the value $\frac{1}{2}\hbar\omega_0$, the vacuum limit in deep tunneling region. The anharmonic terms in $f(x)$ do not contribute significantly. So the integrand in effective potential $\psi(x)$ increases sharply. On the other hand, as temperature increases, D_q increases and also D_q and $f(x)$ compete with each other to merge quantum current to its classical counterpart.

C. *Efficiency of ratchet devices*

1 *General aspect*

To analyze the energetics of directed quantum transport, we now introduce an external load to work against the global motion of the forced thermal ratchet system. From Eq. (4.34) it follows that the dynamics of the particle under overdamped condition is described by the scaled equation (we have dropped the over-bar)

$$\dot{x} = f(x) + \Gamma(t) + A(t) - \frac{\partial V_l}{\partial x}. \tag{4.45}$$

The quantum mechanical mean of the position operator, x represents the state of the energy transducer, that is the state of the ratchet. $\Gamma(t)$ is the internal quantum noise of the thermal bath with the properties as noted earlier. $A(t)$ is an external field with temporal period τ, $A(t+\tau) = A(t)$, in the present problem. We consider $A(t) = A_0 \sin wt$. It is important to note that for a movement of transducer in a preferential direction A_0 must lie between two threshold values, $max_x f(x)$ and -$min_x f(x)$[11]. $\frac{\partial V_l}{\partial x} = l$, is a load against which transducer performs work.

From Sec. IV (B) the expression of current corresponding to Langevin dynamics (4.45)

$$J_{sqr} = \frac{1}{2}[J(A_0) + J(-A_0)]. \tag{4.46}$$

This is average square wave current (assuming external modulation $A(t)$ is a very slow square wave of amplitude A_0) where

$$J(A) = [\exp[\psi(1)] - \exp[\psi(0)]] \left[N\left[\{\exp[\psi(1)]\}\left[\int_0^1 \exp[\psi(x)]dx - C_2\right] + C_2\exp[\psi(0)]\right]\right]^{-1}, \tag{4.47}$$

$$N = \frac{1}{D_q}\int_0^1 \exp[\psi(x)]dx, \tag{4.48}$$

$$C_2 = \frac{\int_0^1 \exp[\psi(x)]dx \int_0^x dy \exp[\psi(y)]}{\int_0^1 \exp[\psi(x)]dx}, \tag{4.49}$$

and

$$\psi(x) = \int_c^x \frac{f(y) + A - l}{D_q}dy;$$

$$\psi(1) = \int_c^1 \frac{f(x) + A - l}{D_q}dx. \tag{4.50}$$

Now we restrict our attention to the analysis of the efficiency of ratchet devices. Depending on the degree and presence of an external load two distinct approaches have been advocated. It has been shown that although in many widely accepted cases efficiency is measured by applying the constant external force, there are situations, where molecular motors are designed not to pull loads (e. g., protein transport within a cell). In such cases a minimum energy input is required to move a particle in a viscous medium. We therefore discuss the two different situations separately.

2 Conventional efficiency in presence of an external load

To discuss the energetics of quantum transport induced by zero mean external derive we consider the energy transducer which interacts with the external derive and the load so that the potential takes the following form

$$U(x,t) = V(x) - \int dx\, Q(x, \langle \delta\hat{x}^n \rangle) + A(t)\, x + lx, \qquad (4.51)$$

where $V(x)$ is the classical potential, second term represents the quantum corrections due to nonlinearity of the classical potential and last two terms are due to external system and the load, respectively. The interaction of transducer with heat bath is assumed to be stochastic, as usual. Thus for the movement of transducer from $x_i(t_i) \to x_f(t_f)$ the total potential energy change (ΔU) and dissipation energy (E_d), during the period $t_i < t < t_f$ are formally given by[64]

$$\Delta U = U(x_f(t_f), t_f) - U(x_i(t_i), t_i), \qquad (4.52)$$

and

$$E_d = \int_{t_i}^{t_f} [-\dot{x} + \Gamma(t)] dx(t) = \int_{t_i}^{t_f} -\left[\frac{\partial U(x,t)}{\partial x}\right] dx(t), \qquad (4.53)$$

respectively.

Because of the conservation law, the sum of the potential energy change and dissipation energy must be equal to the total consumption of energy E_c ($\triangle U + E_d \equiv E_c$), due to the external system $A(t)$.

$$E_c = \int_{t_i}^{t_f} \frac{\partial U(x,t)}{\partial t}\, dt. \qquad (4.54)$$

In the present case the external system is a periodic function of time, so that the ensemble average of total consumption energy (E_c) and dissipation energy (E_d) is given by

$$\langle E_c \rangle = \int_{t_i}^{t_f} dt \int_{space} \frac{\partial U(x,t)}{\partial t} P(x,t)\, dx(t) = \int_{t_i}^{t_f} dt \int_{space} A(t) J(A(t))\, dx(t), \qquad (4.55)$$

and

$$\langle E_d \rangle = \int_{t_i}^{t_f} dt \int_{space} dx \left[-\frac{\partial U(x,t)}{\partial x}\right] J(A(t)) \qquad (4.56)$$

respectively. For the square wave with the amplitude A_0,

$$\langle (E_c)_{sqr} \rangle = \frac{1}{2} A_0 [J(A_0) - J(-A_0)], \qquad (4.57)$$

$$\langle (E_d)_{sqr} \rangle = \frac{1}{2} [A_0 \{J(A_0) - J(-A_0)\} - l\{J(A_0) + J(-A_0)\}]. \qquad (4.58)$$

Hence the work, that the ratchet system extracts from the external system $A(t)$ is given by

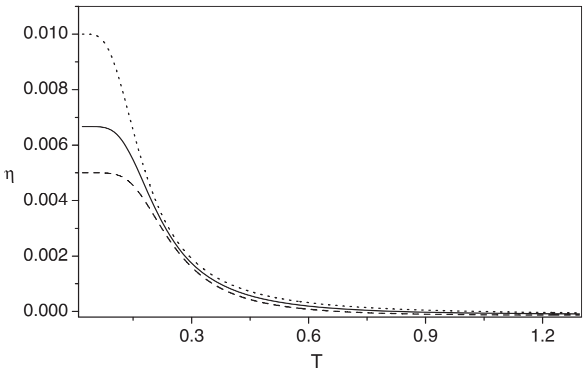
$$\langle W_{sqr} \rangle = \langle (E_c)_{sqr} \rangle - \langle (E_d)_{sqr} \rangle = \frac{1}{2} l [J(A_0) + J(-A_0)] = l \times J_{sqr}. \qquad (4.59)$$

