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Forms of algebraic plane curves whose equations are trinomial

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ON THE FORMS OF ALGEBRAIC PLANE CURVES WHOSE EQUATIONS ARE TRINOMIAL.

BY

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I HEREBY RECOMMEND THAT THE THESIS PREPARED UNDER MY SUPERVISION BY

Geo. Ernest Carscallen

ENTITLED On the Forms of Algebraic Plane Curves Whose Equations are Trinomial

BE ACCEPTED AS FULFILLING THIS PART OF THE REQUIREMENTS FOR THE

DEGREE OF Master of Arts.

Charles H. Sisam.
In Charge of Major Work

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Recommendation concurred in:

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INTRODUCTION.

It is the object of this paper to determine approximately the forms of the real algebraic plane curves, the equations of which are trinomials, that is are of the form;

$$Ax^{n_1}y^{m_1}+Bx^{n_2}y^{m_2}+Cx^{n_3}y^{m_3}=0$$

where A, B, and C are real numbers, n_1, n_2, n_3, m_1, m_2, and m_3 are positive integers and at least one of the numbers n_1, n_2, n_3 and at least one of the numbers m_1, m_2, m_3, is zero.

In determining the approximate forms of these curves we will suppose two curves to have the same form and therefore, for the purpose of this paper, to coincide, if one can be projected into the other by a transformation of the form;

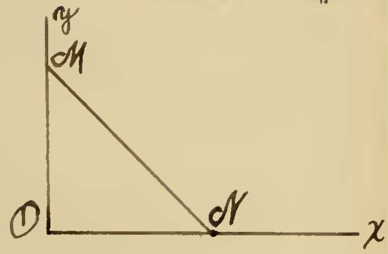
$$x=ax'$$
$$y=by'$$

where a and b are real numbers. Such a transformation consists merely in stretching the plane of the figure in directions parallel to the x- and parallel to the y- axis. Two curves will also be considered as having the same form if the equation of one can be obtained from the equation of the other by first making the equations homogeneous by the introduction of a new variable z and then interchanging the variables x, y and z. Two curves whose equations can be made identical in this manner can be transformed into each other by a real projective transformation.

Finally two curves will be considered as having the same form, to the degree of approximation sought in this paper, if the approximate curves in the neighborhood of the singular points have the same form, and if the curves intersect the axes and the line at infinity in the same points. It will be seen hereafter that if the above data are the same for two curves, then the forms of the two

curves will closely resemble one another.

To determine the behavior of the curves under consideration, especially in the neighborhood of the axes and the line at infinity, the well known theory of the analytic polygon will be employed. #



In an auxiliary figure let each term of a complete equation of the same degree as the given equation be represented by a point whose coordinates are the exponents of x and y in that term. If the degree of the of the equation is n, let N and M be the points whose coordinates are (n,o) and (o,n) respectively. Then ONM is called the analytic triangle. Mark by small circles the points representing terms actually appearing in the given equation.

Draw every line which passes through two or more circled points and has all the remaining circled points on one side of it. The polygon thus formed is called the analytic polygon. Then the terms corresponding to the circled points on any one of these lines which separates the other points from O determine a form of the curve at the origin. Those on O N and O M determine the intersections of the curve with the x- and the y- axis respectively. Those on a line separating the other points from N determine a form of the curves at infinity on the x- axis and those separating from M determine a form at infinity on the y- axis. Those on M N determine the intersections with the line at infinity, that is, the directions of the asymptotes. The linear factors of these terms give the equations of the asymptotes except for the constant term. #

Wieleitner: Therorie der ebenen algebraischen Kurven höherer Ordnung. pp.83 and following.
Johnson: Curve Tracing pp.42 and following.

Johnson: Curve Tracing in Cartesian Coordinates pp. 12.

Since the equations of the curves under consideration are trinomials, the analytic polygon for any one of these curves is a triangle. The case where this triangle degenerates into a straight line may be excluded, since, in this case, the equation of the curve can be factored.

For, if the equation is,

$$Ax^{n_1}y^{m_1}+Bx^{n_2}y^{m_2}+Cx^{n_3}y^{m_3}=0$$

and the three points (n_1, m_1), (n_2, m_2) and (n_3, m_3) are on a straight line we have:

$$\begin{aligned} a\,n_1+b\,m_1 &= c \\ a\,n_2+b\,m_2 &= c \qquad\qquad (1) \\ a\,n_3+b\,m_3 &= c \end{aligned}$$

where a, b and c are rational and integral. That they are rational is readily seen as follows: From the theory of determinants

$$a = k(m_3 - m_2)$$

$$b = k(n_2 - n_3)$$

$$c = k(n_2 m_3 - n_3 m_2)$$

The quanities in parenthesis are integers hence if one of the numbers a, b or c contained an irrational factor it would have to be in k. But if there had been such a factor we would have divided equations (1) through by k thus leaving the coefficients rational and integral. If we substitute $\xi = x^{\frac{1}{a}}$ and $\eta = y^{\frac{1}{b}}$ in the equation of the curve it becomes:

$$\mathcal{A}\,\xi^{an_1}\eta^{bm_1}+\mathcal{B}\,\xi^{an_2}\eta^{bm_2}+\mathcal{C}\,\xi^{an_3}\eta^{bm_3}=0$$

This is a homogeneous binary form in ξ and η with integer exponents. It may therefore be written in the form:

$$(l_1\xi+k_1\eta)(l_2\xi+k_2\eta)\;\cdots\cdots\;(l_p\xi+k_p\eta)=0$$

Hence the given curve having a trinomial equation degenerates into a set of curves each having a binomial equation of the form of:

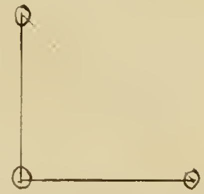
$$(1^{ab})\, x^{b} = (-1)\, y^{a}$$

These curves will be excluded from our discussion. The analytic polygon must therefore be a triangle.

The various positions of the three circled points on the triangle O M N give rise to a number of different cases which, in discussing the forms of these curves, must be considered separately.

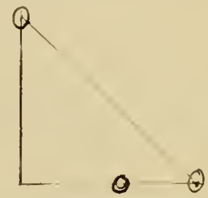
Case I. All three vertices of the analytic triangle are circled. The equation is then:

$$Ax^{m} \pm By^{m} \pm C = 0$$



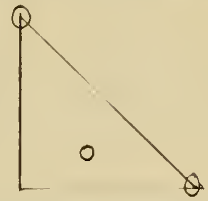
Case II. Two vertices and a point on the side opposite one of them are circled:

$$Ax^{m} \pm By^{m} \pm Cx^{n} = 0$$



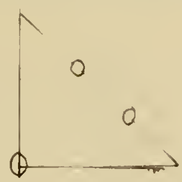
Case III. Two vertices and a point inside the triangle are circled:

$$Ax^{m} \pm By^{m} \pm Cx^{n}y^{s} = 0$$



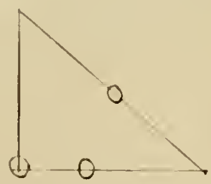
Case IV. One vertex is circled and two points on the opposite side:

$$Ax^{m}y^{m} \pm Bx^{n}y^{s} \pm C = 0$$



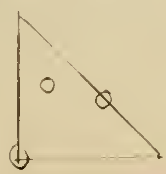
Case V. One vertex, a point on the opposite side and a point on an adjacent side are circled:

$$Ax^{m}y^{m} \pm Bx^{n} \pm C = 0$$



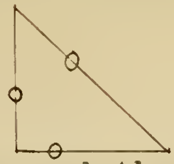
Case VI. One vertex, a point on the opposite side and a point inside the triangle are circled:

$$Ax^{m}y^{m} \pm Bx^{n}y^{s} \pm C = 0$$

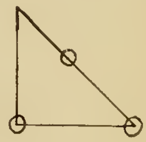


Case VII. No vertex, but one point on each side is circled:

$$Ax^n y^m \pm Bx^\nu \pm Cy^s = 0$$



By making the equation of the curve homogeneous and then interchanging the homogegeous coordinates x, y and z the cases arising from the other positions of the analytic polygon can be reduced to one of these seven. For example consider the case where the analytic polygon becomes: . . . .



The equation of the curve is,

$$Ax^m + Bx^\nu y^{m-\nu} + C = 0$$

Making the equation homogeneous it becomes,

$$Ax^m + Bx^\nu y^{m-\nu} + Cz^m = 0$$

Interchanging y and z and then making the equation non-homogeneous by putting z=1 we obtain:

$$Ax^m + Cy^m + Bx^\nu = 0$$

which is the same as case II.

In a similar manner it may be shown that each of the other cases reduces to one of the above seven cases.

The above seven equations will come out of the following two by making some of the exponents zero.

$$Ax^m y^m \pm Bx^\nu \pm Cy^s = 0 \qquad (2)$$

$$A'x^n y^m \pm B'x^\nu y^s \pm C' = 0 \qquad (3)$$

in which A, B, C, A', B', and C' are positive.

The coefficients A, B, C, A', B', C' can all be reduced to unity by the transformation,

$$x = ax'$$

$$y = by'$$

Equation (2) becomes by this transformation and after dropping primes:

$$Aa^n b^m x^n y^m \pm Ba^\nu x^\nu \pm Cb^s y^s = 0 \qquad (2')$$

If the coefficients are to be equal we have

$$\frac{Aa^{n}b^{m}}{Cb^{s}} = 1 \qquad\qquad \frac{Ba^{r}}{Cb^{s}} = 1$$

From which

$$b^{m-s} = \frac{C}{Aa^{n}} \qquad\qquad b^{s} = \frac{Ba^{r}}{C}$$

Raising both of the first of these equations to the s power and both members of the second to the (m - s) power we obtain:

$$b^{(m-s)s} = \left(\frac{C}{Aa^{n}}\right)^{s} \qquad\qquad b^{(m-s)s} = \left(\frac{Ba^{r}}{C}\right)^{m-s}$$

Equating these two values of $b^{(m-s)s}$;

$$\left(\frac{C}{Aa^{n}}\right)^{s} = \left(\frac{Ba^{r}}{C}\right)^{m-s}$$

Solving for a:

$$a^{(mr+ns-rs)} = \frac{C^{m}}{A^{s}B^{(m-s)}}$$

Solving for b in like manner:

$$b^{(mr+ns-rs)} = \frac{B^{n}}{A^{r}C^{n-r}}$$

The exponent $(rm+ns-rs)$ cannot be zero for its vanishing is the condition that the three points (n, m) (o, s) and (r, o) in the analytic triangle be on a straight line. This is readily shown:

$$\begin{vmatrix} n & m & 1 \\ o & s & 1 \\ r & o & 1 \end{vmatrix} = ns + rm - rs$$

Hence since A, B and C are positive and different from zero, the above equations can be solved in real numbers for a and b. Substituting in (2') the values so found for a and b and dividing by a finite constant the equation becomes

$$x^{n}y^{m} \pm x^{r} \pm y^{s} = 0$$

Let equation (3) be divided through by C' and A and B written for $\frac{A'}{C'}$ and $\frac{B'}{C'}$ respectively.

We then have, $\quad Ax^{n}y^{m} \pm Bx^{r}y^{s} \pm 1 = 0$

With the same transformation as before the equation becomes, after dropping primes:

$$A a^n b^m x^n y^m \pm B a^r b^s x^r y^s \pm 1 = 0 \qquad (3')$$

We must then have

$$A a^n b^m = 1 \qquad\qquad B a^r b^s = 1$$

Solving for a and b

$$a^{ns-mr} = \frac{B^m}{A^s} \qquad\qquad b^{ns-mr} = \frac{A^r}{B^n}$$

Again the exponent (ns - mr) cannot be zero for its vanishing is the condition that the two points (n, m) and (r, s) be in a line with the origin, since:

$$\begin{vmatrix} n & m & 1 \\ r & s & 1 \\ o & o & 1 \end{vmatrix} = n s - r m$$

Hence, since A and B are positive and finite, the above equation can be solved for a and b in real finite numbers. Substituting these values for a and b in (3') the equation becomes:

$$x^n y^m \pm x^r y^s \pm 1 = 0$$

We shall, therefore, always suppose that the coefficients are plus or minus one.

We will now prove that the locus of a trinomial equation such as we are considering can have no double points except at the origin and at the points where the x- and y- axes intersect the line at infinity.

The coordinates of a double point, when the equation is written homogeneously, must satisfy the three partial differential equations: #

$$\frac{\partial f}{\partial x} = 0 \;, \quad \frac{\partial f}{\partial y} = 0 \;, \quad \frac{\partial f}{\partial z} = 0$$

It has been shown that the coefficients can be reduced to unity.

