HOMOGENEOUS OPERATORS ON HILBERT SPACES OF HOLOMORPHIC FUNCTIONS – I

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ABSTRACT. In this paper we construct a large class of multiplication operators on reproducing kernel Hilbert spaces which are *homogeneous* with respect to the action of the Möbius group consisting of bi-holomorphic automorphisms of the unit disc $\mathbb{D}$. For every $m \in \mathbb{N}$ we have a family of operators depending on $m+1$ positive real parameters. The kernel function is calculated explicitly. It is proved that each of these operators is bounded, lies in the Cowen - Douglas class of $\mathbb{D}$ and is irreducible. These operators are shown to be mutually pairwise unitarily inequivalent.

1. INTRODUCTION

A homogeneous operator on a Hilbert space $\mathcal{H}$ is a bounded operator T whose spectrum is contained in the closure of the unit disc $\mathbb{D}$ in $\mathbb{C}$ and is such that $g(T)$ is unitarily equivalent to T for all linear fractional transformations g which map $\mathbb{D}$ to $\mathbb{D}$. This class of operators has been studied in a number of articles [4, 6, 3, 12, 5, 11, 1, 9]. It is known that every homogeneous operator is a block shift, that is, $\mathcal{H}$ is the orthogonal direct sum of subspaces V_n, indexed by all integers, all non-negative integers or all non-positive integers, such that $T(V_n) \subseteq V_{n+1}$ for each n.

The case where $\dim V_n = 1$ for each n is completely known, the corresponding operators have been classified in [5]. The classification in the case where $\dim V_n \leq 2$ and T belongs to the Cowen - Douglas class of $\mathbb{D}$ is complete and the operators are explicitly described in [12]. Beyond this there are only some results of a general nature, and not too many examples are known (cf. [4]).

In the present article we construct a large family of examples. For every natural number m we construct a family depending on $m + 1$ parameters. Each one of the examples is realized as the multiplication operator on a reproducing kernel space of vector-valued holomorphic functions. All of these reproducing kernel Hilbert spaces admit a direct sum decomposition $\oplus_{n \geq 0} V_n$ with $\dim V_n = n + 1$ if $0 \leq n < m$ and $\dim V_n = m + 1$ for $n \geq m$. The reproducing kernels are described explicitly. All our examples are irreducible operators and their adjoints belong to the Cowen - Douglas class.

We have chosen a presentation as elementary as possible, based on explicit computations. This seemed to be appropriate here since our goal was a complete explicit description of the examples. On the other hand, it does not explain the deeper background of the results. To remedy this situation we have added a final section which discusses a more conceptual approach to the examples. In a planned expository article on the subject there will be more details about the various ways in which one can arrive at the construction of our examples.

The more conceptual approach will play a leading role in the sequel to the present article, where a description of all homogeneous Cowen - Douglas operators will be given albeit in a less explicit way than our present examples.

Our results are also the subject of a short note presented to the Comptes Rendus de l'Académie des Sciences, Paris [10].

2. PRELIMINARIES

We denote by $\mathbb{D}$ the open unit disc in $\mathbb{C}$ and by G the group of Möbius transformations $z \mapsto \frac{az+b}{\bar{b}z+\bar{a}}$, $|a|^2 - |b|^2 = 1$. Let G_0 be the group $\mathrm{SU}(1,1) = \left\{ \begin{pmatrix} a & b \\ \bar{b} & \bar{a} \end{pmatrix} : |a|^2 - |b|^2 = 1 \right\}$. So, $G = G_0/\{\pm I\}$. We denote by $\tilde{G}$, the universal covering group of G.

All Hilbert spaces $\mathcal{H}$ considered in this article will be spaces of holomorphic functions $f : \mathbb{D} \to V$ taking their values in a finite dimensional Hilbert space V and possessing a reproducing kernel K. A reproducing kernel is a function $K : \mathbb{D} \times \mathbb{D} \to \mathrm{Hom}(V, V)$ holomorphic in the first variable and anti-holomorphic in the second, such that $K_\omega \zeta$ defined by $(K_\omega \zeta)(z) := K(z, \omega)\zeta$ is in $\mathcal{H}$ for each $\omega \in \mathbb{D}$, $\zeta \in V$, and

$$(2.1) \qquad \langle f, K_\omega \zeta \rangle_{\mathcal{H}} = \langle f(\omega), \zeta \rangle_V$$

for all $f \in \mathcal{H}$.

As is well known, if $\{e_n\}_{n=0}^\infty$ is any orthonormal basis of $\mathcal{H}$, then we have

$$(2.2) \qquad K(z, \omega) = \sum_{n=0}^\infty e_n(z) e_n(\omega)^*$$

with the sum converging pointwise. Here we interpret a formal product $\xi \eta^*$ for $\xi, \eta \in V$ as the transformation $\zeta \mapsto \langle \zeta, \eta \rangle \xi$; when $V = \mathbb{C}^k$, $k \in \mathbb{N}$, and its elements are written as column vectors, $\xi \eta^*$ is just the usual matrix product.

We will be concerned with multiplier representations of $\tilde{G}$ on the Hilbert space $\mathcal{H}$. A *multiplier* is a continuous function $J : \tilde{G} \times \mathbb{D} \to \mathrm{Hom}(V, V)$, holomorphic on $\mathbb{D}$, such that

$$(2.3) \qquad J(gh, z) = J(h, z)J(g, hz)$$

for all $g, h \in \tilde{G}$ and $z \in \mathbb{D}$. For $g \in \tilde{G}$, we define $U(g)$ on on $\mathrm{Hol}(\mathbb{D}, V)$ by

$$(2.4) \qquad (U(g)f)(z) = J(g^{-1}, z)f(g^{-1}(z)).$$

It is easy to see that the multiplier identity (2.3) is equivalent to $U(gh) = U(g)U(h)$.

Suppose that the action $g \mapsto U(g)$, $g \in \tilde{G}$, defined in (2.4) preserves $\mathcal{H}$ and is unitary on it, then we say that U is a *unitary multiplier representation* of $\tilde{G}$.

Also, if the reproducing kernel K transforms according to the rule

$$(2.5) \qquad J(g, z)K(g(z), g(\omega))J(g, \omega)^* = K(z, \omega)$$

for all $g \in \tilde{G}$; $z, \omega \in \mathbb{D}$, then we say that K is *quasi-invariant*.

PROPOSITION 2.1. *Suppose $\mathcal{H}$ has a reproducing kernel K. Then U defined by (2.4) is a unitary representation if and only if K is quasi-invariant.*

Proof. Assume that K is quasi-invariant. We have to show that the linear transformation U defined in (2.4) is unitary. We note, writing $\tilde{\omega} = g^{-1}(\omega)$ and $\tilde{\omega}' = g^{-1}(\omega')$,

$$\begin{aligned}
\langle U(g^{-1})K(\cdot,\omega)\xi, U(g^{-1})K(\cdot,\omega')\eta\rangle &= \langle J(g,\cdot)K(g(\cdot),\omega)\xi, J(g,\cdot)K(g(\cdot),\omega')\eta\rangle \\
&= \langle K(\cdot,\tilde{\omega})J(g,\tilde{\omega})^{*-1}\xi, K(\cdot,\tilde{\omega}')J(g,\tilde{\omega}')^{*-1}\eta\rangle \\
&= \langle K(\tilde{\omega}',\tilde{\omega})J(g,\tilde{\omega})^{*-1}\xi, J(g,\tilde{\omega}')^{*-1}\eta\rangle \\
&= \langle J(g,\tilde{\omega}')^{-1}K(\tilde{\omega}',\tilde{\omega})J(g,\tilde{\omega})^{*-1}\xi, \eta\rangle \\
&= \langle K(\omega',\omega)\xi, \eta\rangle
\end{aligned}$$

and it follows that $U(g^{-1})$ is isometric.