So the work extracted from external system is directly proportional to square wave current. The conventional efficiency of the ratchet system is thus calculated on the basis of external load can be written as

$$\eta = \frac{l J_{sqr}}{E_c} = \frac{l[J(A_0) + J(-A_0)]}{A_0 [J(A_0) - J(-A_0)]}. \qquad (4.60)$$

We now numerically illustrate the behavior of efficiency of the quantum ratchet system as given above. The effect of quantization of the reservoir is apparent in Fig. 13 in the variation of efficiency (η) as a function of temperature for different values of amplitude (A_0) of the external periodic system. The efficiency is a decreasing function of temperature for any value of A_0 and it decreases with increase of A_0 for a fixed value of temperature. The distinctive behavior of efficiency of a quantum system is evident from the nature of quantum current that can attain a maximum value for a finite temperature. On the other hand the maximum efficiency is realized at the zero strength of thermal fluctuation. It is thus apparent that the equilibrium fluctuation due to thermal heat bath is an hindrance for efficient

Figure 13: A plot of quantum efficiency vs T for different strength of external periodic force (i) $A_0 = 1.0$ (dotted line), (ii) $A_0 = 1.5$ (solid line), (iii) $A_0 = 2.0$ (dashed line) and $l = 0.01$. (All the quantities are dimensionless).



extraction of useful work from non-equilibrium fluctuations. To have a closer look at the behavior of efficiency we present in Fig. 14 the dissipation energy and total consumption energy as a function of temperature. At very low temperature the energy loss due to dissipation (E_d) during the movement of energy transducer is small compared to the total energy consumed from the external system, because for a finite net displacement $x_i \rightarrow x_f$ it covers minimum path at low thermal fluctuation. On the other hand at high temperature the path of energy transducer is more chaotic. So for a finite net displacement it covers maximum path and loses a greater amount energy due to dissipation. With increase of the temperature dissipation energy and total energy consumption from external system both are increased and the difference between two energies (E_d and E_c) become insignificant at higher temperature.

The condition for maximum conventional efficiency can be realized by rearranging Eq. (4.60) as a function of $\frac{J(-A_0)}{J(A_0)}$

$$\eta = \frac{l}{A_0}\left(1 - \frac{2|\frac{J(-A_0)}{J(A_0)}|}{1 + |\frac{J(-A_0)}{J(A_0)}|}\right). \tag{4.61}$$

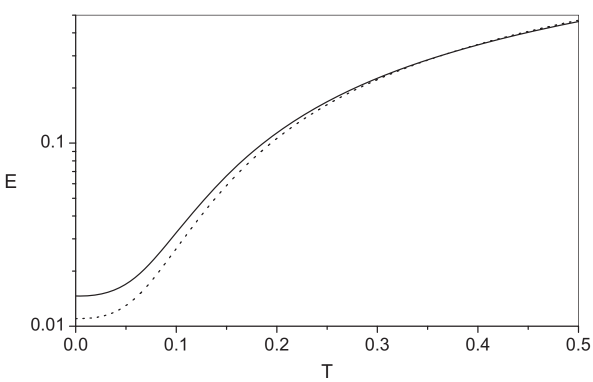
In the limit $|\frac{J(-A_0)}{J(A_0)}| \rightarrow 0$, the maximum efficiency of the energy transform for a given load and force amplitude is given by (the limit can be achieved by suitable adjustment of parameters)

$$\eta_{max} = \frac{l}{A_0}. \tag{4.62}$$

Now we have two important conclusions regarding the maximum efficiency of a ratchet system, (i) it is a simple ratio of load to a parameter of external system (strength of external system) and it is independent of the characteristics of the bath. (ii) η_{max} being independent of the nature of the bath and the system potential, is same both in quantum and classical systems.

In Fig. 15 we compare the conventional efficiency vs temperature profile for the classical and the quantum cases for different values of A_0. We observe that the efficiency of quantum ratchet is significantly lower than classical one and the difference becomes insignificant at higher temperature. Since the vacuum fluctuations tend to be effective in the quantum system as one approaches the zero temperature limit, the transducer loses a higher amount of dissipation energy than the classical one.

Figure 14: A comparison between dissipation energy E_d (dotted line) and total energy consumption E_c (solid line) as a function of temperature for the parameter set $l = 0.05$ and $A_0 = 1.0$. (All the quantities are dimensionless)



3 Efficiency in absence of an external load and generalized efficiency

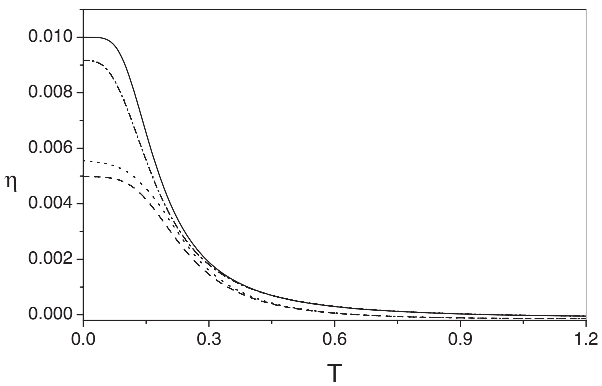
We now consider the situation where the motor works without any external load. The task is not only to translocate the motor over a distance L but also to do this with a given average velocity it must work against the viscous force $\gamma\langle v\rangle$. Now replacing the load by $\gamma\langle v\rangle$ we can define an efficiency (Stokes efficiency)

$$\eta_S = \frac{\gamma\langle v\rangle^2}{E_c}. \tag{4.63}$$

By combining the contribution due to (4.63) and (4.60), it is possible to define further a generalized efficiency for the quantum system

$$\eta_G = \frac{lJ_{sqr} + \gamma\langle v\rangle^2}{E_c}. \tag{4.64}$$

Figure 15: Conventional efficiency (η) vs temperature (T) plot comparing classical (dotted line and solid line) and quantum (dashed line and dashed dot line) limit for different strength of external periodic force (i) $A_0 = 1.3$ (dotted line and dashed line), (ii) $A_0 = 1.0$ (solid line and dashed dot line) and $l = 0.0$. (All the quantities are dimensionless).



The above expression is the quantum generalization of the classical generalized efficiency as given earlier by Suzuki and Munakata[67] and Derényi *et al*[66]. This account for both the work that the motor performs against the external load l as well as the work that is necessary to move the particle over a given distance in a viscous environment at the average velocity $\langle v \rangle$.

In Fig. 16 we the present variation of Stokes efficiency η_S as a function of temperature. It is important to observe that efficiency reaches a maximum at a particular temperature both for classical as well as for the quantum case. However, again at low temperature the efficiency of the classical system drops to zero in sharp contrast to quantum case. At high temperature the system however, tends to classical regime as expected.

