RICHART

Statically Indeterminate Stresses in Stiff Framed Structures

Civil Engineering

M. S.

1915

STATICALLY INDETERMINATE STRESSES IN STIFF FRAMED STRUCTURES

BY

FRANK ERWIN RICHART

B. S. University of Illinois, 1914.

THESIS

Submitted in Partial Fulfillment of the Requirements for the

Degree of

MASTER OF SCIENCE

IN CIVIL ENGINEERING

IN

THE GRADUATE SCHOOL

OF THE

UNIVERSITY OF ILLINOIS

1915

UNIVERSITY OF ILLINOIS
THE GRADUATE SCHOOL

June 1 1915

I HEREBY RECOMMEND THAT THE THESIS PREPARED UNDER MY SUPERVISION BY FRANK ERWIN RICHART

ENTITLED STATICALLY INDETERMINATE STRESSES IN STIFF FRAMED STRUCTURES

BE ACCEPTED AS FULFILLING THIS PART OF THE REQUIREMENTS FOR THE DEGREE OF Master of Science in Civil Engineering

W. M. Wilson
In Charge of Thesis

Ira O. Baker.
Head of Department

Recommendation concurred in:*

Committee on Final Examination*

*Required for doctor's degree but not for master's.

C O N T E N T S .

I. Introduction.

II. Fundamental Equations.

III. Analysis of Stiff Frames.

A. Continuous Beams.

C O N T E N T S (Continued).

C O N T E N T S (Continued).

C. Rectangular Frames.

C O N T E N T S (Continued).

CONTENTS (Continued).

C O N T E N T S (Continued).

T A B L E S .

T A B L E S (Continued).

D I A G R A M S .

STATICALLY INDETERMINATE STRESSES IN STIFF FRAMED STRUCTURES.

I. INTRODUCTION.

1. PRELIMINARY. Various forms of the statically indeterminate frame, which resists distortion by virtue of the stiffness of its members and joints, have been used by engineers and builders for many years, but in most cases the exact determination of stresses has not been attempted. Instead, designers have generally preferred to make assumptions which would lead to safe, though not economical designs, rather than to spend the time and labor required for an analytical treatment of their particular problem. Further, since the statically indeterminate stresses in a structure depend upon the relative sizes of the members, it is necessary to make a preliminary design, and then to redesign the structure after determining the stresses. It is evident that the design of a structure having a large number of members, such as a bent of a building, would entail a great amount of work.

In view of the extensive use of stiff frames, as noted below, it is evident that an exact analysis is very desirable. Structures designed by approximate methods are almost sure to be inadequate at some point and to have an excess of material elsewhere. The saving that might be made in the cost of a structure such as the reinforced concrete viaduct at Richmond, Va., which is 2800 feet long and from 14 to 70 feet high, should be considerable.

The object of this investigation is to devise methods of

determining stresses in stiff frames, which are accurate and which are short enough to be used in the design of structures.

2. ACKNOWLEDGMENT. The writer has used the Slope-Deflection method in this work. This method was first outlined by a German scientist, Professor Otto Mohr, in 1892; but was independently developed recently by Mr. G. A. Maney, formerly Research Fellow in Theoretical and Applied Mechanics at the University of Illinois, and at present Instructor in Structural Engineering at the University of Minnesota. Mr. Maney, in collaboration with Mr. W. M. Wilson, Assistant Professor of Structural Engineering at the University of Illinois, used the method in an investigation of wind stresses in office buildings.

Acknowledgements are due to Professor Wilson, under whose supervision this thesis was written, for helpful suggestions and criticism.

3. IMPORTANCE OF STIFF FRAMED STRUCTURES. The stiff framed structures analyzed in this investigation may be made of steel or reinforced concrete. The most common examples in steel frame construction are: frames of office and mill buildings, intermediate and portal frames of bridges, and riveted trusses.

The use of the stiff frame in reinforced concrete construction has become very extensive, principally within the last eight or ten years. Because of the ease with which a member may be moulded in any desired shape, this type of structure is economical and very adaptable. Examples of this type of construction include: reinforced concrete buildings, elastic arches, railway trestles and viaducts, subways, culverts, open type abutments, hollow dams, reservoirs, coal pockets, craneways, and various

forms of supports for water tanks, coal chutes, signal towers, and like structures.

II. FUNDAMENTAL EQUATIONS.

4. STATICALLY INDETERMINATE SYSTEMS. A system of forces in a vertical plane is said to be statically determinate when each force may be determined in magnitude, direction, and point of application by the three equations of static equilibrium, namely: the algebraic sum of the horizontal forces is equal to zero, the algebraic sum of the vertical forces is equal to zero, and the algebraic sum of the moments of all forces about any point in the plane is equal to zero.

Each member of a structure composed of n members may be treated as a free body, and the three equations of static equilibrium may be applied. Hence 3n equations may be written for the whole structure.

There are three ways in which a member may move with respect to another if unrestrained, namely: by moving horizontally, by moving vertically, and by revolving about the center of the joint. Therefore there will be three or less unknown quantities at each joint, depending upon the conditions of restraint. If one motion is restrained, as in the case of two members connected by a long link, as shown in Fig. 1, there is only one unknown quantity at the joint. If two motions are restrained, as in the case of the pin joint shown in Fig.2, there are two unknown quantities. If all motion is restrained, as in the case of the riveted joint shown in Fig. 3, there are three unknown

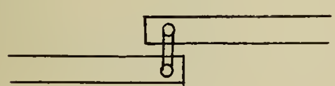


Fig. 1.


Fig. 2.

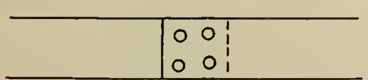


Fig. 3.

quantities. Hence the number of unknown quantities in a given structure is equal to the total number of restraints at all of the joints, including the connections to the foundation or support. Denote the number of unknowns by <u>a</u>. Since there are <u>3n</u> equations, the problem may be solved by statics if <u>a</u> equals or is less than <u>3n</u>. If <u>a</u> is greater than <u>3n</u>, it is necessary to secure <u>a</u>-<u>3n</u> additional equations. The number of additional equations required indicates the degree of indeterminateness, or the degree of indeterminateness of a structure is equal to <u>a</u>-<u>3n</u>. This applies to frames having redundant members as well as to those having incomplete triangular framework.

When more than two members meet at a point the above principle is applied by considering the number of restraints between any one member and the remaining members. For instance, three members might meet at a pin joint A, as shown in Fig. 4a. There are two restraints between the members AB and AC, and two restraints between the members AB and AD. The effect may be considered the same as if the members met at two separate points as shown in Fig. 4b. In general, when the number of members which meet at a joint exceeds two, the effect of an additional joint for each additional member is produced. A few examples will illustrate the rules given above. The degree of indeterminateness found is for the general case, and may be reduced by conditions of symmetry

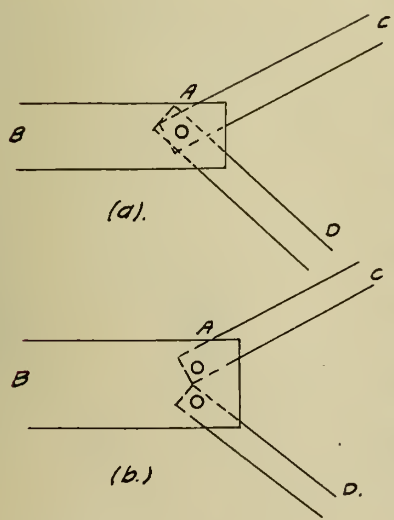


Fig. 4.

of loading and members.

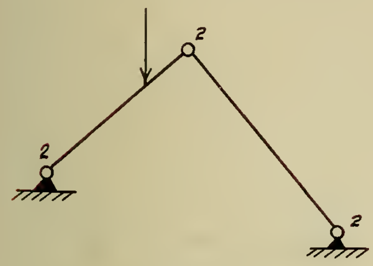


Fig. 5.

Fig. 5 represents one form of a three hinged arch. The number of members considered here is two, so that $n = 2$. There are two unknowns at each of the three joints, so that $a = 6$. $a - 3n = 6 - 6 = 0$. This shows that the structure is statically determinate.

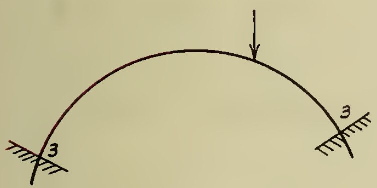


Fig. 6.

Fig. 6 represents a no hinged arch. The arch rib may be considered as one member, so that $n = 1$. There are three unknown quantities at each support, so that $a = 6$. $a - 3n = 6 - 3 = 3$. The no hinged arch is statically indeterminate to the third degree.

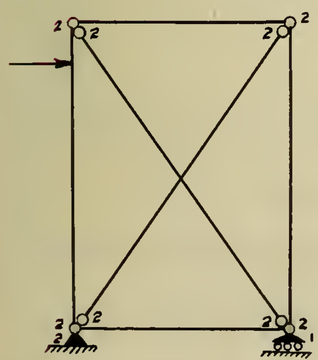


Fig. 7.

Fig. 7 represents a pin connected frame with a redundant member. There is no connection between the two diagonal members at the center. There are six members in the frame, so that $n = 6$. Considering the frame as having two pin joints at each corner where three members meet, and having one pin connection and one sliding connection with the foundation, the number of unknowns is nineteen, so that $a = 19$. $a - 3n = 19 - 18 = 1$. The frame is statically indeterminate to the first degree.

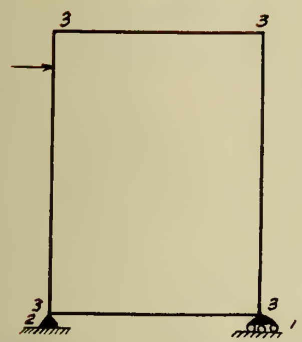


Fig. 8.

Fig. 8 shows a rectangular frame with rigid joints. There is one **hinged** and one sliding connection with the foundation. The number of members here is four, so that $\underline{n} = 4$. As indicated in the figure, the number of unknowns is fifteen, so that $\underline{a} = 15$.
$\underline{a} - \underline{3n} = 15 - 12 = 3$. This frame is statically indeterminate to the third degree.

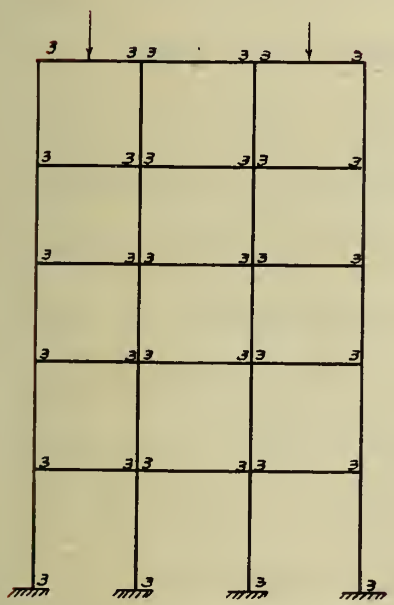


Fig. 9.

Fig. 9 represents a frame of a building, having rigid joints and rigid connections with the foundation. The entire length of each column will be considered as one member. The total number of members is nineteen, so that $n = 19$. The total number of unknown quantities, as indicated in Fig. 9. is 102, so that $\underline{a} = 102$.
$\underline{a} - \underline{3n} = 102 - 57 = 45$. The structure is statically indeterminate to the fortyfifth degree.

There are exceptions to the above rule for the degree of indeterminateness. Fig. 10 represents two elastic members, one a hollow pipe, and the other a solid rod inside the pipe, under compression in a testing machine. It is assumed that the faces of the machine heads are always parallel. $\underline{n}$, the number of members considered, is 2. The unknowns are the four vertical reactions,

Fig. 10.

so that $\underline{a} = 4. \quad \underline{a} - \underline{3n} = 4 - 6 = -2.$ Hence according to the criterion the reactions are statically determinate; but this is not the case since the only equation that can be applied is $\Sigma V = 0$. The reactions depend upon the relative deformations of the members , and hence are statically indeterminate. Such exceptions to the rule for degree of indeterminateness can be easily recognized, and are usually hypothetical rather than practical forms of structures.

From the foregoing discussion it is seen that it is necessary to apply relations in addition to those of statics, in the treatment of a statically indeterminate structure. From the example shown in Fig. 10, it is seen that the action of such a structure depends not only upon the position of the members in the structure, but also upon their relative sections. Similarly, statically indeterminate bending stresses depend upon the relative stiffness of the members.

The methods which have been used in solving statically indeterminate problems include the methods of Least Work, Virtual Velocities, Area Moments, and Slope-Deflections. The last method, which is used in this investigation, is here deduced from the theorem of Area Moments; but it has been derived by other entirely independent mathematical procedure. The principle of Area Moments was first advanced by Prof. Greene of the University of Michigan. A proof of the principle will be presented here in order to make the mathematical procedure as complete as possible.

5. DERIVATION OF THEOREMS OF AREA MOMENTS. If, for a straight beam in flexure, each ordinate of the bending moment diagram is divided by the product of the moment of inertia of the section and the modulus of elasticity of the material at that point, the ordinates of what is termed the $\frac{M}{EI}$ diagram are obtained. Upon the properties of this diagram, the following propositions are based.

Statement of Theorems. 1. Considering any two points A and B on the elastic curve of a beam in flexure, the deviation at B from a tangent to the curve at A is equal to the statical moment of the $\frac{M}{EI}$ diagram between A and B, about the point B.

2. The change in slope of the elastic curve of a beam in flexure between points A and B, is equal to the area of the $\frac{M}{EI}$ diagram between A and B.

Proof. Consider the differential length ds of the elastic curve of a beam in flexure shown in Fig. 11. The unit deformation of a fibre at a distance c from the neutral axis is $c \cdot d\theta/ds$; and the unit stress in the same fibre is equal to Mc/I. By definition, the modulus of elasticity E is the ratio of unit stress to unit deformation. Hence

$$E = \frac{Mc/I}{c\ d\theta/ds} = \frac{M}{I}\frac{ds}{d\theta}.$$

whence
$$d\theta = \frac{M\ ds}{EI}.$$

or
$$\theta = \int \frac{M\ ds}{EI}.$$

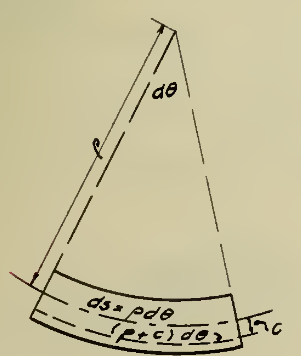


Fig. 11.

Since the radius of curvature for a straight beam is large, it may be

assumed that dx is equal to ds. Then $\theta = \int \frac{M\,dx}{EI}$.

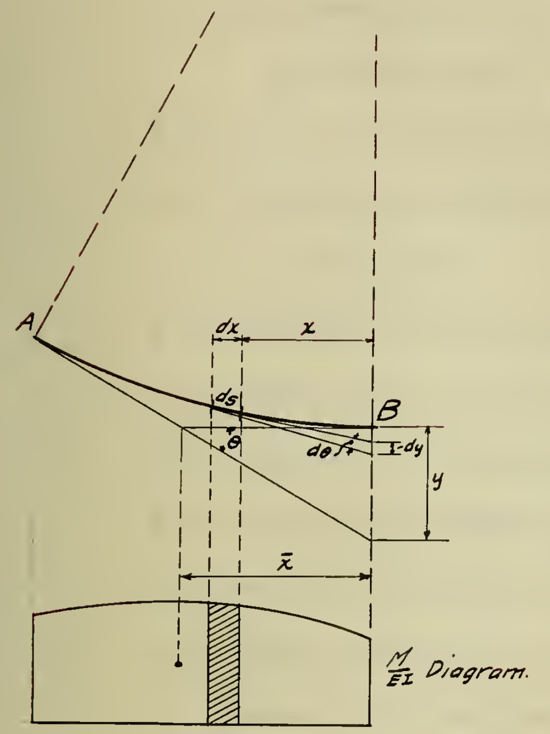


Fig. 12.

In Fig. 12 the curve AB represents the elastic curve of a portion of the beam. By the geometry of the figure, $dy = x\,d\theta$

or, $dy = \frac{Mx\,ds}{EI}$.

Assuming that dx = ds, gives

$$dy = \frac{Mx\,dx}{EI}$$

or, $$y = \int \frac{Mx\,dx}{EI}$$

Now the shaded elementary area of the M/EI diagram is equal to Mdx/EI; and the entire area of the diagram is equal to $\int_A^B \frac{M\,dx}{EI}$.

The statical moment of the shaded area about the point B is equal to x·Mdx/EI; and the statical moment of the entire area of the diagram about B is equal to $\int_A^B \frac{Mx\,dx}{EI}$.

Hence it is seen that the change in slope θ, from A to B $= \int_A^B \frac{M\,dx}{EI}$ = the area of the M/EI diagram between A and B; and that the tangential deviation y, at B $= \int_A^B \frac{Mx\,dx}{EI}$ = the statical moment of the M/EI diagram about B. Both of these theorems are used in the derivation of the fundamental equations which follow.

6. NOTATION. The following notation will be used throughout the work.

a - distance from left end of a member to the point of application of a concentrated load.

b - distance from right end of a member to the point of application of a concentrated load.

C - couple acting upon the end of a member.

D - deflection of one end of a member with respect to the other end.

E - modulus of elasticity of the material.

e - coefficient of linear expansion of the material.

F - area of the bending moment diagram of a simple beam.

H - horizontal reaction or shear.

h - vertical height of a structure.

h_s - slant height of a structure.

I - moment of inertia of the section of a member.

J - twice the sum of the values of K for all members meeting at a joint in a bent of a building.

K - ratio of moment of inertia to length of a member.

k - a fraction of the height or span of a structure.

L - length of a member.

M - bending moment. When used with two subscripts, as M_{AB}, the moment at the end A of the member AB is denoted.

m - ratio of F/L of one span to F/L of the adjacent span of a building frame.

n - ratio of K of top member to K of left hand column in a four sided frame.

P - a concentrated load.

p = ratio of K of top member to K of bottom member in a four sided frame.

R = D/L - deflection of one end of a member with respect to the other end, divided by the length of member.

s = ratio of K of top member to K of right hand column in a four sided frame.

t = change of temperature in degrees.

V = vertical reaction.

W = total load on a member.

w = uniform load per unit length of a member.

Z = 1/K = ratio of length of a member to the moment of inertia of its section.

θ = the change in slope of the tangent to the elastic curve of a member.

μ = ratio of the moment at the end of a member to the moment at the same point with ends of member fixed.

$\alpha = (n^2-2n-2pn-3p)$. For four sided frame.

$\beta = (6n-1-p)$. For four sided frame.

7. <u>ASSUMPTIONS AND CONVENTIONS</u>. The assumptions made in this analysis are as follows:

1. All joints are perfectly rigid.
2. The change in length of a member due to direct stress is equal to zero.
3. The eccentricity of a direct stress due to the deflection of a member is equal to zero.
4. The deflection due to internal shearing stresses is

equal to zero.

5. The length of a member is the distance between the neutral axes of the members it connects, or is the distance from center to center of supports, as the case may be.

The first assumption is the only basis upon which any logical analysis of statically indeterminate stresses can be made. The assumption seems reasonable, and is borne out by the result of tests made on large concrete frames by Prof. M. Abe at the University of Illinois in 1914. Prof Abe makes the following statement: "If a frame is carefully designed and well reinforced, there need be no anxiety as to the rigidity of a joint, and a perfect continuity of members has been proven by these tests." While joints in steel structures are not usually made rigid, it is undoubtedly true that such rigidity can be obtained by proper care in designing the connections.

Assumptions 2,3, and 4 seem justifiable, since nearly all textbooks on the theory of arches and other statically indeterminate structures which consider the theoretical effect of the quantities mentioned, show that they are negligible because they are within the usual limits of accuracy of calculation and design.

The last assumption is best explained by the equations for moments in frames, as given in Section III. It is seen that the moment at a point varies as the ratio of the values of K for certain members, and hence varies as the ratio of the lengths of the members. Although experiments indicate that the clear lengths of members should be considered, the ratio of clear lengths will

be about the same as the ratio of total lengths; so that the use of either will not materially change the value of the moment. For this reason, the length of members will be considered as stated, thus making the equations simpler for use in calculation.

The following conventions will be followed throughout the work, and must be observed in applying all formulas:

1. The change in slope is positive when the tangent to the elastic curve is turned in a clockwise direction.
2. Distances and deflections are positive when they are measured in the same direction from the base line as are positive slopes.
3. The bending moment is positive when the sum of the moments of all forces to the left of a section in a beam is clockwise. Moments will be plotted on the tension, or convex, side of the member.
4. The deflection D is measured from base line to the elastic curve of the member.
5. The distance θL is measured from base line to the tangent to the elastic curve.
6. The tangential deviation y is measured from the tangent to the elastic curve, to the elastic curve.

All deflections are measured normal to the base line, which is the unstrained position of the elastic curve.

8. DERIVATION OF FUNDAMENTAL EQUATIONS.

Case 1. Member in flexure carrying no external load.

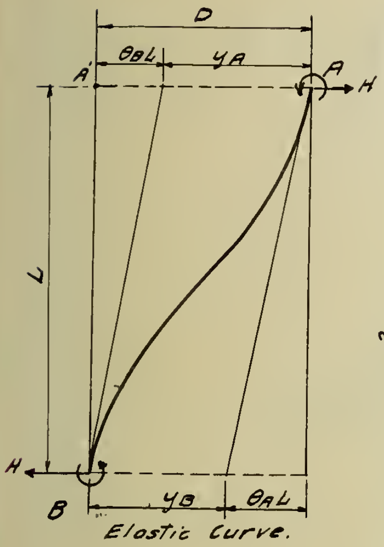


Fig. 13.

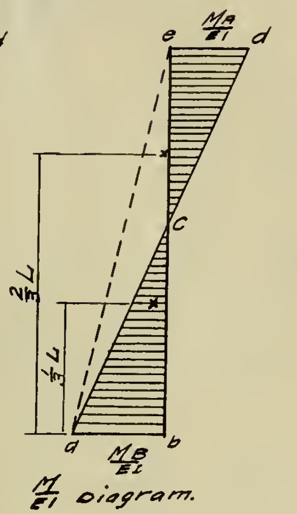


Fig. 14.

In Fig. 13, A'B represents the unstrained position of the elastic curve of a member, and AB represents the strained position of the elastic curve of the same member. The changes in the slopes of the tangents to the elastic curve at A and B are θ_A and θ_B, respectively. The total movement of A normal to A'B is D. The M/EI diagram is shown by Fig. 14, in which E and I are considered constant. Now consider the quantities shown at the point B. The deflection D is composed of two quantities; the tangential deviation y_B, and the displacement due to the change in slope at A, or $\theta_A L$. Before expressing these quantities in terms of the bending moments, it is well to note that the algebraic sum of the areas of the M/EI diagram is equal to the algebraic sum of the areas abe and ade. Hence the position of the point c need not be located. The tangential deviation of B is represented by $D-\theta_A L$, and is equal to the statical moment of the M/EI diagram about B. Therefore

$$D - \theta_A L = \frac{L^2}{EI}\left[\frac{M_A}{3} + \frac{M_B}{6}\right] \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; (a).$$

The change in slope of the elastic curve from B to A is represented

by $\theta_B - \theta_A$, and is equal to the area of the M/EI diagram. Hence

$$\theta_B - \theta_A = \frac{L}{EI}\left[\frac{M_A}{2} + \frac{M_B}{2}\right] \ldots\ldots\ldots\ldots (b).$$

Multiplying equation (a) by 3, and equation (b) by L, and combining, gives

$$3D - 3\theta_A L = \frac{L^2}{EI}\left[\frac{M_A}{1} + \frac{M_B}{2}\right]$$

$$\theta_B L - \theta_A L = \frac{L^2}{EI}\left[\frac{M_A}{2} + \frac{M_B}{2}\right]$$

$$2\theta_A L + \theta_B L - 3D = -\frac{L^2}{EI}\cdot\frac{M_A}{2}, \quad \text{or}$$

$$M_A = -\frac{2EI}{L}\left[2\theta_A + \theta_B - 3D/L\right] \ldots\ldots$$

Similarly,

$$3D - 3\theta_A L = \frac{L^2}{EI}\left[M_A + \frac{M_B}{2}\right]$$

$$2\theta_B L - 2\theta_A L = \frac{L^2}{EI}\left[M_A + M_B\right]$$

$$2\theta_B L + \theta_A L - 3D = \frac{L^2}{EI}\cdot\frac{M_B}{2}, \quad \text{or}$$

$$M_B = \frac{2EI}{L}\left[2\theta_B + \theta_A - 3D/L\right] \ldots\ldots$$

These equations are made more convenient for use by substituting $K = I/L$, and $R = D/L$, whence

$$\left.\begin{aligned} M_A &= -2EK\,(2\theta_A + \theta_B - 3R) \\ M_B &= 2EK\,(2\theta_B + \theta_A - 3R) \end{aligned}\right\} \ldots\ldots (1).$$

Case 2. Member in flexure carrying a concentrated external load P at a distance a from the end B. Fig. 15 represents a member similar to the one shown in Fig. 13, except that a concentrated load P is applied at a distance a from B. The M/EI diagram shown in Fig.16 is similar to the M/EI diagram for the member shown in Fig. 14, with the M/EI diagram for a simple beam carrying the load P, superimposed upon it. Now consider the deflection at the point B. The tangential deviation is equal to

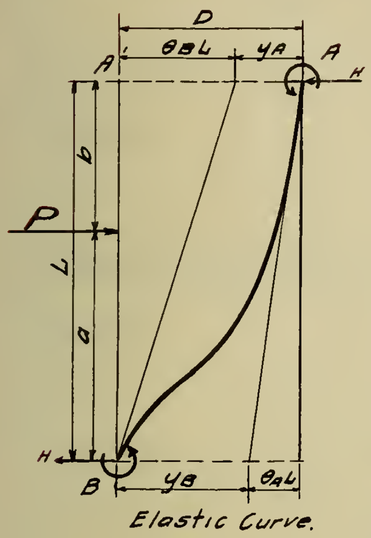


Fig. 15.

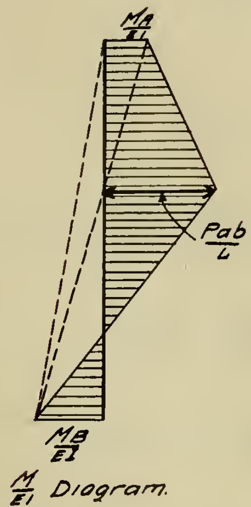


Fig. 16.

$$D - \theta_A L = \frac{L}{EI}\left[\frac{M_B L}{6} + \frac{M_A L}{3}\right] + \frac{Pab}{2EIL}\left[\frac{2a^2}{3} + b\left(a + \frac{b}{3}\right)\right]$$

which reduces to

$$D - \theta_A L = \frac{L}{EI}\left[\frac{M_B L}{6} + \frac{M_A L}{3} + \frac{Pab}{6L}(a+L)\right] \quad \ldots\ldots\ldots\ldots (c).$$

The change in slope from B to A is expressed by $\theta_B - \theta_A$.

$$\theta_B - \theta_A = \frac{L}{EI}\left[\frac{M_B}{2} + \frac{M_A}{2} + \frac{Pab}{2L}\right] \quad \ldots\ldots\ldots\ldots (d).$$

Combining equations (c) and (d) to eliminate M_B, gives

$$2\theta_A L + \theta_B L - 3D = \frac{L}{EI}\left[\frac{-M_A L}{2} - \frac{Pa^2 b}{2L}\right], \text{ or}$$

$$M_A = -\frac{2EI}{L}\left[2\theta_A + \theta_B - 3D/L\right] - \frac{Pa^2 b}{L^2} \quad \ldots$$

Similarly, combining equations (c) and (d) to eliminate M_A ,gives

$$2\theta_B L + \theta_A L - 3D = \frac{L}{EI}\left[\frac{M_B L}{2} + \frac{Pab}{L}\left(L + \frac{a+L}{2}\right)\right] = \frac{L}{EI}\left[\frac{M_B L}{2} + \frac{Pab^2}{2L^2}\right]$$

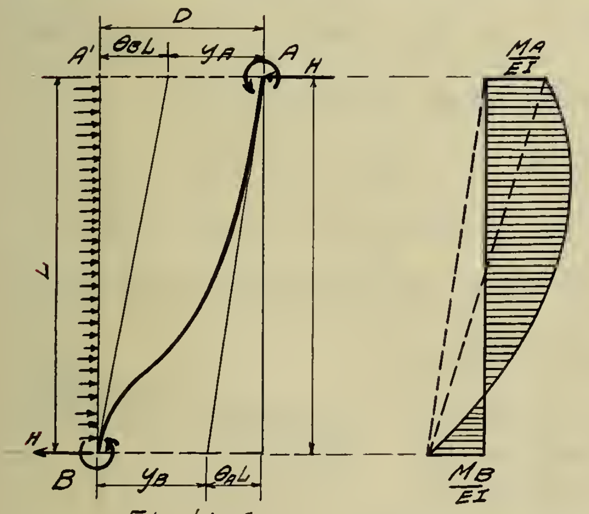
whence

$$M_B = \frac{2EI}{L}\left[2\theta_B + \theta_A - 3D/L\right] - \frac{Pab^2}{L^2} \ . \ . \ .$$

Substituting K= I/L, and R= D/L, the equations for M_A and M_B become

$$\left.\begin{aligned} M_A &= -2EK\,(\,2\theta_A + \theta_B \mp 3R) - Pa^2b/L^2 \\ M_B &= 2EK\,(\,2\theta_B + \theta_A \mp 3R) - Pab^2/L^2 \end{aligned}\right\}\ (2).$$

Case 3. Member in flexure carrying a series of loads symmetrical about the middle of the member. Fig. 17 shows a member carrying a series of loads which is symmetrical about the middle of the member. The M/EI diagram for this member is similar to that of Fig.14,with the M/EI diagram of a simple beam carrying the same loads, superimposed upon it. Let the area of this superimposed diagram be represented by F.



Elastic Curve. $\frac{M}{EI}$ Diagram.

Fig. 17. Fig. 18.

The deflections at the point B will now be considered. The tangential deviation is given by

$$D - \theta_A L = \frac{L}{EI}\left(\frac{M_B L}{6} + \frac{M_A L}{3} + \frac{F}{2}\right) \ . \ . \ . \ . \ . (e).$$

The change in slope from B to A is given by

$$\theta_B - \theta_A = \frac{1}{EI}\left(\frac{M_B L}{2} + \frac{M_A L}{2} + F \right) \quad . \; . \; . \; . \; . \; . \; (f).$$

Combining equations (e) and (f) to eliminate M_A, gives

$$2\theta_B L + \theta_A L - 3D = \frac{L}{EI}\left[\frac{M_B L}{2} \pm \frac{F}{2} \right], \quad \text{whence}$$

$$\left.\begin{aligned} M_B &= 2EK\,(\,2\theta_B + \theta_A - 3R) - F/L \\ \text{and similarly} \qquad M_A &= -2EK(\,2\theta_A + \theta_B - 3R) - F/L. \end{aligned}\right\} \; . \, . (3).$$

It is seen that these equations can be applied to a member carrying a uniform load, concentrated load at middle, equal concentrated loads at the third point, or any loading which is symmetrical about the middle of the beam.

The preceding equations give an expression for the moments at the ends of a member. It may be desirable to know the moment at the middle of a member, or under a concentrated load. From the geometrical construction of the moment diagram, it is seen from

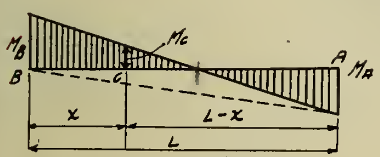


Fig. 19

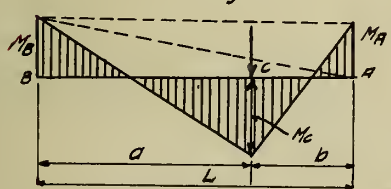


Fig 20.

Fig. 19 that in Case 1, the moment at any point c is equal to $\dfrac{M_A x + M_B(L-x)}{L}$. In Case 2, referring to Fig. 20, the moment under the concentrated load is equal to $\dfrac{Pab + M_A a + M_B b}{L}$.