On the other hand, if U of (2.4) is unitary then the reproducing kernel K of the Hilbert space $\mathcal{H}$ satisfies the transformation rule (2.5). A reproducing kernel K has the expansion (2.2). It follows from the uniqueness of the reproducing kernel that the expansion is independent of the choice of the orthonormal basis. Consequently, we also have $K(z,\omega) = \sum_{\ell=0}(U_{g^{-1}}e_\ell)(z)(U_{g^{-1}}e_\ell)(\omega)^*$ which verifies the equation (2.5). $\qquad\square$

When we are in the situation of the Proposition and if we can prove that the operator M defined by $(Mf)(z) = zf(z)$ is bounded on $\mathcal{H}$, then M is a homogeneous operator. This is well-known and trivial: Clearly, $(g(M)f)(z) = g(z)f(z)$ and hence $(MU(g^{-1})f)(z) = zJ(g,z)f(g(z)) = J(g,z)g^{-1}(g(z))f(g(z)) = (U(g^{-1})(g^{-1}(M))f)(z)$, for all $g \in \tilde{G}$, $f \in \mathcal{H}$, $z \in \mathbb{D}$. If, in addition, $\dim \ker(M - \omega I)^* = n$ and the operator $(M - \omega I)^*$ is bounded below, on the orthogonal complement of its kernel, for every $\omega \in \mathbb{D}$ then M^* is in the Cowen-Douglas class (see [7]) $B_n(\mathbb{D})$.

In the case of reproducing kernel Hilbert spaces of scalar functions (i.e. when $\dim V = 1$) the unitary multiplier representations of $\tilde{G}$ are well-known. We describe them here because they will be used in the next section. They are the elements of the holomorphic discrete series depending on one real parameter $\lambda > 0$. They act on the Hilbert space $A^{(\lambda)}(\mathbb{D})$ characterized by its reproducing kernel $B^\lambda(z,\omega) = (1 - z\bar{\omega})^{-2\lambda}$. Here $B(z,\omega) = (1 - z\bar{\omega})^{-2}$ is the reproducing kernel of the Bergman space $A^2(\mathbb{D})$, the Hilbert space of square integrable (with respect to normalized area measure) holomorphic functions on the unit disc $\mathbb{D}$.

For $g \in \tilde{G}$, $g'(z)^\lambda$ is a real analytic function on the simply connected set $\tilde{G} \times \mathbb{D}$, holomorphic in z. Also $g'(z)^\lambda \neq 0$ since g is one-one and holomorphic. Given any $\lambda \in \mathbb{C}$, taking the principal branch of the power function when g is near the identity, we can uniquely define $g'(z)^\lambda$ as a real analytic function on $\tilde{G} \times \mathbb{D}$ which is holomorphic on $\mathbb{D}$ for all fixed $g \in \tilde{G}$. The multiplier $j_\lambda(g,z) = g'(z)^\lambda$ defines on $A^{(\lambda)}(\mathbb{D})$ the unitary representation D_λ^+ by the formula (2.4), that is,

$$(2.6) \qquad D_\lambda^+(g^{-1})(f) = (g')^{2\lambda}(f \circ g), \ f \in A^{(\lambda)}(\mathbb{D}), \ g \in \tilde{G}.$$

An orthonormal basis of the space is given by $\left\{\sqrt{\frac{(2\lambda)_n}{n!}} z^n\right\}_{n \geq 0}$, where $(x)_n = x(x+1)\ldots(x+n-1)$ is the Pochhammer symbol. The operator M is bounded on the Hilbert space $A^{(\lambda)}(\mathbb{D})$. It is easily seen to be in the Cowen-Douglas class $B_1(\mathbb{D})$.

3. Construction of the Hilbert spaces and representations

Let $\mathrm{Hol}(\mathbb{D}, \mathbb{C}^k)$ denote the vector space of all holomorphic functions on $\mathbb{D}$ taking values in $\mathbb{C}^k$, $k \in \mathbb{N}$. Let λ be a real number and m be a positive integer satisfying $2\lambda - m > 0$. For brevity, we will write $2\lambda_j = 2\lambda - m + 2j$.

For each j, $0 \leq j \leq m$, define the operator $\Gamma_j : A^{(\lambda_j)}(\mathbb{D}) \to \mathrm{Hol}(\mathbb{D}, \mathbb{C}^{m+1})$ by the formula

$$
(\Gamma_j f)_\ell = \begin{cases} \binom{\ell}{j} \frac{1}{(2\lambda_j)_{\ell-j}} f^{(\ell-j)} & \text{if } \ell \geq j \\ 0 & \text{if } \ell < j, \end{cases}
$$

for $f \in A^{(\lambda_j)}(\mathbb{D})$, $0 \leq \ell \leq m$. Here $(\Gamma_j f)_\ell$ denotes the ℓth component of the function $\Gamma_j f$ and $f^{(\ell-j)}$ denotes the $(\ell - j)$th derivative of the holomorphic function f.

We denote the image of Γ_j by $\mathbf{A}^{(\lambda_j)}(\mathbb{D})$ and transfer to it the inner product of $A^{(\lambda_j)}(\mathbb{D})$, that is, we set $\langle \Gamma_j f, \Gamma_j g \rangle = \langle f, g \rangle$, for $f, g \in A^{(\lambda_j)}(\mathbb{D})$. The Hilbert space $\mathbf{A}^{(\lambda_j)}(\mathbb{D})$ is a reproducing kernel space because the point evaluations $f \mapsto (\Gamma_j f)(\omega)$ are continuous for each $\omega \in \mathbb{D}$. Let $\mathbf{B}^{(\lambda_j)}$ denote the reproducing kernel for the Hilbert space $\mathbf{A}^{(\lambda_j)}(\mathbb{D})$.

The algebraic sum of the linear spaces $\mathbf{A}^{(\lambda_j)}(\mathbb{D})$, $0 \leq j \leq m$ is direct. This is easily seen. If $\sum_{j=0}^m \Gamma_j f_j = 0$, $f_j \in A^{(\lambda_j)}(\mathbb{D})$, then $f_0 = (\Gamma_0 f_0)_0 = 0$ since $(\Gamma_j f_j)_0 = 0$ for $j > 0$. Similarly, $f_1 = (\Gamma_1 f_1)_1 = 0$ since $(\Gamma_j f_j)_1 = 0$ for $j > 1$. Continuing in thpositive numbersis fashion, we see that $f_m = 0$. It follows that we can choose m positive numbers, μ_j, $1 \leq j \leq m$, set $\mu_0 = 1$, write $\boldsymbol{\mu} = (\mu_0, \mu_1, \ldots, \mu_m)$, and define an inner product on the direct sum of the $\mathbf{A}^{(\lambda_j)}(\mathbb{D})$ by setting

$$
(3.1) \qquad \langle \sum_{j=o}^m \Gamma_j f_j, \sum_{j=o}^m \Gamma_j g_j \rangle = \sum_{j=0}^m \mu_j^2 \langle f_j, g_j \rangle, \ \ f_j, g_j \in A^{(\lambda_j)}.
$$

We obtain a Hilbert space in this manner which we denote by $\mathbf{A}^{(\lambda, \boldsymbol{\mu})}(\mathbb{D})$. It has the reproducing kernel $\mathbf{B}^{(\lambda, \boldsymbol{\mu})} = \sum_{j=0}^m \mu_j^2 \mathbf{B}^{(\lambda_j)}$.

The direct sum of the discrete series representations $D^+_{\lambda_j}$ on $\oplus_{j=0}^m A^{(\lambda_j)}$ can be transferred to $\mathbf{A}^{(\lambda, \boldsymbol{\mu})}(\mathbb{D})$ by the map $\Gamma = \oplus_{j=0}^m \mu_j \Gamma_j$. It is a unitary representation of the group $\tilde{G}$ which we call U. Its irreducible subspaces are the $\mathbf{A}^{(\lambda_j)}(\mathbb{D})$.

We will show that U is a multiplier representation. For each $\mathbf{A}^{(\lambda_j)}(\mathbb{D})$ separately this is fairly obvious by checking the effect of Γ_j. The important point is that the multiplier is the same on each $\mathbf{A}^{(\lambda_j)}(\mathbb{D})$.