V. Quantum ratchet device under weak damping limit

Inertia plays an important role, in dictating the nature of motion which can be deterministic as well as chaotic, in separation of particles of different mass[104,105,97,106,107], where it has been shown to affect the direction of flux and in several other issues[108]. In view of these developments it is worthwhile to realize an explicit quantum version of an underdamped ratchet. Our aim in this section is to address this specific issue.

We investigate the quantum Brownian dynamics in an underdamped ratchet in which the particles move in a symmetric periodic potential and are simultaneously driven by an asymmetric rectangular wave whose temporal average over a period is zero. In the weak friction limit when the dissipation constant is much small compared to the system frequency but still larger than the inverse of the time period of the rectangular wave, we observe that finite inertia gives rise to a net directed quantum current. While under overdamped condition the motion of the particles is spatial diffusion-limited, the small friction allows the particles to experience the temporal modulation of the rectangular wave many times so that the energy diffusion takes place very slowly for the escape process. We generalize the approach of Risken and Vollmer[98] to take care of the appropriate quantum correction in our formalism and derive the analytical expression for quantum current for an inertial ratchet. Numerical simulations have been carried out to corroborate the theoretical analysis.

A. *Quantum transport induced by slow external modulation in an weakly damped system*

Since equilibrium thermal fluctuations due to heat bath can not break the detailed balance in the quantum stochastic dynamics, we introduce an external derive with zero mean and with sufficient correlation to generate drift motion on an average in one direction. In presence of external derive Eq. (3.17) can be rewritten as

$$\dot{x} \;=\; v, \tag{5.1a}$$

$$m\dot{v} = - \;\gamma v + f(x) + \Gamma(t) + A(t). \tag{5.1b}$$

The quantum mechanical mean of the position operator, x represents therefore the state of a ratchet. $\Gamma(t)$ is the internal quantum noise of the thermal bath with the properties as given earlier. $f(x)$ is the potential force term which in addition to classical potential force $(-V'(x))$ contains quantum

Figure 16: A comparison between classical (dashed dot line and dashed line) and quantum (solid and dotted line) Stokes efficiency for different strength of external periodic force (i) $A_0 = 1.3$ (solid line and dashed dot line), (ii) $A_0 = 1.0$ (dotted line and dashed line) and $l = 0.0$. (All the quantities are dimensionless).

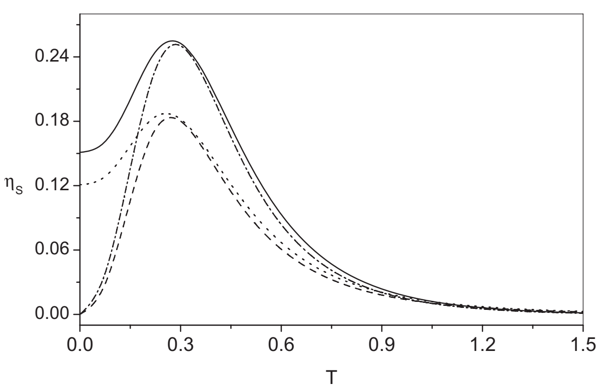
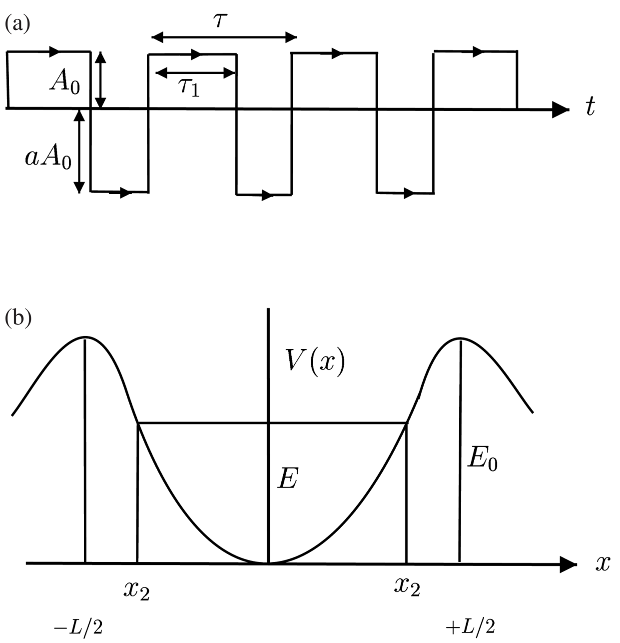


Figure 17: Schematic illustration of (a) an asymmetric rectangular wave (as given by Eq. (5.3)) and (a) a periodic potential (one period of the potential given by Eq. (5.5)).



correction due to nonlinearity of the potential, so that the modified potential felt by the quantum particle is given by

$$U(x) = \int^x f(y)dy = \int^x [-V'(y) + Q(y, \langle \delta \hat{y}^n \rangle)]dy. \quad (5.2)$$

$A(t)$ is an external field with temporal period τ, $A(t+\tau) = A(t)$. In the present problem we consider $A(t)$ is an asymmetric rectangular wave(Fig. 17(a)) of the following form:

$$A(t) = \begin{cases} A_0 & n\tau \leq t < n\tau + \tau_1 \\ -aA_0 & n\tau + \tau_1 \leq t < (n+1)\tau \end{cases} \quad (5.3)$$

The period τ is assumed to be much larger than the time scale of the system in the bath environment. τ and τ_1 are connected though a in such a way that the temporal average of $A(t)$ over a period τ is zero. This implies that we must have

$$\tau_1 = \frac{a\tau}{1+a}. \quad (5.4)$$

Throughout the paper we assume $a = 2$ and a symmetric cosine potential(Fig. 17(b)) with amplitude E_0 and a period $L = 2\pi$

$$V(x) = \frac{E_0}{2}(1 - \cos x). \quad (5.5)$$

The quantum nature of the problem therefore manifests itself in two ways; first, through quantum corrections in $f(x)$ which we consider, in principle, to all orders and secondly in quantum diffusion coefficient D_q for the noise of the bath. The equivalent description for the corresponding probability density is governed by Fokker-Planck equation in two variables of the following form

$$\frac{\partial P(x,v,t)}{\partial t} = -\left\{ -m\frac{\partial v}{\partial x} + \frac{\partial}{\partial v}\left[\gamma v - f(x) - A(t) + D_q \frac{\partial}{\partial v}\right]\right\} P(x,v,t). \quad (5.6)$$