If $a = b = L/2$ in Fig. 20, as in the case of a concentrated load at center, the moment at the center is equal to $Pl/4 + \frac{1}{2}(M_A + M_B)$. For any other loading in Case 3, replace $Pl/4$ by the maximum moment in a simple beam.

9. <u>SUMMARY OF FUNDAMENTAL EQUATIONS</u>. The following tabular summary of equations will be useful for reference.

TABLE I.

Fundamental Equations.

The equations give the moment in sign as well as magnitude.

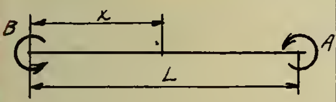
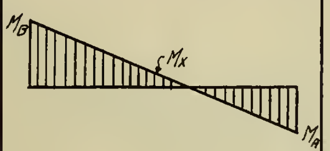
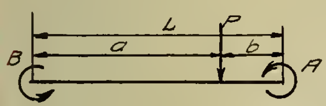
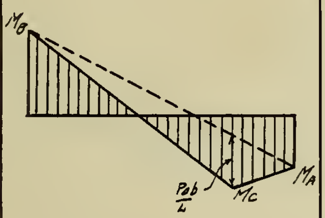
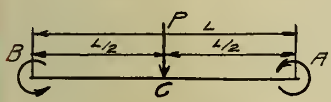
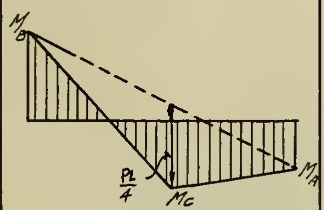
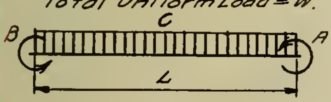
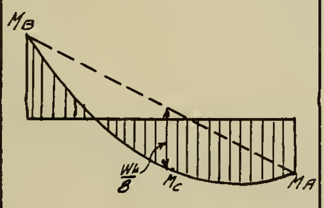
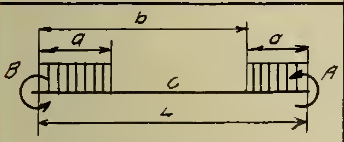
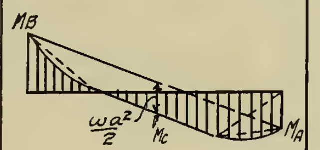
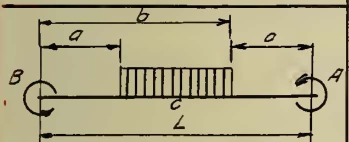
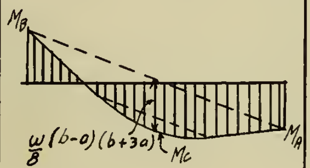
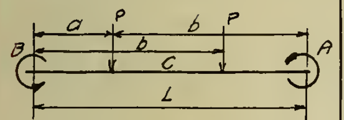
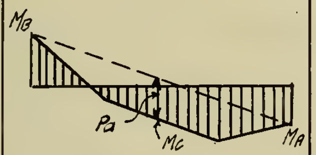
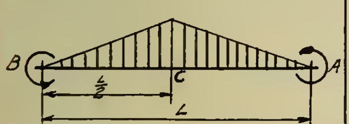
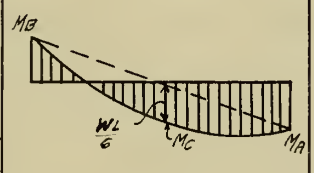
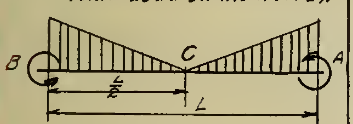
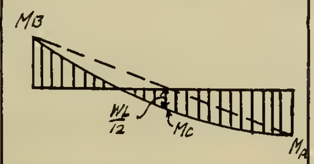
Condition of Loading	Moment Diagram.	Equations.
 No intermediate external loads.		$M_B = 2EK[2\theta_B+\theta_A-3R]$ $M_A = -2EK[2\theta_A+\theta_B-3R]$ (1) $M_x = M_A\frac{x}{L} + M_B(\frac{L-x}{L})$ . . . . (1a)
 Single Concentrated Load at any point.		$M_B = 2EK[2\theta_B+\theta_A-3R] - \frac{Pab^2}{L^2}$ $M_A = -2EK[2\theta_A+\theta_B-3R] - \frac{Pa^2b}{L^2}$ (2) $M_C = \frac{Pab}{L} + \frac{M_A a + M_B b}{L}$ . . . . . (2a).
 Single Concentrated Load at middle.		$M_B = 2EK[2\theta_B+\theta_A-3R] - \frac{PL}{8}$. $M_A = -2EK[2\theta_A+\theta_B-3R] - \frac{PL}{8}$. (3.) $M_C = \frac{PL}{4} + \frac{M_A+M_B}{2}$ . . . . (3a.)
Total Uniform Load = W.  Uniformly Distributed Load over the Entire Length. C = Middle of Member.		$M_B = 2EK[2\theta_B+\theta_A-3R] - \frac{WL}{12}$ $M_A = -2EK[2\theta_A+\theta_B-3R] - \frac{WL}{12}$ (4.) $M_C = \frac{WL}{8} + \frac{M_A+M_B}{2}$ . . . (4a).

TABLE I.- Continued.

Condition of Loading	*Moment Diagram*	*Equations.*
 Uniformly Distributed Load of w lb. per ft, to a distance a from each end. The total load = 2wa. C = Middle of Member.		$M_B = 2EK[2\theta_B+\theta_A-3R] - \frac{wa^2}{6L}(3b+a)$ $M_A = -2EK[2\theta_A+\theta_B-3R] - \frac{wa^2}{6L}(3b+a)$ } (5) $M_C = \frac{wa^2}{2} + \frac{M_A+M_B}{2}$ - - - - - (5a).
 Uniformly Distributed Load on middle of span, to a distance a, from each end. The total load = w(b-a) w = load per. lin. ft.		$M_B = 2EK[2\theta_B+\theta_A-3R] - \frac{w}{L}\left(\frac{(b-a)ab}{2} + \frac{(b-a)^3}{12}\right)$ $M_B = -2EK[2\theta_A+\theta_B-3R] - \frac{w}{L}\left(\frac{(b-a)ab}{2} + \frac{(b-a)^3}{12}\right)$ } (6). A $M_C = \frac{w}{8}(b-a)(b+3a) + \frac{M_A+M_B}{2}$ - - - - (6.)
 Two equal symmetrically spaced concentrated loads.		$M_B = 2EK[2\theta_B+\theta_A-3R] - \frac{Pab}{L}$ $M_A = -2EK[2\theta_A+\theta_B-3R] - \frac{Pab}{L}$ } (7.) $M_C = Pa + \frac{M_A+M_B}{2}$ - - - - (7a.)
Total load on member = W  Distributed Load, increasing uniformly from zero at ends to maximum at center.		$M_B = 2EK[2\theta_B+\theta_A-3R] - \frac{5WL}{48}$ $M_A = -2EK[2\theta_A+\theta_B-3R] - \frac{5}{48}WL$ } (8.) $M_C = \frac{WL}{6} + \frac{M_A+M_B}{2}$ - - - - - (8a.)
Total Load on member = W  Distributed Load, increasing uniformly from zero at the center to a maximum at the ends.		$M_B = 2EK[2\theta_B+\theta_A-3R] - \frac{WL}{16}$ $M_A = -2EK[2\theta_A+\theta_B-3R] - \frac{WL}{16}$ } (9.) $M_C = \frac{WL}{12} + \frac{M_A+M_B}{2}$ - - - - (9a.)

10. OUTLINE OF METHOD OF ANALYZING STRESSES IN STIFF FRAMED STRUCTURES. Any framed structure may be analyzed by breaking it up into its component members, each of which may be acted upon by moments, shear, and direct stress. By applying the fundamental equations of Table I, the bending moments in each member can be expressed in terms of the changes in the slopes and the deflection of the ends of the member. These equations, together with the equations of static equilibrium, give as many equations as there are unknowns. They may be solved by either one or the other of two methods, as follows:

1. The moments may be eliminated and the equations solved for the slopes and the deflections. These slopes and deflections may then be substituted in the original equations, and the moments determined.

2. The slopes and deflections may be eliminated, and the moments determined directly.

The first method can be used to best advantage when a number of members intersect at one point. The second method is especially applicable when only two members intersect at one point, or where certain slopes and deflections are known from the conditions of the structure.

Both methods were used in this thesis. Algebraic expressions for the moments were determined in the case of the simpler frames, where not more than four simultaneous equations were involved. In the remaining problems only the general equations for the structure were written. The solution of these equations is illustrated by numerical examples.

III. ANALYSIS OF STIFF FRAMES.

A. CONTINUOUS BEAMS.

11. BEAM WITH FIXED ENDS. CONCENTRATED LOAD AT ANY POINT. Fig. 21 shows a beam fixed at the ends and carrying a load at any point. The tangents to the elastic curve at the ends are horizontal and the two ends are on the same level. Therefore $\theta_A = 0$, $\theta_B = 0$, and $R = 0$. Equation 2, of Table I, gives

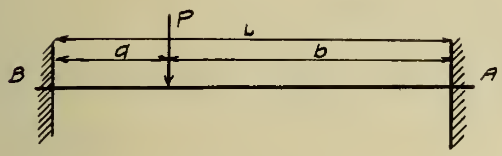


Fig. 21.

$$M_B = -Pab^2/L^2 \quad \ldots\ldots\ldots \quad (10).$$

$$M_A = -Pa^2b/L^2 \quad \ldots\ldots\ldots \quad (11).$$

It follows that $M_A + M_B = -Pab/L$. Hence it is seen that with a concentrated load, the sum of the negative moments at the supports of a beam with fixed ends is numerically equal to the positive maximum moment in a simple beam with the same loading. Also, the moments at the supports vary inversely as their distances from the load, in the same manner as the vertical reactions of a simple beam.

12. BEAM WITH FIXED ENDS. CONCENTRATED LOAD AT ANY POINT. EFFECT OF SETTLEMENT OF SUPPORT. The beam shown in Fig. 22 is similar to the one shown in Fig. 21, except that the support A has settled a distance D. The tangents to the elastic curve at the two ends are horizontal as before. Hence

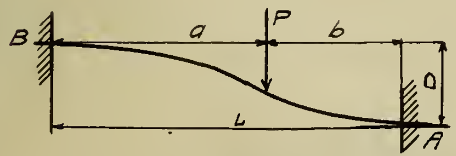


Fig. 22.

$\theta_A = 0$, $\theta_B = 0$, and $R = D/L$. The application of equation 2, gives

$$M_B = -6EID/L^2 - Pab^2/L^2 \quad \ldots\ldots\ldots \quad (12).$$

$$M_A = 6EID/L^2 - Pa^2b/L^2 \quad \ldots\ldots\ldots \quad (13).$$

If the tangents at A and B do not remain horizontal, equation 2 applies directly.

$$M_B = 2EK(2\theta_B + \theta_A - 3R) - Pab^2/L^2, \text{ and}$$

$$M_A = -2EK(2\theta_A + \theta_B - 3R) - Pa^2b/L^2.$$

13. BEAM WITH FIXED ENDS. SYMMETRICAL LOADING.

(a). Single concentrated load at the center. See Fig. 23. As before, $\theta_A=0$, $\theta_B=0$, and $R=0$. Applying equation 3 of Table I, gives

$$M_A = M_B = -PL/8. \quad \cdots\cdots\cdots \quad (14).$$

(b). Uniformly distributed load W. See Fig. 24. Equation 4 of Table I gives

$$M_A = M_B = -WL/12. \quad \cdots\cdots\cdots \quad (15).$$

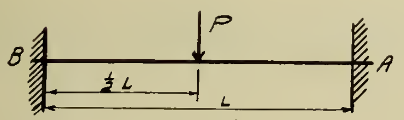


Fig. 23.

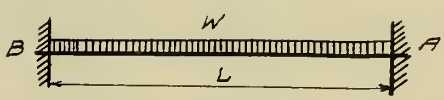


Fig. 24.

From the fundamental equations the following rule may be established. For a beam with fixed ends on the same level, and with loading symmetrical about the middle, the bending moment at the ends is numerically equal to the average bending moment in a simple beam with the same loading. This is seen to be true in the case of the beams shown in Figs. 23 and 24.

14. BEAM CONTINUOUS OVER THREE SUPPORTS. Fig. 25 represents two spans of a beam which is continuous over a number of supports on the same level. The beam carries

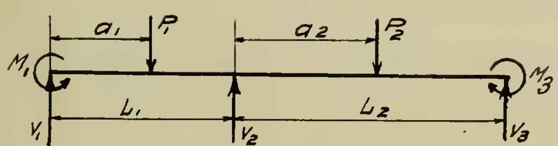


Fig 25

concentrated loads. Equation 2, when applied to each span, gives

$$M_1L_1 = 2EI(2\theta_1 + \theta_2) - (P_1a_1(L_1 - a_1)^2/L_1).$$

$$M_2L_1 = -2EI(2\theta_2 + \theta_1) - (P_1(L_1 - a_1)a_1^2/L_1) .$$

$$M_2L_2 = 2EI(2\theta_2 + \theta_3) - (P_2a_2(L_2 - a_2)^2/L_2).$$

$$M_3L_2 = -2EI(2\theta_3 + \theta_2) - (P_2(L_2 - a_2)a_2^2/L_2) .$$

Combining these equations to eliminate values of θ, gives

$$M_1L_1 + 2M_2L_1 + 2M_2L_2 + M_3L_2 = -\frac{P_1}{L_1}(a_1L_1 - a_1^2 + 2a_1^2)(L_1 - a_1) - \frac{P_2}{L_2}(2a_2L_2 - 2a_2^2 + a_2^2)(L_2 - a_2).$$

If there are several loads on each span, the equation may be written,

$$M_1L_1 + 2M_2(L_1 + L_2) + M_3L_2 = -\sum P_1L_1^2\left(\frac{a_1}{L_1} - \frac{a_1^3}{L_1^3}\right) - \sum P_2L_2^2\left(\frac{2a_2}{L_2} - \frac{3a_2^2}{L_2^2} + \frac{a_2^3}{L_2^3}\right) \quad . \ . \ . \ . \ (16).$$

This is the usual form of the equation of three moments. If the beam carries a uniform load on each span, a similar procedure gives,

$$M_1L_1 + 2M_2(L_1 + L_2) + M_3L_2 = -\tfrac{1}{4}W_1L_1^2 - \tfrac{1}{4}W_2L_2^2 \quad . \ . \ . \ . \ . \ . \ . \ . \ . \ . \ . \ . \ (17).$$

Equations similar to the ones which have been written for the two spans shown in Fig. 25, may be written for the other spans of a beam extending over any number of supports. This will give as many equations as there are unknowns.

15. <u>CONTINUOUS BEAM OVER THREE SUPPORTS</u>. <u>EFFECT OF SETTLEMENT OF SUPPORT</u>. Fig. 26 shows a beam similar to the one shown in Fig. 25, except that the left hand support has settled an amount equal to D. The equations for this beam are

$$M_1L_1 = 2EI(2\theta_1 + \theta_2 - 3D/L_1) - (P_1a_1(L_1-a_1)^2/L_1).$$

$$M_2L_1 = -2EI(2\theta_2 + \theta_1 - 3D/L_1) - (P_1(L_1-a_1)a_1^2/L_1).$$

$$M_2L_2 = 2EI(2\theta_2 + \theta_3) - (P_2a_2(L_2-a_2)^2/L_2).$$

$$M_3L_2 = -2EI(2\theta_3 + \theta_2) - (P_2(L_2-a_2)a_2^2/L_2).$$

Combining these equations to eliminate values of θ, gives

$$M_1L_1 + 2M_2(L_1+L_2) + M_3L_2 = 6EI\frac{D}{L_1} - P_1L_1^2\left(\frac{a_1}{L_1} - \frac{a_1^3}{L_1^3}\right) - P_2L_2^2\left(\frac{2a_2}{L_2} - \frac{3a_2^2}{L_2^2} + \frac{a_2^3}{L_2^3}\right) \quad \ldots\ldots\ldots\ldots (18).$$

The deflection is negative and therefore the value of D, when substituted in the equation, must have a negative sign before it.

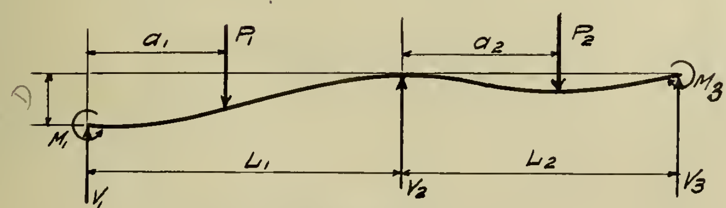


Fig. 26.

B. FRAMES UNDER VERTICAL LOADING. FRAMES AND LOADING SYMMETRICAL ABOUT A VERTICAL CENTER LINE.

16. GENERAL. Fig. 27 represents any frame symmetrical about a vertical center line. If the loads are symmetrical, the horizontal deflections of the points A and B are equal to zero, and the slopes of the tangent to the elastic curve at A and B are equal, but opposite in sign. Therefore , equations 1 and 3 of Table I when applied to the member AB take the form shown in equations 19 and 20 of Table II; and when applied to the member AD, take the form shown in equations 21 and 22 of Table II, with the loadings designated.

TABLE II.

Special Forms of the Fundamental Equations.

Member.	Loading.	Values of θ & D.	Equation.
	No External Load.	$\theta_A = -\theta_B$ $D=0$	$M_A = 2EK\theta_A = M_B$ (19).
	Any Symmetrical Load.	$\theta_A = -\theta_B$ $D=0$	$M_A = 2EK\theta_A - \frac{F}{L} = M_B$ (20.)
	No External Load.	$\theta_D = 0$ $D=0$	$M_A = -4EK\theta_A = -2M_D$ (21). Distance to the Point of Contraflexure, $x = \frac{L}{3}$
	No External Load.	$D=0$	$M_A = -3K\theta_A E.$ (22).

17. THE THREE SIDED FRAME WITH POSTS HINGED AT BASE. Referring again to Fig. 27, from Table II, the moment M_{AB} at the left hand end of AB is given by the equation $M_{AB} = 2EK_1\theta_A - F/L$. Hence,

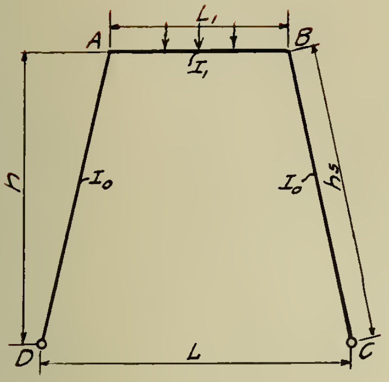


Fig. 27.

$$\theta_A = \frac{M_{AB} + F/L_1}{2EK_1}.$$

Also the moment at the top of AD is given by the equation

$$M_{AD} = -3EK_0\theta_A, \quad \text{whence} \quad \theta_A = -\frac{M_{AD}}{3EK_0}.$$

Equating the two values of θ_A, substituting M_A for M_{AB} and M_{AD}, and $\underline{n}$ for $\frac{K_1}{K_0}$, gives

$$M_A = \frac{F}{L_1}\left(\frac{-3}{3+2n}\right) \quad . \ . \ . \ . (23).$$

Equation 23 is applicable to frames with either inclined or vertical posts. The values of F/L for different conditions of loading, to be used in equation 23 and all following equations of Section B, are given in Table III, page 29.

The direct stress in the posts varies as the secant of the angle of inclination with the vertical. The assumption of no shortening of members due to direct stress is equivalent to saying that the top member is rigidly supported at A and B. Hence, making the angle of inclination equal to 90°, the above equation for M_A holds true in the case of a continuous beam of three spans, with hinged ends. The supports at A and B relieve the direct stress in the members AD and BC.

18. <u>THREE SIDED FRAME WITH POSTS FIXED AT BASE</u>: Referring to Fig. 28, and proceeding as in paragraph 17 :

By equations 20 and 21 of Table II,

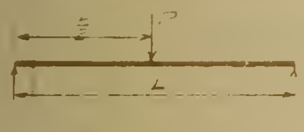
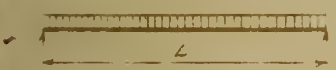
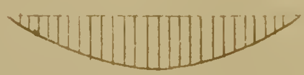
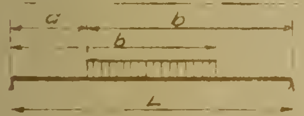
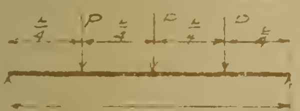
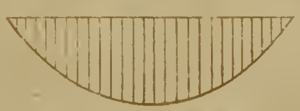
$$M_{AD} = -4EK_0\theta_A, \quad \text{or} \quad \theta_A = -\frac{M_{AD}}{4EK_0}.$$

TABLE III.

Values of $\frac{F}{L}$ for Various Symmetrical Loadings

F = Area of Moment Diagram, Simple Beam

L = Length of Loaded Member

Conditions of Loading. Position of Loads	Moment Diagram.	Maximum Moment	Value of $\frac{F}{L}$.
		$\frac{PL}{4}$	$\frac{PL}{8}$.
Total Load = W. 		$\frac{WL}{8}$	$\frac{WL}{12}$.
 Total Load = 2wa		$\frac{wa^2}{2}$.	$\frac{wa^2}{6}\frac{(3b+a)}{L}$
 Total Load = w(b-)		$\frac{w(b-a)}{8}(b-3a)$ $\frac{W}{x}(L^2-4a^2)$	$\frac{w(b-a)}{12}\left(\frac{6ab+(b-a)^2}{L}\right)$. $\frac{Wa^2}{6L}(3L-2a)$
		Pa.	$\frac{Pab}{L}$.
		$\frac{PL}{3}$.	$\frac{2PL}{9}$
		$\frac{PL}{2}$.	$\frac{5PL}{16}$.
 Total Load		$\frac{WL}{6}$.	$\frac{5WL}{48}$.
 Total Load = W.		$\frac{WL}{12}$.	$\frac{WL}{16}$.

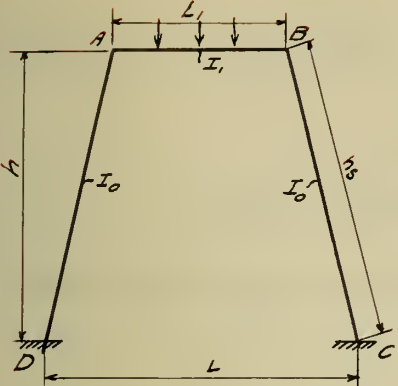


Fig. 28.

$M_{AB} = 2EK_1\theta_A - F/L_1$, whence $\theta_A = \dfrac{M_{AB} + F/L_1}{2EK_1}$.

Equating the values of θ_A, and substituting n for K_1/K_0, gives

$$M_A = \frac{F}{L_1}\left(\frac{-2}{2+n}\right). \quad . \quad . \quad . \quad . \quad . \quad . (24).$$

Values of F/L for different loadings are given in Table III, page 29.

19. <u>BUILDING</u> <u>FRAME</u>. A type of frame which is often encountered in building construction is shown in Fig. 29. Consider all members which meet at the joints A and B. Let AB be the only beam which carries a vertical load. Then certain bending moments are produced in the members meeting at A and B. The far ends of these members are restrained; the degree of restraint being between that of a hinged and that of a fixed end. The moments will now be determined for these two limiting conditions of end restraint.

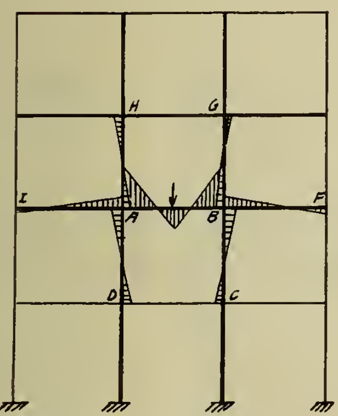


Fig. 29.

(a). Consider the case in which the far ends of the members are hinged. The members taken together as a free body are shown in Fig. 30. Equations 20 and 22 of Table II are applied to the moments at the point A.

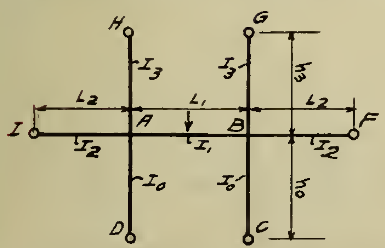


Fig. 30.

$M_{AI} = -EK_2(3\theta_A).$

$M_{AD} = -EK_0(3\theta_A).$

$M_{AH} = -EK_3(3\theta_A).$

Adding, $M_{AI} + M_{AD} + M_{AH} = M_{AB} = -3\theta_A E(K_2 + K_0 + K_3)$.

$$\text{whence} \quad \theta_A = -\frac{M_{AB}}{3E(K_2 + K_0 + K_3)}.$$

$$M_{AB} = 2EK_1\theta_A - \frac{F}{L}, \text{ so } \theta_A = \frac{M_{AB} + \frac{F}{L}}{2EK_1}.$$

Equating values of θ_A, gives.

$$M_{AB} = \frac{F}{L}\left[\frac{-3(K_2 + K_0 + K_3)}{3K_2 + 3K_0 + 3K_3 + 2K_1}\right] \quad \ldots\ldots (25).$$

(b). Consider next the case in which the far ends of the members are fixed. The members taken together as a free body are shown in Fig. 31. From the equations of Table II,

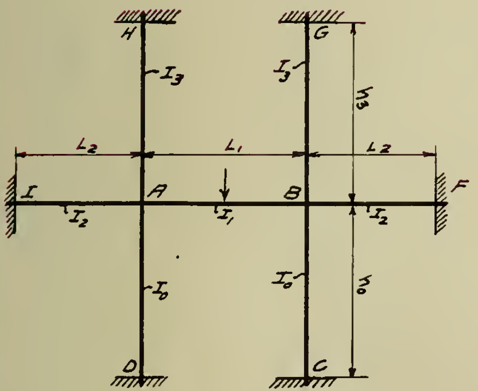


Fig. 31.

$$M_{AI} = -2EK_2(4\theta_A).$$

$$M_{AD} = -2EK_0(4\theta_A).$$

$$M_{AH} = -2EK_3(4\theta_A).$$

$$M_{AI} + M_{AD} + M_{AH} = M_{AB} = 4\theta_A E(K_2 + K_0 + K_3).$$

$$\text{whence} \quad \theta_A = -\frac{M_{AB}}{4E(K_2 + K_0 + K_3)}.$$

$$M_{AB} = 2EK_1\theta_A, \text{ so } \theta_A = \frac{M_{AB} + \frac{F}{L}}{2EK_1}.$$

Equating values of θ_A, gives

$$M_{AB} = \frac{F}{L}\left[\frac{-4(K_2 + K_0 + K_3)}{4K_2 + 4K_0 + 4K_3 + 2K_1}\right] \ldots (26).$$

It is seen that the bending moments at A and B do not vary appreciably in the two cases. Diagram I, page 32, shows curves plot-

Diagram I
Effect of Hinged and Fixed Ends upon the Moment in Member AB
H
G
I
A
B
F
D
C
Let $K_0 = K_2 = K_3$
Ends Fixed
Ends Hinged.
Values of $\frac{K_1}{K_0}$
Moment, M_{AB}, - in Terms of $\frac{E}{L}$
4.5
4.0
3.5
3.0
2.5
2.0
1.5
1.0
.5
0
.20
.40
.60
.80
1.00

ted for equations 25 and 26, with different values of K. From this it is evident that the actual moment in the symmetrical frame of a building subjected to symmetrical loads is determined within quite narrow limits.

20. DEGREE OF RESTRAINT OF THE END OF A MEMBER. As has been briefly noted above, referring to Fig. 29, the resistance to a change in slope at the joint A, which is exerted by a member AD, depends not only upon the stiffness of the member, but also upon the condition of restraint at the end D. When the degree of restraint at D is known, as in the cases of hinged or fixed ends, the restraining effect of this member on the rotation at the point A can be determined. The relative restraining effect of the different members will be proportional to the bending moments produced in them by a change θ_A in the slope at A. Table II, page 27, shows that for a change in slope at A, the relative restraining effects of the members for which equations 19, 21, and 22 are written, are in the ratio of 2K, 4K, and 3K, respectively.

Consider any symmetrical frame, with the member AB carrying a load as shown in Fig. 32. The degree of restraint at A may vary from zero for a simple beam, to unity for a beam with fixed ends; and similarly, the moment M_{AB} may vary from $0 \cdot \frac{F}{L}$ to $1 \cdot \frac{F}{L}$. The degree of restraint at the end A of the member AB may be defined as the ratio of the moment at A to the moment which would exist at A if the end was fixed.

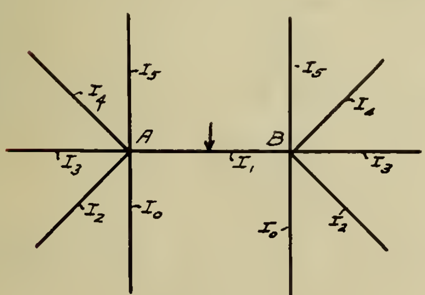


Fig. 32.

Therefore, the degree of restraint may be written as a coefficient of F/L. Equations 25 and 26 show that the degree of restraint at A is equal to the ratio of the restraining effect of the unloaded members, to the restraining effect of all members at the joint. For example, suppose the unloaded members of Fig. 32 are hinged at the far ends. The degree of restraint μ at A is given by the equation

$$\mu = \left[\frac{3K_0 + 3K_5 + 3K_2 + 3K_3 + 3K_4}{3K_0 + 3K_5 + 3K_2 + 3K_3 + 3K_4 + 2K_1}\right]$$

If the same members were fixed at the far ends, the value of μ is given by the equation

$$\mu = \left[\frac{2K_0 + 2K_2 + 2K_3 + 2K_4 + 2K_5}{2K_0 + 2K_2 + 2K_3 + 2K_4 + 2K_5 + K_1}\right]$$

In each case, $M_{AB} = \mu \frac{F}{L}$.

21. BUILDING FRAME. MOMENT IN MEMBERS ADJACENT TO LOADED BEAM. Refer again to Fig. 30, for the frame having members hinged at the ends. The moment M_{AB} is given by equation 25. Consider the moments in the members meeting at A.

$$M_{AD} = -3EK_0\theta_A \qquad \text{or} \qquad \theta_A = -\frac{M_{AD}}{3EK_0}$$

$$M_{AB} = 2EK_1\theta_A - \frac{F}{L} \qquad \text{or} \qquad \theta_A = \frac{M_{AB} + \frac{F}{L}}{2EK_1}$$

Equating values of θ_A, and simplifying,

$$M_{AD} = \frac{F}{L}\left[\frac{-3K_0}{3K_2 + 3K_0 + 3K_3 + 2K_1}\right]. \quad (25a).$$

Similarly,

$$M_{AH} = \frac{F}{L}\left[\frac{-3K_3}{3K_2 + 3K_0 + 3K_3 + 2K_1}\right]. \quad (25b).$$

and

$$M_{AI} = \frac{F}{L}\left[\frac{-3K_2}{3K_2 + 3K_0 + 3K_3 + 2K_1}\right]. \quad (25c).$$

[illegible]

[illegible]

[illegible]

[illegible]

[illegible]

[illegible]

[illegible]

[illegible]

Referring to Fig. 31, for the frame having members fixed at ends, the moments at D, H, and I, are found by a procedure similar to that above. The equations are as follows:

$$M_{AD} = -2M_{DA} = \frac{F}{L}\left[\frac{-2K_0}{2K_2 + 2K_0 + 2K_3 + K_1}\right] \quad . . . . (26a).$$

$$M_{AH} = -2M_{HA} = \frac{F}{L}\left[\frac{-2K_3}{2K_2 + 2K_0 + 2K_3 + K_1}\right] \quad . . . . (26b).$$

$$M_{AI} = -2M_{IA} = \frac{F}{L}\left[\frac{-2K_2}{2K_2 + 2K_0 + 2K_3 + K_1}\right] \quad . . . . (26c).$$

Special forms of the above equations for building frames subjected to symmetrical vertical loads are shown in the summary on page 113.