We need a relation between $g''(z)$ and $g'(z)$. The elements of G_0 are the matrices $\begin{pmatrix} a & b \\ \bar{b} & \bar{a} \end{pmatrix}$, $|a|^2 - |b|^2 = 1$, acting on $\mathbb{D}$ by fractional linear transformations. The inequalities

$$
(3.2) \qquad |a - 1| < 1/2, \ |b| < 1/2
$$

determine a simply connected neighborhood U_0 of e in G_0. Under the natural projections, it is diffeomorphic with a neighborhood U of e in G and with a neighborhood $\tilde{U}$ of e in $\tilde{G}$. So, we may use a, b satisfying (3.2) to parametrize $\tilde{U}$. For $g \in \tilde{U}$, $z \in \mathbb{D}$ we have $g'(z) = (\bar{b}z + \bar{a})^{-2}$ and $g''(z) = -2\bar{b}(\bar{b}z + \bar{a})^{-3}$, which gives a relation

$$
(3.3) \qquad g''(z) = -2c g'(z)^{3/2},
$$

where $c = c_g$ depends on g real analytically and is independent of z; the meaning of $g'(z)^{3/2}$ is as defined earlier. Since both sides are real analytic, (3.3) remains true on all of $\tilde{G} \times \mathbb{D}$.

DEFINITION 3.1. Let $J : \tilde{G} \times \mathbb{D} \to \mathbb{C}^{m+1 \times m+1}$ be the function given by the formula

$$
(3.4) \qquad J(g, z)_{p, \ell} = \begin{cases} \binom{p}{\ell} (-c)^{p-\ell} (g')^{\lambda - \frac{m}{2} + \frac{p+\ell}{2}}(z) & \text{if } p \geq \ell \\ 0 & \text{if } p < \ell, \end{cases}
$$

for $g \in \tilde{G}$. Here c is the constant depending on g as in (3.3)

The following Lemma is used for showing that U is a multiplier representation.

LEMMA 3.1. *For any $g \in \tilde{G}$, we have the formula*

$$\left((g')^\ell(f \circ g)\right)^{(k)} = \sum_{i=0}^{k} \binom{k}{i}(2\ell+i)_{k-i}(-c)^{k-i}(g')^{\ell+\frac{k+i}{2}}\left(f^{(i)} \circ g\right).$$

Proof. The proof is by induction, using the formula (3.3). For $k = 0$, the formula is an identity. Assume the formula to be valid for some k. Then

$$\left((g')^\ell(f \circ g)\right)^{(k+1)}$$

$$= \sum_{i=0}^{k} \binom{k}{i}(2\ell+i)_{k-i}(-c)^{k-i}\left\{(\ell+\frac{k+i}{2})(g')^{\ell+\frac{k+i}{2}-1}g''\left(f^{(i)} \circ g\right) + (g')^{\ell+\frac{k+i}{2}}\left(f^{(i+1)} \circ g\right)g'\right\}$$

$$= \sum_{i=0}^{k} \binom{k}{i}(2\ell+i)_{k-i}(-c)^{k-i}\left\{(2\ell+k+i)(-c)(g')^{\ell+\frac{k+i+1}{2}}\left(f^{(i)} \circ g\right) + (g')^{\ell+\frac{k+i+2}{2}}\left(f^{(i+1)} \circ g\right)\right\}$$

$$= \sum_{i=0}^{k} \binom{k}{i}(2\ell+i)_{k-i}(2\ell+k+i)(-c)^{k+1-i}(g')^{\ell+\frac{k+i+1}{2}}\left(f^{(i)} \circ g\right)$$

$$+ \sum_{i=1}^{k+1} \binom{k}{i-1}(2\ell+i-1)_{k+1-i}(-c)^{k+1-i}(g')^{\ell+\frac{k+i+1}{2}}\left(f^{(i)} \circ g\right).$$

Now, we observe that

$$\binom{k}{i}(2\ell+i)_{k-i}(2\ell+k+i) + \binom{k}{i-1}(2\ell+i-1)_{k+1-i}$$

$$= (2\ell+i)_{k-i}\left\{\binom{k}{i}(2\ell+k+i) + \binom{k}{i-1}(2\ell+i-1)\right\}$$

$$= (2\ell+i)_{k-i}\left\{\left(\binom{k}{i} + \binom{k}{i-1}\right)(2\ell+k) + i\binom{k}{i} + (i-1+k)\binom{k}{i-1}\right\}$$

$$= (2\ell+i)_{k+1-i}\binom{k+1}{i}.$$

Thus $\left((g')^\ell(f \circ g)\right)^{(k+1)} = (2\ell+i)_{k+1-i}\binom{k+1}{i}(-c)^{k+1-i}(g')^{\ell+\frac{k+i+1}{2}}$ completing the induction step.
□

We can now prove the main theorem of this section.

THEOREM 3.1. *The image of $\oplus_0^m D_{\lambda_j}^+$ under Γ is a multiplier representation with the multiplier given by $J(g,z)$ as in (3.4).*

Proof. It will be enough to show

$$\Gamma_j\left(D_{\lambda_j}^+(g^{-1})f\right) = J(g,\cdot)\left((\Gamma_j f) \circ g\right)$$

for each j, $0 \leq j \leq m$. We compute the p'th component on both sides.

For $p < j$, both sides are zero by definition of Γ_j and knowing that $J(g,z)_{p,\ell} = 0$ for $\ell > p$. For $p \geq j$, we have using the Lemma,

$$
\begin{aligned}
\big((\Gamma_j D_{\lambda_j}^+(\varphi^{-1})f)\big)_p &= \binom{p}{j}\frac{1}{(2\lambda_j)_{p-j}}\big((g')^{\lambda_j} f \circ g\big)^{p-j} \\
&= \binom{p}{j}\frac{1}{(2\lambda_j)_{p-j}}\sum_{i=0}^{p-j}\binom{p-j}{i}(2\lambda_j+i)_{p-j-i}(-c)^{p-j-i}(g')^{\lambda_j+\frac{p-j+i}{2}}\big(f^{(i)}\circ g\big) \\
&= \binom{p}{j}\frac{1}{(2\lambda_j)_{p-j}}\sum_{\ell=j}^{p}\binom{p-j}{\ell-j}(2\lambda_j+\ell-j)_{p-\ell}(-c)^{p-\ell}(g')^{\lambda_j-j+\frac{p+\ell}{2}}\big(f^{(\ell-j)}\circ g\big) \\
&= \sum_{\ell=j}^{m}\frac{p!}{j!(\ell-j)!(p-\ell)!}\frac{1}{(2\lambda_j)_{\ell-j}}(-c)^{p-\ell}(g')^{\lambda_j-j+\frac{p+\ell}{2}}\big(f^{(\ell-j)}\circ g\big) \\
&= \sum_{\ell=0}^{m}J(\varphi,\cdot)_{p,\ell}\big((\Gamma_j f)\circ g\big)_\ell.
\end{aligned}
$$

$\square$

4. THE ORTHONORMAL BASIS AND THE OPERATOR M

The vectors $e_n^j(z) := \Gamma_j\big(\sqrt{\frac{(2\lambda_j)_n}{n!}}z^n\big)$ clearly form an orthonormal basis in the Hilbert space $\mathbf{A}^{(\lambda_j)}(\mathbb{D})$. We have, by definition of Γ_j,

$$
(4.1)\qquad \big(e_n^j(z)\big)_\ell = \begin{cases} 0 & \ell < j \text{ or } \ell > n+j \\ \binom{\ell}{j}\frac{\sqrt{n!}}{(n-\ell+j)!}\frac{\sqrt{(2\lambda_j)_n}}{(2\lambda_j)_{\ell-j}}z^{n-\ell+j} & \ell \geq j \text{ and } \ell \leq n+j. \end{cases}
$$