Eq. (5.6) describes a stochastic process where P is a function in a 'quantum' phase space (quantum mechanical mean value space) and the quantum dispersions due to system potential contained in Q of $f(x)$. Since the potential here is periodic in nature we have formally incorporated quantum correction to all orders at this stage. An important approximation, however is the assumption of non-Markovian character of the bath embedded in the nature of D_q. The classical potential $V(x)$ is a periodic function with a length of period L. For simplicity we assume $V(x)$ has only one minimum per period. The amplitude of the potential is $E_0/2$ and the zero of energy is chosen at the minimum of the potential and the points $x = \pm L/2$ correspond to potential maxima as shown in Fig. 17(b). The distribution function $P(x,v,t)$ must be supplemented by the boundary conditions. For $E > E_0$ the distribution follows periodic structure in space, that is

$$P(-L/2, v) = P(+L/2, v).$$

For the energies below the height of the potential well all particles are reflected from the potential barrier to the points $x_{1,2}(E)$. So for the $E < E_0$ regime the boundary condition requires that at the turning points the distribution function of incoming particles is the same as that of reflected particles.
To proceed further we consider the approach of Risken and Vollmer[98]. In the weak friction limit the external forcing must be small otherwise the energy gained by the system due to forcing is not appropriately dissipated and no steady state solution can be obtained. To ensure stationarity Risken and Vollmer have used a scaled force, scaling factor being the dissipation constant. In our present problem where $\gamma \gg 1/\tau$, the limit $\gamma \to 0$ can not realized for a finite τ. However to ensure stationarity we make τ to be very long so that its inverse is much less than γ. At the same time the usual weak friction limit, i.,e., the inequality $\omega_s \gg \gamma$, retains its validity.

And though we do not express the driving force A_0 explicitly scaled by dissipation constant γ as Risken and Vollmer[98], we do assume that A_0 should be small in magnitude for ensuring stationarity. For small friction the energy will slowly vary in course of time. So in the low friction limit the energy will be relevant or slow variable and x or v will become irrelevant or fast variable. Hence it is more convenient to express[98,97] the distribution function $P(x,v,t)$ by space coordinate x and energy E. For this purpose we replace the velocity v by energy variable E and introduce the energy and space distributions W_+ and W_- for positive and negative velocities, respectively,

$$\begin{aligned} v(x,E) &= \sqrt{2m(E-U(x))}, \\ W_+(x,E,t) &= W(x,v(x,E),t), \\ W_-(x,E,t) &= W(x,-v(x,E),t). \end{aligned}$$

For further calculation it is more convenient to introduce the sum and difference of $W_\pm$ and we define

$$W_{S,D} = W_+(x,E,t) \pm W_-(x,E,t). \quad (5.7)$$

Then the boundary conditions for the energy space distribution function $W_{S,D}$ can be rewritten as

$$\begin{aligned} W_S(+L/2,E,t) &= W_S(-L/2,E,t) \\ & \quad E > E_0, \quad (5.8a) \\ W_D(x_1,E,t) &= W_D(x_2,E,t) = 0 \\ & \quad E < E_0. \quad (5.8b) \end{aligned}$$

And the relevant equations in the energy and coordinate space are given by

$$\begin{aligned} \frac{1}{v(x,E)}\frac{\partial W_{S,D}(x,E,t)}{\partial t} &= -\frac{\partial W_{S,D}(x,E,t)}{\partial x} \\ &+ \gamma\frac{\partial}{\partial E}\left[v(x,E)\left(1+D_0\frac{\partial}{\partial E}\right)\right]W_{D,S}(x,E,t) \\ &- \frac{\partial}{\partial E}\left[A(t)W_{S,D}(x,E,t)\right]. \quad (5.9) \end{aligned}$$

where $D_0 = D_q/\gamma$. Both functions $W_{S,D}$ and their energy derivatives are continuous for $E = E_0$. The normalization condition is

$$\int_{E_{min}}^{\infty} dE \int_{x_1(E)}^{x_2(E)} dx \, \frac{W_S(x,E,t)}{v(x,E)} = 1. \quad (5.10)$$

and the expectation value of the drift velocity, the quantity of prime interest is given by

$$v_d = \langle v \rangle = \int_{E_{min}}^{\infty} dE \int_{x_1(E)}^{x_2(E)} dx \, W_D(x,E,t). \quad (5.11)$$

If we choose the forcing frequency of external derive to be very small ($\gamma \gg 1/\tau$), there is enough time for the system to reach the steady state during the period τ and one can obtain the stationary solution of distribution and hence the stationary drift velocity as a function of $A(t)$. So in the stationary limit Eq. (5.9) reduces to

$$\begin{aligned} \frac{\partial W_{S,D}(x,E)}{\partial x} &= \gamma\frac{\partial}{\partial E}\left[v(x,E)\left(1+D_0\frac{\partial}{\partial E}\right)\right] \\ & W_{D,S}(x,E) - \frac{\partial}{\partial E}\left[A(\tau_c)W_{S,D}(x,E)\right]. \end{aligned} \quad (5.12)$$

(To represent a very slow external periodic modulation we replace temporal periodic function $A(t)$ by a quasi-static function $A(\tau_c)$.)

Following Ref[98,97]we use the following ansatz

$$W_{S,D} = \tilde{W}_{S,D}(E) + w_{S,D}(x,E). \quad (5.13)$$

Here $\tilde{W}_{S,D}(E)$ are the slowly varying functions in E only and $w_{S,D}(x,E)$ are rapidly varying in E and slowly varying functions in x that contribute only in a thin boundary layer around $E = E_0$. The role of $w_{S,D}(x,E)$ is to ensure the continuity of $W_{S,D}(x,E)$ for $E = E_0$. Now the modified boundary conditions are

$$\begin{aligned} w_{S,D}(+L/2,E) &= w_{S,D}(-L/2,E) \\ & \quad E > E_0, \quad (5.14a) \\ w_D(\pm L/2) &= \tilde{W}_D(E) = 0 \\ & \quad E < E_0. \quad (5.14b) \end{aligned}$$

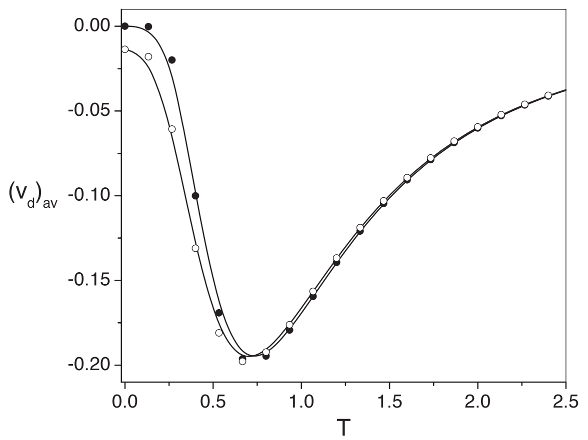
Following[98] one obtains the expression of current as a function of $A(\tau_c)$ as follows

$$v_d(A) = \frac{2NL[\, I_3 + \epsilon\,\chi\, I_4]}{I_0 + I_1 + \epsilon\chi I_2}. \quad (5.15)$$

