Proc. Indian Acad. Sci., Vol. 87 A, No. 2, February 1978, pp. 1-54, © Printed in India.

Geometry of G/P-II
[The work of De Concini and Procesi and the basic conjectures]

V LAKSHMIBAI and C S SESHADRI
School of Mathematics, Tata Institute of Fundamental Research, Bombay 400 005

MS received 22 December 1977

1. Introduction

The main aim of this paper is to show how the work of De Concini and Procesi [5] on classical invariant theory can be interpreted to suggest a generalisation of the Hodge-Young theory (*cf.* Hodge [11]; Hodge and Pedoe [12]) of standard monomials. This generalisation is given as a set of conjectures* (Conjectures I and II) in section 6. On the other hand, we also show that the results of De Concini and Procesi [5] follow as consequences of this generalisation of the Hodge-Young theory (*cf.* section 7).

Let $G=SL(n)$ and P a maximal parabolic subgroup in G so that G/P is the Grassmannian of r-dimensional vector subspaces of an n-dimensional vector space. Let f be a *lowest weight vector* in $H^{\circ}(G/P, L)$, where L denotes the ample generator of Pic G/P. If $W(G)$ denotes the Weyl group of G, one sees that the subgroup of $W(G)$ which fixes the one-dimensional linear space spanned by f, is precisely the Weyl group $W(P)$ of P. The translates of f by $W(G)$ can therefore be indexed by $W(G)/W(P)$ and we set

$$p_{\tau}=\tau . f, \; \tau \in W(G)/W(P).$$

Let us call a *standard monomial of length m on G/P*, an expression of the following form:

$$(*) \qquad p_{\tau_1} p_{\tau_2} \cdots p_{\tau_m}; \; \tau_1 \geqslant \tau_2 \geqslant \cdots \geqslant \tau_m$$

where by $\tau_1 \geqslant \tau_2$ etc. we take the canonical partial order in $W(G)/W(P)$. By the Hodge-Young theory we mean the theorem which states that distinct standard monomials of length m on G/P form a basis of $H^{\circ}(G/P, L^m)$, as well as its generalisation to a Schubert variety in G/P (*cf.* section 2). Let us call an expression of the following form

$$(\tau_1, \ldots, \tau_m), \; \tau_i \in W(G)/W(P), \; \tau_1 \geqslant \tau_2 \geqslant \ldots \geqslant \tau_m$$

*These conjectures have now been proved in collaboration with C. Musili.

a *Young diagram of length m* in $W(G)/W(P)$. We have a canonical identification $W(G)/W(P) \approx I_n(r)$, where

$$I_n(r) = \{(i_1, \ldots, i_r)/1 \leqslant i_1 < i_2 \ldots < i_r \leqslant n\}.$$

One knows that if $\tau_1 = (i_1, \ldots, i_r) = (i)$ and $\tau_2 = (j_1, \ldots, j_r) = (j)$, then

$$\tau_1 \geqslant \tau_2 \text{ in } W(G)/W(P) \Longleftrightarrow i_k \geqslant j_k,\ 1 \leqslant k \leqslant r.$$

Thus Young diagrams in $W(G)/W(P)$ are just certain types of Young diagrams taken in the usual sense (Weyl [23]) and the Hodge-Young theory gives a canonical identification of the set of Young diagrams of length m in $W(G)/W(P)$ with a basis of $H^\circ(G/P, L^m)$. One can define more generally what could be called *Young diagrams in* $W(G)$ (these are just the Young diagrams in the general sense as is usually understood (Weyl [23])), by which we can similarly write a basis of $H^\circ(G/B, M)$, M being an arbitrary line bundle on the *flag variety* G/B of G (Hodge [11]).

It is natural to ask whether the above theory could be carried over to the general case of a semi-simple algebraic group G and P, a parabolic subgroup of G. In (Seshadri [22]) such a generalisation has been done for the case when P is *minuscule* i.e. it is a maximal parabolic subgroup whose associated fundamental weight ω is *minuscule*. Recall Bourbaki [2], Seshadri, [22] that a fundamental weight ω is called *minuscule* if in the irreducible representation V of the group G', with highest weight ω, G' being the semisimple group defined over $\mathbf{C}$ with the same root system as G, *all the weights of V* are translates of ω by the Weyl group $W(G')$ of G'. Let us define $\{p_\tau\}$, $\tau \in W(G)/W(P)$, as well as standard monomials in $\{p_\tau\}$ as above. Then it has been shown (Seshadri [22]) that exactly the same results as above (stated for the case $G = SL(n)$) hold good. This generalisation is, however, not satisfactory enough, since when G is (almost) simple and not of type $\mathbf{A}$, there are not many minuscule weights and there are exceptional G with no minuscule weights at all (Bourbaki [2]; Seshadri [22]).

Let G be a semi-simple algebraic group and P a maximal parabolic subgroup of G. Let $X(\tau)$ denote the Schubert variety associated to $\tau \in W(G)/W(P)$. We have

$$\tau_1 \geqslant \tau_2 \text{ in } W(G)/W(P) \Longleftrightarrow X(\tau_1) \supseteq X(\tau_2).$$

Let $[X(\tau)]$ denote the class determined by $X(\tau)$ in the Chow ring $\mathrm{Ch}(G/P)$ of G/P. Let H denote the Schubert variety of codimension one in G/P. It can be shown that

$$[X(\tau)] \cdot [H] = \sum_{Y \in S_\tau} d_Y [Y],\ d_Y \geqslant 0$$

where centre dot denotes multiplication in $\mathrm{Ch}(G/P)$ and S_τ denotes the set of Schubert varieties of codimension one in $X(\tau)$. We call d_Y the *intersection multiplicity* of Y in $[X(\tau)] \cdot [H]$. If P is minuscule, it can be shown that $d_Y = 1$. Let us call P to be of *classical type* (see section 6) if $d_Y \leqslant 2$ for any pair of Schubert varieties $X(\tau)$ and Y as above. It can be shown that for any G there is always a maximal parabolic subgroup P which is of classical type and that if G is a classical group every maximal parabolic subgroup is of classical type. Let us suppose that P is of classical type.

Let us call a pair of elements (τ, φ) in $W(G)/W(P)$ to be an *admissible pair* in $W(G)/W(P)$ if either $\tau = \varphi$ (in which case it is called a *trivial* admissible pair) or $\tau \neq \varphi$ (in which case it is called a *non-trivial admissible pair*) and there exist $\{\tau_i\}$, $1 \leqslant i \leqslant r$, $\tau_i \in W(G)/W(P)$ such that

(i) $\tau_1 = \tau$, $\tau_m = \varphi$; $\tau_1 \geqslant \tau_2 \geqslant \ldots \geqslant \tau_m$

(ii) $X(\tau_i)$ is of codimension one in $X(\tau_{i-1})$, $2 \leqslant i \leqslant m$.

(iii) the intersection multiplicity of $X(\tau_i)$ in $[X(\tau_{i-1})] \cdot [H]$ is 2, $2 \leqslant i \leqslant m$.

Let (τ_1, φ_1), (τ_2, φ_2) be two admissible pairs in $W(G)/W(P)$. We define

$$(\tau_1, \varphi_1) \geqslant (\tau_2, \varphi_2) \Longleftrightarrow \varphi_1 \geqslant \tau_2.$$

We call a *Young diagram of length m in $W(G)/W(P)$*, a sequence of m admissible pairs in $W(G)/W(P)$ of the following type:

$$(\tau_1, \varphi_1), (\tau_2, \varphi_2), \ldots, (\tau_m, \varphi_m) \text{ with } (\tau_1, \varphi_1) \geqslant (\tau_2, \varphi_2) \geqslant \ldots \geqslant (\tau_m, \varphi_m)$$

where (τ_i, φ_i), for $1 \leqslant i \leqslant m$, are admissible pairs in $W(G)/W(P)$.

Then Conjectures I state (see section 6 for the precise form) that there is a canonical identification of the set of Young diagrams of length m in $W(G)/W(P)$ with a basis of $H^\circ(G/P, L^m)$ (L = the ample generator of Pic G/P); in fact if we denote by $p_{(\tau,\varphi)}$ the element in $H^\circ(G/P, L)$, associated to an admissible pair (τ, φ) in $W(G)/W(P)$, then the canonical basis in $H^\circ(G/P, L^m)$ is given by *standard monomials in $p_{(\tau,\varphi)}$*, of length m, defined in a similar manner as above. In Conjectures II, we take G to be a *classical group*. We then define *Young diagrams in $W(G)$* and state a canonical identification of Young diagrams of a certain type with a basis of $H^\circ(G/B, M)$ where M is *any line bundle* on G/B (B = a Borel subgroup in G).

The importance of this conjectural standard monomial theory (apart from its apparent independent interest) stems from the fact that it would provide a systematic approach towards the proof of the Kodaira type of vanishing theorems for line bundles (in the dominant chamber) on G/B and its Schubert subvarieties (Hochster [9]; Musili [18]; Lakshmibai *et al* [15]; Kempf [13]), the best theorem to date in this connection being due to Kempf [13]. In collaboration with Musili [16], we have proved the Conjectures I in the important particular case when the *fundamental weight ω associated to P is quasi-minuscule* (*equivalently a distinguished weight in the sense of* Kempf [13]) and this case suffices to deduce the results of Kempf [13]. (We call a fundamental weight ω *quasi-minuscule*, if in the irreducible representation V of the group G', with highest weight ω (G' being the semisimple group defined over **C** and having the same root system as that of G), *all the non-zero weights* are translates of ω by the Weyl group $W(G')$ of G').

For classical groups, Conjectures I have been checked for the case $m = 1$.

The connection between the above conjectures and the work of De Concini and Procesi [5] is as follows. Set:

Case (i): $X = \underbrace{W \oplus \ldots \oplus W}_{m \text{ times}} \oplus \underbrace{W^* \oplus \ldots \oplus W^*}_{m \text{ times}}$, $G = GL(W)$ and $\dim W = 2n$

Case (ii): $X = W \oplus \ldots \oplus W$ (m times), $\dim W = 2n$,
$G =$ the orthogonal group $\mathbf{O}(2n)$, $\mathbf{O}(2n) \subset GL(W)$.

Case (iii): $X = W \oplus \ldots \oplus W$ (m times), $\dim W = 2n$,
$G =$ the symplectic group $Sp(2n)$, $Sp(2n) \subset GL(W)$.

We now take the diagonal action of G on X. Set

$$X = \operatorname{Spec} R,\ Y = \operatorname{Spec} R^G,$$

De Concini and Procesi [5] have given a basis of the invariant subring R^G of R by means of certain types of what they call *standard monomials* and have shown that (also Hochster and Eagon [10]) for the linear case and Kutz [14] for the orthogonal case)

$$Y = \begin{cases} \text{determinantal variety in the space } M_m \text{ of } (m \times m) \text{ matrices in case (i)} \\ \quad ,, \qquad ,, \qquad ,, \qquad \text{Sym } M_m \text{ of } (m \times m) \text{ symmetric matrices in case (ii)} \\ \quad ,, \qquad ,, \qquad ,, \qquad \text{Sk } M_m \text{ of } (m \times m) \text{ skew-symmetric matrices in case (iii).} \end{cases}$$

One knows that a determinantal variety in M_m can be identified as an open subset—in fact as the *opposite big cell*—of a Schubert variety in a Grassmannian (*cf* section 2 below and Hochster [9]; Musili [19]). Then it is easily checked that the standard monomials written down by De Concini and Procesi for Case (i) are precisely the restrictions (to the opposite big cell) of the standard monomials on Grassmannians in the Hodge-Young sense as we described above. One then observes (*cf.* section 4) that a determinantal variety in Sym M_m (resp. Sk M_m) is precisely the opposite big cell of a Schubert variety in $Sp\,(2n)/Q$ (resp. $SO\,(2n)/Q$), where Q is the maximal parabolic subgroup associated to a right end root (in the Dynkin diagram, Bourbaki [2]). In the case of $SO(2n)$, it is known that Q is *minuscule* and one finds (ch. Th. 5.1) that the standard monomials of De Concini and Procesi for Case (iii) are again the restrictions to the opposite big cell of standard monomials in the sense of (Seshadri [22]). In the case of $Sp(2n)$, Q is not *minuscule* and the question arose whether the standard monomials of De Concini and Procesi for Case (ii), could be properly interpreted to suggest a good definition of standard monomials for the nonminuscule case. Such an interpretation is given in section 5 and has been the principal motivation behind Conjectures I.

The best way to read this paper is perhaps to go through section 2 first, then the statement of Theorem 4.1 and thereafter sections 5, 6 and 7. The rest of the paper (which is fairly long) is to be referred to, whenever found necessary; probably this material could be written in a more condensed manner so as to reduce the length of the paper.

The second named author is thankful to C. Procesi for the discussions he had with him on his work with De Concini during his stay in Rome in April 1976. The project of this paper took shape during these discussions. We are thankful to N Soundararajan of the Computer group, TIFR for his crucial help in the verification of the above conjectures in low ranks; in fact he has programmed the verifications of Conjectures I for $G = Sp(2n)$, $m = 2$ as well as Conjectures II for $G = Sp(2n)$, $|m| = 2$; in particular, for these cases Conjectures I and II have been verfied for $n \leqslant 7$. We are also thankful to C. Musili for his careful reading of this paper.

1. Basic facts on $SL(n)$

We work with an algebraically closed base field K.

Let V be an m-dimensional vector space over K. We fix a basis $e_1, \ldots, e_m$ of V. We write the elements of V as column vectors of length m with entries in K, so that

$$e_1 = \begin{pmatrix} 1 \\ 0 \\ \cdot \\ \cdot \\ \cdot \\ 0 \end{pmatrix}, \ldots e_i = \begin{pmatrix} 0 \\ \cdot \\ \cdot \\ \cdot \\ 1 \\ 0 \\ 0 \end{pmatrix}, \text{ 1 in the } i\text{th place, 0 elsewhere.}$$

Then multiplication of elements of V by elements of $GL(m)$ on the left, makes V a (left) $GL(m)$ module. We refer to this as the *canonical $GL(m)$ module structure on V.* We set

$H = SL(m)$
B_m = the Borel subgroup in $GL(m)$ formed by upper triangular matrices
T_m = maximal torus in $GL(m)$ formed by diagonal matrices
$B(H)$ = the Borel subgroup $B_m \cap SL(m)$ in H
$T(H)$ = the maximal torus $T_m \cap SL(m)$ in H
L_m = the Lie algebra of $GL(m)$ identified with the set of all $(m \times m)$matrices
Lie H = the Lie algebra of H identified with the set of $(m \times m)$ matrices of trace zero
$N(T_m)$ = normaliser of T_m in $GL(m)$
$N(T(H)) = N(T_m) \cap H$ (it is the normaliser of $T(H)$ in H)
$W(H) = W_m = N(T(H))/T(H) = N(T_m)/T_m$ (Weyl group of H or $GL(m)$)
S_m = symmetric group on the m letters $(1, \ldots, m)$

If θ is an element of S_m such that $\theta(\mathrm{i}) = a_i$, as is customary, we write

$$\theta = \begin{pmatrix} 1 \ldots m \\ a_1 \ldots a_m \end{pmatrix} \text{ or simply } \theta = (a_1, \ldots, a_m).$$

If $M \in N(T_m)$, we check that there is a uniquely determined element $\theta = (a_1, \ldots, a_m) \in S_m$ such that in the ith column of M, the only non-zero element is in the a_ith row. We then write $M = M(\theta)$. The mapping $M \longmapsto \theta$ gives an isomorphism of $W(H)$ or W_m onto S_m, which we call the canonical isomorphism of $W(H)$ or W_m onto S_m. If $t \in T_m$ is of the form

(1) $$t = \begin{pmatrix} t_1 & & & 0 \\ & \cdot & & \\ & & \cdot & \\ 0 & & & t_m \end{pmatrix}$$

we see that

(2) $$M(\theta)t\,(M(\theta))^{-1} = \begin{pmatrix} t_{b_1} & & & 0 \\ & \cdot & & \\ & & \cdot & \\ 0 & & & t_{b_m} \end{pmatrix}, \quad (M(\theta))^{-1}tM(\theta) = \begin{pmatrix} t_{a_1} & & & 0 \\ & \cdot & & \\ & & \cdot & \\ 0 & & & t_{a_m} \end{pmatrix}.$$

where the element $(b_1, \ldots, b_m)$ is the inverse θ^{-1} of θ in S_m.

We set:

$$X_0(T_m) \text{ (resp. } X_0(T(H))) = \operatorname{Hom}(T_m, \mathbf{G}_m) \text{ (resp. } \operatorname{Hom}(T(H), \mathbf{G}_m))$$

$$X(T_m) \text{ (resp. } (X\,T(H))) = X_0(T_m) \otimes_{\mathbf{Z}} \mathbf{R} \text{ (resp. } X_0(T(H)) \otimes_{\mathbf{Z}} \mathbf{R}).$$

The elements $\epsilon_i \in X_0(T_m)$, $1 \leqslant i \leqslant m$, defined by

$$\epsilon_i(t) = t_i,\ t \in T_m \text{ as in (1) above}$$

form a $\mathbf{Z}$-basis of $X_0(T_m)$. We denote the canonical images of ϵ_i in $X(T_m)$ by the same ϵ_i so that $\{\epsilon_i\}$, $1 \leqslant i \leqslant m$, is a basis of the $\mathbf{R}$-vector space $X(T_m)$. The canonical action of $W_m \approx S_m$ on $X_0(T_m)$ (and hence on $X(T_m)$) is defined as follows: for $\theta = (a_1, \ldots, a_m) \in S_m$ and $\chi \in X_0(T_m)$ ($\chi : T \to \mathbf{G}_m$), we define $\theta \cdot \chi$ by

$$(\theta \cdot \chi)(t) = \chi M(\theta)^{-1}\, tM(\theta),\ t \in T_m \text{ and } M(\theta) \in N(T_m)$$

as above. Then by (2) above, we see that

$$\theta(\epsilon_i) = \epsilon_{a_i},\ 1 \leqslant i \leqslant m.$$

We have a canonical surjective linear map

$$\varphi : X(T_m) \to X(T(H)).$$

We see that the restriction of φ to the linear subspace spanned by $\epsilon_i-\epsilon_j$ is an isomorphism and *we identify $X(T(H))$ with this subspace.* One knows that the set

$$R(H)=\{\epsilon_i-\epsilon_j/i\neq j\}$$

can be identified with the set of *roots* of H with respect to $T(H)$ and that

$$R^+(H)=\{\epsilon_i-\epsilon_j/i<j\}$$

is the set of *positive roots* (with respect to $B(H)$). If $\alpha=\epsilon_i-\epsilon_j\in R(H)$, we denote by U_α the unipotent subgroup ($\approx \mathbf{G}_a$) of $GL(m)$ or H defined by:

$U_\alpha=\{I+g/I=(m\times m)$ identity matrix, g an $(m\times m)$ matrix whose only non-zero entry is in the (i,j)th place$\}$.

The action of $T(H)$ on $U_\alpha(\approx\mathbf{G}_a)$ is given by the element of $X_0T(H)$, canonically associated to the root $\alpha=\epsilon_i-\epsilon_j$, since we have

$$t(g_{ij})t^{-1}=(g_{ij}t_it_j^{-1}),\ t \text{ as in (1)},\ (g_{ij})\in GL(m).$$

The $\{U_\alpha\}$, $\alpha>0$, generate the unipotent subgroup $B^u(H)$ of $B(H)$. We see also that the canonical action of $W(H)$ $(=W_m=S_m)$ on $R(H)$ is explicitly given as follows:

$$\theta=(a_1,\ldots,a_m)\in S_m,\ \theta(\epsilon_i-\epsilon_j),=\epsilon_{a_i}-\epsilon_{a_j}.$$

The K-vector space V has a canonical (left) H-module structure induced from the canonical $GL(m)$ module structure on V. Set $l=m-1$, $m\geqslant 2$. Then $\overset{r}{\Lambda}V$, $1\leqslant r\leqslant l$ have canonical H-module structures. One knows that these are the *fundamental representations* of H. We see that the vector $e_1\Lambda\ldots\Lambda e_r\in\overset{r}{\Lambda}V$ is a highest weight vector with respect to $T(H)$ and $B(H)$ since the 1-dimensional space spanned by $e_1\Lambda\ldots\Lambda e_r$ is stable under $B(H)$. If we denote by $\{\alpha_i\}$, $1\leqslant i\leqslant l$, the simple roots $\alpha_i=\epsilon_i-\epsilon_{i+1}$ $1\leqslant i\leqslant l$ in $R(H)$ and by $\{\omega_i\}$, $1\leqslant i\leqslant l$ the fundamental weights, having the property, $\langle\omega_i,\alpha_j^{\vee}\rangle=\delta_{ij}$ (Kronecker delta) (notations as in Bourbaki [2]) we find that the weight of $e_1\Lambda\ldots\Lambda e_r$ is ω_r (we see that the weight of $e_1\Lambda\ldots\Lambda e_r$ under T_m is $\epsilon_1+\ldots+\epsilon_r-(r/m)\left(\Sigma_{i=1}^{m}\epsilon_i\right)$ which is precisely ω_r, see Bourbaki [2]). Thus the highest weight of the H-module $\overset{r}{\Lambda}V$ is ω_r.