We compute the reproducing kernel $\mathbf{B}^{(\lambda_j)}$ for the Hilbert space $\mathbf{A}^{(\lambda_j)}(\mathbb{D})$. We have

$$
\begin{aligned}
\mathbf{B}^{(\lambda_j)}(z,\omega) &= \sum_{n=0}^{\infty}\big((\Gamma_j e_n^j)(z)\big)\big((\Gamma_j e_n^j)(\omega)\big)^* \\
&= \big(\Gamma_j\sum_{n=0}^{\infty}e_n^j(z)\big)\big(\Gamma_j\sum_{n=0}^{\infty}e_n^j(\omega)\big)^* \\
(4.2)\qquad &= \Gamma_j^{(z)}\Gamma_j^{(\bar\omega)}B^{\lambda_j}(z,\omega),
\end{aligned}
$$

since the series converges uniformly on compact subsets. Explicitly,

$$
(4.3)\qquad \mathbf{B}^{(\lambda_j)}(z,\omega)_{p,\ell} = \begin{cases} \binom{\ell}{j}\binom{p}{j}\frac{1}{(2\lambda_j)_{\ell-j}}\frac{1}{(2\lambda_j)_{p-j}}\partial^{(p-j)}\bar\partial^{(\ell-j)}B^{\lambda_j}(z,\omega) & \text{if } \ell,p\geq j \\ 0 & \text{otherwise.} \end{cases}
$$

In particular, it follows that $\mathbf{B}^{(\lambda_j)}(0,0)$ is diagonal, and

$$
(4.4)\qquad \mathbf{B}^{(\lambda_j)}(0,0)_{\ell,\ell} = \begin{cases} 0 & \text{if } \ell < j \\ \binom{\ell}{j}^2\frac{(\ell-j)!}{(2\lambda_j)_{\ell-j}} & \text{if } \ell \geq j. \end{cases}
$$

Then

$$
(4.5)\qquad \mathbf{B}^{(\lambda,\boldsymbol{\mu})}(0,0)_{\ell,\ell} = \sum_{j=0}^{m}\mathbf{B}^{(\lambda_j)}(0,0)\,\mu_j^2.
$$

A more general formula for $\mathbf{B}^{(\lambda,\boldsymbol{\mu})}(z,\omega)$ can be easily obtained using (2.5). For $z \in \mathbb{D}$, we set $p_z = \frac{1}{\sqrt{(1-|z|^2)}}\begin{pmatrix} 1 & z \\ \bar{z} & 1 \end{pmatrix} \in \mathrm{SU}(1,1)$. We also write p_z for the corresponding element of $\tilde{G}$ such that p_z depends contnuously on $z \in \mathbb{D}$ and $p_0 = e$. Then $p_z(0) = z$; $p_z^{-1} = p_{-z}$. By Theorem 3.1, formula (2.5) holds for $\mathbf{B}^{\lambda,\boldsymbol{\mu}}$ and gives

$$(4.6) \qquad J_{p_{-z}}(z)\mathbf{B}^{\lambda,\boldsymbol{\mu}}(0,0)J_{p_{-z}}(z)^* = \mathbf{B}^{\lambda,\boldsymbol{\mu}}(z,z).$$

We have $p'_{-z}(\zeta) = \frac{1-|z|^2}{(1-\bar{z}\zeta)^2}$; $p'_{-z}(z) = (1-|z|^2)^{-1}$. The $-c$ of (3.3) corresponding to p_{-z} is $\frac{\bar{z}}{1-|z|^2}$. So (3.4) gives

$$J_{p_{-z}}(z)_{p,\ell} = \begin{cases} (1-|z|^2)^{-\lambda-\frac{m}{2}}\binom{p}{\ell}\bar{z}^{p-\ell}(1-|z|^2)^{m-p} & p \geq \ell \\ 0 & p < \ell, \end{cases}$$

which can be written in matrix form as

$$(4.7) \qquad J_{p_{-z}}(z) = (1-|z|^2)^{-\lambda-\frac{m}{2}}D(|z|^2)\exp(\bar{z}S_m),$$

where $D(|z|^2)_{p,\ell} = (1-|z|^2)^{m-\ell}\delta_{p,\ell}$ is diagonal and S_m is the forward shift on $\mathbb{C}^{m+1}$ with weight sequence $\{1,\ldots,m\}$, that is, $(S_m)_{\ell,p} = \ell\delta_{p+1,\ell}$, $0 \leq p, \ell \leq m$. Substituting (4.7) into (2.5) and polarizing we obtain

$$(4.8) \qquad \mathbf{B}^{(\lambda,\boldsymbol{\mu})}(z,\omega) = (1-z\bar{\omega})^{-2\lambda-m}D(z\bar{\omega})\exp(\bar{\omega}S_m)\mathbf{B}^{(\lambda,\boldsymbol{\mu})}(0,0)\exp(zS_m^*)D(z\bar{\omega}).$$

In general, let $\mathcal{H}$ be a Hilbert space consisting of holomorphic functions on the open unit disc $\mathbb{D}$ with values in $\mathbb{C}^{m+1}$. Assume that $\mathcal{H}$ possesses a reproducing kernel $K : \mathbb{D} \times \mathbb{D} \to \mathbb{C}^{(m+1)\times(m+1)}$. The set of vectors $\mathcal{H}_0 = \{K_\omega\xi : \omega \in \mathbb{D}, \xi \in \mathbb{C}^{m+1}\}$ span the Hilbert space $\mathcal{H}$. On the dense set of vectors $\mathcal{H}_0$, we define a map T by the formula $TK_\omega\xi = \bar{\omega}K_\omega\xi$ for $\omega \in \mathbb{D}$. The following Lemma gives a criterion for boundedness of the operator T.

LEMMA 4.1. *The densely defined operator T is bounded if and only if for some positive constant c and for all $n \in \mathbb{N}$*

$$\sum_{i,j=1}^{n}\langle(c-\omega_j\bar{\omega}_i)K(\omega_j,\omega_i)x_i, x_j\rangle \geq 0$$

for $x_1,\ldots,x_n \in \mathbb{C}^{m+1}$ and $\omega_1,\ldots,\omega_n \in \mathbb{D}$. If the map $T : \mathcal{H}_0 \to \mathcal{H}_0 \subseteq \mathcal{H}$ is bounded then it is the adjoint of the multiplication operator on $\mathcal{H}$.

The proof is well-known and easy in the scalar case. We omit the obvious modifications required in the general case.

It is known and easy to verify that for every $\epsilon > 0$, the multiplication operator $M^{(\epsilon)}$, defined by $(M^{(\epsilon)}f)(z) = zf(z)$, is bounded on $A^{(\epsilon)}$. Consequently, the kernel B^ϵ satisfies the positivity condition of the Lemma above for $\epsilon > 0$. Fix $m \in \mathbb{N}$. Consider the reproducing kernel $\mathbf{B}^{(\lambda,\boldsymbol{\mu})}$. We recall that $\mathbf{B}^{(\lambda,\boldsymbol{\mu})}$ is a positive definite kernel on the unit disc $\mathbb{D}$ if and only if $\lambda > m/2$.

THEOREM 4.1. *The multiplication operator $M^{(\lambda,\boldsymbol{\mu})}$ on the Hilbert space $\mathbf{A}^{(\lambda,\boldsymbol{\mu})}$ is bounded for all $\lambda > m/2$.*

Proof. Let ϵ be a positive real number such that $\lambda-\epsilon > m/2$. Let us find $\boldsymbol{\mu}'$ with $\mu'_j > 0$, $0 \leq j \leq m$, such that

$$(4.9) \qquad \mathbf{B}^{(\lambda,\boldsymbol{\mu})}(z,\omega) = (1-z\bar{\omega})^{-2\epsilon}\mathbf{B}^{(\lambda-\epsilon,\boldsymbol{\mu}')}(z,\omega).$$

Since the multiplication operator is bounded on the Hilbert space whose reproducing kernel is $(1 - z\bar{\omega})^{-2\epsilon}$ for every $\epsilon > 0$, it follows that we can find $r > 0$ such that $(r - z\bar{\omega})(1 - z\bar{\omega})^{-2\epsilon}$ is positive definite. Assuming the existence of $\boldsymbol{\mu}'$ as above, we conclude that $(r - z\bar{\omega})\mathbf{B}^{(\lambda,\boldsymbol{\mu})}(z,\omega)$ is positive definite finishing the proof. To find such a $\boldsymbol{\mu}'$, it is enough to prove $\mathbf{B}^{(\lambda,\boldsymbol{\mu})}(0,0) = \mathbf{B}^{(\lambda-\epsilon,\boldsymbol{\mu}')}(0,0)$, because then (4.6) and (4.7) (or (4.8)) immediately imply (4.9).