Salmon: Higher Plane Curves, third edition, Art. 69.

We will suppose this done and write equation (2) in the following form:

$$x^n y^m + Ax^r + By^s = 0$$

where A and B are plus or minus one.

To show that the theorem holds in case VII, write the equation of the curve homogeneously. It becomes:

$$x^n y^m + Ax^r z^{n+m-r} + By^s z^{n+m-s} = 0$$

The three partial differential equations are then:

$$\frac{\partial f}{\partial x} = nx^{n-1}y^m + Arx^{r-1}z^{n+m-r} = 0 \qquad (4)$$

$$\frac{\partial f}{\partial y} = mx^n y^{m-1} + Bsy^{s-1}z^{n+m-s} = 0 \qquad (5)$$

$$\frac{\partial f}{\partial z} = A\,(n+m-r)\,x^r z^{n+m-r-1} + B(n+m-s)\,y^s z^{n+m-s-1} = 0 \qquad (6)$$

If a point on one axis satisfies these equations it must also be on another axis. For if x = 0 then equation (4) is satisfied but in order to satisfy (5) and (6), y or z must be zero. If y = 0 then (5) is satisfied but x or z must be zero to satisfy (4) and (6). If z = 0 then (6) is satisfied but x or y must be zero to satisfy (4) and (5). For the consideration of points not on any of the three axes let the equations be divided through by x^{r-1}, y^{s-1} and $z^{n+m-r-1}$ respectively. They then become:

$$nx^{n-r}y^m + Arz^{n+m-r} = 0 \qquad (7)$$

$$mx^n y^{m-s} + Bsz^{n+m-s} = 0 \qquad (8)$$

$$A(n+m-r)x^r + B(n+m-s)y^s z^{r-s} = 0 \qquad (9)$$

Since none of the exponents n, m, r and s are zero:

From (7) $$x^{n-r} = \frac{-Arz^{n+m-r}}{ny^m}$$

From (9) $$x^r = \frac{-B(n+m-s)y^s z^{r-s}}{A(n+m-r)}$$

Multiplying these together:

$$x^{n} = \frac{Br\ (n+m - s)y^{s-m}z^{n+m-s}}{n(n\ m-r)}$$

Equating this to the value of x^{n} obtained from equation (8) we have:

$$\frac{Br\ (n+m - s)y^{s-m}z^{n+m-s}}{n(n+m-r)} = \frac{- Bsz^{n+m-s}}{m\ y^{m-s}}$$

or
$$\frac{r(n+m - s)}{n(n\ \ m - r)} = - \frac{s}{m}$$

Since the letters in this equation are all positive integers and $r < n+m > s$ the left side is positive and the right side negative which is impossible. This proves the theorem for Case VII.

For Case II which comes from the above when $s = n$, $m = 0$ and $n > r$, equations (4), (5) and (6) become:

$$\frac{\partial f}{\partial x} = n\ x^{n-1} + A\ r\ x^{N-1} z^{n-N} = 0$$

$$\frac{\partial f}{\partial y} = B\ s\ y^{s-1} = 0$$

$$\frac{\partial f}{\partial z} = A\ (n - r)\ x^{N} z^{n-N-1} = 0$$

The curve cannot therefore, in this case, have a mode except at the origin or at the point at infinity on the x- axis.

To show that the theorem holds in cases I, IV, V and VI we may write the equation of the curve in the form:

$$x^{n}y^{m} + Ax^{N}y^{s} + B = 0$$

and making it homogeneous:

$$x^{n}y^{m} + Ax^{N}y^{s}z^{n+m-N-s} + Bz^{n+m} = 0$$

Forming the three partial differ en tial equations:

$$\frac{\partial f}{\partial x} = n\ x^{n-1}y^{m} + A\ r\ x^{N-1}y^{s}\ z^{n+m-N-s} = 0 \qquad (10)$$

$$\frac{\partial f}{\partial y} = m\ x^{n}y^{m-1} + A\ s\ x^{N}y^{s-1}z^{n+m-N-s} = 0 \qquad (11)$$

$$\frac{\partial f}{\partial z} = A\ (n+m-r-s)\ x^r y^s z^{n+m-r-s-1}\quad B(n+m) z^{n+m-1} = 0 \qquad (12)$$

If x = 0 equations (10) and (11) are satisfied but for (12) to be satisfied z must also be zero. If z = 0 equation (12) is satisfied but for (10) and (11) to be satisfied either x or y must be zero. Hence, if a point on one axis satisfies these equations it must also be on another axis, therefore at the intersection of the two.

For the consideration of a point on neither axis let the three equations be divided through by $x^{r-1}y^s$, $x^r y^{s-1}$ and $z^{n+m-r-s-1}$ respectively which gives:

$$n\ x^{n-r} y^{m-s} + A\ r\ z^{n+m-r-s} = 0 \qquad (13)$$

$$m\ x^{n-r} y^{m-s} + A\ s\ z^{n+m-r-s} = 0 \qquad (14)$$

$$A(n+m-r-s)\ x^r y^s + B(n+m)\ z^{r+s} = 0 \qquad (15)$$

From (13) and (14) since x, y, z and A are finite

$$\begin{vmatrix} n & r \\ m & s \end{vmatrix} = 0$$

This is the condition that the points (n, m) and (r, s) be collinear with the origin. But the origin is a circled point, hence this is the condition that the circled points be collinear, which is impossible since we have excluded such cases. This proves the theorem for cases I, IV, V and VI. Case III must be considered alone.

Its equation is:

$$x^n + A\ y^n + B\ x^r y^s = 0 \qquad n > (r+s) > 1$$

Writing it homogeneously:

$$x^n + A\ y^n + B\ x^r y^s z^{n-r-s} = 0$$

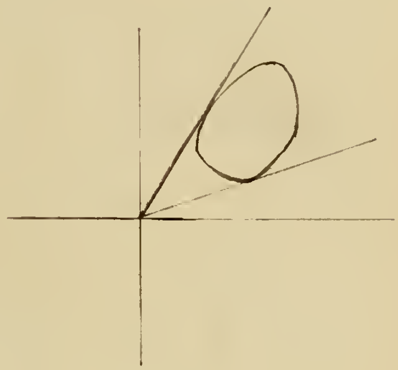
$$\frac{\partial f}{\partial x} = nx^{n-1} + Brx^{r-1} y^s z^{n-r-s} = 0$$

$$\frac{\partial f}{\partial y} = Any^{n-1} + Bsx^r y^{s-1} z^{n-r-s} = 0$$

$$\frac{\partial f}{\partial z} = B\ (n-r-s)\ x^r y^s z^{n-r-s-1} = 0$$

To satisfy the third of these equations, one of the variables x, y or z must be zero. If $x = 0$ then in order for the second equation to be satisfied y must equal zero, and if $y = 0$ or $z = 0$ it follows from the first equation that $x = 0$. This proves the theorem for case III.

There can be no isolated circuits of these curves which lie entirely in the finite part of the plane and do not intersect the axis. For suppose such a circuit to exist. Two tangents could be drawn to it through the origin. Let the equation be written homogeneously and let one of the variables, z say, occur in only one term. We then have:



$$f(x, y, z) = x^{\alpha_1}y^{\beta_1} + A\, x^{\alpha_2}y^{\beta_2} + Cx^{\alpha_3}y^{\beta_3}z^{\gamma} = 0$$

The equation of a tangent at a point (x_1, y_1, z_1) on the curve is:

$$\frac{\partial f}{\partial x_1}x + \frac{\partial f}{\partial y_1}y + \frac{\partial f}{\partial z_1}z = 0 \quad \#$$

If this is to pass through the origin we have:

$$\frac{\partial f}{\partial z_1} = C\gamma\, z_1^{\gamma-1}\, x_1^{\alpha_3} y_1^{\beta_3} = 0$$

Since neither C nor γ is zero, either, x, y, or z must be zero; that is the point of tangency must be on one of the lines $x = 0$, $y = 0$ or $z = 0$ which is contrary to the hypothesis that the circuit does not cross either of these lines. This proves the theorem for all cases in which one variable is lacking from two terms; that is for all cases in which in the analytic triangle two circled points are on the same side of the triangle. These are the first five cases.

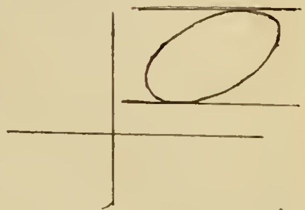
Salmon: Higher Plane Curves, third edition, Art. 64.

For cases VI and VII we proceed as follows:

By interchanging y and z in case VI it is seen that the equation of the curve can in both cases be written in the form:

$$x^n y^m + Ax^r + By^s = 0 \tag{16}$$

If the curve has a closed branch which is entirely in the finite part of the plane and does not touch or cross either axis, then it is possible to draw two tangents to it parallel to the x- axis. The points of tangency must satisfy the relation:



$$\frac{\partial f}{\partial x} = nx^{n-1}y^m + Arx^{r-1} = 0 \tag{17}$$

and since x is not zero we may write

$$nx^{n-r}y^m + Ar = 0 \tag{18}$$

Multiply (16) by r and (17) by x and subtract.

$$rx^n y^m + Arx^r + Bry^s = 0$$

$$nx^n y^m + Arx^r = 0$$

$$(r-n)x^n y^m + Bry^s = 0 \tag{19}$$

If $(r-n) = 0$ we obtain since $B \neq 0$, $r \neq 0$, $y = 0$ which is contrary to hypothesis.

Suppose $r - n \neq 0$. Then since y is not zero we can divide by y^s and solving for y^{m-s} have

$$y^{m-s} = \frac{B\,r}{(n-r)x^n}$$

Raising both members of this equation to the m power:

$$y^{(m-s)m} = \left[\frac{B\,r}{(n-r)x^n}\right]^m \tag{20}$$

From (18) $\qquad y^m = \dfrac{-A\,r}{n\,x^{n-r}}$ and raising both members to the (m - s) power:

$$y^{(m-s)m} = \left[\frac{-A\,r}{n\,x}\right]^{m-s} \tag{21}$$

Equating the two values of $y^{(m-s)m}$ from equations (20) and (21) and solving for x we obtain:

$$x^{mn+ms-ns} = \left(\frac{-n}{A\ r}\right)^{m-s} \cdot \left(\frac{B\ r}{n\ -\ r}\right)^{m}$$

Solving for y in like manner we obtain:

$$y^{mn+ms-ns} = \left(\frac{-Ar}{n}\right)^{n} \cdot \left(\frac{Bn}{(r-n)A}\right)^{m-n}$$

We have already seen that the exponent $mr + ns - rs$ is not zero. If it is odd, there is only one real point in the plane; and if is even there is at most one real point in any one quadrant of the plane, for which $\frac{\partial f}{\partial x} = 0$. But if there were a finite circuit lying entirely in any one quadrant, there would be at least two points in that quadrant for which $\frac{\partial f}{\partial x} = 0$. Hence, there exists no such circuit.

This proves the theorem for cases VI and VII. The theorem is, therefore, true in all cases.

It is easily seen that these curves cannot intersect a line parallel to one of the axes, or a line through the origin in more than two points in any one quadrant. For let one variable, y say, be given a constant value. We then have a trinomial equation in x. According to Descartes rule of signs this equation can have no more positive roots than there are changes of sign and no more negative roots than there are changes of sign when -x is substituted for x. Since the equation is trinomial there can be at most two changes of sign, hence not more than two roots of the same sign and therefore not more than two intersections in one quadrant. For the consideration of a line through the origin let the equation be written in polar coordinates. For a given value of θ we then have a trinomial in ρ which can have, at most, two positive and two negative roots,

hence at most two intersections in any one quadrant.

In discussing the forms of these curves we shall use primes to denote the number and distribution of negative signs. We will use a single prime when the last term is negative, a double prime when the middle term is negative and three primes when the last two terms are negative. For example:

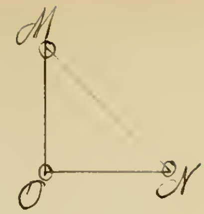
Case I is $x^n + y^n + 1 = 0$

Case I' is $x^n + y^n - 1 = 0$

Case I'' is $x^n - y^n + 1 = 0$

Case I''' is $x^n - y^n - 1 = 0$

CASE I.



$$x^n \pm y^n \pm 1 = 0$$

There will be two sub-cases a and b according as n in even or odd

Ia

$$x^n + y^n + 1 = 0 \qquad n, \text{ even}$$

This is clearly imaginary

Ib

$$x^n + y^n + 1 = 0 \qquad n, \text{ odd}$$

The curve intersects the x-axis at $x = -1$ and the y-axis at $y = -1$ and at no other points.