Let $\mathbf{P}(\overset{r}{\Lambda}V)$ denote the projective space of 1-dimensional subspaces of $\overset{r}{\Lambda}V$ (in Grothendieck's sense, this should be denoted by $\mathbf{P}((\overset{r}{\Lambda}V)^{\vee})$, $(\overset{r}{\Lambda}V)^{\vee}$=dual of $(\overset{r}{\Lambda}V)$. Then H operates on $\mathbf{P}(\overset{r}{\Lambda}V)$. We denote by P_r the isotropy subgroup of H at the point of $\mathbf{P}(\overset{r}{\Lambda}V)$ corresponding to $e_1\Lambda\ldots\Lambda e_r$. Then P_r is a maximal *parabolic subgroup* in H, $P_r\supset B(H)$ and H/P_r can be canonically identified with the Grassmannian of r dimensional linear subspaces in V. We see that P_r is of the form

$$\left(\begin{array}{c|c} (r\times r) & * \\ \hline 0 & (m-r)\times(m-r)\end{array}\right)$$

In fact P_r is the *maximal parabolic subgroup in H canonically associated to the simple root* α_r (Borel [1]; Lakshmibai *et al* [15]) i.e., if M_r is the *reductive part of* P_r, then M_r is generated by T_m and all U_α such that the root α is spanned by the simple roots $\{\alpha_i\}$, $i \neq r$(i.e. $\alpha_i \neq \alpha_r$) (or that the unipotent part of the radical of P_r is generated by all U_α, such that $\alpha > 0$ and $\alpha = \Sigma n_i \alpha_i$ with $n_r \neq 0$).

Let $W_r = W_{P_r}$ be the Weyl group of P_r. One knows that W_r is the isotropy subgroup of $W(H)$ at $e_1 \Lambda \ldots \Lambda e_r$ (in the sense that $\theta \in W(H) = S_m$ is in W_r if and only if $M(\theta)$ leaves stable the 1-dimensional subspace through $e_1 \Lambda \ldots \Lambda e_r$). Note that if $\theta = (a_1, \ldots, a_m)$, then

$$M(\theta)e_i = m_{a_i, i}\, e_{a_i},\ M(\theta) = (m_{ij}).$$

This implies that

$$M(\theta)(e_1 \Lambda \ldots \Lambda e_r) = \lambda e_{a_1} \Lambda \ldots \Lambda e_{a_r},\ \lambda \neq 0. \tag{3}$$

Let $I_m(r)$ denote the following set:

$$I_m(r) = \{(i_1, \ldots, i_r)/1 \leqslant i_1 < i_2 < \ldots < i_r \leqslant m\}.$$

Then from (3) above, we see that we have a canonical identification of $W(H)/W_r$ with $I_m(r)$; in fact, if $\theta = (a_1, \ldots, a_m) \in S_m$, then the canonical image of θ in $I_m(r)$ is obtained by taking the first r elements $a_1, \ldots, a_r$ of θ and arranging them in the increasing order. We have a canonical *partial order in* $W(H)$ (resp. $W(H)/W_r$). Given w_1, w_2 in $W(H)$ (resp. $W(H)/W_r$), this partial order can be defined by saying that given a *reduced decomposition* of w_2 (with respect to the simple reflections s_i in $W(H)$, s_i being the reflection with respect to the simple root α_i), it contains some reduced decomposition of w_1 as a 'subword' or equivalently as follows: Let $X(w_i, H/B(H))$ (resp. $X(w_i, H/P_r)$) denote the *Schubert variety* in $H/B(H)$ (resp. H/P_r) associated to w_i i.e. this variety is the Zariski closure of the *Schubert cell* $B(H)\, w_i e_{B(H)}$ (resp. $B(H)\, w_i\, e_{P_r}$) in $H/B(H)$ (resp. H/P_r), where we denote by $e_{B(H)}$ (resp. e_{P_r}), the point in $H/B(H)$ (resp. H/P_r) represented by the class $B(H)$ (resp. P_r) (note also that when we write $w_i\, e_{B(H)}$ (resp. $w_i\, e_{P_r}$), we use a representative of w_i in $N(T(H))$ and that $w_i\, e_{B(H)} \in H/B(H)$ (resp. $w_i\, e_{P_r} \in H/P_r$) is independent of the choice of a representative). Then

$$w_1 \leqslant w_2 \text{ in } W(H) \text{ (resp. } W(H)/W_r) \Longleftrightarrow X(w_1, H/B(H)) \subseteq (Xw_2, H/B(H))$$
$$\text{(resp. } X(w_1, H/P_r) \subseteq X(w_2, H/P_r)).$$

We have a *canonical partial order* in $I_m(r)$, namely, if $(i) = (i_1, \ldots, i_r)$ and $(j) = (j_1, \ldots, j_r)$ are in $I_m(r)$, we write

$$(i) \leqslant (j) \Longleftrightarrow i_1 \leqslant j_1, \ldots, i_r \leqslant j_r.$$

Then we have the following:

(i) The canonical identification of $W(H)/W(P_r)$ with $I_m(r)$ preserves the canonical partial orders in each.

(ii) If $w \in W(H)/W(P_r)$ and $w=(i_1, \ldots, i_r) \in I_m(r)$ (identifying $W(H)/W(P_r)$ with $I_m(r)$), then

$$\dim X(w, H/P_r) = \sum_{k=1}^{r} i_k - \frac{r(r+1)}{2}$$

(iii) Let $w = (i_1, \ldots, i_r) \in I_m(r)$. Let $(j_1, \ldots, j_s)$ be the complement of $(i_1, \ldots, i_r)$ in $(1, 2, \ldots, m)$ arranged in the increasing order. Then $w_1 = (i_1, \ldots, i_r, j_1, \ldots, j_s)$ is the *minimal representative* in $W(H)$ of $w \in W(H)/W(P_r)$, in the sense that among all the representatives of w in $W(H)$, w_1 is of minimum length in $W(H)$ (see Bourbaki [2] for the definition of length).

(iv) For $w \in W(H)$, one knows that the dimension of the Schubert variety $X(w, H/B(H))$ is equal to the *length of* w *in* $W(H)$ denoted $l(w, W(H))$. For $w \in W(H)/W(P_r)$, we call the length of w in $W(H)/W(P_r)$, denoted $l(w, W(H)/W(P_r))$ the length in $W(H)$ of the minimal representative of w in $W(H)$. One knows that the dimension of $X(w, H/P_r)$, $w \in W(H)/W(P_r)$ is equal to $l(w, (H)/W(P_r))$.

Let more generally G be a semi-simple group (defined over K), B a Borel subgroup in G and P a parabolic subgroup in G, $P \supset B$. We denote by $W(G)$ (resp. $W(P)$) the Weyl group of G (resp. P). For $w \in W(G)$ (resp. $W(G)/W(P)$), we denote by $X(w, G/B)$ (resp. $X(w, G/P)$), the Schubert variety in G/B (resp. G/P) associated to w. Let $w \in W(G)/W(P)$; then we call an element $w_1 \in W(G)$ a *minimal* (resp. *maximal*) representative of w if (i) w_1 represents w and (ii) if w_2 is any representative of w in $W(G)$, then $l(w_2) \geqslant l(w_1)$ (resp. $l(w_2) \leqslant l(w_1)$). A minimal representative w_1 of w is characterized by the following property (Bourbaki [2]).

$$l(w_1\lambda, W(G)) = l(w_1, W(G)) + l(\lambda, W(G)), \lambda \in W(P).$$

The following can be checked easily (and we require them later):

(i) the minimal and maximal representatives in $W(G)$ of $w \in W(G)/W(P)$ are uniquely determined. Let w_i, $i = 1, 2$ be respectively the minimal and maximal representatives in $W(G)$ of w. Then we have:

(ii) $X(w_2, G/B)$ is the inverse image of $X(w, G/P)$ under the canonical morphism $G/B \rightarrow G/P$. Further, under this morphism, $X(w_1, G/B)$ maps on to $X(w, G/P)$ and the morphism $X(w_1, G/B) \rightarrow X(w, G/P)$ is birational.

(iii) $w_2 = w_1 \cdot w_0(P)$, where $w_0(P)$ is the element of maximal length in $W(P)$.

(iv) Let $w' \in W(G)/W(P)$ and w'_1, w'_2 be respectively the minimal and maximal representatives in $W(G)$ of w'. Then

$$w \leqslant w' \text{ in } W(G)/W(P) \Longleftrightarrow w_1 \leqslant w'_1 \text{ in } W(G) \Longleftrightarrow w_2 \leqslant w'_2 \text{ in } W(G).$$

(v) if w'_1, w'_2 are two representatives in $W(G)$ of w such that

$$l(w'_2, W(G)) = l(w'_1, W(G)) + l(w_0(P), W(G))$$

then $w_1 = w'_1$ and $w_2 = w'_2$.

Let L_r denote the (very) ample generator of Pic $H/P_r \approx \mathbf{Z}$. Then $H^0(H/P_r, L_r)$ acquires a canonical (left) H-module structure and, in fact, it is the dual of $\overset{r}{\Lambda} V$. Let us denote by

$$p_{(j)} = p_{(j_1, \ldots, j_r)}, (j) = (j_1, \ldots, j_r) \in I_m(r)$$

the canonical dual basis in $H^0(H/P_r, L_r)$, so that we have

$$\langle e_{(i)}, p_{(j)} \rangle = \delta_{(i), (j)} \text{ (Kronecker delta)}, e_i = e_{i_1} \Lambda \ldots \Lambda e_{i_r}.$$

The $p_{(j)}$ are known as the *Plücker coordinates*. We observe the following:

(i) $p_{(m-r+1, \ldots, m)}$ is a *highest weight vector in* $H^\circ(H/P_r, L_r)$ and its weight is $i(\omega_r)$, where i denotes the Weyl involution, i.e. $i = -w_0$, w_0 being the element of largest length in $W(H)$.

(ii) $p_{(1, \ldots r)}$ is a *lowest weight vector in* $H^\circ(H/P_r, L_r)$ and its weight is $-\omega_r$.

Now we have two natural ways of indexing in a more intrinsic manner the weight vectors in $H^\circ(H/P_r, L_r)$. For this we observe that the subgroup of $W(H)$ which fixes the 1-dimensional subspace spanned by $p_{(m-r+1, \ldots, m)}$ is $W(i(P_r))$, $i(P_r)$ being the maximal parabolic subgroup, canonically associated to $i(\omega_r)$, in the sense described above. We write this property as follows (by abuse of notation):

$$\tau \in W(H), \ \tau \cdot p_{(m-r+1, \ldots, m)} = p_{(m-r+1, \ldots, m)} \Longleftrightarrow \tau \in W(i(P_r)).$$

Similarly, one finds that

$$\tau \in W(H), \tau \cdot p_{(1, \ldots, r)} = p_{(1, \ldots, r)} \Longleftrightarrow \tau \in W(P_r).$$

Hence, we write

$$\{f_\tau\}, \tau \in W(H)/W(i(P_r)), \quad f_\tau = \tau \cdot p_{(m-r+1, \ldots, m)}$$

$$\{p_\tau\}, \tau \in W(H)/W(P_r); \quad p_\tau = \tau \cdot p_{(1, \ldots, r)}$$

$$\{e_\tau\}, \ \tau \in W(H)/W(P_r), \quad e_\tau = \tau \cdot e_{(1, \ldots, r)} = \tau \cdot (e_1 \Lambda \ldots \Lambda e_r)$$

(the elements f_τ, p_τ, e_τ are well-defined up to scalar multiples). The $\{p_\tau\}$, $\{f_\tau\}$ are two natural ways of indexing the weight vectors in $H^\circ(H/P_r, L_r)$. We see that

$$\langle e_\tau, p_{\tau'} \rangle = \delta_{\tau, \tau'} \text{ (Kronecker delta)}.$$

One has now the following properties, which hold in a more general context (Seshadri [22] where the indexing similar to $\{f_\tau\}$ is considered):

(i) For $\tau =$ identity (i.e. the class $W(i(P_r))$, f_τ is a *highest weight vector.*

(ii) $f_{\tau_1} \mid X(\tau_2 w_0, H/P_r)$ is not identically zero $\Longleftrightarrow \tau_1 \geqslant \tau_2$ in $W(H)/W(i(P_r))$ (one observes that the Schubert varieties in H/P_r can be written as $X(\tau w_0, H/P_r)$, $\tau \in (W(H)/W(i(P_r))$, but then

$$\tau_1 \geqslant \tau_2 \text{ in } W(H)/W(i(P_r)) \Longleftrightarrow X(\tau_1 w_0, H/P_r) \subseteq X(\tau_2 w_0, H/P_r)$$

(a) For $\tau =$ identity, p_τ is a *lowest weight vector.*

(b) $\tau_1 \leqslant \tau_2$ in $W(H)/W(P_r) \Longleftrightarrow p_{\tau_1} \mid X(\tau_2, H/P_r) \neq 0.$

Now (b) takes the explicit form as follows: Let $\tau \in W(H)/W(P_r) \approx I_m(r)$ and $\tau = (i) = (i_1, \ldots, i_r) \in I_m(r)$. Then if $p_{(j)}$, $(j) \in I_m(r)$ denotes a Plücker coordinate as above, we have

$$p_{(j)} \mid X(\tau, H/P_r) \neq 0 \Longleftrightarrow (j) \leqslant (i).$$

We see that the set

$$\{x \in H/P_r \mid p_{(m-r+1, \ldots, m)}(x) \neq 0\}$$

is the *big cell* in H/P_r. We call the set

$$\{x \in H/P_r \mid p_{(1, \ldots, r)}(x) \neq 0\}$$

the opposite big cell in H/P_r.

2. Determinantal varieties and Grassmannians (Hochster [9]; Musili [19])

We keep the basic notations of section 1. Let $M_{m,r}$ denote the set of $(m \times r)$ matrices with entries in K, $m>r$. Let V be as in section 1 i.e. an m-dimensional vector space over K with a standard basis $e_1, \ldots, e_m$, such that its elements are denoted by column vectors etc. Let $Z = (Z_1, \ldots, Z_r) \in M_{m,r}$, Z_i being the ith column of Z. Then we can identify Z_i with an element of V. Let φ_1 be the mapping

$$\varphi_1 : M_{m,r} \to \overset{r}{\Lambda}\, V, \; Z \to Z_1 \wedge \ldots \wedge Z_r.$$

Let $p_{(i)}$, $(i) = (i_1, \ldots, i_r) \in I_m(r)$, be as in section 1 the canonical dual basis in the dual of $\overset{r}{\Lambda}\, V$ (Plücker coordinates). We see that

$$q_{(i)} = (p_{(i)} \circ \varphi_1)(Z) = \begin{cases} \text{determinant of the } (r \times r) \text{ minor of } Z \text{ formed by} \\ \text{the rows corresponding to the indices } i_1, \ldots, i_r. \end{cases}$$

Multiplication by an $(m\times m)$ matrix (resp. $(r\times r)$ matrix) on the left (resp. on the right) gives a left (resp. right) action of $GL(m)$ (resp. $GL(r)$) on $M_{m,r}$. We see that φ_1 is $GL(m)$ equivariant. Let $M^s_{m,r}$ denote the subset of $M_{m,r}$ formed by matrices of rank r. Then φ_1 induces a surjective morphism

$$\varphi_2 : M^s_{m,r} \to H/P_r \ (H/P_r \text{ as in section 1})$$

and in fact the Grassmannian H/P_r is the orbit space $M^s_{m,r}/GL(r)$. We see that the inverse image by φ_2 of the *opposite big cell* in H/P_r is the following set

$$\left\{Z \in M^s_{m,r} \middle| q_{(1,\ldots,r)}(Z) \neq 0\right\}.$$

From this it is immediate that the restriction of φ_2 to the following subset of $M^s_{m,r}$, i.e.

$$\begin{pmatrix} Id_r \\ Y \end{pmatrix}, \quad \begin{cases} Id_r = r\times r \text{ identity matrix} \\ Y \in M_{m-r,r} \end{cases}$$

is an isomorphism onto the opposite big cell in H/P_r.

Let $m = n+r$ and $n \geqslant r$. Let ν be the closed immersion

$$\nu : M_{n,r} \to M_{n+r,r}$$

defined by

$$X \in M_{n,r},\ X \to \begin{pmatrix} Id_r \\ JX \end{pmatrix},\ J = \begin{pmatrix} 0 & & 1 \\ & .\cdot^{1} & \\ 1 & & 0 \end{pmatrix} \ (n\times n \text{ matrix}).$$

We note that $\nu(M_{n,r}) \subset M^s_{n+r,r}$ and that $\varphi_2 \circ \nu$ is an isomorphism of $M_{n,r}$ on to the opposite big cell in H/P_r.

Definition 2.1. (i) Let $(i) = (i_1, \ldots, i_s)$ and $(j) = (j_1, \ldots, j_t)$ be two sets of integers. We write $(i) \leqslant (j)$ if $s \geqslant t$ and $i_k \leqslant j_k$, $k \leqslant t$, i.e.

$$\begin{array}{l} i_1 \leqslant j_1 \\ \vdots \quad\ \ \vdots \\ i_t \leqslant j_i \\ \vdots \\ i_s \end{array}$$

If we have pairs of sets of integers $((i), (j))$ and $((i)', (j)')$, $(i) = (i_1, \ldots, i_s)$, $(i)' = (i_1', \ldots, i_{s'}')$, $(j) = (j_1, \ldots, j_t)$, $(j)' = (j_1', \ldots, j_{t'}')$, we write $((i), (j)) \leqslant ((i)', (j)')$ if $(i) \leqslant (i)'$ and $(j) \leqslant (j)'$.

(ii) Let $(i)=(i_1, \ldots, i_k) \in I_n(k)$, $(j)=(j_1, \ldots, j_k) \in I_r(k)$ with $0 \leqslant k \leqslant r$ (if $k=0$, (i) and (j) are supposed to be empty). We denote by $p_{(i),(j)} : M_{n,r} \to K$ the function defined by $Z \to p_{(i),(j)}(Z)$ where

$$p_{(i),(j)}(Z) = \begin{cases} \text{determinant of the } (k\times k) \text{ minor of } Z \text{ formed by the rows} \\ \text{and columns corresponding to the indices in (i) and} \\ (j) \text{ respectively. If } k=0,\ p_{(i),(j)}=1. \end{cases}$$

Definition 2.2. Let $(\lambda)=(\lambda_1, \ldots, \lambda_r) \in W(H)/W(P_r) \approx I_m(r)$ where $I_m(r)=\{(i_1, \ldots i_r) \mid 1 \leqslant i_1 < i_2 < \ldots < i_r \leqslant m\}$.

We call the pair $((\alpha), (\beta))$ (resp. $((i), (j))$), $(\alpha) \in I_n(k)$, $(\beta) \in I_r(k)$, $0 \leqslant k \leqslant r$ (resp. $(i) \in I_n(k')$, $(j) \in I_r(k')$), $0 \leqslant k' \leqslant r$) defined below, as the *canonical pair* (resp. the *canonical dual pair*) *associated to* (λ):

Firstly the integers k and k' are defined by

$$\lambda_k \leqslant n,\ \lambda_{k+1} > n:$$

$$\lambda_{r-k'} \leqslant r,\ \lambda_{r-k'+1} > r \text{ (recall that } m=n+r). \quad \text{Now set}$$

$(\alpha) =$ $(\lambda_1, \ldots, \lambda_k)$

$(\beta) =$ complement of $(m+1-\lambda_{k+1})$, $(m+1-\lambda_{k+2}), \ldots, (m+1-\lambda_r)$ in $(1, \ldots, r)$ arranged in the ascending order.

$(i) =$ $(m+1-\lambda_{r-k'+1}), (m+1-\lambda_{r-k'+2}), \ldots, (m+1-\lambda_r)$ arranged in the ascending order.

$(j) =$ complement of $(\lambda_1, \ldots, \lambda_{r-k'})$ in $(1, \ldots, r)$ arranged in the ascending order.

Note: (i) k (resp. k') is $=0$ means that (α) and (β) (resp. (i) and (j)) are empty.