By (4.5), writing $L(\lambda)_{\ell j} = \mathbf{B}^{(\lambda_j)}(0,0)_{\ell\ell}$, the question becomes whether we can find positive numbers μ_j' satisfying the equations

$$(4.10) \qquad \sum_j L(\lambda)_{\ell j}\, \mu_j^2 = \sum_j L(\lambda - \epsilon)_{\ell j}\, \mu_j'^{\,2}.$$

By (4.4) each $L(\lambda)_{\ell j}$ is continuous in λ,; also $L(\lambda)_{\ell j} = 0$ for $\ell < j$, and $L(\lambda)_{00} = 1$. It follows that for small $\epsilon > 0$, the system (4.10) has solutions satisfying $\mu_0'^{\,2} = 1$, $\boldsymbol{\mu}_j'^{\,2} > 0$ ($1 \le j \le m$). $\square$

Next we compute the matrix of M with respect to the orthonormal basis $\{\mu_j e_n^j(z) : n \ge 0; 0 \le j \le m\}$. Let $\mathcal{H}(n)$ be the linear span of the vectors $\{e_{n-j}^j(z) : 0 \le j \le \min(m,n)\}$. It is clear that M maps the space $\mathcal{H}(n)$ into $\mathcal{H}(n+1)$. (The subspace $\mathcal{H}(n)$ of $\mathbf{A}^{(\lambda,\boldsymbol{\mu})}(\mathbb{D})$ is a "K-type" of the representation U.) We therefore have

$$M\mu_j e_{n-j}^j = \sum_{k=0}^m M(n)_{k,j}\, \mu_k\, e_{n+1-k}^k.$$

Let $E(n)$ be the matrix, determined by (4.1), such that $\left(e_{n-j}^j(z)\right)_\ell = E(n)_{\ell,j} z^{n-\ell}$, $n \ge j$, $0 \le j \le m$. In this notation,

$$E(n)_{\ell,j}\mu_j = \sum_{k=0}^m M(n)_{k,j} E(n+1)_{\ell,k}\mu_k.$$

In matrix form, this means

$$E(n)D(\boldsymbol{\mu}) = E(n+1)D(\boldsymbol{\mu})M(n), \text{ which gives}$$
$$M(n) = D(\boldsymbol{\mu})^{-1}E(n+1)^{-1}E(n)D(\boldsymbol{\mu}),$$

where $D(\boldsymbol{\mu})$ is the diagonal matrix with $D(\boldsymbol{\mu})_{\ell,\ell} = \mu_\ell$. (These are the blocks of M regarded as a "block shift" with respect to the orthogonal decomposition of $\mathbf{A}^{(\lambda,\boldsymbol{\mu})}(\mathbb{D}) = \oplus_{n=0}^\infty \mathcal{H}(n)$.)

To get information about $M(n)$, we note that, as $n \to \infty$, Stirling's formula gives, for any fixed $b \in \mathbb{R}$,

$$\Gamma(n+b) \sim \sqrt{2\pi}(n+b)^{n+b-1/2}e^{-(n+b)} \sim \sqrt{2\pi}n^{n+b-1/2}(1+\tfrac{b}{n})^n e^{-(n+b)} \sim e^b n^{n+b-1/2}.$$

Applying this we immediately get, by (4.1),

$$E(n)_{\ell,j} \sim n^\ell n^{\lambda-m/2-1/2} E_{\ell,j},$$

where E is the matrix with entries

$$E_{\ell,j} = \begin{cases} \binom{\ell}{j}\dfrac{\sqrt{\Gamma(2\lambda-m+2j)}}{\Gamma(2\lambda-m+\ell+j)} & \ell \ge j \\ 0 & \ell < j \end{cases}$$

independent of n. Using the diagonal matrix $\mathbf{d}(n)$ with $\mathbf{d}(n)_{\ell,\ell} = n^\ell$, we can write

$$E(n) \sim n^{\lambda-m/2-1/2}\mathbf{d}(n)E.$$

It follows that

$$M(n) = D(\boldsymbol{\mu})^{-1} E(n+1)^{-1} E(n) D(\boldsymbol{\mu})$$
$$\sim \big(\frac{n}{n+1}\big)^{\lambda - m/2 - 1/2} D(\boldsymbol{\mu})^{-1} E^{-1} \mathbf{d}(n+1)^{-1} \mathbf{d}(n) E D(\boldsymbol{\mu}).$$

Since $\frac{n}{n+1} = 1 + O(\frac{1}{n})$, this implies

$$M(n) = I + O(\tfrac{1}{n}),$$

where I is the identity matrix of order $m+1$ and $O(\frac{1}{n})$ stands for a $(m+1) \times (m+1)$ matrix each of whose entries is $O(\frac{1}{n})$.

We denote by U_+ the operator on $\mathbf{A}^{(\lambda, \boldsymbol{\mu})}(\mathbb{D})$ defined by $U_+ e^j_{n-j} = e^j_{n+1-j}$ ($0 \le j \le \min(m, n)$, $n - j \ge 0$).

THEOREM 4.2. *The operator M on $\mathbf{A}^{(\lambda, \boldsymbol{\mu})}(\mathbb{D})$ is the sum of U_+ and of an operator in the Hilbert-Schmidt class. In particular, M is bounded and its adjoint belongs to the Cowen-Douglas class.*

5. IRREDUCIBILITY

Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be two reproducing kernel Hilbert spaces consisting of holomorphic functions on $\mathbb{D}$ taking values in $\mathbb{C}^{m+1}$. Suppose that the multiplication operator M on these two Hilbert spaces are bounded. Furthermore, assume that the standard set of $m+1$ orthonormal vectors $\varepsilon_0, \ldots, \varepsilon_m$ in $\mathbb{C}^{m+1}$, thought of as constant functions on $\mathbb{D}$, are in both $\mathcal{H}_1$ and $\mathcal{H}_2$. Since $\big(\sum_{i=0}^m p_i(M) \varepsilon_i \big)(z) = \sum_{i=0}^m p_i(z) \varepsilon_i$ for polynomials p_i with scalar coefficients, it follows that the polynomials $\mathbf{p}(z) = \sum_{i=0}^m p_i(z) \varepsilon_i$ belong to these Hilbert spaces. We assume that the polynomials $\mathbf{p}$ are dense in both of these Hilbert spaces.

Suppose that there is a bounded operator $X : \mathcal{H}_1 \to \mathcal{H}_2$ satisfying $MX = XM$. Then

$$(X\mathbf{p})(z) = \big(X \sum_{i=0}^m p_i \varepsilon_i \big)(z) = \big(X \sum_{i=0}^m p_i(M) \varepsilon_i \big)(z) = \big(\sum_{i=0}^m p_i(M) X \varepsilon_i \big)(z) = \big(\sum_{i=0}^m p_i(M) X \varepsilon_i \big)(z).$$

Now, if we let $(X\varepsilon_i)(z) = \sum_{j=0}^m x_i^j(z) \varepsilon_j$, then $(X\mathbf{p})(z) = \Phi_X(z) \mathbf{p}(z)$, where $\Phi_X(z) = ((x_i^j(z)))_{j,i=0}^m$. Since the polynomials $\mathbf{p}$ are dense, it follows that $(Xf)(z) = \Phi_X(z) f(z)$ for all $f \in \mathcal{H}_1$.

