CARSCALIMN

Forms of algelaxaic plame carves
whose equations are fincmial

## Fathematics

## A. M.

1910

UNIVERSITY OF ILLINOIS


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ON THE FORMS OF ALGEBRAIC PLANE CURVES WHOSE EQUATIONS ARE TRINOMIAL

BY

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A. B. Wabash College, 1906

## THESIS

> Submitted in Partial Fulfillment of the Requirements for the Degree of
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IN MATHEMATICS

IN

THE GRADUATE SCHOOL

OF THE

UNIVERSITY OF ILLINOIS

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UNIVERSITY OF ILLINOIS THE GRADUATE SCHOOL

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\text { May 31 } 19
$$

I HEREBY RECOMMEND THAT THE THESIS PREPARED UNDER MY SUPERVISION BY

$$
L_{i-a}^{h} \text { Ornest l arscailen }
$$

Entitled Gm the Forms of Algebraic Plane Curves AShose Equations are Trinomial

BE ACCEPTED AS FULFILLING THIS PART OF THE REQUIREMENTS FOR THE

DEGREE OF


In Charge of Major Work


Head of Department

Recommendation concurred in:


UNA:

It is the object of this peper to determine approximetely the forms of the real algebraic plane curves, the nquations of which are trinomials, that is sire of the form;

$$
A x^{m_{1}} y^{m_{1}}+B x^{n_{2}}+m_{2}+C x^{m_{3}} y^{m_{3}}=0
$$

where $A, 3$, and $C$ are roul numbers; $n_{1}, n_{2}, n_{3}, m_{1}, m_{2}$, and mare nositive integers and at least one of the numbers $n, n_{2}, n_{3}$ and at Ierst, onn of the numbers $m_{1}, a_{2}, m_{3}$, is zero.

In determining the apnroximate forms of these ourves we will sunpose two curves to have the seme form and therefore, for the purnose of this nather, to coincide, if one can be proinctad into the other ber a transformation of the form;

$$
\begin{aligned}
& X=8 X^{\prime} \\
& J=b Y^{\prime}
\end{aligned}
$$

Whern a and b are real numbers. Such a transformetion consists mereIJ in streftehing the plane of the figure in dirnctions narallel to the $x$ - and raryilel to the $y$ - axis. "wo curves rijl also he considreas as having the same form if the equation of one can be obtaincd from the equation of the other br first makine the equations homogenenas by the introduction of a new variable $z$ and then interchanging the variphles $x, y$ and $z$. Two curues whose equatinns can be made identical in this manner can be transformed into each other by a real projectimn transformation.

Einally troo curyns will be considnred as heving the same form, to the degren of approximation sought in this paper, if the approximate curves in the nfighborhood of the singalar rointis have the same form, and if the cirves intersect the axfs and the line at infinitur in the same pointis. Tt will ho seon heresfifer tihet, ift the ahove data arn the same for two curves, then the forms of the two
curves will closely resemble one another.
To determine the hehavior of the curves macor consideration, cspecially in the noiphborhood of the axes and the line at infinity, the well known theory of the analytic polygon will he emnloyed. \# In an alriliary figure lot each term of a complete equation of the same degren os the given cquation be renresentad hy a roint whose coordinates are the exponents of

$x$ and $\pi$ in that torm. If the degree of the of the equation is $n$, let $T$ and $M$ be the noints whose coordinates are $(n, 0)$ and $(0, n)$ respectivfly. Then ONM is called the analytic triangle. Mark hy small circles the noints representing terms actually appearins in the given equation.

Draw pyery line which passes through trio or more circled points and has gll the romaining circled pointis on one sidn ofit. The polygon this formed is called the analytic nolycon. "hen the terms corresnondine to the eircied points on any one of these lines which spparates the other moints from $C$ determine a form of the curve at the origin. Those on $0:-1$ and 0 determine the intersections of the curve wit? the $x$ - and the $y$ - axis respectively. chose on a line separotine the other points from II deterine a form of the curves at infinits on the $x$ - axis and those snparating frov $:$ determine a form at infinjty on the $y$ - axis. Those on $\because N$ determinc the intersections with the line at infinity, that is, the directions of the asymptotes. The linear factors of these terms give the equations of the asymptotes excent, for the constant term. 4

A Wieleitner: Therorie der ebenen alechraischen Kurven höherer orinume. pp . 33 and folloring.

Johnson: Curve Tracing pp. 42 and followine.
\# Johnson: Curve Tracing in Cartesisn Coordinates pp. 12.

Since the equations of the curves under consideration are trimnomials, the analytic polygon for any one of these curves is a trioangle. The case where this triangle degenerates into a straight line my be excluded, since, in this case, the equation of the curve can be factored.

For, if the equation is,

$$
A x^{m_{1}} y^{m_{1}}+B X^{m_{2}} y^{m_{2}}+C X^{M_{13}} \tau^{m_{3}}=0
$$

and the three points $\left(n_{1}, m_{1}\right),\left(n_{2}, m_{2}\right)$ and $\left(n_{3}, m_{3}\right)$ are on a straight line we have:

$$
\begin{align*}
& \text { a } n_{1}+b m_{1}=c \\
& \text { a } n_{2}+b m_{2}=c  \tag{I}\\
& \text { a } n_{3}+b m_{3}=c
\end{align*}
$$

where $a, b$ and $c$ are rational and integral. That they are rational is readily seen as follows: From the theory of determinants

$$
\begin{aligned}
& a=k\left(m_{3}-m_{2}\right) \\
& b=k\left(n_{2}-n_{3}\right) \\
& c=k\left(n_{2} m_{3}-n_{3} m_{2}\right)
\end{aligned}
$$

The guanitios in parenthesis sere integers hence if one of the nomhers $a, b$ or contained an irrational foetor it would have to be in $k$. But if there had been such a factor we would have divided equal tions (I) through by $k$ thus leaving the coefficients rational and intecral. If we suhstituto $\dot{\xi}=x^{\frac{1}{n}}$ and $\eta=y^{\frac{1}{4}}$ in the equation of the curve it becomes:

$$
A \xi^{a m_{1}} \eta^{h m_{1}}+B \xi^{a m n_{2}} \eta^{\operatorname{lm}_{2}}+b \xi^{a n_{3}} \eta^{h m m_{3}}=0
$$

This is a homogeneous binary form in $\xi$ and $\eta$ with integer exponents. It may therefore be written in the form:

$$
\left(l_{1} \xi+k_{1} \eta\right)\left(l_{2} \xi+k_{2} \eta\right)
$$

$$
\left(k_{q} \xi+k_{p} \eta\right)=0
$$

Hence the gite curve having a trinomial equation degenerates into a set of curves each having a binomial equation of the form of:
$\left(11^{b-}\right) x^{b}=(-1) y^{a}$
Thesc curves will be excluded from our discussion. The analytic polyeon must therefore be a triangle.

The various positions of the three circled points on the triangle 0 N N give rise to a number of different cascs which, in discussing the forms of those curves, must be considered separately.

Case I. All three vertices of the analytic triangle are circled. The equation is then:

$$
A x^{m} \pm B y^{m} \pm C=0
$$



Case II. Two vertices and a point on the side opposite one of them are circled:

$$
A x^{m} \pm B y^{m} \pm C x^{N}=0
$$



Case III. Two vertices and a point inside the triangle are circled:

$$
A x^{n} \pm B y^{\prime n} \pm C x^{n} y^{s}=0
$$



Case IV. Cne vertex is circled and two points on the opposite side:

$$
A x^{m} y^{m} \pm B x^{N} y^{s} \pm C=0
$$



Case 7. One vertex, a point on the onposite sine and a point on an adjacent, side are circied:

$$
A x^{M} y^{m} \pm B x^{N} \neq C=0
$$



Case VI. One vertex, a point on the opposite side and a point inside the triangle are circled:

$$
A x^{m} y^{m} \pm 3 x^{2} y^{s} \neq C=0
$$



Case TII. No vertex, but one point on each side is circled:

$$
\Lambda x^{m} y^{m} \pm B x^{r} \pm C y^{s}=0
$$



3y makins the equation of the curve homogeneous and then interchanring the homogegeous coordinates $x$, and $z$ the ceses arising from the other rositions of the analytic nolygon can he reduced to one of these seven. For example consider the case where the analytic rolzgon becomes:

The equation of the curve is,


$$
A x^{m}+3 x^{n} y^{m-N}+C=0
$$

l"akine the equation homomeneous it hecomes,

$$
i x^{m}+3 x^{n} y^{m-n}+C z^{m}=0
$$

Interchanging $y$ and $z$ and then making the equation non-homogeneous by putting $z=1$ we obtain:

$$
A x^{4}+C y^{2}+B x^{N}=0
$$

which is the same as case II.
In e similur manner it may be shovin thet exch of the other cases reduces to one of the above seven cuses.

The quove seven equations will come out of the following two hy miking some of the exponents zero.

$$
\begin{align*}
& A x^{m} y \pm B x^{N} \pm C y^{s}=0  \tag{2}\\
& A^{\prime} x^{m} \Psi^{m} \pm B^{\prime} x^{N} y^{s} \pm C^{\prime}=0 \tag{3}
\end{align*}
$$

in which $A, 5, C, A^{\prime}, B^{\prime}$, and $C^{\prime}$ are positive.
The coefficients $A, B, C, A^{\prime}, B^{\prime}, C^{\prime}$ can all be reduced to unity by the transformation,

$$
\begin{aligned}
& x=a x^{\prime} . \\
& z=h y^{\prime}
\end{aligned}
$$

Fauation ! 2: bncomes by this transformation and aftier aropping primes:

$$
A a^{n} b^{m} x^{m} y^{m} \pm 3 a^{N} x^{\nu} \pm C b^{s} y^{s}=0
$$

If the coefficients are to be equal we have

$$
\frac{A s^{m} b^{m}}{C b^{s}}=1
$$

$$
\frac{B a^{r}}{C b^{5}}=1
$$

From which

$$
b^{m-s}=\frac{c}{A a^{m}} \quad n^{s}=-\frac{3 a^{N}}{c}
$$

Raising both of the first of these equations to the s rower and both members of the second to the ( $m-s$ ) power we obtain:

$$
f^{(m-s)^{s}}=\left(\frac{C}{A a^{m}}\right)^{s} \quad b^{(m-s)^{S}}=\left(\frac{H a^{N}}{C}\right)^{m-S}
$$

Equating these two values of $b^{(m-s)^{s}}$;

$$
\left(\frac{C}{H 8^{m}}\right)^{S}=\left(\frac{B \varepsilon^{N}}{C}\right)^{m-S}
$$

Solving for a :

$$
a^{(m N+m s-v S)}=\frac{c^{m}}{A^{s} B^{(m-s)}}
$$

Solving for $b$ in like manner:

$$
b^{(m \sim+m s-N S)}=\frac{3^{m}}{A^{n} C^{m-N}}
$$

The exponent (mans - rsi cannot he zero for its vanishing is the condition that the three points $(n, m)(0, s)$ and $(r, 0)$ in the analytic triangle be on a straight line. This is readily shown:

$$
\left|\begin{array}{lll}
n & m & 1 \\
0 & s & 1 \\
r & 0 & 1
\end{array}\right|=n s+r m-r s
$$

Hence since $A, 3$ and $C$ are positive and different from zero, the above equations can be solved in real numbers for a and $b$. Substituting in (2') the values so found for $a$ and $b$ and dividing by a finite constant the equation becomes

$$
x_{y^{n}}^{m} \pm x^{\prime \prime} \pm y^{s}=0
$$

Ie equation ( 3 ) be divided through by $C^{\prime}$ and $A$ and 3 written for $\frac{A^{1}}{C^{r}}$ and $\frac{3^{\prime}}{C^{\prime}}$ respectively.
We then have,

$$
A x^{n} y^{m} \pm x^{N} y^{s} \pm 1=0
$$

with the same transformation as before the equation becomes, after dropping primes:

$$
\begin{equation*}
A a^{n} b^{m} x^{m} y^{m} \pm 3 a^{N} b^{s} x^{N} y^{s} \pm 1=0 \tag{31}
\end{equation*}
$$

Te must then have

$$
\therefore a^{m} b^{m}=1 \quad B a^{N} b^{s}=1
$$

Solving for a and $b$

$$
a^{m s-m N}=\frac{3^{m}}{A^{s}}
$$

$$
b^{m s-m N}=\frac{A^{N}}{\bar{B}^{m}}
$$

Again the exponent (ns - mr) cannot be zero for its vanishing is the condition that the two points $(n, m)$ and $(r, s)$ be in a line with the origin, since:

$$
\left|\begin{array}{lll}
n & m & I \\
r & s & 1 \\
0 & 0 & 1
\end{array}\right|=n s-I m
$$

Hence, since $A$ and $B$ are positive and finite, the $\varepsilon_{i b o v e ~ e q u a t i o n ~}^{n}$ can be solved for a and $b$ in real finite numbers. Substituting these values for a and $h$ in ( $3^{\prime}$ ) the equation becomes:

$$
x^{m} y^{m} \pm x^{n} \nabla^{s} \pm 1=0
$$

Te shall, therefore, elweys suppose that the coefficients are plus or minus one.

We will nov prove that the locus of a trinomial equation such as we are considering can have no double points except at the origin and at the points where the $x$ - and $y$-axes intersects the line at infinity.