22. <u>BUILDING FRAME. COLUMN AND BEAM ENDS HINGED. ALL BEAMS LOADED</u>. Referring to Fig. 33, let F_1 be the area of the moment diagram for AB, considered as a simple beam; and F_2 be the area of a similar diagram for BF and IA. Consider the member IA, which is under a symmetrical load. From equation 3,

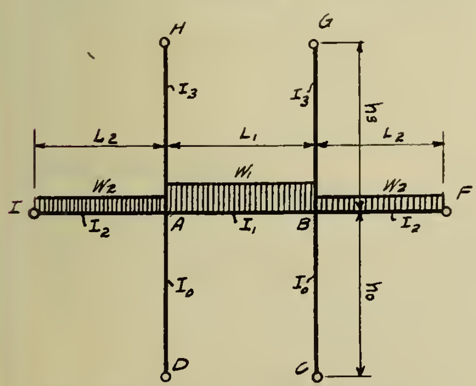


Fig. 33.

$$M_{IA} = 0 = 2EK_2(2\theta_I + \theta_A) - F_2/L_2.$$

$$M_{AI} = -2EK_2\ (2\theta_A + \theta_I) - F_2/L_2.$$

Eliminating θ_I from these equations, and applying the equations of Table II to the other members of the frame, gives

$$M_{AI} = -3EK_2\,\theta_A - \frac{3F_2}{2L_2} \;.\;.\;.\;.\;.\;.\;.\;.\;.\;(a).$$

$$M_{AD} = -3EK_0\,\theta_A \;.\;.\;.\;.\;.\;.\;.\;.\;.\;.\;(b).$$

$$M_{AH} = -3EK_3\,\theta_A \;.\;.\;.\;.\;.\;.\;.\;.\;.\;.\;(c).$$

Adding, $M_{AI} + M_{AD} + M_{AH} = M_{AB} = -3E\theta_A(K_2 + K_0 + K_3) - \frac{3}{2}\frac{F_2}{L_2}$.

whence $$\theta_A = -\frac{\left(M_{AB} + \frac{3}{2}\frac{F_2}{L_2}\right)}{3E(K_2 + K_0 + K_3)}.$$

Also $M_{AB} = 2EK_1\theta_A - \frac{F_1}{L_1}$ or $\theta_A = \frac{M_{AB} + \frac{F_1}{L_1}}{2EK_1}$. . . . . . . (d).

Equating values of θ_A, and substituting $mF_1/L_1 = F_2/L_2$, gives

$$M_{AB} = \frac{F_1}{L_1}\left[\frac{-3(K_0 + K_3 + K_2 + mK_1)}{3K_0 + 3K_3 + 3K_2 + 2K_1}\right] \;.\;.\;.\;(27).$$

It is interesting to note that if m is equal to 2/3, M_{AB} will become $-F_1/L_1$, as in the case of a beam with fixed ends, and θ_A will become equal to zero. If $m > 2/3$, $M_{AB} > -F_1/L_1$, and θ_A will be negative.

M_{AD}, M_{AH}, and M_{AI} may be expressed in terms of F_1/L_1, as follows: From equations (b) and (d), above

$$\theta_A = \frac{-M_{AD}}{3EK_0} = \frac{M_{AB} + \frac{F_1}{L_1}}{2EK_1}$$

Substituting the value of M_{AB} from equation 27, gives

$$M_{AD} = \frac{F_1}{L_1}\left[\frac{\frac{3}{2}K_0(3m-2)}{3K_0 + 3K_3 + 3K_2 + 2K_1}\right] \;(27a)$$

Similarly, $$M_{AH} = \frac{F_1}{L_1}\left[\frac{\frac{3}{2}K_3(3m-2)}{3K_0 + 3K_3 + 3K_2 + 2K_1}\right] \;(27b).$$

From equations (a) and (d), above

$$\theta_A = -\frac{M_{AI} + \frac{3mF_1}{2L_1}}{3EK_2} = \frac{M_{AB} + \frac{F_1}{L_1}}{2EK_1}$$

Substituting the value of M from equation 27, as before, gives

$$M_{AI} = \frac{F_1}{L_1}\left[\frac{4.5mK_0 + 4.5mK_3 + 3K_2 + 3mK_1}{3K_0 + 3K_3 + 3K_2 + 2K_1}\right] \;.\;.\;.\;.\;.\;.\;.\;.\;.\;.\;.\;.\;(27c).$$

It is seen from equations 27a, 27b, and 27c, that if $m = 2/3$, $M_{AD} = M_{AH} = 0$, and $M_{AI} = -F_1/L_1$. It should be noted that while Fig. 33 shows a uniform loading on the beams, the equations apply to any symmetrical loading.

23. <u>BUILDING FRAME. COLUMN AND BEAM ENDS FIXED. ALL BEAMS LOADED.</u> This frame is represented by Fig. 34. Applying the equations of Table II, a procedure similar to that used in paragraph 22, gives

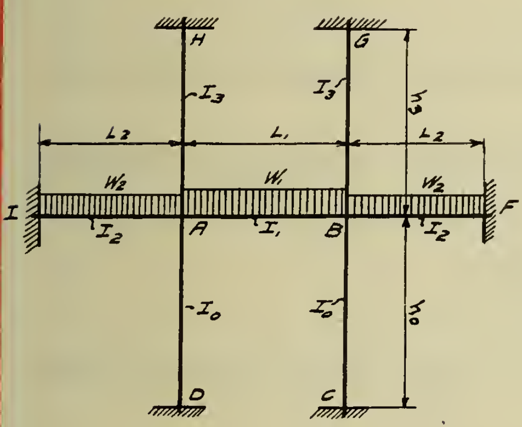


Fig. 34

$$M_{AB} = \frac{F_1}{L_1}\left[\frac{-(2K_0+2K_3+2K_2+mK_1)}{2K_0+2K_3+2K_2+K_1}\right]. \quad (28).$$

$$M_{AI} = \frac{F_1}{L_1}\left[\frac{-(2mK_0+2mK_3+2K_2+mK_1)}{2K_0+2K_3+2K_2+K_1}\right]. \quad (28a).$$

$$M_{AD} = \frac{F_1}{L_1}\left[\frac{2(m-1)K_0}{2K_0+2K_3+2K_2+K_1}\right]. \quad (28b).$$

$$M_{AH} = \frac{F_1}{L_1}\left[\frac{2(m-1)K_3}{2K_0+2K_3+2K_2+K_1}\right]. \quad (28c).$$

$M_{AD} = 2M_{DA}$ and $M_{AH} = 2M_{HA}$.

As in paragraph 22, <u>m</u> is the ratio of F_2/L_2 to F_1/L_1. From the above equations it is seen that when <u>m</u> $= 1$, $M_{AB} = M_{AI} = -F_1/L_1$, and $M_{AD} = M_{AH} = 0$. It is also seen that the equations may be modified to the form of the equations of paragraphs 19 and 21, by making $F_2/L_2 = 0$, or <u>m</u> $= 0$. These equations can also be applied to simpler frames by letting the value of K of any member equal zero. Thus by letting K_3 and K_0 equal zero, equation 28 applies to the continuous beam of three spans, with fixed ends. Several special forms of these equations are given the summary on page 114.

24. <u>FOUR SIDED FRAME</u>. <u>SYMMETRICAL LOAD ON TOP BEAM</u>. See Fig. 35. Applying equations 1, 19, and 20, and substituting $n = K_1/K_0$, and $p = K_1/K_3$, gives

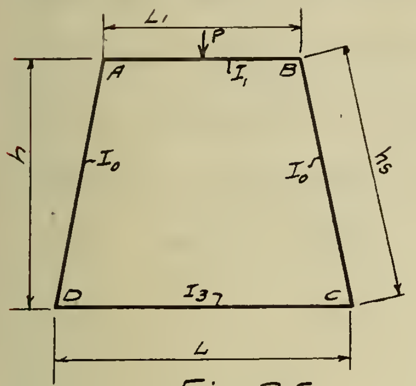


Fig. 35.

$$M_{DC}\, p = -2E\theta_D \qquad \text{(a).}$$

$$M_{DA}\, n = 2E(2\theta_D + \theta_A) \qquad \text{(b).}$$

$$M_{AB} = 2E\theta_A - \frac{F}{L} \qquad \text{(c).}$$

$$M_{AD}\, n = -2E(2\theta_A + \theta_D) \qquad \text{(d).}$$

From Equations (a) and (b), $-n\theta_D = 2p\theta_D + p\theta_A$ or $\theta_D = \frac{-p\theta_A}{n+2p}$.

Substituting this value of θ_D in (d), gives

$$M_{AD}\, n = -2E\theta_A\left(2 - \frac{p}{n+2p}\right) \quad \text{or} \quad \theta_A = -\frac{M_{AD}\, n}{K_1 2E}\left(\frac{n+2p}{2n+3p}\right).$$

Substituting this value of θ_A in (c), and writing $M_A = M_{AB} = M_{AD}$, gives

$$M_A = \frac{F}{L}\left(\frac{-(2n+3p)}{n^2+2pn+2n+3p}\right). \quad \text{Let } n^2+2pn+2n+3p = \alpha,$$

Then $$M_A = \frac{F}{L}\left[\frac{-(2n+3p)}{\alpha}\right] \qquad \text{(29).}$$

From Equation (a), $M_D = -\frac{2E\theta_D}{p} = +\frac{2E\theta_A}{n+2p}$

From Equation (c), $\theta_A = \frac{M_A + \frac{F}{L}}{2E}$

Combining $M_D = \frac{M_A + \frac{F}{L}}{n+2p}$ whence

$$M_D = \frac{F}{L}\left[\frac{n}{\alpha}\right] \qquad \text{(29a).}$$

25. <u>THREE SPAN VIADUCT BENT. COLUMNS HINGED AT BASE. LOAD ON MIDDLE SPAN ONLY.</u> This type of frame is used , especially in Europe, in reinforced concrete viaduct construction; and also in the open type concrete bridge abutment, which is being developed to a considerable extent in this country. The same type of structure has been used for reinforced concrete traveling crane runways.

Such a frame is shown in Fig. 36. W represents any vertical loading symmetrical about the vertical center line of the frame. Owing to the symmetry of loading and frame, there will be no horizontal deflection at the upper ends of the columns. Consider the points I and A. Applying the equations of Table II, gives

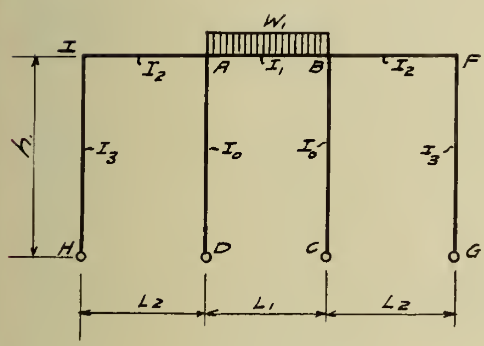


Fig. 36.

$M_{IA} = 2EK_2(2\theta_I + \theta_A)$ . . . . (a).

$M_{IH} = -3EK_3\,\theta_I$ . . . . (b).

$M_{AI} = -2EK_2(2\theta_A + \theta_I)$ . . . . . . (c).

$M_{AD} = -3EK_0\,\theta_A$ . . . . (d).

$M_{AB} = 2EK_1\theta_A - \frac{F}{L}$ . . . . (e).

From Equations (a) and (b) $\theta_I = -\theta_A\left(\frac{2K_2}{3K_3+4K_2}\right)$.

Substituting this value of θ_I in (c), gives $M_{AI} = -4EK_2\,\theta_A\left(\frac{3K_3+3K_2}{3K_3+4K_2}\right)$.

From equation 22, if there was a hinge at I, M_{AI} would equal $3EK_2\theta_A$. From equation 21, if there was a fixed end at I, M_{AI} would equal $4EK_2\theta_A$. From the above expression for M_{AI}, it is seen that its value may vary between these limits. At the point A, the degree of

restraint, μ, may be expressed by the equation

$$\mu_{AB} = -\left(\frac{4K_2\left(\frac{3K_3+3K_2}{3K_3+4K_2}\right)+3K_0}{4K_2\left(\frac{3K_3+3K_2}{3K_3+4K_2}\right)+3K_0+2K_1} \right)$$

Hence,

$$M_{AB} = \frac{F}{L}\left[\frac{-4K_2\left(\frac{3K_3+3K_2}{3K_3+4K_2}\right)-3K_0}{4K_2\left(\frac{3K_3+3K_2}{3K_3+4K_2}\right)+3K_0+2K_1} \right] \; . \; . \; (30).$$

From equation (e), $\theta_A = \frac{F}{L}\left(\frac{\mu_{AB}+1}{2EK_1}\right)$. Therefore, expressing the moments at A and I in terms of Θ_A, and substituting the value of μ_{AB}, gives

$$M_{AD} = \frac{F}{L}\left[\frac{-3K_0}{4K_2\left(\frac{3K_3+3K_2}{3K_3+4K_2}\right)+3K_0+2K_1} \right] \; . \; . \; (30a).$$

$$M_{AI} = \frac{F}{L}\left[\frac{-4K_2\left(\frac{3K_3+3K_2}{3K_3+4K_2}\right)}{4K_2\left(\frac{3K_3+3K_2}{3K_3+4K_2}\right)+3K_0+2K_1} \right] \; . \; . \; (30b).$$

$$M_{IH} = \frac{F}{L}\left[\frac{\left(\frac{+6K_2K_3}{3K_3+4K_2}\right)}{4K_2\left(\frac{3K_3+3K_2}{3K_3+4K_2}\right)+3K_0+2K_1} \right] \; . \; . \; (30c).$$

26. <u>THREE SPAN VIADUCT BENT. COLUMNS FIXED AT BASE. LOAD ON MIDDLE SPAN ONLY</u>. See Fig. 37. <u>W</u> represents any loading, symmetrical about the vertical center line of the bent. The equations of Table II which are applied are as follows:

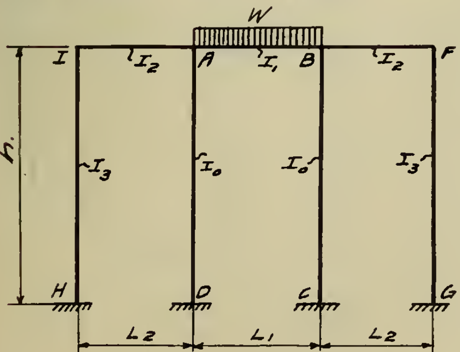


Fig. 37.

$M_{IH} = -4EK_3\,\theta_I \; = -2M_{HI}$ . . . . . (a).

$M_{IA} = 2EK_2(2\theta_I+\theta_A)$ . . . . (b).

$M_{AI} = -2EK_2(2\theta_A+\theta_I)$ . . . . . . (c).

$$M_{AD} = -4EK_0\theta_A = -2M_{DA} \quad \text{. . . . (d).}$$

$$M_{AB} = 2EK_1\theta_A - \frac{F}{L} \quad \text{. . . . . . . (e).}$$

By a procedure similar to that of paragraph 25, the following expressions for the moments in the various members are obtained.

$$M_{AB} = \frac{F}{L}\left[\frac{-K_2\left(\frac{4K_3+3K_2}{K_3+K_2}\right) - 4K_0}{K_2\left(\frac{4K_3+3K_2}{K_3+K_2}\right) + 4K_0 + 2K_1}\right] \quad \text{. . . (31).}$$

$$M_{AI} = \frac{F}{L}\left[\frac{-K_2\left(\frac{4K_3+3K_2}{K_3+K_2}\right)}{K_2\left(\frac{4K_3+3K_2}{K_3+K_2}\right) + 4K_0 + 2K_1}\right] \quad \text{. . . (31a).}$$

$$2M_{DA} = M_{AD} = \frac{F}{L}\left[\frac{-4K_0}{K_2\left(\frac{4K_3+3K_2}{K_3+K_2}\right) + 4K_0 + 2K_1}\right] \quad \text{. . . . (31b).}$$

$$2M_{HI} = M_{IH} = \frac{F}{L}\left[\frac{\left(\frac{2K_3K_2}{K_3+K_2}\right)}{K_2\left(\frac{4K_3+3K_2}{K_3+K_2}\right) + 4K_0 + 2K_1}\right] \quad \text{. . . . (31c).}$$

27. <u>THREE SPAN VIADUCT. COLUMNS HINGED AT BASE. LOAD ON ALL SPANS</u>. See Fig. 38. W_1 represents any loading symmetrical about the vertical center line of the middle span. W_2 represents any loading symmetrical about the vertical center line of the outside span. Applying equations 3, 20, and 22, gives

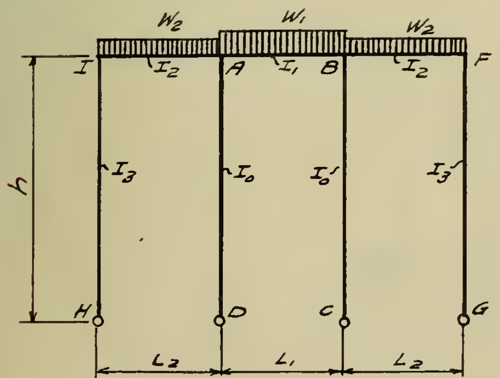


Fig. 38.

$$M_{IH} = -3EK_3\theta_I \quad \text{. . . . . (a).}$$

$$M_{IA} = 2EK_2(2\theta_I + \theta_A) - \frac{F_2}{L_2} \quad \text{. . . . (b).}$$

$$M_{AI} = -2EK_2(2\theta_A + \theta_I) - \frac{F_2}{L_2} \quad \text{. . . . (c).}$$

$$M_{AD} = -3EK_0\theta_A \quad \text{. . . . . (d).}$$

$$M_{AB} = 2EK_1\theta_A - \frac{F_1}{L_1} \quad \text{. . . . . (e).}$$

Combining equations (a) and (b), and solving for θ_I, gives

$$\theta_1 = -\theta_A\left(\frac{2K_3}{3K_3+4K_2}\right) + \frac{F_2}{L_2}\left(\frac{1}{E(3K_3+4K_2)}\right) \quad . \; . \; . \; . \; . \; (f).$$

Substituting this value of θ_I in (c), gives

$$M_{AI} = -2EK_2\left(2\theta_A - \frac{2\theta_A K_2}{3K_3+4K_2}\right) - \frac{F_2}{L_2}\left(1 + \frac{2K_2}{3K_3+4K_2}\right) . \; (g).$$

Combining equations (d) and (g), since $M_{AI} + M_{AD} = M_{AB}$, gives

$$M_{AB} = -\theta_A\left[4EK_2\left(\frac{3K_3+3K_2}{3K_3+4K_2}\right) + 3EK_0\right] - \frac{F_2}{L_2}\left[\frac{3K_3+6K_2}{3K_3+4K_2}\right]$$

whence

$$\theta_A = -\left[\frac{M_{AB} + \frac{F_2}{L_2}\left(\frac{3K_3+6K_2}{3K_3+4K_2}\right)}{4EK_2\left(\frac{3K_3+3K_2}{3K_3+4K_2}\right) + 3EK_0}\right]$$

Substituting this value of θ_A in equation (e), and letting mF_1/L_1 equal F_2/L_2, gives

$$M_{AB} = \frac{F_1}{L_1}\left[\frac{-4K_2\left(\frac{3K_3+3K_2}{3K_3+4K_2}\right) - 3K_0 - 2mK_1\left(\frac{3K_3+6K_2}{3K_3+4K_2}\right)}{4K_2\left(\frac{3K_3+3K_2}{3K_3+4K_2}\right) + 3K_0 + 2K_1}\right] \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; (32).$$

From equations (e) and (32),

$$\theta_A = \frac{M_{AB} + \frac{F_1}{L_1}}{2EK_1}$$

Substituting this value of θ_A in equations (a), (c), (d), and (f), the values of the remaining moments are found.

$$M_{AI} = \frac{F_1}{L_1}\left[\frac{-4K_2\left(\frac{3K_3+3K_2}{3K_3+4K_2}\right) - m(3K_0+2K_1)\left(\frac{3K_3+6K_2}{3K_3+4K_2}\right)}{4K_2\left(\frac{3K_3+3K_2}{3K_3+4K_2}\right) + 3K_0+2K_1}\right] \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; (32a)$$

$$M_{AD} = \frac{F_1}{L_1}\left[\frac{-3K_0\left(1 - m\left[\frac{3K_3+6K_2}{3K_3+4K_2}\right]\right)}{4K_2\left(\frac{3K_3+3K_2}{3K_3+4K_2}\right) + 3K_0 + 2K_1}\right] \; . \; . \; . \; (32b)$$

$$M_{IH} = \frac{F_1}{L_1}\left[\left(\frac{-3K_3}{3K_3+4K_2}\right)\left(\frac{2K_2(3m-1) + 3mK_0 + 2mK_1}{4K_2\left(\frac{3K_3+3K_2}{3K_3+4K_2}\right) + 3K_0 + 2K_1}\right)\right] \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; (32c)$$

If $m = \frac{3K_3 + 4K_2}{3K_3 + 6K_2}$, in the equations 32, 32a, and 32b, it is seen that $M_{AB} = M_{AI} = -F_1/L_1$, and $M_{AD} = 0$. Also, putting m equal to zero for the case in which there is no load on the outer spans, equations 32, 32a, 32b, and 32c take the same form as equations 30, 30a, 30b, and 30c of paragraph 25.

28. THREE SPAN VIADUCT BENT. COLUMNS FIXED AT BASE. LOAD ON ALL SPANS. See Fig. 39. W_1 represents any load symmetrical about the vertical center line of the middle span. W_2 represents any load symmetrical about the vertical center line of the outside span.

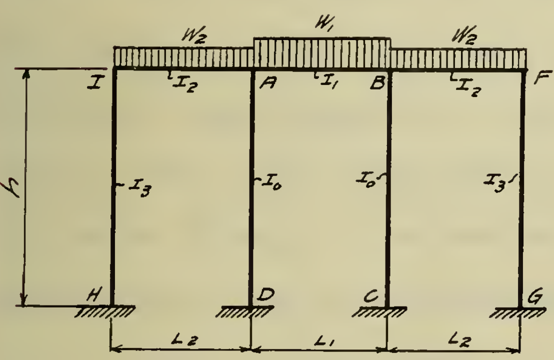


Fig. 39.

Equations 3, 20, and 21, as applied here, give

$$M_{IH} = -4EK_3\theta_I \ldots\ldots (a).$$

$$M_{IA} = 2EK_2(2\theta_I + \theta_A) - \frac{F_2}{L_2} \ (b).$$

$$M_{AI} = -2EK_2(2\theta_A + \theta_I) - \frac{F_2}{L_2} \ (c).$$

$$M_{AD} = -4EK_0\theta_A \ldots\ldots (d).$$

$$M_{AB} = 2EK_1\theta_A - \frac{F_1}{L_1} \ldots (e).$$

The entire procedure is similar to that of paragraph 27, and gives the following expressions for the moments at different points of the frame.

$$M_{AB} = \frac{F_1}{L_1}\left[\frac{-K_2\left(\frac{4K_3+3K_2}{K_3+K_2}\right) - 2mK_1\left(\frac{2K_3+3K_2}{2K_3+2K_2}\right) - 4K_0}{K_2\left(\frac{4K_3+3K_2}{K_3+K_2}\right) + 2K_1 + 4K_0}\right] \quad (33).$$

$$M_{AI} = \frac{F_1}{L_1}\left[\frac{-K_2\left(\frac{4K_3+3K_2}{K_3+K_2}\right) - m(2K_1+4K_0)\left(\frac{2K_3+3K_2}{2K_3+2K_2}\right)}{K_2\left(\frac{4K_3+3K_2}{K_3+K_2}\right) + 2K_1 + 4K_0}\right] \quad (33a).$$

$$M_{AD} = \frac{F_1}{L_1}\left[\frac{-4K_0\left(1 - m\left[\frac{2K_3+3K_2}{2K_3+2K_2}\right]\right)}{K_2\left(\frac{4K_3+3K_2}{K_3+K_2}\right) + 2K_1 + 4K_0}\right] \ldots (33b).$$

$$M_{IH} = \frac{F_1}{L_1}\left[\left(\frac{-K_3}{K_3+K_2}\right)\left(\frac{2K_2[3m-1]+2mK_1+4mK_0}{K_2\left(\frac{4K_3+3K_2}{K_3+K_2}\right)+2K_1+4K_0}\right]\right. \quad (33c).$$

$M_{IH} = 2M_{HI}$, and $M_{AD} = 2M_{DA}$.

If $m = \dfrac{2K_3 + 2K_2}{2K_3 + 3K_2}$, in equations 33, 33a, and 33b, it is seen that $M_{AB} = M_{AI} = -F_1/L_1$, and $M_{AD} = 0$. Also, putting m equal to zero for the case in which there is no load on the outer spans, equations 33, 33a, 33b, and 33c take the same form as equations 31, 31a, 31b, and 31c, of paragraph 26.

In these last two paragraphs it should be remembered that the loading on the two spans need not be of the same kind. For instance, there might be a uniformly distributed load W_1, on the middle span, and concentrated loads whose sum equals W_2 on each of the outer spans; or concentrated loads whose sum equals W_1 on the middle span, and a uniformly distributed load W_2 on each of the outer spans; or any other combination of loads so placed that the loading on a span is symmetrical about the center of the span. Values of F/L for different loadings are given in Table III, page 29.

The foregoing analysis should prove to be of value in the design of viaduct bents. It will be supplemented in Section F by an analysis of the stresses due to horizontal traction and wind loads on the structure.

C. RECTANGULAR FRAMES.

29. USES OF RECTANGULAR FRAMES. A large number of structures are rectangular frames. As has been stated before, there is no complete analysis of the stresses in such structures in suitable form for practical use in designing.

Examples of this kind of framing are seen in the intermediate and portal frames of through bridges, concrete bridges, tunnels, subways, culverts, sewers, aqueducts, reservoirs, and other structures.

The writer has attempted, in analyzing these frames, to put all formulas in as simple a form as possible for use in computation. Several numerical examples are given at the end of this Section, which illustrate the use of these formulas.

30. RECTANGULAR FRAME WITH VERTICAL CONCENTRATED LOAD ON TOP. (a). Load at any Point. In Fig. 40, let the load P act at a distance a from the point A. For convenience in algebraic procedure, substitute 1/Z for K in the fundamental equations, and transfer Z to the left hand side of the equations. Applying equations 1 and 2, gives

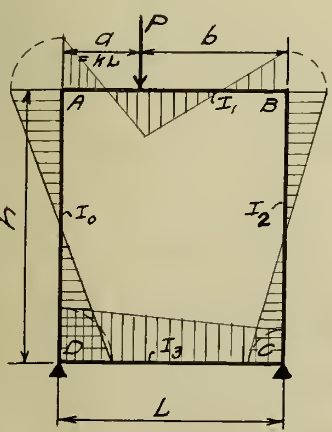


Fig. 40.

$M_{DA}Z_0 = 2E(2\theta_D+\theta_A-3R)$. . . . .(a).

$M_{AD}Z_0 = -2E(2\theta_A+\theta_D-3R)$. . . . .(b).

$M_{AB}Z_1 = 2E(2\theta_A+\theta_B)- Pab^2Z_1/L^2$ . . . .(c).

$M_{BA}Z_1 = -2E(2\theta_B+\theta_A)- Pa^2bZ_1/L^2$. . . .(d).

[illegible]

[illegible]

[illegible]

[illegible]

[illegible]

$$M_{BC}Z_2 = 2E(2\theta_B+\theta_C-3R) \quad \ldots\ldots (e).$$

$$M_{CB}Z_2 = -2E(2\theta_C+\theta_B-3R) \quad \ldots\ldots (f).$$

$$M_{CD}Z_3 = 2E(2\theta_C+\theta_D) \quad \ldots\ldots (g).$$

$$M_{DC}Z_3 = -2E(2\theta_D+\theta_C) \quad \ldots\ldots (h).$$

Let H represent the horizontal shear in AD and BC. Then

$$M_{DA} - M_{AD} = Hh, \text{ and}$$

$$M_{CB} - M_{BC} = Hh, \text{ or}$$

$$M_{DA} - M_{AD} - M_{CB} + M_{BC} = 0. \quad \ldots\ldots (i).$$

Equations (a) to (h) will now be combined to eliminate all values of θ and R. Adding equations (a) to (h), and letting $M_A=M_{AD}=M_{AB}$, and similarly for M_B, M_C, and M_D, gives

$$M_A(Z_0+Z_1) + M_B(Z_1+Z_2) + M_C(Z_2+Z_3) + M_D(Z_3+Z_0) = -\frac{PabZ_1}{L} \quad (j).$$

Adding equations (b) and (e); two times equations (a) and (f); and three times equations (g) and (h), gives

$$M_AZ_0 + M_BZ_2 + M_C(2Z_2+3Z_3) + M_D(2Z_0+3Z_3) = 0. \quad \ldots\ldots (k).$$

Adding equations (d) and (g); two times equations (e) and (f); and subtracting equations (a) and (b), gives

$$-M_AZ_0 + M_B(2Z_2+Z_1) + M_C(2Z_2+Z_3) - M_DZ_0 = -\frac{PabZ_1}{L}\cdot\frac{a}{L} \quad \ldots (l).$$

This gives the four equations, (i), (j), (k), and (l), containing the four unknown moments. Let $n=Z_0/Z_1$, $p=Z_3/Z_1$, $s=Z_2/Z_1$, and $k=a/L$. When these values are substituted in equations (i), (j), (k), and (l), they take the form shown in Table IV, on the next page.

TABLE IV.

Equations for the Rectangular Frame
with Concentrated Load at any Point on Top.

Equation	M_A	M_B	M_C	M_D	Known Term.
j	n + 1	1 + s	s + p	p + n	- Pab/L.
k	n	s	2s + 3p	2n + 3p	0
l	- n	1 + 2s	2s + p	- n	- Pabk/L.
i	- 1	+ 1	- 1	+ 1	0

In the above table, the quantities in each column are the coefficients of the moment indicated at the head of the column, and the quantities in the last column are the right hand members of the equations.

Solving these equations simultaneously for the moments, gives

$$M_A = \frac{Pab}{L}\left(\frac{[3p^2+2s^2+6p+2s+2n+11sn+5pn+17sp]-K[sn-np+5sp+2n+2s+6p+s^2]}{\Delta}\right) \quad \ldots\ldots\ldots (34).$$

$$M_B = \frac{Pab}{L}\left(\frac{[3p^2+2s^2+6p+2s+2n+11sn+5pn+17sp]-(1-K)[sn-np+5sp+2n+2s+6p+s^2]}{\Delta}\right) \quad \ldots\ldots (34a).$$

$$M_C = -\frac{Pab}{L}\left(\frac{[-5sp-2s^2-2s-3p+4np+7sn+n]+(1-K)[2np+5s+6p+s^2+2sp+sn-n]}{\Delta}\right) \quad \ldots\ldots\ldots (34b).$$

$$M_D = -\frac{Pab}{L}\left(\frac{[-5sp-2s^2-2s-3p+4np+7sn+n]+K[2np+5s+6p+s^2+2sp+sn-n]}{\Delta}\right) \quad \ldots\ldots\ldots (34c).$$

where $\Delta = -[22(spn+sp+sn+np)+2(sp^2+s^2p+n^2p+p^2n+s^2+s+n^2+n)+6(sn^2+s^2n+p^2+p)]$

If $I_2 = I_0$, so that the frame is symmetrical about the vertical center line, s = n, and equations 34, 34a, 34b, and 34c become

$$M_A = \frac{Pab}{L}\left[-\frac{2n+3p}{2\alpha} + \left(\frac{k}{\beta} - \frac{1}{2\beta}\right)\right] \quad \ldots\ldots\ldots (35)$$

$$M_B = \frac{Pab}{L}\left[-\frac{2n+3p}{2\alpha} - \left(\frac{k}{\beta} - \frac{1}{2\beta}\right)\right] \quad \ldots\ldots\ldots (35a).$$

$$M_C = \frac{Pab}{L}\left[\frac{n}{2\alpha} - \left(\frac{k}{\beta} - \frac{1}{2\beta}\right)\right] \quad \ldots\ldots\ldots (35b).$$

$$M_D = \frac{Pab}{L}\left[\frac{n}{2\alpha} + \left(\frac{k}{\beta} - \frac{1}{2\beta}\right)\right] \quad \ldots\ldots\ldots (35c).$$

where $\alpha = n^2 + 2pn + 2n + 3p.$

and $\beta = 6n + 1 + p.$

(b). <u>Load at Middle</u>. <u>Frame symmetrical about the vertical center line</u>. With the load at the middle, $k = \frac{1}{2}$, and $\left(\frac{k}{\beta} - \frac{1}{2\beta}\right) = 0$. Equations 35, 35a, 35b, and 35c then become

$$M_A = M_B = \frac{PL}{8}\left(-\frac{2n+3p}{\alpha}\right) \quad \ldots\ldots\ldots (36).$$

$$M_C = M_D = \frac{PL}{8}\left(\frac{n}{\alpha}\right) \quad \ldots\ldots\ldots (36a)$$

From equations 29 and 29a of paragraph 24,

$$M_A = M_B = \frac{F}{L}\left(-\frac{2n+3p}{\alpha}\right) \quad \ldots\ldots\ldots (29).$$

$$M_C = M_D = \frac{F}{L}\left(\frac{n}{\alpha}\right) \quad \ldots\ldots\ldots (29a)$$

The last two equations are essentially the same as equations 36 and 36a.