We calculate the adjoint of the intertwining operator X. We have

$$\langle X K_1(\cdot, \omega) \xi, K_2(\cdot, u) \eta \rangle = \langle \Phi_X(\cdot) K_1(\cdot, \omega) \xi, K_2(\cdot, u) \eta \rangle = \langle \Phi_X(u) K_1(u, \omega) \xi, \eta \rangle$$
$$= \langle K_1(u, \omega) \xi, \overline{\Phi_X(u)}^{\mathrm{tr}} \eta \rangle = \langle K_1(\cdot, \omega) \xi, K_1(\cdot, u) \overline{\Phi_X(u)}^{\mathrm{tr}} \eta \rangle$$

for all $\xi, \eta \in \mathbb{C}^{m+1}$, that is,

(5.1) $$X^* K_2(\cdot, u) \eta = K_1(\cdot, u) \overline{\Phi_X(u)}^{\mathrm{tr}} \eta,$$

for all $\eta \in \mathbb{C}^{m+1}$ and $u \in \mathbb{D}$. Hence the intertwining operator X is unitary if and only if there exists an invertible holomorphic function $\Phi_X : \mathbb{D}_0 \to \mathbb{C}^{(m+1) \times (m+1)}$ for some open subset $\mathbb{D}_0$ of $\mathbb{D}$ satisfying

(5.2) $$K_2(z, \omega) = \Phi_X(z) K_1(z, \omega) \overline{\Phi_X(\omega)}^{\mathrm{tr}}.$$

Let $\mathcal{H}$ be a Hilbert space consisting of $\mathbb{C}^n$ - valued holomorphic functions on $\mathbb{D}$. Assume that $\mathcal{H}$ has a reproducing kernel, say K. Let Φ be a $n \times n$ invertible matrix valued holomorphic function on $\mathbb{D}$ which is invertible. For $f \in \mathcal{H}$, consider the map $X : f \mapsto \tilde{f}$, where $\tilde{f}(z) = \Phi(z) f(z)$. Let

$\tilde{\mathcal{H}} = \{\tilde{f} : f \in \mathcal{H}\}$. The requirement that the map X is unitary, prescribes a Hilbert space structure for the function space $\tilde{\mathcal{H}}$. The reproducing kernel for $\tilde{\mathcal{H}}$ is clearly

$$(5.3) \qquad \tilde{K}(z,\omega) = \Phi(z)K(z,w)\Phi(\omega)^*.$$

It is easy to verify that XMX^* is the multiplication operator $M : \tilde{f} \mapsto z\tilde{f}$ on the Hilbert space $\tilde{\mathcal{H}}$. Suppose we have a unitary representation U given by a multiplier J acting on $\mathcal{H}$ according to (2.5). Transplanting this action to $\tilde{\mathcal{H}}$ under the isometry X, it becomes

$$\big(\tilde{U}_{g^{-1}}\tilde{f}\big)(z) = \tilde{J}(g,z)\tilde{f}(g \cdot z),$$

where the new multiplier $\tilde{J}$ is given in terms of the original multiplier J by

$$(5.4) \qquad \tilde{J}(g,z) = \Phi(z)J(g,z)\Phi(g \cdot z)^{-1}.$$

Of course, now $\tilde{K}$ transforms according to (2.5), with the aid of $\tilde{J}$.

LEMMA 5.1. *Suppose that the operator M acting on the Hilbert space $\mathcal{H}$ with reproducing kernel K is bounded, the constant vectors $\varepsilon_0, \ldots, \varepsilon_m$ are in $\mathcal{H}$, and that the polynomials $\mathbf{p}$ are dense in $\mathcal{H}$. If there exists a (self adjoint) projection X commuting with the operator M then*

$$\Phi_X(z)K(z,\omega) = K(z,\omega)\overline{\Phi_X(\omega)}^{\mathrm{tr}}$$

for some holomorphic function $\Phi_X : \mathbb{D} \to \mathbb{C}^{(m+1)\times(m+1)}$ with $\Phi_X^2 = \Phi_X$.

Proof. We have already seen that any such operator X is multiplication by a holomorphic function Φ_X. To complete the proof, note that

$$\Phi_X(\cdot)K(\cdot,\omega)\xi = XK(\cdot,\omega)\xi = X^*K(\cdot,\omega)\xi = K(\cdot,\omega)\overline{\Phi_X(\omega)}^{\mathrm{tr}}\xi$$

for all $\xi \in \mathbb{C}^{m+1}$.   □

From the Lemma, putting $\omega = 0$, we see that $\Phi_X(z) = K(z,0)\overline{\Phi(0)}^{\mathrm{tr}}K(z,0)^{-1}$ for any self adjoint intertwining operator X. Furthermore, $X_0 := \Phi_X(0)$ is an ordinary projection on $\mathbb{C}^{m+1}$, if $K(0,0) = I$. The multiplication operator on the two Hilbert spaces $\mathcal{H}$ with reproducing kernel K and $\mathcal{H}_0$ with reproducing kernel $K_0(z,\omega) = K(0,0)^{-1/2}K(z,\omega)K(0,0)^{-1/2}$ are unitarily equivalent via the unitary map $f \mapsto K(0,0)^{-1/2}f$. The reproducing kernel K_0 has the additional property that $K_0(0,0) = I$. Therefore, we conclude that M is reducible if and only if there exists a projection X_0 on $\mathbb{C}^{m+1}$ satisfying

$$(5.5) \qquad X_0 K_0(z,0)^{-1}K_0(z,\omega)K_0(0,\omega)^{-1} = K_0(z,0)^{-1}K_0(z,\omega)K_0(0,\omega)^{-1}X_0.$$

This is the same as requiring the existence of a projection X_0 which commutes with all the coefficients in the power series expansion of the function $\widehat{K_0}(z,\omega) := K_0(z,0)^{-1}K_0(z,\omega)K_0(0,\omega)^{-1}$ around 0. We also point out that $\widehat{K_0}$ is the normalized kernel in the sense of [8] and is characterized by the property $\widehat{K_0}(z,0) \equiv 1$.

For the rest of this section, we set $B := \mathbf{B}^{(\lambda,\boldsymbol{\mu})}(0,0)$ and $S := S_m$, as in Section 3.

LEMMA 5.2. *The operator $M := M^{(\lambda,\boldsymbol{\mu})}$ on the Hilbert space $A^{(\lambda,\boldsymbol{\mu})}$ is irreducible if and only if there is no projection X_0 on C^{m+1} commuting with all the coefficients in the power series expansion of the function*

$$(1 - z\bar{\omega})^{-2\lambda-m}B^{1/2}\exp\left(-zS^*\right)B^{-1}D(z\bar{\omega})\exp\left(\bar{\omega}S\right)B\exp(zS^*)D(z\bar{\omega})B^{-1}\exp(-\bar{\omega}S)B^{1/2},$$

around 0.

Proof. From (4.8), we have $\mathbf{B}_0^{(\lambda,\boldsymbol{\mu})}(z,0) = B^{1/2}\exp(zS^*)B^{-1/2}$, where $\mathbf{B}_0^{(\lambda,\boldsymbol{\mu})} := B^{-1/2}\mathbf{B}^{(\lambda,\boldsymbol{\mu})}B^{-1/2}$. To complete the proof, using (4.8), we merely verify that

$$
\begin{aligned}
\widehat{\mathbf{B}_0}(z,\omega) \\
= \quad & \big(\mathbf{B}_0^{(\lambda,\boldsymbol{\mu})}(z,0)\big)^{-1}\mathbf{B}_0^{(\lambda,\boldsymbol{\mu})}(z,\omega)\big(\mathbf{B}_0^{(\lambda,\boldsymbol{\mu})}(0,\omega)\big)^{-1} \\
= \quad & (1-z\bar\omega)^{-2\lambda-m}B^{1/2}\exp\big(-zS^*\big)B^{-1}D(z\bar\omega)\exp\big(\bar\omega S\big)B\exp(zS^*)D(z\bar\omega)B^{-1}\exp(-\bar\omega S)B^{1/2}.
\end{aligned}
$$

$\square$

Let D_s denote the coefficient of $(-1)^s z^s\bar\omega^s$ in the expansion of $D(z\bar\omega)$ and $\tilde D_s = B^{-1}D_s$. (The choice of D_s ensures that the diagonal sequence in $\tilde D_s$ is positive.)