The integrals of the expression for current are given by

$$\begin{aligned} I_0 &= \sqrt{\frac{2\pi D_q}{m}} \int_{-L/2}^{+L/2} \exp\left[-\frac{f(x)}{D_0}\right], \quad (5.16) \\ I_1(A) &= 2L\int_{E_0}^{\infty} \bar{v}'(E)\exp\left[-\frac{E}{D_0}\right] \\ & \left\{\cosh\left[\frac{\mu A(\tau_c)g(E)}{D_q}\right]-1\right\}dE, \quad (5.17) \\ I_2(A) &= 2L\int_{E_0}^{\infty} \bar{v}'(E)\exp\left[-\frac{E}{D_0}\right] \end{aligned}$$

Figure 18: Drift velocity $(v_d)_{av}$ vs temperature T plot (using Eq. (3,21)) comparing quantum (the profile with open circle) and classical (the profile with solid circle) cases for the parameter set $A_0 = 0.1$, $\gamma = 0.1$, $m = 1.0$ and the barrier height $E_0 = 6.0$.



Wigner canonical thermal distribution shows how the quantum mechanical mean values of coordinate and momentum operator of a Harmonic oscillator are thermally distributed (Gaussian), where the width is given by a sum of energies due to mean photon number and vacuum fluctuation.

$$\sinh\left[\frac{\mu A(\tau_c) g(E)}{D_0}\right] dE, \qquad (5.18)$$

$$I_3(A) = \int_{E_0}^{\infty} \exp\left[-\frac{E}{D_0}\right] \sinh\left[\frac{\mu A(\tau_c) g(E)}{D_0}\right] dE, \qquad (5.19)$$

$$I_4(A) = \int_{E_0}^{\infty} \exp\left[-\frac{E}{D_0}\right] \cosh\left[\frac{\mu A(\tau_c) g(E)}{D_0}\right] dE, \qquad (5.20)$$

where $\mu = 1/\gamma$, χ is a perturbation as given in the Ref[98] whose approximate value is 0.84. The expressions for $g(E)$, $\nu(E)$ and ϵ are given by

$$g(E) = \int_{E_0}^{E} \frac{dE'}{\bar{\nu}(E')}; \quad \bar{\nu}(E) = \frac{1}{L}\int_{-L/2}^{+L/2} \nu(x,E)\, dx$$

$$\text{and} \quad \epsilon = \left[\frac{\sqrt{2}\mu A(\tau_c)}{\sqrt{\bar{\nu}(E_0) D_0}}\right]\sqrt{\gamma},$$

respectively. Here the point to be noted is that the x-dependent solutions enter in our calculation of current and normalization constant via the continuity condition in such a way that the corrected x-independent solutions deviate from pure energy dependent distribution functions by terms of the order $\sqrt{\gamma}$. The average velocity over a forcing period is given by

$$(v_d)_{av} = \frac{1}{\tau}\int_{n\tau}^{(n+1)\tau} v_d(A(\tau_c))\, d\tau_c$$

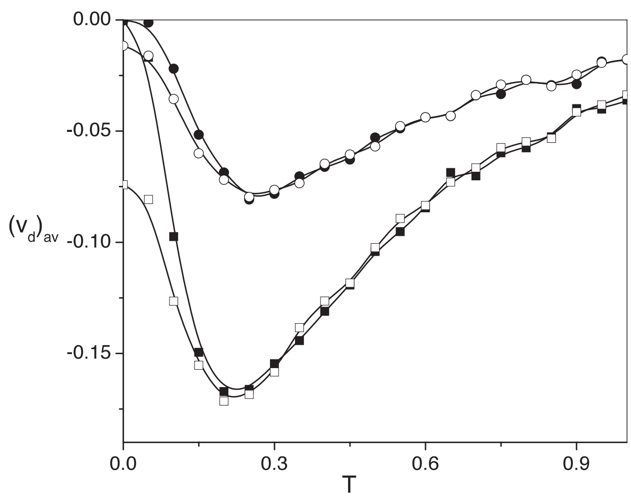
$$= \left(\frac{a}{1+a}\right) v_d(+A_0) + \left(\frac{1}{1+a}\right) v_d(-aA_0). \qquad (5.21)$$

The above expression has a standard form one expects for the current in an adiabatic ratchet; classical[101] or quantum alike. For a symmetric potential and symmetric external derive i,e,. $a = 1$ $\tau_1 = \frac{1}{2}\tau$ the period average drift velocity is zero since $I_3(+A) = -I_3(-A)$ and $\epsilon(+A)\, I_4(+A) = -\epsilon(-A)\, I_4(-A)$. Therefore to obtain a net flow of quantum particles in a preferential direction the potential or the temporal external derive must be asymmetric in nature. To understand the quantum feature of the current it is pertinent to note that the quantum characteristics due to nonlinearity of the system potential is reflected in I_0 through $f(x)$. Furthermore all the integrals I_i $(i = 0, 4)$ contain D_0 the effective quantum temperature as expressed in (3.15) $(D_0 = D_q/\gamma)$. Its origin may be traced to the Wigner canonical thermal distribution[94] used to describe the harmonic oscillator bath. As $T \to 0$ this distribution goes over to a pure state distribution and as a result it renders validity of the expression for quantum current due to non-equilibrium fluctuations even in the deep tunneling regime. We believe that this validity is a necessary requirement for an inertial quantum ratchet for which the quantum fluctuations due to surrounding have to be appropriately accounted for. The analytical expression (5.21) for the current due to underdamped quantum ratchet is illustrated in a typical drift velocity vs. temperature plot in Fig. 18 which exhibits a bell-shaped profile. With increase of noise strength D_q of the bath which is also measure of temperature the magnitude of drift velocity increases to a maximum followed by a decrease. At high temperature all the particles are thermalized as a result of which their organized motion in a preferential direction gets hindered.

B. *Numerical simulations of drift velocity*

Equation (5.21) is the central analytical result of this paper. To analyze the essential features of the directed motion of underdamped quantum particles we now resort to direct numerical simulation of Eq. (5.1a)–(5.1b) and the equations for quantum correction (5.26)–(5.32) simultaneously using standard Heun's algorithm. We use in our numerical simulation a slowly varying time dependent quantity $A(\tau_c)$ to correspond to the stationary solution of the distribution function $W_{S,D}(x,E)$ of our analytical scheme. For the entire calculation we set $\tau = 3 \times 10^3$ and barrier height $\Delta V = 6.0$ and take care on the inequality $\omega_s \gg \gamma \gg 1/\tau$ where

Figure 19: Drift velocity $(v_d)_{av}$ vs temperature T plot (numerical results) comparing quantum (the profiles with open circles and boxes) and classical (the profiles with solid circles and boxes) cases for different strength of external modulation (i) $A_0 = 1.0$ (the profiles with solid and open boxes) and (ii) $A_0 = 0.8$ (the profiles with solid and open circles) and the other parameter set $\gamma = 1.0$, $m = 1.0$, $\tau = 1500$ and $E_0 = 6.0$.