There is an asymptote whose equation is given except for the constant term, by the linear factor of $x^n + y^n$.

To obtain the constant term let the equation of the curve be written:

$$x + y = \frac{-1}{x^n - x^{n-1}y + \cdots + y^n}$$

As a point moves out on the curve and approaches this asymptote, its coordinates $x \doteq \infty$, $y \doteq \infty$ and $\frac{x}{y} \doteq -1$. Hence the equation of the asymptote is:

$$x + y = \lim_{\substack{x=\infty,\ y=\infty \\ \frac{x}{y} \doteq -1}} \frac{-1}{x^n - x^{n-1}y \cdots + y^n} = 0$$

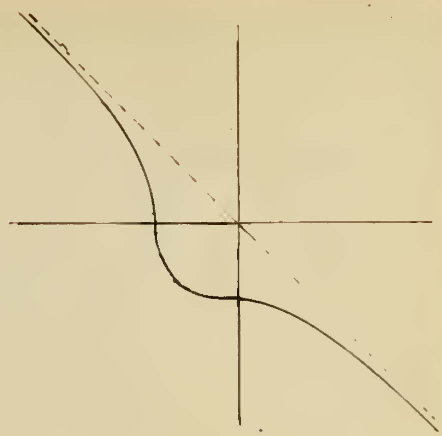
The asymptote does not cut the curve in any finite point, for elimenating y between the two equations we have:

$$x^n + (-x)^n + 1 = 0$$

or since n is odd $\qquad 1 = 0$

Aline parallel to either axis can be cut the curve in but one real point. For the equation may be written $\qquad y^n = 1 - x^n$

and for any given value of x there is but one real value of y, since there is but one real odd root of a number. Hence we conclude, since the curve is continuous, that it has approximately the form shown on the right.



I'_a

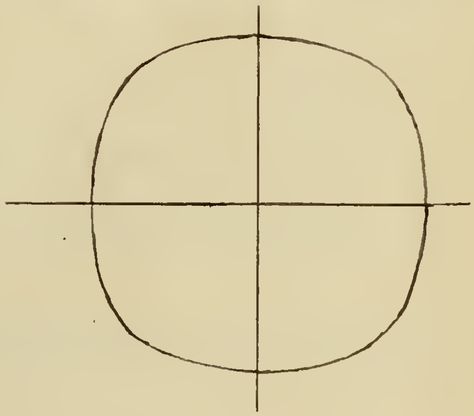
$$x^n + y^n - 1 = 0 \qquad n, \text{ even}$$

The curve intersects the x-axis at $x = \pm 1$ and the y-axis at $y = \pm 1$. It is symmetrical with respect to both axes and is a circle when $n = 2$.

Writing the equation in the form:

$$y^n = 1 - x^n$$

we see that it is imaginary for $|x| > 1$ and also for $|y| > 1$. As n is made larger and larger the curve approaches the form of a square. Hence we conclude that the curve has approximately the accompanying form:



I_b

$$x^n + y^n - 1 = 0 \qquad n, \text{ odd}$$

This is reduced to Case I in which all the coefficients have the same sign by the transformation:

$$x = -x, \qquad y = -y$$

I''_a

$$x^n - y^n + 1 = 0 \qquad n, \text{ even}$$

This curve intersects the y-axis at $y = \pm 1$ but does not cut the x-axis at all.

Since the terms $x^n - y^n$ contain two real linear factors there are two asymptotes whose constant terms we will find by the same method as before. The equations of the asymptotes will be therefore:

$$x + y = \lim_{\substack{x= \;,\; y= \\ \frac{x}{y} = -1}} \frac{-1}{x \quad - x \quad y \quad -y}$$

$$x + y = 0$$

$$x - y = \lim_{\substack{x= \;,\; y= \\ \frac{x}{y} = 1}} \frac{-1}{x \quad x \quad y \quad y}$$

$$x - y = 0$$

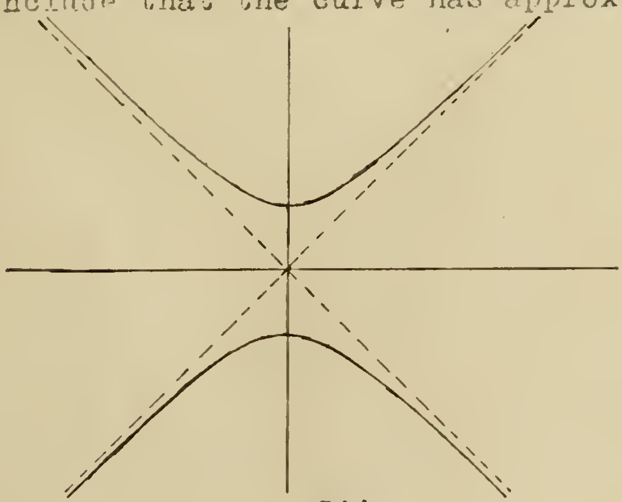
Neither of these cut the curve in finite points for eliminating y between the curve and the asymptote we have in either case:

$$x^n - x^n + 1 = 0$$

$$1 = 0$$

The curve is symmetrical with respect to both axes. If $n=2$ it is the ordinary hyperbola. For $n > 2$ it falls between the hyperbola $x^2 - y^2 + 1 = 0$ and its asymptotes.

Hence we conclude that the curve has approximately the following form:



16'

$x^n - y^n + 1 = 0$ n, odd

This is changed to I_b where all the coefficients are positive by the transformation $y = -y$.

$$I_a'''$$

$$x^m - y^m - 1 = 0 \qquad n, \text{ even}$$

This is changed to I_a'' by the transformation

$$x = y$$

$$y = x$$

$$I_b'''$$

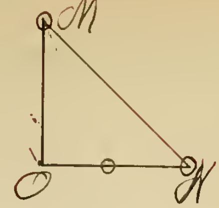
$$x^m - y^m - 1 = 0 \qquad n, \text{ odd}$$

This is changed to I_b by the transformation

$$x = -x$$

Case II

$x^n \pm y^n \pm x^r = 0 \qquad n > r$



There will be four different cases according as the exponents are even or odd.

a) n, r even

b) n even, r odd

c) n odd, r even

d) n, r odd

II a

$x^n + y^n + x^r = 0 \qquad n, r \text{ even}$

The curve is imaginary

II b

$x^n + y^n + x^r = 0 \qquad n \text{ even}, r \text{ odd}$

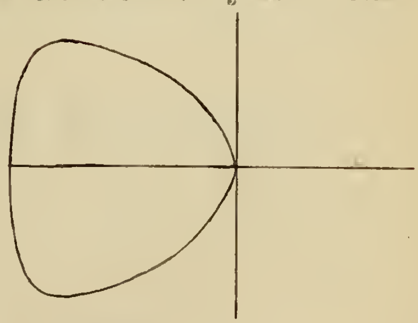
The curve intersects the x-axis at the origin and at $x = -1$

The form at the origin is given by the two terms $y^n + x^r = 0$.

Writing the equation in the form:

$$y^n = -x^n - x^r = -x^r(x^{n-r}+1)$$

we see that the curve is imaginary for $0 < |x| < -1$. It is symmetrical with respect to the x-axis. It does not meet the line at infinity nor the y-axis except at the origin. Hence we conclude that its form is approximately as shown in the figure.



II c

$x^n + y^n + x^r = 0 \qquad n, \text{ odd}, r, \text{ even}$

The curve intersects the x-axis at the origin and at $x = -1$

The form at the origin is given by the two terms

$y^n + x^r = 0$

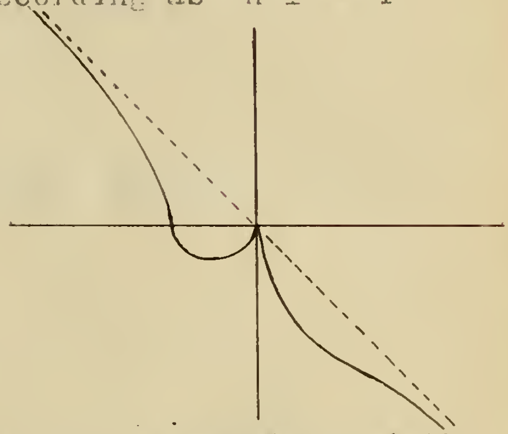
There is a rectilinear asymptote which we will find by the same method as before.

$$x + y = \lim_{\substack{x \doteq \infty,\, y \doteq \infty \\ \frac{x}{y} \doteq -1}} \frac{-x^{r}}{x^{n-1} - x^{n-2}y - \cdots + y^{n-1}}$$

$$x + y = 0 \qquad n - 1 > r$$

$$x + y = -1/n \qquad n - 1 = r$$

Hence there are two forms of the curve according as $n-1 \gtreqqless r$



The asymptote $x + y = 0$ does not cut the curve in any finite point except at the origin. We therefore conclude that its form is approximately as shown on the right when $n-1 > r$

To find the intersections of the asymptote, $x + y = -1/n$, with the curve write the equation of the curve in the form:

$$(x + y)\left[x^{n-1} - x^{n-2}y + x^{n-3}y^{2} \cdots\cdots + y^{n-1}\right] + x^{r} = 0$$

Substituting $-1/n$ for $x + y$ and remembering that $n - 1 = r$ we have:

$$(-n + 1)\, x^{n-1} - x^{n-2}y + x^{n-3}y^{2} - \cdots\cdots y^{n-1} = 0$$

Dividing this equation through by y^{n-1}, substituting z for x/y and changing signs we obtain:

$$(n - 1)\, z^{n-1} + z^{n-2} - z^{n-3} + z^{n-4} \cdots\cdots - 1 = 0 \qquad (1)$$

Transform this to another equation having the same roots with opposite signs:

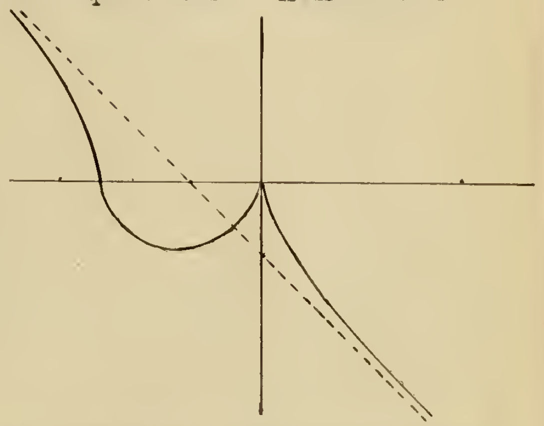
$$(n - 1)\, z^{n-1} - z^{n-2} - z^{n-3} - z^{n-4} \cdots\cdots - 1 = 0 \qquad (2)$$

By Descartes rule of signs equation (2) can have but one positive root and it is easily seen that this root is 1. Equation (1) then has but one negative root -1. A negative value of $z = x/y$ corres-

ponds to an intersection in the second or fourth quadrant and for $x/y = -1$ this intersection is at infinity. Therefore the only finite intersections are in the third quadrant. To find the number of intersections in this quadrant consider the slope of the tangent:

$$\frac{dy}{dx} = \frac{-1}{y^{n-1}} \quad x^{n-2}\left[x + \frac{n-1}{n}\right]$$

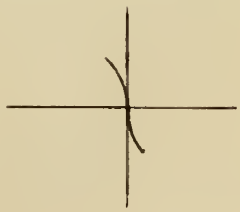
In following the curve to the left from the origin, the slope of the curve is downward for positive values of $\frac{dy}{dx}$ and upward for negative values. Since n is odd it follows that the expression outside the brackets is positive for points in the third quadrant. Hence the curve turns up at $x = \frac{n-1}{n} = 1-1/n$. Since this is beyond the point where the asymptote cuts the x-axis and since the function is single-valued it follows that there is but one intersection of the curve and asymptote in the third quadrant. Therefore we conclude that the curve has approximately the form shown.