LEMMA 5.3. *If* $\big(S^{*i}\tilde D_s S^p B S^{*q}\tilde D_t S^j\big)_{k,n} \neq 0$ *for some choice of* i, j, s, t, p, q *in* $\{0, 1, \ldots, m\}$ *then*

$$0 \le s \le m - k - i, \quad 0 \le t \le m - n - j;$$
$$0 \le p \le k + i, \qquad 0 \le q \le n + j;$$

and $k + i - p = n + j - q.$

Proof. Recall that

$$
S : \begin{cases} e_\ell \mapsto (\ell+1)e_{\ell+1} & \text{if } 0 \le \ell \le m-1 \\ e_m \mapsto 0 & \text{otherwise.} \end{cases}
$$

So

$$
S^p : \begin{cases} e_\ell \mapsto \iota e_{\ell+p} & \text{if } 0 \le i \le m-1 \\ e_m \mapsto 0 & \ell > m-p, \end{cases}
$$

where $\iota = (\ell+1)\ell\cdots(\ell-p)$. Also,

$$
\tilde D_s : \begin{cases} e_\ell \mapsto c e_\ell & \text{if } 0 \le \ell \le m-s \\ e_m \mapsto 0 & \ell \ge m-s+1, \end{cases}
$$

where c is a non-zero constant depending on ℓ, s. Therefore

$$
Q := S^{*i}\tilde D_s S^p : \begin{cases} e_\ell \mapsto c' e_{\ell+p-i} & \text{if } 0 \le i \le m-p-s \text{ and } \ell+p-i \ge 0 \\ e_m \mapsto 0 & \text{otherwise} \end{cases}
$$

for some non-zero constant c'. Hence the full condition for $Q_{k,\ell} \neq 0$ is

$$(5.6) \qquad i - p \le \ell \le m - p - s, \ k = \ell + p - i.$$

Let $R := S^{*q}\tilde D_t S^j$. By what we have just proved, it follows that $R_{\ell,n} \neq 0$ if and only if

$$(5.7) \qquad q - j \le n \le m - j - t, \ n = \ell - j + q.$$

The conditions (5.6) and (5.7) simplify as follows:

$$(5.8) \qquad 0 \le \ell = k + i - p = n + j - q = \ell \le m, \ k + i \le m - s \text{ and } n + j \le m - t.$$

$\square$

Let $a(\ell)$ denote the coefficient of $z^{m+\ell+1}\bar\omega^{m+\ell}$ in the polynomial A with

$$
\begin{aligned}
A(z,\omega) = \quad & \exp\big(-zS^*\big)B^{-1}D(z\bar\omega)\exp\big(\bar\omega S\big)B\exp(zS^*)D(z\bar\omega)B^{-1}\exp(-\bar\omega S) \\
= \quad & \sum (-1)^i \frac{S^{*i}}{i!}z^i (-1)^s \tilde D_s z^s\bar\omega^s \frac{S^p}{p!}\bar\omega^p B \frac{S^{*q}}{q!}z^q (-1)^t \tilde D_t z^t\bar\omega^t (-1)^j \frac{S^j}{j!}\bar\omega^j,
\end{aligned}
$$

where the sum is over $0 \le i, j, p, q, s, t \le m.$

LEMMA 5.4. *For* $0 \le \ell \le m - 1$, $a(\ell)_{k,n} = \begin{cases} \text{not zero} & \text{if } k = m - \ell - 1 \text{ and } n = m - \ell \\ \text{zero} & \text{if } k - n \ne 1 \text{ or } k > m - \ell - 1 \end{cases}$.

Proof. Clearly, $A(z,w) = \sum A_{ijpqst} z^{i+s+q+t} \bar{\omega}^{s+p+t+j}$, where the sum is over $0 \le i,j,p,q,s,t \le m$. Therefore, $a(\ell) = \sum c S^{*i} \tilde{D}_s S^p B S^{*q} \tilde{D}_t S^j$, where the sum is over all i,j,p,q,s,t such that $s+t+i+q = m+\ell+1$ and $s+t+p+j = m+\ell$; $c = \frac{(-1)^{i+j+s+t}}{i!\,j!\,p!\,q!}$.

It follows from the preceding Lemma that if $a(\ell)_{k,n} \ne 0$, then $i - j + q - p = n - k$. However, for the terms occuring in the sum, we now have $i - j + q - p = (s+t+i+q) - (s+t+p+j) = 1$. Thus if $a(\ell)_{k,n} \ne 0$ then $n - k = 1$.

Furthermore, if $a(\ell)_{k,n} \ne 0$, then we also have $m+\ell+1 = (s+t+i+q)$. Hence $m+\ell+1-(s+t+i) = q \le n + j$ from the last inequality of the preceding Lemma, that is, $s + t + i + j \ge m + \ell + 1 - n$. This along with $s + t + i + j \le 2m - k - n$, which is obtained by adding the first two inequalities of the preceding Lemma, gives $k \le m - \ell - 1$

The proof of the second part of the Lemma is now complete.

If $k = m-\ell-1$ and $n = m-\ell$, for the terms occuring in the sum for $a(\ell)$, we have $s+t+i+j = 2\ell+1$. It follows that $a(\ell)_{m-\ell-1,m-\ell}$ is a sum of negative numbers. This proves the first part of the Lemma. $\square$

THEOREM 5.1. *The multiplication operator* $M := M^{\lambda,\boldsymbol{\mu}}$ *on the Hilbert space* $\mathbf{A}^{(\lambda,\boldsymbol{\mu})}$ *is irreducible.*

Proof. Suppose there exists a non-trivial projection P commuting with $\widehat{\mathbf{B}_0}(z,\omega)$ for all $z,\omega \in \mathbb{D}$. Then by Lemma 5.2 such a projection must commute with $B^{1/2}A(z,\omega)B^{1/2}$ for all $z,\omega \in \mathbb{D}$. However, Lemma 5.4 shows that there is no non-trivial projection commuting with the set of matrices $\{B^{1/2}a(\ell)B^{1/2} : 0 \le \ell \le m - 1\}$. This completes the proof. $\square$

6. INEQUIVALENCE

Let pr $: E_T \to \mathbb{D}$ be the holomorphic vector bundle corresponding to an operator $T \in \mathrm{B}_k(\mathbb{D})$. The operator T is homogeneous if and only if for any $g \in G$, there exists an automorphism $\hat{g}$ of the

$$E_T \xrightarrow{\hat{g}} E_T$$

bundle E_T covering g, that is, the diagram $\ \downarrow \mathrm{pr} \qquad \downarrow \mathrm{pr}\ $ is commutative.

$$\mathbb{D} \xrightarrow{g} \mathbb{D}$$

THEOREM 6.1. *If* T *is a homogeneous operator in* $\mathrm{B}_k(\mathbb{D})$ *then the the universal covering group* $\tilde{G}$ *of* G *acts on* E_T *by automorphisms.*

Proof. Let $\hat{G}$ be the group of automorphisms of E_T. This is a Lie group. Let $p : \hat{G} \to G$ be the natural homomorphism. Let $N = \ker p$, the automorphisms fixing all the points of $\mathbb{D}$. Then $\hat{G}/N \simeq G$, and for the corresponding Lie algebras, we have $\hat{\mathfrak{g}}/\mathfrak{n} \simeq \mathfrak{g}$. Since $\mathfrak{g}$ is semisimple, by the Levi decomposition, there is a subalgebra $\hat{\mathfrak{g}}_0 \subseteq \hat{\mathfrak{g}}$ such that $\hat{\mathfrak{g}} = \hat{\mathfrak{g}}_0 + \mathfrak{n}$, where the sum is a vector space direct sum. Let $\hat{G}_0$ be the corresponding analytic subgroup.