ω_s is the characteristic frequency of the system. A very small time step (Δt) of 0.001 for numerical integration has been used. For the initial conditions we have assumed that at $t = 0$ all the particles are in the potential minimum at $x = 0$ with zero velocity. Throughout our simulation, we focus on the asymptotic, periodic regime where the effects due to the influence of initial conditions and transient processes have been smoothed out. The time homogeneous statistical properties are obtained in the long time limit after the temporal (period of driving) and the ensemble averaging are performed. The calculated velocities are averaged over 5,000–10,000 trajectories depending on the values of parameters. The trajectories are allowed to evolve over a time of the order of 150–1000 periods depending on length of the period and the strength of the periodic external force and dissipation of the medium. We present our numerical results in Fig. (19–21) for different parameters, such as, mass (m) and amplitude of external drive (A_0).

The effect of quantization of a classical inertial ratchet is shown in Fig. 19, where we present a comparison of the drift velocity vs temperature profile for the classical and the quantum cases. One observes that in the low temperature region the classical current is significantly lower in magnitude than the quantum current and at higher temperature the effect of quantization becomes insignificant. This may be interpreted in terms of an interplay between quantum diffusion coefficient D_q and the potential force term $f(x)$. It has been pointed out earlier that $f(x)$ contains quantum correction arising due to nonlinearity of the potential. As temperature $T \to 0$, D_q approaches to $\frac{1}{2}\hbar\omega_0$, the vacuum limit in deep tunneling region. The anharmonic terms in $f(x)$ do not contribute significantly. On the other hand, as the temperature increases, D_q increases and also D_q and $f(x)$ compete with each other to reduce the quantum current to its classical counterpart.

Our theoretical and numerical calculations show that depending upon their mass the drift velocity of the particles changes markedly. This is shown in the Fig. 20. For increasing mass of the particles the drift velocity decreases and the maxima are slightly shifted. With appropriate control of the time period of the external derive one can amplify the current due to mass dependence of the drift velocity of the quantum particles. This dependence of the drift velocity of quantum particles on their masses can be used to separate the mesoscopic particles with different masses. Apart from the size dependent separation of the particles due to different friction coefficients, the separation by mass by periodic driving is thus a new and independent possibility. In Fig. 21 we make a comparison between numerical simulation and the corresponding theoretical result. The theoretical results on the variation of current on temperature for varied strength of drive agree fairly well with the corresponding numerical results in the very slow modulation regime.

Figure 20: Drift velocity $(v_d)_{av}$ vs temperature T plot for different values of the mass of the particle (i) $m = 0.4$ (the profiles with open triangles) and (ii) $m = 1.0$ (the profiles with open circles) (iii) $m = 1.6$ (the profiles with solid boxes) and the other parameter set $\gamma = 1.0$, $A_0 = 1.0$, $\tau = 1500$ and $E_0 = 6.0$.

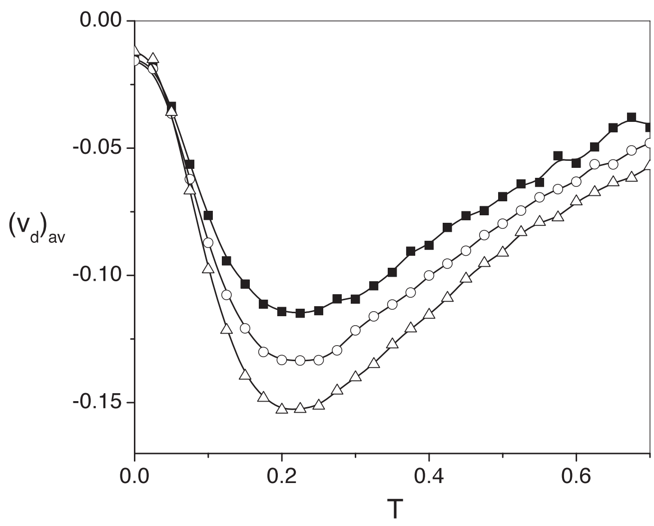
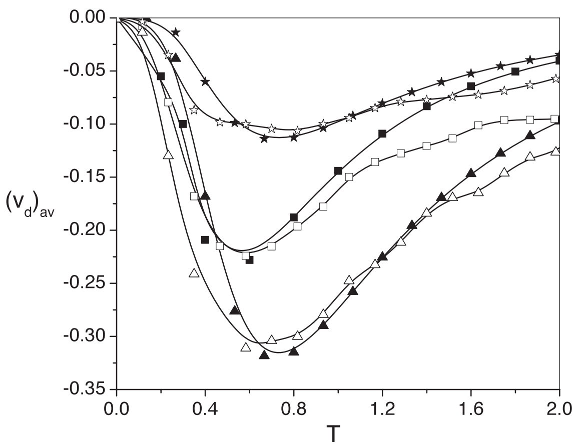


Figure 21: A comparison between numerical (the profiles with blank stars, boxes and triangles) and analytical (the profiles with dark stars, boxes and triangles) results of drift velocity v_d vs temperature T plot for different values of driving strength ($A_0 = 0.08$ (star); $A_0 = 0.10$ (box); $A_0 = 0.12$ triangle) and the other parameter set $\gamma = 0.1$, $m = 1.0$, $\tau = 3000$ and $E_0 = 6.0$.