II*d*

$$x^n + y^n + x^r = 0 \qquad (n, \ r \ \text{odd})$$

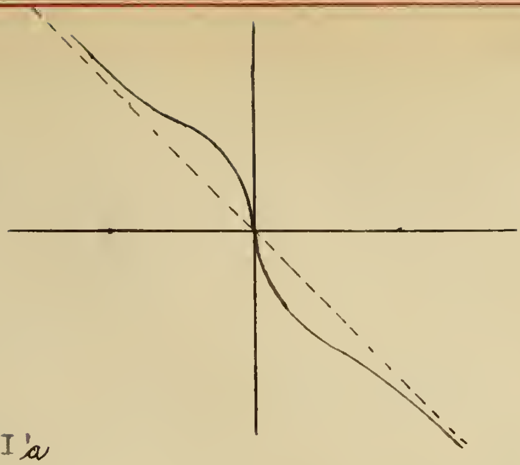
The only intersections of the curve with the axes are at the origin. The form at the origin is given by the two terms $y^n + x^r = 0$ which is the form of a cubical parabola. There is one asymptote:



$$x + y = \lim. \frac{-x^r}{x^{n-1} - x^{n-2}y \ldots + y^{n-1}} = 0$$

but it does not intersect the curve except at the origin. Hence we conclude that the curve has approximately the form shown on next page at the top.

(figure for pp.21)



II'a

$$x^n + y^n - x^r = 0 \qquad n,\ r \text{ even}$$

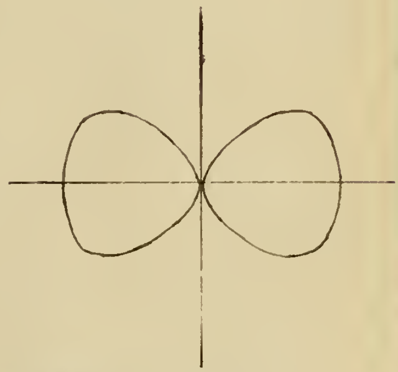
The x-intercepts are ±1 Form at the origin is given by the two terms $y^n - x^r = 0$

The curve is symmetrical with respect to both axes and writing the equation in the form:

$$y^n = x^r - x^n = x^r(1 - x^{n-r})$$

we see that it is imaginary for $|x| > 1$

Hence we conclude that the curve has approximately the accompanying form.



II'b

$$x^n + y^n - x^r = 0 \qquad n \text{ even},\ r \text{ odd}$$

This is changed to IIb by the transformation x=-x

II'c

$$x^n + y^n - x^r = 0 \qquad n \text{ odd},\ r \text{ even}$$

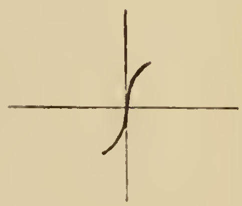
This is changed to IIc by the transformation x = - x, y = - y.

II'd

$$x^n + y^n - x^r = 0 \qquad n,\ r \text{ odd}$$

The curve intersects the x-axis at x = ±1

Form at the origin is given by $y^n - x^r = 0$

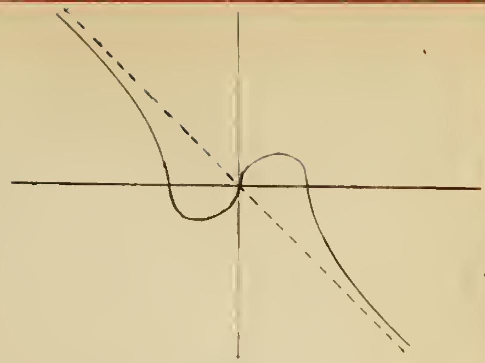


There is one asymptote:

$$x + y = \lim_{\substack{x \doteq \infty \ y \doteq \infty \\ \frac{x}{y} \doteq -1}} \frac{x^r}{x^{n-1} - x^{n-2}y \ldots\ldots + y^{n-1}} = 0$$

which has no finite intersection with the curve except at the origin.

We therefore conclude that the curve has approximately the form shown on the right.



II'a

$$x^n - y^n + x^r = 0 \qquad n, r \text{ even.}$$

The form at the origin is given by $-y^n + x^r = 0$

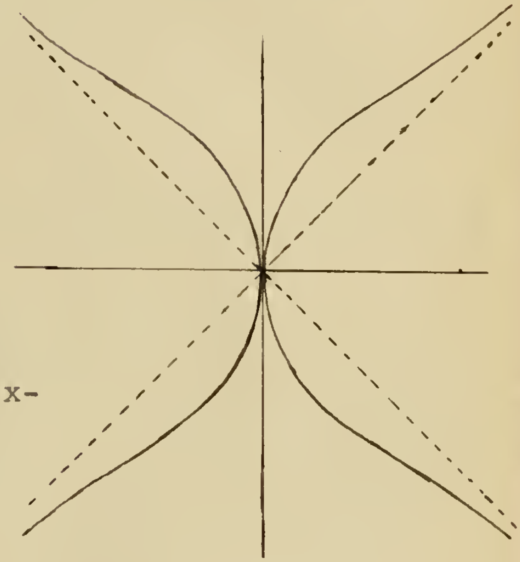
There are two asymptotes:

$$x+y = \lim_{\substack{x \doteq \infty, y \doteq \infty \\ \frac{x}{y} \doteq -1}} \frac{-x^r}{x^{n-1} - x^{n-2}y + \cdots - y^{n-1}} = 0$$

$$x-y = \lim_{\substack{x \doteq \infty, y \doteq \infty \\ \frac{x}{y} \doteq 1}} \frac{-x^r}{x^{n-1} + x^{n-2}y + \cdots + y^{n-1}} = 0$$

Neither asymptote intersects the curve in finite points except at the origin.

Hence we conclude that the curve has approximately the form shown on the right.



II'b

$$x^n - y^n + x^r = 0 \qquad n \text{ even, } r \text{ odd}$$

The x-intercepts are at the origin and at $x = -1$.

Form at the origin is given by $y^n = x^r$

There are two asymptotes:

$$x+y = \lim_{\substack{x \doteq \infty, y \doteq \infty \\ \frac{x}{y} \doteq -1}} \frac{-x^r}{x^{n-1} - x^{n-2}y - \cdots - y^{n-1}}$$

$$= 0 \qquad n - 1 > r$$

$$= -1/n \qquad n - 1 = r$$

$$x-y = \lim_{\substack{x \doteq \infty, y \doteq \infty \\ \frac{x}{y} \doteq 1}} \frac{-x^r}{x^{n-1} + x^{n-2}y \cdots + y^{n-1}}$$

$$= 0 \qquad n - 1 > r$$

$$= -1/n \qquad n - 1 = r$$

The two asymptotes $x + y = 0$ and $x - y = 0$ cut the curve at the

origin but at no other finite points.

We will prove that the line $x - y = -1/n$ cuts the curve in no finite points and from considerations of symmetry it will follow for $x + y = -1/n$

Writing the equation in the form:

$$(x - y)(x^{m-1} + x^{m-2}y \ldots\ldots\ldots y^{m-1}) + x^{n} = 0$$

Substituting $-1/n$ for $x-y$:

$$-1/n\ (x^{m-1} + x^{m-2}y \ldots\ldots\ldots + y^{m-1}) + x^{n} = 0$$

Clearing of fractions, collecting and dividing through by y^{m-1} we obtain:

$$(1 - n)\left(\frac{x}{y}\right)^{m-1} + \left(\frac{x}{y}\right)^{m-2} + \ldots\ldots\ldots + 1 = 0$$

This equation has a positive root 1 which corresponds to the intersection at infinity.

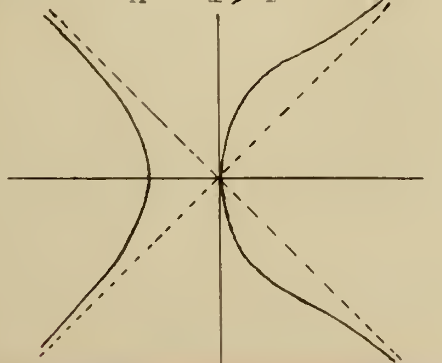
Moreover by Descartes rule of signs there can not be more than one positive root. Hence the only intersection of the curve and asymptote in the first or third quadrant, that is where $\frac{x}{y}$ is positive is at infinity. Writing the equation of the curve in the form:

$$y^{m} = x^{m} + x^{n} = x^{n}(x^{m-n} + 1)$$

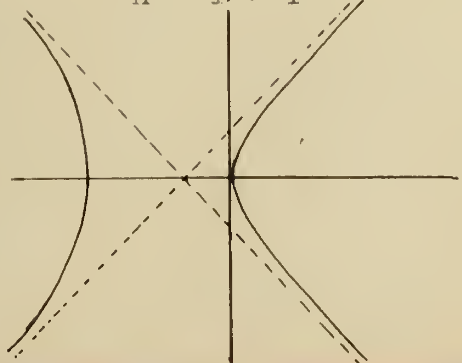
we see that for values of x between 0 and -1 the quanity on the right is negative, hence y is imaginary. Since that part of the asymptote, $x-y=-1/n$, which lies in the third quadrant is between $x=0$ and $x=-1/n$ it cannot cut the curve in this quadrant.

Hence we conclude that the curve has approximately the following forms:

$n - 1 > r$



$n - 1 = r$



II''c

$x^n - y^n + x^r = 0$ n odd r even

This is changed to IIc by the transformation $y = -y$

II''d

$x^n - y^n + x^r = 0$ n, r odd

This is changed to IId by putting $y = -y$

II''a'

$x^n - y^n - x^r = 0$ n, r even

The intercepts are $x = \pm 1$

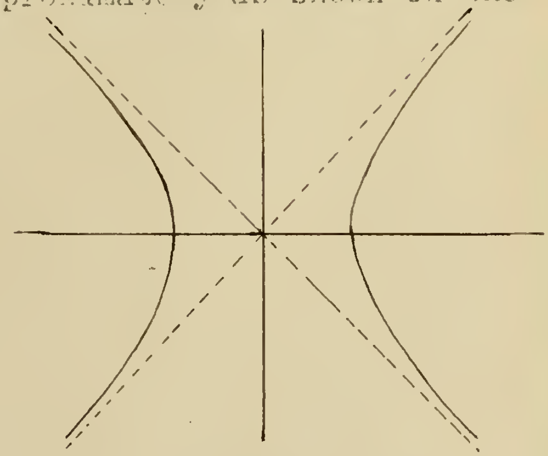
Form at the origin is given by $y^n + x^r = 0$ which is imaginary. Hence the origin is a congugate point.

There are two asymptotes:

$$x + y = \lim_{\substack{x \doteq \infty,\, y \doteq \infty \\ \frac{x}{y} \doteq -1}} \frac{x^r}{x^{n-1} - x^{n-2}y - \cdots - y^{n-1}} = 0$$

$$x - y = \lim_{\substack{x \doteq \infty,\, y \doteq \infty \\ \frac{x}{y} \doteq 1}} \frac{x^r}{x^{n-1} + x^{n-2}y + \cdots + y^{n-1}} = 0$$

Neither of these intersect the curve in finite points. We conclude then that the form of the curve is approximately as shown on the right.



II''b'

$x^n - y^n - x^r = 0$ n even, r odd

This becomes II'b' by the transformation $x = -x$

II''c'

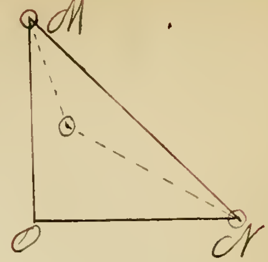
$x^n - y^n - x^r = 0$ n odd, r even

This becomes II by the transformation x = - x

$$II'''_d$$

$$x^n - y^n - x^r = 0 \qquad n, r \text{ odd}$$

This becomes II' by the transformation y = - y



Case III

$$x^n \pm y^n \pm x^N y^s \qquad (n > r+s)$$

We will divide this into sub-cases a and b according as n is odd or even and each of these into four cases according as r and s are odd or even.

a) n is odd

1 r is odd
s is odd

2 r even
s even

3 r odd
s even

4 r even
s odd

b) n is even

1 r is odd
s is odd

2 r even
s even

3 r odd
s even

4 r even
s odd

III a_1

$$x^n + y^n + x^N y^s = 0 \qquad n, r, s \text{ odd.}$$

The only intersections of any of the curves of case III with the axes are at the origin.