The coordinates of \& double point, when the equation is written homoerneously, must satisfy the three partial differential equations

$$
\frac{\partial f}{\partial x}=0, \frac{\partial f}{\partial y}=0, \quad \frac{\partial f}{\partial z}=0
$$

It has been shown that the coefficients can he reduced to unity.
\# SAlmon: Higher Elan Curves, third edition, Art. 69.
"e will suppose this done and rite equation (2) in the fo"loring form:

$$
x^{n} y^{m}+\Delta x^{N}+B y^{S}=0
$$

where $A$ and $B$ are plus or minus one.
To show that the theorem holds in case "II, write the equation of the curve homogeneously. It becomes:

$$
x^{m} y^{m}+\Delta x^{\nu} z^{m+m-\nu}+B y^{s} z^{m+m-s}=0
$$

The three partial differential equations are then:

$$
\begin{align*}
& \frac{\partial f}{\partial x}=n x^{m-1} \nabla+A r x^{n-1} z^{n+m-N}=0  \tag{4}\\
& \frac{\partial f}{\partial y}=m x^{m} y^{m-1}+E s v^{s-1} z^{n+m-s}=0  \tag{5}\\
& \frac{\partial f}{\partial f}=A(n+m-r) x^{N} z^{m+m-N-1}+B(n+m-s) y_{s}^{s} z^{m+m-s-1}=0 \tag{6}
\end{align*}
$$

If a point on one axis satisfies these equations it must also be on another axis. For if $x=0$ then equation (4) is satisfied but in order to satisfy '5) and (6), y or mast the zero. If $y=0$ then (5) is satisfied but $x$ or $z$ must be zero to satisfy (4) and (6). If $z=0$ then (6) is satisfied hut $x$ or $y$ must be zero to satisfy '4) and (5). For the consideration of points not on any of the three axes let the equations be divided through by $x^{\sim-1}$, yo and $z^{\text {sim }}$ an -1 respectively. They then become:

$$
\begin{align*}
& n x^{m-N y}+A r z^{m+m-N}=0  \tag{7}\\
& m x^{m} y^{m-s}+B s z^{m+m-s}=0  \tag{8}\\
& A(n+m-r) x^{n}+B(n+m-s) y^{s} z^{n-s}=0 \tag{19}
\end{align*}
$$

Since none of the exponents $n, m, r$ and s are zero:
From (7)

$$
x^{n-N}=\frac{-A x^{i n+m-N}}{n y m}
$$

from (9)

$$
x^{N}=\frac{-B(n+m-s) y^{s} z r-s}{A(n+m-r)}
$$

Multiplying these together:

$$
x^{n}=\frac{B r(n+m-S) y^{s-m} z^{n+(n-s}}{n!n m-r!}
$$

Equating this to the value of $x^{m}$ obtained from equation (8) we have:

$$
\frac{B r\left(n+m-s^{\prime} V^{s-m} z^{m+m-s}\right.}{n(n+m-r)}=\frac{-3 s z^{m+m-s}}{m y^{m-s}}
$$

or

$$
\frac{r(n+m-s)}{n(n-r)}=-\frac{S}{m}
$$

Since the letters in this equation are all positive integers on n $r<n+m>s$ the left, side is positive and the right side negative which is impossible. This proves the theorem for case III.

For Case II which comes from the above when $s=n, m=0$ and $n>r$, equations (4), (5) and (6) become:

$$
\begin{aligned}
& \frac{\partial f}{\partial \chi}=n x^{M-1}+A r x^{\mu-1} z^{M-N}=0 \\
& \frac{\partial f}{\partial y}=\sum s y^{s-1}=0 \\
& \frac{\partial f}{\partial z}=A\left(n-r!x^{N} z^{M-N-1}=0\right.
\end{aligned}
$$

The curve cannot therefore, in this case, have a mode except at the origin or at the point at infinity on the x- axis.

To show that the theorem holds in cases I, IV, Y and VI we may Write the equation of the curve in the form:

$$
x^{m} y^{M}+A x^{N} y^{s}+B=0
$$

and mekinç it homogeneous:

$$
x^{M} y^{m}+a x^{N} y^{s} z^{M+m-N-s}+5 z^{n+m}=0
$$

Forming the three partial differ en tidal equations:

$$
\begin{align*}
& \frac{\partial f}{\partial x}=n x^{M-1} y^{M}+A r x^{N-1} y^{s} z^{M+M-N-s}=0  \tag{10}\\
& \frac{\partial f}{\partial y}=\pi x^{M} y^{m-1}+A s x^{N} y^{s-1} n^{M+M-N-s}=0 \tag{11}
\end{align*}
$$

(-

$$
\begin{equation*}
\frac{\partial f}{\partial g}=A(n+m-r-s) x^{N} y^{s} z^{n+m-N-s-1} \quad B\left(n+m i z^{n+m-1}=u\right. \tag{IL}
\end{equation*}
$$

If $\mathrm{x}=0$ equations (IO) and (II) are satisfied but for (I2) to be satisfied z rust also be zero. If $z=0$ equation (12) is satisfied but for (10) and (11) to bo satisfied either $x$ or y must be zero. Hence, if a point on one axis satisfies these equations it must also be on another axis, therefore at the intersection of the two.

For the consideration of a point on neither axis let the three equations be divided through by $x^{n-1} y^{s}$, $x^{N} y^{s-1} a n a ̉ z^{m+m-N-s-1}$ respectively which gives:

$$
\begin{align*}
& n x^{m-N m-s}+A \text { r } z^{m+m-N-s}=0  \tag{13}\\
& m x^{m-N y m} y^{m-s}+A \text { s } z^{m+M-N-s}=0 \\
& A(n+m-r-s) \quad x^{N} y^{s}+s(n+m) z^{N+s}=0
\end{align*}
$$

From (13) and (14) since $x, y, z$ and $A$ are finite

$$
\left|\begin{array}{ll}
n & r \\
m & s
\end{array}\right|=0
$$

This is the condition that the points ( $n, m$ ) and ( $r, s$ ! be collinear with the origin. But the origin is a circled point, hence this is the condition that the circled points be collinear, which is imppossible since vo have excluded such cases. This proves the theorem for cess I, IV, $V$ and $Y I$. Case III must be considered alone. Its equation is:

$$
x^{m}+A y^{m}+B x^{N} y^{s}=0 \quad n>(r+s)>1
$$

":ritine it homogeneously:

$$
\begin{aligned}
& x^{n}+A y^{m}+B x^{N} y^{s} z^{m-N-s}=0 \\
& \frac{\partial f}{\partial x}=n x^{m-1}+B r x^{N-1} y^{s} z^{m-N-s}=0 \\
& \frac{\partial f}{\partial y}=A n y^{m-1}+B s x^{N} y^{s-1} z^{m-N-s}=0 \\
& \frac{\partial}{\partial z}=B(n-r-s) x^{N} y^{s} z^{m-N-s-1}=0
\end{aligned}
$$

To satisfy the third of these equations, one of the variables $x$, y or $z$ must, be sern. If $x=0$ then in crder for tine second equation to br satisfied y must equal zero, and if $y=0$ or $z=0$ it folIows from the first equation thet $x=0$. This proves the theorem for case III.

There can be no isoleted circuits of these curves vihich Iie entirely in the finite part of the plane and do not intersect the exis. For supnose such a circuit to exist. Two tangents could be argwn to it through the origin. Let the equition be written nomogeneously and lnt one of the variables, z. say, occur in only one term. We then
 heve :

$$
f(x, y, z)=x^{\alpha_{1}} y^{\beta_{1}}+\AA x^{\alpha} y_{y} \beta_{2}+C x^{\alpha_{3} \beta_{3}} z_{z}^{r}=0
$$

The equation of i tancent at, a point ( $x_{1}, \nabla_{1}, z_{1}$, ) on the curve is:

$$
\frac{\partial f}{\partial x_{1}} x+\frac{\partial f}{\partial y_{1}} y+\frac{\partial f}{\partial g_{1}} z=0
$$

If this is to pass tihromeh the origin me have:

$$
\frac{\partial f}{\partial q_{1}}=c \gamma_{2}^{\gamma_{1}^{-1}} x_{1}^{\alpha_{3}} \nabla_{1}^{\beta_{3}}=0
$$

Since neither $C$ nor $r$ is zero, either, $X, y$ or a fust be zero; that is the noint of taneency must, be on one of the lines $x=0$, $y=0$ or $z=0$ which is contrary to the hynothesis that the circuit dons not ernss fither of these lines. This proves the theorem for all cases in which one variable is lacking from tro terms; that is for all casse in which in the analytic triangle two circlod points sue on the asme slde of the triancle. Thase sre the first IVe cases.
\# Salmon: Hioher Plene Curves, third ndition, Art. 64.
$=$
$x$

I

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For ecesis ri and rit we nrocend as follows:
By interchanging $T$ and $z$ in case $T$ it is seen that the equation of the curve can in hot cases be written in the form:

$$
x^{m} y^{m}+A x^{\nu}+3 y^{s}=0
$$

If the curve has a closed hranch which is entirely in the finito part of the plane and does not touch or cross either axis, then it is possible to dram two tangents to it parallel to the $x$-axis. The points of tangency must satisfy the relation:

$$
\begin{equation*}
\frac{\partial f}{\partial x}=n x^{m-1} y^{m}+\Delta r x^{n-1}=0 \tag{17}
\end{equation*}
$$

and since $x$ is not zero we may write

$$
\begin{equation*}
n x^{m-N} y^{m}+A r=0 \tag{18}
\end{equation*}
$$

Multiply (16) by $r$ and (17) by $x$ and subtract.

$$
\begin{align*}
& r x^{m} y^{m}+A r x^{N}+B r y^{s}=0 \\
& \frac{n x^{n} V^{m}+A r X^{N}=0}{(x-n) x^{n} y^{m}+B r y}=0
\end{align*}
$$

If $(r-n)=0$ we obtain since $B \neq 0, r \neq 0, \pi=0$ rich is contrary to hypothesis.

Suppose $r-n \neq 0$. Then since $y$ is not zero we can divide by $y^{s}$ and solving for $y^{m-s}$ have

$$
y^{m-s}=\frac{B r}{(n-r)^{n}}
$$

Raising hot members of this equation to the $m$ power:

$$
\begin{equation*}
y^{(m-s) m}=\left[\frac{n r}{(n-r) x^{m}}\right]^{m} \tag{120}
\end{equation*}
$$

From (18)

$$
y^{m}=\frac{-A r}{n x^{m-n}}
$$

and raising both members
to the $(m-s)$ power:

$$
\begin{equation*}
\mathrm{y}(m-s) \stackrel{m}{=}\left[\frac{-H r}{n \mathrm{x}}\right]^{m-s} \tag{21}
\end{equation*}
$$

Equating the two values of $y^{(m-s) m}$ from equations ! 20! and (2I) and solving for x re obtain:

$$
x^{m \nu+m s-r s}=\left(\frac{-n}{A r}-\right)^{m-s} \cdot\left(\frac{3 r}{n-r}\right)^{m}
$$

Solving for $y$ in like manner we obtain:

$$
\nabla^{m N+n s-N s}=\left(\frac{-A r}{n}-\right)^{\nu} \cdot\left(\frac{R n}{(r-n) A}\right)^{M-N}
$$

We have already seen that the exponent mr ts - rs is not zero. If it is odd, there is only one real point in the plane; and if is even there is at most one real point in any one quadrant of the plane, for which $\frac{\partial t}{\partial x}=0$. But if there were a finite circuit lying entirely in any one quadrant, there would be at least, two points in that quadrant for which $\frac{\partial f}{\partial x}=0$. Hence, there exists no such circult.

This proves the theorem for cases TI and "II. The theorem is, therefore, true in all cases.

It is easily seen that these curves cannot intersect a line nora?lel to one of the aves, or a line through the origin in more then two points in any one nuadrent. Tor let one verifohle, say, be given a constant value. -e then have a trinomial? equation in $x$. $\therefore$ cording to Descartes rule of signs this equation can have no more positive roots than there are chances of sign and no more negative roots than there fire changes of sign when $-x$ is substituted for $x$. Since r the equation is trinomial there can br $\varepsilon^{+}$mont two changes of sign, hence not more than tivo roots of the same sign encl therefore not more then two intersections in one quadrant. For the considerration of g line thrnush the origin lot the equation berritton in polar coordinates. For a given value of $\theta$ vie then have a trinomial in $P$ which can have, at most, two positive and two negative roots,
hence at most two intersections in any one quadrant.
In discussing the forms of these curves we shall use primes to denote the number and distribution of negative signs. :ie will use a single prime when the last term is negative, a double prime when the middle term is negative and three primes when the last two terms are negative. For example:

$$
\begin{array}{r}
\text { Case } I \text { is } x^{m}+y^{m}+1=0 \\
\text { Case } I^{\prime} \text { is } x^{m}+y^{m}-I=0 \\
\text { Case } I^{\prime \prime} \text { is } x^{n}-y^{m}+1=0 \\
\text { case I'' is } x^{n}-y^{n}-I=0
\end{array}
$$

$$
\begin{gathered}
\text { CAST I. } \\
x^{m} \pm y^{m} \neq 1=0 \\
\text { There wilt? hon tiro sun-casess a and } b \text { accord- }
\end{gathered}
$$


ing as $n$ in even or odd

$$
x^{n}+y^{n}+1=0^{\text {Ias }} \quad n \text {, even }
$$

This is clearly imaginary
Ib

$$
x^{m}+y^{m}+1-0 \quad n, \text { odd }
$$

The curve intersects the $x-a x i s$ at $x=-1$ and the $y-a x i s$ at $y=-1$ and it no other points.

Where is an asymptote vihose equation is given except for the constant term, by the linear factor of $x^{m}+\pi^{m}$. mo obtain the constant term let the nquetion of the cire be written:

$$
x+y=\frac{-1}{x^{n}-x^{m-1} y+\cdots \cdots+y^{m}}
$$

As a point; mores out on the curve and approaches this asymptote, its coordinates $x \doteq \infty, y \doteq \infty$ and $-\frac{x}{y} \doteq-1$. Hence the equation of the asymptote is:

$$
\begin{aligned}
x+y= & \int_{x^{m}-x^{m-1}}^{y \cdots \cdots+y^{m}}=0 \\
& x=\frac{-1}{}=\frac{x}{y} \doteq-1
\end{aligned}
$$

The asymptote dons not out the curve in any finite point, for slimemanating $y$ betwan the two equations ma hare.

$$
\begin{aligned}
x^{n}+(-x)^{n}+1 & =0 \\
1 & =0
\end{aligned}
$$

or since $n$ is od त
Aline parallel to either axis can be cut the curve in but one real point. For the equation may be written

$$
y^{M}=1-x^{M}
$$

end for any givan vilun of $x$ there is but one rfal walun of $y$, since there is but one real ode root of a number. Hence we conclude, since the curve is continuous, that it has approximately the form shovin on the ripht.

I'a
$x^{m}+y^{m}-1=0 \quad n$, חifn


The curve intersects the $x$-aris at $x= \pm 1$ and the $\bar{y}$-gxis at $\bar{y}= \pm 1$ It is smmetrical with rnspect to both wxes and is a circle when $n=2$.
"ritine the equation in the form:

$$
y^{n}=1-x^{n}
$$

We ses that it is imagingry for $|x|>7$ and arao for $|y|>1$. As $n$ is made lerger and larger the curve approaches the form of a square. Hence vie conclude that the curve has approvimate? m the accompanying form:

## I'b

$$
x^{m}+y^{n}-1=0
$$


n, odd

This is reducod to case $I$ in which all the coffficients have the same sign by the transformation:

$$
\begin{gathered}
\mathrm{x}=-\mathrm{x}, \quad \mathrm{y}=-\bar{y} \\
I^{\prime} \dot{a} \\
\mathrm{x}^{M}-\mathrm{y}^{M}+1=0 \quad n, \text { fyen }
\end{gathered}
$$

This curve intorsects the $y$-axis at, $y= \pm 1$ but anes not cut the $x$-axis at sll.