C. *Efficiency of ratchet device in underdamped limit*

Finally we proceed to analyze the effectiveness of the rectification process and identify a suitable parameter that tells us how noisy the rectification signal is and how long one has to be wait until an appreciable net signal can be detected with sufficient confidence. To this end we define rectification efficiency in absence of an external bias (load)[102] as

$$\eta = \frac{(v_d)^2_{av}}{|(v_d^2)_{av} - D_q|}. \tag{5.22}$$

η denotes rectification efficiency, $\gamma(v_d)^2_{av}$ is the dissipated power due to directed motion of the particle against friction and the total input power from the time periodic forcing is $\gamma|(v_d^2)_{av} - D_q|$ (corresponding to kinetic energy in presence of forcing—kinetic energy in absence of forcing). We calculate numerically η and $(v_d)_{av}$ as a function of the external derive A_0 and it is shown in Fig. 22(a,b) for $D_q = 0.5$ and 0.1 for the parameter set $\gamma = 1.0$, $m = 1.0$, $E_0 = 6.0$ and $\tau = 50$. It is apparent that for an amplitude $A_0 \simeq 0.12$, the directed inertia transport for $D_q = 0.5$ sets in before the lower threshold of the ratchet force is reached. Below this threshold at $A_0 \simeq 0.12$ (for the case $D_q = 0.5$), the system mainly dwells in the locked state. Upon further increasing the amplitude of driving, $A > 0.12$ the Brownian motor generates net directed current. This type of behavior of current physically implies that at low value of A_0 (< 0.12), jumps between the neighboring wells are rare due to the fact that input energy from external source is pumped into the kinetic energy of intrawell motion and eventually dissipate (since our observation is in presence of small noise). As A_0 is increased further, the Brownian motor starts working and some part of input energy contributes to net motion of the particle. Above $A_0 \simeq 0.3$, input force crosses the upper threshold value of potential force, the average drift velocity of the particle decreases because of the weakening influence of the ratchet potential at large rocking amplitude. Rectification efficiency shows a qualitatively similar behavior as the average drift velocity.

D. *Appendix A: calculation of* $Q(x, \langle\delta\hat{x}^n\rangle)$

It may recalled that our analytical expression for current (5.21) contains quantum corrections Q up to all orders. For numerics we, however, proceed to estimate Q order by order as follows. From Eq. (3.6) quantum correction of the potential due to nonlinearity is given by

$$Q(x, \langle\delta\hat{x}^n\rangle) = -\frac{1}{2!}V'''(x)\langle\delta\hat{x}^2\rangle - \frac{1}{3!}V''''(x)\langle\delta\hat{x}^3\rangle - \ldots\ldots \tag{5.23}$$

The quantum correction terms can be determined as follows. We return to the operator equation (3.2) and put

$$\hat{x}(t) = x(t) + \delta\hat{x}(t) \quad \text{and} \quad \hat{p}(t) = p(t) + \delta\hat{p}(t),$$

where $x(t) = \langle\hat{x}(t)\rangle$ and $p(t) = \langle\hat{p}(t)\rangle$ are the quantum mechanical mean values of the operators $\hat{x}$ and $\hat{p}$ respectively. By construction $[\delta\hat{x}, \delta\hat{p}] = i\hbar$ and $\langle\delta\hat{x}\rangle = \langle\delta\hat{p}\rangle = 0$. We then obtain the quantum correction equations (Markovian limit)

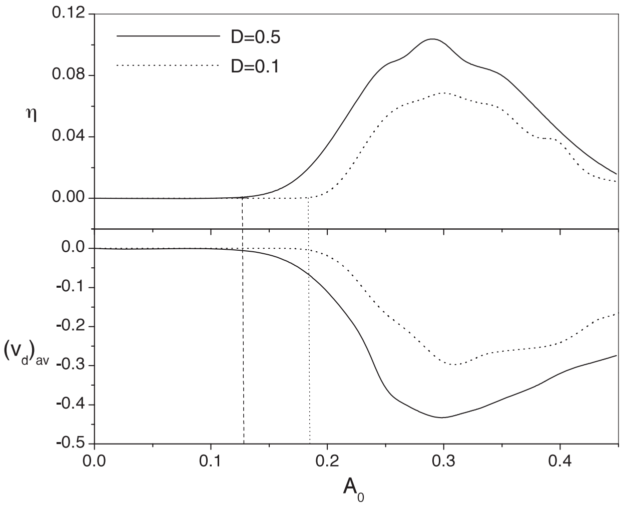
$$m\delta\ddot{\hat{x}} + \gamma\delta\dot{\hat{x}} + V''(x)\delta\hat{x} + \sum_{n\geq 2}\frac{1}{n!}V^{n+1}(x)(\delta\hat{x}^n - \langle\delta\hat{x}^n\rangle) = \hat{\Gamma}(t) - \Gamma(t). \tag{5.24}$$

We then perform a quantum mechanical average with initial product separable coherent states of the oscillators of the bath only to get rid of the internal noise term and to obtain the reduced operator equations for the system as

$$m\delta\dot{\hat{x}} = \delta\hat{p}, \tag{5.25a}$$

$$\delta\dot{\hat{p}} = -\gamma\delta\hat{p} - V''(x)\delta\hat{x} + \sum_{n\geq 2}\frac{1}{n!}V^{n+1}(x)(\delta\hat{x}^n - \langle\delta\hat{x}^n\rangle). \tag{5.25b}$$

Figure 22: (a) Efficiency vs driving strength plot at two different temperature and for the parameter set $\gamma = 1.0$, $m = 1.0$, $\tau = 50$ and $E_0 = 6.0$. (b) Drift velocity vs driving strength plot for the same parameter set



With the help of (5.25a) and (5.25b) we then obtain the equations for $\langle \delta \hat{x}^n(t) \rangle$

$$\frac{d}{dt}\langle \delta \hat{x}^2 \rangle = \frac{1}{m}\langle \delta \hat{p}\, \delta \hat{x} + \delta \hat{x}\, \delta \hat{p} \rangle, \tag{5.26}$$

$$\begin{aligned} \frac{d}{dt}\langle \delta \hat{p}\, \delta \hat{x} + \delta \hat{x}\, \delta \hat{p} \rangle &= -\gamma \langle \delta \hat{p}\, \delta \hat{x} + \delta \hat{x}\, \delta \hat{p} \rangle \\ &\quad - V''(x) \langle \widehat{\delta x^2} \rangle \\ &\quad - V'''(x) \langle \widehat{\delta x^3} \rangle + \frac{1}{m} \langle \widehat{\delta p^2} \rangle, \end{aligned} \tag{5.27}$$

$$\begin{aligned} \frac{d}{dt}\langle \delta \hat{p}^2 \rangle &= -2\gamma \langle \widehat{\delta p^2} \rangle - 2V''(x) \langle \delta \hat{p}\, \delta \hat{x} \\ &\quad + \delta \hat{x}\, \delta \hat{p} \rangle \\ &\quad - V'''(x) \langle \delta \hat{x}\, \delta \hat{p}\, \delta \hat{x} \rangle, \end{aligned} \tag{5.28}$$

$$\frac{d}{dt}\langle \delta \hat{x}^3 \rangle = \frac{3}{m}\langle \delta \hat{x}\, \delta \hat{p}\, \delta \hat{x} \rangle, \tag{5.29}$$