(ii) when $r=n$, we have $k'=n-k$ and

$(\beta) =$ complement of $(i_1, \ldots, i_{n-k})$ in $(1, \ldots, n)$ arranged in the increasing order.

$(\alpha) =$ complement of $(j_1, \ldots j_{n-k})$ in $(1, \ldots, n)$ arranged in the increasing order.

Lemma 2.1. (i) Let $(\lambda) \in I_m(r)$. Then if $q_{(\lambda)}: M_{n+r,r} \to K$ is the function defined by: $Z \in M_{n+r}$, $q_{(\lambda)}(Z) =$ determinant of the $(r\times r)$ minor of Z formed by rows corresponding to the indices $\lambda_1, \ldots, \lambda_r$, we have

$$q_{(\lambda)} \circ \nu = p_{(i),(j)},$$

where $((i), (j))$ is the *canonical dual pair* associated to (λ).

(ii) Let $(\lambda), (\lambda)' \in I_m(r)$ and $((\alpha), (\beta))$ (resp. $((i), (j))$), $((\alpha)', (\beta)')$ (resp. $((i)', (j)')$) the *canonical pairs* (resp. canonical dual pairs) associated respectively to $(\lambda), (\lambda)'$. Then we have a bijective map φ (resp. φ') of $W(H)/W(P_n)$ on to the set of canonical pairs (resp. canonical dual pairs) given by

$$\varphi((\lambda)) = ((\alpha), (\beta)) \text{ (resp. } \varphi'((\lambda)) = ((i), (j))\text{)}.$$

Further

$$((\alpha), (\beta)) \leqslant ((\alpha)', (\beta)') \Longleftrightarrow (\lambda) \leqslant (\lambda)'$$

$$\text{resp. } ((i), (j)) \leqslant ((i)', (j)') \Longleftrightarrow (\lambda)' \leqslant (\lambda)).$$

i.e., φ (resp. φ') is order preserving (resp. order reversing).

Proof. (i) Let $W \in M_{n+r,\, r}$ be the element defined by

$$W = \begin{pmatrix} Y \\ J_2 \end{pmatrix}, \; Y \in M_{n,\, r}, \; J_2 = \begin{pmatrix} 0 & & & 1 \\ & & 1 & \\ & 1 & & \\ 1 & & & 0 \end{pmatrix} \quad (r \times r \text{ matrix}).$$

We claim that we have the following:

$$(*) \quad \begin{cases} \text{Let } (\lambda) = (\lambda_1, \ldots, \lambda_r) \in I_{n+r}(r); \text{ then } q_{(\lambda)}(W) = \pm\, p_{(\alpha),\,(\beta)}(Y), \text{ where} \\ ((\alpha), (\beta)) \text{ is the canonical pair associated to } (\lambda) \text{ (cf. Definition 2.2 above).} \end{cases}$$

The relation (*) states that the determinant of the minor of W formed by the rows corresponding to the indices of (λ) is equal to the determinant (upto sign) of the minor of Y formed by the rows and columns corresponding to the indices in (α) and (β) respectively. We leave the verification of (*) as an exercise.

$$\text{Let us set } Y = J_1 Z, \; J_1 = \begin{pmatrix} 0 & & 1 \\ & 1 \cdot & \\ \cdot \cdot & & \\ 1 & & 0 \end{pmatrix} \quad (n \times n \text{ matrix}). \quad \text{We see that}$$

$$\begin{pmatrix} I \\ Z \end{pmatrix} = \begin{pmatrix} 0 & J_2 \\ J_1 & 0 \end{pmatrix} \begin{pmatrix} Y \\ J_2 \end{pmatrix}, \; \begin{pmatrix} 0 & J_2 \\ J_1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & \cdot & \cdot & \cdot & 1 \\ & & & 1 & \\ & & 1 & & \\ 1 & & & & 0 \end{pmatrix} \quad (m \times m \text{ matrix})$$

so that the matrix $\begin{pmatrix} I \\ Z \end{pmatrix}$ is obtained from $\begin{pmatrix} Y \\ J_2 \end{pmatrix}$ by changing the lth row to the $(n+r+1-l)$th row, $1 \leqslant l \leqslant n+r$. Thus (*) and this observation prove (i).

The assertion (ii) is left as an easy exercise.

Definition 2.3. (i) Let $R_{n,\, r}$ denote the polynomial algebra over K formed by nr indeterminates so that $M_{n,\, r}$ is the set of K-valued points of Spec $R_{n,\, r}$ and we refer to $R_{n,\, r}$ as the coordinate ring of $M_{n,\, r}$. Let $\triangle_k$ denote the ideal (called a *determinantal* *ideal*) in $R_{n,r}$ generated by $p_{(i),(j)}$ (*cf.* Def. 2.1 for the definition of $p_{(i),(j)}$) such that the number of indices in (i) (= that of (j)) is $> k$ or equivalently

$$((1, \ldots k), (1, \ldots, k)) \not\leqslant ((i), (j)).$$

We denote by D_k the zero set (in $M_{n,r}$) of the ideal $\triangle_k$; D_k is called a *determinantal variety* (we shall see that D_k is a variety). We see that D_k is defined by the vanishing of all $(d \times d)$ minors, $d \geqslant k+1$.

(ii) We call an element in $R_{n,r}$ of the form

$$p_{(i),(j)}\, p_{(i)',(j)'}\, p_{(i)'',(j)''} \ldots$$

a *standard monomial in* $R_{n,r}$ *if*

$$((i), (j)) \leqslant ((i)', (j)') \leqslant ((i)'', (j)'') \ldots$$

$$\text{i.e.} \quad \begin{cases} (i) \leqslant (i)' \leqslant (i)'' \ldots \\ (j) \leqslant (j)' \leqslant (j)'' \ldots \end{cases}$$

The number of $p_{(i),(j)}$ in this expression is called the *length* of this standard monomial.

Theorem 2.1. We keep the notations above as well as the basic notations of section 1.

Then we have the following:

(i) We have a canonical identification of $W(H)/W(P_r) \approx I_m(r)$ with the set of pairs of elements of the form $((\alpha), (\beta))$ (resp. $(i), (j)$)

$$(\alpha) \in I_n(k), (\beta) \in I_r(k), 0 \leqslant k \leqslant r$$
$$(\text{resp. } (i) \in I_n(k'), (j) \in I_r(k'), 0 \leqslant k' \leqslant r) \text{ (recall } m = n + r).$$

In fact if $\tau = (\lambda) = (\lambda_1, \ldots, \lambda_r) \in W(H)/W(P_r)$, then the pair $((\alpha), (\beta))$ (resp. $((i), (j))$) is the *canonical pair* (resp. *canonical dual pair*) associated to τ (*cf.* Def. 2.2). This identification preserves (resp. reverses) the partial orders.

(ii) For $\tau \in W(H)/W(P_r)$, let $D(\tau) = D((i), (j))$ denote the subvariety of $M_{n,r}$, obtained as the intersection of $X(\tau, H/P_r)$ with the opposite big cell in H/P_r, identified with $M_{n,r}$ as above, $((i), (j))$ being the canonical dual pair associated to τ. Then the ideal $I(D(\tau))$ of the subvariety $D(\tau)$ of $M_{n,r}$ is the ideal in $R_{n,r}$ generated by

$$\{p_{(i)',(j)'}\}, \ ((i), (j)) \not\leqslant ((i)', (j)').$$

In particular, the determinantal ideal $\triangle_k = I(D((i), (j)))$, $(i) = (j) = (1, \ldots, k)$, is a prime ideal in $R_{n,r}$ and $D_k = D\,((1, \ldots, k), (1, \ldots, k))$ is a variety.

(iii) Let $R(\tau)=R((i),(j))$ be the coordinate ring of the subvariety $D(\tau)=D((i),(j))$ of $M_{n,r}$ (as in (ii) above) so that

$$R(\tau)=R_{n,r}/I(D(\tau)).$$

Then distinct standard monomials of the form

$$p_{(i)',(j)'}\,p_{(i)'',(j)''}\,p_{(i)''',(j)'''}\cdots,$$
$$((i),(j))\leqslant((i)',(j)')\leqslant((i)'',(j)'')\leqslant\ldots$$

form a basis of (the underlying vector space of) $R(\tau)$. In particular $R(\tau)$, where the canonical dual pair associated to τ is $((1,\ldots k),(1,\ldots k))$, has a basis formed of distinct standard monomials

$$p_{(i)',(j)'}\,p_{(i)'',(j)''}\cdots$$

such that the number of indices in $(i)'$ (and hence in $(j)'$, $(i)''$, $(j)''$ ... etc.) is $\leqslant k$.
Proof. The assertion (i) is just a restatement of Lemma 2.1 above and the identification of $W(H)/W(P_n)$ mentioned at the end of section 1.

One knows (*cf.* section 1) that the weight vectors of $H^0(H/P_r, L_r)$ (L_r—the ample generator of Pic H/P_r) can be indexed by p_τ, $\tau\in W(H)/W(P_r)$, such that the restriction of p_τ to $X(\rho, W(H)/W(P_r))$, $\rho\in W(H)/W(P_r)$ is zero if and only if $\tau\not\leqslant\rho$ (*cf.* section 1). We call a *standard monomial of* (*length* l) in $\{p_\tau\}$ an expression of the following form

$$p_{\tau_1}p_{\tau_2}\cdots p_{\tau_l},\ \tau_1\geqslant\tau_2\geqslant\ldots\geqslant\tau_l$$

Let A be the homogeneous coordinate ring of H/P_r with respect to its canonical imbedding in $\mathbf{P}(\overset{r}{\Lambda}V)$. We have

$$A=\sum_{l=0}^{\infty}H^{\circ}(H/P_r, L_r^{\,l}).$$

Then the basic theorem of the Hodge-Young theory on Grassmannians (Hodge and Pedoe [12]; Musili [18]; Seshadri [22]) states that distinct standard monomials form a basis of A (as a vector space). Recall that $f=p_{(1,\ldots,r)}$, is a lowest weight vector, so that the set

$$U=\{x\in H/P_r\,|\,f(x)\neq 0\}$$

is the opposite big cell in H/P_r. Let $A_{(f)}$ denote the 'homogeneous localisation' of A with respect to f, i.e. $A_{(f)}$ is the subring of A_f (localisation of A with respect to f) generated by the elements $\{p_\tau/f\}$, $\tau\in W(H)/W(P_r)$. Let us call a *standard monomial in* $A_{(f)}$ an element which is either 1 or an expression of the form

$$\frac{p_{\tau_1}}{f}\cdot\frac{p_{\tau_2}}{f}\cdots\frac{p_{\tau_l}}{f},\ \tau_1\geqslant\tau_2\geqslant\ldots\geqslant\tau_l,\ p_{\tau_i}\neq f,\ 1\leqslant i\leqslant l.$$

We claim that distinct standard monomials in $A_{(f)}$ form a basis of $A_{(f)}$. It suffices to prove that they are linearly independent. To prove this we observe that if θ is a standard monomial in A, then θf^k is also a standard monomial in A, since $f = p_{(1, \ldots, r)}$ and $(1, \ldots, r)$ identified with an element of $W(H)/W(P_r)$ is the least element in $W(H)/W(P_r)$. Hence, if we have a linear combination of distinct standard monomials in $A_{(f)}$, multiplying it by a suitable power of f, makes it a linear combination of distinct standard monomials in A and from this it is immediate that distinct standard monomials in $A_{(f)}$ are linearly independent.

We have a canonical identification of $M_{n,r}$ with the opposite big cell U in H/P_r and $U = \operatorname{Spec} A_{(f)}$. Hence we have a canonical identification of $A_{(f)}$ with the coordinate ring $R_{n,r}$ of $M_{n,r}$. In this identification, we see that p_τ identifies with $\pm p_{(i),(j)} \in R_{n,r}$, where $((i),(j))$ is the canonical dual pair associated to τ (*cf.* Def. 2.2 and Lemma 2.1). Thus this identification, together with the fact that distinct standard monomials in $A_{(f)}$ form a basis, proves the assertion (iii) of the above theorem for the case $D(\tau) = M_{n,r}$. The Hodge-Young theory implies in fact (Hodge [11]; Hodge and Pedoe [12]; Musili [18]) that in the homogeneous coordinate ring of the Schubert variety $X(\tau, W(H)/W(P_r))$, $\tau \in W(H)/W(P_r)$, distinct standard monomials of the form

$$p_{\tau_1} \cdots p_{\tau_l}, \ \tau \geqslant \tau_1$$

form a basis and that the homogeneous ideal of $X(\tau, W(H)/W(P_r))$ is generated by $\{p_\alpha\}$, $\alpha \not\leqslant \tau$. From these the remaining claims in (iii) as well as (ii) follow easily. This proves the theorem.

Remark 2.1. We have deduced the above theorem as a consequence of the Hodge-Young theory on Grassmannians (Hodge and Pedoe [12]; Musili [18]; Seshadri [22]). We see in fact that the above theorem is essentially equivalent to the Hodge-Young theory on Grassmannians i.e. conversely we can deduce the Hodge-Young theory on Grassmannians from the above theorem. For example, in the proof of the theorem, we should have noted that we have in fact:

Distinct Standard monomials in A form a basis $\Longleftrightarrow$

Distinct Standard monomials in $A_{(f)}$ form a basis.

The assertion (iii) of the above theorem for the case $D(\tau) = M_{n,r}$ is proved in Doubilet-Rota-Stein (Doubilet *et al* [7]) and the couples

$$((i)', (j)') \leqslant ((i)'', (j)'') \leqslant \cdots$$

which figure in the definition of a standard monomial in $R_{n,r}$ (*cf.* Definition 2.1) are called by them 'double standard tableaux' generalizing the usual Young's notion of a standard tableau.

3. Some basic facts about the symplectic and special orthogonal groups.

We keep the basic notations of sections 1 and 2. Hereafter we take $m = 2n$ so that

$$H = SL(2n).$$

Let E_1, E_2 denote the $(2n \times 2n)$ matrices

$$E_1 = \begin{pmatrix} 0 & J \\ -J & 0 \end{pmatrix}, \ E_2 = \begin{pmatrix} 0 & J \\ J & 0 \end{pmatrix}, \ J = \begin{pmatrix} 0 & & 1 \\ & \cdot & \\ 1 & & 0 \end{pmatrix} \ (n \times n \text{ matrix}).$$

We set

$$G_1 = Sp(2n) = \{A \in SL(2n) \text{ (or } GL(2n))/ {}^tAE_1A = E_1\}$$

$$G_2 = SO(2n) = \{A \in SL(2n)/ {}^tAE_2A = E_2\}, \text{ char } K \neq 2.$$

We see that

$${}^tAE_1A = E_1 \Longleftrightarrow A = E_1^{-1} ({}^tA)^{-1} E_1 \text{ (note } E_1^{-1} = -E_1)$$

$${}^tAE_2A = E_2 \Longleftrightarrow A = E_2 ({}^tA)^{-1} E_2 \text{ (note } E_2 = E_2^{-1}).$$

Let σ_1, σ_2 denote the involutory automorphisms in $GL(2n)$ (as well as their restrictions to $SL(2n)$) defined by

$$\sigma_1(A) = E_1 ({}^tA)^{-1} E_1^{-1}, \ \sigma_2(A) = E_2 ({}^tA)^{-1} E_2.$$

We see that

$$G_1 = SL(2n)^{\sigma_1} = GL(2n)^{\sigma_1}$$

$$G_2 = SL(2n)^{\sigma_2} = GL(2n)^{\sigma_2} \cap SL(2n)$$

i.e., G_i is the fixed point set in $SL(2n)$, under σ_i, $i = 1, 2$. It is easily checked that the groups T_{2n}, $T(H)$ and $B(H)$ are stable under σ_i, $i = 1, 2$. We set

$$T(H)^{\sigma_i} = T(G_i), \ B(H)^{\sigma_i} = B(G_i), \ i = 1, 2.$$

One knows that $T(G_i)$ is a maximal torus in G_i and that $B(G_i)$ is a Borel subgroup in G_i, $i = 1, 2$. We see that $T(G_i)$ has the following form:

$$T_{2n}^{\sigma_1} = T_{2n}^{\sigma_2} = T(G_1) = T(G_2) = \begin{pmatrix} t_1 & & & & 0 \\ & \cdot & & & \\ & & t_n & & \\ & & & t_n^{-1} & \\ & & & & \cdot \\ 0 & & & & t_1^{-1} \end{pmatrix}$$

Let Lie G_i denote the Lie algebra of $G_i (i=1, 2)$. We see that

$$\text{Lie } G_1 = \{A \in M_{2n} \mid E_1^{-1}\ {}^tAE_1 + A = 0\}.$$

$$\text{Lie } G_2 = \{A \in M_{2n} \mid E_2\ {}^tAE_2 + A = 0\}.$$

The adjoint representation of G_i on $L(G_i)$ is given by

$$g \in G_i,\ A \in \text{Lie } G_i, \text{ then } g\ o\ A = g\,A\,g^{-1}.$$

We set, for $i = 1, 2$

(i) $N(T(G_i)) =$ normaliser of $T(G_i)$ in G_i

(ii) $W(G_i) = N(T(G_i))/T(G_i)$.

(iii) (a) $X_0(T(G_i)) = \text{Hom }(T(G_i), G_m)$.

(b) $X(T(G_i)) = X_0(T(G_i)) \otimes_{\mathbf{Z}} \mathbf{R}$.

We note that the following hold:

I. $N(T(G_i)) \subset N(T(H)),\ i = 1, 2.$

This is a consequence of the fact that if $D \in GL(2n)$ is a diagonal matrix with *distinct* diagonal elements, then for $A \in SL(2n)$

$$ADA^{-1} \text{ is diagonal} \Longrightarrow A \in N(T(H)).$$

II. $N(T(H))$ is stable under σ_i and

$$N(T(G_i)) = N(T(H))^{\sigma_i},\ i = 1, 2$$

III. The canonical map

$$N(T(G_i))/T(G_i) \to N(T(H))/T(H)\ (i = 1, 2)$$

is an inclusion i.e. the Weyl group $W(G_i)$ of G_i $(i = 1, 2)$ can be identified canonically as a subgroup of the Weyl group of H.

IV. The involution σ_i induces an involution on $W(H)$ since σ_i leaves $N(T(H))$ stable as well as the subgroup $T(H)$ and this induced involution on $W(H)$ is the same for $i = 1, 2$. Let us denote by σ this induced involution on $W(H)$. It is checked easily that if $w = (a_1, \ldots, a_{2n}) \in W(H)$, then

$$\sigma(w) = (c_1, \ldots, c_{2n}),\ c_i = 2n+1-a_{2n+1-i}$$

Let us denote by $\tilde{\sigma}$ the permutation i.e. the element of S_{2n}, defined by

$$\tilde{\sigma}(i) = 2n+1-i,\ 1 \leqslant i \leqslant 2n.$$

Then we have

$$\sigma(w) = \tilde{\sigma}\ w\ \tilde{\sigma}\ (\tilde{\sigma} = \tilde{\sigma}^{-1}), \text{ so that}$$

$$w \in (W(H))^{\sigma} \Longleftrightarrow w = \tilde{\sigma}\ w\ \tilde{\sigma}$$

$$\Longleftrightarrow a_i = 2n+1-a_{2n+1-i}$$

One sees that

(a) $W(G_1) = (W(H))^{\sigma}$ i.e. if $w = (a_1, \ldots, a_{2n}) \in W(H)$, then $w \in W(G_1) \Longleftrightarrow a_i = 2n+1-a_{2n+1-i}$.

(b) $W(G_2) = \{w \in (W(H))^{\sigma} = W(G_1)/w$ is an *even* permutation in $W(H)\}$.

V. The canonical involution on $X(T_{2n})$ (resp. $X_0(T_{2n})$, $X_0(H)$, $X(H)$) is independent of $i = 1, 2$ and we denote this by σ. With respect to the canonical basis $\{\epsilon_i\}$, $1 \leqslant i \leqslant 2n$, of $X(T_{2n})$ (section 1), σ takes the form

$$\sigma\epsilon_i = -\epsilon_{2n+1-i},\quad 1 \leqslant i \leqslant 2n.$$

Recall that we have identified $X(T(H))$ as the subspace of $X(T_{2n})$, spanned by $\epsilon_i - \epsilon_j$ (section 1). We have a canonical surjective linear map

$$\varphi : X(T_{2n}) \to X(T(G_1)) = X(T(G_2)).$$

We see that

$$\varphi(\epsilon_i) = -\varphi(\epsilon_{2n+1-i}),\ 1 \leqslant i \leqslant 2n.$$

We have then the following properties:

(a) σ leaves $R(H)$ (resp. $R^+(H)$) stable (identifying $R(H)$ with $\epsilon_i - \epsilon_j$, $i \neq j$)

(b) $\alpha, \beta \in R(H)$, then

$$\varphi(\alpha) = \varphi(\beta) \Longleftrightarrow \alpha = \sigma(\beta)$$

(c) φ is equivariant for the canonical actions of $W(G_1)$ on $X(T_{2n})$ and $X(T(G_1))$.