Since the terms $x^{m}-y^{m}$ contain two real linear factors there are two asymptotes whose constant terms we will find by the same method as before. The equations of the asymptotes will he therefore:


$$
\begin{aligned}
x+y= & 0 \\
x-y= & \frac{1}{x}=\frac{1}{x}=1 \\
& =\frac{x}{y}=1 \\
x-y= & 0
\end{aligned}
$$

Neither of thane cut the curve in finite points for eliminating y between the curve and the amplote we have in either case:

$$
\begin{array}{r}
x^{m}-x^{m}+1=0 \\
1=0
\end{array}
$$

The curve is symmetrical with respect to both axes. If $n=2$ it is the ordinary hyperbola. For $n>2$ it falls between the hyrerbola $x^{2}-y^{2}+1=0$ and its asymptotes.

Hence we conclude that the curve has approximately the following form:

$I^{\prime}{ }^{\prime}$

$$
x^{m}-y^{m}+1=0 \quad n, 0 d d
$$

This is chenged to If where all the coefficients are positive by the transformation $y=-y$.

$$
\begin{gathered}
I^{\prime} a^{\prime} \\
x^{m}-y^{n}-1=0 \quad n, \text { even }
\end{gathered}
$$

This is changed to $I$ 'a' by the transformation

$$
\begin{gathered}
x=y \\
y=x \\
I_{b^{\prime}}^{\prime} \\
x^{m}-y^{m}-I=0 \quad \text { n, odd }
\end{gathered}
$$

This is chanera to If by the transformation

$$
x=-x
$$

$$
x^{n} \pm y^{n} \pm x^{\nu}=0 \quad n>r
$$

There will be four different cases accord-
 ing is the exponents are even or nd d.
a) $n, r$ even
b; $n$ even, $r$ ocd
c! $n$ odd, $r$ even
a) $n, r$ กतส

II a

$$
x^{n}+y^{n}+x^{N}=0 \quad n, r \text { even }
$$

The curve is imaginary
II b

$$
x^{n}+y^{n}+x^{N}=0 \quad n \text { eVen, } r \text { ode }
$$

The curve intersects the $x-a \pi i s$ at the origin and at $x=-1$
The form at the origin is given by the two terms $y^{m}+x^{N}=0$. Triting the equation in the form:

$$
x^{\prime \prime}=-x^{M}-x^{N}=-x^{N}\left(x^{M-N}+1\right)
$$

We see that the curve is imaginary for $0<|x|<-1$. It is symmetrical with rr-
spent, to the r-axis. It does not meet the I inf et infinity nor the $y$-axis except, at the origin. Hence we conclude that its form is approximately as shown in the figure.

$$
x^{n}+y^{n}+x^{n}=0 \quad n \text {, arid, } r \text {, fen }
$$


II

The cure intersects the $x$ wis wt the origin and ot $x=-1$ The form ot the origin is given by the two terms $7^{m}+x^{N}=0$


There is o rectilinear gsymptote which we will find by the seme method is before.

$$
\begin{array}{ll}
x+y= & \lim _{\substack{x=\infty \\
\frac{x}{y} \\
y}} \frac{-x^{n}-1}{n} \\
x+y=0 & n-1>r \\
x+y=-1 / n & n-1=r
\end{array}
$$

Hence there are two forms of the curve accordine as $n-1 \geqslant r$ The asymptote $x+y=0$ does not cut the curve in sin finite point except at, the orisin. Te therefore conclude that its form is approximately as shown on the richt whan $n-1>r$
To find the intersections of the
 asymptote, $x+y=-I / n$, with the curve write the equetion of the curve in the form:

$$
(x+y)\left[x^{n-1}-x^{n-2} y+x^{n-3} y^{2} \cdots \cdot . .+y^{m-1}\right]+x^{N}=0
$$

Substituting $-1 / n$ for $x+y$ and ramombaring thet $n-I=r$ we have:

$$
(-n+1) x^{m-1}-x^{m-2} y+x^{m-3} y^{2} \cdots \cdot \cdot y^{m-1}=0
$$

Diviaine this equetion through by $y^{m-1}$, suhstituting $z$ for $x / y$ and chancing sions we obtain:

$$
\begin{equation*}
(n-1) z^{n-1}+z^{m-2}-z^{n-3}+z^{m-4} \ldots . . . .-7=0 \tag{1}
\end{equation*}
$$

Transform this to another equation having the same foots with opposite signs:

$$
\begin{equation*}
(n-1) z^{M-1}-z^{M-2}-7^{M-3}-2^{M-4} \cdots \cdots-1=0 \tag{2}
\end{equation*}
$$

By Descartes rule of sims equation (2) can have but one positive root and it is easily sren that this root is I. Equation (1) then has but one nergetive root - I. A neputivn vo?ue of $z=x^{\prime}$ y corres-
ponds to an intirrsection in the second or fourth quadrant and for $x^{\prime} y=-1$ this intersection is at infinity. Therefore the only finite intersections are in the third quadrant. To find the number of intersfctions in this quadrant consider the s?ope of the tangent:

$$
\frac{d y}{d x}=\frac{-1}{y^{n-1}} x^{n-2}\left[x+\frac{n-1}{n}\right]
$$

In following the curve to the left from the oripin, the slope of the curve is downvard for positite velues of $\frac{d y}{d x}$ and upward for negative values. Since $n$ is odd it follows that the expression outside the brackets is positive for points in the thire quadrant. Hence the curve turns un at $x=\frac{n-I}{n}=1-1 / n$. Since this is beyond tre point Vhere the asymptote cuts the x -axis and since the function is singlevalued it follow: that there is but one intersection of the curve and asymntote in the third quadrant. Therefore re conclude that the curve has approximately the form shown.

## IId

$$
x^{n}+y^{n}+x^{n}=0 \quad(n, r \text { oda })
$$

The only intersceti:ns of the curve rith the axes are et the origin. The form at the origin is given by the two terms $y^{M}+x^{N}=0$ which is the form of $\varepsilon$ cubica? norabola. There is one gsymntote:

$$
x+y=\lim \cdot \frac{-x^{N}}{x^{M-1}-x^{M-2} y \cdots+y^{m-1}}=0
$$


but it dons not intersect the curve excert at the origin. Hence we conclude that, the curve has approximately the form shown on next name at the ton.
(figure for pr.2.2)


$$
x^{m}+y^{m}-x^{\mu}=0 \quad n, r \text { even }
$$

The $x$-intercepts are $\neq 1$ form at the origin is given by the two terms $y^{n}-x^{n}=0$

The curve is symmetrical with respect to both ayes
 and viriting the equation in the form:

We see that it is imaginary for $|x|>$ ?
Hence pic conclude that the curve hes approximately the accompanying form.

$$
y^{n}=x^{N}-x^{M}=x^{N}\left(1-x^{M-N}\right)
$$



$$
x^{n}+y^{n}-x^{n}=0
$$

$$
\text { n even, } r \text { odd }
$$

This is changed to If by the transformation $x=-x$
IIi

$$
x^{m}+y^{m}-x^{N}=0
$$

$$
n \text { odd, } r \text { even }
$$

This is chanced to IIC br the transformation $x=-X, J=-\Psi$. TI'd

$$
x^{M}+y^{M}-x^{N}=0 \quad n, r \text { oud }
$$

The curve intersects the $x-a x i s$ at $x= \pm 1$ Form at the origin is given by $y^{M}-x^{N}=0$ There is one asymptote:


$$
x+y=\lim _{\substack{x=\infty \\ \frac{x}{y}=-1}} \frac{x^{\mu}}{}=0
$$

Which has no finite intersection with the curve except at the origin.

Tie therefore conclude that the curve has approximately the form shown on the right.


$$
x^{n}-y^{n}+x^{N}=0
$$

$$
n, r \text { even. }
$$

The form at the origin is given by $-y^{\prime}+x^{N}=0$ There are two asymptotes:
$x+y=\lim _{\substack{x=\infty, y=\infty \\ \frac{x}{y}=-1}} \frac{-x^{\mu}}{x^{m-1}-x^{m-2} y+\ldots \ldots-y^{m-1}}=0$
$x-y=\lim _{x=\infty, y \div \infty}$

$$
\frac{x}{y}=1
$$

$$
\frac{-x^{n}}{x^{n-1}+x^{m-2} y+\ldots+y^{n-1}}=0
$$

Neither asymptote intersects the curve in finite points except at the origin.

Hence we conclude that the curve has approximately the form shown on the right.

IIi

$n$ cen, $r$ odd

The $x$-intercepts are at the origin and $a t x=-1$.
Form at the oriein is given by $y^{m}=x^{\nu}$
There are two asymptotes:


The two asymptotes $x+y=0$ and $x-y=0$ cut the curve at the

$$
\begin{aligned}
& x+y=\lim _{x=\infty, y=\infty} \frac{-x^{N}}{x^{m-1}-x^{m-2} y-\cdots-y^{n-1}} \\
& \frac{x}{y}=-1 \\
& =0 \quad n-l>r \\
& =-1 / n \quad n-1=r
\end{aligned}
$$

$$
\begin{aligned}
& =0 \quad n-1>r \\
& =-1 / n \quad n-1=r
\end{aligned}
$$

oriegin but at no other finite points.
Fe will prove that the line $x-y=-1 / n$ cuts the curre in no finite points and from considerations of symmetry it will follow for $x+y=-1 / n$
liriting the equation in the form:

$$
(x-y)\left(x^{m-1}+x^{m-2} y \ldots y^{m-1}\right)+x^{N}=0
$$

Substituting - I/ for $x-y$ :

$$
-1!n\left(x^{m-1}+x^{n-2} y \cdots+y^{n-1}\right)+x^{n}=0
$$

Clearing of fractions, collnctino and diriding throum by $y^{m-1}$ we obtain:

$$
\left(1-n!\left(\frac{x}{y y}\right)^{n-1}+\left(\frac{x}{y y}\right)^{m-2}+\cdots \cdots+1=0\right.
$$

This equation has a positive root 1 which corresponds to the intersection at infinity.

Moreover by Descartes rule of signs there can not be more than one positive root. Hence the only intorspction ol the curve and asymptote in the first or third quadrant, that is where $\frac{x}{y}$ is positive is at infinity. writing the equation of the cmrve in the form:

$$
y^{m}=x^{n}+x^{N}=x^{n}\left(x^{M-N}+1\right)
$$

we see that for values of $x$ between 0 and-I the quanitir on the right is nepative, hence $y$ is imapinary. Since thut nart of the asymptote, $x-y=-1 / n$, which lios in the third quadrant is between $x=0$ and $x=-1$ In it cannot cut the curve in this quadrant.

Hence we conclucie that the curve has approximately the following forms:


II ' ${ }^{\prime}$

$$
x^{m}-y^{M}+x^{n}=0 \quad n \text { odd } r \text { even }
$$

This is chanced to II $b_{0}$ the transformation $y=-y$

$$
\begin{gathered}
I I^{\prime} d \\
x^{m}-y^{n}+x^{N}=0 \quad n, r \text { odea }
\end{gathered}
$$

This is chanced to II by putting $y=-\pi$

$$
\begin{gathered}
I I^{\prime} a^{\prime} \\
x^{n}-y^{n}-x^{N}=0 \quad n, r \text { even }
\end{gathered}
$$

The intercepts ire $x= \pm 1$
Norm at the origin is eimen by $y^{n}+x^{\mu}=0$ which is imaginary. Hence the origin is a congugate point.

There are two asymptotes:

$$
\begin{aligned}
& x+\pi=\lim _{\substack{\chi=\infty, y \dot{y} \\
\frac{x}{y} \doteq-1}} \frac{x^{n}}{x^{m-1}-x^{m-2} y \cdots \cdots+y^{m-1}}=0 \\
& x-y=\lim _{\substack{x=\infty, y \pm \infty}} \frac{x^{n}}{x^{m-1}+x^{m-2} y \cdots \cdots+y^{m-1}}=0
\end{aligned}
$$

Neither of these intersect the curve in finite points. We conclude then that the form of the curve is apnenvimate? y as shown on the right.

II' $b^{\prime}$

$$
x^{n}-y^{n}-x^{N}=0
$$



This becomes II $f^{\prime}$ bur the transformation $x=-x$
II' ${ }^{\prime}$

$$
x^{m}-y^{m}-x^{r}=0 \quad n \text { odder even }
$$

This becomes II by the transformation $x=-x$

$$
\begin{gathered}
I I^{\prime} d^{\prime} \\
x^{n}-y^{n}-x^{n}=0 \quad n, r \text { odd }
\end{gathered}
$$

This becomes II' by the transfornation $y=-y$

$$
x^{m} \pm y^{\prime n} \pm x^{n} y^{s} \quad(n>r+s)
$$

We will divide this into sub-ceses and $b$ according as $n$ is odd or even and each of these
 into four cases according as $r$ and s are nd i or even.
a)


## III $\infty_{1}$

$$
x^{n}+y^{n}+x^{n} y^{s}=0 \quad n, x, s \text { odd. }
$$

The only intersections of any of the curves of case II with the axes are at the origin.
There are two forms gt the origin given by $x^{m}+x^{N} y^{s}=0$

and $y^{M}+x^{N_{Y} s}=0$
The curve has one asymptotic:

$$
\begin{aligned}
x+y & =\lim ^{x} \frac{-x^{n} y^{s}}{\frac{x}{y}=-1} \frac{-x^{m-1}-x^{m-2} y^{y} y \ldots+y^{n-1}}{} \\
& =0 \quad n-1>r+s \\
& =1 / n \quad n-1=r+s \quad \because
\end{aligned}
$$

If $n-I>r+s$ the asymptote $x+y=0$ intersects the curve only $s$ t the origin.

To find the finite intersections of the curve with the asymptote, $x+y=I / n$, when $n-i=r+s$ let the equation be
written in the form:

$$
\begin{equation*}
(x+y)\left(x^{n-1}-x^{n-2} y^{7} \cdots \cdot . .+y^{n-1}\right)+y_{y^{n}}^{s}=0 \tag{.1}
\end{equation*}
$$

Substitute $I / n$ for $X+y$ in this equation, divide throideh by $y^{m-1}$, substitute z for x/t and collect.