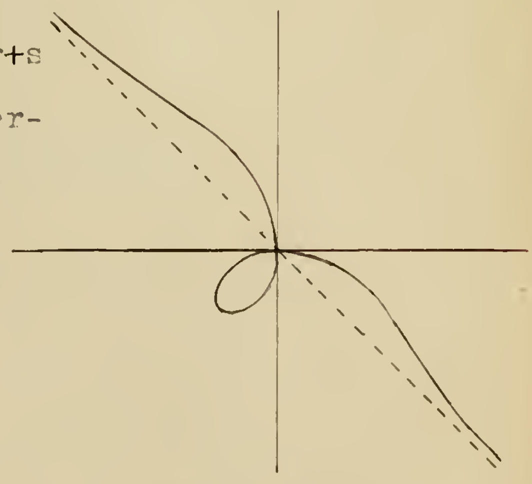
There are two forms at the origin given by $x^n + x^N y^s = 0$ and $y^n + x^N y^s = 0$

The curve has one asymptote:

$$x + y = \lim_{\substack{x \doteq \infty,\, y \doteq \infty \\ \frac{x}{y} \doteq -1}} \frac{-x^N y^s}{x^{n-1} - x^{n-2} y \cdots\cdots + y^{n-1}}$$

$$= 0 \qquad n - 1 > r+s$$

$$= 1/n \qquad n - 1 = r+s$$



If $n - 1 > r + s$ the asymptote $x+y = 0$ intersects the curve only at the origin.

To find the finite intersections of the curve with the asymptote, $x + y = 1/n$, when $n-1 = r+s$ let the equation be written in the form:

$$(x+y)(x^{n-1} - x^{n-2}y \cdots\cdots\cdots + y^{n-1}) + x^N y^s = 0 \qquad (1)$$

Substitute $1/n$ for $x+y$ in this equation, divide through by y^{m-1}, substitute z for x/y and collect.

The result is:

$$z^{m-1} - z^{m-2} + z^{m-3} - \ldots\ldots + (n-1)z^{n} \ldots\ldots + 1 = 0 \qquad (2)$$

Since the curve cannot extend into the first quadrant we are interested only in the negative roots of this equation, that is, for x and y of opposite signs. Transforming (2) into another equation having the same roots with opposite signs, we obtain:

$$z^{m-1} + z^{m-2} + z^{m-3} \ldots\ldots + (1-n)z^{n} \ldots\ldots + 1 = 0 \qquad (3)$$

This equation, by Descartes rule of signs, can have at most two positive roots. By substitution we find that 1 is a root.

Hence -1 is a root of (2) and this corresponds to an intersection at infinity. Since (3) has one positive root and the last term is positive there must be another positive root. Dividing equation (3) by x - 1 by Horners method we have:

$$\begin{array}{cccccccc|c} 1z^{m-1} & +1z^{m-2} & +1z^{m-3} & \ldots & +1z^{n+1} & +(1-n)z^{n} & 1z^{n-1} \ldots & +1 & 1 \\ & 1 & 2 & & (s-1) & s & -n+s+1 & & \\ \hline & 2 & 3 & & s & -n+s+1 & -n+s+2 & 0 & \end{array}$$

The depressed equation is then:

$$z^{m-2} + 2z^{m-3} + 3z^{m-4} \ldots\ldots + sz^{n} - \{n-(s+1)\} z^{n-1} - \{n-(s+2)\} z^{n-2} \ldots\ldots - 1 = 0$$

Since, in this equation, $f(0) = -1$ the equation will have a root between 0 and 1 if;

$$1 + 2 + 3 \ldots\ldots + s > 1 + 2 + 3 \ldots\ldots + \{n - (s+1)\}.$$

That is, if;

$$(s/2)(s+1) > \frac{\{n-(s+1)\}(n-s)}{2}$$

$$s^2 + s > n^2 - 2ns \quad s^2 - n - s$$

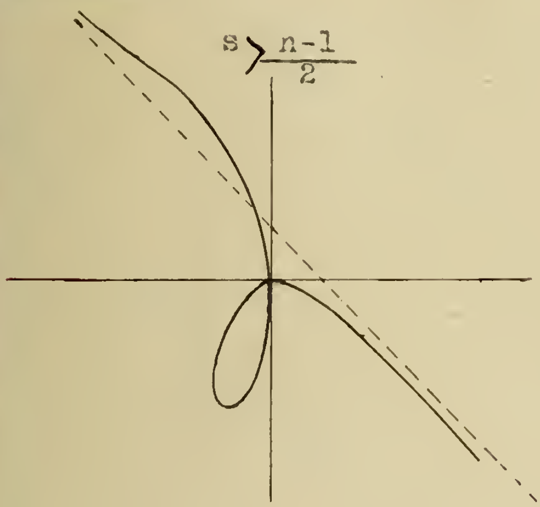
$$0 > n^2 - 2ns - n$$

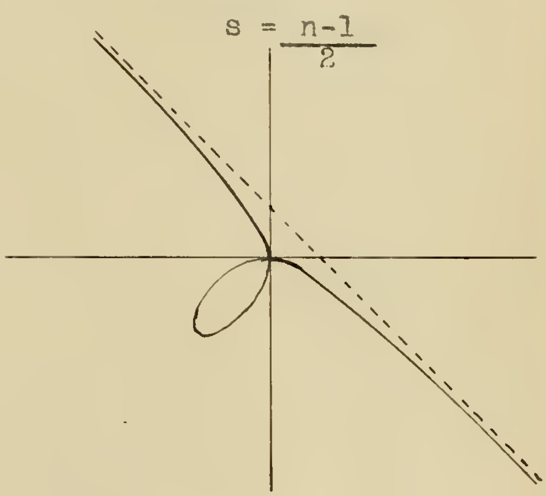
or $$s > \frac{n-1}{2}$$

If this inequality holds it means that at the point of intersection of the curve and asymptote $\left|\frac{x}{y}\right| < 1$ or $|x| < |y|$ and therefore corresponds to an intersection in the second quadrant.

If $s < \frac{n-1}{2}$ then $\left|\frac{x}{y}\right| > 1$ and the intersection is in the fourth quadrant, but the curve can in this case be projected into the other where $s > \frac{n-1}{2}$ by an interchange of x and y. If $s = \frac{n-1}{2}$ then 1 is a double root of equation (3), -1 is a double root of equation (2) and the only intersections of the curve and asymptote are at infinity.

Hence we conclude that the curve has approximately the following forms:





III$_{a_2}$

$$x^n + y^n + x^r y^s = 0 \qquad n, \text{ odd} \quad r, s \text{ even}$$

Forms at the origin are given by $x^n + x^r y^s = 0$ and $y^n + x^r y^s = 0$

There is one asymptote:

$$x + y = \lim_{\substack{x \doteq \infty,\ y \doteq \infty \\ \frac{x}{y} \doteq -1}} \frac{-x^r y^s}{x^{n-1} - x^{n-2} y + \cdots + y^{n-1}}$$

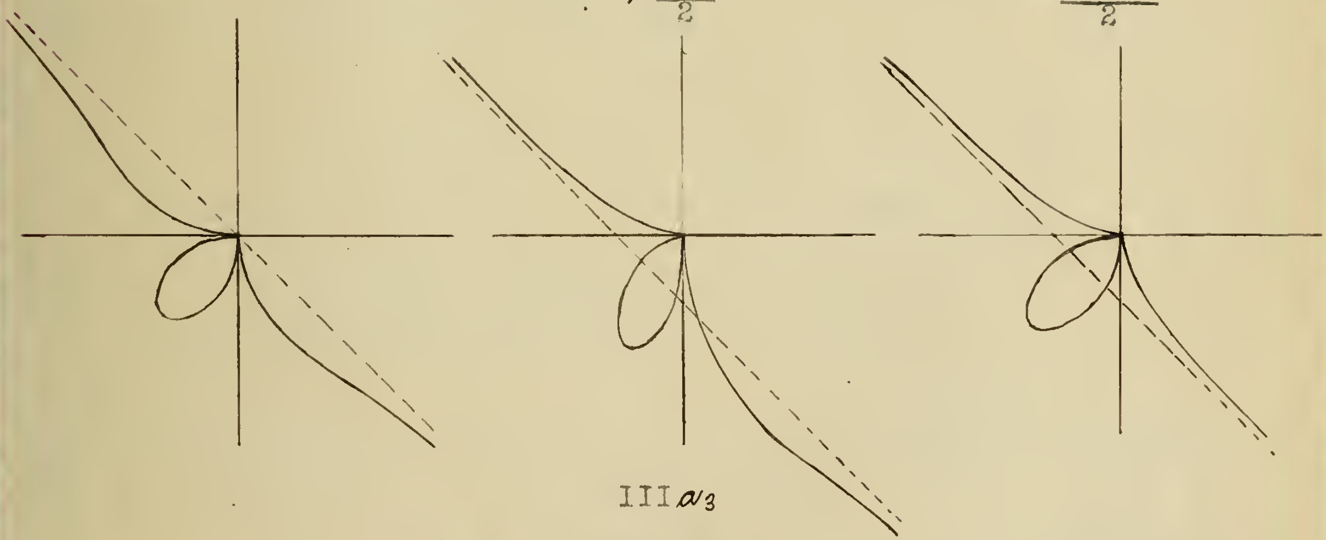
$$= 0 \qquad (n - 1 > r + s)$$

$$= -1/n \qquad (n - 1 = r + s)$$

The asymptote $x + y = 0$ does not intersect the curve at any finite point except the origin. If $n-1=r+s$ then it can be shown in the

same manner as in the previous case that if $s > \frac{n-1}{2}$, the asymptote $x+y = -1/n$ intersects the curve in a finite point in the fourth quadrant and if $s < \frac{n-1}{2}$ the intersection is in the second quadrant. This latter case, however, will be changed to the former if we interchange x and y. If $s = \frac{n-1}{2}$ we can show as in case IIIα_2 that there are no finite intersections in the second or fourth quadrants. If $n-1=r+s$ these curves, for all values of n, pass through the point $(-\frac{1}{2}, -\frac{1}{2})$. The distance of the asymptote from the origin is $\frac{\sqrt{2}}{2n}$ hence that part of the curve in the third quadrant always intersects the asymptote. Hence we conclude that the curve has approximately the following forms:

$n - 1 > r+s$ $\qquad$ $n - 1 = r+s$: $s > \frac{n-1}{2}$, $\quad s = \frac{n-1}{2}$



IIIα_3

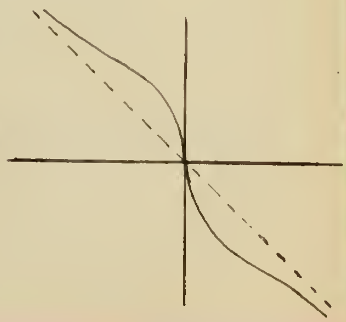
$$x^n + y^n + x^r y^s = 0 \qquad n, r \text{ odd}, s \text{ even}$$

Of the two forms at the origin $x^n + x^r y^s = 0$ and $y^n + x^r y^s = 0$ the first is imaginary and the second is like a cubical parabola.

The asymptote is:

$$x+y = \lim_{\substack{x \doteq \infty, y \doteq \infty \\ \frac{x}{y} \doteq -1}} \frac{-x^r y^s}{x^{n-1} - x^{n-2}y \ldots\ldots + y^{n-1}} = 0$$

It's only finite intersection with the curve is at the origin.



IIIa_4

$x^n + y^n + x^r y^s = 0$ n,s odd, r even

This becomes the same as IIIa_3 by an interchange of x and y.

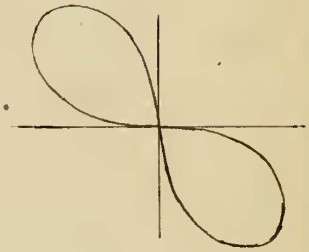
IIIb_1

$x^n + y^n + x^r y^s = 0$ n even, r, s odd

The two forms at the origin $x^n + x^r y^s = 0$ and $y^n + x^r y^s = 0$ are like cubical parabolas.

There are no asymptotes.

The curve is symmetrical with respect to the origin.



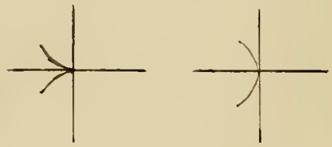
IIIb_2

$x^n + y^n + x^r y^s = 0$ n, r, s even

The curve in this case is imaginary.

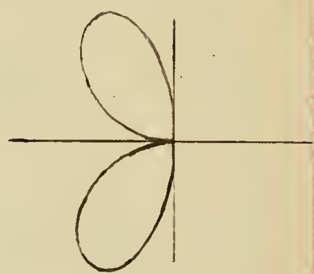
IIIb_3

$x^n + y^n + x^r y^s = 0$ n, s even r, odd



The forms at the origin $x^n + x^r y^s = 0$ and $y^n + x^r y^s = 0$ are like the semi-cubical parabola the ordinary parabola respectively. There are no asymptotes.

The curve is symmetrical with respect to the x-axis.



IIIb_4

$x^n + y^n + x^r y^s = 0$ n, r odd, s even

This becomes the same as IIIb_3 by an interchange of x and y.