(d) the elements of $R(H)$ which are *fixed by* σ are

$$\pm(\epsilon_i - \epsilon_{2n+1-i}),\ 1 \leqslant i \leqslant n.$$

From these considerations, as well as the explicit nature of the adjoint representation of G_i on Lie G_i, $i = 1, 2$, we deduce that

(e) $R(G_1)$ (resp. $R^+(G_1)) = \varphi(R(H))$ (resp. $\varphi\ (R^+(H))$)

$$R(G_2) \text{ (resp. } R^+(G_2)) = \varphi\ (R(H) - R(H)^\sigma) \text{ (resp. } \varphi\ R^+(H) - (R^+(H))^\sigma))$$

where $R(G_i)$ (resp. $R^+(G_i)$) denotes the set of roots (resp. positive roots) of G_i with respect to $T(G_i)$ and $B(G_i)$, $i = 1, 2$. We can state this relation as saying that $R(G_1)$ (resp. $R^+(G_1)$) can be identified with the orbit space $R(H)$ (resp. $R^+(H)$) modulo the action of σ and that $R(G_2)$ (resp. $R^+(G_2)$) can be identified with the orbit space under σ of $R(H)$ (resp. $R^+(H)$) minus the fixed point set under σ. We see now that $R^+(G_i)$ can be identified through φ with the following subsets of $X(T_{2n})$:

$$R^+(G_1) = \begin{cases} \epsilon_i - \epsilon_j, & 1 \leqslant i < j \leqslant n \\ \epsilon_i + \epsilon_j, & 1 \leqslant i < j \leqslant n \\ \quad 2\epsilon_i & 1 \leqslant i \leqslant n \end{cases}$$

$$R^+(G_2) = \begin{cases} \epsilon_i - \epsilon_j, & 1 \leqslant i < j \leqslant n \\ \epsilon_i + \epsilon_j, & 1 \leqslant i < j \leqslant n \end{cases}$$

The canonical action of $W(G_1)$ on $R(G_1)$ can then be written explicitly as follows:

$$w = (a_1, \ldots, a_{2n}), \quad \begin{cases} w(\epsilon_i - \epsilon_j) = \eta_{a_i} - \eta_{a_j} \\ w(\epsilon_i + \epsilon_j) = \eta_{a_i} + \eta_{a_j} \\ w(2\epsilon_i) \quad = 2\eta_{a_i} \end{cases}$$

where

$$\eta_i = \epsilon_i,\ 1 \leqslant i \leqslant n;\ \eta_i = -\ \epsilon_{2n+1-i},\ n+1 \leqslant i \leqslant 2n.$$

The explicit description of the canonical action of $W(G_2)$ on $R(G_2)$ follows from above, since $W(G_2) \subset W(G_1)$ and $R(G_2) \subset R(G_1)$.

VI. Recall (section 1) that we have denoted by $\{s_i\}$, $1 \leqslant i \leqslant 2n-1$, the simple reflections in H, i.e. $\{s_i\}$ are the reflections with respect to the simple roots $\epsilon_i - \epsilon_{i+1}$, $1 \leqslant i \leqslant 2n-1$, so that s_i is the transposition $(i, i+1)$. One sees easily that the simple roots in $R(G_1)$ and $R(G_2)$ (with the identification as subsets of $X(T_{2n})$ as above) are given as follows:

$$\epsilon_1 - \epsilon_2,\ \epsilon_2 - \epsilon_3,\ \ldots,\ \epsilon_{n-1} - \epsilon_n,\ 2\epsilon_n \text{ in } R(G_1)$$
$$\epsilon_1 - \epsilon_2,\ \epsilon_2 - \epsilon_3,\ \ldots,\ \epsilon_{n-1} - \epsilon_n,\ \epsilon_{n-1} + \epsilon_n \text{ in } R(G_2).$$

Let us denote by $\{\theta_i\}_{1 \leqslant i \leqslant n}$ the simple reflections in $W(G_1)$ as well as $W(G_2)$; to be more precise, set:

$$\left.\begin{aligned} \theta_i &= \text{reflection with respect to } \epsilon_i - \epsilon_{i+1}, \quad 1 \leqslant i \leqslant n-1 \\ &= \quad\text{,,} \qquad\qquad \text{,,} \qquad\quad 2\epsilon_n,\ i = n \end{aligned}\right\} \text{ in } W(G_1)$$

$$\begin{aligned}\theta_i &= \text{reflection with respect to } \epsilon_i-\epsilon_{i+1}, \quad 1\leqslant i\leqslant n-1 \\ &= \quad \text{''} \qquad\qquad \text{''} \qquad \epsilon_{n-1}+\epsilon_n, \quad i=n\end{aligned}\Bigg\} \text{ in } W(G_2).$$

It is an easy exercise to deduce the following (Bourbaki [2], p. 53).

(a) $\left.\begin{aligned}\theta_i &= s_i\,\sigma(s_i)\\ &= s_i\, s_{2n-i}\end{aligned}\right\}, \ 1\leqslant i\leqslant n-1$ in $W(G_1)$ and $W(G_2)$

(b) $\theta_n = s_n$ in $W(G_1)$ $(s_n=\sigma(s_n))$
$\left.\begin{aligned}&= \text{product of the transpositions: } (n-1,\, n+1)\,(n,\, n+2) \text{ or}\\ &\quad\text{equal to } s_{n-1}\, s_n\, s_{n-1}\, s_n\, s_{n+1}\, s_n\end{aligned}\right\}$ in $W(G_2)$.

Definition 3.1. (i) Let $w\in W(G_1)$ (resp. $W(G_2)$). Then we denote by $l(w, W(G_1))$ (resp. $l(w, W(G_2))$, the *length* of w *in* $W(G_1)$ (*resp.* $W(G_2)$) i.e. the length of a reduced decomposition of w with respect to the simple reflections $\{\theta_i\}$ in $W(G_1)$ (resp. $W(G_2)$). Since $W(G_i)\subset W(H)$, $i=1, 2$, we denote by $l(w, W(H))$ the length of w in $W(H)$.

(ii) For $w=(a_1,\ldots,a_{2n})\in W(H)$, we set

$$m(w) = \#\ \{i \mid i\leqslant n,\ w(i)=a_i>n\}$$

Lemma 3.1. An element $w\in W(G_1)$ is in $W(G_2)$ if and only if $m(w)$ is *even* i.e.

$$W(G_2)=\{w \mid w\in W(H),\ w \text{ is } \sigma\text{-invariant and } m(w) \text{ is even}\}.$$

Proof. We leave this as an exercise.

Proposition 3.1. We have the following

(i) $l(w, W(H)) = 2l(w, W(G_1)) - m(w)$, $w\in W(G_1)$

(ii) $l(w, W(H)) = 2l(w, W(G_2))+m(w)$, $w\in W(G_2)$.

Proof. This is an easy consequence of the fact given in V(e) above, namely that $R(G_1)$ (resp. $R(G_2)$) can be identified with the orbit space of $R(H)$ (resp. $R(H)-R(H)^\sigma$) under the action of σ etc. For w as in the proposition, we set

$$S(w, G_i)=\{\beta\in R^+(G_i) \mid w(\beta)<0\},\ i=1 \text{ or } 2.$$

$$S(w, H)=\{\beta\in R^+(H) \mid w(\beta)<0\}.$$

We have

$$l(w, W(H)) = \#\ S(w, H),\ l(w, W(G_i)) = \#\ S(w, G_i),\ i=1, 2.$$

Consider the canonical map

$$\varphi: X(T_{2n}) \to X(T(G_1)) = X(T(G_2)) \ (cf.\ \text{V above}).$$

Then φ induces *surjective* maps

$$\varphi : R(H) \to R(G_1) \text{ (resp. } \varphi : R(H) - R(H)^{\sigma} \to R(G_2)).$$

The mapping φ has the property:

$$\alpha \in R(H), \text{ then } \alpha > 0 \Longleftrightarrow \varphi(\alpha) > 0.$$

From this we see that φ induces a surjective map

$$\varphi : S(w, H) \to S(w, G_1)$$

$$\text{(resp. } \varphi : S(w, H) - S(w, H)^{\sigma} \to S(w, G_2)).$$

We see easily that σ leaves $S(w, H)$ stable. Besides,

$$(S(w, H))^{\sigma} = \{\alpha/\alpha = \epsilon_i - \epsilon_{2n+1-i}, 1 \leqslant i \leqslant n \text{ and } w(\alpha) < 0 \text{ in } R(H)\}.$$

Now if $\alpha = \epsilon_i - \epsilon_{2n+1-i}$, then $w(\alpha) = \epsilon_{a_i} - \epsilon_{a_{2n+1-i}}$. Hence

$$w(\alpha) < 0 \text{ in } R(H) \Longleftrightarrow a_{2n+1-i} < a_i, \ i \leqslant n.$$

But $a_{2n+1-i} = 2n+1-a_i$ since $w \in W(G_1)$ (*cf.* IV above).

Thus

$$w(\alpha) < 0 \Longleftrightarrow 2n+1 < 2a_i, \ 1 \leqslant i \leqslant n$$

$$\text{i.e. } n < a_i, \quad 1 \leqslant i \leqslant n$$

so that

$$\#\, (S(w, H))^{\sigma} = m(w) \text{ (\textit{cf.} (ii), Definition 3.1).}$$

We see that

$$\#\, S(w, H) = 2(\#\, S(w, G_1)) - \#\, (S(w, H))^{\sigma}.$$

$$\text{(resp. } 2(\#\, S(w, G_2)) + \#\, (S(w, H))^{\sigma}).$$

The proposition is now an immediate consequence.

For $w \in W(H)$, we denote by $C(w, H/B(H))$ the *Schubert cell* in $H/B(H)$ defined by w i.e., the subset $B(H)w\, e_{B(H)}$ in $H/B(H)$. If $w \in W(G_1)$ (resp. $W(G_2)$), we denote by $C(w, G_1/B(G_1))$ (resp. $C(w, G_2/B(G_2))$) the Schubert cell in $G_1/B(G_1)$ (resp. $G_2/B(G_2)$),

Lemma 3.2. For $w \in W(G_i)$ $(i = 1, 2)$, the Schubert cell $C(w, H/B(H))$ is stable under σ_i and we have

$$C(w, H/B(H))^{\sigma_i} = C(w, G_i/B(G_i)).$$

Proof. The fact that $C(w, H/B(H))$ is stable under σ_i ($i = 1, 2$) is immediate. Let $B(H)^u$ denote the unipotent part of $B(H)$. Let B_1 denote the isotropy subgroup of $B(H)^u$ at $w \in C(w, H/B(H))$. Then since $\sigma_i(w) = w$, we see that σ_i leaves B_1 stable. We have

$$(*) \qquad B(H)^u = \prod_{\alpha \in R^+(H)} U_\alpha, \ U_\alpha \approx \mathbf{G}_a$$

U_α being the 1-dimensional subgroup of $B(H)^u$ canonically associated to $\alpha \in R^+(H)$. We see that

$$B_1 = \prod_{\substack{\alpha \in R^+(H),\\ w^{-1}(\alpha) > 0}} U_\alpha$$

Let

$$B_2 = \prod_{\substack{\alpha \in R^+(H),\\ w^{-1}(\alpha) < 0}} U_\alpha$$

By the 'uniqueness' of the decomposition (*) (Chevalley [4]), we see that σ_i leaves B_2 stable. Now if $x \in C(w, H/B(H))$, it has a representation $x = bw$, $b \in B_2$, b unique. Hence $\sigma_i(x) = x$ is equivalent to $\sigma_i(b) = b$ and the lemma follows.

4. The varieties G/Q, G of type C_n or D_n and Q the maximal parabolic subgroup associated to a right end root

As in section 1, let P_n denote the maximal parabolic subgroup in H, $H = SL(2n)$ associated to the simple root $\alpha_n = \epsilon_n - \epsilon_{n+1}$. Then P_n is of the form

$$\left(\begin{array}{c|c} (n\times n) & * \\ \hline 0 & (n \times n) \end{array}\right).$$

We see that the involutions σ_1, σ_2 leave P_n stable (the other maximal parabolic subgroups of H are *not* left stable by σ_i, in fact $\sigma_1(P_r)$ (resp. $\sigma_2(P_r)) = i(P_r)$, i as in section 1), so that σ_i induces canonical involutions (denoted by the same σ_i) on H/P_n, $i = 1, 2$.

Let Z be the subgroup of H of the form

$$Z = \left\{ \begin{pmatrix} Id_n & 0 \\ Y & Id_n \end{pmatrix} \middle/ \begin{array}{l} Id_n = (n \times n) \text{ identity matrix} \\ Y \in M_n \text{ (space of } n \times n \text{ matrices)} \end{array} \right\}$$

The canonical morphism $H \to H/P_n$ induces a canonical morphism

$$\psi : Z \to H/P_n.$$

It is now easy to check the following:

(i) ψ is an open immersion and ψ (Z) identifies with the *opposite big cell* in H/P_n. We see also that if we identify Z canonically with the following subset of $M_{2n,\, n}$.

$$Z \approx \left\{ \begin{pmatrix} Id_n \\ Y \end{pmatrix} \Big/ \ Y \in M_n \right\},$$

then $Z \subset M^s_{2n,\, n}$ (notation of section 2) and the morphism $Z \to H/P_n$ induced by the canonical morphism $\varphi_2 : M^s_{2n,\, n} \to H/P_n$ coincides with the morphism ψ above.

(ii) $$\sigma_1 \begin{pmatrix} Id_n & 0 \\ Y & Id_n \end{pmatrix} = \begin{pmatrix} Id_n & 0 \\ J\,{}^tYJ & Id_n \end{pmatrix}$$

$$\sigma_2 \begin{pmatrix} Id_n & 0 \\ Y & Id_n \end{pmatrix} = \begin{pmatrix} Id_n & 0 \\ -J\,{}^tYJ & Id_n \end{pmatrix}$$

In particular, the involution $\sigma_i (i=1, 2)$ leaves stable Z and the opposite big cell $\psi(Z)$ in H/P_n. Further

$$Z^{\sigma_1} = \left\{ \begin{pmatrix} Id_n & 0 \\ Y & Id_n \end{pmatrix} \middle| \begin{matrix} Y \in M_n \\ J\,{}^tYJ = Y \end{matrix} \right\},$$

$$Z^{\sigma_2} = \left\{ \begin{pmatrix} Id_n & 0 \\ Y & Id_n \end{pmatrix} \middle| \begin{matrix} Y \in M_n \\ -J\,{}^tYJ = Y \end{matrix} \right\}.$$

If we set $Y = JX$, then we have

(a) $J\,{}^tYJ = Y \Longleftrightarrow {}^tX = X$

(b) $-J\,{}^tYJ = Y \Longleftrightarrow {}^tX = -X.$

Thus if we identify Z or M_n with the opposite big cell in H/P_n by the mapping $M_n \to H/P_n$ defined by

$$X \in M_n,\ X \to \psi \begin{pmatrix} Id_n & 0 \\ JX & Id_n \end{pmatrix} \in H/P_n,$$

we see that the set of fixed points of the opposite big cell in H/P_n under σ_1 (resp. σ_2) can be identified with the set of symmetric (resp. skew-symmetric) matrices in M_n.

We see immediately that

$$Y = J\,{}^tYJ \Longleftrightarrow (Id_n,\ {}^tY) \begin{pmatrix} 0 & J \\ -J & 0 \end{pmatrix} \begin{pmatrix} Id_n \\ Y \end{pmatrix} = 0$$

$$Y = -J\,{}^tYJ \Longleftrightarrow (Id_n,\ {}^tY) \begin{pmatrix} 0 & J \\ J & 0 \end{pmatrix} \begin{pmatrix} Id_n \\ Y \end{pmatrix} = 0.$$

Thus the condition that $Y = J\,{}^tYJ$ (resp. $Y = -J\,{}^tYJ$) means that the n-dimensional linear subspace of the $2n$-dimensional linear space V, represented by the following point of H/P_n, namely

$$\psi\begin{pmatrix} Id_n & 0 \\ Y & Id_n \end{pmatrix} = \begin{cases} \text{the } n\text{-dimensional linear subspace spanned by the} \\ \text{columns of } \begin{pmatrix} Id_n \\ Y \end{pmatrix} \in M_{2n,\,n} \text{ the columns representing points} \\ \text{of } V \text{ endowed with its standard basis } e_1, \ldots, e_{2n} \text{ (section 2)} \end{cases}$$

is a maximal *totally isotropic subspace* for the skew-symmetric (resp. symmetric) form on V represented by E_1 (resp. E_2). One knows that the set of points in H/P_n, represented by maximal totally isotropic subspaces with respect to E_1, is a closed subvariety isomorphic to G_1/Q, where Q is the maximal parabolic subgroup corresponding to the 'right end root' in the Dynkin diagram of G_1 (i.e. the root $2\epsilon_n$). Further we note that

$$\dim G_1/Q = \frac{n(n+1)}{2} (= \text{dimension of the set of } (n \times n) \text{ symmetric matrices})$$

This dimension can be calculated, for example, by noting that

$$\dim G_1/Q = \# R^+(G_1) - \# R^+(Q_0) \; R^+(Q_0) \text{ denotes the set of positive roots of the semi-simple part } Q_0 \text{ of } Q\text{; note that } Q_0 \text{ is of type } \mathbf{A}_{n-1}).$$

Similarly, one knows that the set of points in H/P_n, represented by maximal totally isotropic subspaces with respect to E_2, is a union of two closed subvarieties, each of which is isomorphic to G_2/Q, Q being a maximal parabolic subgroup in Q corresponding to one of the right end roots in the Dynkin diagram of G_2 (i.e. the roots $\epsilon_{n-1} - \epsilon_n$ and $\epsilon_{n-1} + \epsilon_n$). Further

$$\dim G_2/Q = \frac{n(n-1)}{2} = \text{dimension of the set of } (n \times n) \text{ skew-symmetric matrices.}$$

From these facts, we conclude easily the following:

Let Sym M_n (resp. *Sk* M_n) denote the set of symmetric (resp. skew-symmetric) in the set M_n of $(n \times n)$ matrices. Let Δ_0 and Δ denote the morphisms

$$\Delta_0 : M_n \to H/P_n$$

$$\Delta : \mathrm{Sym}\, M_n \to H/P_n \text{ (resp. } Sk\, M_n \to H/P_n)$$

defined by

$$X \in M_n,\; X \longmapsto \psi\begin{pmatrix} Id_n & 0 \\ J & Id_n \end{pmatrix},$$

$$X \in \mathrm{Sym}\, M_n \text{ (resp. } Sk\, M_n),\; X \longmapsto \begin{pmatrix} Id_n & 0 \\ J & Id_n \end{pmatrix}.$$

Then the image of Δ is an *open* subvariety of the closed subvariety G_1/Q (resp. G_2/Q) of H/P_n, Q being the maximal parabolic in G_1 corresponding to the right end root (resp. to a right end root of G_2).

Recall that for $(\lambda) = (\lambda_1, \ldots, \lambda_n) \in I_{2n}(n)$, we had denoted by $p_{(\lambda)}$ the 'Plücker coordinate' associated to (λ) and it is an element of $H^\circ(H/P_n, L_n)$, L_n being the ample generator of Pic H/P_n. If we take the canonical order preserving isomorphism $I_{2n}(n) \approx W(H)/W(P_n)$, we have seen (section 1) that if $\tau = (\lambda) \in W(H)/W(P_n)$, then

$$p_\tau = p_{(\lambda)} = \tau \cdot p_{(1, \ldots, n)}.$$

Let ((i), (*j*)) be the *canonical dual pair* associated to (λ) (*cf.* Def. 2.2). Then we have seen (*cf.* Lemma 2.1 and proof of Th. 2.1) that the restriction of $p_{(\lambda)}$ to M_n (identified as the opposite big cell of H/P_n through Δ_0) or to be more precise $p_{(\lambda)}/f(f = p_{(1, \ldots, n)})$, can be identified with the function $p_{(i), (j)}$ on M_n (*cf.* Def. 2.1).

Then we set the following:

(i) L'_n = restriction of the line bundle L_n on H/P_n to G_1/Q (resp. G_2/Q)

(ii) $p'_{(\lambda)}$ for $(\lambda) \in I_{2n}(n)$, is the restriction of $p_{(\lambda)}$ to G_1/Q (resp. G_2/Q), so that $p'_{(\lambda)}$ is a section of the line bundle L'_n.