(c). Uniformly Distributed Load. Frame Symmetrical about the Vertical Center Line. The uniform load is equal to W/L pounds per lineal foot. In this case F/L equals WL/12, and equations 29 and 29a become

$$M_A = M_B = \frac{WL}{12}\left(- \frac{2n+3p}{\alpha}\right) \dots \dots \dots (37).$$

$$M_C = M_D = \frac{WL}{12}\left(\frac{n}{\alpha}\right) \dots \dots \dots (37a).$$

(d). Two Point Loading. Frame and Loads Symmetrical about the Vertical Center Line. See Fig. 41. Here F/L equals Pab/L. Equations 29 and 29a become

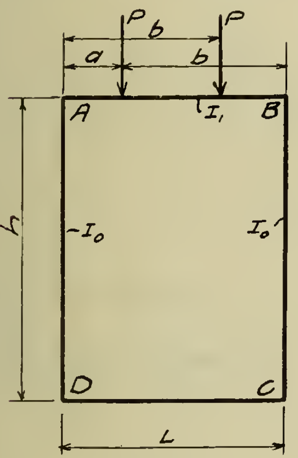


Fig. 41.

$$M_A = M_B = \frac{Pab}{L}\left(- \frac{2n+3p}{\alpha}\right) \dots \dots (38).$$

$$M_C = M_D = \frac{Pab}{L}\left(\frac{n}{\alpha}\right) \dots \dots \dots (38a).$$

If a = L/3 , these equations become

$$M_A = M_B = \frac{2L}{9}\left(- \frac{2n+3p}{\alpha}\right) \dots \dots (39).$$

$$M_C = M_D = \frac{2L}{9}\left(\frac{n}{\alpha}\right) \dots \dots \dots (39a).$$

For other symmetrical loadings, see values of F/L in Table III, page 29, to be substituted in equations 29 and 29a.

31. FRAME WITH HORIZONTAL LOAD ON COLUMNS.

(a). Concentrated Load at any Point. Fig. 42 shows a rectangular frame with a concentrated load applied on member AD, at a

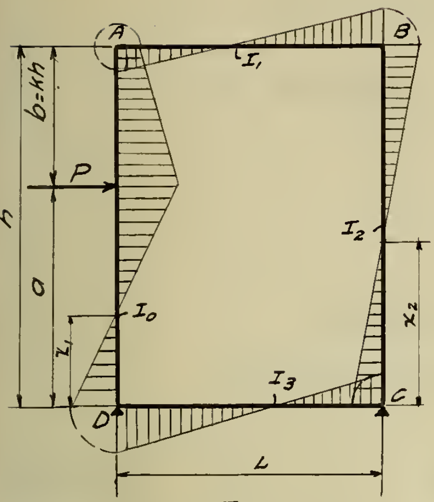


Fig. 42.

distance a from the bottom. Applying equations 1 and 2, substituting Z for 1/K, and transferring it to the left hand member of the equation, gives

$$M_{DA}Z_0 = 2E(2\theta_D+\theta_A-3R) - \frac{PabZ_0}{h}.\frac{b}{h} \quad . . (a).$$

$$M_{AD}Z_0 = -2E(2\theta_A+\theta_D-3R) - \frac{PabZ_0}{h}.\frac{a}{h} \quad . . (b).$$

$$M_{AB}Z_1 = 2E(2\theta_A+\theta_B) \; . . . . (c).$$

$$M_{BA}Z_1 = -2E(2\theta_B+\theta_A) \; . . . . (d).$$

$$M_{BC}Z_2 = 2E(2\theta_B+\theta_C-3R). \; . . . . (e).$$

$$M_{CB}Z_2 = -2E(2\theta_C+\theta_B-3R). \; . . . . (f).$$

$$M_{CD}Z_3 = 2E(2\theta_C+\theta_D) \; . . . . (g).$$

$$M_{DC}Z_3 = -2E(2\theta_D+\theta_C) \; . . . . (h).$$

Let the shear in AD below P be represented by H_1, and the shear in BC by H_2. Then

$$M_D = -H_1x_1,$$

$$M_A = H_1(h-x_1) - Pb,$$

$$M_B = -H_2(h-x_2).$$

$$M_C = H_2x_2.$$

Adding these four equations, gives

$$M_A - M_B + M_C - M_D = Ph - Pb = Pa. \; . . . (i).$$

Combining equations (a) to (h), to eliminate all values of θ and R, gives

$$M_A(Z_0+Z_1) + M_B(Z_1+Z_2) + M_C(Z_2+Z_3) + M_D(Z_3+Z_0) = -\frac{PabZ_0}{h} \quad (j).$$

$$M_AZ_1 + M_BZ_1 - M_C(2Z_3+Z_2) - M_D(2Z_3+Z_0) = \frac{PabZ_0}{h}.\frac{b}{h} \; . . . . (k).$$

$$M_A Z_1 + M_B(2Z_1+3Z_2) + M_C(2Z_3+3Z_2) + M_D Z_3 = 0. \quad . \; . \; . \; . \; . \; . \qquad (1).$$

The last four equations are rewritten in Table V.

TABLE V.

Equations for the Rectangular Frame
with Concentrated Load at any Point on Column.

Equation	M_A	M_B	M_C	M_D	Known Term.
i	1	- 1	1	- 1	Pa.
j	n + 1	1 + s	s + p	p + n	- Pabn/h.
k	1	1	-(2p + s)	-(2p + n)	Pabnk/h.
l	1	2 + 3s	2p + 3s	p	0.

In the above table $n = Z_0/Z_1$, $p = Z_3/Z_1$, $s = Z_2/Z_1$, and $k = b/h$. The four equations can be solved simultaneously, but the resulting expressions for the moments are very long. It is simpler to substitute the numerical values of the quantities in Table V, and determine the moments by a process of elimination.

If the frame is symmetrical about the vertical center line, $I_0 = I_2$, and equations (i), (j), (k), and (l) are much simplified. Letting $s = n$, a solution of the above equations gives

$$M_A = Pa\left[\frac{k^2 n(n+p)}{2\alpha} - kn\left(\frac{n+2p}{2\alpha} + \frac{3}{2\beta}\right) + \frac{3n+p}{2\beta}\right]. \quad (40).$$

$$M_B = Pa\left[\frac{k^2 n(n+p}{2\alpha} - kn\left(\frac{n+2p}{2\alpha} - \frac{3}{2\beta}\right) - \frac{3n+p}{2\beta.}\right]. \quad (40a)$$

$$M_C = Pa\left[-\frac{k^2n(n+1)}{2\alpha} + kn\left(\frac{3}{2\beta} - \frac{1}{2\alpha}\right) + \frac{3n+1}{2\beta}\right] \quad . \ . \ . \ (40b).$$

$$M_D = Pa\left[-\frac{k^2n(n+1)}{2\alpha} - kn\left(\frac{3}{2\beta} + \frac{1}{2\alpha}\right) - \frac{3n+1}{2\beta}\right] \quad . \ . \ . \ (40c).$$

in which $\alpha = n^2 + 2pn + 2n + 3p.$

and $\beta = 6n + 1 + p.$

(b). Concentrated Load at Top. Frame Symmetrical about the Vertical Center Line. In this case $k = 0$, and $a = h$. Hence, equations 35, 35a, 35b, and 35c become

$$M_A = Ph\left(\frac{3n+p}{2\beta}\right) \quad . \ . \ . \ . \ . \ . \ . \ . \ . \ . \ . \ . \ . \ . \ (41).$$

$$M_B = Ph\left(-\frac{3n+p}{2\beta}\right) \quad . \ . \ . \ . \ . \ . \ . \ . \ . \ . \ . \ . \ . \ . \ (41a).$$

$$M_C = Ph\left(\frac{3n+1}{2\beta}\right) \quad . \ . \ . \ . \ . \ . \ . \ . \ . \ . \ . \ . \ . \ . \ (41b).$$

$$M_D = Ph\left(-\frac{3n+1}{2\beta}\right) \quad . \ . \ . \ . \ . \ . \ . \ . \ . \ . \ . \ . \ . \ . \ (41c).$$

(c). Concentrated Load at Middle of Column. Frame Symmetrical about the Vertical Center Line. In this case $a = \frac{1}{2}h$, and $k = \frac{1}{2}$. Equations 35, 35a, 35b, and 35c become

$$M_A = \frac{Ph}{8}\left[\frac{3n+2p}{\beta} - \frac{n(n+3p)}{2\alpha}\right] \quad . \ . \ . \ . \ . \ . \ . \ . \ (42).$$

$$M_B = \frac{Ph}{8}\left[-\frac{3n+2p}{\beta} - \frac{n(n+3p)}{2\alpha}\right] \quad . \ . \ . \ . \ . \ . \ . \ . \ (42a).$$

$$M_C = \frac{Ph}{8}\left[\frac{9n+2}{\beta} - \frac{n(n+3)}{2\alpha}\right] \quad . \ . \ . \ . \ . \ . \ . \ . \ (42b).$$

$$M_D = \frac{Ph}{8}\left[-\frac{9n+2}{\beta} - \frac{n(n+3)}{2\alpha}\right] \quad . \ . \ . \ . \ . \ . \ . \ . \ (42c).$$

(d). Uniformly Distributed Load on one Column. Frame Symmetrical about the Vertical Center Line. Referring to Fig. 43, let the uniform load equal W, and the unit load will equal W/h pounds per lineal foot. In equations 40, 40a, 40b, and 40c, replace P by the elementary load Wdy/h; let a be replaced by the variable distance y, and k will equal (h-y)/h. This gives

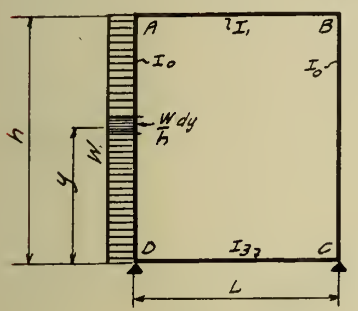


Fig. 43.

$$M_A = \int_0^h \frac{Wy}{h}\left[\left(\frac{h^2-2hy+y^2}{h^2}\right) \frac{n(n+p)}{2\alpha} - n\frac{(h-y)}{2h}\left(\frac{n+2p}{\alpha}+\frac{3}{\beta}\right) + \frac{3n+p}{2\beta}\right] dy.$$

$$M_C = \int_0^h \frac{Wy}{h}\left[\frac{3n+1}{2\beta} + \frac{(h-y)}{h} n\left(\frac{3}{2\beta}-\frac{1}{2\alpha}\right) - \frac{(h^2-2hy+y^2)}{h^2}\, \frac{n(n+1)}{2\alpha}\right] dy.$$

Similar expressions are obtained for M_B and M_D. Performing the integrations indicated, and simplifying, gives

$$M_A = \frac{Wh}{12}\left[-\frac{n(n+3p)}{2\alpha} + \frac{3(2n+p)}{\beta}\right] \quad . \; . \; . \; . \; . \; . \quad (43).$$

$$M_B = \frac{Wh}{12}\left[-\frac{n(n+3p)}{2\alpha} - \frac{3(2n+p)}{\beta}\right] \quad . \; . \; . \; . \; . \; . \quad (43a).$$

$$M_C = \frac{Wh}{12}\left[-\frac{n(n+3)}{2\alpha} + \frac{3(4n+1)}{\beta}\right] \quad . \; . \; . \; . \; . \; . \; . \quad (43b).$$

$$M_D = \frac{Wh}{12}\left[-\frac{n(n+3)}{2\alpha} - \frac{3(4n+1)}{\beta}\right] \quad . \; . \; . \; . \; . \; . \; . \quad (43c).$$

(e). Uniformly Distributed Load on Both Columns. Frame Symmetrical about the Vertical Center Line. Consider the load W applied to each column. The total moment at A is equal to the moment at that point due to the load on the left hand column plus the

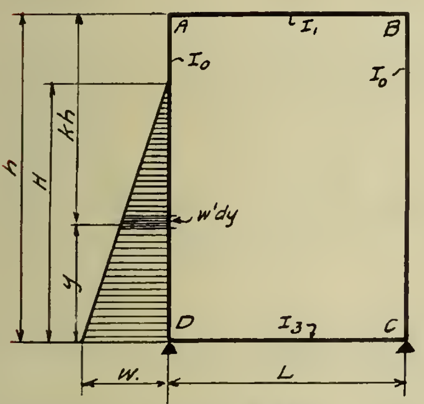
moment at the same point due to the load on the right hand column. These two moments are given by equations 43 and 43a, respectively. Adding the two quantities, gives

$$M_A = M_B = \frac{Wh}{12}\left[-n\left(\frac{n+3p}{\alpha}\right)\right] \ldots\ldots\ldots\ldots (44).$$

Similarly,

$$M_C = M_D = \frac{Wh}{12}\left[-n\left(\frac{n+3}{\alpha}\right)\right] \ldots\ldots\ldots\ldots (44a).$$

(f). Hydraulic Pressure on one Column to a Height H. Frame Symmetrical about the Vertical Center Line. See Fig. 44. Let the total pressure on the column equal W, and the maximum unit pressure will equal w. Also, the unit pressure at a distance y from D will be designated by w'. Then $w = 2W/H$, and $w' = w\left(\frac{H-y}{H}\right) = \frac{2W}{H^2}(H-y)$. Replace P in equations 40, 40a, 40b, and 40c, by the differential load, w'dy; and let $a = y$ and $k = (h-y)/h$. Taking the summation of the moments due to the elementary loads, gives



$$M_A = \int_0^H \frac{2Wy}{H^2}(H-y)\left[\frac{(h-y)^2}{h^2}\,\frac{(n+p)n}{2\alpha} - \frac{n(h-y)}{2h}\left(\frac{n+2p}{\alpha} + \frac{3}{\beta}\right) + \frac{3n+p}{2\beta}\right]dy.$$

$$M_C = \int_0^H \frac{2Wy}{H^2}(H-y)\left[\frac{3n+1}{2\beta} + \frac{(h-y)}{h}\,n\left(\frac{3}{2\beta} - \frac{1}{2\alpha}\right) - \left(\frac{h-y}{h}\right)^2 \frac{n(n+1)}{2\alpha}\right]dy.$$

The expressions for M_B and M_D are similar to those for M_A and M_C. Integrating the right hand members of these equations and simplifying, gives

$$M_A = \frac{WH}{30}\left[\frac{n(n+p)}{2\alpha}\left(10 - 10\frac{H}{h} + 3\left(\frac{H}{h}\right)^2\right) - \frac{n}{2}\left(\frac{n+2p}{\alpha} + \frac{3}{\beta}\right)\left(10 - 5\frac{H}{h}\right) + 10\left(\frac{3n+p}{2\beta}\right)\right]$$

$$\ldots\ldots\ldots\ldots (45).$$

$$M_B = \frac{WH}{30}\left[\frac{n(n+p)}{2\alpha}\left(10 - 10\frac{H}{h} + 3\left(\frac{H}{h}\right)^2\right) - n\left(\frac{n+2p}{2\alpha} - \frac{3}{2\beta}\right)\left(10 - 5\frac{H}{h}\right) - 10\left(\frac{3n+p}{2\beta}\right)\right] \quad . \ (45a).$$

$$M_C = \frac{WH}{30}\left[\frac{10(3n+1)}{2\beta} + \left(10 - 5\frac{H}{h}\right)\left(\frac{3}{2\beta} - \frac{1}{2\alpha}\right)n - \left(10 - 10\frac{H}{h} + 3\left[\frac{H}{h}\right]^2\right)\left(\frac{n(n+1)}{2\alpha}\right)\right] \quad . \ . \ . (45b).$$

$$M_D = \frac{WH}{30}\left[-10\left(\frac{3n+1}{2\beta}\right) - \left(10 - 5\frac{H}{h}\right)\left(\frac{3}{2\beta} + \frac{1}{2\alpha}\right)n - \left(10 - 10\frac{H}{h} + 3\left[\frac{H}{h}\right]^2\right)\left(\frac{n(n+1)}{2\alpha}\right)\right] \quad . \ . \ . (45c).$$

(g). Hydraulic Pressure on Both Columns to a Height H. Frame Symmetrical about the Vertical Center Line. Let W equal the total pressure on one column. Applying the same method as used in paragraph (e), the following equations are obtained.

$$M_A = M_B = \frac{WH}{30}\left[\frac{n(n+p)}{\alpha}\left(10 - 10\frac{H}{h} + 3\left[\frac{H}{h}\right]^2\right) - n\left(\frac{n+2p}{\alpha}\right)\left[10 - 5\frac{H}{h}\right]\right] \quad . \ . \ . (46).$$

$$M_C = M_D = \frac{WH}{30}\left[-\frac{n}{\alpha}\left(10 - 5\frac{H}{h}\right) - n\left(\frac{n+1}{\alpha}\right)\left(10 - 10\frac{H}{h} + 3\left[\frac{H}{h}\right]^2\right)\right] \quad . \ . \ . \ . (46a).$$

(h). Hydraulic Pressure on Entire Height of one Column. Frame Symmetrical about the Vertical Center Line. Let W equal the total pressure on the column. By the equations of paragraph (f), making H = h,

$$M_A = \frac{Wh}{12}\left[-n\left(\frac{2n+7p}{5\alpha}\right) + \left(\frac{3n+2p}{\beta}\right)\right] \quad \ldots\ldots\ldots\ldots \ (47).$$

$$M_B = \frac{Wh}{12}\left[-n\left(\frac{2n+7p}{5\alpha}\right) - \left(\frac{3n+2p}{\beta}\right)\right] \quad \ldots\ldots\ldots\ldots \ (47a).$$

$$M_C = \frac{Wh}{12}\left[-\frac{n(3n+8)}{5\alpha} + \left(\frac{9n+2}{\beta}\right)\right] \quad \ldots\ldots\ldots\ldots \ (47b).$$

$$M_D = \frac{Wh}{12}\left[-n\left(\frac{3n+8}{5\alpha}\right) - \left(\frac{9n+2}{\beta}\right)\right] \quad \ldots\ldots\ldots\ldots \ (47c).$$

(i). Hydraulic Pressure on Entire Height of Both Columns. Frame Symmetrical about the Vertical Center Line. Letting H = h, the equations of paragraph (g) become

$$M_A = M_B = \frac{Wh}{12}\left[-2n\left(\frac{2n+7p}{5\alpha}\right)\right] \quad \ldots\ldots (48).$$

$$M_C = M_D = \frac{Wh}{12}\left[-2n\left(\frac{3n+8}{5\alpha}\right)\right] \quad \ldots\ldots (48a).$$

32. FRAME WITH VERTICAL LOAD ON BOTTOM MEMBER. FRAME SYMMETRICAL ABOUT THE VERTICAL CENTER LINE.

(a). Any Symmetrical Loading. Fig. 45 shows a rectangular frame with loads on the member CD. Applying equations 1 and 3 to the members of the frame, and substituting Z = 1/K, these equations are then combined as in paragraph 30, to eliminate all values of θ and R. Four equations containing the four unknown moments are obtained, as shown in Table VI. As before $n = Z_0/Z_1$, and $p = Z_3/Z_1$.

TABLE VI.

Equations for the Rectangular Frame
with Symmetrical Loading on the Bottom Member.

Equation	M_A	M_B	M_C	M_D	Known Term.
a	1	2 + 2n	n	0	0.
b	1 + n	0	n + p	0	- pF/L.
c	2 + 2n	1	0	n	0.
d	0	1 + n	0	n + p	- pF/L.

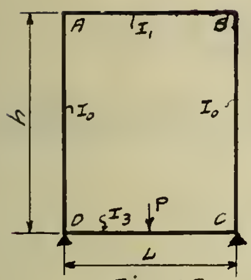


Fig. 45.

A solution of the equations of Table VI, gives

$$M_A = M_B = \frac{F}{L}\left(\frac{-pn}{n^2+2pn+2n+3p}\right) = \frac{F}{L}\left(\frac{-pn}{\alpha}\right). \quad .(49).$$

$$M_C = M_D = \frac{F}{L}\left(\frac{p[2n+3]}{n^2+2pn+2n+3p}\right) = \frac{F}{L}\left(\frac{p[2n+3]}{\alpha}\right) (49a)$$

(b). Concentrated Load at Middle of Beam. For this kind of loading, F/L is equal to PL/8. Therefore equations 49 and 49a take the following form.

$$M_A = M_B = \frac{PL}{8}\left(\frac{-pn}{\alpha}\right) \quad . \ . \ . \ . \ . \ . \ . \ . \ . \quad (50).$$

$$M_C = M_D = \frac{PL}{8}\left(\frac{p(2n+3)}{\alpha}\right). \quad . \ . \ . \ . \ . \ . \ . \quad (50a)$$

(c). Uniformly Distributed Load on Beam. For this kind of loading, F/L is equal to WL/12, where W is the total uniform load. Hence equations 49 and 49a take the following form .

$$M_A = M_B = \frac{WL}{12}\left(\frac{-pn}{\alpha}\right) \quad . \ . \ . \ . \ . \ . \ . \ . \ . \quad (51).$$

$$M_C = M_D = \frac{WL}{12}\left(\frac{p(2n+3)}{\alpha}\right) \quad . \ . \ . \ . \ . \ . \ . \quad (51a)$$

(d). Two Equal Symmetrical Loads on Beam. See Fig. 46. Here F/L is equal to Pab/L, and equations 49 and 49a become

$$M_A = M_B = \frac{Pab}{L}\left(\frac{-pn}{\alpha}\right) \quad . \ . \ . \ . \ . \ . \ . \quad (52).$$

$$M_C = M_D = \frac{Pab}{L}\left(\frac{p[2n+3]}{\alpha}\right) \quad . \ . \ . \ . \ . \quad (52a)$$

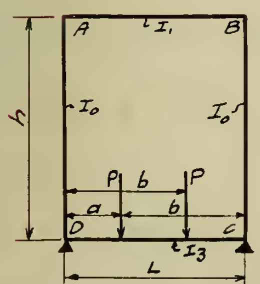


Fig 46.

For values of F/L for other symmetrical loadings, see Table III, page 29.

33. NUMERICAL EXAMPLES OF THE APPLICATION OF FORMULAS FOR RECTANGULAR FRAMES.

(a). Intermediate Cross Frame of a Through Railway Bridge.

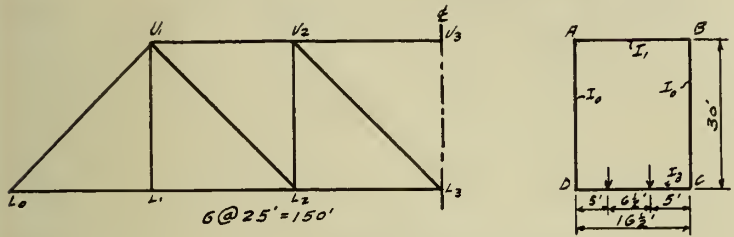


Fig. 47.

Fig.47 shows a half elevation and a section of a through bridge. Consider the bending stress which is transmitted from the floor beam to the verticals by rigid joints at C and D. Such stresses are not usually considered, and are termed secondary stresses.

The members of the frame at U_1L_1 have the following sections;

Verticals , 4∠s, 6"x4"x$\frac{3}{8}$", laced. 20" back to back. I_0= 992 in.^{4}

Floor beam, 50"x ½"pl.,4∠s 6"x 6"x$\frac{9}{16}$". I_3= 8856 in.^{4}

Top strut, same as verticals. I_1= 992 in.^{4}

h = 30', and L = 16½'. Hence $n = \frac{992 \times 30}{992 \times 16\frac{1}{2}} = 1.818$

$$p = \frac{992}{8856} = .112$$

α =(1.818 x 1.818) + 3.636 +(3.636 x.112) + .336 = 7.678

2n + 3 = 3.636 + 3 = 6.636

Referring to Fig. 48, the maximum stringer reaction is 188,000 lb. The maximum moment in the floor beam, considered as a

simple beam is <u>Pa</u>. Pa = 188,000 x 60 = 11,284,000 in.lb.

$$Pa \times b/L = 11,284,000 \times \frac{11.5}{16.5} = 7,850,000 \text{ in.lb.}$$

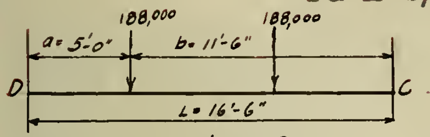


Fig 48.

From paragraph 32, $M_A = \frac{Pab}{L}(\frac{pn}{\alpha})$.

Hence, $M_A = 7,850,000(\frac{.112 \times 6.636}{7.678}) = 208,000$ in.lb. This moment would cause a maximum unit stress of $\frac{208,000 \times 10}{992}$, or 2100 lb. per sq. in. both in the verticals and in the top strut. Also from paragraph 32, $M_D = \frac{Pab}{L}(\frac{p(2n+3)}{\alpha}) = 7,850,000(\frac{.112 \times 6.636}{7.678}) = 760,000$ inch pounds. This moment would cause a maximum unit stress of $\frac{760,000 \times 10}{992} = 7650$ lb. per sq. in. in the verticals.

Since the vertical in this case is a hanger, it is designed to carry 16,000 lb. per sq. in. on the net area of the section; hence it is seen that if the connection at D is perfectly rigid, the stress due to bending is 48% of the primary stress for which the member is designed. Further, it is evident that the maximum primary stress in the hanger will occur simultaneously with the maximum secondary or bending stress. Since the hanger is designed to carry only tension, it follows that it would be best to make the member as flexible as possible in the plane of the floor beam, in order to reduce the secondary stress. In any case, the bending stress increases with the depth of the member; so that if two members have the same moment of inertia, but different depths, the maximum unit stresses due to bending will be in the ratio of the two depths. Verticals other than hangers are not likely to have maximum primary and secondary stresses at the same time.

(b). <u>Cross Section of a Subway Bent</u>. The loading on a subway bent shown in Fig.49 is that used on the New York Subway System as described in the Engineering Record, April 10, 1915. For the numerical case, assume that a = 9', c = 8', and that the section of the subway is 15' square.

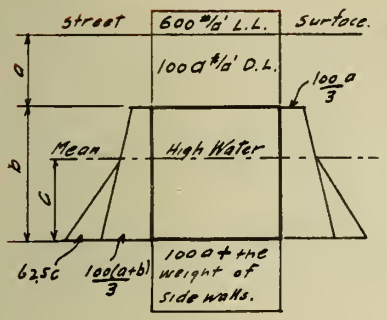


Fig. 49.

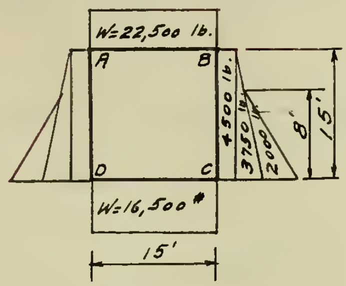


Fig. 50.

The lateral earth pressure on the sidewalls is taken as that of a liquid weighing 33 1/3 lb. per cu. ft. The dead load, including the weight of the sidewalls, is assumed to be uniformly distributed over the bottom of the invert. The dead load is taken at 100a, and is equal to 900 lb. per sq. ft. for this case. The weight of side walls is taken at 200 lb per sq.ft. of the surface of the invert.

The total loads on the bent are shown in Fig. 50. Let $n = 1$, and $p = 1.5$, whence $\alpha = (1+2+3+4\frac{1}{2}) = 10\frac{1}{2}$. Consider the five different kinds of loading on the bent.

1. For the vertical loads on the top, W = 22,500 lb. Applying equations 37 and 37a, gives

$$M_A = \frac{WL}{12}\left(-\frac{2n+3p}{\alpha}\right) = -17,400 \text{ ft.lb.}$$

$$M_D = \frac{WL}{12}\left(\frac{n}{\alpha}\right) = 2,680 \text{ ft.lb.}$$

2. For the upward thrust on the invert, W = 16,500 lb. Applying equations 51 and 51a,

$$M_A = \frac{WL}{12}\left(\frac{pn}{\alpha}\right) = 2,970 \text{ ft.lb.}$$

$$M_D = \frac{WL}{12}\left(\frac{-p(2n+3p)}{\alpha}\right) = -19,200 \text{ ft.lb.}$$

3. For the uniform load on both sidewalls, $W = 4500$ lb. Applying equations 44 and 44a,

$$M_A = \frac{Wh}{12}\left(\frac{-n(n+3p)}{\alpha}\right) = -2940 \text{ ft.lb.}$$

$$M_D = \frac{Wh}{12}\left(\frac{-n(n+3)}{\alpha}\right) = -2140 \text{ ft.lb.}$$

4. For the hydraulic load on full height of sidewalls, W is equal to 3750 lb. Applying equations 48 and 48a,

$$M_A = \frac{Wh}{12}\left(\frac{-2n(2n+7p)}{5\alpha}\right) = -2230 \text{ ft.lb.}$$

$$M_D = \frac{Wh}{12}\left(\frac{-2n(3n+8)}{5\alpha}\right) = -1960 \text{ ft.lb.}$$

5. For the hydraulic load to a height of 8' on the sidewalls $W = 2000$ lb., and $H/h = 8/15$. Applying equations 46 and 46a,

$$M_A = \frac{WH}{30}\left(\frac{2.5}{10.5}(5.52) - \frac{4.0}{10.5}(7.33)\right) = -790 \text{ ft.lb}$$

$$M_D = \frac{WH}{30}\left(-\frac{1.0}{10.5}(7.33) - \frac{2.0}{10.5}(5.52)\right) = -935 \text{ ft.lb}$$

Hence M_A for the combined loading is equal to the algebraic sum of these five moments, which is equal to -23,290 ft.lb. Similarly $M_D = -21,555$ ft.lb. The moment at the middle of the top

member = $PL/8 + M_A = \frac{22,500 \times 15}{8} - 23,290 = + 18,850$ ft.lb.

The moment at the middle of the invert = $\frac{16,500 \times 15}{8} - 21,555$, and is equal to 9,400 ft.lb. The moments at the joints are seen to be of great relative importance. The shears and direct stresses in all members may now be found by the usual methods of statics.

D. SYMMETRICAL THREE SIDED FRAMES WITH COLUMNS HINGED AT THE BASE.

34. GENERAL. The three sided frame , with either hinged or fixed column ends, is used to a considerable extent in culverts, viaducts, building construction, and in concrete bridges and abutments. There is often a great deal of uncertainty as to the degree of restraint of the columns at the base. Usually the ends of the columns are restrained, but they are not held perfectly rigid. However, action as a hinged joint may be secured by using a pin joint; or, in the case of concrete columns, by having the column rest in a socket in the foundation. The last method was used on the Richmond Viaduct which was referred to in the introduction.

35. FRAME WITH LOADS ON TOP.

(a). Concentrated Load at any Point. Fig. 51 shows a three sided frame with a load P on AB, at a distance a from the point A.

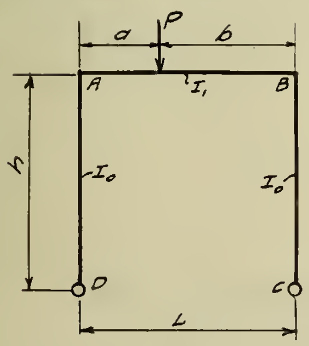


Fig. 51.