The result is:

$$
\begin{equation*}
z^{n-1}-z^{m-2}+z^{n-3} \ldots . . .+(n-I) z^{n} \ldots+1=0 \tag{2}
\end{equation*}
$$

Since the curve cannot extend into the first quadrant, we are interester only in the negative roots of this equation, that is, for $x$ and $\bar{y}$ of opposite signs. Transforming (2) into another equation having the same roots with opposite signs, we obtain:

$$
\begin{equation*}
z^{m-1}+z^{m-2}+z^{m-3} \cdots \cdots+(1-n) z^{N} \cdots \cdots+1=0 \tag{3}
\end{equation*}
$$

This equation, by Descartes rule of signs, can have et, most two positive roots. By substitution we find that ? is a root. Hence -1 is a root $0 f(2)$ and this corresponds to an intersection at infinity. Since (3) has one positive root, and the last, term is positive there must be another positive root. dividing equation (3) by $x-1$ by Horners method re have:


The depressed equation is then:

$$
\left.z^{m-2}+2 z^{m-3}+3 z^{m-4} \ldots \ldots+s z^{n}-\{n-!s+1!\} z^{N-1}\{n-1 s+2)\right\} z^{n-2} \ldots-1=0
$$

Since, in this equation, $f(0)=-1$ the equation will have is root between 0 and I if;

$$
1+2+3 \ldots+s>1+2+3 \ldots+\{n-\{s+1!\}
$$

That is, if;

$$
\begin{aligned}
(s / 2)(s+1)> & \{n-(s+1)\}(n-s) \\
s^{2}+s & >n^{2}-2 n s \quad s^{2}-n-s \\
0 & >n^{2}-2 n s-n \\
s & >\frac{n-1}{2}
\end{aligned}
$$

If this inequality holds it means that at, the point of intersection of the curve and asymptotes $\left|\frac{x}{y}\right|<1$ or $|x|<|y|$ and therefore corresponds to an intersection in the second quadrant.
If $s<\frac{n-1}{n}$ then $\left|\frac{x}{y}\right|>1$ and the intersection iss in the fourth quadrant, hut the curve can in this case be projected into the other where $s>\frac{n-1}{2}$ by an interchange of $x$ and $y$. If $s=\frac{n-1}{2}$ then $I$ is a double root of equation (3), -1 is a double root of equation (2); and the only intersections of the curve and asymptote are at infinitu.

Hence we conclude that the curve has approximately the following forms:



III $a_{2}$
$x^{n}+y^{m}+x^{n} y^{s}=0$
n, odd $x, s$ cen
Forms at the origin are given by $x^{m}+x^{N} y^{s}=0$
and $y^{n}+x^{N} y^{s}=0$
There is one asymptote:

$$
\begin{aligned}
x+y & =7 \lim _{x} \frac{-x^{N} y^{s}}{\frac{x}{y}=y=-\infty} \frac{d}{x^{n-1}-x^{m-2}} \frac{1}{y+\cdots+y^{n-1}} \\
& =0 \quad(n-1>r+s) \\
& =-1 / n \quad(n-1=r+s)
\end{aligned}
$$



The asymptote $x+y=0$ does not intersect the curve at any finite point except the origin. If $n-l=r+s$ then it can be shove in the
same manner as in the previous case that if $s>\frac{n-1}{2}$, the asymptote $x+y=-1$ in intersects the curve in a finite point in the fourth quadrant and if $s<\frac{n-1}{2}$ the intersection is in the second quadrant. This latter case, however, will be changed to the former if wo interchance $x$ and $y$. If $s=\frac{n-1}{2}$ we can show $u$ in case II Io that there are no finite intersections in the second or fourth quadrants. If $n-l=r+s$ these curves, for all values of $n$, pass through the point $\left(-\frac{1}{2},-\frac{1}{2}\right)$. The distance of the asymptote from the origin is $\frac{\sqrt{2}}{2 n}$ hence that, part of the curve in the third quadrant always intersects the asymptote. Hence we conclude that the curve hes approximately the following forms:


Ce the two forms at tine orixin $x^{n}+x^{2} y^{s}=0$ and $y^{n}+x^{2} y^{s}=0$ the first: is imaginary and the second is like a cubical parabola.


The asymptote is:


It's only finite intersection with the curve is at the origin.

$$
\begin{aligned}
& \mathrm{IIIa}_{4} \\
& x^{n}+y^{n}+x^{N} s=0 \quad n, s \text { odd, } r \text { oven }
\end{aligned}
$$

This becomes the same as III $a_{3}$ by an interchange of $x$ and $J$.

$$
\begin{array}{r}
\text { III } b_{1} \\
x^{n}+y^{n}+x^{n} y^{s}=0
\end{array}
$$

$$
n \text { even, } r, s \text { odd }
$$

The two forms at the origin $x^{M}+x^{N} y^{s}=0$ and $y^{M}+x^{N} y^{S}=0$ are like cubical parabolas.
 There are no asymptotes.

The curve is symmetrical with respect to the origin.

$$
\begin{aligned}
& \text { III bs } \\
& x^{n}+y^{n}+x^{n} y^{s}=0 \quad n, r, s \text { eTon }
\end{aligned}
$$

The curve in tints case is imaginary.

$$
\begin{aligned}
& \text { III }_{3} \\
& x^{m}+y^{n}+x^{v_{y}^{s}}=0 \text { n, s even } \quad r, \text { odd }
\end{aligned}
$$

The forms at the origin $x^{M}+x^{N} y^{s}=0$ and $\mathrm{y}^{n}+\mathrm{x}_{V^{S}}^{S}=0$ are like the semi-cubical parabola

 the ordinary parabola respectively. The fe are no タ.symntotes.

The curve is symmetrical with respect to the $x$-axis.

$$
\begin{array}{r}
I I I b_{4} \\
x^{n}+y^{n}+x^{N_{v}}=0
\end{array}
$$



This becomes the same gs III, by an inter chance of $x$ and $d$. III' ${ }^{\prime}$,

$$
x^{m}+y^{m}-x^{N} y^{s}=0 \quad n, r, s \text { odd }
$$

This is chanced to III, $3 y$ the transformation $x=-x, y=-y$.
III ${ }^{\prime} / 2$

$$
x^{m}+y^{m}-x^{N} y^{s}=0 \quad n \text { odd, } r \text {, } s \text { even }
$$

This is chanced to $I I I a_{2}$ by the transformation $x=-x, y=-y$.

$$
x^{n}+y^{n}-x^{n} y^{s}=0
$$

$$
n, r \text { odd, } s \text { even. }
$$

The curve is symmetrical with respect to the origin. The two forms at tine origin, $x^{n}-x^{\nu} y^{s}=0$ and $y^{n}-x^{\nu} y^{s}=0$
 The curve has one asymptote:

$$
x+y=\operatorname{im}_{x=\infty} \frac{x^{n} y^{s}}{x^{n-1}, x^{m-2} y \ldots+\ldots+y^{m}=1} \equiv 0
$$

which has no $\frac{x}{x}$ finite intersection with the curve except it the origin.

$$
\begin{array}{r}
\text { III' } a_{4} \\
x^{m}+y^{n}-x^{N_{y} s}=0
\end{array}
$$



This becomes the same as II I' $a_{3}$ br an interchange of $x$ and $y$.
III'b,

$$
x^{n}+y^{n}-x^{N} y^{s}=0 \quad n \text { oven, } r, s \text { odd. }
$$

This is transformed to III, by putting $\mathrm{x}=-\mathrm{x}$.

$$
\begin{gathered}
I T I^{\prime} b_{2} \\
x^{m}+y^{m}-x^{\nu_{V}^{s}}=0
\end{gathered}
$$

$$
n, r, s \text { even. }
$$

The two forms at the origin $x^{m}-x^{N} y^{s}=0$ and $\pi^{n}-x^{2} y^{s}=0$ are as shown on the riot. The curve is symmetrical with respect to bot, axes.

$$
\begin{gathered}
I^{\prime} b_{3} \\
x^{m}+r^{m}-x^{N} y^{S}=0
\end{gathered}
$$



$n$, s even, $r$ odd

This is transformed to IJIfs by nutting $x=-x$.

$$
\begin{gathered}
\text { III }_{4} \\
\mathrm{x}^{m}+\mathrm{y}^{m}-\mathrm{x}^{N_{y}^{s}}=0
\end{gathered}
$$

$$
n, r \text { even, s odd }
$$

This is transformed to IIIf3 by nutting $x=\pi, y=-x$.

$$
\begin{aligned}
& \text { III'á, } \\
& x^{n}-\pi^{n}+x^{N} \mathrm{y}^{s}-0 \quad \Omega, r, s, \text { od a }
\end{aligned}
$$

This is transformed to III by matting $x=-x$.
III' $\dot{a}_{2}$

$$
x^{n}-y^{n}+x^{N} y^{s}=0 \quad n \text { nad, } r \text {, s even }
$$

This is transformed to III by putting $y=-y$.

$$
\text { III' }^{\prime} \dot{a}_{3}
$$

$$
x^{m}-y^{m}+x^{N} y^{s}=0 \quad n, r \text { odd } s \text { even }
$$

This is transformed to III by putting $\bar{y}=-y$.

## III ${ }^{\prime} \dot{u}_{4}$

$$
x^{m}-y^{n}+x^{N} y^{s}=0
$$

$$
n, s \text { odd, return }
$$

This is transformed to III' by putting $x=-x$.

## ITI'B,

$$
x^{n}-y^{n}+x^{\nu} y^{s}=0 \quad n \text { even, } r, s \text { odd }
$$

The curve is symmetrical with respect, to tho oricoin. Forms at the origin $x^{m}+x^{\nu} y^{s}=0$ sid $y^{M}-x^{2} y^{s}=0$ are
 like cubical parabolas.

There are two asymptotes:
$x+y=\lim _{\substack{x=\infty \\ \frac{x}{y}=y=1}} \frac{-x^{N} y^{s}}{x^{m-1}-x^{m-2} y \ldots \ldots-y^{m-1}}=0$
$x-y=\lim _{\substack{x=\infty \\ \frac{x}{y}=1}}^{\lim _{\substack{ } \infty}=1} \frac{x^{N} y^{S}}{x^{m-1}+x^{m-2} y \ldots+y^{m-1}}=0$
Neither of these intersect the curve
except at the origin. ines vie
conclude that the curve has approximately the form shown.

$$
\begin{array}{r}
\text { III' } b_{2}^{\prime} \\
x^{m}-y^{m}+x^{N} s=0
\end{array}
$$

$$
n, r, s e v e n
$$

Of the two forms at the origin $x^{n}+x^{N} y^{s}=0$
and $y^{m}-x^{N} y^{s}=0$ the first is imaginary and the
second is as shown on the right.
There are two asymptotes:

Neither of these intersect the curve except at the orion. Hence we mes y conclude that the form is approximate? 7 ye shown on the rimht.


$$
\text { III' }_{3}^{\prime}
$$

$$
x^{m}-y^{m}+x^{N} v^{s}=0 \quad n, \text { s even, } r \text { odd }
$$

The two forms at the origin $x^{m}+x^{N} y^{s}=0$ and $\nabla^{n}-x^{N_{y} s}=0$ are as shown on the right. There are
 twin esurmntotos:

The asymptotes $x+y=0$ and $x-y=0$ intersect the curve in no finite points except the origin.

By the same method that vas used in case II Ia, it can be shown that, $x+y=-1 / n$ has a finite intersection with the curve in the second nusargnt if $r>\frac{n-1}{2}, s<\frac{n-1}{2}$ and from symratyvit follows the in this case $x-y=-1 / n$ has a finite intersection in the tinird quadrant. If $r<\frac{n-1}{Z}, s>\frac{n-1}{2}$ and $\operatorname{thnn} x+y=-I \ln$ has a finite intersection in the fourti quadrant sine $x-y=-1 / n$ in the first quadrant. Fence we conclude that the forms are approximately

$$
\begin{aligned}
& x+y=\operatorname{im}_{\substack{x=\infty \\
\frac{x}{y}=-1}} \frac{-x^{N} y^{s}}{x^{n-1}-x^{n-2} y \cdots \cdots-y^{n-1}} \\
& =0 \quad n-1>r+s \\
& =-\frac{1}{n} \quad n-1=r+s \\
& x-y=\lim _{\substack{x \\
\frac{x}{y}, y^{ \pm}=1}} \frac{-x^{N} y^{s}}{x^{m-1}+\pi^{m-2} y \ldots+y^{m-1}} \\
& =0 \quad n-I>r+s \\
& =-\frac{1}{n} \quad n-1=r+s
\end{aligned}
$$

$$
\begin{aligned}
& x+y=\lim _{\substack{x=\infty, y=\infty \\
\frac{x}{y}=-1}} \frac{-x^{n} y^{s}-x^{m-2}}{y^{n}-1}=0
\end{aligned}
$$

is shown in the following: fisures.

$$
n-1>x+s
$$




$$
x^{m}-y^{m}+x^{N} y_{j}^{s}=0
$$

n, raven, s oad
This is changed to III' $b_{3}^{\prime}$ by putting $x=y, y=-x$
III!''
$x^{n}-y^{n}-x^{r} y^{s}=0$
This becomes the same as III' by an interchange of $x$ and $y$.

$$
x^{m} y^{m} \pm x^{N} y^{s} \pm 1=0 \quad n+m=r+s
$$

Te will aiviae this into four sub-cases: $a, b, c$ and $d$ according as $n$ and $m$ are even or ocd and reich
 of these into two others according as $r$ and s are even or odea.
a) $n, m$ oven
b! $n$ even, $m$ od त
r, even
s, even
r, oven
1 s , odd
$2 \begin{aligned} & r, \operatorname{odã} \\ & s, \operatorname{dad}\end{aligned}$
$2 \begin{aligned} & \text { r, olid } \\ & \text { s, even }\end{aligned}$
c) $n$, oud , m, even
d) $n, m$ odd
$1 \begin{aligned} & \text { r, ode } \\ & \text { s, evan }\end{aligned}$
$1 \begin{aligned} & \mathrm{r} \text {, even } \\ & \mathrm{s} \text {, even }\end{aligned}$
$2 \begin{aligned} & \text { r, even } \\ & s, \text { odd }\end{aligned}$
$2 \begin{aligned} & r, \text { add } \\ & s, \text { odd }\end{aligned}$

We shall suppose throughout that $n>r$ and therefore mes.

$$
\begin{aligned}
& \text { ITal } \\
& x^{m} y^{m}+x^{n} y^{s}+1=0 \quad \text { n.m,r, } \quad \text { even }
\end{aligned}
$$

The curve is imaginary.

$$
\begin{aligned}
& I V a_{2} \\
& x^{m} y^{m}+x^{N} y^{s}+1=0 \quad n, m \text { fven, } r, s \text { oud a }
\end{aligned}
$$

Where are no finite intersections with the axes.
The form at infinity on the $x$-axis $x^{m} y^{m}+1=0$ is imaginary. The form at infinity on the $y-a x i s x^{2} y^{5}+1=0$ is as shown on the right.

There is one asymptote:

$$
x+y=\underset{x=\infty, y \div \infty}{x}
$$



$$
\frac{x}{y}=-1
$$

It has no finite intrysnctions with the curve. Since the terms gre ql of oren degree the curve is symmetrical with respect, to the origin.


$$
x^{m} y^{m}+x^{N} y^{s}+1=0
$$

$n, r$ cyan, $r$, $\Omega$ odd
Form at infinity on the $x-a x i s i s$ given by $x^{n} y^{m}+1=0$ ans at infinitjun the $y$-axis by $x^{v} y_{y}^{s}+1=0$. The curve cannot cross the axe is and lies entirely in the third and fourth quadrants. Hence vie conclude theist it has anproximately the form inãicatjea.

$$
\begin{array}{r}
\text { IVb2 } \\
x^{m} y^{m}+x^{y} y^{s}+1=0
\end{array}
$$

$$
n, \text { s even, in, r odd }
$$

Form at infinity on the $x$-axis is $x^{m} y^{m}+1-0$ Form at infinity on the $\bar{y}$-axis is $x^{N} y^{s}+1=0$ These are as shown on the right.