(iii) $p'_{(i), (j)}$ is the restriction of $p_{(i), (j)}$ to Sym M_n (resp. Sk M_n).

(iv) f = a lowest weight vector in $H^\circ(H/P_n, L_n)$ and f = the restriction of f' to G_1/Q (resp. G_2/Q).

Then we have the following:

(i) The section $p'_{(\lambda)} \in H^\circ(G_1/Q, L'_n)$, $(\lambda) \in I_{2n}(n)$ is non-zero. This happens because the restriction of $p'_{(\lambda)}$ to Sym M_n (identified as an open subset of G_1/Q through Δ) is $p'_{(i), (j)}$, ((i), (*j*)) being the canonical dual pair associated to (λ) and $p'_{(i), (j)} \neq 0$. Since f is a *lowest weight* vector in $H^\circ(H/P_n, L_n)$, the one dimensional space spanned by f is $B(H)^-$ stable, where $B(H)^-$ is the Borel subgroup of H *opposite to* $B(H)$ (consisting of the lower triangular matrices in H). One sees that $B(H)^-$ is stable under σ_1 and that $(B(H)^-)^{\sigma_1} = B(H)^- \cap G_1$ is a Borel subgroup in G_1 and one concludes easily that $B(H)^- \cap G_1$ is in fact the Borel subgroup $B(G_1)^-$, opposite to $B(G_1)$. We see then that f' is a lowest weight vector in $H^\circ(G_1/Q, L_n')$. The weight of f is $-(\epsilon_1 + \ldots + \epsilon_n)$. Hence the weight of f' is also $-(\epsilon_1 + \ldots + \epsilon_n)$ (identifying $X(T(G_1))$ canonically as a subspace of $X(T_{2n})$). We see that

$$-(\epsilon_1 + \ldots + \epsilon_n) = (w_0)_{G_1}(\omega_n)$$

where ω_n is the fundamental weight associated to the right end root i.e. $\langle \omega_n, 2\epsilon_n^v \rangle = 1$ and $(w_0)_{G_1}$ is the element of $W(G_1)$ of largest length [Bourbaki [2]; note that $(w_0)_{G_1} = -1$ on $X(T(G_1))$]. Thus we conclude that the highest weight vector in

$H^\circ(G_1/Q, L'_n)$ has weight ω_n. We see then that L'_n is the ample generator of Pic G_1/Q so that $H^\circ(G_1/Q, L'_n)$ is the fundamental representation with highest weight $\omega_n = i(\omega_n)$. Since Sym M_n is precisely the set of points where f' does not vanish, we see that Sym M_n can be identified (through $\triangle$) with the opposite big cell in G_1/Q,

(ii) Let $p'_{(\lambda)} \in H^\circ(G_2/L_n')$ and $p'_{(i),(j)}$ the restriction of $p'_{(\lambda)}$ to Sk M_n. Suppose that $(i) = (j)$. Then we have $q^2_{(i)} = p'_{(i),(j)}$, where

$$q_{(i)} : \text{Sk } M_n \to K$$

is the function obtained by taking the Pfaffian of the minor with rows and columns corresponding to the indices in (i). If $(i) \in I_n(r)$, then $q_{(i)} = 0$ if r is odd and $q_{(i)} \neq 0$ if r is even. We see again, as for the case of G_1 above, that f' is a lowest weight vector in $H^\circ(G_2/Q, L'_n)$ and that its weight is

$$-(\epsilon_1 + \ldots + \epsilon_n) = -2\omega_n,$$

where ω_n is the fundamental weight associated to the right end root $\epsilon_{n-1} + \epsilon_n$. Hence the highest weight of $H^\circ(G_2/Q, L'_n)$ is $2i(\omega_n)$. From these considerations, we conclude easily the following:

(i) Let F be the ample generator of Pic G_2/Q. Then $F^2 = L'_n$.

(ii) Let $p'_{(\lambda)} \in H^\circ(G_2/Q, L'_n)$ such that if $((i), (j))$ is the canonical dual pair associated to (λ), then $(i) = (j)$. If $(i) \in I_n(r)$, then $q_{(i)} = 0$ if r is odd, and if r is even, we have a well-determined element $g_{(\lambda)} \in H^\circ(G_2/Q, F)$ such that $g^2_{(\lambda)} = p'_{(\lambda)}$.

(iii) the highest weight of $H^\circ(G_2/Q, F)$ is $i(\omega_n)$ so that Q is the maximal parabolic subgroup associated to the right end root $(\epsilon_{n-1} + \epsilon_n)$.

(iv) Sk M_n is the opposite big cell in G_2/Q.

Note that $i(\omega_n) = \omega_n$ if n is *even* and $i(\omega_n) = \omega_{n-1}$ if n is odd, ω_{n-1} being the fundamental weight associated to the right end root $(\epsilon_{n-1} - \epsilon_n)$.

Let us denote by $W(Q)$ the Weyl group of the maximal parabolic subgroup Q of G_1 (resp. G_2), Q being as above. We claim that

$$W(Q) = W(G_1) \cap W(P_n) \text{ (resp. } W(G_2) \cap W(P_n)).$$

In fact $W(Q)$ is the subgroup of $W(G_1)$ (resp. $W(G_2)$) which fixes the point $e(Q)$ in G_1/Q (resp. G_2/Q), $e(Q)$ being the point associated to the class Q. Now under the canonical immersion

$$G_1/Q \text{ (resp. } G_2/Q) \hookrightarrow H/P_n,$$

$e(Q)$ goes to the point $e(P_n)$ and $W(P_n)$ is the subgroup of $W(H)$ which fixes $e(P_n)$ ($e(P_n)$ is the canonical image of $e_1\Lambda\ldots\Lambda e_n$ in $\mathbf{P}(\overset{n}{\Lambda} V)$). Thus $W(Q) = W(H)\cap W(G_1)$ (resp. $W(H)\cap W(G_2)$) and we get a canonical inclusion map

$$W(G_1)/W(Q) \hookrightarrow W(H)/W(P_n) \text{ (resp. } W(G_2)/W(Q) \hookrightarrow W(H)/W(P_n)).$$

We have then

Lemma 4.1. (i) Let $w \in W(H)/W(P_n)$ and w_1, w_2 be respectively the *minimal and maximal representatives* in $W(H)$ of w (section 1). Suppose that $w \in W(G_1)/W(Q)$ (resp. $W(G_2)/W(Q)$). Then w_1, w_2 are in $W(G_1)$ (resp. $W(G_2)$) and are in fact, respectively the minimal and maximal representatives in $W(G_1)$ (resp. $W(G_2)$) of w.

(ii) Let us identify $W(H)/W(P_n)$ canonically with $I_{2n}(n)$ (order preserving isomorphism, section 1). Let $(\lambda) \in I_{2n}(n)$ and $((\alpha), (\beta))$ (resp. $((i), (j))$) be the canonical pair (resp. the canonical dual pair) associated to (λ). Then we have

(a) $(\alpha) \in I_n(r)$, $(\beta) \in I_n(r)$ (resp. $(i) \in I_n(s)$, $(j) \in I_n(s)$), $r+s=n$ (the number of indices in (α) and (β) are the same and similarly for (i) and (j)).

(b) $(\lambda) \in W(G_1)/W(Q) \Longleftrightarrow (\alpha)=(\beta)$, (resp. $(i)=(j)$), so that we have a canonical bijection: $W(G_1)/W(Q) \approx \bigcup_{0\leqslant r\leqslant n} I_n(r)$.

(c) $(\lambda) \in W(G_2)/W(Q) \Longleftrightarrow (\alpha)=(\beta)$, and $(n-r)$ is

even, r being the integer such that $(\alpha) \in I_n(r)$ (resp. $(i)=(j)$ and s is *even* where s is the integer such that $(i) \in I_n(s)$).

(d) $(\lambda) \in W(G_1)/W(Q) \Longrightarrow (j)$ is the complement of (α) in $(1, 2, \ldots, n)$.

(e) the mapping $(\lambda) \longmapsto ((\alpha), (\alpha)), ((\beta), (\beta))$ defines an injection

$$\varphi\colon W(H)/W(P_n) \hookrightarrow W(G_1)/W(Q) \times W(G_1)/W(Q)$$

and the image of φ can be identified with the set of elements (w_1, w_2) such that $m(w_1) = m(w_2)$ (*cf.* Def. 3.1).

(iii) Let $w \in W(G_1)/W(Q)$ (resp. $W(G_2)/W(Q)$) and $w=(i_1, \ldots, i_r) \in I_n(r)$ (canonical representation as in (ii) above) Then we have

$$l\,(w, W(G_1)/W(Q)) = \sum_{k=1}^{r} i_k+(n+1)(n-r)-\frac{n(n+1)}{2}$$

$$\text{(resp. } l\,(w, W(G_2)/W(Q)) = \sum_{k=1}^{r} i_k+n(n-r)-\tfrac{1}{2}n(n+1)).$$

(iv) Let $w_1, w_2 \in W(G_1)/W(Q)$ (resp. $W(G_2)/Q$) and

$$w_1 = (i) = (i_1, \ldots, i_r) \in I_n(r) \text{ and } w_2 = (j) = (j_1, \ldots, j_s) \in I_n(s)$$

(representations as in (ii) above). Then

$$w_1 \leqslant w_2 \text{ in } W(G_1)/W(Q) \text{ (resp. } W(G_2)/W(Q)) \Longleftrightarrow (i) \leqslant (j) \text{ (cf. Def. 2.1.)}$$
$$\Longleftrightarrow w_1 \leqslant w_2 \text{ in } W(H)/W(P_n).$$

(v) Let $w \in W(G_1)/W(Q)$ (resp. $W(G_2)/W(Q)$). Then

$$X(w, G_1/Q) = X(w, H/P_n) \cap G_1/Q$$

$$(\text{resp. } X(w, G_2/Q) = X(w, H/P_n) \cap G_2/Q).$$

Proof: (i) Since w_2 is the maximal representative in $W(H)$ of w, $X(w_2, H/B(H))$ is the inverse image of $X(w, H/P_n)$ under the canonical morphism $H/B(H) \to H/P_n$. We have seen that the involution σ_i $(i = 1, 2)$ on $H/B(H)$ goes down to an involution (denoted by the same σ_i) on H/P_n. Hence the morphism $H/B(H) \to H/P_n$ is equivariant for σ_i. Since $w \in W(G_1)/W(Q)$ (resp. $W(G_2)/W(Q)$), $X(w,H/P_n)$ is stable under the involution under σ_i and it follows that $X(w_2, H/B(H))$ is stable under σ_i. This implies that $w_2 \in W(G_1)$ (resp. $W(G_2)$). We see also that w_2 is a representative in $W(G_1)$ (resp. $W(G_2)$) of $w \in W(G_1)/W(Q)$ (resp. $W(G_2)/W(Q)$). Since w_1 is a minimal representative in $W(H)$ of w, as we mentioned in section 1, we have

$$w_2 = w_1 . w_0(P_n),$$

where $w_0(P_n)$ is the element of largest length in $W(P_n)$. We claim now that $w_0(P_n) \in W(G_1)$ (resp. $W(G_2)$) and is in fact equal to $w_0(Q)$, which is the element of largest length in $W(Q)$. To prove the claim, it suffices to show that $w_0(Q)$, which is afortiori in $W(P_n)$, is of maximal length in $W(P_n)$. This is now an easy consequence of Prop. 3.1. We observe first that $m(w_0(Q)) = 0$ (notation as in Prop. 3.1). Then Prop. 3.1. shows that

$$l(w_0(Q), W(H)) = 2l(w_0(Q), W(G_i)), \; i = 1, 2.$$

One knows that

$$l(w_0(Q), W(G_i)) = \dim \; Q_i/B(Q_i) \;\; (i = 1, 2),$$

$$(B(Q_i) = \text{a Borel subgroup in } Q_i).$$

It is easy to check that

$$\dim P_n/B(P_n) = 2 \dim Q_i/B(Q_i) \; (i = 1, 2)$$

$$(B(P_n) = \text{a Borel subgroup in } P_n).$$

It follows then that

$$l(w_0(Q),\ W(H)) = \dim P_n/B(P_n).$$

This implies that $w_0(Q)$ is the element of maximal length in $W(P_n)$ and the above claim that $w_0(P_n) = w_0(Q)$ follows. Thus we have

$$w_2 = w_1 . w_0(Q).$$

This relation implies in particular that $w_1 \in W(G_1)$ (resp. $W(G_2)$) and is a representative of w. Since w_1, w_2 are respectively the minimal and maximal representatives in $W(H)$ of w, as mentioned in section 1, we have

$$l(w_2,\ H/B(H)) = l(w_1,\ H/B(H)) + l(w_0(Q),\ H/B(H)).$$

It is easy to check that $m(w_1) = m(w_2)$ $(m(w_0(Q)) = 0$ as mentioned above), so that by Prop. 3.1. this formula implies that

$$l(w_2,\ G_i/B(G_i)) = l(w_1,\ G_i/B(G_i)) + l(w_0(Q), G_i/B(G_i)),\ i = 1, 2.$$

As we saw in section 1, this relation implies that w_1, w_2 are respectively the minimal and maximal representatives in $W(G_1)$ (resp. $W(G_2)$) of w.

(ii) Let $(\lambda) = (\lambda_1, \ldots, \lambda_n) \in W(H)/W(P_n) = I_{2n}(n)$. A minimal representative w_1 of (λ) in $W(H)$ is given by

$$w_1 = (\lambda_1, \ldots, \lambda_n;\ \mu_1, \ldots, \mu_n)$$

$(\mu_1, \ldots, \mu_n)$ being the complement of $(\lambda_1, \ldots, \lambda_n)$ in $(1, 2, \ldots, 2n)$, arranged in the increasing order. Thus to write down the condition for (λ) to be in $W(G_1)/W(Q)$ (resp. $W(G_2)/W(Q)$), on account of (i) above, we have only to express the condition that w_1 is σ-invariant (resp. σ-invariant and $m(w)$ is even, *cf.* Lemma 3.1). The fact that w_1 is σ-invariant is equivalent to saying that

$$\mu_i = 2n+1-\lambda_{n+1-i},\ 1 \leqslant i \leqslant n.$$

Let now r, $0 \leqslant r \leqslant n$, be such that $\lambda_r \leqslant n$ and $\lambda_{r+1} > n$. Then we see that $(\mu_1, \ldots, \mu_{n-r})$ is the complement of $(\lambda_1, \ldots, \lambda_r)$ in $(1, \ldots, n)$ and that

$$(\lambda_{r+1}, \ldots, \lambda_n) = (2n+1-\mu_{n-r}, \ldots, 2n+1-\mu_1).$$

Now the assertions (ii) follow easily and we leave them as exercise.

(iii) Let $w \in W(G_1)/W(Q)$ (resp. $W(G_2)/W(Q)$) and be represented canonically by $w = (i_1, \ldots, i_r) \in I_n(r)$. Let w_1 be a minimal representative of w in $W(H)$. Then we have

$$w_1 = (\underbrace{i_1, \ldots, i_r, 2n+1-j_{n-r}, \ldots, 2n+1-j_1}_{\text{first } n \text{ elements}}, \ldots)$$

where $(j_1, \ldots, j_{n-r})$ is the complement of $(i_1, \ldots, i_r)$ in $(1, \ldots, n)$, arranged in the increasing order (*cf.* (ii) above). Then by (i)

$$l(w_1, W(G_i)/W(Q)) = l(w_1, W(G_i)). \ (i = 1, 2).$$

Further, we have

$$l(w_1, W(H)/W(P_n)) = l(w_1, W(H))$$
$$= \sum_{k=1}^{r} i_k + \sum_{k=1}^{n-r} (2n+1-j_k) - \frac{n(n+1)}{2} \text{ (section 1).}$$

and $m(w_1) = (n-r)$.

Then applying Prop. 3.1, we get

(a) $$l(w, W(G_2)/W(Q)) = l(w, W(G_1)/W(Q)) - m(w), \text{ and}$$
$$l(w, W(G_1)/W(Q)) = \tfrac{1}{2}\, l(w_1, W(H)) + \tfrac{1}{2}\, m(w_1).$$

This gives

(b) $$l(w, W(G_1)/W(Q)) = \tfrac{1}{2}\left\{(n-r)(2n+1) + \sum_{k=1}^{r} i_k - \left(\frac{n(n+1)}{2} - \sum_{k=1}^{r} i_k\right)\right.$$
$$\left. - \frac{n(n+1)}{2}\right\} + \tfrac{1}{2}(n-r) = \left(\sum_{k=1}^{r} i_k\right) + (n+1)(n-r) - \frac{n(n+1)}{2}.$$

Now (a) and (b) together prove (iii).

(iv) We have seen that $f = p_{(1, \ldots, n)}$ is a lowest weight vector in $H^\circ(H/P_n, L_n)$. Then as we saw in section 1 (towards the end), the family

$$\{p_\tau\}, \ \tau \in W(H)/W(P_n), \ p_\tau \text{ being defined as } p_\tau = \tau . f, \text{ has the property:}$$

(*) $$p_{\tau_1} \mid X(\tau_2, H/P_n) \neq 0 \Longleftrightarrow \tau_1 \leqslant \tau_2 \text{ in } W(H)/W(P_n).$$

We have seen that the restriction of f to G_1/Q is a lowest weight vector in $H^\circ(H/P'_n, L'_n)$ $L'_n = L_n$ restricted to G_1/Q. Further, we have an element $g \in H^\circ(G_2/Q, F)$, $F^2 = L'_n$, $L'_n = L_n$ restricted to G_2/Q, such that $g^2 = f$ restricted to G_2/Q and g is a lowest weight vector in $H^\circ(G_2/Q, F)$. Let us denote by g_0 the restriction of f to G_1/Q (resp. $g = g_0$, the element $g \in H^0(G_2/Q, F)$ as above). Then the family

$$\{g_\tau\}, \ g_\tau = \tau . g_0, \ \tau \in W(G_1)/W(Q) \text{ (resp. } W(G_2)/W(Q))$$

has the property (*cf.* section 1)

$$\tau_1 \leqslant \tau_2 \text{ in } W(G_1)/W(Q) \text{ (resp. } W(G_2)/W(Q)) \Longleftrightarrow g_{\tau_1} \mid X(\tau_2, G_1/Q) \neq 0$$
$$\text{(resp. } \mid g_{\tau_1} X(\tau_2, G_2/Q) \neq 0).$$

Then from (*), we conclude that

$$\tau_1 \leqslant \tau_2 \text{ in } W(G_1)/W(Q) \text{ (resp. } W(G_2)/W(Q)) \Longleftrightarrow \tau_1 \leqslant \tau_2 \text{ in } W(H)/W(P_n)$$

since, because of $X(\tau_2, G_1/Q)$ (resp. $X(\tau_2, G_2/Q)) \subset X(\tau_2, H/P_n)$, one has

$$\left.\begin{array}{r} g_{\tau_1} \mid X(\tau_2, G_1/Q) \neq 0 \\ \text{(resp. } g_{\tau_1} \mid X(\tau_2, G_2/Q \neq 0)) \end{array}\right\} \Longrightarrow f_{\tau_1} \mid X(\tau_2, H/P_n) \neq 0$$

Thus to conclude the proof of (iv), we have only to prove the following implication:

$$w_1, w_2 \in W(G_1)/W(Q) \text{ (resp. } W(G_2)/W(Q) \text{ and } w_1 \neq w_2, \text{ then}$$

$$w_1 \leqslant w_2 \text{ in } W(H)/W(P_n) \Longrightarrow w_1 \leqslant w_2 \text{ in } W(G_1)/W(Q) \text{ (resp. } W(G_2)/W(Q)).$$

We shall now show that we can find $w'_1 \in W(G_1)/W(Q)$ (resp. $W(G)_2/W(Q)$) such that

(a) $w_1 \leqslant w_1' \leqslant w_2$ in $W(H)/W(P_n)$.

(b) $l(w_1', W(G_1)/W(Q))$ (resp. $l(w_1', W(G_2)/W(Q))$

$= l(w_2, W(G_1)/W(Q)) - 1$ (resp. $l(w_2, W(G_2)/W(Q)) - 1$)

(c) $w_1' = w_2 r_\alpha$, where r_α is the reflection with respect to a root $\alpha \in R^+(G_1)$ (resp. $R^+(G_2)$ (in this identity, we take for w_1' and w_2 their respective minimal representatives in $W(H)$). These assertions would imply that

$$w_1' \leqslant w_2 \text{ in } W(G_1) \text{ (resp. } W(G_2))$$

and thus to complete the proof of (iv), it suffices to prove the above assertions. We set

$$w_1 = (i_1, \ldots, i_r) \in I_n(r),\ w_2 = (j_1, \ldots, j_s) \in I_n(s).$$

We have $r \geqslant s$ since $w_1 \leqslant w_2$ in $W(H)/W(P_n)$ is equivalent to $(i) \leqslant (j)$ in the sense of Definition 2.1.