$$\begin{aligned} \frac{d}{dt}\langle \delta \hat{x}\, \delta \hat{p}\, \delta \hat{x} \rangle &= -\gamma \langle \delta \hat{x}\, \delta \hat{p}\, \delta \hat{x} \rangle - V''(x) \langle \widehat{\delta x^3} \rangle \\ &\quad + \frac{2}{m} \langle \delta \hat{p}\, \delta \hat{x}\, \delta \hat{p} \rangle, \end{aligned} \tag{5.30}$$

$$\begin{aligned} \frac{d}{dt}\langle \delta \hat{p}\, \delta \hat{x}\, \delta \hat{p} \rangle &= -2\gamma \langle \delta \hat{p}\, \delta \hat{x}\, \delta \hat{p} \rangle \\ &\quad - 2V''(x) \langle \delta \hat{x}\, \delta \hat{p}\, \delta \hat{x} \rangle \\ &\quad + \frac{1}{m} \langle \widehat{\delta p^3} \rangle, \end{aligned} \tag{5.31}$$

$$\begin{aligned} \frac{d}{dt}\langle \widehat{\delta p^3} \rangle &= -3\gamma \langle \delta \hat{p}^3 \rangle \\ &\quad - 3V''(x) \langle \delta \hat{p}\, \delta \hat{x}\, \delta \hat{p} \rangle, \end{aligned} \tag{5.32}$$

(in the quantum correction equations we take up to third order terms, since the higher order contributions are small). We solved the above quantum correction equations numerically to calculate quantum correction terms and hence Q.

VI. Conclusion

We have formulated the quantum stochastic dynamics of Brownian particle and analyzed the problem of a ratchet device in the overdamped and the underdamped limits. The quantization of the dynamics is manifested in two different ways. First, the harmonic oscillator reservoir is quantum mechanical in character and its internal noise characteristics and the fluctuation-dissipation relation are described by the canonical thermal Wigner distribution. Secondly, the nonlinearity of the potential brings in additional quantum contribution since the nonlinear terms of the potential beyond the harmonic one are entangled with quantum corrections. Therefore the system experiences an effective force term $f(x)$ comprising a classical $-V'(x)$ plus a quantum correction term $Q(x, \langle \delta \hat{x}^n \rangle)$ as $f(x) = -V'(x) + Q(x, \langle \delta \hat{x}^n \rangle)$. This consideration leads us to the form of a generalized equilibrium distribution in terms of a nonlocal potential. The implication of this factor in ratchet effect or in Landauer blow-torch effect has been thoroughly examined[11,32,36,93] by Van Kampen, Büttiker and others. A close look into the expression for current in Eq. (4.18) or Eq. (4.31) reveals that the origin of fluctuation induced current essentially rests on this factor and therefore the contribution of the nonlinearity induced quantum effect on this current becomes quite apparent. It is important to note that the essential requirements demanded by symmetry considerations and thermodynamic consistency condition have to be fulfilled in the treatment.We also have analyzed the efficiency of the quantum ratchet device under various conditions. We now summarize the main conclusions:

(a) We note that quantization can not break the symmetry of the ratchet device but, in general, may change the superposition of amplitudes of the periodic nonlinear function so that the current is significantly affected. This is apparent from the structure of the force term $f(x)$ pointed out earlier. For example, for a pendulum potential $V(x) = \frac{1}{2}(\cos x + 1)$, which has been used for superionic conductors, $f(x)$ $[= -V'(x) - \Delta_q V'''(x) \left(V'(x)\right)^2$ for the leading order quantum correction where $\Delta_q \sim O(h^2)]$ is a superposition of $\sin x$ and $\Delta_q \sin^3 x$, a classical and a quantum part, respectively. It therefore follows from the previous discussions that the quantum part of potential affects $\psi(x)$ and consequently the current.

(b) We observe that while at low temperature quantization significantly enhances the classical current, at higher temperature the difference is insignificant. This may be interpreted in terms of an interplay between the quantum diffusion coefficient D_q and the force term $f(x)$ appearing in the effective potential $\psi(x)$ as $\int_0^x \frac{f(x')}{D_q} dx'$. As the temperature $T \to 0$, D_q approaches to the value $\frac{1}{2}\hbar\omega_0$, the vacuum limit and also in the deep tunneling region the anharmonic terms in $f(x)$ do not contribute significantly, the integrand increases sharply. On the other hand, as the temperature increases, D_q increases resulting in a decrease of $\psi(x)$. Since for the classical current, the quantum contribution to $f(x)$ is absent, one observes a crossover of the of the classical and the quantum current in an intermediate region of the temperature, where D_q and $f(x)$ compete with each other and beyond which the quantum current is marginally lower than the classical current. For a further increase of temperature the classical and the quantum current merge identically, as expected.

(c) In contrast to the behavior of quantum current at low temperature, the conventional efficiency of a classical ratchet in presence of a load is higher at low temperature as compared to its quantum counterpart and again the efficiency in the two cases tends to merge at higher temperature. Furthermore the maximum efficiency is independent of the nature of the system potential and the bath and is thus independent of quantization. We have also examined a quantum version of Stokes efficiency in absence of load where energy due to frictional resistance is considered as a part of expenditure of useful energy. A significant quantum enhancement of Stokes efficiency at low temperature has been observed. The careful consideration of the total energy consumption and dissipation reveals that the generation of higher current and Stokes efficiency may not always imply the higher efficiency of thermal ratchet in a conventional sense although the generic features of the device in its classical and quantum versions remain the same.

(d) We have demonstrated that finite inertia significantly affects the magnitude of quantum current, an issue which has found importance in separation of particles according to their masses. We believe that the observations on quantum ratchet operating in the energy diffusion regime as well as overdamped regime would be of significance in applications on devices on mesocopic scale.

Motivated by a number of experimental observations on molecular motors, ratchet models have been exploited in the quantum realm in several issues. Mennerat-Robilliard *et. al*[62], have been demonstrated directed motion of cold Rubium atom evolving in an asymmetric optical bipotential, exemplifying a link between statistical physics and laser cooling. It has been agrued[63] that the phase across an asymmetric DC SQUID in presence of a magnetic flux can admit an effective ratchet potential. In order to probe its selected characteristics of tunneling dynamics, fluctuation-induced voltage rectification in a Josephson junction[61] has been analyzed where one uses a point contact with a defect tunneling incoherently between two states as a source asymmetric dichotomous noise. The study of these evidences illustrates how the quantum effect in ratchet motion are to be looked into in the related contexts involving mechanical motion on a microscale e. g., in electroconformational coupling, fluxinalrotational motion on metal surface[109–111]. The ratchet motion is thus likely to remain an active area for some more years to come.

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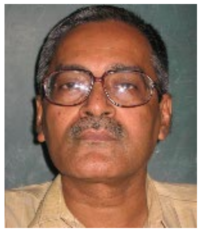

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