This frame may be considered as a special case of the rectangular frame, in which the moment of inertia of the bottom member is zero. In equations 35, 35a, 35b, and 35c, for the rectangilar frame, if $I_3 = 0$, $p = I_1/I_3 = \infty$. Hence, dividing both numerator and denominator of each equation by p, and putting p equal to zero, gives

$$M_C = M_D = 0$$

$$M_A = \frac{Pab}{2L}\left(\frac{-3}{2n+3}\right) \quad . \ . \ . \ . \ . \ . \ (53)$$

(b). Concentrated Load at Middle. In equation 53 of paragraph 35, let $a = b = \frac{1}{2}L$. This gives

$$M_A = M_B = \frac{PL}{8}\left[\frac{-3}{2n+3}\right] = \frac{F}{L}\left[\frac{-3}{2n+3}\right] \quad \ldots\ldots (54).$$

$$M_C = M_D = 0$$

This agrees with the equations of paragraph 17. For any other symmetrical vertical loading, see Table III, page 29, for values of F/L.

36. FRAME WITH HORIZONTAL LOADS ON COLUMN.

(a). Concentrated Load at any Point. Fig. 52 represents a frame with a concentrated load P on the column AD, at a distance a from the point D. Reducing equations 40, 40a, 40b, and 40c, for the rectangular frame as in paragraph 35, by making $p = \infty$, gives

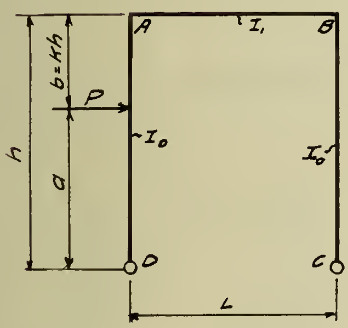


Fig 52.

$$M_A = \frac{Pa}{2}\left[\frac{(k-2)kn}{2n+3} + 1\right] \quad \ldots\ldots (55).$$

$$M_B = \frac{Pa}{2}\left[\frac{(k-2)kn}{2n+3} - 1\right] \quad \ldots\ldots (55a).$$

$$M_C = M_D = 0$$

(b). Concentrated Load at Top of Column. In equations 55 and 55a, let $k = 0$, and $a = h$. This gives

$$M_A = \frac{Ph}{2} \quad \ldots\ldots (56).$$

$$M_B = -\frac{Ph}{2} \quad \ldots\ldots (56a).$$

$$M_C = M_D = 0.$$

(c). Uniformly Distributed Load on one Column. The total load on the column is equal to W. Substituting $p = \infty$ in equations 43, 43a, 43b, and 43c, gives

$$M_A = \frac{Wh}{12}\left(\frac{9(n+2)}{2(2n+3)}\right) \quad \dots\dots\dots (57).$$

$$M_B = \frac{Wh}{12}\left(\frac{-3(5n+6)}{2(2n+3)}\right) \quad \dots\dots\dots (57a).$$

$$M_C = M_D = 0$$

(d). Uniformly Distributed Load on Both Columns. Each column carries a load W. The loadings on the two columns are opposite in direction. Substituting $p = \infty$ in equations 44 and 44a, gives

$$M_A = M_B = \frac{Wh}{12}\left[\frac{-3n}{2n+3}\right] \quad \dots\dots\dots (58).$$

$$M_C = M_D = 0$$

(e). Hydraulic Load on one Column. See Fig 53. The total load on the column is represented by W. $W = \frac{1}{2}wh$, where w is the maximum unit pressure. Substituting $p = \infty$ in equations 47, 47a, 47b, and 47c, gives

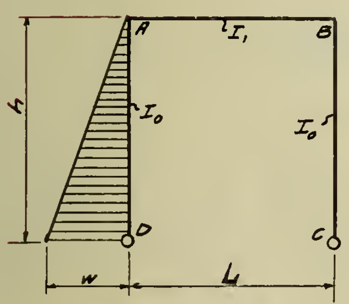


Fig. 53

$$M_A = \frac{Wh}{12}\left[\frac{13n+30}{5(2n+3)}\right] \quad \dots\dots\dots (59).$$

$$M_B = \frac{Wh}{12}\left[\frac{-(27n+30)}{5(2n+3)}\right] \quad \dots\dots\dots (59a).$$

$$M_C = M_D = 0$$

(f). Hydraulic Load on Both Columns. W represents the load on each column. Substituting $p = \infty$ in equations 48 and 48a, gives

$$M_A = M_B = \frac{Wh}{12}\left[\frac{-14n}{5(2n+3)}\right] \quad \dots\dots (60).$$

$$M_C = M_D = 0$$

37. INFLUENCE LINES FOR REACTIONS AND MOMENTS. HORIZONTAL LOADS ON THE COLUMNS. A very interesting study of the effect of the position of a horizontal load, and also the effect of variations in the value of n, upon the stresses in a frame, can be made by the use of equations 55 and 55a. Values of n will be taken at ½, 1, 2, 5, 10, and ∞, while the values of k are varied from 0 to 1.0. The equations will be rewritten here for reference. They are

$$M_A = \frac{Ph}{2}(1-k)\left[k(k-2)\frac{n}{2n+3} + 1\right] = \frac{Ph}{2}\mu_A.$$

$$M_B = \frac{Ph}{2}(1-k)\left[k(k-2)\frac{n}{2n+3} - 1\right] = \frac{Ph}{2}\mu_B.$$

Also, $H_C = \frac{M_B}{h} = \frac{P}{2}\mu_B.$

and $H_D = P - H_C.$

The calculations are arranged in tabular form. Let $C = \frac{n}{2n-3}$. Values of C for different values of n are given in Table VII.

TABLE VII.

Values of C for Different Values of n.

n	.5	1	2	5	10	∞
C	.1250	.2000	.2857	.3846	.4348	.5000

Values of μ for different values of k are given in Table VIII. Substituting values of C from Table VII in Table VIII gives values of μ for different values of n and k. From the equations above it is seen that the stresses in the frame vary directly with μ. The values of μ are shown in Table IX.

TABLE VIII.

Values of μ for Different Values of k.

k	1-k	k-2	(k-2)kC	μ_A	μ_B
0	1.0	-2.0	0	-1.0	-1.0
.1	.9	-1.9	-.19C	-.171C -.9	-.171C -.9
.2	.8	-1.8	-.36C	-.288C -.8	-.288C -.8
.3	.7	-1.7	-.51C	-.357C -.7	-.357C -.7
.4	.6	-1.6	-.64C	-.384C -.6	-.384C -.6
.5	.5	-1.5	-.75C	-.375C -.5	-.375C -.5
.6	.4	-1.4	-.84C	-.336C -.4	-.336C -.4
.7	.3	-1.3	-.91C	-.273C -.3	-.273C -.3
.8	.2	-1.2	-.96C	-.192C -.2	-.192C -.2
.9	.1	-1.1	-.99C	-.099C -.1	-.099C -.1
1.0	0	-1.0	-C	0	0

TABLE IX.

Values of μ for Different Values of k and n.

μ	k	Values of n.					
		.5	1	2	5	10	∞
	0	1.0	1.0	1.0	1.0	1.0	1.0
	.1	.8784	.8658	.8512	.8343	.8255	.8145
	.2	.7640	.7424	.7179	.6890	.6748	.6560
μ_A	.3	.6554	.6286	.5980	.5630	.5450	.5215
	.4	.5520	.5232	.4905	.4520	.4330	.4080
	.5	.4531	.4250	.3930	.3560	.3360	.3125

TABLE IX. - Continued.

Values of μ for Different Values of k and n.

μ	k	Values of n.					
		.5	1	2	5	10	∞
μ_A	.6	.3580	.3328	.3042	.2708	.2540	.2320
	.7	.2659	.2454	.2220	.1950	.1810	.1635
	.8	.1760	.1616	.1452	.1263	.1165	.1040
	.9	.0876	.0802	.0717	.0619	.0570	.0505
	1.0	.0	0	0	0	0	0
	Uniform Load.	.4688	.4500	.4286	.4022	.3920	.3750
$-\mu_B$.	0	1.0	1.0	1.0	1.0	1.0	1.0
	.1	.9214	.9342	.9488	.9657	.9745	.9855
	.2	.8360	.8576	.8821	.9110	.9252	.9440
	.3	.7446	.7714	.8020	.8370	.8550	.8785
	.4	.6480	.6768	.7095	.7480	.7670	.7920
	.5	.5469	.5750	.6070	.6440	.6630	.6875
	.6	.4420	.4672	.4958	.5292	.5460	.5680
	.7	.3341	.3546	.3780	.4050	.4190	.4365
	.8	.2240	.2384	.2548	.2737	.2835	.2960
	.9	.1124	.1198	.1283	.1381	.1430	.1495
	1.0	.0	0	0	0	0	0
	Uniform Load.	.5312	.5500	.5714	.5978	.6080	.6250

The data in Table IX are plotted as influence lines in Diagram II. This diagram shows that a variation of n from $\frac{1}{2}$ to ∞ has comparatively little influence on the moment or reaction

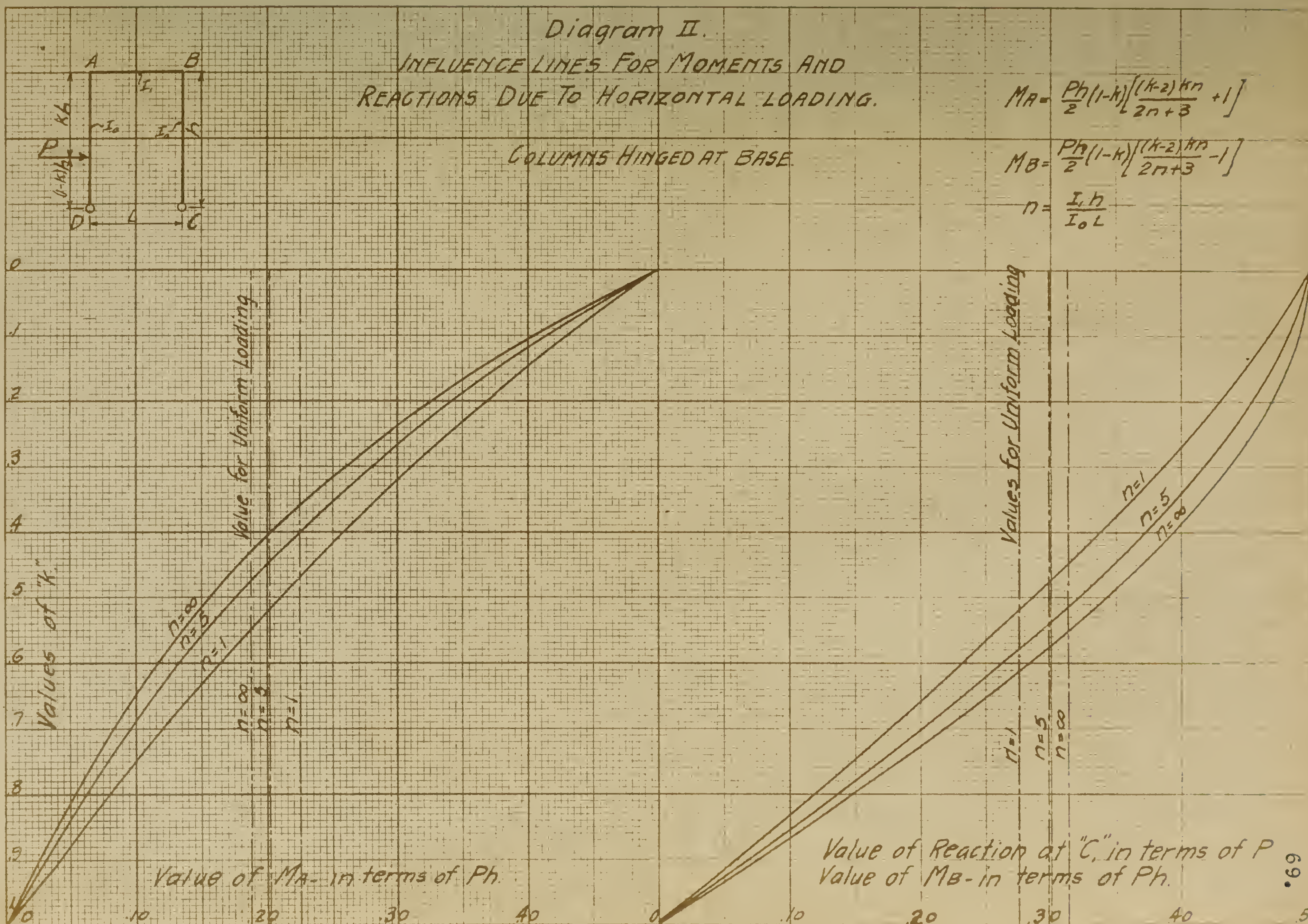


Diagram II.
INFLUENCE LINES FOR MOMENTS AND
REACTIONS DUE TO HORIZONTAL LOADING.
COLUMNS HINGED AT BASE.
$M_A = \frac{Ph}{2}(1-k)\left[\frac{(k-2)kn}{2n+3}+1\right]$
$M_B = \frac{Ph}{2}(1-k)\left[\frac{(k-2)kn}{2n+3}-1\right]$
$n = \frac{I_1 h}{I_0 L}$
A
B
C
D
P
kh
(1-k)h
I_1
I_0
L
Values of "k"
Value for Uniform Loading
Values for Uniform Loading
n=∞
n=5
n=1
Value of M_A in terms of Ph.
Value of Reaction at "C" in terms of P
Value of M_B in terms of Ph.
0
.1
.2
.3
.4
.5
.6
.7
.8
.9
.10
.20
.30
.40
.5

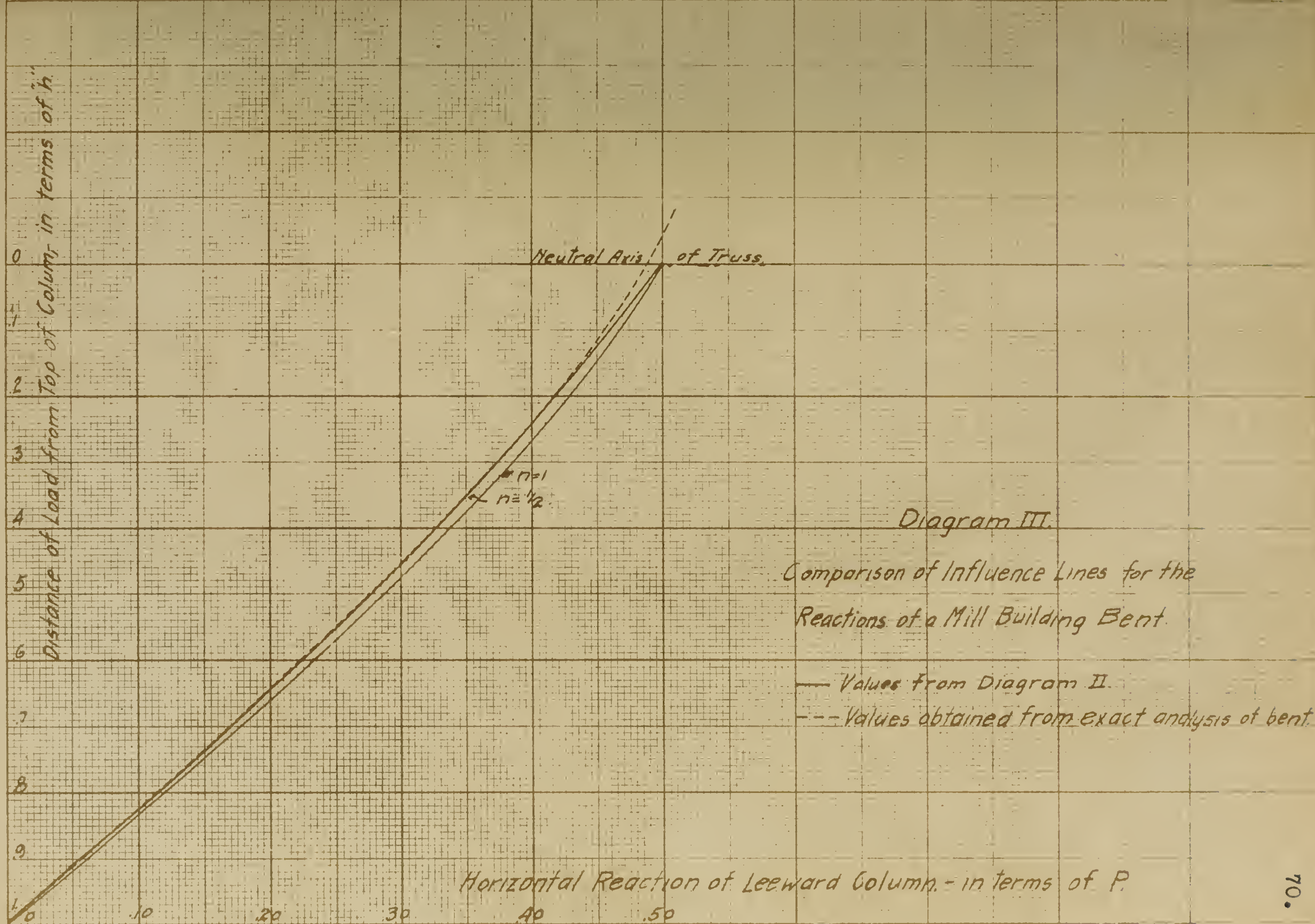


Diagram III.
Comparison of Influence Lines for the
Reactions of a Mill Building Bent.
Values from Diagram II.
Values obtained from exact analysis of bent.
Neutral Axis of Truss.
n=1
n=½
Distance of Load from Top of Colum- in terms of h.
Horizontal Reaction of Leeward Column - in terms of P.
0
.1
.2
.3
.4
.5
.6
.7
.8
.9
1.
0
.10
.20
.30
.40
.50

produced by a given loading, the average variation being only ten to fifteen per cent. On the other hand, a slight change in k produces a large change in the moments and reactions.

These results are of interest when applied to the bent of a mill building. In Fig. 54, if the kneebrace is a solid gusset plate, as is often the case, a rigid connection exists between the truss and the columns; and the bent may be considered as a three sided frame with a top girder of variable moment of inertia. Since the least value of n is undoubtedly greater than $\frac{1}{2}$,

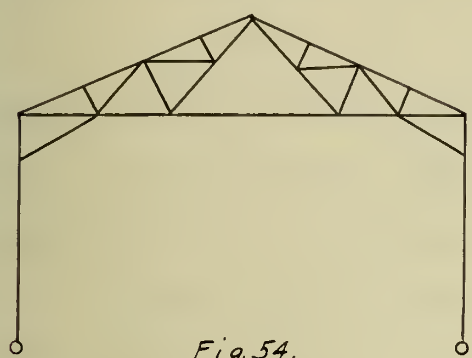


Fig. 54.

and since a change in n does not have very much effect on the value of H, it follows that the probable value of H will lie between the curves for which $n = \frac{1}{2}$ and $n = \infty$, on Diagram II. Hence the leeward reaction due to a uniform horizontal wind load W on the side of the building will be about .3 W, and the reaction on the windward column will be about .7 W. This is quite different from the customary assumption that the horizontal reactions on the columns of a mill building are equal. Since the maximum moment usually occurs at the upper end of the leeward column, its value is only about .3Wh instead of .5Wh as usually assumed. If the sides of the building were open, so that the only horizontal load is that on the roof, the usual assumption would be fairly accurate.

The conditions are somewhat different in the bent shown in Fig. 55. The column is not rigidly connected to the truss, but is

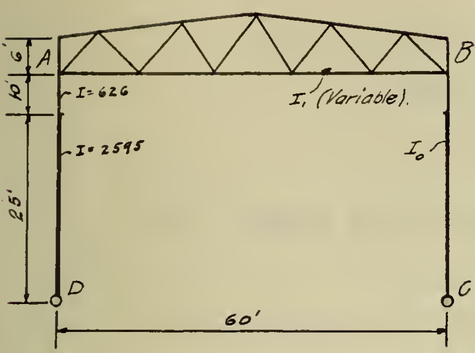


Fig. 55.

free to bend between A and B. At some point between A and B, however, the changes in slope of the axis of the column and of the truss are equal. Hence the length of the column to be considered is greater than BC and less than AC. This will affect the value of n somewhat, but since a truss of this kind is usually quite deep, the value of n will in all cases be large; and the conclusions as to the reactions of the bent considered above will also apply to this bent.

A mill building bent is susceptible of exact analysis, by the use of the slope deflection method applied to the column, combined with a displacement diagram of the truss. The writer made such an analysis of a bent shown in Fig. 55, and obtained an influence line for the reactions. This influence line is shown on Diagram III. The line is seen to agree very closely with the line for n - ½, as determined by the equations for a three sided frame. The irregularity of the upper part of the curve is due to the fact that the column section changes at a point 25 feet from the base, and also that the shortening of the truss is considered in the exact analysis. The vertical component of the wind load on a roof with steep pitch would be considerable. Hence the influence lines for reactions due to vertical loads should also be of use in the analysis of a bent.

The example given above indicates that the influence lines for the reactions of the three sided frame may be applied to any mill building bent, with a considerable degree of accuracy.

E. SYMMETRICAL THREE SIDED FRAMES WITH COLUMNS FIXED AT THE BASE.

38. FRAME WITH LOADS ON TOP.

(a). Concentrated Load at any Point. Fig. 56 shows a three sided frame with a load P on AB, at a distance a from the point A.

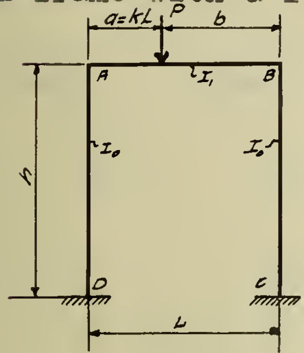


Fig. 56.

This frame may be considered as a special case of the rectangular frame, in which the moment of inertia of the bottom member is infinite. In equations 35, 35a, 35b, and 35c, for the rectangular frame, if $I_3 = \infty$, $p = I_1/I_3 = 0$. Substituting $p = 0$ in these four equations, gives

$$M_A = \frac{Pab}{2L}\left[-\frac{2}{n+2} + \left(\frac{2k-1}{6n+1}\right)\right] \quad \ldots\ldots (61).$$

$$M_B = \frac{Pab}{2L}\left[-\frac{2}{n+2} - \left(\frac{2k-1}{6n+1}\right)\right] \quad \ldots\ldots (61a).$$

$$M_C = \frac{Pab}{2L}\left[\frac{1}{n+2} - \left(\frac{2k-1}{6n+1}\right)\right] \quad \ldots\ldots (61b).$$

$$M_D = \frac{Pab}{2L}\left[\frac{1}{n+2} + \left(\frac{2k-1}{6n+1}\right)\right] \quad \ldots\ldots (61c).$$

(b). Concentrated Load at Middle. In the equations of the preceding paragraph, let $a = b = \frac{1}{2}L$, and $k = \frac{1}{2}$. This gives

$$M_A = M_B = \frac{PL}{8}\left[\frac{-2}{n+2}\right] = \frac{F}{L}\left[\frac{-2}{n+2}\right] \ldots\ldots (62).$$

$$M_C = M_D = \frac{PL}{8}\left[\frac{1}{n+2}\right] = \frac{F}{L}\left[\frac{1}{n+2}\right] \ldots\ldots (62a).$$

For any other symmetrical vertical loading, see Table III, page 29, for values of F/L.

39. FRAME WITH HORIZONTAL LOADS ON COLUMN.

(a). Concentrated Load at any Point. Fig. 57 represents a frame with a concentrated load P on the column AD, at a distance a from the point D. Reducing equations 40, 40a, 40b, and 40c, for the rectangular frame as in paragraph 38, by making p=0, gives

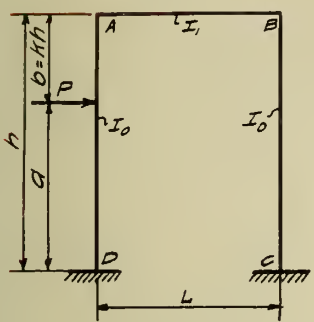


Fig. 57.

$$M_A = \frac{Pa}{2}(1-k)\left[\frac{3n}{6n+1} - \frac{kn}{n+2}\right] \quad \ldots\ldots\ldots (63).$$

$$M_B = \frac{Pa}{2}(1-k)\left[-\frac{3n}{6n+1} - \frac{kn}{n+2}\right] \quad \ldots\ldots\ldots (63a).$$

$$M_C = \frac{Pa}{2}\left[\frac{3n(1+k)+1}{6n+1} - \frac{k(1+k+kn)}{n+2}\right] \quad \ldots\ldots (63b).$$

$$M_D = \frac{Pa}{2}\left[-\frac{3n(1+k)+1}{6n+1} - \frac{k(1+k+kn)}{n+2}\right] \quad \ldots\ldots (63c).$$

(b). Concentrated Load at Top of Column. In equations 63, 63a, 63b, and 63c, let k = 0, and a = h. This gives

$$M_A = \frac{Ph}{2}\left[\frac{3n}{6n+1}\right] \quad \ldots\ldots\ldots\ldots (64).$$

$$M_B = \frac{Ph}{2}\left[\frac{-3n}{6n+1}\right] \quad \ldots\ldots\ldots\ldots (64a).$$

$$M_C = \frac{Ph}{2}\left[\frac{3n+1}{6n+1}\right] \quad \ldots\ldots\ldots\ldots (64b).$$

$$M_D = \frac{Ph}{2}\left[-\frac{3n+1}{6n+1}\right] \quad \ldots\ldots\ldots\ldots (64c).$$

(c). Uniformly Distributed Load on one Column. The total load on the column = W = wh, where w is the load per lineal foot. Substituting p = 0 in equations 43, 43a, 43b, and 43c, gives

$$M_A = \frac{Wh}{12}\left[\frac{-n}{2(n+2)} + \frac{6n}{6n+1}\right] \quad \ldots\ldots (65).$$

$$M_B = \frac{Wh}{12}\left[-\frac{n}{2(n+2)} - \frac{6n}{6n+1}\right] \quad \ldots\ldots (65a).$$

$$M_C = \frac{Wh}{12}\left[\frac{3(4n+1)}{6n+1} - \frac{n+3}{2(n+2)}\right] \quad \ldots (65b).$$

$$M_D = \frac{Wh}{12}\left[-\frac{3(4n+1)}{6n+1} - \frac{n+3}{2(n+2)}\right] \quad \ldots (65c).$$

(d). Uniformly Distributed Load on Both Columns. Each column carries a load W. The loadings on the two columns are opposite in direction. Substituting p = 0 in equations 44 and 44a, gives

$$M_A = M_B = \frac{Wh}{12}\left[\frac{-n}{n+2}\right] \quad \ldots\ldots\ldots (66).$$

$$M_C = M_D = \frac{Wh}{12}\left[-\frac{n+3}{n+2}\right] \quad \ldots\ldots\ldots (66a).$$

(e). Hydraulic Load on one Column. See Fig. 58. The unit pressure varies from w lb. per lineal ft. as the base to zero at the top. The total load $W = \frac{1}{2}wh$. Letting p = 0 in equations 47, 47a, 47b, and 47c, gives

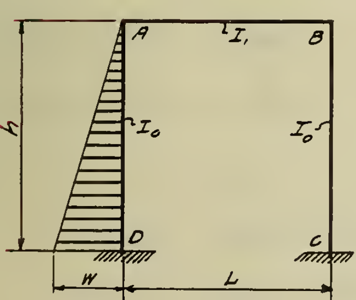


Fig. 58.

$$M_A = \frac{Wh}{12}\left[-\frac{2n}{5(n+2)} + \frac{3n}{6n+1}\right] \quad \ldots\ldots (67).$$

$$M_B = \frac{Wh}{12}\left[-\frac{2n}{5(n+2)} - \frac{3n}{6n+1}\right] \quad \ldots\ldots (67a).$$

$$M_C = \frac{Wh}{12}\left[-\frac{3n+8}{5(n+2)} + \frac{9n+2}{6n+1}\right] \quad . . . (67b).$$

$$M_D = \frac{Wh}{12}\left[-\frac{3n+8}{5(n+2)} - \frac{9n+2}{6n+1}\right] \quad . . . (67c).$$

(f). Hydraulic Load on Both Columns. W represents the load on each column. Substituting p = 0 in equations 48 and 48a, gives

$$M_A = M_B = \frac{Wh}{12}\left[-\frac{4n}{5(n+2)}\right] \quad . . . . . (68).$$

$$M_C = M_D = \frac{Wh}{12}\left[-\frac{2(3n+8)}{5(n+2)}\right] \quad . . . (68a).$$

40. INFLUENCE LINES FOR REACTIONS AND MOMENTS . HORIZONTAL LOADS ON THE COLUMNS. Influence lines similar to those of paragraph 37 are given for this type of frame in Diagrams IV and V. As in the previous case, the effect of a variation of n from 1 to ∞ is seen to be small in comparison to the effect of a slight change in the position of the load. The variation in the position of the point of contraflexure in each column is also shown in Diagram V . Equations 63, 63a, 63b, and 63c were used in calculating the moments; and from the values obtained, the horizontal reactions at the point of contraflexure and the positions of the points of contraflexure were obtained by applying the relations of static equilibrium. Table X gives the value of all moments in terms of Ph/2; Table XI gives the Value of the horizontal reactions in terms of $\frac{1}{2}$P; and Table XII gives the height to the point of contraflexure of each column in terms of the total height, h.

[illegible]

TABLE X.

Values of Moments for Different Values of k and n.

Moment in terms of $\frac{Ph}{2}$	k	Values of n.				
		1	2	5	10	∞
	.0	.428	.461	.483	.491	.5000
	.10	.320	.333	.333	.331	.324
	.20	.232	.232	.219	.208	.192
	.30	.161	.152	.131	.120	.098
	.40	.106	.096	.073	.059	.036
M_A	.50	.065	.053	.031	.023	.000
	.60	.037	.025	.007	-.002	-.016
	.70	.018	.010	-.001	-.007	-.018
	.80	.006	.003	-.003	-.007	-.012
	.90	.001	.000	-.001	-.002	-.004
	1.00	.000	.000	.000	.000	.000
	Uniform Load.	.115	.112	.101	.095	.083
	0	.428	.561	.483	.491	.500
	.10	.374	.415	.450	.465	.486
	.20	.318	.360	.401	.421	.448
	.30	.259	.300	.342	.363	.392
	.40	.202	.238	.276	.297	.324
$-M_B$	.50	.149	.178	.210	.223	.250
	.60	.100	.122	.146	.159	.176
	.70	.069	.073	.089	.095	.108
	.80	.028	.034	.042	.046	.052
	.90	.007	.009	.011	.012	.014
	1.00	.000	.000	.000	.000	.000
	Uniform Load.	.171	.196	.221	.233	.250

TABLE X,-Continued.

Value of Moments for Different Values of k and n.

Moment in terms of $\frac{Ph}{2}$	k	Values of n. 1	2	5	10	∞
M_C.	.0	.571	.539	.517	.509	.500
	.10	.517	.496	.488	.486	.486
	.20	.450	.440	.440	.443	.448
	.30	.378	.374	.379	.382	.392
	.40	.302	.301	.309	.314	.324
	.50	.226	.228	.236	.241	.250
	.60	.154	.158	.166	.169	.176
	.70	.094	.096	.100	.104	.108
	.80	.044	.046	.048	.050	.052
	.90	.012	.012	.013	.013	.014
	1.00	.000	.000	.000	.000	.000
	Uniform Load	.246	.242	.243	.246	.250
$-M_D$	.0	.571	.539	.517	.509	.500
	.10	.589	.555	.529	.518	.504
	.20	.599	.568	.540	.528	.512
	.30	.603	.574	.548	.535	.518
	.40	.590	.565	.542	.530	.516
	.50	.559	.541	.523	.513	.500
	.60	.508	.495	.481	.474	.464
	.70	.430	.421	.412	.408	.402
	.80	.322	.317	.313	.311	.308
	.90	.180	.179	.177	.177	.176
	1.00	.000	.000	.000	.000	.000
	Uniform Load.	.468	.450	.434	.426	.417

TABLE XI.

Value of Horizontal Reactions for Different Values of k and n.