III'a_1

$x^n + y^n - x^r y^s = 0$ n, r, s odd

This is changed to IIIa_1 by the transformation x=-x, y=-y.

III'a_2

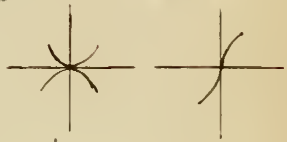
$x^n + y^n - x^r y^s = 0$ n odd, r, s even

This is changed to IIIa_2 by the transformation x=-x, y=-y.

III$'a_3$

$x^n + y^n - x^r y^s = 0$ n, r odd, s even.

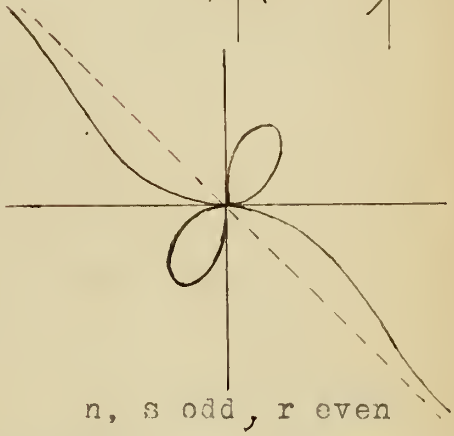
The curve is symmetrical with respect to the origin. The two forms at the origin, $x^n - x^r y^s = 0$ and $y^n - x^r y^s = 0$ are as shown on the right.



The curve has one asymptote:

$$x + y = \lim_{\substack{x \doteq \infty,\, y \doteq \infty \\ \frac{x}{y} \doteq -1}} \frac{x^r y^s}{x^{n-1} - x^{n-2} y \ldots\ldots + y^{n-1}} = 0$$

which has no finite intersection with the curve except at the origin.



III$'a_4$

$x^n + y^n - x^r y^s = 0$ n, s odd, r even

This becomes the same as III$'a_3$ by an interchange of x and y.

III$'b_1$

$x^n + y^n - x^r y^s = 0$ n even, r, s odd.

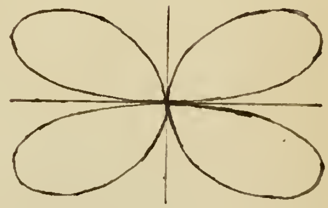
This is transformed to IIIb_1 by putting x = - x.

III$'b_2$

$x^n + y^n - x^r y^s = 0$. n, r, s even.

The two forms at the origin $x^n - x^r y^s = 0$ and $y^n - x^r y^s = 0$ are as shown on the right.

The curve is symmetrical with respect to both axes.



III$'b_3$

$x^n + y^n - x^r y^s = 0$ n, s even, r odd

This is transformed to IIIb_3 by putting x = - x.

III$'b_4$

$x^n + y^n - x^r y^s = 0$ n, r even, s odd

This is transformed to IIIb_3 by putting x = y, y = - x.

$$III''_{a_1}$$

$$x^n - y^n + x^r y^s = 0 \qquad n,\ r,\ s,\ \text{odd}$$

This is transformed to III by putting $x = -x$.

$$III''_{a_2}$$

$$x^n - y^n + x^r y^s = 0 \qquad n\ \text{odd},\ r,\ s\ \text{even}$$

This is transformed to III by putting $y = -y$.

$$III''_{a_3}$$

$$x^n - y^n + x^r y^s = 0 \qquad n,\ r\ \text{odd}\quad s\ \text{even}$$

This is transformed to III by putting $y = -y$.

$$III''_{a_4}$$

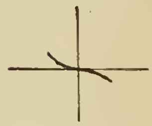
$$x^n - y^n + x^r y^s = 0 \qquad n,\ s\ \text{odd},\ r\ \text{even}$$

This is transformed to III' by putting $x = -x$.

$$III''_{b_1}$$

$$x^n - y^n + x^r y^s = 0 \qquad n\ \text{even},\ r,\ s\ \text{odd}$$

The curve is symmetrical with respect to the origin. Forms at the origin $x^n + x^r y^s = 0$ and $y^n - x^r y^s = 0$ are like cubical parabolas.

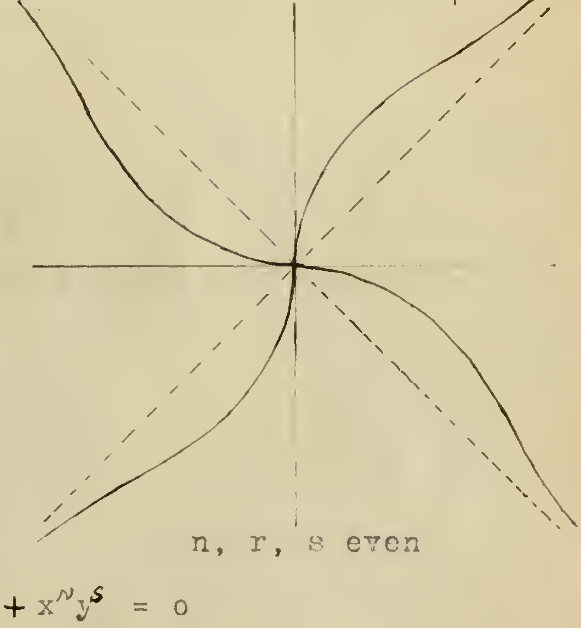


There are two asymptotes:

$$x + y = \lim_{\substack{x \doteq \infty,\ y \doteq \infty \\ \frac{x}{y} \doteq -1}} \frac{-x^r y^s}{x^{n-1} - x^{n-2}y - \cdots - y^{n-1}} = 0$$

$$x - y = \lim_{\substack{x \doteq \infty\ y \doteq \infty \\ \frac{x}{y} \doteq 1}} \frac{-x^r y^s}{x^{n-1} + x^{n-2}y + \cdots + y^{n-1}} = 0$$

Neither of these intersect the curve except at the origin. Hence we conclude that the curve has approximately the form shown.



$$III''_{b_2}$$

$$x^n - y^n + x^r y^s = 0 \qquad n,\ r,\ s\ \text{even}$$

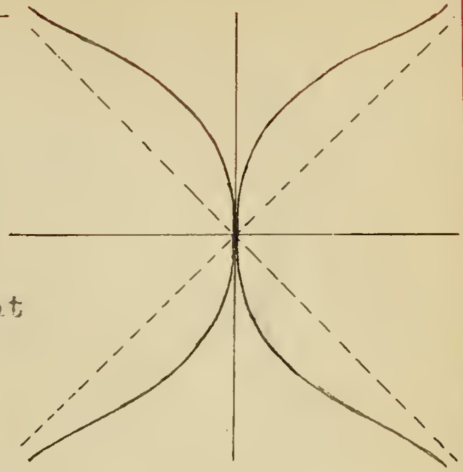
Of the two forms at the origin $x^n + x^r y^s = 0$ and $y^n - x^r y^s = 0$ the first is imaginary and the

second is as shown on the right.

There are two asymptotes :

$$x + y = \lim_{\substack{x \doteq \infty,\, y \doteq \infty \\ \frac{x}{y} \doteq -1}} \frac{-x^{r} y^{s}}{x^{n-1} - x^{n-2}y \qquad -y^{n-1}} = 0$$

$$x - y = \lim_{\substack{x \doteq \infty,\, y \doteq \infty \\ \frac{x}{y} \doteq 1}} \frac{-x^{r} y^{s}}{x^{n-1} + x^{n-2}y \qquad +y^{n-1}} = 0$$



Neither of these intersect the curve except at the origin. Hence we may conclude that the form is approximately as shown on the right.

III'b'_3

$$x^n - y^n + x^r y^s = 0 \qquad n,\ s \text{ even},\ r \text{ odd}$$

The two forms at the origin $x^n + x^r y^s = 0$ and $y^n - x^r y^s = 0$ are as shown on the right. There are two asymptotes:

$$x + y = \lim_{\substack{x \doteq \infty,\, y \doteq \infty \\ \frac{x}{y} \doteq -1}} \frac{-x^{r} y^{s}}{x^{n-1} - x^{n-2}y \,.\,.\,.\,.\,.\,. -y^{n-1}}$$

$$= 0 \qquad n - 1 > r + s$$

$$= -\frac{1}{n} \qquad n - 1 = r + s$$

$$x - y = \lim_{\substack{x \doteq \infty,\, y \doteq \infty \\ \frac{x}{y} \doteq 1}} \frac{-x^{r} y^{s}}{x^{n-1} + x^{n-2}y \,.\,.\,.\,.\,. +y^{n-1}}$$

$$= 0 \qquad n - 1 > r + s$$

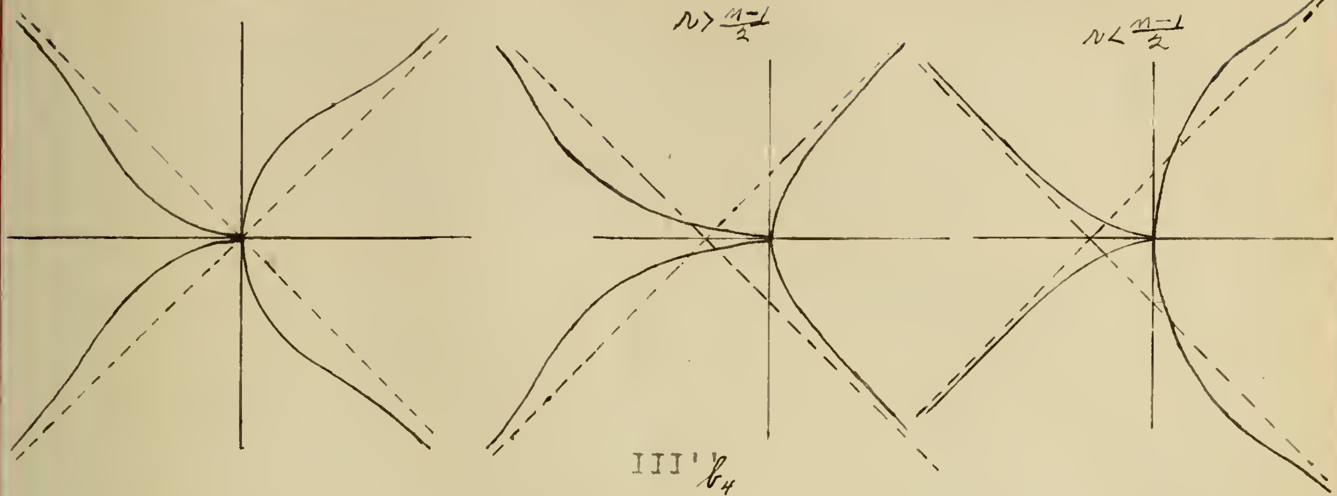
$$= -\frac{1}{n} \qquad n - 1 = r + s$$

The asymptotes $x + y = 0$ and $x - y = 0$ intersect the curve in no finite points except the origin.

By the same method that was used in case IIIa_1 it can be shown that $x + y = -1/n$ has a finite intersection with the curve in the second quadrant if $r > \frac{n-1}{2}$, $s < \frac{n-1}{2}$ and from symmetry it follows that in this case $x - y = -1/n$ has a finite intersection in the third quadrant. If $r < \frac{n-1}{2}$, $s > \frac{n-1}{2}$ and then $x + y = -1/n$ has a finite intersection in the fourth quadrant and $x - y = -1/n$ in the first quadrant. Hence we conclude that the forms are approximately

as shown in the following figures.

$n - 1 > r + s$ $\qquad\qquad$ $n - 1 = r + s$



III'$_{b_4}$

$$x^n - y^n + x^r y^s = 0 \qquad n,\ r \text{ even},\ s \text{ odd}$$

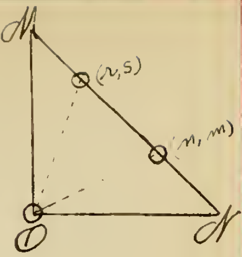
This is changed to III'$_{b_3}$ by putting $x = y$, $y = -x$

III'''

$$x^n - y^n - x^r y^s = 0$$

This becomes the same as III'' by an interchange of x and y.

Case IV.



$$x^n y^m \pm x^r y^s \pm 1 = 0 \qquad n+m = r+s$$

We will divide this into four sub-cases: a, b, c and d according as n and m are even or odd and each of these into two others according as r and s are even or odd.

a) n, m even

1. r, even; s, even
2. r, odd; s, odd

b) n even, m odd

1. r, even; s, odd
2. r, odd; s, even

c) n, odd, m, even

1. r, odd; s, even
2. r, even; s, odd

d) n, m odd

1. r, even; s, even
2. r, odd; s, odd

We shall suppose throughout that $n > r$ and therefore $m < s$.