There is an asymptote rinich $n^{\prime}$
the same method as before te
find to be:

$$
x+y=0
$$

It has no finite intersections with the curve.

IV e,

$$
x^{m} y^{m}+x^{n s} y^{s}+1=0
$$


$n, r$ odin, m, s even

This is transformed to IVf, by an interchange of $x$ and $y$.

$$
I V e_{2}
$$

$$
x^{n} y^{m}+x^{N} y^{s}+1=0 \quad n, s \text { odd, } m, r \text { even }
$$

Mores st infinity on the $x$-and $y$-axes are given by $x^{m} y^{m}+I=0$ and $x^{N} y^{s}+I=$ cresrectiveI..

 These sure as shove on the right.

There is one asymptote which, $b_{i}$ the method used before, is found to bn:

$$
x+y=0
$$



It does not cut the curve in any Iinite point. Hence re concluac that, the form of the curve is approximatoly gs indicatiod.

$$
\begin{array}{r}
\text { IVd, } \\
x^{m} y^{m}+x^{v} y^{s}+1=0
\end{array}
$$

$$
n, m \text { oत่ , } r, e \text { evon }
$$

This is transformod to IVa, by an interchange of $x$ and $y$.

$$
\begin{array}{r}
\text { IVd/2 } \\
x^{n} y^{m}+x^{\nu} y^{s}+1=0
\end{array}
$$

$$
n, m, r, \text { is oid }
$$

The axes are tion on?y asprrtotos. The forms at infinity on the $x$ - and $z-$ axis are
 given hy $x^{n} y^{m}+1=0$ and $x^{N} y^{5}+1=0$ respectively. Irese are as shoun on the right.

Mhe curve is summetrical with resrect
to tire orimin. hence re conclude that it has approximatinly the form incicaten.

$$
\begin{array}{r}
\text { IV'a } a_{1} \\
x^{m} y^{m}+x y-1=0
\end{array}
$$

$$
n, m, r, s \text { even }
$$

The axes refe tre only usimptotes. The fores at infinity on the $x$ - and $y$-axis, $x^{m} y^{m}-l=0$ anu $x$ y $-\bar{l}=0 \quad$ as indicatin on the rirnt. The curve is symmetrical with respect to both axes.

$$
\begin{array}{r}
\text { IV'ana } \\
x y \times y-I=0
\end{array}
$$

The forer gt infinity on tho x-Exis is eniven i) $y^{n} y^{m}-1=0$ and on the $y-a x i s$ by $x^{N} y^{s}-1-0$. These are is shovin on the riont. There is one asymptote and by then method used bofore its -quation is easily found to be:

$$
x+y=0
$$

It intorsocts the curve in no finite points.



The curve is symmetrical with respect to i，he origin．

$$
\begin{aligned}
& \text { IVll } \\
& x^{m} y_{1}^{\prime}+x^{n} y^{s}-1=0 \quad n, r \text { nven, } m, \text { s odd }
\end{aligned}
$$

Whis is changea to IVl，ba the transformation $y=-y$
IV $b_{2}$

$$
x^{n} y^{m}+x^{n} y^{s}-1=0 \quad n, s \text { even } m, r \text { odd }
$$

This is changnd to $I V l_{2}$ by the transformation $x=-x, y=-y$
IVジン

$$
x^{n} y^{m}+x^{v} y^{s}-1=0 \quad n \text {, r osid, } m \text {, s even }
$$

This ie chaned to IV by tref trensformation $x=-x, y=-y$ ． IV ${ }^{\prime} \mathrm{C}_{2}$

$$
x^{m} y^{m}+x^{n} y^{s}-1-0 \quad n, s \text { odd, m,reven }
$$

This is cheancea to IYU by the transformation $x=-x, y=-y$ ．
IV＇d，

$$
x^{m} y^{m}+x^{n} y^{s}-1=0 \quad n, m \text { odd } r, s \text { even }
$$

This hecomes the same sis TV＇oz by an interchange of $x$ and $y$ ．

$$
\begin{gathered}
\text { IV'd/2 } \\
x^{n} y_{y}^{m}+x^{n} y^{s}-I=0 \quad n, m, r, \mathrm{~S} \text { odd }
\end{gathered}
$$

This is trensformed to IV $d_{2} b y$ nutting $x=-x$ ．

$$
\begin{gathered}
I y^{\prime} \dot{a}, \\
x^{m} y^{m}-y^{n} y^{s}+1=0 \quad \text { n,m,r,g مyคn }
\end{gathered}
$$

The form at infinity on the $x-a x i s x^{n} y^{m}+1=0$ is imacinary．The form ut infinity on the $y$－axis $x^{n} y^{s}-1=0$ is as shown on the right．

There are two asymptotes which，by the same method as before，vie find to be：

$$
\begin{aligned}
& x+y=0 \\
& x-y=0
\end{aligned}
$$

Neither of these intersect the curve in finite．

points. Hence we conclude that the curye has approyimately the form indicated.

$$
\begin{gathered}
I V \dot{a}_{2}^{\prime} \\
x^{m} y^{m}-x^{N} y^{s}+1=0 \quad n, m n v e n, r, s \text { odd }
\end{gathered}
$$

This is transformed to $I V a_{2}$ by nutting $x=-x$.

$$
\begin{gathered}
\text { IV'́', } \\
x^{n} y^{m}-x^{N} y^{s}+1=0 \quad n, r \text { even, } m, \text { s odd }
\end{gathered}
$$

The forms at infinity on the $x$-and $y$-axes given hy $x^{m} y^{m}+1=0$ and $\mathrm{x}^{N_{y}}-1=0$ respectively are as shown on the rigint.


There are two asymptotes, found by the same method as hefore to be:

$$
\begin{aligned}
& x+y=0 \\
& x-y=0
\end{aligned}
$$

Neither of these intersect the curve in finite noints. Hence wo conclude that the curve hes apnroximately the form indicated
 on the rieht.

$$
\text { IV } b_{2}^{\prime}
$$

$$
x^{n} y^{m}-x^{N} y^{s}+1=0 \quad n \text {, s even, } m, r \text { odd }
$$

This is chanoed to ITb by putiting $x=-x$.

$$
\begin{gathered}
I V^{\prime} c_{1}^{\prime} \\
x^{n} y^{m}-x^{N} y^{s}+1=0 \quad n, r \text { odd, } m, \text { s even }
\end{gathered}
$$



$$
\begin{gathered}
I V^{\prime} e_{2}^{\prime} \\
x^{m} y^{m}-x^{s} y^{s}+1=0 \quad n, s \text { odd, m, }=0 \text { even }
\end{gathered}
$$

This is chanced to $I V_{2}$ by the transformation: $x=-J, y=x$.

$$
\begin{gathered}
I V d_{1}^{\prime} \\
x^{n} \nabla^{m}-x^{\nu} y^{s}+1=0 \quad n, m \text { odd, r, s even }
\end{gathered}
$$

This is changed to IV'd, by the transformation $x=y, y=-x$.

$$
\begin{gathered}
\text { IV' } d_{2}^{\prime} \\
x^{n} y^{m}-x^{2 v} y^{s}+1=0 \quad n, m, r, s \text { odd }
\end{gathered}
$$

The forms at infinity on the axes given by $x^{m} y^{m}+1=0$ and $x^{2} y^{s}-1=0$ are as shown on the right; There are two asymptotes:


$$
\begin{aligned}
& x+y=0 \\
& x-y=0
\end{aligned}
$$

Neither of thees intersect, the curve in finite points. The curve is symmetrical With respect to the origin.

$$
x^{m} y^{m}-x^{2} y^{s}-1=0
$$



This becomes the same as IV'' by an interchange of $x$ and $y$.

Te will diviue this into two sub－cases a and b accordine gis $n$ is even or odd and agch of tinnse into
 four other aconrding as $m$ and $r$ are cten or odd．
a）$n$ even
b）$r_{1}$ ode
$1 \begin{aligned} & m \text { nten } \\ & r \\ & \text { puen }\end{aligned}$
2 moven
$r$ odd
$m$ oda
$m$ aren
1）metrn
$2 \begin{aligned} & \text { m even } \\ & \text { goda }\end{aligned}$
z． $\begin{aligned} & m \text { nda } \\ & r \operatorname{ran}\end{aligned}$
$4 \underset{m}{r}$ odd
$4 \begin{aligned} & m \text { のnd } \\ & r \text { のतd }\end{aligned}$
none of thesc curves rave any finite intersections with the v－ruxis．The slope of the tingent is

$$
\frac{d y}{u x}=-\frac{n x^{n-1} y^{m} \pm r x^{n-1}}{m x^{m} y^{m-1}}
$$

Which is intinitn for a noint at which the curye erosses the x－sxis unloss $\underline{E}=1$ ．Wence $t_{1} e$ chuve crosses the x－uxis at right anglos unless $m=1$ ．In draving the curvos ver shal？suppoen that $m>I$ ．

$$
\begin{gathered}
\nabla a_{1} \\
x^{m} y^{m}+v^{N}+1=0 \quad n, m, r \text { orfn }
\end{gathered}
$$

This curve is cifcarly imacinary．

$$
T a_{2}
$$

$$
x^{n} \cdot y^{m}+x^{N}+1=0 \quad n, m \text { eyen, } r \text { odi }
$$

Mre curve interesocts the $x$－uxis at，$x=-1$ ．
－凹he form et infintiy on tihe－axis $x^{m} y^{m}+1=0$ is imasingiry，sinco $n$ and marr even．The form at intinitar
 on the $x$－axis is mivnn by $x^{m} y^{y^{m}}+x^{N}=0$ or sincen $x$ in not zero we may deviac throngh by $x^{N}$ and writa $y^{m}-\frac{-?}{\mathrm{x}^{n-N}}$ ．


This has to forms according as nmr. They are shown on first page.
Writing the equation in the form:

$$
y^{m}=\frac{-\left(x^{N}+I\right)}{x^{m}}
$$

we see that for $x>-1$ the curve is imaginary. Hence we conclude that th form is annmoimat, ely as follows.


The form at infinity on the $y$-axis is $x^{m} y^{m}+1=0$. The form at infinity on the $x$-axis is $x^{m} y^{m}+x^{2}=0$. This takes thee forms according as $n \geqslant r$.





The curve is symmetrical with respect to the $y$-axis.

$x^{m} y^{m}+x^{2}+1=0$

n even, $n, r$ od a

The curve intersects the $x$-axis at $x=-1$.

The form at infinity on the $y$-axis is given by $x^{m} y^{m}+1=0$, and at infinity on the $x-a x i s$ by $x^{m} y^{m}+x^{N}=0$. The letter form defends upon Whether $\eta \gtrless r$.


"in conclude that the form of the curve is amroximatoly as follows.


m, $r$ morn, $n$ odd
There are no finite intersections with the exes.
The curve is e, metrical with respect to the x-axia and cannot extend into the first or fourth quadrants.
The form at, infinity on the $y$-axis is citron by r $x^{n} y^{\prime m}+I=0$. The form at, infinity on the $x-a x i s$ $x^{n} y^{m}+x^{N}=0$, divides into two cases gecoraine as
$n \geqslant r$. These forme arr sis si ova on then right.


Fence se conclude the the curve has the following forms.


$$
\begin{gathered}
V b_{2} \\
x^{m} y^{m}+x^{n}+1=0 \quad n, x \text { di, } t \in V \in n
\end{gathered}
$$

The curve cuts the $x-a x i s s$ at $x=-1$. Forms at,
infinite on the $x$ anis is $x^{m} y^{\prime m}+x^{N}=0$ or $y^{m}=-1 / x^{m-10}$
 Since mana $n-x$ are furn this is imaginary. Form
$a t$ infinity on the $y$-axis is $x^{m} y^{m}+1=0$.
"rifting the equation in the form:

$$
y^{m}=-\frac{7}{x^{n}}-x^{n-1}
$$

We sop that the curve is imaginary for $-I>v>0$. The curve is symmetrical with respect to the $x$-axis.


$$
\begin{gathered}
V b_{3} \\
x^{m} y^{m}+x^{N}+1=0 \quad m, n \text { odd, } r \text { evan. }
\end{gathered}
$$

Where arm no finite intersections with the aves and tho curve is surmetricst with respect to the origin. Form ut infinity on the $y$-axis is unpen by $x^{m} y^{m}+1=0$ and at infinity on lir $x$-axis by $x^{m} y^{m}+x^{N}=0$. The lest form appends upon whether $n \geqslant x$




Hence we conclude that the curve has armpoximetnly the following forms:



$$
\begin{gathered}
V b_{4} \\
x^{m} y^{m}+x^{n}+1=0 \quad n, m, r \text { odd }
\end{gathered}
$$

The curve intersects the $x$-axis at $x-1$. The form at infinity on the $\tilde{y}$-axis is given by $x^{m} y^{m}+1=0$ and at infinity on the $x-a x i s$ by $x^{m} y^{m}+x^{N}=0$. The lattor has three forms according as $n \equiv r$.


When $n=r$ the gsymitote $y=-1$ aoces not interscot the curve in any finite noints.

Hence we conclude that the curve has the following forme:


$$
x^{m} y^{m}+x^{n}-I=0 \quad n, m, r \text { eromen }
$$

The intercepts are $x= \pm 1$. The form at infinity on the $x$-axis $x^{n} y^{m}+x^{n}=0$ is imaginary. The form at in-

finity on the $y$-axis is $x^{n} y^{m}-1=0$ which is as shown on the right; The curve is symmetrical with respnct, to the $x$-qxis and is imagingry for $|x|>7$ as is seen by vriting the equation $y^{m}=\frac{1-x^{N}}{x^{n}}$.


$$
\begin{gathered}
\text { Your } \\
x^{M} y^{m}+x^{N}-1=0 \quad n, \text { siren, } r \text { odd }
\end{gathered}
$$

The $x$-in trecept is $x=1$
The forms at infinity on the two axes riven by $x^{n} y_{r}^{m}-x=0$ and $x^{m}{ }^{m}+x^{N}=0$ are as follovis;


The curve is symmetrical with respect to the $x$-axis and writing
the equation in the form $y^{m}=\frac{1-x^{N}}{3^{m}}$ we see that it is imaginary for $x>1$


$V^{\prime} a_{3}$
$x^{m}:^{m}+x^{n}-1=0$

$$
n, r \text { coven, m odd }
$$

The curve intersects the $x$-axis at, $x= \pm 1$
The forms at infinity on the $x-$ and $y-a x e s$ given $b_{y} x^{n} y^{m}+x^{N}=0$ and $\mathrm{x}^{m} y^{m}$ - ? = 0 resnnctitroly are as follows.