H. in terms of $\frac{1}{2}$P	k	Values of n. 1	2	5	10	∞
	0	1.0	1.0	1.0	1.0	1.0
	.1	1.109	1.089	1.062	1.048	1.028
	.2	1.232	1.200	1.159	1.136	1.104
	.3	1.363	1.326	1.279	1.255	1.216
	.4	1.496	1.461	1.415	1.389	1.352
H_D	.5	1.625	1.594	1.554	1.536	1.500
	.6	1.746	1.720	1.688	1.672	1.648
	.7	1.847	1.831	1.811	1.801	1.784
	.8	1.928	1.920	1.910	1.904	1.896
	.9	1.981	1.972	1.976	1.975	1.972
	1.0	2.000	2.000	2.000	2.000	2.000
	Uniform Load.	1.583	1.562	1.536	1.521	1.500
	0	1.0	1.0	1.0	1.0	1.0
	.1	.891	.911	.938	.952	.972
	.2	.768	.800	.841	.864	.896
	.3	.637	.674	.721	.745	.784
	.4	.504	.539	.585	.611	.648
H_C	.5	.375	.406	.446	.464	.500
	.6	.254	.280	.312	.328	.352
	.7	.153	.169	.189	.199	.216
	.8	.0718	.080	.090	.096	.104
	.9	.019	.021	.024	.025	.028
	1.0	0	0	0	0	0
	Uniform Load.	.417	.438	.464	.479	.500

TABLE XII.

Position of Points of Contraflexure for Different Values of k and n.

Position in terms of h.	k	Values of n. 1	2	5	10	∞
In Column AD.	.0	.571	.539	.517	.509	.500
	.10	.531	.510	.497	.494	.490
	.20	.485	.473	.466	.465	.464
	.30	.442	.433	.428	.426	.426
	.40	.395	.386	.384	.383	.381
	.50	.344	.339	.337	.334	.333
	.60	.291	.288	.285	.283	.281
	.70	.233	.230	.228	.227	.225
	.80	.167	.165	.164	.163	.1625
	.90	.091	.090	.089	.089	.089
	1.00	.000	.000	.000	.000	.000
	Uniform Load.					
In Column BC.	.0	.571	.539	.517	.509	.500
	.10	.579	.544	.520	.511	.500
	.20	.586	.550	.522	.513	.500
	.30	.593	.554	.525	.513	.500
	.40	.600	.558	.528	.514	.500
	.50	.602	.562	.529	.515	.500
	.60	.606	.564	.530	.516	.500
	.70	.614	.567	.530	.517	.500
	.80	.615	.569	.530	.517	.500
	.90	.619	.571	.532	.518	.500
	1.00	.000	.000	.000	.000	.000
	Uniform Load.	.590	.552	.524	.513	.500

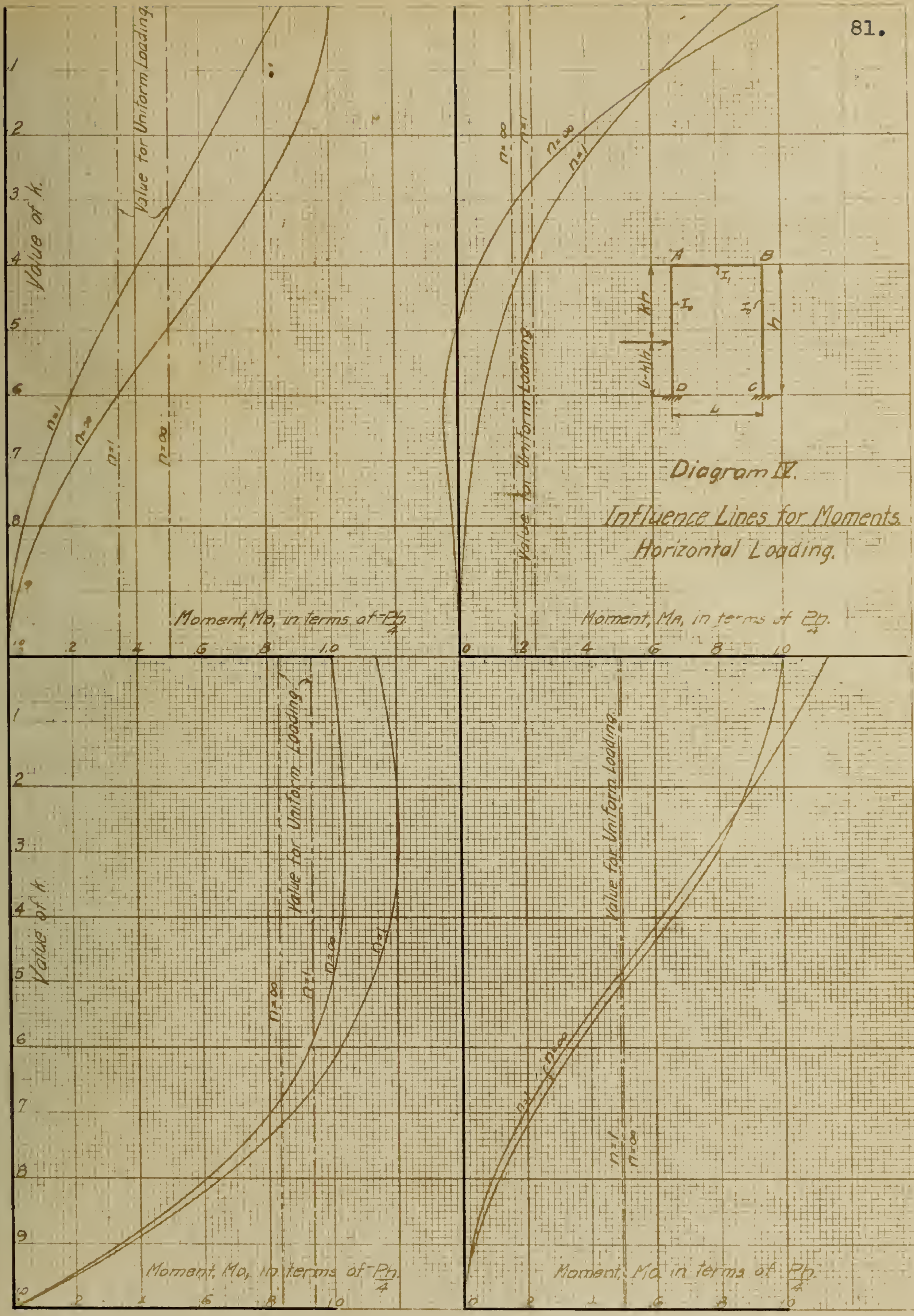


Diagram IV.
Influence Lines for Moments.
Horizontal Loading.
Value of k.
Value for Uniform Loading.
Moment, Mo, in terms of Ph/4
Moment, MA, in terms of Ph/4
Moment, Mo, in terms of Ph/4
Moment, Mo in terms of Ph/4
n=1
n=∞
A
B
C
D
I1
I0
kh
(1-k)h
h
L

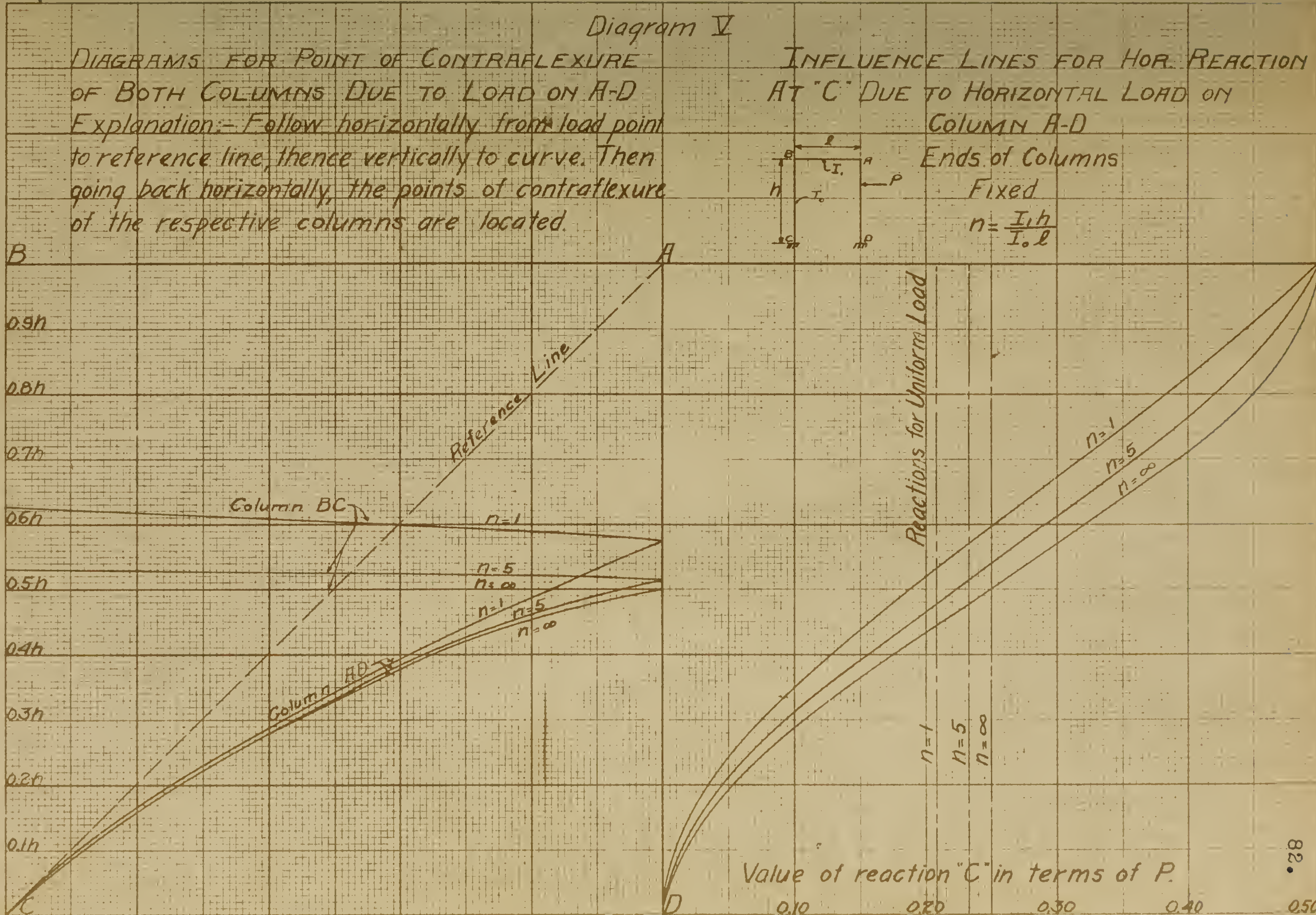
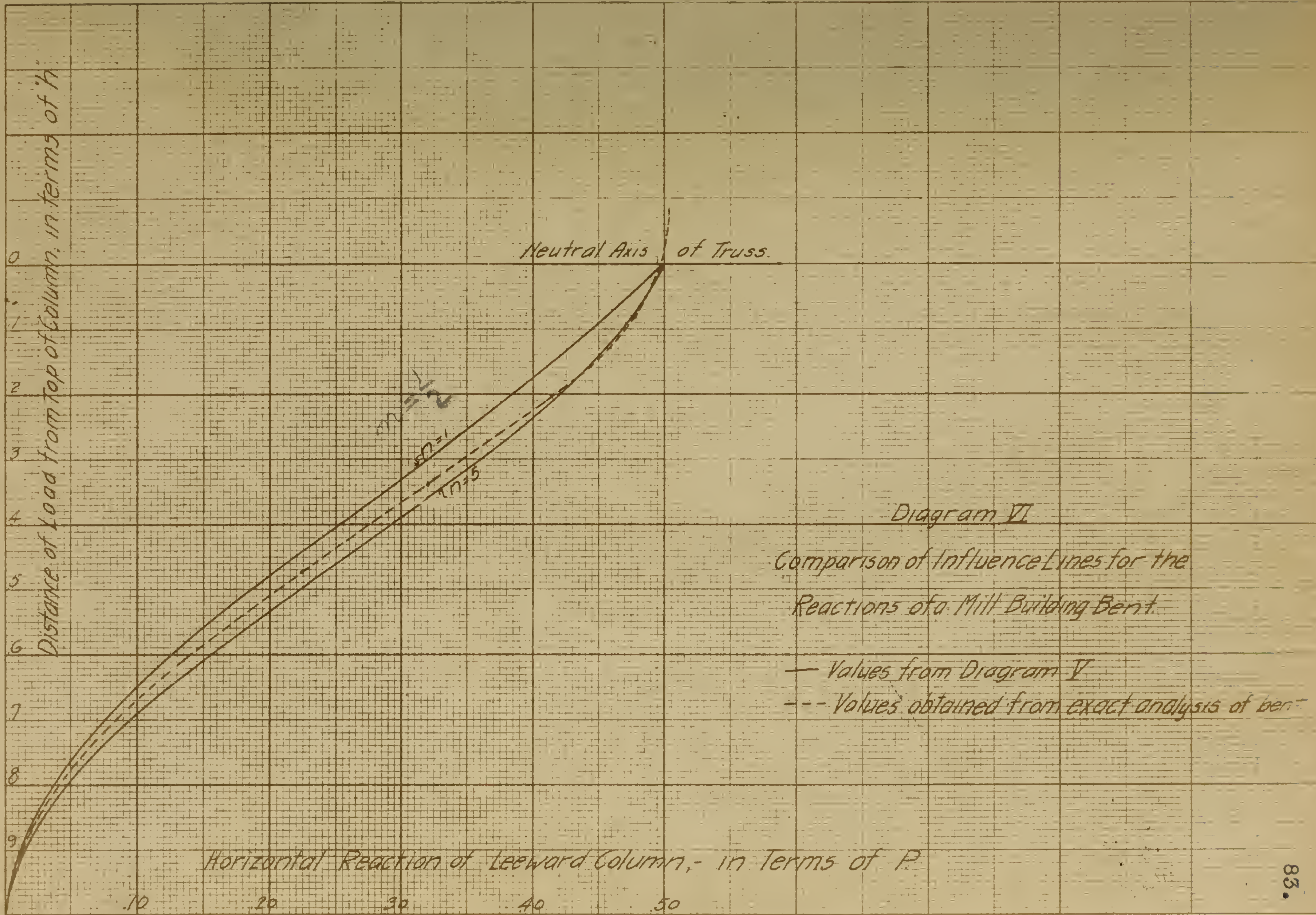


Diagram V
Diagrams for Point of Contraflexure of Both Columns Due to Load on A-D
Explanation:- Follow horizontally from load point to reference line, thence vertically to curve. Then going back horizontally, the points of contraflexure of the respective columns are located.
Influence Lines for Hor. Reaction At "C" Due to Horizontal Load on Column A-D
Ends of Columns Fixed
$n = \frac{I_1 h}{I_0 l}$
B
A
C
D
l
h
P
I_1
I_0
0.9h
0.8h
0.7h
0.6h
0.5h
0.4h
0.3h
0.2h
0.1h
Reference Line
Column BC
Column AD
n=1
n=5
n=∞
Reactions for Uniform Load
Value of reaction "C" in terms of P.
0.10
0.20
0.30
0.40
0.50


Neutral Axis of Truss.
Diagram VI
Comparison of Influence Lines for the
Reactions of a Mill Building Bent.
— Values from Diagram V
- - - Values obtained from exact analysis of ben
n=1
n=5
Distance of Load from Top of Column, in terms of h.
Horizontal Reaction of Leeward Column,- in Terms of P.
0
1
2
3
4
5
6
7
8
9
.10
.20
.30
.40
.50

Diagrams IV and V show the relative magnitude of the moments and reactions, and also the position of the point of contraflexure, as calculated in Tables X, XI, and XII. The value of moments and reactions due to a uniform load, as calculated by equations 65, 65a, 65b, and 65c, are also shown. It is seen that the lines for uniform loading represent the average abscissae of the influence lines, for the respective quantities.

These results may be applied to the bent of a mill building, as was done in paragraph 37. From Diagram V, the maximum moment shown for a uniform loading is is M_D, which has a value of about .24 Ph. P is the total load and h is the height of the frame. In the design of a mill building, it is often assumed that the two horizontal reactions are each equal to $\frac{1}{2}$P, and that the point of contraflexure is at a distance h/3 from the base of the windward column. From this the maximum moment occurs at the point B, and is equal to .333 Ph, which is nearly 40% greater than that found above.

An exact analysis of the bent shown in Fig. 55 was made, considering the columns to be fixed at the bases. Influence lines for the horizontal reactions are shown on Diagram VI, and compared with similar lines from Diagram IV. As in paragraph 37, it follows that the influence lines for the three sided frame may be applied to the bent of a mill building with a very satisfactory degree of accuracy. With roofs of steep pitch, it may be desirable to use influence lines for vertical as well as horizontal loads, to get the best results.

41. NUMERICAL EXAMPLES OF THREE SIDED FRAMES. Fig. 59 shows a bent of an elevated railroad structure, under vertical loads due to two trains. The dimensions and loads are shown in the figure.

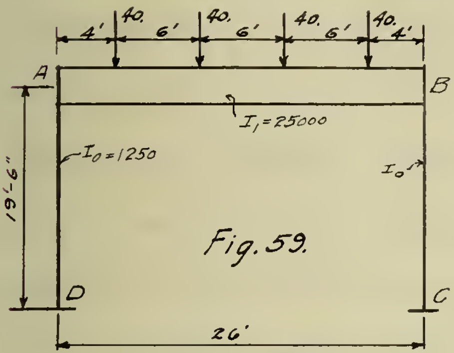


Fig. 59.

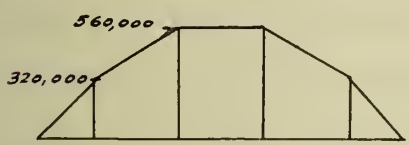


Fig. 60.

$I_1 = 25,000 \text{ in}^4$, and

$I_0 = 1,250 \text{ in}^4$, whence

$n = I_1h/I_0l = 15.$

Referring to Fig. 60, the average ordinate to the moment diagram for a simple beam is equal to F/L. In this case,

$$F/L = \frac{3x560+6x440+4x160}{.013} = 381,500 \text{ ft.lb.}$$

(1). Considering the columns to be hinged at the bases,

$$M_A = \frac{-3}{2n+3}\,\frac{F}{L} = \frac{-3 \times 381,500}{33} = -34,700 \text{ ft.lb.}$$

(2). Considering the columns to be fixed at the bases,

$$M_A = \frac{-2}{n+2}\,\frac{F}{L} = \frac{-2 \times 381,500}{17} = -44,900 \text{ ft.lb.}$$

$$M_D = -\tfrac{1}{2}M_A = 22,450 \text{ ft.lb.}$$

Hence for vertical loads, the greatest bending stress in the columns occurs when the ends of the columns are fixed.

F. THREE LEGGED BENT.

42. THE OPEN TYPE ABUTMENT. The following analysis applies especially to the open type abutment, which has unsymmetrical members and is subjected to horizontal loads. The framed abutment of reinforced concrete is a recent development in railway bridge construction in this country. This structure affords great economy of material and minimum obstruction to the waterway, and also has a very pleasing appearance. Since the abutment has an open cross-section, it does not have to support the lateral pressure of the earth fill behind it; and therefore it is especially suited to conditions requiring a very high abutment. Such abutments used on the Lind Viaduct near Lind, Wash., were made 77 feet in height. These structures are usually unsymmetrical because the columns supporting the bridge seat are necessarily much larger than the rest. The bases of columns may be either hinged or partially fixed.

Paragraphs 25 to 28 give equations for the three legged bent under vertical loads, so the following analysis will consider only the case of a horizontal traction load on the top girder.

43. BENT WITH COLUMNS HINGED AT BASE. HORIZONTAL LOAD AT TOP OF COLUMN. See Fig. 61. It is evident that the horizontal deflections at A, B, and F, are equal. Hence $D = R_0h_0 = R_2h_2 = R_4h_4$. Applying equation 1 to all members , and substituting $Z = 1/K$, gives

$$M_{DA}Z_0 = 2E(2\theta_D+\theta_A-3R_0) = 0. \quad . \quad . \quad . \quad . (a).$$

$$M_{AD}Z_0 = -2E(2\theta_A+\theta_D-3R_0) \quad . \quad . \quad . \quad . \quad . \quad . \quad (b).$$

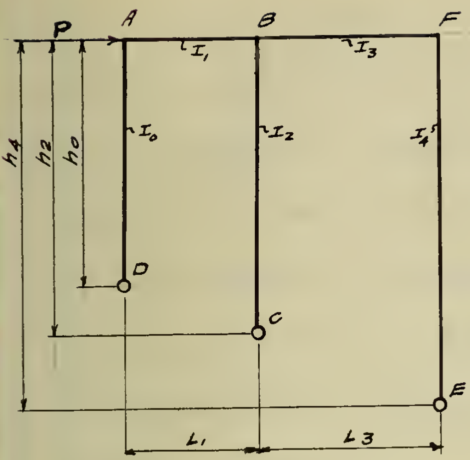


Fig. 61.

$M_{AB}Z_1 = 2E(2\theta_A+\theta_B)$ . . . . (c).

$M_{BA}Z_1 = -2E(2\theta_B+\theta_A)$ . . . . (d).

$M_{CB}Z_2 = 2E(2\theta_C+\theta_B-3R_2) = 0$. . . . . . (e).

$M_{BC}Z_2 = -2E(2\theta_B+\theta_C-3R_2)$ . . . . (f).

$M_{BF}Z_3 = 2E(2\theta_B+\theta_F)$ . . . . (g).

$M_{FB}Z_3 = -2E(2\theta_F+\theta_B)$ . . . . (h).

$M_{FE}Z_4 = 2E(2\theta_F+\theta_E-3R_4)$ . . . . (i).

$M_{EF}Z_4 = -2E(2\theta_E+\theta_F-3R_4) = 0$. . . . . . (j).

Since $M_{AD}/h_0 = H_0$, $M_{BC}/h_2 = H_2$, and $-M_{FE}/h_4 = H_4$,

$$M_{AD}/h_0 + M_{BC}/h_2 - M_{FE}/h_4 = P. \quad . . . . (k).$$

Combining equations (a) to (j), to eliminate all values of θ and R, gives

$M_{AB}Z_1 + 2M_{BA}Z_1 + 2M_{BF}Z_3 + M_{FB}Z_3 = 0$. . . . . (l).

$2M_A h_0(Z_0+Z_1) + M_{BA}Z_1h_0 - 2M_{BC}Z_2h_2 - 2M_{BF}Z_3h_2 - M_{FB}Z_3h_2 = 0$. . (m).

$2M_A h_0(Z_0+Z_1) + M_{BA}Z_1h_0 + 2M_F h_4(Z_3+Z_4) + M_{BF}Z_3h_4 = 0$. . . . . (n).

Substituting $M_{BC} = M_{BF} - M_{BA}$, reduces the number of unknowns in equations (k), (l), (m), and (n) to four. The equations are rewritten in Table XIII.

TABLE XIII.

Equations for Moments in Three Legged Bent.

Columns Hinged at Base.

Equation No.	MA	MBA	MBF	MF	Known Term.
n.	$2h_0(Z_0+Z_1)$	Z_1h_0	Z_3h_4	$2h_4(Z_3+Z_4)$	0
l.	Z_1	$2Z_1$	$2Z_3$	Z_3	0
m.	$2h_0(Z_0+Z_1)$	$Z_1h_0+2Z_2h_2$	$-2h_2(Z_2+Z_3)$	$-Z_3h_2$	0
k.	$\frac{1}{h_0}$	$-\frac{1}{h_2}$	$\frac{1}{h_2}$	$-\frac{1}{h_4}$	P

It is possible to solve these four equations algebraically, but the resulting equations would be quite long and complicated. It is simpler to substitute numerical values of the terms involved, and then to solve the four equations by a process of elimination.

A numerical example is given here, using data given by Mr. N. M. Stineman in a treatment of the same problem by the method of Least Work. See Proc. Western Society of Engineers, Sept, 1914. In Fig. 61, let

$h_4 = 42.5'$	$I_4 = 10.$	$Z_0 = 29.75.$
$h_2 = 34.0'$	$I_2 = 1.$	$Z_2 = 34.00.$
$h_0 = 29.75'$	$I_1 = I_3 = 8.$	$Z_1 = Z_3 = 2.125.$
$L_1 = L_3 = 17.0'$	$P = 120,000.$	$Z_4 = 4.25.$

Putting these values in the equations of Table XIII, the equations are solved as shown in Table XIV, below.

TABLE XIV.

Solution of Numerical Example of Three Legged Bent.

Columns Hinged at Base.

Equation No.	M_A	M_{BA}	M_{BF}	M_F	Known Term.
n	1896.5625	63.2188	90.3125	541.8750	0
l	1.0	2.0	2.0	1.0	0
m	1896.5625	2375.2188	-2456.5000	-72.2500	0
k	.0336	-.0294	+.0294	-.0235	120,000
n	1.0	+.0333	+.0476	+.2857	+3,570,000
l	1.0	+2.0000	+2.0000	+1.0000	0
m	1.0	+1.2524	-1.2952	-.0381	0
k	1.0	-.8750	+.8750	-.7000	0

TABLE XIV.- Continued.

Equation	M_A	M_{BA}	M_{BF}	M_F	Known Term.
l-k (o).	0	+ 2.8750	+ 1.1250	+ 1.7000	-3,570,000
l-n (p).	0	+ 1.9667	+ 1.9524	+ .7143	0
m-n (q).	0	+ 1.2190	- 1.3429	-.3238	0
o		+ 1.0	+ .3913	+.5913	-1,241,740
p		+ 1.0	+ .9927	+ .3632	0
q		+ 1.0	- 1.1015	-.2656	0
o-p (r)		0	+ 1.4928	+ .8569	-1,241,740
p-q (s)		0	+ 2.0942	+.6288	0
r			+ 1.0	+.5740	- 831,785
s			+ 1.0	+.3002	
r-s (t)				+.2738	- 831,785
t				1.0	-3,038,040

From equation (t), M_F = -3,038,040 ft.lb.

From equation (s), M_{BF}= 912,140 ft.lb.

From equation (p), M_{BA}= 197,875 ft.lb.

From equation (l), M_A = 818,010 ft.lb.

Since $M_{BC}=M_{BF}-M_{BA}$, M_{BC}= 714,265 ft.lb.

44. BENT WITH COLUMNS FIXED AT BASE. HORIZONTAL LOAD AT TOP OF COLUMN. Fig. 62 shows a bent similar to that of Fig. 61, but having the columns fixed at the base. The horizontal deflection at A, B, and F, is equal to D. $D=R_0h_0=R_2h_2=R_4h_4$. The slopes at the fixed ends C, D, and E, are equal to zero. Applying equation 1 to each member of the bent, gives the following equations.

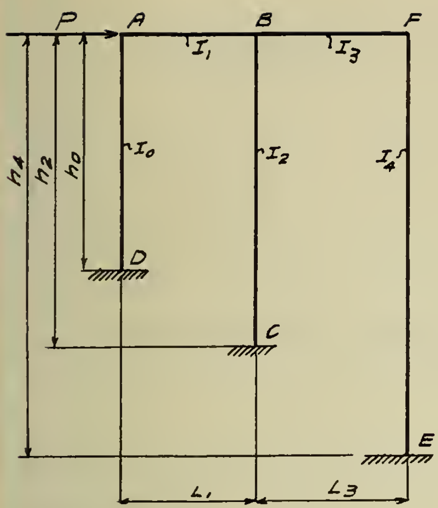


Fig. 62.

$M_{DA} = \ 2EK_0(\theta_A - 3R_0)$ . . . . (a).

$M_{AD} = -2EK_0(2\theta_A - 3R_0)$ . . . . (b).

$M_{AB} = \ 2EK_1(2\theta_A + \theta_B)$ . . . . (c).

$M_{BA} = -2EK_1(2\theta_B + \theta_A)$ . . . . (d).

$M_{CB} = \ 2EK_2(\theta_C - 3R_2)$ . . . . (e).

$M_{BC} = -2EK_2(2\theta_C - 3R_2)$ . . . . (f).

$M_{BF} = \ 2EK_3(2\theta_B + \theta_F)$ . . . . (g).

$M_{FB} = -2EK_3(2\theta_F + \theta_B)$ . . . . (h).

$M_{EF} = \ 2EK_4(\theta_F - 3R_4)$ . . . . (i).

$M_{FE} = -2EK_4(2\theta_F - 3R_4)$ . . . . (j).

Since $M_{AD} - M_{DA} = H_0h_0$, $M_{BC} - M_{CB} = H_2h_2$, and $M_{EF} - M_{FE} = H_4h_4$,

$$\frac{M_{AD} - M_{DA}}{h_0} + \frac{M_{BC} - M_{CB}}{h_2} + \frac{M_{EF} - M_{FE}}{h_4} = P. \quad . \quad . \quad . \quad . \quad (k).$$

Since there are seven unknown moments, the easiest method of solving these equations is to equate moments at each of the joints, and solve for the four unknowns, θ_A, θ_B, θ_C, and D.

At the point A, $M_{AB} - M_{AD} = 0$, whence

$$2(K_0+K_1)\theta_A + K_1\theta_B - (3K_0/h_0)D = 0. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (l).$$

At the point B, $M_{BC} - M_{BF} + M_{BA} = 0$, whence

$$2K_2\theta_B - 3(K_2/h_2)D + 2K_3\theta_B + K_3\theta_F + 2K_1\theta_B + K_1\theta_A = 0. \quad (m).$$

At the point F, $M_{FE} - M_{FB} = 0$, whence

$$K_3\theta_B + 2(K_3+K_4)\theta_F - (3K_4/h_4)D = 0. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (n).$$

Expressing equation (k) in terms of θ and R, gives

$$3(K_0/h_0)\theta_A - 6(K_0/h_0^2)D + 3(K_2/h_2)\theta_B - 6(K_2/h_2^2)D$$

$$+ 3(K_4/h_4)\theta_F - 6(K_4/h_4^2)D = -P/2E. \quad . \quad . \quad . \quad (o).$$

Equations (l), (m), (n), and (o), are written below in Table XV.

TABLE XV.

Equations for Slopes and Deflections in Three Legged Bent.

Columns Fixed at Base.

Equation No.	θ_A	θ_B	θ_F	D	Known Term.
o.	$\frac{K_0}{h_0}$	$\frac{K_2}{h_2}$	$\frac{K_4}{h_4}$	$-2\left[\frac{K_0}{h_0^2}+\frac{K_2}{h_2^2}+\frac{K_4}{h_4^2}\right]$	$-\frac{P}{6E}$
m.	K_1	$2[K_1+K_2+K_3]$	K_3	$-3\ \frac{K_2}{h_2}$	0
l.	$2[K_1+K_0]$	K_1		$-3\ \frac{K_0}{h_0}$	0
n.		K_3	$2[K_3+K_4]$	$-3\ \frac{K_4}{h_4}$	0

These equations are to be solved by putting in the numerical values of K and h, and solving the equations by elimination. After finding the values of θ_A, θ_B, θ_F, and D, it remains to substitute these values in equations (a) to (j) to obtain the values of the moments.

A numerical example for the case of fixed column ends is given here. The dimensions of the bent and the loading are the same as those used in paragraph 43, page 87. In addition to the dimensions given on page 87, the following quantities are used.

$K_0 = .03361$. $K_2 = .02941$. $K_4 = .2353$. $K_1 = K_3 = .4706$.

For convenience in calculation let E = 20,000. After substituting these values in the equations of Table XV, the resulting equations are solved by elimination, as shown in Table XVI.

TABLE XVI.

Solution of Numerical Example of Three Legged Bent.

Columns Fixed at Base.

Equation No.	θ_A	θ_B	θ_F	D	Known Term.
o.	1.1297	.8650	5.5360	-.3873	-1.0
m.	.4706	1.9412	.4706	-.0026	0
l.	1.0084	.4706		-.0034	0
n.		4706	1.4118	-.0166	0
o.	1.0	+.7656	+4.9004	-.3428	-.8852
m.	1.0	+4.1249	+ 1.0000	-.0055	0
l.	1.0	+ .4666		-.0034	0
o-m. (p)	0	- 3.3593	+ 3.9004	-.3373	-.8852
m-l. (q)	0	+ 3.6583	+ 1.0000	-.0021	0
p.		+ 1.0	- 1.1611	+.1004	+.2635
q.		+ 1.0	+ .2733	-.0006	0
n.		+ 1.0	+ 3.0000	-.0353	0
n-p. (r).		0	+ 4.1611	-.1357	-.2635
n-q. (s.)		0	+ 2.7267	-.0347	0
r.			+ 1.0	-.0326	-.0633
s.			+ 1.0	-.0127	0
r-s. (t)			0	-.0199	-.0633
t.				+ 1.0	+3.1833
s.			+1.0		+.0405
n.		+1.0			-.0091
o.	+1.0				+.0148

Substituting these values of θ and R in equations (a) to (j), gives

M_{DA} = -411,000 ft.lb. M_{AD} = 392,100 ft.lb.