Case I. Suppose that

$$i_1 = j_1,\ i_2 = j_2,\ \ldots,\ i_{k-1} = j_k - 1 \text{ and } i_k \neq j_k,\ k \leqslant s.$$

Then we set

$$w_1' = (j_1, \ldots, j_{k-1}, j_k - 1, j_{k+1}, \ldots, j_s).$$

We observe that

$$(j_k - 1) \notin (j_1, \ldots, j_s).$$

Case II. We have

$$i_1 = j_1,\ i_2 = j_2,\ \ldots,\ i_s = j_s.$$

Then we have $r > s$ since otherwise $w_1 = w_2$. Then we set

Case G_1: $w_1' = (j_1, \ldots, j_s, n)$

Case G_2: $w_1' = (j_1, \ldots, j_s, n-1, n)$ (note that in this case $r - s \geqslant 2$).

The required property (*a*) above is immediate. The property (*b*) above also follows immediately from (iii) of Lemma 4.1, proved above. It remains to prove (c) to conclude the proof of (iv).

Case I (as above): Let

$w_2 = (b_1, \ldots, b_{2n})$—minimal representative of w_2. Let p be the integer such that

$$b_{2n+1-p} = (j_k - 1).$$

Since $(j_k - 1) \notin (j_1, \ldots, j_s)$, we see that $(s+1) \leqslant p \leqslant n$. Let r_α denote the reflection (in $W(G_1)$, resp. $W(G_2)$) with respect to the positive root $\alpha = \epsilon_k + \epsilon_p$, which is in $R^+(G_1)$ as well as $R^+(G_2)$, identified as subsets of $X(T_{2n})$ (*cf.* section 3). We see that r_α as an element of $W(H)$ is the product of transpositions

$$r_\alpha = (k, 2n+1-p)\,(p, 2n+1-k) \text{ (note } k \leqslant s,\ s+1 \leqslant p).$$

It is now a simple exercise to deduce that

$$w_1' = w_2\, r_\alpha.$$

Case II (as above): We set

Case G_1: r_α = reflection with respect to $\alpha = 2\epsilon_{s+1} \in R^+(G_1)$ (note $s+1 \leqslant n$).

Case G_2: r_α = reflection with respect to $\alpha = \epsilon_{s+1} + \epsilon_{s+2} \in R^+(G_2)$ (note $s+2 \leqslant n$).

As an element of $W(H)$, we observe that

Case G_1: r_α = transposition $(s+1, 2n-s)$.

Case G_2: r_α = product of transpositions $(s+1, 2n-s-1)\,(s+2, 2n-s)$. It is now a simple exercise to deduce that $w_1' = w_2\, r_\alpha$.

This concludes the proof of (iv) of Lemma 4.1.

(v) This is an immediate consequence of Lemma 3.2 and (iv) of Lemma 4.1. In fact for $w \in W(G_1)/W(Q)$ (resp. $W(G_2)/W(Q)$), set

$$Y = X(w, H/P_n) \cap G_1/Q \text{ (resp. } X(w, H/P_n) \cap G_2/Q).$$

We see that Y is a union of Schubert varieties in G_1/Q (resp. G_2/Q) and by Lemma 3.2, it follows $X(w,G_1/Q)$ (resp. $X(w,G_2/Q)$) is an irreducible component of Y. Suppose that there is *another* irreducible component of Y, then it is a Schubert variety in G_1/Q (resp. G_2/Q) and it is of the form $X(w', G_1/Q)$ (resp. $X(w', G_2/Q')$), $w' \in W(G_1)/W(Q)$ (resp. $W(G_2)/W(Q)$) with $w' \leqslant w$. We have, a fortiori,

$$w' \leqslant w \text{ in } W(H)/W(P_n)$$

so that by (iv) of Lemma 4.1, it follows that

$$w' \leqslant w \text{ in } W(G_1)/W(Q) \text{ (resp. } W(G_2)/W(Q)).$$

This leads to a contradiction. This proves (v) and the proof of Lemma 4.1 is complete.

The basic results in section 4 can now be summarized in the following:

Theorem 4.1. Let Q denote the maximal parabolic subgroup in $G_1 = Sp(2n)$ (resp. $G_2 = SO(2n)$) associated to the right end root α_n in the Dynkin diagram (notations as in Bourbaki [2]). Then we have the following:

(i) We have a canonical imbedding of G_1/Q (resp. G_2/Q) in H/P_n (Grassmannian of n planes in a $2n$ dimensional vector space) such that the *opposite big cell* in H/P_n restricts to the opposite big cell in G_1/Q (resp. G_2/Q); further if we take the canonical identification of the opposite big cell in H/P_n with M_n = Space of $(n \times n)$ matrices (*cf.* Th. 2.1. and section 2), the opposite big cell in G_1/Q (resp. G_2/Q) can be identified with Sym M_n = Space of $(n \times n)$ symmetric matrices (resp. Sk $M_n = (n \times n)$ skew symmetric matrices)

(ii) Let $W(H)$, $W(G_1)$,.. etc. denote the Weyl groups of H, G_1, ..., etc. Then we have canonical inclusions

$$\text{(a) } W(G_1) \text{ (resp. } W(G_2)) \hookrightarrow W(H)$$

$$\text{(b) } W(G_1)/W(Q) \text{ (resp. } W(G_2)/W(Q)) \hookrightarrow W(H)/W(P_n).$$

The inclusion (b) preserves the partial orders in each. For $(\lambda) \in I_{2n}(n) \approx W(H)/W(P_n)$, let $((\alpha), (\beta))$ (resp. $((i), (j))$) denote the canonical (resp. canonical dual) pair associated to (λ) (*cf.* Def. 2.2). Then $(\lambda) \in W(G_1)/W(Q)$ (resp. $WG_2)/W(Q)$) if and only if $(\alpha) = (\beta)$ (resp. $(\alpha) = (\beta)$ and the number of elements in (i) or (j) is even).

(iii) Let L_n denote the ample generator of Pic H/P_n and L'_n the restriction of L_n to G_1/Q (resp. G_2/Q). Then L'_n is the ample generator of Pic G_1/Q (resp. $L'_n = F^2$, where F is the ample generator of Pic G_2/Q). Let $f = p_{(1,\ldots,r)}$ be a lowest weight vector in $H^0(H/P_n, L_n)$. Then the restriction f' of f to G_1/Q (resp. G_2/Q) is a lowest weight vector in $H^\circ(G_1/Q, L_n')$ (resp. $f' = g^2$, where g is a lowest weight vector in $H^\circ(G_2/Q,F)$). If $\{p_\tau\}$, $\tau \in W(H)/W(P_n)$, is as usual the family defined by $p_\tau = \tau . f$, then the restriction of p_τ (or to be more precise p_τ/f) to Sym M_n, identified as the opposite big cell in G_1/Q, is the function $p'_{(i),(j)}$: Sym $M_n \to K$, defined as follows: $p'_{(i),(j)}(Z)$ = determinant of the minor of Z with rows and columns associated to the indices of (i), (j) respectively, $Z \in$ Sym M_n.

$((i), (j))$ = canonical dual pair associated to τ.

Further the restriction of p_τ to Sk M_n, identified as the opposite big cell in G_2/Q, is the function $q^2_{(i)}$: Sk $M_n \to K$, defined as follows:

$q_{(i)}(Z)$ = Pfaffian of the square minor of Z with rows associated to the indices of (i), $Z \in$ Sk M_n,

$((i), (j))$ = the canonical dual pair associated to τ.

Remark 4.1. (1) We see that the big cell in H/P_n restricts to the big cell in G_1/Q. This is also the case for G_2/Q, if n is even; however if n is odd, the restriction of the big cell in H/P_n to G_2/Q is *empty*.

(2) We have seen (*cf.* Th. 4.1) that the partial order in $W(G_1)/W(Q)$ (resp. $W(G_2)/W(Q)$) is induced from $W(H)/W(P_n)$. It is also true that the partial order in $W(G_1)$ is induced from $W(H)$; however this is not the case for $W(G_2)$.

5. Interpretation of the standard monomial theory of De Concini and Procesi [5] on the space of symmetric and skew symmetric matrices

We keep the basic notations of the previous sections.

Definition 5.1. (i) Let D_k denote the determinantal variety in M_n as in Def. 2.3. Set

$$D_k(G_1) \text{ (resp. } D_k(G_2)) = D_k \cap \text{Sym } M_n \text{ (resp. } D_k \cap \text{Sk } M_n \text{ and } k \text{ is even).}$$

We call these *determinantal varieties in* Sym M_n (*resp.* Sk M_n). We observe that if $w \in W(G_1)/W(Q)$ (resp. $W(G_2)/W(Q)$) is such that the canonical *dual* pair associated to w is $((1, \ldots, k)\ (1, \ldots, k))$, then $D_k(G_1)$ resp. $(D_k(G_2))$ is the *opposite big cell* in $X(w, G_1/Q)$ (resp. $X(w, G_2/Q)$), in particular we see that $D_k(G_1)$ (resp. $D_k(G_2)$) is irreducible.

(ii) Let A_n denote the coordinate ring of Sym M_n. For a pair $((i), (j))$, $(i) \in I_n(r)$, $(j) \in I_n(r)$, $0 \leqslant r \leqslant n$, let $p'_{(i),(j)}$: Sym $M_n \to K$ denote the function defined in section 4 (*cf.* iii, Th. 4.1), so that $p'_{(i),(j)} \in A_n$. We call a *standard monomial in* A_n, an expression of the following type:

$$p'_{(i),(j)}\, p'_{(i)',(j)'}\, p'_{(i)'',(j)''}\, p'_{(i)''',(j)'''} \cdots$$

$$(i) \leqslant (j) \leqslant (i)' \leqslant (j)' \leqslant (i)'' \leqslant (j)'' \leqslant \ldots \; (cf.\ \text{Def. } 2.1)$$

and the number of $p_{(i),(j)}$ in this expression is called the *length* of this standard monomial.

(iii) Let B_n denote the coordinate ring of Sk M_n. For (i) $\in I_n(r)$, $0 \leqslant r \leqslant n$, let $q_{(i)}$: Sk $M_n \to K$ denote the function defined in section 4 (*cf.* (iii), Th. 4.1.), so that $q_{(i)} \in B_n$. We call a *standard monomial in* B_n, an expression of the following type:

$$q_{(i)'}\, q_{(i)''}\, q_{(i)'''} \cdots$$

$$(i)' \leqslant (i)'' \leqslant (i)''' \leqslant \ldots \; (cf.\ \text{Def. } 2.1).$$

Theorem 5.1. (i) Let $w \in W(G_2)/W(Q)$. Then the Schubert variety $X(w, G_2/Q)$ in G_2/Q is normal (in fact, it is also Cohen-Macaulay). In particular, the determinantal varieties $D_k(G_2)$ are normal.

(ii) Let g be a lowest weight vector in $H^\circ(G_2/Q, F)$, $F =$ the ample generator of Pic G_2/Q. Set

$$q_\tau = \tau.g,\ \tau \in W(G_2)/W(Q).$$

We call a *standard monomial of length m* on G_2/Q, an element in $H^\circ(G_2/Q, F^m)$ of the form:

$$q_{\tau_1} q_{\tau_2} \cdots q_{\tau_m},\ \tau_1 \geqslant \tau_2 \geqslant \ldots \geqslant \tau_m.$$

Then distinct standard monomials of length m of the form

$$q_{\tau_1} q_{\tau_2} \cdots q_{\tau_m},\ \tau_1 \geqslant \tau_2 \geqslant \ldots \geqslant \tau_m,\ w \geqslant \tau_1$$

form a basis of the vector space $H^\circ(X(w, G_2/Q), F^m)$.

(iii) Distinct standard monomials in B_n of the form

$$q_{(i)'}\, q_{(i)''}\, q_{(i)'''} \cdots$$

$(1, \ldots, k) \leqslant (i)' \leqslant (i)'' \leqslant \ldots$ (i.e. the number of elements in $(i)'$, $(i)''$ is $\leqslant k$) form a basis of the coordinate ring of $D_k(G_2)$ (as a vector space).

Proof. We note that the fundamental weight ω_n is *minuscule*, in the sense that if M is the irreducible representation with highest weight ω_n, when the base field is of characteristic zero, then all the weights in M are of the form $\tau(\omega_n)$, $\tau \in W(G_2)$ (Bourbaki [2]; Seshadri [22]). Then the assertions (i) and (ii) are just special cases of Seshadri [22] (note that in Seshadri [22] the weight vectors in $H^\circ(G_1/Q, F)$ are indexed as $\tau g'$, g' a highest weight vector), where the Hodge-Young theory of standard monomials is generalized to G/P, G a semi-simple group and P is a maximal parabolic subgroup associated to a minuscule weight. Then the assertion (iii) follows from (ii), in the same way as we had done in the proof of Th. 2.1. for the case H/P_r.

Definition 5.2. Let f' be a lowest weight vector in $H^\circ(G_1/Q, L'_n)$, where L'_n is the ample generator of Pic G_1/Q. Set

$$\{q_\tau\},\ \tau \in W(G_1)/W(Q),\ q_\tau = \tau . f'.$$

Recall that in section 4, for $\tau \in W(H)/W(P_n)$, we had defined p'_τ = restriction of $p_\tau \in H^\circ(H/P_n, L_n)$ ($p_\tau = \tau \cdot f$, f = Plücker coordinate $p_{(1,\ldots,n)}$). Hence $q_\tau = p'_\tau$, when $\tau \in W(G_1)/W(Q)$. For $\tau \in W(H)/W(P_n)$, let (τ_1, τ_2) be the pair of elements in $W(G_1)/W(Q)$, which is the image of τ in the canonical injection.

$$W(H)/W(P_n) \hookrightarrow W(G_1)/W(Q) \times W(G_1)/W(Q) \quad (cf.\ (e),\ \text{(ii) of Lemma 4.1}).$$

Let us call τ *admissible*, if $\tau_1 \geqslant \tau_2$. By Lemma 4.1, we see easily that given a pair of elements $(\tau_1, \tau_2) \in W(G_1)/W(Q)$, (τ_1, τ_2) comes from an admissible element τ if and only if

$$(*) \qquad \begin{cases} \text{(i) } \tau_1 \geqslant \tau_2 \text{ in } W(G_1)/W(Q), \\ \text{(ii) } m(\tau_1) = m(\tau_2)\ (cf.\ \text{Def. 3.1}). \end{cases}$$

Thus the set of admissible elements can be canonically identified with the set of pairs (τ_1, τ_2) in $W(G_1)/W(Q)$, satisfying (i) and (ii) of (*) and we call such a pair (τ_1, τ_2) an *admissible pair* in $W(G_1)/W(Q)$ and the element $\tau \in W(H)/W(P_n)$, the *admissible element in* $W(H)/W(P_n)$ associated to τ_1, τ_2. Let (τ_1, τ_2) be an admissible pair in $W(G_1)/W(Q)$ and τ the associated admissible element in $W(H)/W(P_n)$. We set

$$q_{\tau_1, \tau_2} = p'_\tau.$$

Let $(\alpha_1\ \alpha_2)$, (β_1, β_2) be two pairs of admissible elements in $W(G_1)/W(Q)$ and α, β respectively the associated admissible elements in $W(H)/W(P_n)$. Then we define $(\alpha_1, \alpha_2) \geqslant (\beta_1, \beta_2)$ by

$$(\alpha_1, \alpha_2) \geqslant (\beta_1, \beta_2) \Longleftrightarrow \alpha_2 \geqslant \beta_1\ (\Longleftrightarrow \alpha \geqslant \beta).$$

We can identify $W(G_1)/W(Q)$ with pairs of (admissible) elements of the form (τ, τ), $\tau \in W(G_1)/W(Q)$ and the order in $W(G_1)/W(Q)$ is consistent with the order introduced above. If (τ_1, τ_2) is an admissible pair in $W(G_1)/W(Q)$, we call it a *non-trivial pair* (resp. *trivial pair*) if $\tau_1 \neq \tau_2$ (resp. $\tau_1 = \tau_2$). We now observe

$$q_{\tau, \tau} = q_\tau.$$

We call a *standard monomial of length m on* G_1/Q, an element in $H^\circ(G_1/Q, (L'_n)^m)$ of the form

$$q_{\alpha_1, \alpha_2}\, q_{\beta_1, \beta_2}\, q_{\gamma_1, \gamma_2} \ldots,$$

$(\alpha_1, \alpha_2) \geqslant (\beta_1, \beta_2) \geqslant \ldots; (\alpha_1, \alpha_2), (\beta_1, \beta_2), \ldots$ being admissible pairs in $W(G_1)/W(Q)$.

For $\tau \in W(H)/W(P_n)$, let $p'_{(i),(j)}$ be the function

$$p'_{(i),(j)} : \mathrm{Sym}\, M_n \to K$$

which is the restriction of p'_τ to Sym M_n. Then we see that

$$(i) \leqslant (j) \Longleftrightarrow \tau \text{ is admissible.}$$

Then we have

$$q_{\tau_1, \tau_2} \mid \mathrm{Sym}\, M_n = p'_{(i),(j)},$$

(τ_1, τ_2) being essentially the *canonical pair* associated to τ and $((i), (j))$ the canonical dual pair associated to τ. We see that the restriction to Sym M_n of a standard monomial of length m on G_1/Q is standard in the sense of (ii), Def. 5.1 and conversely if the restriction of

$$p'_{\theta_1} \ldots p'_{\theta_m}, \quad \theta_i \in W(H)/W(P_n)$$

to Sym M_n is standard, then it is standard on G_1/Q.

Theorem 5.2. Let $w_k \in W(G_1)/W(Q)$ be the element such that the canonical dual pair associated to w_k is $((1, \ldots, k), (1, \ldots k))$ $(0 \leqslant k \leqslant n)$. Then we have:

(i) Distinct standard monomials of length m of the form (*cf.* Def. 5.2)

$$q_{\tau_1, \tau'_1} \ldots q_{\tau_m, \tau'_m}; \; w_k \geqslant \tau_1$$

form a basis of $H^\circ(X, (w_k, G_1/Q), (L'_n)^m)$ (to be precise, we take the restrictions to $X(w_k, G_1/Q)$ of these elements in $H^\circ(G_1/Q, (L'_n)^m)$.

(ii) Distinct standard monomials in A_n (*cf.* Def. 5.1) of the form

$$p'_{(i),(j)}\, p'_{(i)',(j)'}\, p'_{(i)'',(j)''} \ldots$$

$(1, \ldots k) \leqslant (i)$ i.e. the number of elements in $(i), (j), (i)', (j)'$ etc. is $\leqslant k$.

form a basis of the coordinate ring (considered a vector space) of the determinantal variety $D_k(G_1)$ in Sym M_n.

Proof. The assertion (ii) is proved in De Concini-Procesi [5]. Since $D_k(G_1)$ is the opposite big cell in $X(w_k, G_1/Q)$, the assertion (i) follows from (ii), in fact as in the proof of Th. 2.1 and Remark 2.1, we see that the assertions (i) and (ii) are equivalent.

Remark 5.1. (i) The assertion (i) of Th. 5.2 does not follow from (Seshadri [22]) as the fundamental weight ω_n is *not* minuscule (we are in type C_n).

(ii) It can be shown that the varieties $D_k(G_1)$ are normal and Cohen-Macaulay. This is proved in Kutz [14] and from this one can deduce that $X(w_k, G_1/Q)$ is normal and Cohen-Macaulay.

We shall now give an intrinsic description of admissible pairs in $W(G_1)/W(Q)$, which will suggest how the theory of standard monomials developed in (Seshadri [22]) for G/P when P is minuscule, could be generalized to the nonminuscule case.

Suppose now that G is a semi-simple algebraic group for which we fix a maximal torus T and a Borel subgroup B, $T \subset B$. Let P be a maximal parabolic subgroup in G, $P \supset B$. Let $Ch(G/P)$ (resp. $Ch(G/B)$) denote the Chow ring of G/P (resp. G/B). If X is a Schubert variety in G/P (resp. G/B), we denote by $[X]$, the class in $Ch(G/P)$ (resp. $Ch(G/B)$) represented by X. One knows that $Ch(G/P)$ (resp. $Ch(G/B)$) is a free **Z**-module and that the classes represented by distinct Schubert varieties form a **Z**-basis of $Ch(G/P)$ (resp. $Ch(G/B)$). Hence if X, Y are Schubert varieties in G/P (resp. G/B), the intersection product $[X]\cdot[Y]$ can be expressed as

$$[X]\cdot[Y] = \Sigma_Z\, d_Z\, [Z]$$

where Z runs through the set of all Schubert varieties in G/P (resp. G/B). We call d_z the intersection multiplicity of Z in $[X]\cdot[Y]$ (or simply $X\cdot Y$).