For $n=r$ the asymptote $\quad=-1$ does not intersect the curve in any finite points. Nh e curve is symmet, Tical wish respect to the y-axis.


This is changed to $\forall_{4} h_{4}$ the transformation $x=-x, y-y$

$$
v b_{1}
$$

$$
x^{m}=m^{m}+x^{n}-I=0 \quad n \text { odd, m, } r \text { even }
$$

The intipreepts are $x= \pm 1$
The forms at infinity on tine $x$ - and $y$ - axes given Dy $x^{m} y^{m}+\underset{y}{m}=0$ sind $x^{m} y^{m}-1$ mi $\sim$ respectively are as follows.


The curve is symmericui with respect $1 ; 0$ the $x$-axis. "rifting the equation in the form $y^{m}=\frac{1-x^{N}}{x^{m}}$ we sen that, the curve is imaginary for $x>1$. Hence we conclude that the curve has approximation y the following forms.


## 4

1

$\qquad$


$$
\begin{aligned}
& { }^{V} \dot{l}_{2} \\
& x^{m} y^{m}+x^{N}-1-0 \quad \text { n, r odd, m oven }
\end{aligned}
$$

This is transformed to Vbaby rutting $x=-\mathrm{z}$.

The curve interepets the $x$-axis at $x= \pm l$
The forms est infinity on the $x$ - and y-axeiz Ere riven by $x^{m} y^{m}+x^{n}=0$ and $x^{m} y^{m}-I=0 \operatorname{resicctircia}$.


The curve is surmetricai with moises to time orion.



This is chanced to vf by puttitine $x=-x$.

$$
T{ }^{\prime} \alpha_{1}^{\prime}
$$

$$
x^{m} \cdot j^{m}-x^{N}+1=0
$$

$$
n, \dot{e}, x \text { even }
$$

Intercepts on the $x$-axis are $x=\neq 1$. The form at infinit on the $y$-axis $x^{m} y^{m}+1=0$ is imaginary since $n$ and mere even. The form at infinities on the $x$-axis given ty $x^{m} y^{m}-x^{N}=0$ is as shown on the right. The curve is symmetrical with roerect to both axes. Writing the equation in the form $y^{m}=\frac{x^{N}-I}{x^{m}}$ wo sec

$$
\begin{aligned}
& \text { Y'b3 }
\end{aligned}
$$

that, tite curve is imaseingrer for $|x|<1$


This is transformed to yaz nutifing $\mathrm{x}=-\mathrm{x}$.
Y'á3

$$
x^{n} y^{m}-x^{2}+1=0 \quad n, r \text { ayrn, } m \text { orत }
$$

Whis is t, ransformed to Vás by puttinu y = - I.

$$
x^{n} y^{m}-x^{r}+1=0 \quad n \text { even, } m, r \text { óáa }
$$

Lhis is treansformed to Vay bJ nutting $x=-x$.

$$
x^{m} y^{m}-x^{2}+1=0 \quad{ }^{\prime} b_{1}^{\prime} \quad n \text { nda, m, r even }
$$

This is transformea io Vb, ny putting $x=-7$.

$$
\begin{gathered}
V_{b_{2}^{\prime}}^{\prime} \\
x^{m} y^{m}-x^{n}+1=0
\end{gathered}
$$

$$
\eta \text {, } r \text { oud, m even }
$$

The curve chits the $x$-sixis at $x=1$



If $n=r$ ine asymptotes $y= \pm ?$ do not interesect the cirruc in any finnite points. The curve is symmetricsl vitite rescect to the x-axis.


This is chanced to $T b_{3}$ by rutting $\pi=-{ }^{T}$.

$$
\begin{gathered}
V_{b_{4}^{\prime}}^{\prime} \\
x^{m} y^{m}-x^{2}+z=0 \quad n, m, r \text { od l }
\end{gathered}
$$

This is changed to $V f_{4}$ by the transformation $x=-x, y-y$.

$$
x^{n} y^{m}-x^{N}-l=0 \quad n, m, r \text { even }
$$

The forms at infinity: on the $x$ - and y-elxns maven by $x^{m} y^{m}-x^{N}=0$ and $x^{m} y^{m}-I$ - 0 respectively are as fol? aws:



If $n=r$ the asymptotes $y= \pm I$ an not intorefect, the curve in finite points. The ane iss sumptrica? with respect to both suns.


$$
x^{m} y^{m}-x^{N}-1=0
$$

$$
n, m \text { ovna, } r \text { odd }
$$

Tr is broomes Y'ar by puttane $x=-x$.

$$
\underline{x}^{m} y^{m}-x^{N} A_{3}^{\prime} N_{3}^{\prime}=0
$$

$$
n, r \text { even, mode }
$$

This broomes "o $\alpha_{3} b_{e}$ muttin: $y=-v \cdot$

$$
x^{M_{y} m}-x^{N}-1=0
$$

n even, m, roda


$$
\mathrm{x}^{M} \mathrm{y}^{m}-\mathrm{x}^{N}-1=0
$$

Whis becomes Th, by ututinE゙ $x=-x$.

$$
x^{m_{y}^{m}-x^{N}-1=0 \quad n, r \text { odè }, b_{2}^{\prime \prime} \text { even }}
$$

Mhis becomes T'b2 by putting $x=-x$.

$$
\begin{gathered}
Y_{f_{3}^{\prime}}^{\prime} \\
x^{n_{y}} y^{m}-x^{N}-1=0 \quad n, m, \text { nad, } r \text { even }
\end{gathered}
$$

This becomes "fy hy muttinow $=$ - T.
"'b'A

$$
x^{n_{1}, m_{-}} x^{N}-1=1 \quad n, m, r o d d
$$

Wils incomes Yb bJ puttine $y=-\mathbb{N}$.

## Case VI

$$
x^{m} y^{m} \pm x^{N} y^{s} \pm 1=0 \quad(n+m>r+s)
$$

If the point $p_{2}(x, s)$ falls within the triangle CMP, there will be tro forme at infinity on the y-uxis
 and if it falls within the trisngle OMy thore will be two forms at infinity on the $\bar{x}$-axis. Since $x$ and $y$ are simidarily inernaved it is eridently sufficient to consider only one of these cuses. lie shall therefore sunpose that P2falls in the trianole crip, Hence it will follow that $\left.\frac{n}{r}\right\rangle-\frac{m}{5}, n>r$ and $m \geqslant s$.
(.:e will msike the following classification according as the cxponente are even or odd.
a $n$ evon, m คทกn
b a evrn, m oñ
$1 \begin{aligned} & r \text { ruen } \\ & \text { s even }\end{aligned}$
$2 \begin{aligned} & r \text { erien } \\ & 2 \text { ociù }\end{aligned}$
$2 \quad \begin{aligned} & \text { s even } \\ & \text { s oda }\end{aligned}$
$3 \begin{aligned} & \text { r odd } \\ & \text { s eyrn }\end{aligned}$
: $\quad \begin{aligned} & r \text { odd } \\ & s \\ & \text { riven }\end{aligned}$
$4 \begin{array}{r}r \\ \text { g ond } \\ \text { ond }\end{array}$
$4 \begin{aligned} & \text { s ond } \\ & \text { s odd }\end{aligned}$
c $n$ oda, meyfen
d $n$ odd, m odd
$1 \begin{aligned} & x \text { etren } \\ & s \text { evnn }\end{aligned}$
$1 \begin{array}{ll}r & \text { runn } \\ \text { s } & \text { bvan }\end{array}$
$2 \begin{aligned} & r \text { even } \\ & \text { s ocid }\end{aligned}$
2 s even

3 s oda
4

$$
\begin{aligned}
& r \text { ond } \\
& s \text { ore }
\end{aligned}
$$

$$
4 \begin{aligned}
& r \text { nuid } \\
& \text { s nû̃ }
\end{aligned}
$$

None of tinese curves intersect the axes in finite points.

$$
\begin{gathered}
\text { YI N, } \\
x^{m} y^{m}+x^{N} y_{1}^{s}+1=0 \quad n, m, r, s \text { even }
\end{gathered}
$$

The curve is imagrinary.

$$
\begin{gathered}
\text { VI } \mu_{2} \\
x^{m_{y} m^{m}}+x^{N_{y} s}+I=0 \quad n, m, r \text { even, } s \text { odd }
\end{gathered}
$$

The form at infinity on the $x-\varepsilon x i s, x^{m} y^{m}+1=0$ is imarinary. The form at infinity on the $y$-axis given by $x^{m} y^{m}+y^{N} y^{s}=0$ and $x^{n} y^{s}+1=0$ are as shown belovian<s




The curve cannot extent? into the first or second quadrants. It is symmetrical with respect to the $y$-axis. Hence we conciuac that the form is approximate, $\begin{aligned} & \text { as follows. }\end{aligned}$
$\frac{m>S}{()_{1}^{\infty}}$

$$
x^{m} y^{m}+x^{n} y^{s}+1=0
$$

$$
n, m, s \text { AYin, } r \text { odd }
$$

Form at infinity on the $x$-axis is imaginary as before. The firms at, infinity on the $y$-axis given by $x^{n} y^{m}+x^{N} y^{s}=0$ and $x^{N} y^{s}+1=0$ are as follows.





If $m=s$ the asymptote $x=-1$ does not, intersect tire curve in any finite points. The curve is symmetrical with respect to the $x-s x i s$.

Hence we conclude that the form is approximately as follovis.




VI $\mathrm{an}_{4}$

$$
x^{m} y^{m}+x^{n} y^{s}+1=0
$$

$$
n, m \text { even, } r, s \text { odd }
$$

The form gt infinity on the $x$-axis is imaginary. The forms at infinity on the $y$-axis given by $x^{m} y^{m}+x^{N} y^{s}=0$ and $x^{N} y^{s}+I=0$ are as follows.


The curve is symmetrical with respect to the origin.



$$
x^{m} y^{m}+x^{N} y^{s}+1=0
$$

$$
n, r, s \text { Tn, } m \text { odd }
$$

The form at infinity on the $x$-axis, is given by $x^{m} y^{m}+1=0$ Of the two forms at infinity on the $y$-axis. $x^{N} y^{s}+1=0$ and $x^{m} y^{m}+x^{N} y^{s}=0$
the first is imaginary and the second depends upon whether $m \gtrless s$.




The curve is symatricsl with respect to the y-axis ana lies entireIT in the third and fourth quadrants. Hence we conclude that the form is approximately? as follows.

$$
\xlongequal{\text { ( }}
$$


$n, x$ even, $m, ~ s$ odd

The form at infinity on the $x$-axis is t, he same as in VIl, shove. Cf the two forms at infinite on the $y$-axis, $x^{m} y^{m}+x^{N} y^{s}$ and $x^{\nu} y^{s}+I=0$ the first is imaginary and the second is as shoran on the right. The curve is symmetrical with respect to the $y$-axis. Hance re conclude that it has approximation. the accompanying form.

$$
V I b_{3}
$$

$$
x^{m} \nabla^{m}+y^{N} y^{s}+1=0 \quad n, s \cap \pi n n, m, r \text { odd }
$$

The form at infinity on the x-axis is the same as in riff, above. The forms at infinity on the $y$-axis given by $x^{m} y^{m}+x^{N} y^{s}=0$ and
$x^{2} y^{5}+1=0$ are us shown $n \in 10 \%$.


We conclude that the curve has ghoroximately the following


$$
\begin{array}{r}
V_{I} b_{4} \\
x^{m} y^{m}+x^{n} y^{s}+I=0
\end{array}
$$

$$
n, \text { nun, } m, r, s \text { ocd }
$$

The forms at infinity on the two axes given by $x^{m}{ }^{m}+1=0$, $x^{m} y^{m}+x^{n} d^{s}=0$ and $x_{m i s}^{n} 1=0$ are ass follows.






If $m=$ s the asymptote $x=-1$ does not intersect the curve in finitem points.



保

$$
\begin{gathered}
\text { VI } \boldsymbol{e}_{1} \\
x^{m} y^{m}+x^{N_{y} s}+1=0 \quad n \text { oud, } m, r, \text { s oven }
\end{gathered}
$$

The form at infinity on the $x$-axis is given by $x^{m} y^{m}+1=0$ Of the tyro forms at infinity on the $y-a x i s x^{2} y^{s}+1=0$ and $x^{M} y^{m}+x^{N} y^{s}=$ (the first is imaginary.





If $m=5$ tho asymptote $x=-1$ has no finite intorecetions with the curve. The curve is symmetrical with respect to the x-axis.


$$
\begin{array}{r}
V I e_{2} \\
x^{m} y^{m}+x^{\nu} y_{y^{5}}^{s}+1=0
\end{array}
$$

The forms at infinity on the two axes riven by $x^{m} y^{m}+l=0$, $x^{m} y^{m}+x^{N} y_{y}^{s}=0$ s. nd $x^{\nu} y^{s}+1=0$ are as follows




:..e concenter that the curve hens anmroximuta?y the following forms. (Son figures on next nape:


$$
\begin{aligned}
V \perp \mu_{3} \\
¥ \quad y \quad I=0
\end{aligned}
$$

$$
n, r \text { odd, re, s pren }
$$

The form st infinition the $y$-axis oiven $b_{y} x^{m} y^{m}+x^{N} y^{s}=0$ is imuginary. The other form

 shown on the risht. The ourve is symmetricar with rnsnerot to the x-axis. ifnce we conciune thit it, has annroximately the form indicetsou.

$$
Y I e_{4}
$$

$$
x^{m} T^{m}+x^{n} y^{s}+7=0 \quad n, r, S \text { ond, } n \text {-y<n }
$$

The forms it infinity on the trio rxats, given by $x^{m} y^{m}+1=0$, $x^{m} y^{m}+x^{n} y^{s}=0$ and $x^{N} y^{s}+1=\underset{m>s}{0}$ are ax follors $\sum_{m}<5$





Hence vif concludn that, the curve heas approximatoly thr following forns. (sfe noxt pame)



$$
\begin{aligned}
& \text { Vi } d_{1} \\
& x^{m} y^{m}+x^{N} j^{s}+i=0 \quad n, m \text { od, } r, s \text { oven }
\end{aligned}
$$

The form at infinity on the $y$-axis given by $x^{N} y^{5}+$ ? $=0$ is imamincry. The other given by $x^{m} y^{m}+x^{N} y^{s}=n$ and the form at infinity on the $x$-axis given by $x^{m} y^{m}+1=0$ are as shown below. The same form at infinity on the r-axis cons through the next four cessna.