IVa_1

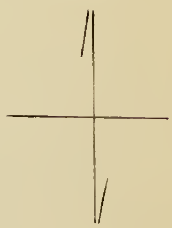
$$x^n y^m + x^r y^s + 1 = 0 \qquad n,\ m,\ r,\ s \text{ even}$$

The curve is imaginary.

IVa_2

$$x^n y^m + x^r y^s + 1 = 0 \qquad n,\ m \text{ even},\ r,\ s \text{ odd}$$

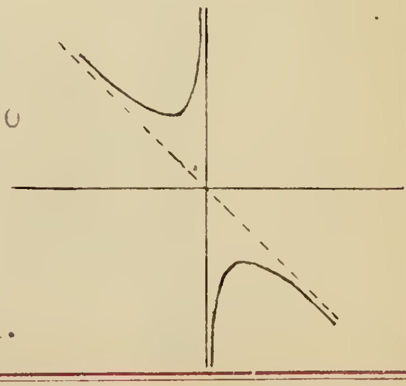
There are no finite intersections with the axes.



The form at infinity on the x-axis $x^n y^m + 1 = 0$ is imaginary. The form at infinity on the y-axis $x^r y^s + 1 = 0$ is as shown on the right.

There is one asymptote:

$$x + y = \lim_{\substack{x \doteq \infty,\ y \doteq \infty \\ \frac{x}{y} \doteq -1}} \frac{-1}{x^{n-1}y^m - x^{n-2}y^{m+1} \qquad + x^r y^{s-1}} = 0$$

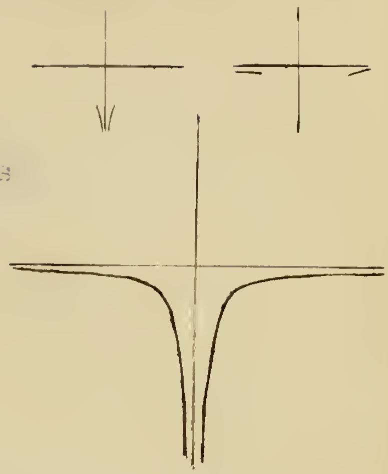


It has no finite intersections with the curve. Since the terms are all of even degree the curve is symmetrical with respect to the origin.

IVb_1

$x^n y^m + x^N y^s + 1 = 0$ n, r even, m, s odd

Form at infinity on the x-axis is given by $x^n y^m + 1 = 0$ and at infinity on the y-axis by $x^\nu y^s + 1 = 0$. The curve cannot cross the axes and lies entirely in the third and fourth quadrants. Hence we conclude that it has approximately the form indicated.



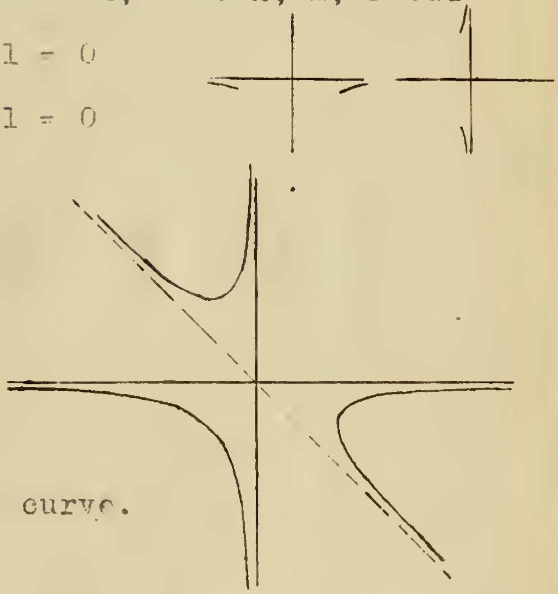
IVb_2

$x^n y^m + x^N y^s + 1 = 0$ n, s even, m, r odd

Form at infinity on the x-axis is $x^n y^m + 1 = 0$
Form at infinity on the y-axis is $x^N y^s + 1 = 0$
These are as shown on the right.
There is an asymptote which by the same method as before we find to be:

$$x + y = 0$$

It has no finite intersections with the curve.



IVc_1

$x^n y^m + x^N y^s + 1 = 0$ n, r odd, m, s even

This is transformed to IVb_1 by an interchange of x and y.

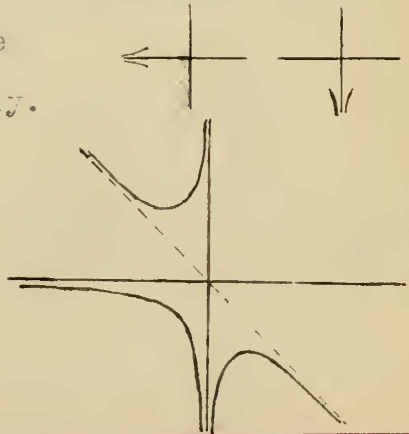
IVc_2

$x^n y^m + x^N y^s + 1 = 0$ n, s odd, m, r even

Forms at infinity on the x-and y-axes are given by $x^n y^m + 1 = 0$ and $x^N y^s + 1 = 0$ respectively. These are as shown on the right.

There is one asymptote which, by the method used before, is found to be:

$$x + y = 0.$$



It does not cut the curve in any finite point. Hence we conclude that the form of the curve is approximately as indicated.

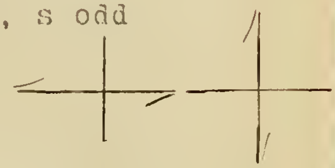
IVd_1

$$x^n y^m + x^r y^s + 1 = 0 \qquad n,\ m \text{ odd},\ r,\ s \text{ even}$$

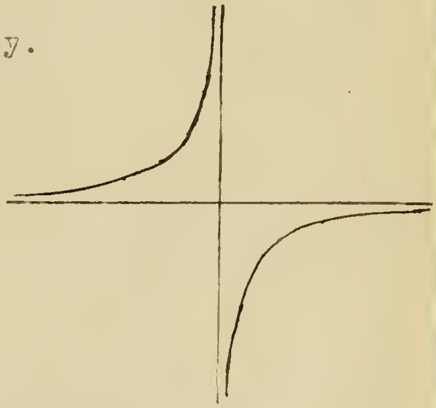
This is transformed to IVa_1 by an interchange of x and y.

IVd_2

$$x^n y^m + x^r y^s + 1 = 0 \qquad n,\ m,\ r,\ s \text{ odd}$$



The axes are the only asymptotes. The forms at infinity on the x- and y-axis are given by $x^n y^m + 1 = 0$ and $x^r y^s + 1 = 0$ respectively. These are as shown on the right.

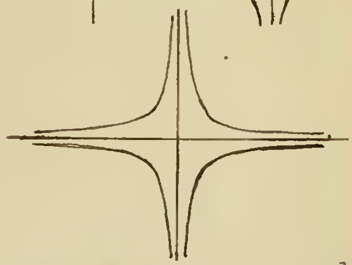


The curve is symmetrical with respect to the origin. Hence we conclude that it has approximately the form indicated.

IV'a_1

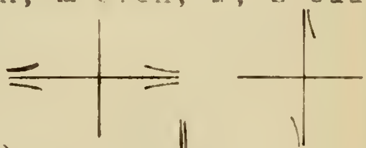
$$x^n y^m + x\ y - 1 = 0 \qquad n,\ m,\ r,\ s \text{ even}$$

The axes are the only asymptotes. The forms at infinity on the x- and y-axis, $x^n y^m - 1 = 0$ and $x\ y - 1 = 0$ are as indicated on the right. The curve is symmetrical with respect to both axes.



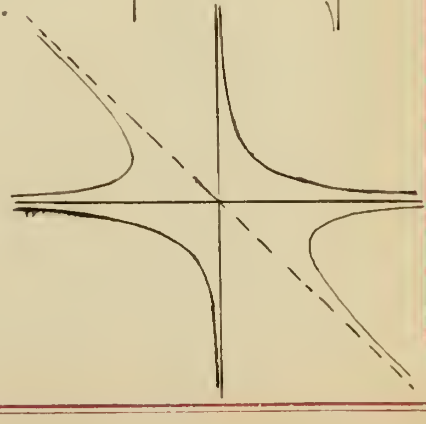
IV'a_2

$$x\ y\ \ x\ y - 1 = 0 \qquad n,\ m \text{ even},\ r,\ s \text{ odd}$$



The form at infinity on the x-axis is given by $x^n y^m - 1 = 0$ and on the y-axis by $x^r y^s - 1 - 0$. These are as shown on the right. There is one asymptote and by the method used before its equation is easily found to be:

$$x + y = 0$$



It intersects the curve in no finite points.

The curve is symmetrical with respect to the origin.

IV'b_1

$x^n y^m + x^r y^s - 1 = 0$ n, r even, m, s odd

This is changed to IVb_1 by the transformation $y = -y$

IV'b_2

$x^n y^m + x^r y^s - 1 = 0$ n, s even m, r odd

This is changed to IVb_2 by the transformation $x = -x, y = -y$

IV'c_1

$x^n y^m + x^r y^s - 1 = 0$ n, r odd, m, s even

This is changed to IV by the transformation $x = -x$, $y = -y$.

IV'c_2

$x^n y^m + x^r y^s - 1 - 0$ n, s odd, m, r even

This is changed to IVc_2 by the transformation $x = -x$, $y = -y$.

IV'd_1

$x^n y^m + x^r y^s - 1 = 0$ n, m odd r, s even

This becomes the same as IV'a_2 by an interchange of x and y.

IV'd_2

$x^n y^m + x^r y^s - 1 = 0$ n, m, r, s odd

This is transformed to IVd_2 by putting $x = -x$.

IV''a_1

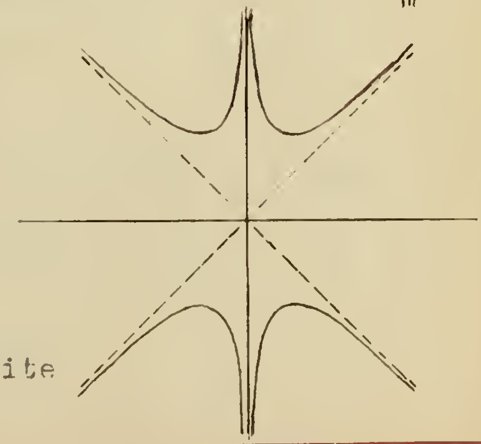
$x^n y^m - x^r y^s + 1 = 0$ n, m, r, s even

The form at infinity on the x-axis $x^n y^m + 1 = 0$ is imaginary. The form at infinity on the y-axis $x^r y^s - 1 = 0$ is as shown on the right.

There are two asymptotes which, by the same method as before, we find to be:

$x + y = 0$

$x - y = 0$



Neither of these intersect the curve in finite

points. Hence we conclude that the curve has approximately the form indicated.

IV'_{a_2}

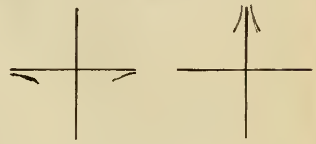
$$x^n y^m - x^r y^s + 1 = 0 \qquad n, m \text{ even}, r, s \text{ odd}$$

This is transformed to IV_{a_2} by putting x = - x.

IV'_{b_1}

$$x^n y^m - x^r y^s + 1 = 0 \qquad n, r \text{ even}, m, s \text{ odd}$$

The forms at infinity on the x- and y-axes given by $x^n y^m + 1 = 0$ and $x^r y^s - 1 = 0$ respectively are as shown on the right.

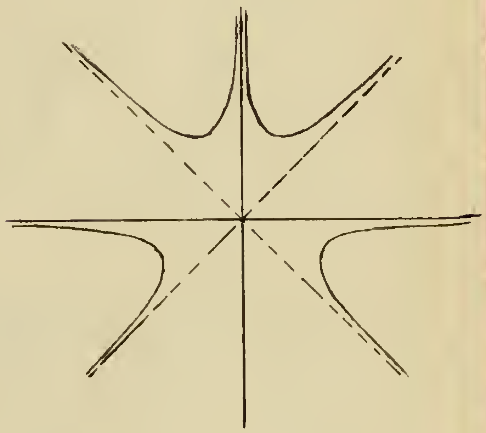


There are two asymptotes, found by the same method as before to be:

$$x + y = 0$$

$$x - y = 0$$

Neither of these intersect the curve in finite points. Hence we conclude that the curve has approximately the form indicated on the right.