We denote by $[H]$ the element in $Ch(G/P)$, represented by the unique Schubert variety H of codimension one in G/P. This is the class determined by a hyperplane intersection of G/P in its canonical projective imbedding. We denote by the same $[H]$ the element in $Ch(G/B)$, determined by the Schubert variety of codimension one in G/B, which is the inverse image of H by the canonical morphism $G/B \to G/P$.

Let $w_1, w_2 \in W(G)$ (Weyl group of G) such that we have

(i) $X(w_1, G/B) \subset X(w_2, G/B)$ i.e. $w_1 \leqslant w_2$ in $W(G)$

(ii) $X(w_1, G/B)$ is of codimension one in $X(w_2, G/B)$. Then one knows that
$w_1 = w_2 r_\alpha$ (see for example Demazure [6])

where r_α is the reflection with respect to a positive root α (with reference to T, B). Then Chevalley's formula (Chevalley [4]; Demazure [6]) states that the intersection multiplicity of $X(w_1, G/B)$ in $[X(w_2, G/B)]\cdot[H]$ is $\langle \omega, \alpha^V \rangle$, where ω is the fundamental weight to which P is associated. Suppose now that $u, v \in W(G)/W(P)$ are such that

(i) $X(u, G/P) \subset X(v, G/P)$ i.e. $u \leqslant v$ in $W(G)/W(P)$.

(ii) $X(u, G/P)$ is of codimension one in $X(v, G/P)$.

Let w_1, w_2 be respectively *minimal representatives* in $W(G)$ of u, v. Then again we have $w_1 = w_2 r_\alpha$, where r_α is the reflection with respect to a positive root α. From the preceding Chevalley's formula, it follows immediately (using the projection formula) that the intersection multiplicity of $X(u,G/P)$ in $[X(v, G/P)] \cdot [H]$ is $\langle \omega, \alpha^v \rangle$.

Now the crucial result is

Lemma 5.1: Let v, u be a pair of elements in $W(G_1)/W(Q)$ such that $u \leqslant v$ and $X(u,G_1/Q)$ is of codimension one in $X(v, G_1/Q)$. Then (v, u) is an *admissible pair* if and only if the intersection multiplicity of $X(u, G_1/Q)$ in $[X(v, G_2/Q)] \cdot [H]$ is 2.

Proof. Let w_1, w_2 be the minimal representatives in $W(G_1)$ of u and v respectively. Then $w_1 = w_2 r_\alpha$, where r_α is the reflection with respect to a positive root α.

Suppose now that (v, u) is an *admissible pair*. Then since $X(u, G_1/Q)$ is of codimension one in $X(v, G_1/Q)$, we see by (iii) of Lemma 4.1. that u and v are of the following form.

$$u = (j_1, j_2, \ldots, j_{k-l}, j_k{-}1, j_{k+1} \ldots, j_s), \quad u \in I_n(s), \ v \in I_n(s), \ s \leqslant n.$$

$$v = (j_1, j_2, \ldots, j_{k-1}, j_k, j_{k+1}, \ldots, j_s), \quad (j_k - 1) \notin (j_1, \ldots, j_s)$$

(u and v differ only in the kth index and the kth index of u is one less than that of v).

Let $w_2 = (b_1, \ldots, b_{2n}) \in W(G_1)$ and p be the integer such that

$$b_{2n+1-p} = (j_k - 1)$$

Then as in the proof of (iv), Lemma 4.1, we see that $s+1 \leqslant p \leqslant n$ and that α is the positive root

$$\alpha = \epsilon_k + \epsilon_p \in R^+(G_1).$$

Then by Chevalley's formula (mentioned above), it follows that the intersection multiplicity of $X(u, G_1/Q)$ in $[X(v, G_1/Q)] \cdot [H]$ is given by $\langle \omega, (\epsilon_k + \epsilon_p)^V \rangle$. We have

$$\langle \omega, (\epsilon_k + \epsilon_p)^V \rangle = \frac{2(\omega, \epsilon_k + \epsilon_p)}{(\epsilon_k + \epsilon_p, \epsilon_k + \epsilon_p)} = (\omega, \epsilon_k + \epsilon_p) = 2$$

$$(\omega = \epsilon_1 + \ldots + \epsilon_n).$$

Suppose on the other hand that the intersection multiplicity of $X(u, (G_1/Q)$ in $[(X(v, G_1/Q)] \cdot [H]$ is 2. Then to prove that (v, u) is an admissible pair, we ha only to show that

$$m(u) = m(v) \text{ i.e. } m(w_1) = m(w_2) \ (cf.\ \text{Def. } 5.2).$$

Let

$$v = (i_1, \ldots, i_s) \in I_n(s),\ s \leqslant n.$$

Suppose that $m(u) \neq m(v)$. Then the hypothesis that $u \leqslant v$ and $X(u, G_1/Q)$ is of codimension one in $X(v, G_1/Q)$ implies easily (by (iii) of Lemma 4.1) that

$$u = (i_1, \ldots i_s, n)$$

i.e. $m(u) = m(v) + 1 = s+1$, the first s elements of u and v are the same and the last element of u is n. Then again as we saw in the proof of (iv), Lemma 4.1, α is the positive root $2\epsilon_{s+1}$. Then by Chevalley's formula, the intersection multiplicity of $X(u, G_1/Q)$ in $[X(v, G_2/Q)] \cdot [H]$ is given by $\langle \omega, (2\epsilon_{s+1})^V \rangle$. We have

$$\langle \omega, (2\epsilon_{s+1})^V \rangle = \frac{2(\omega, 2\epsilon_{s+1})}{(2\epsilon_{s+1}, 2\epsilon_{s+1})} = (\omega, \epsilon_{s+1}) = 1.$$

$$(\omega = \epsilon_1 + \ldots . \epsilon_n).$$

This leads to a contradiction and hence $m(u) = m(v)$. This completes the proof of Lemma 5.1.

Corollary: Let (τ, φ) be a pair of elements in $W(G_1)/W(G_1)$. Then it is a *non-trivial admissible pair* if and only if we can find a chain of elements $\{\tau_i\}, 1 \leqslant i \leqslant m, \tau_i \in W(G_1)/W(Q)$ having the following properties:

(i) $\tau_l = \tau,\ \tau_m = \varphi$ and $\tau_1 \geqslant \tau_2 \geqslant \ldots \geqslant \tau_m$

(ii) $X(\tau_i, G_1/Q)$ is of codimension one in $X(\tau_{i-1}, G_1/Q)$ and the intersection multiplicity of $X(\tau_i, G_1/Q)$ in $[X(\tau_{i-1}\ G_1/Q)] \cdot [H]$ is 2 $(2 \leqslant i \leqslant m)$.

Proof. If the conditions (i) and (ii) are satisfied, it is clear by the above lemma that (τ, φ) is an admissible pair, since it follows that

$$m(\tau_1) = m(\tau_2) = \ldots = m(\tau_m).$$

Suppose then that on the other hand (τ, φ) is an admissible pair in $W(G_1)/W(Q)$. Then we have

$$\left.\begin{array}{l} \tau = (j_1, \ldots, j_s),\ \tau \in I_n(s) \\ \varphi = (i_1, \ldots, i_s),\ \varphi \in I_n(s) \end{array}\right\} 0 \leqslant s \leqslant n.$$

Suppose that we have

$$i_1 = j_1,\ i_2 = j_2, \ldots, i_{k-1} = j_{k-1} \text{ and } i_k < j_k,\ k \leqslant s.$$

Then we set

$$\varphi' = (j_1, \ldots, j_{k-1}, j_k - 1, j_{k+1} \ldots, j_s).$$

Then we have

(a) $\tau \geqslant \varphi' \geqslant \varphi$.

(b) (τ, φ') and (φ', φ) are admissible pairs.

(c) $X(\varphi', G_1/Q)$ is of codimension one in $X(\tau, G_1/Q)$.

Then by Lemma 5.1 we construct by an easy inductive argument, a chain which satisfies the properties (i) and (ii) above. This completes the proof of the corollary.

Remark 5.1. It can be checked that the elements in $H^\circ(G_1/Q, L'_n)$

$\{q_{\tau_1, \tau_2}\}$, (τ_1, τ_2) admissible pair in $W(G)/W(Q)$ have the following properties:

(i) The restriction of q_{τ_1, τ_2} to $X(w, G_1/Q)$, $w \in W(G_1)/W(Q)$ is not identically zero if and only if

$$w \geqslant \tau_1, \text{ i.e., } (w, w) \geqslant (\tau_1, \tau_2).$$

(ii) the weight of q_{τ_1, τ_2} is

$$\tfrac{1}{2}((\tau_1(\omega') + \tau_2(\omega')) = -\tfrac{1}{2}(\tau_1(\omega_n) + \tau_2(\omega_n)).$$

ω' being the lowest weight of the G_1-module $H^\circ(G_1/Q, L_n')$ (note that $\omega' = \omega_0(i(\omega_n)) = -\omega_n$).

6. The basic conjectures

The results of section 5 as well as computations in low rank suggest the following conjectures:

Let G be a semi-simple, simply-connected, algebraic group defined over an algebraically closed field K. Fix a maximal torus T and a Borel subgroup B, $T \subset B$. We refer to roots, Bruhat decomposition etc. with respect to this choice of T, B. Let P be a *maximal parabolic subgroup*, $P \supset B$, associated to a fundamental weight ω. We say that P or ω is of *classical type* if

$$\langle \omega, \alpha^V \rangle = 0, \pm 1 \text{ or } \pm 2, \forall \text{ root } \alpha \text{ of } G.$$

The hypothesis that P is of classical type implies (on account of Chevalley's formula, section 5, also Chevalley [4]; Demazure [6] that if X, Y are Schubert varieties in G/P such that Y is of codimension one in X and H is the codimension one Schubert variety in G/P, then the intersection multiplicity of Y in $(X)\cdot(H)$ is utmost 2 (*cf* section 5 for the notations $X, \ldots,$ etc.). In fact it can be seen that this property is an equivalent description of P to be of classical type.

Let $W(G)$ (resp. $W(P)$) denote the Weyl group of G (resp. P). We denote by $X(w, G/B)$ (resp. $X(w, G/P)$) the Schubert variety in G/B (resp. G/P) associated to $w \in W(G)$ (resp. $W(G)/W(P)$).

Definition 6.1. (i) Let (τ, φ) be a pair of elements in $W(G)/W(P)$. We call it an *admissible pair* if

(a) Either $\tau = \varphi$, in which case we call it a *trivial admissible pair*, or

(b) we can find $\tau_1, \ldots, \tau_m \in W(G)/W(P)$ such that

(1) $X(\tau) = X(\tau_1) \supset X(\tau_2) \supset \ldots \supset X(\tau_m) = X(\varphi)$

$$\text{i.e. } \tau_1 = \tau \geqslant \tau_2 \geqslant \ldots \geqslant \tau_m = \varphi,$$

(2) $X(\tau_i)$ is of codimension one in $X(\tau_{i-1})$ and the intersection multiplicity of $X(\tau_i)$ in $[X(\tau_{i-1})] \cdot [H]$ is 2 $(2 \leqslant i \leqslant m)$.

In case (b), we say that (τ, φ) is a *non-trivial admissible pair.*

(ii) Let (τ_1, φ_1), (τ_2, φ_2) be two admissible pairs in $W(G)/W(P)$. Then we define

$$(\tau_1, \varphi_1) \geqslant (\tau_2, \varphi_2) \text{ if } \varphi_1 \geqslant \tau_2 \text{ in } W(G)/W(P).$$

(Note that this relation $\geqslant$ is *not* a partial order in the set of admissible pairs in $W(G)/W(P)$. It satisfies the axiom of transitivity but not of reflexivity).

Conjectures I. Let G be as above and suppose that P is a maximal parabolic subgroup $(P \supset B)$ of classical type. Let ω be the fundamental weight associated to P. Let L be the ample generator of Pic G/P. Then given an *admissible pair* (τ, φ) in $W(G)/W(P)$, we can find a $p_{\tau, \varphi} \in H^\circ(G/P, L)$ having the following properties:

(i) $p_{\tau, \varphi}$ does not vanish on the Schubert variety $X(w, G/P)$, $w \in W(G)/W(P)$, if and only if

$$w \geqslant \tau \text{ i.e. } (w, w) \geqslant (\tau, \varphi)$$

(ii) the one dimensional space spanned by $p_{\tau, \varphi}$ is stable under T and the weight of $p_{\tau, \varphi}$ is

$$-\tfrac{1}{2}(\tau(\omega) + \varphi(\omega))$$

(iii) let us call an element of $H^0(G/P, L^m)$ of the form

$$p_{\tau_1, \varphi_1}\, p_{\tau_2, \varphi_2} \cdots p_{\tau_m, \varphi_m},$$

$$(\tau_1, \varphi_1) \geqslant (\tau_2, \varphi_2) \geqslant \ldots \geqslant (\tau_m, \varphi_m)$$

a *standard monomial of length m*. Then distinct standard monomials of length m form a basis of $H^\circ(G/P, L^m)$, $m \geqslant 1$. In fact distinct standard monomials of length m as above such that $w \geqslant \tau_1$, form a basis of $H^\circ(X(w, G/P), L^m)$, $m \geqslant 1$.

Let us call a *Young tableau* (or *diagram*) *of length m* in $W(G)/W(P)$, a sequence of m admissible pairs in $W(G)/W(P)$ of the following type

$$\begin{cases} (\tau_1, \varphi_1), (\tau_2, \varphi_2), \ldots, (\tau_m, \varphi_m), \\ (\tau_1, \varphi_1) \geqslant (\tau_2, \varphi_2) \geqslant \ldots \geqslant (\tau_m, \varphi_m). \end{cases}$$

The above conjectures imply in particular that

$$\dim H^\circ(G/P, L^m) = \# \{\text{Young tableaux of length } m\}.$$

Conjectures II. Let us suppose that G is a *classical group* i.e. it is of type **A, B, C** or **D.** Note that this is equivalent to supposing that every maximal parabolic subgroup of G is of classical type in the sense defined above. Let l be the rank of G and let us index the maximal parabolic subgroups (containing B) by P_i, $1 \leqslant i \leqslant l$, so that P_i, is associated to the fundamental weight ω_i in the notation of Borel [1]. Suppose that

$$u \in W(G)/W(P_{i_1}),\ v \in W(G)/W(P_{i_2}),\ i_1 \leqslant i_2.$$

Then we define $u \geqslant v$, if

(i) either $i_1 = i_2$; in which case $u \geqslant v$ is just in the sense of the partial order in $W(G)/W(P_{i_1})$, or

(ii) $i_1 \neq i_2$; then if v_1 is the minimal representative in $W(G)$ of v and $\bar{v}_1$ is the canonical image of v_1 in $W(G)/W(P)$, we have

$$u \geqslant \bar{v}_1 \text{ in } W(G)/W(P_{i_1}).$$

Let us denote by $(\tau(i), \varphi(i))$ an *admissible pair* in $W(G)/W(P_i)$ and $P_{(\tau(i),\, \varphi(i))}$ the element in $H^\circ(G/P_i, L_i)$ associated to $(\tau(i), \varphi(i))$ in conjectures I (L_i being the ample generator of Pic G/P_i). We denote by the same L_i, the line bundle on G/B, which is the pull back of L_i by the canonical morphism $G/B \to G/P_i$). We define $(\tau(i), \varphi(i)) \geqslant (\tau(j), \varphi(j))$ if $i \leqslant j$ and $\varphi(i) \geqslant \tau(j)$, as defined above.

Let $(a) = (a_1, \ldots, a_r)$, $1 \leqslant a_1 < a_2 < \ldots < a_r \leqslant l$, $m = (m_1, \ldots m_r)$, $m_i \geqslant 1$ and $m = \Sigma_{i=1}^{r} m_i$. We call a *standard monomial of type* $((a), m)$ (*and total degree* $|m|$) an expression of the following type:

$$(*) \quad \left(p_{\tau_1(a_1),\,\varphi_1(a_1)}\, p_{\tau_2(a_1),\,\varphi_2(a_1)} \cdots p_{\tau_{m_1}(a_1),\,\varphi_{m_1}(a_1)}\right) \cdots \left(p_{\tau_1(a_r),\,\varphi_1(a_r)} \cdots p_{\tau_{m_r}(a_r),\,\varphi_{m_r}(a_r)}\right)$$

such that

$$(**) \quad (\tau_1(a_1), \varphi_1(a_1)) \geqslant (\tau_2(a_1), \varphi_2(a_1)) \geqslant \ldots \geqslant (\tau_{m_1}(a_1), \varphi_{m_1}(a_1)) \geqslant (\tau_1(a_2), \varphi_1(a_2)) \ldots$$

The element in (*) can be identified with an element of

$$H^\circ\left(G/B, L_{a_1}^{m_1} \otimes \ldots \otimes L_{a_r}^{m_r}\right)$$

Then the conjecture is that distinct standard monomials as in (*) *form a basis of* $H^\circ\left(G/B, L_{a_1}^{m_1} \otimes \ldots \otimes L_{a_r}^{m_r}\right)$, *when* G *is of type* **A, B** *or* **C.** *Let us call an expression as in* (**) *a Young diagram in* $W(G)$ *of type* $((a), m)$ and total degree $|m|$: The above conjecture implies that

$$\dim H^\circ\left(G/B, L_{a_1}^{m_1} \otimes \ldots \otimes L_{a_r}^{m_r}\right) = \# \{\text{Young diagrams in } W(G) \text{ of type } ((a).m)\}$$

We note that an expression as in (**) is precisely a Young diagram in the usual sense when $G=SL(l+1)$ and we identify $W(G)/W(P_j) \approx I_{l+1}(j)$. We observe also that the Conjectures II hold good for type **A**: this is a classical fact when the general field is of characteristic zero (Weyl [23]) further, because of the vanishing theorem it is valid in arbitrary characteristics also.

The Conjectures II can also be stated for G of type **D.** The above definition of standard monomials has to be suitably modified in this case†.

As in I above, a stronger version of these conjectures can be formulated for any Schubert variety in G/B but this has to be done with greater care.

7. Classical invariant theory

We shall now indicate how the results of De Concini and Procesi [5] concerning classical invariant theory can be obtained from the results of section 2 and 5. We shall now work out this method in detail for the case of invariants under the symplectic group. The other cases are treated in a similar way. We shall suppose in the sequel, for simplicity of treatment that char $K \neq 2$ ($K =$ ground field).

Let $X=\operatorname{Spec} R$ be an *affine variety* on which a *reductive algebraic group* G operates. Let $Y=\operatorname{Spec} R^G$ and φ the canonical morphism $\varphi : X \to Y$. One knows, thanks to Haboush's proof of Mumford's conjecture, that we have the following: (Haboush [8]; Seshadri [21])

(i) Y is an *affine variety* (i.e. R^G is a K-algebra of finite type).

†This modification has been done in collaboration with Musili.

(ii) φ is a *good quotient*, in particular, φ is surjective and $\varphi(x_1)=\varphi(x_2)$, $x_i\in X$ if and only if $\overline{0(x_1)}$ and $\overline{0(x_2)}$ intersect, where $\overline{0(x_i)}$ denotes the closure of the G-orbit $0(x_i)$ in X.

(iii) φ is a *categorical quotient* (Mumford [17]; Seshadri [20] and [21].

Let now V denote as in section 2 above, a $2n$ dimensional vector space with basis $e_1, \ldots, e_n, e_{n+1}, \ldots, e_{2n}$. As in sections 3-5, let G_1 denote the symplectic group Sp$(2n)$ i.e. the group which leaves the following form invariant:

$$\begin{pmatrix} 0 & J \\ -J & 0 \end{pmatrix}, \; J=\begin{pmatrix} 0 & & 1 \\ & \cdot^{\cdot^{\cdot}} & \\ 1 & & 0 \end{pmatrix} \; (n\times n \text{ matrix})$$

We denote by $\langle,\rangle$ the skew-symmetric form on V given by the above matrix. Let X denote the affine space

$$X = V\oplus \ldots \oplus + V\,(m \text{ times})$$

and take the diagonal action of G_1 on X. Let SkM_m denote the space of skew-symmetric matrices of rank m (as in sections 4 and 5) and let

$$\varphi \; : \; X \to \mathrm{Sk}M_m$$

denote the G_1 'invariant' morphism (i.e. for the trivial action of G_1 on SkM_m, φ is a G_1 morphism) defined as follows:

$$\begin{cases} x=(x_1, \ldots, x_m) \in X, \; x_i \in V. \\ \varphi(x) \text{ is the skew symmetric matrix whose } (i,j)\text{th entry is } \langle x_i, x_j\rangle. \end{cases}$$

Then we have the following (parts of which are the known classical theorems of invariants under the symplectic group, when the ground field is of characteristic zero (Weyl [23]).