The curve is symmetrical with respects to the origin.



$$
x^{m} i^{\prime \prime \prime}+x^{N} y^{s}+1=0
$$

$$
n, m, s \text { odd, } r \text { even }
$$

The form at infinity on the $x$-axis is the same as in YIdrabovo.
The forms at infinity on the $y$-axis are given by $x^{m} y^{m}+7^{N} y^{s}=0$ and
$x^{v} n^{s}+1=0$ ．The first divine into three cures recording as m麦s．





If $m=s$ the asymptote $x=-I$ has no finite intersections with
the curve．Hence we conclude that the curve has approximately the following forms．




$$
x^{m} y^{m}+x^{N} y^{s}+1=0
$$

$$
n, m, r \text { od, s oven }
$$

Form at infinity on the x －axis is the same as in T Th，about．
The form at infinity on tine $y$－axis given $b_{v} x^{m} y^{m}+x^{N} y^{5}=0$ and
$x^{n} v^{s}+1=0$ sro as shown be love：
 Hence en conclude that the curve hes two forms ermoximately as follows，demanding upon whether $m \geqslant s$ ．



$$
\begin{aligned}
& V I d y \\
& x^{m} v^{m}+N_{y}^{s}+1=0 \quad n, c, x, \Rightarrow \text { ocd }
\end{aligned}
$$

The form at, infinity on the $y-a x i s$ given by $x^{m} y^{m}+x^{N}=0$ is imaginary. The other, driven by $x^{2} y^{s}+1=0$ is as shown on the right. The form
at infinity on the $x$-axis is the same as above $x^{2} y^{s}+1=0$ is as shown on the right. The form
at infinity on the $x$-axis is the same as above in II $d_{1}$. The curve is symmetrical with respect to the orion. Hence in conclude the it has approxiamtaly the form indicatca.


$$
\begin{gathered}
\text { VI'a, } \\
x^{m} y^{m}+x^{N} y^{s}-\tau=0 \quad n, m, r, s \text { evan }
\end{gathered}
$$

The form at infinity on the y-axis given by $x^{M} y^{m}+x^{N} y^{s}-0$ is imaginary. The other form $x^{2} i^{5}-I=0$ is shoving on the right. The form



 at infinity on the $x-y x i s, x^{m} m-1=0$ is also shown on the right and this form goes through the next four cases. The carve is symmetrical with respect to both saros. "'e conclusion that its form is anproxireatoly as indicated on the right.


$$
\begin{aligned}
& \text { VI' } a_{2} \\
& x^{m} y^{m}+x^{N} y^{s}-1=0 \quad n, m, r \text { in, s odd }
\end{aligned}
$$

The tron forme at infinite on the $y$-axis $x^{n} y^{m}+x^{N} y^{s}=0$ and $x^{n} y^{5}-i=0$ Even as shove, $n$ on tire right. The curve iss sumetrical with respect; to the $y$-axis.


The forms at infinity of the y -axis $x^{m} y^{m}+x^{N} y^{s}=0$ and $x^{N} y^{s}-1=0$ gre $\because$ s shown bo love At infinity on the $x$-axis the form is the same as in VIa, above.





If $m=s$ the sisymntote $x=-1$ does not intersect the curve in finito points. The cuman is symmetrical with respect to tine x-ayis. Hence we conclude that the form is auncoximately as forlorn.



$$
x^{M} y^{m}+x^{N}, s-7-0
$$


$n, m \cdots i, r, s$ odd

The forms at antinitu on the $\bar{y}$-axis fire as indicated below, at infinity on the x-axis the same as in whoa, above.



The curve is symmetrical with respect to the origin. "n conclude
that the curve ias is arnroxiriotelo the following forms．


$$
x^{M} y^{m}+z^{N} y^{s}-1=0
$$

$$
n, r, \text { s preen, m odd }
$$

The form at，infinity on the $x$－axis is $x^{m} y^{m}-1=0$ and this goes throwers the next four cases．This together with the forms at infinity on then $y$－axis given by $x^{M} y^{m}+y^{N} y^{s}$ a and $x^{N} y^{s}-j=0$ are as indicated below．



The curve is sumetricul with respect，to tine origin．Hence we con－ clung that it，has the fol owing forms．



$$
x^{m} y^{m}+x^{N} y^{s}-1=0
$$

$$
n, r \in v n \Omega, m, s \text { oct }
$$

This becomes the same rs Vita by putting $\mathrm{y}_{\mathrm{y}}=-\mathrm{y}$ ．
！！his becomes ibo by nutting X－－v，シ－－iv．

$$
\begin{aligned}
& \text { YI暔3 } \\
& x^{m} y_{y}^{m}+x^{N}{ }_{0}^{s}-7=0 \\
& n \text {, is nun, m, r odd }
\end{aligned}
$$

$$
x^{m} y^{m}+x^{n} 7^{s}-7=0 \quad n, n \nabla r n, m, r, s \text { oda }
$$

-his hecomes it Ify hu rutiting $\quad$ H $=$ - y.

$$
\begin{gathered}
V+C_{1} \\
x^{n} y^{m}+x^{n} y^{s}-1=0 \quad n, \text { odd, } m, r, s \text { cven }
\end{gathered}
$$

The form at infinition the x-axis is Giten buy $y^{m} y^{m}-I=0$ and this form is the same for the noxt four casns. This tomether with the forms at, infinity on the $y$-axisujum hy $x^{m} y^{m}+x^{r} y^{s}=0$ and $x^{n} \pi^{s}-1=0$ are 8, shown bre or.






The curve is sommetricen with respect ton the v-evin. Tf $m=s$ the asumptoter $=-1$ dons not intomanct then ourve in finitir monntis.


$\because V_{2}$
$x^{m} j^{m}+x^{n} e^{s}-i=0 \quad n, s$ odi, m, reven
Nhis is trinsformed tio TI $\boldsymbol{N}_{2}$ hy putting $\mathrm{x}=-\mathrm{x}, \mathrm{y}=-\mathrm{y}$.

This is transforius to YIe hy buthine $x=-x$.
VIery

$$
x^{m} y^{m}+x^{r} y^{5}-1=0 \quad n, r, s \text { onc, } E \text { oven }
$$

This is transformed to To. O/4 by matting $x=-v$.

$$
\begin{aligned}
& { }^{+T} \dot{c}_{3} \\
& x^{m} \cdot m+x^{n} y^{s}-I=0 \quad n \text {, } r \text { oìi, } m, \text { s cyen }
\end{aligned}
$$

## VI ide

$x^{M} v^{m} x^{N} s-1=0$ n, m uni, $r$, s cvon

The form at infinity on the $x-a \bar{x} i=$ is riven $y_{0} x^{m} u^{n n}-1=0$ and it is tine sare for the next four cears. Whis and the forms at infinity on the $y$-qxis aiven hry $x^{m}+x^{N} y^{s}-0$ and $x^{N} y^{s}-7=0$ are as shown br? nw.





Tho curve is symmetrical with resnect to the orisin. Lrace we concluke that th, form is arproximatoly ak follows.


$$
\begin{aligned}
& I I d_{2} \\
& y^{m}-i^{m}+x^{N} y^{6}-I=0 \quad n, \text { ta, s ode r rven }
\end{aligned}
$$


YT $d_{3}$

$$
x^{m} y^{m}+x^{n} N_{y}-1=0 \quad n, \text { za, } r \text { oude, s ciran }
$$

"his hecomos iI $d_{3}$ by the ireansformation $\bar{x}=-x$.

$$
\begin{gathered}
T I d^{M} d_{4} \\
x^{m} y^{m}+x^{n} s+1=0 \quad n, m, r, i s \text { oiic }
\end{gathered}
$$

This becomes Viduby the transformation $x=-x$.

$$
\begin{gathered}
\text { VI'á, } \\
x \pi-x y+1=0 \quad n, m, r, s \operatorname{even}
\end{gathered}
$$

The form at infinity on the $x$-axis étinn hy $x^{x} y^{m}+7=0$ isimag insry. The forus at infinily on the J-asis siven hy $x^{m} y^{m}-x^{N} y^{s}=0$ ant $\nabla^{N} y^{s}-1-0$ are as follows.


The asumytotnes $x= \pm 1$ do not intersect tine curyo in finite noints when $m=$ s. llenon we conclude that the curwn hass arrroximatifly the followinc forms.




$$
x^{m} y^{m}-x^{2} y^{5}+1=0
$$

This hrcci Ti $\mu_{2} b_{0}$ the transformation $y=-y$.

$$
\begin{gathered}
Y I^{\prime} a_{3}^{\prime} \\
x^{M_{y} m}-x^{N} y^{s}+1=0 \quad n, \text { s swen, } r \text { odd }
\end{gathered}
$$

This brenmes vian ha the transformation $x=-x$

$$
\begin{gathered}
Y T a_{4} \\
x^{m} y^{m}-x^{N} y^{s}+1=0 \quad n, m n+\infty n, r, s \text { odd }
\end{gathered}
$$

Whis becomas "Jaty by the transformation $\mathrm{x}=$ - z

$$
\begin{gathered}
\text { YI' } f_{1}^{\prime} \\
x^{m} y^{\prime} m_{-}-x^{N} y^{s}+1=0 \quad n, r, s \text { even, m oid }
\end{gathered}
$$

This necomes Vib, by ther triansformation $v=-$ y

$$
\begin{gathered}
Y I b_{2}^{\prime} \\
x^{n} y^{m}-x^{r} y^{s}+I=0 \quad \text { n, reven, m, } s \text { odd }
\end{gathered}
$$

The forms at infinity on the wxes riven by $x^{m} y^{m}+1=0, x^{m} y^{m}-x^{N} y^{s}=0$ $\operatorname{and} x^{n} y^{5}-1=0$ are as follovis.





The astrmptotins $x= \pm 1$ do not intersect the curyf in fintife noints when $m=s$. The curve is wimmetrical with rospect to the J-axis. Hance we conmand that it has ampoximately tirn followince forms.



$$
x^{m} y_{r}^{m}-x^{N} v_{r}^{s}+1-n \quad 2 \text {, s.aven, } r!\text {, } r \text { odd }
$$

Whis becomas VIf b duting $x=-x$.

$$
\begin{gathered}
T l b_{y}^{\prime} \\
x^{M} y^{m}-x^{\mu} j^{s}+l=0 \quad n \text {, even }, m, r \text {, is nad }
\end{gathered}
$$

Whis broorres VIby $\mathrm{baj}_{\mathrm{j}}$ ruttin $n_{2 j} x=-\mathrm{x}$.

$$
\begin{gathered}
\text { II' } e^{\prime}, \\
x^{M} y^{m}-x^{N} y^{s}+I=0 \quad n, \text { odd, m, r, seven }
\end{gathered}
$$

This bocomes Ti $e$, puttine $\bar{x}=-x$.

$$
x_{y^{m}}^{m}-x^{N} y^{v}+1=0
$$

$$
n, s \text { ond, } m, x \in \nabla f n
$$

This bncomes "T. $\boldsymbol{U}_{2}$ ny nutting ${ }_{2}=-y$.

$$
\begin{gathered}
T I^{\prime} \dot{C}_{3} \\
x^{M} y^{m}-x^{n} y^{s}+1=0 \quad \text { m, avon, } n, r \text { odd }
\end{gathered}
$$

The form at infinity on the exes given by $x^{m} z^{m}+1-n$, $x^{m} y^{m} x^{N} v^{s}=0$ and $x^{N} y^{s}-1=0$ are $m>5010$ s.






The asymptotes $= \pm 7$ lo not intersect the curve in finite points when $m=s$. The curve is symmetrical with respect, to tho taxis. Hance we onncliade theist it, has anprovimstely the fiollowinom forms.




$$
y^{m} y^{m}-x^{N} y^{s}+7=0
$$

$$
n, r \text {, s ode, } m \text { furn }
$$

This is changer to "IC, by the transformation $\bar{z}=-\tilde{y}$.

$$
\begin{aligned}
& \text { VI'd'd } \\
& x^{m} y^{m}-x^{n} y^{s}+1=0 \quad n, m \text { odd }, r, \text { s even }
\end{aligned}
$$

This is changed to yId, by the transformation $x=-x$.

$$
\begin{aligned}
& \text { VII' } \\
& x^{m} y^{m}-x^{N} y^{s}+1=0 \quad n, m, s \text { od }, r \text { even }
\end{aligned}
$$

This is chanced to VI $d_{2}$ b the transformation $x=-\mathrm{x}, \mathrm{y}=-\mathrm{y}$.
VI ${ }^{\prime} d_{3}^{\prime}$

$$
x^{m} y^{m}-x^{v} y^{s}+1=0 \quad n \text {, fa, r once, s oven }
$$

This is chanced to Yidaby the transformation $x=-x, y=-y$.


$$
\begin{gathered}
Y I d_{4}^{\prime} \\
x^{m} y^{m}-x^{N} y^{s}+7=0 \quad \text { n, m, r, ord }
\end{gathered}
$$

The forms it infinity on the axes riven by $x^{m} y^{m}+1=0, x^{m} y^{m}-x^{\nu} y^{5}=0$ and $x^{2} y^{5}-1=0$ are as follows.


| $m$ | $=s$ |
| :---: | :---: |
| $h_{1}$ |  |
| $\vdots$ | $\vdots$ |
|  | $\vdots$ |
|  | 1 |




If $m=s$ the asymntotifs $x= \pm 1$ do not intersect the curve in finite points. The curve is symmetrical within respect to the origin.




VI 'ai,'

$$
x^{m} y^{m}-x^{N} y^{s}-1=0 \quad n, m, r, s \text { even }
$$

The form at infinity on the $\bar{y}$-axis given $h y x^{N} y^{5}+1=0$ is imageinury. The form at infinity on the $x$-axis $x^{m} y^{m}-1=0$ and the other form gt infinity on the $y^{r-a x i s} x^{M} y^{m}-x^{N} y^{s}=0$ are as follows.





If $m=r$ the asymptotes $x=\neq 1$ do not intersect tine curve in finite points. The curve is symmetrical with respect, to both axes. Hence vie conclude that it has approximately the following forms.