There is a neigbourhood U of $e \in \hat{G}_0$ such that $p_{|U}$ is a homeomorphism onto a neighbourhood $p(U)$ of $e \in G$. But then $p(\hat{g}U) = p(\hat{g})p(U)$. So, p is a homeomorphism of a neighbourhood of any point $\hat{g} \in \hat{G}_0$ to a neighbourhood of $p(\hat{g})$ in G. It follows that the image of p is an open subgroup and so must equal G. Therefore, $\hat{G}_0$ is a covering group of G.

Now, $\hat{G}_0$ acts on E_T by automorphisms and projects to G. The universal cover $\tilde{G}$ now also acts on E_T.

□

REMARK 6.1. With slightly more work one can prove that $\hat{\mathfrak{g}}_0$ is an ideal and therefore the $\tilde{G}$ action on E_T is unique. If T is irreducible it is known independently (cf. [4]) that the $\tilde{G}$ action is unique.

THEOREM 6.2. *For every $m \geq 1$, the operators $M^{(\lambda,\boldsymbol{\mu})}$, $\lambda > \frac{m}{2}; \mu_1, \ldots, \mu_m > 0$ are mutually unitarily inequivalent.*

Proof. Suppose $M^{(\lambda,\boldsymbol{\mu})}$ and $M^{(\lambda',\boldsymbol{\mu}')}$ are unitarily equivalent. Then the corresponding Hermitian holomorphic bundles are isomorphic [7]. Hence the multipliers J and J' giving the $\tilde{G}$ action on $\mathbb{A}^{(\lambda,\boldsymbol{\mu})}$ and $\mathbb{A}^{(\lambda',\boldsymbol{\mu}')}$ are equivalent in the sense that there exists a invertible matrix function $\phi(z)$, holomorphic in z, such that

$$\Phi(z)J(g,z)\Phi(gz)^{-1} = J'(g,z)$$

on $\tilde{G} \times \mathbb{D}$ which is nothing but (5.4). Setting here $g = p_{-z}$, (4.7) gives

$$\Phi(z) = (1 - |z|^2)^{\lambda - \lambda'} D(|z|^2) \exp(-\bar{z}S_m) D(|z|^2) F(0) \exp(\bar{z}S_m) D(|z|^2)^{-1}.$$

The right hand side is real anlytic in $z, \bar{z}$ on $\mathbb{D}$. Since Φ is holomorphic, $\Phi(z) = \Phi(0)$ identically. Looking at the Taylor expansion, we obtain

$$S_m\Phi(0) = \Phi(0)S_m.$$

This implies that $\Phi(0) = p(S_m)$, a polynomial in S_m. (Note that S_m is conjugate to S, the unweighted shift with entries $S_{\ell p} = \delta_{p+1\,\ell}$, which is its Jordan canonical form. For S the corresponding property is easy to see.) We write

$$D^{1,1} = \frac{\partial^2}{\partial z \partial \bar{z}}\bigg|_0 D(|z|^2) = -\begin{pmatrix} m & & & \\ & m-1 & & \\ & & \ddots & \\ & & & 1 \\ & & & & 0 \end{pmatrix},$$

and for the Taylor coefficient of $z\bar{z} = |z|^2$ we obtain

$$(\lambda - \lambda')\Phi(0) + D^{1,1}\Phi(0) - \Phi(0)D^{1,1} = 0.$$

Consider the diagonal of this matrix equality. All diagonal elements of $\Phi(0) = p(s_m)$ are the same number $x \neq 0$ (since $p(S_m)$ is triangular and invertible). Hence $\lambda - \lambda' = 0$. Now, since the diagonal entries of $D^{1,1}$ are all different, $\Phi(0)$ must be diagonal. So, $\Phi(0) = xI_{m+1}$. Also, $\Phi(0)$ intertwines the operators $M^{(\lambda,\boldsymbol{\mu})}$ and $M^{(\lambda',\boldsymbol{\mu}')}$, hence $\Phi(0)\mathbf{B}^{(\lambda,\boldsymbol{\mu})}(z,\omega)\Phi(0)^* = \mathbf{B}^{(\lambda',\boldsymbol{\mu}')}(z,\omega)$ as in (5.2). Using this with $z = \omega = 0$ and using (4.4), (4.5) we get $|x|^2\mu_j^2 = \mu_j'^2$ for all j. Suince $\mu_0 = 1 = \mu_0'$, it follows that $|x|^2 = 1$ and $\mu_j = \mu_j'$ for $1 \leq j \leq m$. □

7. SOME FURTHER REMARKS

We presented the operators $M^{(\lambda,\boldsymbol{\mu})}$ in as elementary a way as possible, but this presentation hides the natural way in which these operators can be found to begin with. One such way, which was actually followed by the authors, is to start with an irreducible finite dimensional representation ϱ_m of $\mathrm{SU}(1,1)$ (it is well known that there is exactly one for every natural number m), observe that $J(g,z) = \varrho_m(g^{-1})$ can be used as a multiplier, then transform this multiplier to a more convenient form, to construct a representation of $\tilde{G}$ and proceed from there. This was also the procedure of

Wilkins [12] who worked with the identical (2- dimensional) representation of SU(1, 1). The authors are planning to write an expository article in which there will be some details of this approach.

Another way, which is probably the most natural one, is to start with the process of holomorphic induction to construct the homogeneous vector bundles which are to be the Cowen - Douglas bundles of our operators. It is well known that every finite dimensional representation ϱ of the triangular subalgebra $\mathfrak{t}$ of $\mathfrak{sl}(2, \mathbb{C})$ (the Lie algebra of the stabilizer of 0 in $\mathrm{SL}(2, \mathbb{C})$ acting on the extended complex plane) gives rise to a $\tilde{G}$-homogeneous holomorphic vector bundle, from which the ϱ can be reconstructed. Refining this statement, it is easy to see that the $\tilde{G}$-homogeneous holomorphic Hermitian vector bundles are in one-to-one correspondence with the unitary equivalence classes of representations ϱ of $\mathfrak{t}$ on on the finite dimensional Hilbert spaces $\mathbb{C}^n$ with the added property that ϱ is skew Hermitian on the (one-dimensional;) subalgebra $\mathfrak{k}$, the Lie algebra of the stabilizer of 0 in $\mathrm{SU}(1, 1)$.

If we start with the restriction to $\mathfrak{t}$ of the $(m + 1)$-dimensional representation ϱ_m of $\mathfrak{sl}(2, \mathbb{C})$ and we put on the representation space all possible inner products so that the requirement concerning $\mathfrak{k}$ is satisfied then we obtain a family of bundles parametrized by $\lambda \in \mathbb{R}$ and $\mu_1, \ldots, \mu_m > 0$. It can then be shown that these bundles correspond to Cowen - Douglas operators if and only if $\lambda > m/2$, and in this case corresponding operator is $M^{(\lambda, \boldsymbol{\mu})}$.

One can use this approach starting with any finite dimensional representation ϱ of $\mathfrak{t}$. Such a ϱ can always be written as $\varepsilon_\lambda \otimes \varrho_0$, where ε_λ ($\lambda \in \mathbb{R}$) is a one dimensional representation of $\mathfrak{k} \cong \mathbb{R}$ extended trivially to $\mathfrak{t}$ and ϱ_0 is normalized in a certain way. There is always a corresponding homogeneous Hermitian vector bundle and a number λ_ϱ such that for $\lambda > \lambda_\varrho$ the bundle corresponds to a homogeneous Cowen - Douglas operator.

In this generality one cannot expect as explicit results as in the present paper, but one can proceed to still make fairly precise statements. In this way one gets a kind of classification of all homogeneous Cowen - Douglas operators. This will be the subject of a second article in this series.

Finally we mention that many of our arguments extend without change to the case of operator tuples and holomorphic vector bundles over bounded symmetric domains in several complex variables. There are, of course, a number of new features (cf. [2, 1]) as well in this general situation which still have to be explored in the future.

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