Theorem 7.1. The image of φ is the determinantal variety $D_{2n}(G_2)$ in SkM_m (*cf.* Def. 5.1); further, the canonical map

$$\varphi : X \to D_{2n}(G_2) \text{ (denoted by the same } \varphi)$$

can be identified with the canonical morphism $X\to Y$, $Y=$Spec R^G, $X=$Spec R described above. In particular, R^G is *generated* by the G-invariant functions $\langle x_k, x_l\rangle$ and by Theorem 5.1, we get a basis of R^G as follows: Let $q_{(i)}$, $(i)=(i_1, \ldots, i_r)$, r *even*, $r\leqslant 2n$, be the Pfaffian of the minor of the skew-symmetric matrix $\varphi(x)$ with rows associated to (i). Then R^G has a basis consisting of the standard monomials in $q_{(i)}$ (*cf.* (iii), Def. 5.1):

$$q_{(i)}q_{(i)'}q_{(i)''}\ldots, \; (i)\leqslant(i)'\leqslant(i)'' \text{ (note length of (i), (i)}', \ldots, \leqslant 2n).$$

Proof. For this we make use of the following:
Lemma 7.1. Let Z=Spec R be an affine space on which there is given a *linear action* of a reductive algebraic group G (Seshadri [20] and [21] for the definition of linear actions) and

$$\psi : Z \to \mathbf{A}^N (\mathbf{A}^N = N \text{ dimensional affine space})$$

a *graded G-invariant morphism* (graded means $\psi(tz)=t^r\psi(z)$, $t \in K$). Let S=Spec R^G and D a *closed subvariety* of $\mathbf{A}^N$ such that $\psi(Z) \subset D$. Then in order that the canonical morphism $\psi : Z \to D$ be identified with the canonical morphism $\pi : Z \to S$, it *suffices* that the following conditions hold:

(i) for $z \in Z^{ss}$ (set of *semi-stable* points in Z, (Seshadri [20] and [21]) $\psi(z) \neq (0)$.

(ii) $\exists$ a non-empty G-stable open subset U of Z such that G operates freely on U, $U \to U \bmod G$ is a G-principal fibre space and ψ induces an *immersion* of $U \bmod G$ into D (or $\mathbf{A}^N$).

(iii) D is *normal*

(iv) $\dim U/G = \dim D$.

Proof of lemma. Let $\mathbf{A}^N = \operatorname{Spec} P$. We write

$$Z_1 = \operatorname{Proj} R,\ S_1 = \operatorname{Proj} R^G,\ \mathbf{P}^{N-1} = \operatorname{Proj} P.$$

Since ψ is G-invariant, we get a canonical morphism $\rho : S \to \mathbf{A}^N$ such that the following diagram is commutative

(1)

$$\begin{tikzcd} Z \arrow[d, "\pi"'] \arrow[dr, "\psi"] & \\ S \arrow[r, "\rho"'] & \mathbf{A}^N \end{tikzcd}$$

since $\pi : Z \to S$ is a *categorical quotient.* Let Z_1^{ss} denote the open subset of semi-stable points in Z_1. Then we have a canonical morphism $\pi_1 : Z_1^{ss} \to S_1$, which is a categorical quotient (Seshadri [z0] and [21]. Since ψ is graded, ψ defines a rational morphsim $\psi_1 : Z_1 \to \mathbf{P}^{N-1}$ and the hypothesis (i) above implies that ψ_1 is *regular* in Z_1^{ss}. Since $\pi_1 : Z_1^{ss} \to S_1$is a categorical quotient and ψ_1 is G-invariant, we deduce again a *morphism* $\rho_1 : S_1 \to \mathbf{P}^{N-1}$ and a commutative diagram

(2)

$$\begin{tikzcd} Z_1^{ss} \arrow[d, "\pi_1"'] \arrow[dr, "\psi_1"] & \\ S_1 \arrow[r, "\rho_1"'] & \mathbf{P}^{N-1} \end{tikzcd}$$

We claim now that ρ_1 is a *finite morphism*. In fact we can find a homogeneous element s in P such that its canonical image $\psi^*(s)$ is a non-zero element of R^G. Now $\psi_1{}^*(s)$ can be identified with a section of an ample line bundle M on S_1 (a power of the tautological line bundle on Z_1^{ss} descends to M). From these considerations it follows easily that the inverse image by ρ_1 of the tautological line bundle on $\mathbf{P}^{N-1}$ is *ample* on S_1. By a familiar argument, this implies that ρ_1 is a finite morphism. The fact that ρ_1 is finite, implies that the morphism ρ itself is finite and this is a consequence of *Lemma* 7.2. Let $\delta : \mathrm{Spec}\, B \to \mathrm{Spec}\, K[X_1, \ldots, X_r]$ be a morphism induced by a graded homomorphism (B a graded K-algebra of finite type). Suppose that the rational morphism $\delta_1 : \mathrm{Proj} B \to \mathrm{Proj} K[X_1, \ldots, X_r]$ is a *finite morphism*. Then δ is itself a finite morphism.

Proof of Lemma 7.2. By classical arguments one finds that $\bigoplus_{j \geqslant j_0} B_j$ is a $K[X_1, \ldots, X_r]$ module of finite type for suitable j_0. Since $\bigoplus_{j < j_0} B_j$ is finite dimensional, it follows that B is a $K[X_1, \ldots, X_r]$ module of finite type, which means that δ is finite. This proves Lemma 7.2.

Let us return to the proof of Lemma 7.1. From Lemma 7.2 it follows then that ρ is *finite*. The hypotheses (ii) and (iv) of Lemma 7.1 imply that ρ induces a birational morphism $\rho : S \to D$. Now S is normal and D is normal by (iii). Hence it follows that $\rho : S \to D$ is an isomorphism. This concludes the proof of Lemma 7.1.

We now go back to the proof of Theorem 7.1. We will now check that $\varphi(Z) \subset D_{2n}(G_2)$ and that the conditions of Lemma 7.1 hold (taking $D = D_{2n}(G_2)$, $Z = X$, $\psi = \varphi$ etc.). This is done in the following steps:

1. Let $x^0 = (x_1{}^0, \ldots, x_m{}^0) \in X$. Then we claim the following:

$$x^0 \notin X^{ss} (\text{i.e. } x^0 \text{ is } \textit{not} \text{ semi-stable})$$

$\Longleftrightarrow$ if W is the subspace of V spanned by $x_i{}^0$, then W is *totally isotropic* i.e.

$$\langle x_i{}^0, x_j{}^0 \rangle = 0 \;\forall\, i, j,\; 1 \leqslant i, j \leqslant m$$

First of all we observe that the above claim implies that the hypothesis (i) of Lemma 7.1 is satisfied for φ.

To prove the above claim, we first note that if W is *not* totally isotropic, then $\langle x_k^0, x_l^0\rangle \neq 0$ for some k, l with $1 \leqslant k \leqslant m$, $1 \leqslant l \leqslant m$. Let $F_{kl} : X \to K$ be the function

$$F_{kl}(x) = \langle x_k, x_l\rangle, \; x = (x_1, \ldots, x_m).$$

Then F_{kl} is G-invariant and $F_{kl}(x^0) \neq 0$ from which it follows easily that $x^0 \in X^{ss}$. Hence to prove the above claim, it remains to show that

$$W \text{ totally isotropic} \Longrightarrow x^0 \notin X^{ss}.$$

Now the set of maximal isotropic subspaces is a homogeneous space under $G_1 = Sp(2n)$, so that we can find $g \in G_1$ such that $W \subset W_0$ where W_0 is the maximal isotropic subspace of V spanned by the first n coordinates $e_1, \ldots, e_n$. This means that if $gx^0 = (y_1, \ldots, y_m)$, the subspace of V spanned by y_i is contained in W. Since

$$x^0 \in X^{ss} \Longleftrightarrow gx^0 \in X^{ss}$$

it follows, we can suppose, without loss of generality, that $\{x_i^0\}$, $1 \leqslant i \leqslant m$, are in W_0. Let now $\lambda = \lambda(t)$ be the 1-PS (one parameter subgroup) of the maximal torus T of G_1 of the form

$$\begin{pmatrix} t & & & & 0 \\ & \ddots & & & \\ & & t & & \\ & & & t^{-1} & \\ & & & & \ddots \\ 0 & & & & & t^{-1} \end{pmatrix}$$

We have

$$x_i^0 = (y_{i1}, \ldots, y_{in}, 0, \ldots, 0), \; y_{ij} \in K$$

and

$$(y_{i1}, \ldots, y_{in}, \ldots 0, , 0) \begin{pmatrix} t & & & & 0 \\ & \ddots & & & \\ & & t & & \\ & & & t^{-1} & \\ & & & & \ddots \\ 0 & & & & & t^{-1} \end{pmatrix} = t(y_{i1}, \ldots, y_{in}, 0, \ldots, 0).$$

$$= x^0\lambda.$$

(we take the action of G_1 on X to be on the right).

We deduce that $\lim_{t\to 0} x^0\lambda = (0)$ which implies that $x^0 \notin X^{ss}$. This proves claim I.

II. In order to prove the theorem, we observe first that we can take m to be sufficiently large. In fact, set

$$\text{Spec } R_m = V \oplus \ldots \oplus V (m \text{ times})$$

Then if $m_1 > m_2$, we have canonical G_1-morphisms

$$j : X_{m_1} \to X_{m_2},\ i : X_{m_2} \hookrightarrow X_{m_1}$$

where j is the projection on to the first m_2 factors and i is the inclusion by putting the last m_1–m_2 factors to be zero. If i^*, j^* denote the canonical induced homomorphisms

$$i^* : R_{m_1} \to R_{m_2},\ j^* : R_{m_2} \to R_{m_1},$$

we get $i^* \circ j^* =$ identity. Using these maps, it is seen easily that

$$\text{Theorem 7.1 true for } m_1 \Longrightarrow \text{Theorem 7.1 true for } m_2.$$

Hence to prove theorem, *we shall hereafter suppose that* $m \geqslant 2n = \dim V$.

Let U be the open subset of X defined by

$$U = \{x \in X \mid \text{if } x = (x_1, \ldots, x_m),\ x_i \in V, \text{ then the } x_i,\ 1 \leqslant i \leqslant 2n, \text{ are linearly independent in } V\}.$$

Obviously U is G_1-stable and by 1, $U \subset X^{ss}$. We claim now that G_1 operates *freely* on U, $U \to U \bmod G_1$ is a principal fibre space and that φ induces an immersion of $U \bmod G_1$ into Sk M_m.

To prove this claim, note that we have a G_1-isomorphism

$$U \approx GL(V) \times \underbrace{V \times \ldots \times V}_{\text{m}-2n \text{ times}}.$$

From this identification, it follows easily that G_1 operates freely on U. Further one sees that $U \bmod G_1$ can be identified with the fibre space, with base ($GL(V) \bmod G_1$) and fibre ($V \times \ldots \times V$) ($m-2n$ times), associated to the principal fibre space $GL(V) \to GL(V) \bmod G$. Thus to prove the above claim, it suffices to show that φ induces an *immersion* of $U \bmod G_1$ into Sk M_n.

Let us first show that φ induces an *injective* map of $U \bmod G_1$ into SkM_m. Let $x, y \in U$. Because of the identification $U \approx GL(V) \times (V \times \ldots \times V)$, we can write

$$x = (Y_1, Z_1),\ y = (Y_2, Z_2),\ Y_i \in GL(V).$$

Suppose that $\varphi(x) = \varphi(y)$. If $x = (x_1, \ldots, x_m)$, $y = (y_1, \ldots, y_m)$, $x_i, y_j \in V$ this implies, in particular that

$$\langle x_i, x_j \rangle = \langle y_i, y_j \rangle;\ 1 \leqslant i, j \leqslant 2n.$$

This can be written in matrix notation as follows:

$$ {}^tY_1J'Y_1={}^tY_2J'Y_2,\ J'=\begin{pmatrix}0 & J\\ -J & 0\end{pmatrix}. $$

$$ \text{i.e.}\quad {}^t(Y_1Y_2^{-1})\,J'(Y_1\ Y_2^{-1})=J' $$

$$ \text{i.e.}\quad Y_1Y_2^{-1}\in G_1 \text{ which} \Longrightarrow \exists\, g\in G \text{ such that } Y_1=gY_2. $$

Hence to prove that φ induces an injection of $U \bmod G_1$ into $\mathrm{Sk}M_m$, we can suppose that

$$ x=(Y,Z_1),\ y=(Y,Z_2),\ Y=(x_1,\ldots,x_{2n}). $$

Let $Z_1=(v_1,\ldots)$, $Z_2=(v_2,\ldots)$, $v_i\in V$; Then $\varphi(x)=\varphi(y)$ implies, in particular, that

$$ \langle x_i\ v_1\rangle=\langle x_i,v_2\rangle,\ i=1,\ldots,2n. $$

Since the form $\langle\ ,\ \rangle$ is non-degenerate, it follows that $v_1=v_2$. This implies then that $Z_1=Z_2$ and it follows that φ induces an injection of $U \bmod G_1$ into $\mathrm{Sk}M_m$.

We see now easily that the above argument, in fact, proves the stronger fact that if A is *any K-algebra*, φ, induces an *injective map of A-valued points*

$$ (U \bmod G_1)(A)\to \mathrm{Sk}_m(A). $$

This implies that φ induces an immersion of U mod G_1 into $\mathrm{Sk}M_n$ and the above claim in II is proved.

III. It remains to show, since $D_{2n}(G_2)$ is normal (*cf.* Theroem 5.1) that

(a) $\dim U/G=\dim D_{2n}(G_2)$

(b) $\varphi(X)\subset D_{2n}(G_2)$.

By Theorem 5.1 the ideal of $D_{2n}(G_2)$ in $\mathrm{Sk}M_m$ is generated by $q_{(i)}$, $(i)=(i_1,\ldots,i_{2r})$, $2r>2n$. Hence to prove (*b*), it suffices to show that all the $(2r\times 2r)$ minors of $\varphi(x)\in \mathrm{Sk}M_m$, $x\in X$, vanish for $2r>2n$. We leave this as an exercise. The proof of (*a*) is also easy, as the left side of (*a*) is ($\dim X-\dim G$) and the right side of (*a*) is also easily calculated by taking the Schubert variety $X(\tau, G_2/Q)$ in $\mathrm{SO}(2n)/Q$ of which $D_{2n}(G_2)$ is the opposite big cell and calculating $l(\tau w_0)$ in $W(G_2)/W(Q)$. We leave this also as an exercise.

This completes the proof of the theorem.

Remark 7.1. A generalisation similar to Theorem 7.1 for the classical theorems on invariants for $GL(n)$ and $\mathrm{O}(2n)$ can be obtained (DeConcini and Procesi [5] Hocheter and Eagon [10]; Kutz [14]) by the same arguments as above and their statements are as follows:

(i) *Case* $GL(n)$: Let V be the n-dimensional vector space and $G=GL(V)$. Set

$$X=\underbrace{V\oplus\ldots\oplus V}_{m\text{ times}}\oplus\underbrace{V^*\oplus\ldots\oplus V^*}_{m\text{ times}},\ V^*=\text{dual of } V.$$

For $x\in X$, $x=(x_1,\ldots,x_m,\xi_1,\ldots,\xi_m)$ let $\varphi(x)=\|\langle x_i,\xi_j\rangle\|\in M_m$ ($m\times m$ matrices), $\langle\,,\rangle$ being the canonical bilinear form on $V\times V^*$. Let

$$p_{(i),(j)}=\begin{cases}\text{determinant of the minor of } \varphi(x), \text{corresponding to}\\ (i),(j);\,(i)=(i_1,\ldots,i_r),\,(j)=(j_1,\ldots,j_r),\ r\leqslant m.\end{cases}$$

Then the morphism $\varphi: X\to M_m$ is G-invariant, φ maps X on to the *determinantal variety* D_n in M_m (opposite big cell of a Schubert variety in $G_{m,\,2m}$, section 1). Further φ identifies the categorical quotient $X \bmod G$ with D_n. Applying Theorem 2.1, we get also a basis of R^G, $X=\operatorname{Spec} R$, by *standard monomials* in $p_{(i),(j)}$ with $r\leqslant n$, $(i)=(i_1,\ldots,i_r)$ as in that theorem.

(ii) Case $\mathbf{O}(2n)$: (De Concini and Procesi [5]. Let V be a $2n$-dimensional vector space and $\mathbf{O}(2n)$ the orthogonal group leaving $\begin{pmatrix}0 & J\\ J & 0\end{pmatrix}$ invariant. We write $G=\mathbf{O}(2n)$. Set

$$X=V\oplus\ldots\oplus V\ (m\text{ times}),\ X=\operatorname{Spec} R$$

For $x\in X$, $x=(x_1,\ldots,x_m)$, let $\varphi(x)=\|\langle x_i,x_j\rangle\|\in\operatorname{Sym} M_m=$ space of symmetric $(m\times m)$, matrices $\langle\,,\rangle$ being the scalar product on V. Let

$$p_{(i),(j)}=\begin{cases}\text{determinant of the minor of } \varphi(x) \text{ corresponding}\\ \text{to } (i),(j),\,(i)=(i_1,\ldots,i_r),\,(j)=(j_1,\ldots,j_r),\ r\leqslant m.\end{cases}$$

Then the morphism $\varphi: X\to\operatorname{Sym} M_m$ is G-invariant, φ maps X on to the *determinantal variety* $D_{2n}(G_1)$ in $\operatorname{Sym} M_m$ (opposite big cell of a Schubert variety in $\mathrm{Sp}(2n)/Q$, Q maximal parabolic subgroup defined by the right end root, *cf.* Remark 2.1) and φ identifies the categorical quotient $X \bmod G$ with $D_{2n}(G_1)$. Applying Th. 5.2, we get a basis of R^G by standard monomials in $p_{(i),(j)}$ (*cf.* Def. (ii), 5.1).

References

1. Borel A 1966 *Proc. Symp. Pure Math.* **9** 3
2. Bourbaki N 1968 *Groupes et algebres de Lie* (Paris: Herman) Chap. 4-6
3. Chevalley C C 1958 *Classification de groupes de Lie algebriques* (Paris: Secretariat Mathematique)
4. Chevalley C C; Sur les decompositions cellularies des espaces G/B (Manuscrit non publíe)
5. De Concini C and Procesi C 1976 *Adv. Math.* **21** 330
6. Demazure M 1974 *Ann. Sc. Éc. Norm. Sup.* **7** 58
7. Doubilet P, Gian-Carlo Rota and Joel Stein 1974 *Stud. Appl. Math.* 185
8. Haboush W 1975 *Ann. Math.* **102** 67

9. Hochster M 1973 *J. Algebra* **25** 40
10. Hochster M and Eagon J A 1971 *Am. J. Math.* **93** 1020
11. Hodge W V D 1943 *Proc. Cambridge Philos. Soc.* **39** 22
12. Hodge W V D and Pedoe D 1952 *Methods of Algebraic Geometry* (Cambridge Univ. Press) Vol. 2
13. Kempf G 1976 *Ann. Math.* **103** **557**
14. Kutz D E 1974 *Trans. Am. Math. Soc.* **194** 115
15. Lakshmibai V, Musili C and Seshadri C S 1974 *Ann. Sc. Éc. Norm. Sup.* **7** 89
16. Lakshmibai V, Musili C and Seshadri C S; Geometry of G/P-III (to be published)
17 Mumford D 1965 *Geometric Invariant Theory*, *Ergebrisse der Math. und Iher Grenzgebiete*, (Springer Verlag) p. 34
18. Musili C 1972 *J. Indian Math. Soc.* **36** 143
19. Musili C 1974 *J. Indian Soc.* **38** 131
20. Seshadri C S 1975 *Proc. Symp. Pure Math.* **29** 263
21. Seshadri C S 1977 *Adv. Math.* **26** 225
22. Seshadri C S: Geometry of G/P—I. C P Ramanujam: A Tribute (Tata Institiute of Fundamental Res.) (to be published.)
23. Weyl H 1946 *The Classical Groups* (New Jersey: Princeton Univ. Press)