M_{CB} = -341,820 ft.lb. M_{BA} = 64,860 ft.lb.

M_{BF} = 417,600 ft.lb. M_{BC} = 352,080 ft.lb.

M_{FE}=-1,353,000 ft.lb. M_{EF}=-1731,000 ft.lb.

G. SYMMETRICAL BENT OF A BUILDING UNDER VERTICAL LOADS.

45. THE SYMMETRICAL BENT OF A BUILDING. The bent of a building several stories in height is statically indeterminate to a high degree. The bending stresses in columns of steel frames due to vertical loads are not generally considered, because of the somewhat flexible girder connections. However, in monolithic concrete structures, the joints are known to be rigid. It has been recognized that a loading of alternate girders in a building produces large bending stresses in the columns, but the writer has seen no exact analysis of the problem. In the following paragraph, the moments due to vertical loads in a bent of four columns and five stories are determined. The method may be applied to any type of bent.

46. GENERAL EQUATIONS FOR A FOUR COLUMN, FIVE STORY BENT. MEMBERS AND LOADS SYMMETRICAL ABOUT VERTICAL CENTER LINE OF BENT. Because of the complexity of the frame, the usual conventions for the sign of bending moments are hard to apply to this case. Hence, the resisting couple acting upon the end of a member will be used as the statically indeterminate quantity to be determined. The moment of the couple is considered positive when it acts in a clockwise direction. With this convention, the fundamental equation 4 may be rewritten for a beam BA, carrying a uniformly distributed load, W. See Fig. 63. The couple at B is equal to

the bending moment at that point, but the couple at A is equal to M_A, but of opposite sign. The equations for the couples at A and B, are

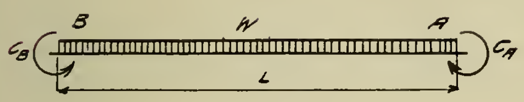


Fig. 63.

$$C_B = 2EK(2\theta_B + \theta_A - 3R) - WL/12.$$

$$C_A = 2EK(2\theta_A + \theta_B - 3R) + WL/12.$$

The general equations for the building are written in such a form as to be applicable to different conditions of loading. The numerical subscripts shown in Fig.64. indicate different intensities of loading. By making certain of these intensities equal to zero, the equations for the different cases of loading are determined.

In Fig. 64, the columns are fixed at their bases, so that the slopes at these points are equal to zero. The notation used to designate the different joints of the frame is shown in the figure. Writing the equations for the resisting couples acting upon each member, and equating the sum of the couples at each joint to zero, gives

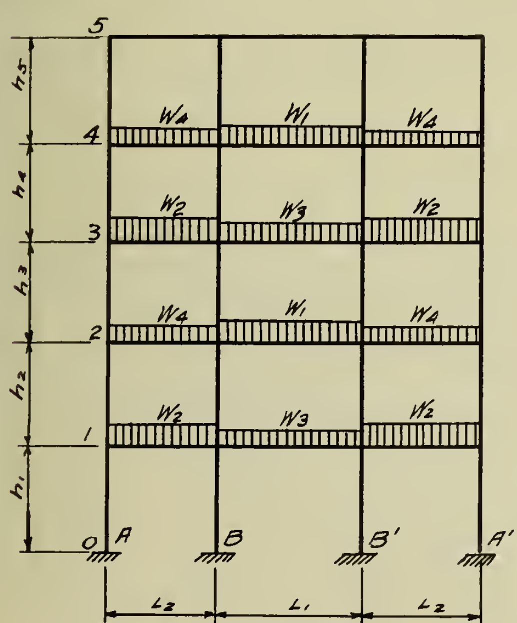


Fig. 64.

At the point A_1, $C_{A10} + C_{AB1} + C_{A12} = 0 = 2K_{A1}\theta_{A1} + K_{AB1}(2\theta_{A1} + \theta_{B1}) - \frac{W_2 L_2}{24E} + K_{A2}(2\theta_{A1} + \theta_{A2})$.

or $(2K_{A1} + 2K_{A2} + 2K_{AB1})\theta_{A1} + K_{AB1}\theta_{B1} + K_{A2}\theta_{A2} = \frac{W_2 L_2}{24E}$. (a.)

Similarly, at A_2, $(2K_{A2}+2K_{AB_2}+2K_{A3})\theta_{A2} + K_{A2}\cdot\theta_{A1} + K_{AB_2}\cdot\theta_{B_2} + K_{A3}\cdot\theta_{A3} = \frac{W_4L_2}{24E}$. (b).

At A_3, $(2K_{A3}+2K_{AB_3}+2K_{A4})\theta_{A3} + K_{A3}\cdot\theta_{A2} + K_{AB_3}\cdot\theta_{B_3} + K_{A4}\cdot\theta_{A4} = \frac{W_2L_2}{24E}$ (c).

At A_4, $(2K_{A4}+2K_{AB4}+2K_{A5})\theta_{A4} + K_{A4}\theta_{A3} + K_{AB4}\theta_{B4} + K_{A5}\cdot\theta_{A5} = \frac{W_4L_2}{24E}$ (d).

At A_5, $(2K_{A5}+2K_{AB5})\theta_{A5} + K_{A5}\cdot\theta_{A4} + K_{AB5}\cdot\theta_{B5} = 0$ (e).

Similarly, at B_1, $(2K_{B_1}+2K_{AB_1}+2K_{B_2}+2K_{BB_1'})\theta_{B_1} + K_{B_1}\theta_{B0} + K_{AB_1}\theta_{A_1} + K_{B_2}\theta_{B_2} + K_{BB_1'}\theta_{B_1'} = \frac{W_3L_1-W_2L_2}{24E}$ (f).

At B_2, $(2K_{B_2}+2K_{AB_2}+2K_{B_3}+2K_{BB_2'})\theta_{B_2} + K_{B_2}\theta_{B_1} + K_{AB_2}\theta_{A_2} + K_{B_3}\theta_{B_3} + K_{BB_2'}\theta_{B_2'} = \frac{W_1L_1-W_4L_2}{24E}$ (g).

At B_3, $(2K_{B_3}+2K_{AB_3}+2K_{B_4}+2K_{BB_3'})\theta_{B_3} + K_{B_3}\theta_{B_2} + K_{AB_3}\theta_{A_3} + K_{B_4}\theta_{B_4} + K_{BB_3'}\theta_{B_3'} = \frac{W_3L_1-W_2L_2}{24E}$ (h).

At B_4, $(2K_{B4}+2K_{AB4}+2K_{B5}+2K_{BB_4'})\theta_{B4} + K_{B4}\theta_{B3} + K_{AB_4}\theta_{A4} + K_{B5}\theta_{B5} + K_{BB_4'}\theta_{B_4'} = \frac{W_1L_1-W_4L_2}{24E}$ (i).

And at B_5, $(2K_{B5}+2K_{AB5}+2K_{BB_5'})\theta_{B5} + K_{B5}\theta_{B4} + K_{AB5}\theta_{A5} + K_{BB_5'}\theta_{B_5'} = 0$. (j)

From the symmetry of the frame and loading, $\theta_{B'} = -\theta_B$ Also, letting twice the sum of the K's for all members meeting at a joint equal a constant, J, equations (a) to (j) may be written in simpler form, as shown in Table XVII.

TABLE XVII.

General Equations for Five Story Symmetrical Bent. Vertical Loads.

Equation No.	θ_{A1}	θ_{B1}	θ_{A2}	θ_{B2}	θ_{A3}	θ_{B3}	θ_{A4}	θ_{B4}	θ_{A5}	θ_{B5}	Known Term.
a.	J_{A1}	K_{AB1}	K_{A2}								$\frac{W_2L_2}{24E}$
f.	K_{AB1}	$J_{B1}-K_{BB_1'}$		K_{B2}							$\frac{W_3L_1-W_2L_2}{24E}$
b.	K_{A2}		J_{A2}	K_{AB2}	K_{A3}						$\frac{W_4L_2}{24E}$
g.		K_{B2}	K_{AB2}	$J_{B2}-K_{BB_2'}$		K_{B3}					$\frac{W_1L_1-W_4L_2}{24E}$
c.			K_{A3}		J_{A3}	K_{AB3}	K_{A4}				$\frac{W_2L_2}{24E}$
h.				K_{B3}	K_{AB3}	$J_{B3}-K_{BB_3'}$		K_{B4}			$\frac{W_3L_1-W_2L_2}{24E}$
d.					K_{A4}		J_{A4}	K_{AB4}	K_{A5}		$+\frac{W_4L_2}{24E}$
i.						K_{B4}	K_{AB4}	$J_{B4}, -K_{BB_4'}$		K_{B5}	$\frac{W_1L_1-W_4L_2}{24E}$
e.							K_{A5}		J_{A5}	K_{AB5}	0
j.								K_{B5}	K_{AB5}	$J_{B5}, -K_{BB_5'}$	0

47. SOLUTION OF NUMERICAL CASE OF BUILDING BENT. The general equations of paragraph 46 will now be applied to the numerical case of a bent under five different conditions of loading. Since the values of K for the columns and girders of both steel and reinforced concrete buildings are often about equal, all values of K in this numerical case will be considered equal. Further, since only the relative values of K affect the distribution of bending moments in the structure, all values of K will equal unity. In this numerical case the intensity of loading will be considered equal on all floors, so that $W_1=W_2=W_3=W_4$; and the spans L_1 and L_2 will also be considered equal. Hence, the right hand terms in the equations of Table XVII will be expressed in terms of $W_1L_1/24E$, and all values of θ determined from these equations will be in terms of $W_1L_1/24E$.

The five cases of loading used are represented in Figs.65, 66, 67, 68, and 69. Since the variation of loadings affects only the right hand terms of the general equations, all computations may be carried along together. Substituting numerical values in the equations of Table XVII, it is seen that the values of J are $J_{A1}=J_{A2}=J_{A3}=J_{A4}=6$, $J_{A5}=4$, $J_{B1}=J_{B2}=J_{B3}=J_{B4}=8$, and $J_{B5}=6$. The numerical solution of equations by a process of elimination is given in Table XVIII.

[illegible]

CASE I.

Fig. 65

CASE 2.

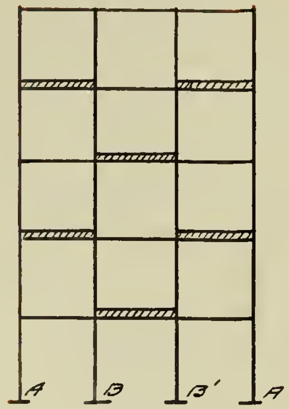


Fig. 66.

CASE 3.

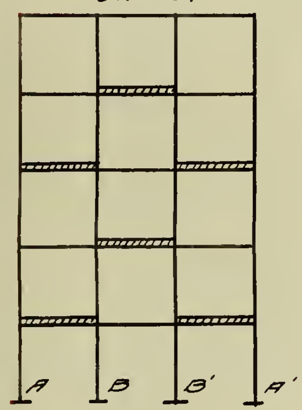


Fig. 67.

CASE 4.

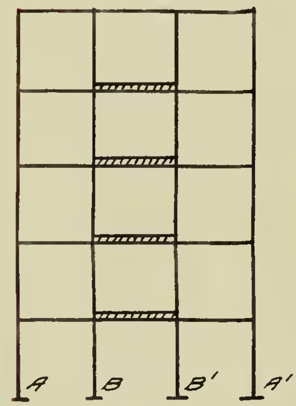


Fig 68.

CASE 5.

Fig. 69.

TABLE XVIII.

Solution of Equations for Building Bent.

Eq. No	θ_{A_1}	θ_{B_1}	θ_{A_2}	θ_{B_2}	θ_{A_3}	θ_{B_3}	θ_{A_4}	θ_{B_4}	θ_{A_5}	θ_{B_5}	Known Terms. Coefficient of $W.L./24E$ Case 1	Case 2.	Case 3.	Case 4	Case 5
1	6	1	1								1.0	0	1.0	0	1.0
2	1	7		1							0	1.0	-1.0	1.0	-1.0
3	1		6	1	1						1.0	1.0	0	0	1.0
4		1	1	7		1					0	-1.0	1.0	1.0	-1.0
5			1		6	1	1				1.0	0	1.0	0	1.0
6				1	1	7		1			0	1.0	-1.0	1.0	-1.0
7					1		6	1	1		1.0	1.0	0	0	1.0
8						1	1	7		1	0	-1.0	1.0	1.0	-1.0
9							1		4	1	0	0	0	0	0
10								1	1	5	0	0	0	0	0
1	1.0	+.1667	+.1667								+.1667	0	+.1667	0	+.1667
2	1.0	+7.0		1.0							0	1.0	-1.0	1.0	-1.0
3	1.0		6.0	1.0	1.0						1.0	1.0	0	0	1.0
1-2	0	-6.8333	+.1667	-1.0							+.1667	-1.0	+1.1667	-1.0	+1.1667
1-3	0	+.1667	-5.8333	-1.0	-1.0						-.8333	-1.0	+.1667	0	-.8333
a		-1.0	+.0244	-.1463							+.0244	-.1463	+.1707	-.1463	+.1707
b		+1.0	-35.0	-6.0	-6.0						-5.0	-6.0	+1.0	0	-5.0
4		+1.0	+1.0	+7.0		+1.0					0	-1.0	+1.0	+1.0	-1.0
a+b		0	-34.9756	-6.1463	-6.0						-4.9756	-6.1463	+1.1707	-.1463	-4.8293
a+4		0	+1.0244	+6.8537		+1.0					+.0244	-1.1463	+1.1707	+.8537	-.8293
c			-1.0	-.1757	-.1716						-.1423	-.1757	+.0335	-.0042	-.1381
d			+1.0	+6.6904		+.9762					+.0238	-1.1190	+1.1428	+.8333	-.8095
5			+1.0		+6.0	+1.0	+1.0				+1.0	0	+1.0	0	+1.0
c+d			0	+6.5147	-.1716	+.9762					-.1185	-1.2947	+1.1763	+.8291	-.9476
c+5			0	-.1757	+5.8284	+1.0	+1.0				+.8577	-.1757	+1.0335	-.0042	+.8619
e				+1.0	-.0263	+.1498					-.0182	-.1987	+.1805	+.1272	-.1454
f				-1.0	+33.1724	+5.6915	+5.6915				+4.8816	-1.000	+5.8822	-.0239	+4.9055
6				+1.0	+1.0	+7.0		+1.0			0	+1.0	-1.0	+1.0	-1.0
e+f				0	+33.1461	+5.8413	+5.6915				+4.8634	-1.1987	+6.0627	+.1033	+4.7601

TABLE XVIII Continued.

Solution of Equations for Building Bent.

Eq. No	θ_{A_1}	θ_{B_1}	θ_{A_2}	θ_{B_2}	θ_{A_3}	θ_{B_3}	θ_{A_4}	θ_{B_4}	θ_{A_5}	θ_{B_5}	Known Term. Coefficient of W.L./24E. Case 1	Case 2	Case 3	Case 4	Case 5
e-6				0	-1.0263	-6.8502		-1.0			-.0182	-1.1987	+1.1805	-.8728	+.8546
g					+1.0	+.1762	+.1717				+.1467	-.0362	+.1829	+.0030	+.1436
h					-1.0	-6.6746		-.9744			-.0177	-1.1683	+1.1503	-.8505	+.8326
7					+1.0		+6.0	+1.0	+1.0		+1.0	+1.0	0	0	+1.0
g+h					0	-6.4984	+.1717	-.9744			+.1287	-1.2045	+1.3332	-.8474	+.9762
g-7					0	+.1762	-5.8283	-1.0	-1.0		-.8533	-1.0362	+.1829	+.0030	-.8564
i						-1.0	+.0264	-.1499			+.0199	-.1854	+.2052	-.1304	+.1502
j						+1.0	-33.0766	-5.6754	-5.6754		-4.8428	-5.8808	+1.0380	+.0170	-4.8604
8						+1.0	+1.0	+7.0		+1.0	0	-1.0	+1.0	+1.0	-1.0
i+j						0	-33.0502	-5.8253	-5.6754		-4.8229	-6.0662	+1.2432	-.1134	-4.7102
i+8						0	+1.0264	+6.8501		+1.0	+.0199	-1.1854	+1.2052	+.8696	-.8498
k							-1.0	-.1763	-.1717		-.1459	-.1836	+.0376	-.0034	-.1423
l							+1.0	+6.6739		+.9743	+.0194	-1.1549	+1.1742	+.8492	-.8279
9							+1.0	0	+4.0	+1.0	0	0	0	0	0
k-l							0	+6.4976	-.1717	+.9743	-.1265	-1.3385	+1.2118	+.8438	-.9702
k-9							0	-.1763	+3.8283	+1.0	-.1459	-.1836	+.0376	-.0034	-.1423
m								+1.0	-.0264	+.1499	-.0195	-.2060	+.1865	+.1298	-.1493
n								-1.0	+21.7147	+5.6721	-.8276	-1.0414	+.2133	-.0193	-.8071
10								+1.0	+1.0	+5.0	0	0	0	0	0
m+n								0	+21.6883	+5.8220	-.8471	-1.2474	+3998	+1105	-9564
n-10								0	-1.0264	-4.8501	-.0195	-.2060	+.1865	+.1298	-.1493
o									+1.0	+.2684	-.0391	-.0575	+.0185	+.0051	-.0441
p									-1.0	-4.7253	-.0190	-.2007	+.1817	+.1264	-.1457
o-p									0	-4.4569	-.0581	-.2582	+2002	+.1315	-.1898
										+1.0	+.0130	+.0579	-.0447	-.0295	+.0426
											See Table XIX for Values of θ				

Values of θ, as determined for the five different cases of loading, are given below in Table XIX. Substituting these values of θ back in the original equations gives the values of the resisting couples at the ends of all members. The numerical values thus found are coefficients of $W_1L_1/12$. Hence , for all girders, these coefficients express the degree of restraint at the ends of the member.

The values of the resisting couples in columns and girders of the bent are given in Tables XX and XXI.

Diagrams VII and VIII show the bending moments in columns A and B, for all stories and all cases of loading.

TABLE XIX.

Values of θ, in terms of $W_1L_1/24E$.

θ	Case 1	Case 2	Case 3	Case 4	Case 5
θ_{B5}	.0130	.0579	-.0447	-.0295	.0426
θ_{A5}	-.0426	-.0731	.0305	.0130	-.0559
θ_{B4}	-.0225	-.2166	.1940	.1345	-.1572
θ_{A4}	.1572	.2344	-.0770	-.0225	.1796
θ_{B3}	-.0124	.2241	-.2363	.1096	-.1219
θ_{A3}	.1219	-.1159	.2377	-.0124	.1343
θ_{B2}	-.0131	-.2353	.2222	.1105	-.1236
θ_{A2}	.1237	.2369	-.1133	-.0131	.1365
θ_{B1}	-.0195	.1865	-.2061	.1298	-.1493
θ_{A1}	.1494	-.0706	.2200	-.0194	.1688

TABLE XX

Resisting Couples in Columns.

Resisting Couple.	Value of Resisting Couple in Terms of $W_1L_1/12$				
	Case 1	Case 2	Case 3	Case 4	Case 5
C_A 5-4	.0720	.0882	-.0160	.0035	.0678
C_A 4-5	.2718	.3957	-.1235	-.0320	.3033
C_A 4-3	.4363	.3529	.0837	-.0574	.4935
C_A 3-4	.4010	.0026	.3984	-.0473	.4482
C_A 3-2	.3675	.0051	.3621	-.0379	.4051
C_A 2-3	.3683	.3579	.0111	-.0386	.4073
C_A 2-1	.3968	.4032	-.0066	-.0456	.4418
C_A 1-2	.4225	.0957	.3267	-.0519	.4741
C_A 1-0	.2988	-.1412	.4400	-.0388	.3376
C_A 0-1	.1494	-.0706	.2200	-.0194	.1688
C_B 5-4	.0035	-.1008	.1046	.0755	-.0720
C_B 4-5	-.0320	-.3753	.3433	.2395	-.2718
C_B 4-3	-.0574	-.2091	.1517	.3786	-.4363
C_B 3-4	-.0473	.2316	-.2786	.3537	-.4010
C_B 3-2	-.0379	.2129	-.2504	.3297	-.3674
C_B 2-3	-.0386	-.2465	.2081	.3306	-.3691
C_B 2-1	-.0457	-.2841	.2383	.3508	-.3965
C_B 1-2	-.0521	.1377	-.1900	.3701	-.4222
C_B 1-0	-.0390	.3730	-.4122	.2596	-.2986
C_B 0-1	-.0195	.1865	-.2061	.1298	-.1493

TABLE XXI.

Resisting Couples in Girders.

Resisting Couple.	Value of Resisting Couple in Terms of $W_1L_1/12$.				
	Case 1	Case 2	Case 3	Case 4	Case 5
C_{AB5}	-.0722	-.0883	.0163	-.0035	-.0692
C_{BA5}	-.0166	.0427	-.0589	-.0460	.0293
C_{AB4}	-.7081	-.7478	.0400	.0895	-.7980
C_{BA4}	1.1122	.8012	.3110	.2465	.8652
C_{AB3}	-.7686	-.0077	-.7609	.0848	-.8533
C_{BA3}	1.0971	.3323	.7651	.2068	.8905
C_{AB2}	-.7657	-.7615	-.0044	.0843	-.8506
C_{BA2}	1.0975	.7663	.3311	.2079	.8893
C_{AB1}	-.7207	.0453	-.7661	.0910	-.8117
C_{BA1}	1.1104	.3024	.8078	.2402	.8702
C_{BB5}	.0130	.0579	-.0447	-.0295	.0426
C_{BB4}	-1.0225	-.2166	-.8060	-.8655	-.1572
C_{BB3}	-1.0124	-.7759	-.2363	-.8904	-.1219
C_{BB2}	-1.0131	-.2353	-.7778	-.8895	-.1236
C_{BB1}	-1.0195	-.8135	-.2061	-.8702	-.1493

Diagram VII.

Bending Moments in Exterior Column "A", in Terms of $\frac{W_1 L_1}{12}$

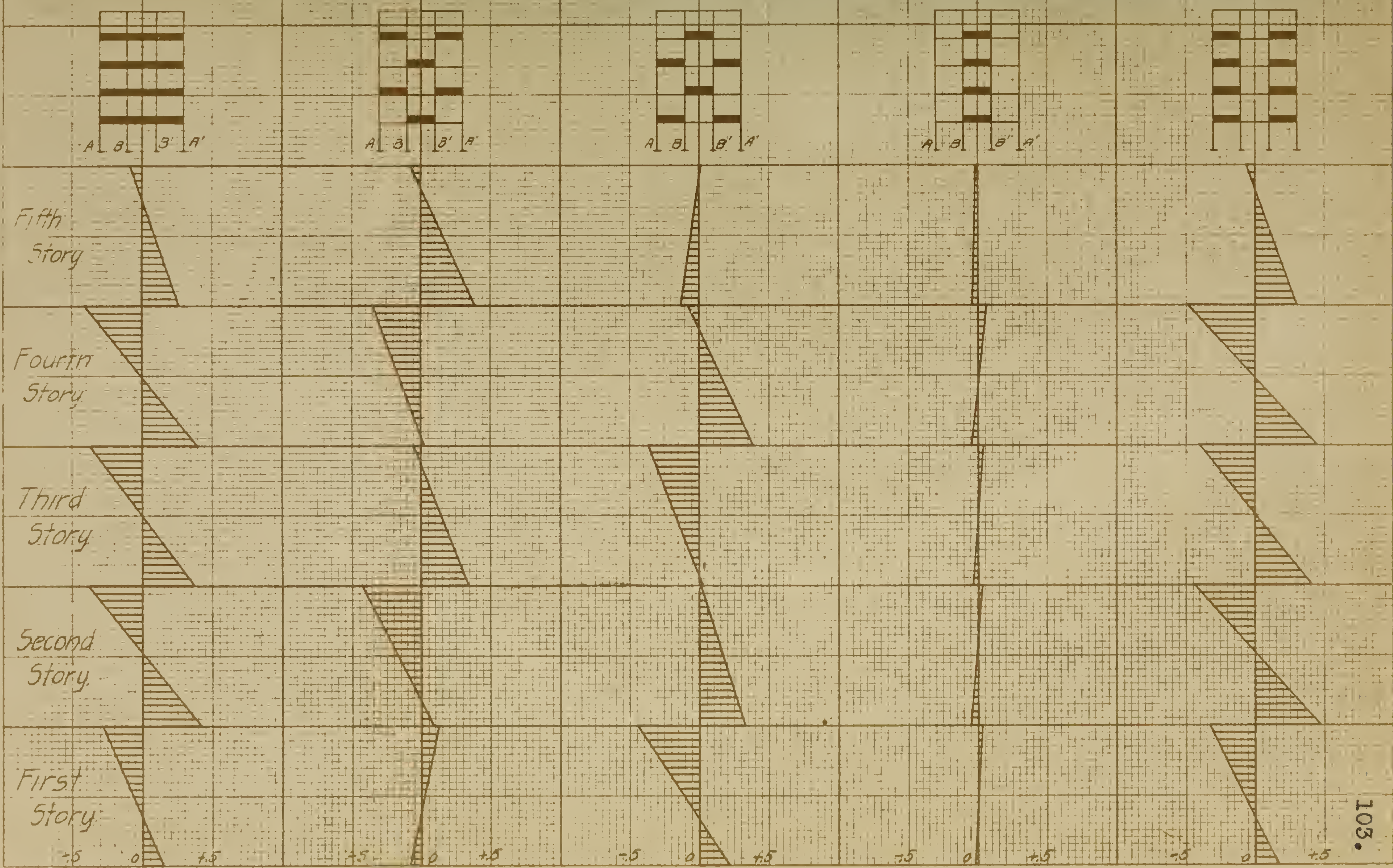
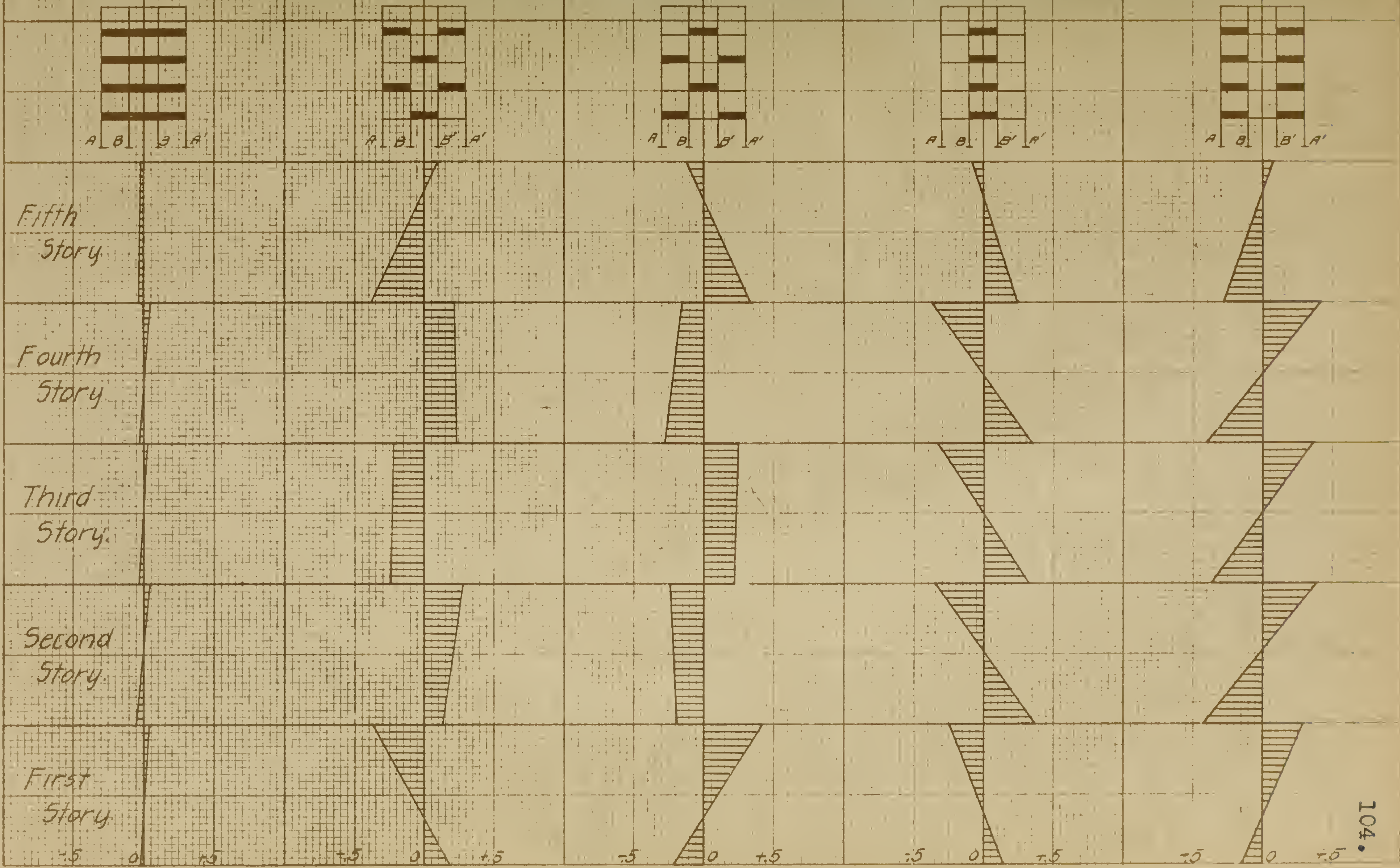


Diagram VIII

Bending Moments in Interior Column "B", in Terms of $\frac{W.L.}{12}$.



48. <u>EFFECT OF ECCENTRIC LOADS ON COLUMNS.</u> For the particular bent analyzed, the following conclusions are drawn:

1. The effect of fixed column ends at the base, and of no load on the roof, is noticeable upon the moments in the columns at the first and fifth stories; but the third story may be regarded as a typical story for a bent having four columns.

2. The exterior columns receive a little greater moment than the interior columns in most cases.

3. The maximum bending moment in any column is about WL/24, or one half as much as the moment which exists at the end of a fixed girder.

4. The moments in the columns change abruptly at the joints, which would cause high bond stresses in a reinforced concrete column.

5. The loading of Cases 4 and 5 produces slightly greater and more variable moments in the columns than that of Cases 2 and 3.

6. The moments in the interior columns, with all spans loaded, are negligible.

7. The negative moments at the ends of loaded girders vary from .7WL/12 to 1.11WL/12. The greatest values occur at joints with the interior columns, when all spans are loaded.

8. While further analysis of bents with unequal spans and varying values of K are needed for a complete treatment of this subject, the above conclusions should apply to ordinary structures of this type.

G. MISCELLANEOUS APPLICATIONS OF FUNDAMENTAL EQUATIONS.

49. DEFLECTIONS. Since the fundamental equations of Table I express the relation between bending moments, slopes, and deflections, it follows that when the moments in a structure have been determined, the equations can be applied to determine the slope and deflection at any point. The few examples which follow illustrate the method to be followed.

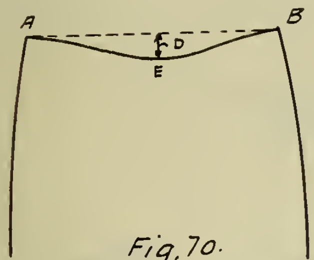


Fig. 70.