This becomes VI $a_{2}$ by the transformation $y=-y$.

$$
\begin{gathered}
\text { TI' } a_{3}^{\prime}{ }^{\prime} \\
x y-y y-z=0 \quad n, m, s \text { nyคn, } r \text { oda }
\end{gathered}
$$

This becomins Yiak by the transformetion $\mathrm{x}=-\mathrm{x}$.

$$
\begin{gathered}
{\text { VI' } a_{4}^{\prime}}_{\prime} \\
x^{m} z^{m}-x^{N} y_{j}^{s}-I=0 \quad n, m \text { even }, r, \mathbb{S} \text { odd }
\end{gathered}
$$

This brcomns "Iay by the traneformation $x=-x$.

$$
\begin{gathered}
\text { "I' } b_{1}^{\prime} \\
x^{M}, m-x^{N} y^{s}-I=0 \quad n, r, s \text { syen, mi odu }
\end{gathered}
$$

This becomes 7If, ly the transformation $:=-\mathrm{J}$.

$$
\begin{gathered}
V I b_{2}^{\prime \prime} \\
x^{m} y^{m}-x_{2}^{N} y^{s}-I=0 \quad n, r \text { NURn, m, s od } \bar{d}
\end{gathered}
$$

This hecomes "Iffi by the transformation y - - "

$$
\begin{aligned}
& V I^{\prime} f_{3}^{\prime} \\
& x^{m_{y} m}-x^{N} y^{s}-I=0 \quad n, s \text { sven, m, r odz }
\end{aligned}
$$

This brcomes TIf ${ }_{3}$ by the transformation $:=-v \cdot$

$$
\begin{aligned}
& \text { YI' }{ }_{8}^{\prime \prime} \text { ' } \\
& x^{m} y^{m}-x^{N_{y}} v^{5}-1=0 \quad n \text {, ntonn, } m, r, s \text { odd }
\end{aligned}
$$

This becomes TIf ${ }_{4}$ by the transformation $\mathrm{x}=-\mathrm{x}$.

$$
\begin{gathered}
Y I^{\prime} \prime_{\prime}^{\prime} \prime \\
x^{n} y^{m}-x^{n} y^{s}-I=0 \quad n, \text { odd, } m, r, \text { S } \operatorname{Han}
\end{gathered}
$$

This becomes YIGbu the traneformation $x$ - $x$.

$$
\begin{aligned}
& \text { VI' } e_{2}^{\prime \prime} \\
& x^{M} y^{m}-x^{N} y^{s}-1=0 \quad n, s \text { odar, reynn }
\end{aligned}
$$



$$
x^{M} y^{m}-x^{n} y^{s}-7=0 \quad n, r \text { odd, } m, \text { a cucn }
$$

This becomes Yif by the transformation $x=-x$.

$$
\begin{gathered}
\text { VI' ' ' }_{4}^{\prime} \\
x^{M} y^{m}-x^{N} y^{s}-I=0 \quad n, r, \text { oida, m \&VCiI }
\end{gathered}
$$

Mhis hecomes YIé4 ber the transformation $\mathbb{T}=-\mathrm{y}$.

$$
\begin{aligned}
& \text { TI'di' } \\
& x^{M} y^{\prime}-X^{\nu} y^{g}-I=0 \quad n, m \text { oda, } r, \text { is cven }
\end{aligned}
$$

This becores VId, hy the transformation $x=-x$.

$$
\begin{aligned}
& y I d_{2}^{\prime} \\
& x^{n} y^{\prime}-x^{n} y^{s}-1=0 \quad \text { n, } m, s \text { nad, } r \text { nyen }
\end{aligned}
$$

This becorras "Id by the transformeition $v=-x$.

$$
\begin{gathered}
\text { VI'd } d_{3}^{\prime \prime} \\
x^{M_{y} m}-x^{N} y^{s}-I=0 \quad n, m, r o d e, \approx \operatorname{cvfn}
\end{gathered}
$$

This becomes vidzby the transfincmation $y=-y$.

$$
\begin{aligned}
& \text { YI } d_{\mu}^{\prime \prime \prime} \\
& x^{m} y^{m}-x^{N} y_{r}^{s}-1=0 \quad n, m, r, s \text { odd }
\end{aligned}
$$

This becores TI'dy hy the transformation $x=-x$.

$$
x^{m} y^{m} \neq x^{\nu} \neq y^{s}=0 \quad \text { s }\langle n+\infty>r
$$

He vill divide this into the folloving suh－casns．

ョ ユ，m fv゙ャn

$2 \begin{aligned} & \text { s nupa } \\ & \text { s oda }\end{aligned}$
$3 \begin{aligned} & \text { r odd } \\ & \text { s fucn }\end{aligned}$
4 n odd
is
odd
u $n, m$ nes
$1 \begin{aligned} & r \text { cyen } \\ & \text { is rven }\end{aligned}$
$2 \begin{aligned} & \text { r evrn } \\ & \text { s odd }\end{aligned}$
is $\begin{aligned} & \text { r odd } \\ & \text { s even }\end{aligned}$
$4 \begin{aligned} & r \\ & \text { is oid }\end{aligned}$
b $n$ even，modd
$1 \begin{aligned} & \text { If AFFn } \\ & \text { is fifn }\end{aligned}$
$2 \begin{gathered}r \text { synn } \\ \text { s nad }\end{gathered}$
3 is nid
$4 \begin{array}{r}r \\ \text { is odd }\end{array}$


1 reven s even
$2 \begin{aligned} & r \text { nyen } \\ & \text { g oid }\end{aligned}$
3 s ond
$4 \begin{gathered}\text { s ocid } \\ \text { s odd }\end{gathered}$

These sixteen cese cin he reducna to six however，by the follow－ ing transformations after the equations hate been mado homogencous by the introanction of a nc．variable z．

VIIa hecomes VIIa $a_{2}$ by the transformation

$$
\text { Hence te have onlur to consicinc the sub-ceses a, } u_{2}, a_{4}, b_{1}, u_{3}
$$ and $d_{4}$ with eall the conrlingations of sion．

$$
x^{m} y^{m}+x^{n}+y^{s}=0 \quad n, m, r, \text { si even }
$$

The curve is imaginary but hes on f ra? point, the oričia.

## VII $a_{2}$

$x^{m} y^{m}+x^{2}+y^{s}=0$
$n, m, r$ nerf $n, s$ odd
The form at the origin is riven by $x^{N}+y^{s}=0$, at infinity on the $\bar{y}$-axis by $x^{m} y^{m}+z^{s}=0$. Frack of those take two lorna according as $r \gtrless s$ and $m \gtrless s$. The form at infinity on tine $x-a x i s$ given by $x^{m} y^{m}+x^{N}=0$ is imaginary. The antroximatr curves are as follows.

:ie conclude that the curve has approximately the following forms. $m>s$
$m<s$


The torlu ut the origin is given by $x^{2}+0^{-s}=$ O. Whin tones three
forms accoraing as $r$ es. We will omit the cause which $r<s$ for this can be transformed to the one in when $r>$ sher on interchange of $x$ EnG $y$. The forms at infinity on the $x$-and araxes riven by $x^{m} y^{m}+x^{n}=0$ and $x^{m} y^{m}+y=0$ rospectirnty reach take two forms scontifo as $n \geqslant r$ and moI. These ir proximate forms are as indicatra bol nr:。


Tencen we conclude that the curve has approximstoly thr following forms.










> VII $b_{1}$
> $x^{m} y^{m}+x^{2}+y^{s}=0 \quad n, r, s$ even, $m$ oda

The from at the oriein rivon by $x^{2}+y^{s}=0$ is imarinary . Honce
since the oriain is on the curve it is u convugote point. Fin form ret infinity on the $y$-axis given by $x^{m} y+y^{s}=0$ has tro forms according as m k s and the form int infinity on the - -axis given by $x^{n} y^{m}+x^{N}=0$ takes three forms according as $n \gtreqless r$.






If $n=r$ the asyrmpoter $=-1$ does not intersect the curve in finitfe pointis. "he curve is eymmetricel with resrinct, to the y-avis. Hencer we conclude that the curve has arprozimately the following forms.



The form at the origin is given by $x^{2}+y^{s}-n$, at infinity on the $x-\operatorname{cisis} b_{y} x^{m} y^{m}+x^{N}=0$ and at infinity on the axis by $x^{m} y^{m}+y^{s}=0$. Fsh of these take tivo forms as indicated bo? ow.


Hence wa conclude that the curve has rarroximetriy the following forms
$\mu>S$


## $\mu<S$



$$
x^{m} y^{m}+x^{2}+y^{s}=0 \quad n, m, r, \Omega o d d
$$

The forin at tha origin $x^{2}+y^{s}=0$ depends mpon whether $r$ 引 . The case in when $r<s$ can be transformed into the case in which $r>$ s by an intorchaner of $x$ and $y$. The form at infinity on lifn x-atis is given hy $x^{m} y^{m}+x^{\nu}=0$ and on the $y$-axis by $x^{m} y^{m}+y^{5}=0$. Tach of these take thrno forms drfending ipon the relative walue of the nxponerts. By an interchanef of $x$ wnk y the three caseu

$$
\begin{aligned}
& (1) m=s=r, n>r \\
& (3) m<s, r=s, n>r \\
& (3) m<s, r=s, n=r
\end{aligned}
$$

are transformen respectively into thr throe cases

$$
\begin{aligned}
& \left(1^{\prime}\right) n=r=s, m>s \\
& \left(2^{\prime}\right) n<r, r=s, m>s \\
& \left(3^{\prime}\right) n<r, r=s, m=s
\end{aligned}
$$

Tha arnroximate forms arn as follows.








The asymntotos $y=-1$ and $x=-1$ an not intinrsect the curve in finIte noints in the casns in which $n=r$ and $m=s$ respectively. Hence, we conclude that tine curve has approximately the following forms.


$m>s$
$m>N$



$m>s$
$m>N$





$$
r=S
$$




$$
\begin{aligned}
& \text { VII' } a, \\
& x^{m} y^{m}+X^{n}-y^{s}=0 \quad n, m, r, s \text { even } .
\end{aligned}
$$

The form at infinity on the $\bar{x}$-axis $x^{m} y^{m}+x^{N}=0$ is imaginary. The
 and $x^{m} y^{m}-y^{s}=0$ respectively are an indicatifed below:


| $m=5$ |  |  |
| :---: | :---: | :---: | :---: |
| $\vdots$ | 1 | 1 |
| $\vdots$ | 1 | 1 |
| $\vdots$ | 1 |  |



If $m=s$ the asymptotes $x= \pm 1$ do not internet, the curve in any finits points. Hence win conclude that the curve has approximately the following forms.
$m>s$

$$
\begin{gathered}
\text { VII } \alpha_{2} \\
x^{m} y^{m i n}+x^{N}-y^{n}=0 \quad n, m, r \text { even, } s \text { odd }
\end{gathered}
$$

This is transformed to VII by putting $y=-\nabla \cdot$

$$
\begin{array}{ll}
\text { VII' } a_{y} \\
x^{m} y^{m}+x^{r}-y^{s}=0 & n, m \text { even, } r, s \text { odd }
\end{array}
$$

This is transformed to VII by putting $\mathrm{y}=-\mathrm{y}$.

$$
\begin{aligned}
& \text { VII } b_{1} \\
& x^{M} y^{N / 4}+x^{N}-y^{s}=0 \quad n, r, s \text { even, } m \text { cad }
\end{aligned}
$$

There are throe forms at the origin depending upon whether $r$ es, three at infinity on the $x$-axis depending upon whether $n$ 友r and two at infinity on the taxis derenaing upon whether m m . Prese approximatie forms are as follows.


Fence we conclude that, the curve has approximately the following
finrıาs.



## $M>N$







$m<N$






$$
x^{m} y^{m}+x^{N}-y^{s}=0 \quad n \text {, s even, } m, r \text { ode }
$$



## VII＇dy

$$
x^{M} y^{M}+x^{N}-y^{s}=0 \quad n, m, r, s \text { odd }
$$

？his is transformed to＂IIdyby mating $x=-x$ ．

$$
\begin{array}{r}
\text { VII } a_{1}^{\prime} \\
x^{m} y^{m}-x^{n}+y^{5}=0
\end{array}
$$

$$
n, m, r, s e v \cap n
$$

This is transformed to ViI发，by an interchange of $x$ and $i^{T}$ ．
VII＇AU

$$
x^{m} y^{m}-x^{r}+y^{s}=0 \quad n, m, r \text { cen, s odd }
$$

The form at the origin given by $女^{\nu}-y^{5}=0$ ，the form at infinity on the $x$－axis given by $x^{m} y^{m}-x^{N}=0$ and form at infinity on the $y$－axis －liven $b_{y} x^{m} y^{m}+y^{s}=0$ are as follows．







Hence we conclude that the curve has apmenximatnly the following forms． $n>N$



$$
m=N
$$





$$
x^{n} y^{m}-x^{n}+y^{s}=0
$$

$$
n, \text { n nven, } r, s \text { oda }
$$

Thiss hecomes \#IIa, by the transformation $x=-x$.
YII'解

$$
x^{m} y^{m}-y^{n}+y^{s}=0 \quad n, r, \text { s even, m odd }
$$

This becomes TIIf, by the transformation $y=-y \cdot$

$$
\begin{aligned}
\text { VII } b_{3}^{\prime} \\
x^{m} y^{m}-x^{N}+y^{s}=0 \quad n, s \text { evon, } m ; r \text { odd }
\end{aligned}
$$

This is transformed to ritb by putting $x=-x$.

$$
\begin{gathered}
\text { TII'd } \\
x^{n} y_{m}^{m}-x^{n}+r^{s}=0 \quad n, m, r, s \text { nad }
\end{gathered}
$$

This becomes VIId, by the transformation $y=-J \cdot$

$$
\begin{aligned}
& V^{T I} d_{1}^{\prime} \\
& x^{m_{y}^{m}}-x^{N}-y^{s}=0 \quad n, m, r, s \text { even }
\end{aligned}
$$

This hecomes YI'a, by the trunsformation $x=z ; z=x$.

$$
x^{m} y^{m}-x^{N}-y^{s}=0 \quad n, m, r \text { orren, } A_{2}^{\prime}
$$

-his becomes YII'áa by the transformation $y=-\mathrm{J}$.

$$
\begin{aligned}
& \text { TII' } \mathfrak{a}_{4}^{\prime \prime} \\
& x^{m^{\prime}} y^{m}-x^{N}-y^{s}=0 \quad n, m \text { even, } r, s \text { odd }
\end{aligned}
$$

This hecomos wIany tha transformation $\dot{x}=-\mathrm{y}, \mathrm{y}=-\mathrm{y}$.

$$
\begin{aligned}
& \text { TII' } f_{1}^{\prime \prime} \\
& x^{m} y^{m}-x^{\nu}-y^{s}=0 \quad n, r, \text { s nven, m odd }
\end{aligned}
$$

This Decomes TIIbby the transformation $\%=-\nabla \cdot$

$$
\begin{aligned}
& \text { VII' } f_{3}^{\prime \prime} \\
& x^{n} y^{m}-x^{\nu}-y^{s}=0 \quad n, s \text { even, m, r odd }
\end{aligned}
$$

This hecomes TTTf ber the transformation ${ }_{0}=$ -

$$
\begin{gathered}
\text { VTI'dí' } \\
x^{n} y^{m}-x^{N}-y^{s}=0 \quad n, \text { m, } r, s \text { odd }
\end{gathered}
$$

This becomes TIId, by the trensformation $X=-x, y=-y$.