Fig. 70 represents any frame in which the members are symmetrical about the vertical center line. Consider the effect of a uniform load W, on the member AB. If the point E represents the middle of the member, θ_E is equal to zero, and the maximum deflection will occur at this point. Since $\theta_A = -\theta_B$, equation 20 of Table III applies to the member AB, so that

$$M_A = 2EK_1\theta_A - F/L, \quad \text{or} \quad \theta_A = \frac{M_A + F/L}{2EK_1} \quad . . . . (a).$$

Applying equation 4 to the portion of the member, AE, gives

$$M_{AE} = \frac{2EI}{(L/2)}\left(2\theta_A - \frac{3D}{(L/2)}\right) - \frac{WL/4}{12}, \text{ which simplifies to}$$

$$M_A = 4EK_1(2\theta_A - 6D/L) - WL/48, \text{ from which}$$

$$D = -\frac{L}{6}\left(\frac{M_A + WL/48}{4EK_1}\right) + \theta_A L/3 \quad . . . . (b).$$

Combining equations (a) and (b), and substituting $F/L = WL/12$, and $M_A = \mu WL/12$, gives the maximum deflection at the middle of the member.

$$D = \frac{WL^2(4\mu + 5)}{384EK_1} \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; (69).$$

For a concentrated load at the middle of the member, a similar procedure is followed. Applying equations 1 and 20, and substituting $PL/8 = F/L$, and $\mu PL/8 = M_A$, gives for the maximum deflection at the center of the member

$$D = \frac{PL^2(3\mu + 4)}{192EK_1} \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; (70).$$

Consider a member carrying no external loads , but acted upon at the ends by two couples of equal magnitude and opposite direction. Then the slopes at the two ends are equal, but opposite in sign, and the maximum deflection D, occurs at mid span. In a manner similar to that of the preceding case, applying equations 1 and 19, gives

$$D = M_A L/8EK_1 \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; (71).$$

Applying equations 69 and 70 to the case of a simple beam, $\mu = 0$, and the deflections are $5WL^2/384EK_1$, and $4PL^2/192EK_1$, for the two conditions of loading. For a beam with fixed ends $\mu = 1$, and the deflections are given by $WL^2/384EK_1$, and $PL^2/192EK_1$, respectively, for the two conditions of loading. The above equations are

especially applicable to the frames of Section B, in which values of μ may be obtained from equations 23 to 33, for the various frames.

The conditions of symmetry of frame and loading made the solution of the above cases comparatively simple. However, when the frame and loading are not symmetrical, a solution of the problem in general terms would be quite difficult. Instead, since the application of the fundamental equations will always provide as many equations as there are unknown quantities, a solution of any particular case may be made by substituting numerical values in the equations, and solving by elimination.

50. INFLUENCE OF CHANGE IN TEMPERATURE. The effect of a change in temperature upon the structures which have been considered in this thesis is very important. It is evident that the expansion of members in a stiff frame will produce bending moments in the members , unless all elongate proportionately; and even then stresses will be produced if the frame is attached rigidly to the foundation. Furthermore, there may be a variation of 20 or 30 degrees in the temperature at various parts of a structure, so that the question becomes still more complex. It is evident that since temperature stresses often amount to quite a large percentage of the dead load stresses, any great refinement in the calculation of the latter may be useless. The point to be emphasized in the application of the fundamental equations to this subject is that

the change in length of a member is taken into account by the term R, or D/L, in the equations for the adjoining members of the frame. The three sided frame will be considered here as an illustration .

Case 1. Symmetrical Three Sided Frame with Columns Hinged at Base. See Fig. 71. Let e represent the coefficient of linear expansion , and let t represent a change of temperature, in degrees. Then etL is the change in the length of the member AB, and eth is the change in the lengths of AD and BC. This change in length will be too small to affect the value of K, in the fundamental equations which will now be applied. From the symmetry of the frame, the deflections at A and B will each be ½etL. Applying equations 1 and 19, gives

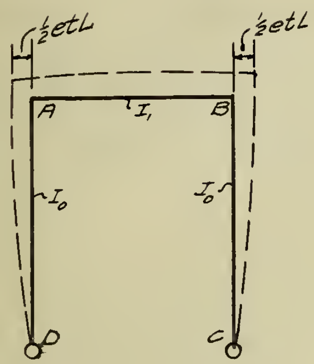


Fig. 71.

$$M_{AB} = 2EK_1\theta_A \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (a).$$

$$M_{AD} = -2EK_0(2\theta_A + \theta_D - 3D/h) \quad . \quad . \quad . \quad (b).$$

$$M_{DA} = 2EK_0(2\theta_D + \theta_A - 3D/h) \quad . \quad . \quad . \quad (c).$$

Combining equations (b) and (c), and solving for θ_A, gives

$$-\theta_A = M_{AD}/3EK_0 + D/h.$$

From equation (a), $\theta_A = M_{AB}/2EK_1$.

It is evident that $M_{AD} = M_{AB} = M_A$. Equation the two values of θ_A, and solving for M , gives the following equation.

$$M_A = \frac{+6EK_1K_0D}{h(2K_1+3K_0)}$$. Substituting $\frac{1}{2}etL = -D$, and

$n = K_1/K_0$, gives

$$M_A = \frac{(EI_1et)}{h}\left(\frac{-3}{2n+3}\right) \quad \ldots\ldots\ldots\ldots (72).$$

Case 2. Symmetrical Three Sided Frame with Columns Fixed at Base. In this case the slopes at the bases of the columns are equal to zero. Applying equations 1 and 19, a procedure similar to that of Case 1, gives

$$M_A = \frac{(EI_1et)}{h}\cdot\left(\frac{-3}{n+2}\right) \quad \ldots\ldots\ldots\ldots (73).$$

and

$$M_D = \frac{(EI_1et)}{h}\cdot\left(\frac{3(n+1)}{n(n+2)}\right) \quad \ldots\ldots\ldots\ldots (74).$$

51. INFLUENCE OF SETTLEMENT OR SLIDING OF A SUPPORT. The distribution of stresses in a stiff frame may be greatly influenced by a slight movement of one support. For example, consider the effect of a vertical movement of one support of a three sided frame. Fig.72 represents a frame with the columns hinged at the bases, the base C having settled an amount CC'. The frame DA'B'C' may be considered as an upright frame, in which the supports have spread apart through a distance Δ, and which is acted upon by the components of the force P, normal and parallel to A'B'. The moments produced by these components

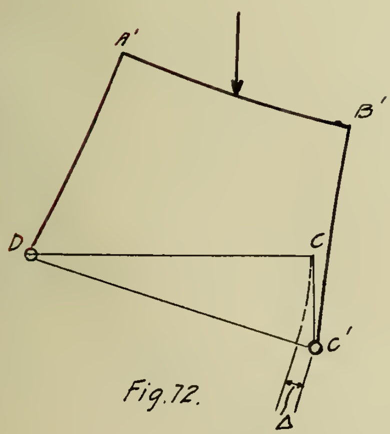


Fig.72.

may be determined by the use of equations 54, 56, and 56a, page 64. The effect of the bases spreading apart the distance Δ is exactly the same as that due to a change in length etL, of the member AB in paragraph 50. Hence, substituting Δ for etL in equation 72, the moment at A due to the movement Δ of the base C, is equal to

$$M_A = \frac{3EK_1\Delta}{h(2n+3)} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (75).$$

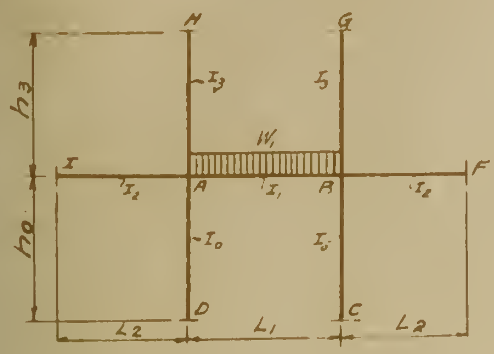
If, instead of settling, the support C moves horizontally through a distance Δ , equation 75 gives the value of the moment produced at the point A.

Similar procedure may be followed for any case of the settlement of a part of a structure, although a general solution will be very difficult for the more complex frames.

IV. SUMMARY.

52. <u>SUMMARY OF EQUATIONS</u>. The equations for some of the more important frames considered in this thesis are given in the tabular summary on pages 113 to 120. This summary should prove very useful for reference.

SUMMARY OF EQUATIONS FOR BUILDING FRAMES



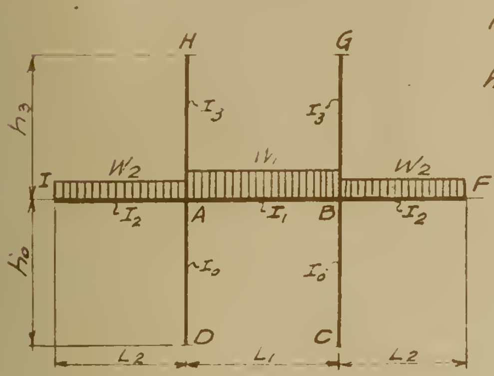
$K = \frac{I}{L}$, or $\frac{I}{h}$, for any member.

For a given symmetrical loading on a member, $\frac{F}{L}$ represents the mean ordinate to the moment diagram for a simple beam under this loading. Values of $\frac{F}{L}$ are given in Table III, page 29.

W is any series of loads which are symmetrical about the center of AB.

Moment	Hinged Ends at C, D, F, G, H, and I.	Fixed Ends at C, D, F, G, H, and I
M_{AB}	$\frac{F}{L}\left[\frac{-3(K_2+K_0+K_3)}{3K_2+3K_0+3K_3+2K_1}\right]$	$\frac{F}{L}\left[\frac{-2(K_2+K_0+K_3)}{2K_2+2K_0+2K_3+K_1}\right]$
M_{AI}	$\frac{F}{L}\left[\frac{-3K_2}{3K_2+3K_0+3K_3+2K_1}\right]$	$\frac{F}{L}\left[\frac{-2K_2}{2K_2+2K_0+2K_3+K_1}\right]$
M_{AD}	$\frac{F}{L}\left[\frac{-3K_0}{3K_2+3K_0+3K_3+2K_1}\right]$	$\frac{F}{L}\left[\frac{-2K_0}{2K_2+2K_0+2K_3+K_1}\right]$
M_{AH}	$\frac{F}{L}\left[\frac{-3K_3}{3K_2+3K_0+3K_3+2K_1}\right]$	$\frac{F}{L}\left[\frac{-2K_3}{2K_2+2K_0+2K_3+K_1}\right]$
M_{HA}	0	$-\frac{1}{2} M_{AH}$
M_{DA}	0	$-\frac{1}{2} M_{AD}$

SUMMARY OF EQUATIONS FOR BUILDING FRAMES.- Continued.



K and $\frac{F}{L}$ are defined on the preceding page.

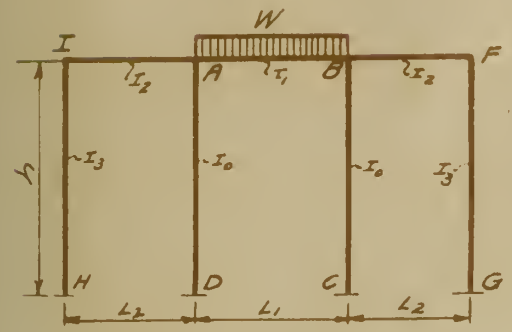
W_1 represents any series of loads, symmetrical about the center of AB.

W_2 represents any series of loads symmetrical about the center of AI or BF.

m = ratio of $\frac{F_2}{L_2}$ to $\frac{F_1}{L_1}$.

Moment.	Hinged Ends at C, D, F, G, H, and I.	Fixed Ends at C, D, F, G, H, and I.
M_{AB}	$\frac{F_1}{L_1}\left[\frac{-3(K_0+K_3+K_2+mK_1)}{3K_0+3K_3+3K_2+2K_1}\right]$.	$\frac{F_1}{L_1}\left[\frac{-(2K_0+2K_3+2K_2+mK_1)}{2K_0+2K_3+2K_2+K_1}\right]$.
M_{AI}	$\frac{F_1}{L_1}\left[\frac{4\frac{1}{2}mK_0+4\frac{1}{2}mK_3+3K_2+3mK_1}{3K_0+3K_3+3K_2+2K_1}\right]$.	$\frac{F_1}{L_1}\left[\frac{-(2mK_0+2mK_3+2K_2+mK_1)}{2K_0+2K_3+2K_2+K_1}\right]$.
M_{AD}	$\frac{F_1}{L_1}\left[\frac{K_0(4\frac{1}{2}m-3)}{3K_0+3K_3+3K_2+2K_1}\right]$.	$\frac{F_1}{L_1}\left[\frac{2K_0(m-1)}{2K_0+2K_3+2K_2+K_1}\right]$.
M_{AH}	$\frac{F_1}{L_1}\left[\frac{K_3(4\frac{1}{2}m-3)}{3K_0+3K_3+3K_2+2K_1}\right]$.	$\frac{F_1}{L_1}\left[\frac{2K_3(m-1)}{2K_0+2K_3+2K_2+K_1}\right]$.
M_{HA}	0	$-\frac{1}{2}M_{AH}$.
M_{DA}	0	$-\frac{1}{2}M_{AD}$.

SUMMARY OF EQUATIONS FOR VIADUCT BENTS



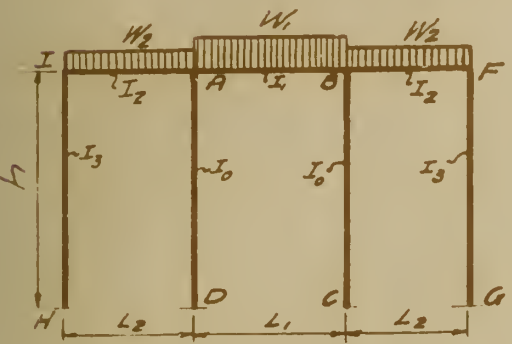
$K = \frac{I}{L}$, or $\frac{I}{h}$, for any member.

For a given symmetrical loading on a member, $\frac{F}{L}$ represents the mean ordinate to the moment diagram, for a simple beam under this loading. Values of $\frac{F}{L}$ are given in Table III, page 29.

W is any series of loads which are symmetrical about the center of AB.

Moment	Ends Hinged at C, D, G, and H.	Ends Fixed at C D G, and H
M_{AB}	$\frac{F}{L}\left[\frac{-4K_2\left(\frac{3K_3+3K_2}{3K_3+4K_2}\right)-3K_0}{4K_2\left(\frac{3K_3+3K_2}{3K_3+4K_2}\right)+3K_0+2K_1}\right]$	$\frac{F}{L}\left[\frac{-K_2\left(\frac{4K_3+3K_2}{K_3+K_2}\right)-4K_0}{K_2\left(\frac{4K_3+3K_2}{K_3+K_2}\right)+4K_0+2K_1}\right]$
M_{AI}	$\frac{F}{L}\left[\frac{-4K_2\left(\frac{3K_3+3K_2}{3K_3+4K_2}\right)}{4K_2\left(\frac{3K_3+3K_2}{3K_3+4K_2}\right)+3K_0+2K_1}\right]$	$\frac{F}{L}\left[\frac{-K_2\left(\frac{4K_3+3K_2}{K_3+K_2}\right)}{K_2\left(\frac{4K_3+3K_2}{K_3+K_2}\right)+4K_0+2K_1}\right]$
M_{AD}	$\frac{F}{L}\left[\frac{-3K_0}{4K_2\left(\frac{3K_3+3K_2}{3K_3+4K_2}\right)+3K_0+2K_1}\right]$	$\frac{F}{L}\left[\frac{-4K_0}{K_2\left(\frac{4K_3+3K_2}{K_3+K_2}\right)+4K_0+2K_1}\right]$
M_{IH}	$\frac{F}{L}\left[\frac{\left(\frac{6K_2K_3}{3K_3+4K_2}\right)}{4K_2\left(\frac{3K_3+3K_2}{3K_3+4K_2}\right)+3K_0+2K_1}\right]$	$\frac{F}{L}\left[\frac{\left(\frac{2K_2K_3}{K_3+K_2}\right)}{K_2\left(\frac{4K_3+3K_2}{K_3+K_2}\right)+4K_0+2K_1}\right]$
M_{HI}	0	$-\frac{1}{2}M_{IH}$.
M_{DA}	0	$-\frac{1}{2}M_{AD}$.

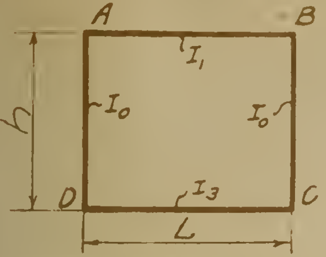
SUMMARY OF EQUATIONS FOR VIADUCT BENTS, Continued.



K and $\frac{F}{L}$ are defined on the preceding page.
W_1 represents any series of loads, symmetrical about the center of AB.
W_2 represents any series of loads symmetrical about the center of AI or BF.
m = ratio of $\frac{F_2}{L_2}$ to $\frac{F_1}{L_1}$.

Moment	Ends Hinged at C, D, G, and H.	Ends Fixed at C, D, G, and H.
M_{AB}	$\frac{F_1}{L_1}\left[\frac{-4K_2\left(\frac{3K_3+3K_2}{3K_3+4K_2}\right)-3K_0-2mK_1\left(\frac{3K_3+6K_2}{3K_3+4K_2}\right)}{4K_2\left(\frac{3K_3+3K_2}{3K_3+4K_2}\right)+3K_0+2K_1}\right].$	$\frac{F_1}{L_1}\left[\frac{-K_2\left(\frac{4K_3+3K_2}{K_3+K_2}\right)-4K_0-2mK_1\left(\frac{2K_3+3K_2}{2K_3+2K_2}\right)}{K_2\left(\frac{4K_3+3K_2}{K_3+K_2}\right)+4K_0+2K_1}\right].$
M_{AI}	$\frac{F_1}{L_1}\left[\frac{-4K_2\left(\frac{3K_3+3K_2}{3K_3+4K_2}\right)-m[3K_0+2K_1]\left(\frac{3K_3+6K_2}{3K_3+4K_2}\right)}{4K_2\left(\frac{3K_3+3K_2}{3K_3+4K_2}\right)+3K_0+2K_1}\right].$	$\frac{F_1}{L_1}\left[\frac{-K_2\left(\frac{4K_3+3K_2}{K_3+K_2}\right)-m[4K_0+2K_1]\left(\frac{2K_3+3K_2}{2K_3+2K_2}\right)}{K_2\left(\frac{4K_3+3K_2}{K_3+K_2}\right)+4K_0+2K_1}\right].$
M_{AD}	$\frac{F_1}{L_1}\left[\frac{-3K_0\left(1-m\left[\frac{3K_3+6K_2}{3K_3+4K_2}\right]\right)}{4K_2\left(\frac{3K_3+3K_2}{3K_3+4K_2}\right)+3K_0+2K_1}\right].$	$\frac{F_1}{L_1}\left[\frac{-4K_0\left(1-m\left[\frac{2K_3+3K_2}{2K_3+2K_2}\right]\right)}{K_2\left(\frac{4K_3+3K_2}{K_3+K_2}\right)+4K_0+2K_1}\right].$
M_{IH}	$\frac{F_1}{L_1}\left[\left(\frac{-3K_3}{3K_3+4K_2}\right)\left(\frac{2K_2(3m-1)+3mK_0+2mK_1}{4K_2\left(\frac{3K_3+3K_2}{3K_3+4K_2}\right)+3K_0+2K_1}\right)\right].$	$\frac{F_1}{L_1}\left[\left(\frac{-K_3}{K_3+K_2}\right)\left(\frac{2K_2(3m-1)+4mK_0+2mK_1}{K_2\left(\frac{4K_3+3K_2}{K_3+K_2}\right)+4K_0+2K_1}\right)\right].$
M_{HI}	0	$-\frac{1}{2}M_{IH}.$
M_{DA}	0	$-\frac{1}{2}M_{AD}.$

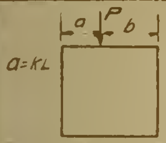
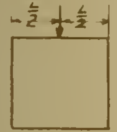
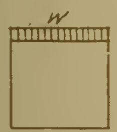
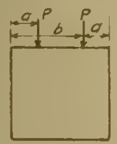
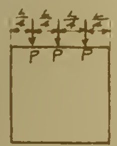
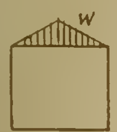
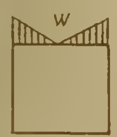
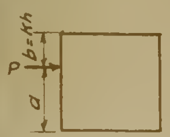
SUMMARY OF EQUATIONS FOR RECTANGULAR FRAMES



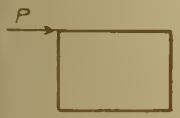
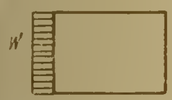
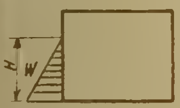
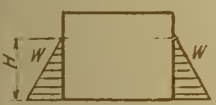
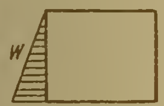
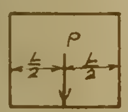
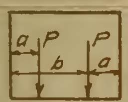
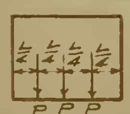
$$n = \frac{I_1 h}{I_0 L} \qquad p = \frac{I_1}{I_3}$$

$$\alpha = n^2 + 2np + 2n + 3p$$

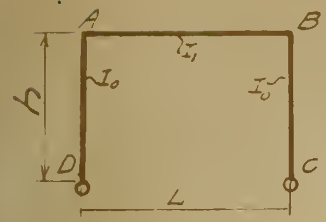
$$\beta = 6n + p + 1$$

Loading.	Values of M_A and M_B Use Upper Signs for M_A, Lower for M_B	Values of M_C and M_D Use Upper Signs for M_C, Lower for M_D
	$\frac{Pab}{L}\left[-\frac{2n+3p}{2\alpha} \pm \left(\frac{k}{\beta} - \frac{1}{2\beta}\right)\right]$	$\frac{Pab}{L}\left[\frac{n}{2\alpha} \mp \left(\frac{k}{\beta} - \frac{1}{2\beta}\right)\right]$
	$\frac{PL}{8}\left[-\frac{2n+3p}{\alpha}\right]$	$\frac{PL}{8}\left[\frac{n}{\alpha}\right]$
	$\frac{WL}{12}\left[-\frac{2n+3p}{\alpha}\right]$	$\frac{WL}{12}\left[\frac{n}{\alpha}\right]$
	$\frac{Pab}{L}\left[-\frac{2n+3p}{\alpha}\right]$	$\frac{Pab}{L}\left[\frac{n}{\alpha}\right]$
	$\frac{5PL}{16}\left[-\frac{2n+3p}{\alpha}\right]$	$\frac{5PL}{16}\left[\frac{n}{\alpha}\right]$
	$\frac{5WL}{48}\left[-\frac{2n+3p}{\alpha}\right]$	$\frac{5WL}{48}\left[\frac{n}{\alpha}\right]$
	$\frac{WL}{16}\left[-\frac{2n+3p}{\alpha}\right]$	$\frac{WL}{16}\left[\frac{n}{\alpha}\right]$
	$Pa\left[\frac{k^2 n(n+p)}{2\alpha} - kn\left(\frac{n+2p}{2\alpha} \pm \frac{3}{2\beta}\right) \pm \frac{3n+p}{2\beta}\right]$	$Pa\left[-\frac{k^2 n(n+1)}{2\alpha} \pm kn\left(\frac{3}{2\beta} \mp \frac{1}{2\alpha}\right) \pm \frac{3n+1}{2\beta}\right]$
	$\frac{Ph}{8}\left[\pm\frac{3n+2p}{\beta} - \frac{n(n+3p)}{\alpha}\right]$	$\frac{Ph}{8}\left[\pm\frac{9n+2}{\beta} - \frac{n(n+3)}{2\alpha}\right]$

SUMMARY OF EQUATIONS FOR RECTANGULAR FRAMES.-Continued.

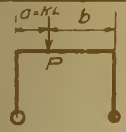
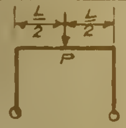
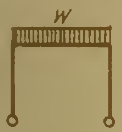
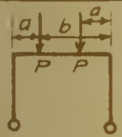
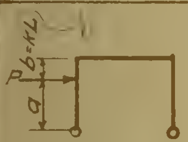
Loading	Values of M_A and M_B. Use Upper Signs for M_A, Lower for M_B	Values of M_C and M_D. Use Upper Signs for M_C, Lower for M_D.
	$Ph\left[\pm \frac{3n+p}{2\beta}\right]$	$Ph\left[\pm \frac{3n+1}{2\beta}\right]$
	$\frac{Wh}{12}\left[-\frac{n(n+3p)}{2\alpha} \pm \frac{3(2n+p)}{\beta}\right]$	$\frac{Wh}{12}\left[-\frac{n(n+3)}{2\alpha} \pm \frac{3(4n+1)}{\beta}\right]$
	$\frac{Wh}{12}\left[-\frac{n(n+3p)}{\alpha}\right]$	$\frac{Wh}{12}\left[-\frac{n(n+3)}{\alpha}\right]$
	$\frac{WH}{30}\left[\frac{n(n+p)}{2\alpha}\left(10-10\frac{H}{h}+3\left[\frac{H}{h}\right]^2\right) - \frac{n}{2}\left(\frac{n+2p}{\alpha} \pm \frac{3}{\beta}\right)\left(10-5\frac{H}{h}\right) \pm 10\left(\frac{3n+p}{2\beta}\right)\right]$	$\frac{WH}{30}\left[\pm \frac{10(3n+1)}{2\beta} \pm \frac{n}{2}\left(\frac{3}{\beta} \mp \frac{1}{\alpha}\right)\left(10-5\frac{H}{h}\right) - \frac{n(n+1)}{2\alpha}\left(10-10\frac{H}{h}+3\left[\frac{H}{h}\right]^2\right)\right]$
	$\frac{WH}{30}\left[\frac{n(n+p)}{\alpha}\left(10-10\frac{H}{h}+3\left[\frac{H}{h}\right]^2\right) - \frac{n(n+2p)}{\alpha}\left(10-\frac{5H}{h}\right)\right]$	$\frac{WH}{30}\left[-\frac{n(n+1)}{\alpha}\left(10-10\frac{H}{h}+3\left[\frac{H}{h}\right]^2\right) - \frac{n}{\alpha}\left(10-5\frac{H}{h}\right)\right]$
	$\frac{Wh}{12}\left[-n\left(\frac{2n+7p}{5\alpha}\right) \pm \frac{3n+2p}{\beta}\right]$	$\frac{Wh}{12}\left[-n\left(\frac{3n+8}{5\alpha}\right) \pm \frac{9n+2}{\beta}\right]$
	$\frac{Wh}{12}\left[-2n\left(\frac{2n+7p}{5\alpha}\right)\right]$	$\frac{Wh}{12}\left[-2n\left(\frac{3n+8}{5\alpha}\right)\right]$
	$\frac{PL}{8}\left[-\frac{pn}{\alpha}\right]$	$\frac{PL}{8}\left[p\left(\frac{2n+3}{\alpha}\right)\right]$
	$\frac{Pab}{L}\left[-\frac{pn}{\alpha}\right]$	$\frac{Pab}{L}\left[p\left(\frac{2n+3}{\alpha}\right)\right]$
	$\frac{WL}{12}\left[-\frac{pn}{\alpha}\right]$	$\frac{WL}{12}\left[p\left(\frac{2n+3}{\alpha}\right)\right]$
	$\frac{5PL}{16}\left[-\frac{pn}{\alpha}\right]$	$\frac{5PL}{16}\left[p\left(\frac{2n+3}{\alpha}\right)\right]$

SUMMARY OF EQUATIONS FOR THREE SIDED FRAMES.



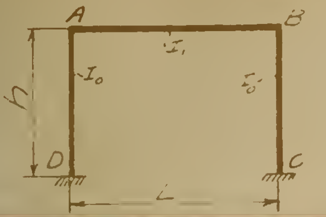
COLUMNS HINGED AT BASE.

$$n = \frac{I_1 h}{I_0 L}$$

Loading	Value of M_A.	Value of M_B.
	$\frac{Pab}{2L}\left[-\frac{3}{2n+3}\right].$	$\frac{Pab}{2L}\left[\frac{-3}{2n+3}\right].$
	$\frac{PL}{8}\left[\frac{-3}{2n+3}\right].$	$\frac{PL}{8}\left[\frac{-3}{2n+3}\right].$
	$\frac{WL}{12}\left[\frac{-3}{2n+3}\right].$	$\frac{WL}{12}\left[-\frac{3}{2n+3}\right].$
	$\frac{Pab}{L}\left[\frac{-3}{2n+3}\right].$	$\frac{Pab}{L}\left[\frac{-3}{2n+3}\right].$
	$\frac{Pa}{2}\left[\frac{(k-2)kn}{2n+3}+1\right].$	$\frac{Pa}{2}\left[\frac{(k-2)kn}{2n+3}-1\right].$
	$\frac{Ph}{2}.$	$-\frac{Ph}{2}.$
	$\frac{Wh}{12}\left[\frac{9(n+2)}{2(2n+3)}\right].$	$\frac{Wh}{12}\left[-\frac{3(5n+6)}{2(2n+3)}\right].$
	$\frac{Wh}{12}\left[-\frac{3n}{2n+3}\right].$	$\frac{Wh}{12}\left[\frac{-3n}{2n+3}\right].$
	$\frac{Wh}{12}\left[\frac{13n+30}{5(2n+3)}\right].$	$\frac{Wh}{12}\left[-\frac{(27n+30)}{5(2n+3)}\right].$
	$\frac{Wh}{12}\left[-\frac{14n}{5(2n+3)}\right].$	$\frac{Wh}{12}\left[-\frac{14n}{5(2n+3)}\right].$

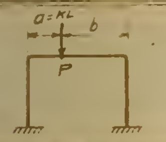
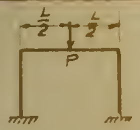
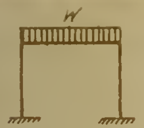
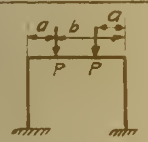
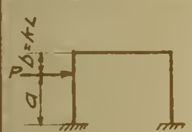
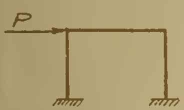
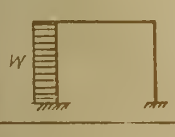
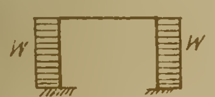

d

SUMMARY OF EQUATIONS FOR THREE SIDED FRAMES.



COLUMNS FIXED AT BASE.

$$n = \frac{I_1 h}{I_0 L}.$$

Loading	Values of M_A and M_B Use Upper Signs for M_A, Lower for M_B.	Values of M_C and M_D. Use Upper Signs for M_C, Lower for M_D.
	$\frac{Pab}{2L}\left[-\frac{2}{n+2} \pm \left(\frac{2k-1}{6n+1}\right)\right]$	$\frac{Pab}{2L}\left[\frac{1}{n+2} \mp \left(\frac{2k-1}{6n+1}\right)\right]$
	$\frac{PL}{8}\left[\frac{-2}{n+2}\right]$	$\frac{PL}{8}\left[\frac{+1}{n+2}\right]$
	$\frac{WL}{12}\left[\frac{-2}{n+2}\right]$	$\frac{WL}{12}\left[\frac{1}{n+2}\right]$
	$\frac{Pab}{L}\left[\frac{-2}{n+2}\right]$	$\frac{Pab}{L}\left[\frac{1}{n+2}\right]$
	$\frac{Pa}{2}(1-k)\left[\pm \frac{3n}{6n+1} - \frac{kn}{n+2}\right]$	$\frac{Pa}{2}\left[\pm \frac{3n(1+k)+1}{6n+1} - \frac{k(1+k+kn)}{n+2}\right]$
	$\frac{Ph}{2}\left[\pm \frac{3n}{6n+1}\right]$	$\frac{Ph}{2}\left[\pm \frac{3n+1}{6n+1}\right]$
	$\frac{Wh}{12}\left[-\frac{n}{2(n+2)} \pm \frac{6n}{6n+1}\right]$	$\frac{Wh}{12}\left[\pm \frac{3(4n+1)}{6n+1} - \frac{n+3}{2(n+2)}\right]$
	$\frac{Wh}{12}\left[-\frac{n}{n+2}\right]$	$\frac{Wh}{12}\left[-\frac{n+3}{n+2}\right]$
	$\frac{Wh}{12}\left[-\frac{2n}{5(n+2)} \pm \frac{3n}{6n+1}\right]$	$\frac{Wh}{12}\left[-\frac{3n+8}{5(n+2)} \pm \frac{9n+2}{6n+1}\right]$
	$\frac{Wh}{12}\left[-\frac{4n}{5(n+2)}\right]$	$\frac{Wh}{12}\left[-\frac{2(3n+8)}{5(n+2)}\right]$