IV'_{b_2}

$$x^n y^m - x^r y^s + 1 = 0 \qquad n, s \text{ even}, m, r \text{ odd}$$

This is changed to IV_{b_2} by putting x = - x.

IV'_{c_1}

$$x^n y^m - x^r y^s + 1 = 0 \qquad n, r \text{ odd}, m, s \text{ even}$$

This is changed to IV'_{b_1} by the transformation: x = - y, y = x.

IV'_{c_2}

$$x^n y^m - x^r y^s + 1 = 0 \qquad n, s \text{ odd}, m, r \text{ even}$$

This is changed to IV'_{b_2} by the transformation: x = - y, y = x.

IV''_{d_1}

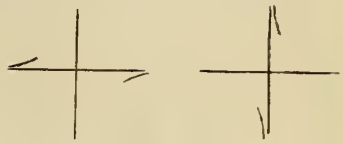
$x^n y^m - x^r y^s + 1 = 0$ n, m odd, r, s even

This is changed to IV''_{d_2} by the transformation x = y, y = - x.

IV''_{d_2}

$x^n y^m - x^r y^s + 1 = 0$ n, m, r, s odd

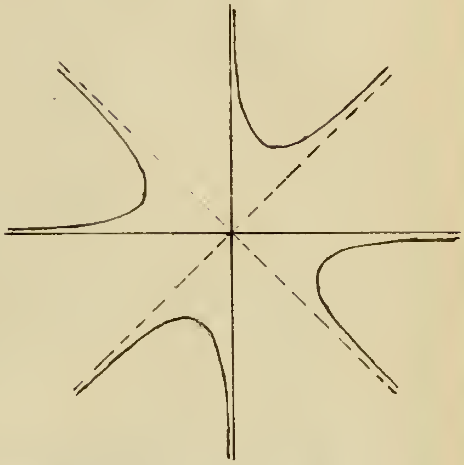
The forms at infinity on the axes given by $x^n y^m + 1 = 0$ and $x^r y^s - 1 = 0$ are as shown on the right.



There are two asymptotes:

$$x + y = 0$$

$$x - y = 0$$



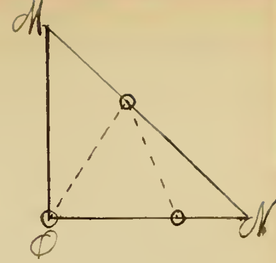
Neither of thses intersect the curve in finite points. The curve is symmetrical with respect to the origin.

IV'''

$x^n y^m - x^r y^s - 1 = 0$

This becomes the same as IV'' by an interchange of x and y.

Case V.



$$x^n y^m \pm r \pm 1 = 0$$

We will divide this into two sub-cases a and b according as n is even or odd and each of these into four other according as m and r are even or odd.

a) n even		b) n odd	
1	m even, r even	1	m even, r even
2	m even, r odd	2	m even, r odd
3	m odd, r even	3	m odd, r even
4	m odd, r odd	4	m odd, r odd

None of these curves have any finite intersections with the y-axis. The slope of the tangent is

$$\frac{dy}{dx} = -\frac{n x^{n-1} y^m \pm r x^{r-1}}{m x^n y^{m-1}}$$

which is infinite for a point at which the curve crosses the x-axis unless m = 1. Hence the curve crosses the x-axis at right angles unless m = 1. In drawing the curves we shall suppose that $m > 1$.

$\mathrm{V}a_1$

$$x^n y^m + x^r + 1 = 0 \qquad n,\ m,\ r \text{ even}$$

This curve is clearly imaginary.

$\mathrm{V}a_2$

$$x^n y^m + x^r + 1 = 0 \qquad n,\ m \text{ even},\ r \text{ odd}$$

The curve intersects the x-axis at $x = -1$.

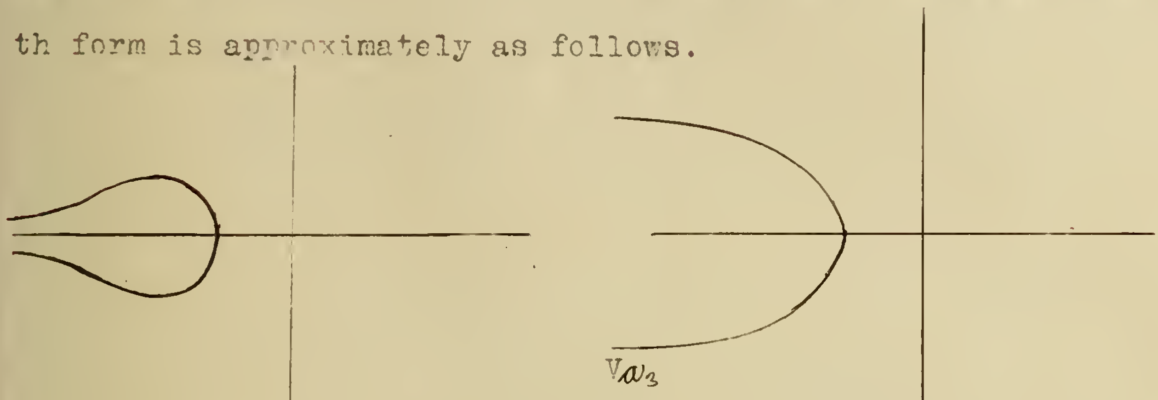
The form at infinity on the y-axis $x^n y^m + 1 = 0$ is imaginary, since n and m are even. The form at infinity on the x-axis is given by $x^n y^m + x^r = 0$ or since x is not zero we may devide through by x^r and write $y^m = \dfrac{-1}{x^{n-r}}$.

This has two forms according as $n \gtrless r$. They are shown on first page.

Writing the equation in the form:

$$y^m = \frac{-(x^r+1)}{x^n}$$

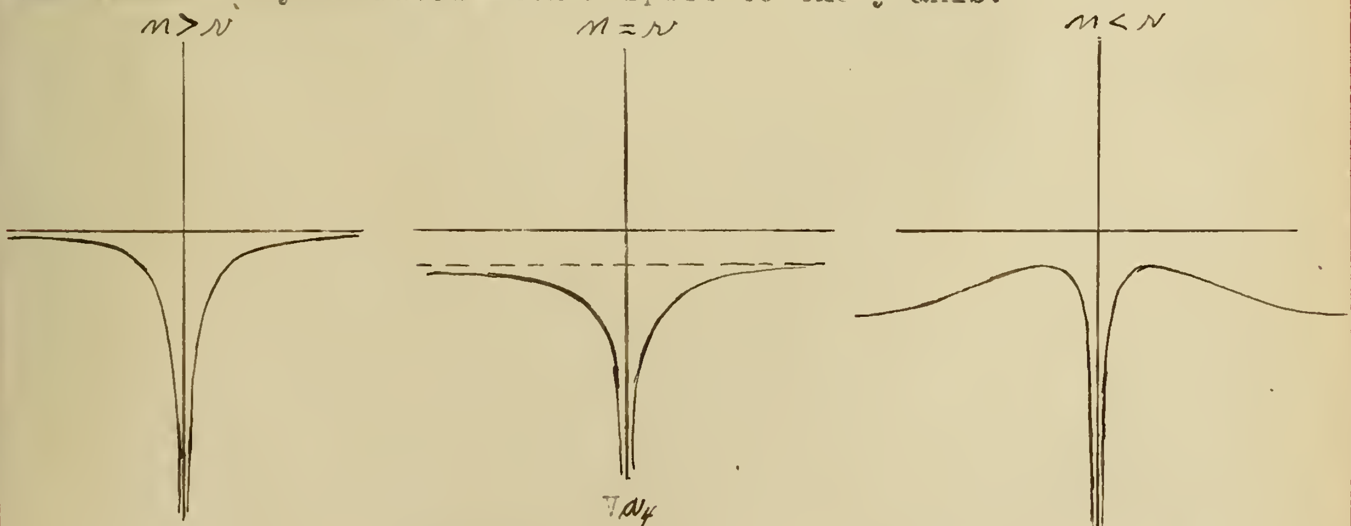
we see that for $x > -1$ the curve is imaginary. Hence we conclude that th form is approximately as follows.



$x^n y^m + x^r + 1 = 0$ n, r even, m odd

The form at infinity on the y-axis is $x^n y^m + 1 = 0$. The form at infinity on the x-axis is $x^n y^m + x^r = 0$. This takes three forms according as $n \gtreqless r$.

The curve is symmetrical with respect to the y-axis.



$x^n y^m + x^r + 1 = 0$ n even, m, r odd

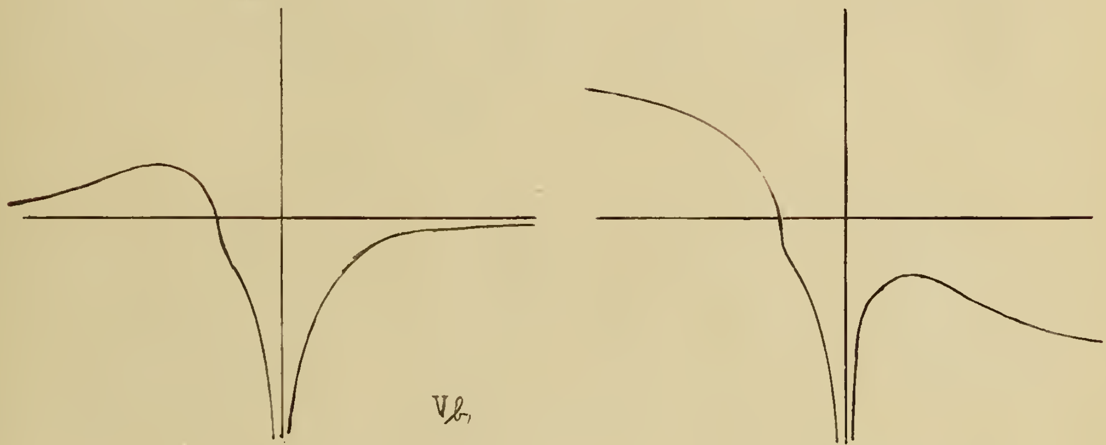
The curve intersects the x-axis at x = - 1.

The form at infinity on the y-axis is given by $x^n y^m + 1 = 0$, and at infinity on the x-axis by $x^n y^m + x^N = 0$. The latter form depends upon whether $n \gtrless r$.

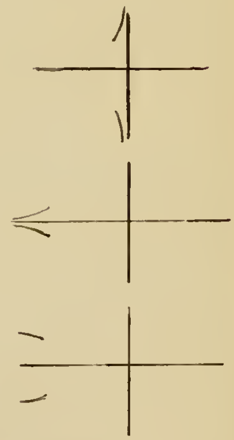
$n > r$ $\qquad$ $n < r$

We conclude that the form of the curve is approximately as follows.

$n > r$ $\qquad$ $n < r$



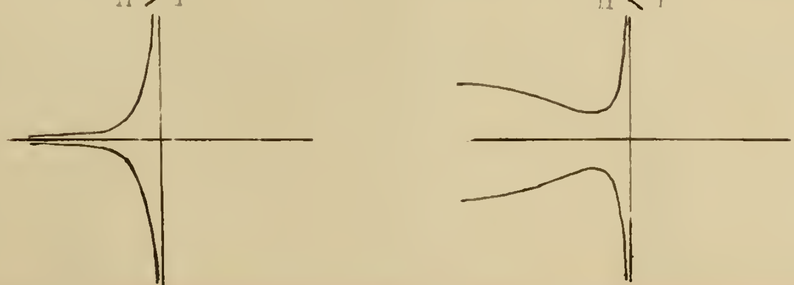
$x^n y^m + x^N + 1 = 0$ $\qquad$ m, r even, n odd



There are no finite intersections with the axes. The curve is symmetrical with respect to the x-axis and cannot extend into the first or fourth quadrants. The form at infinity on the y-axis is given by $x^n y^m + 1 = 0$. The form at infinity on the x-axis $x^n y^m + x^N = 0$, divides into two cases according as $n \gtrless r$. These forms are as shown on the right.

Hence we conclude that the curve has the following forms.

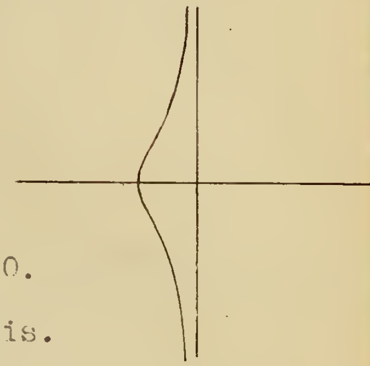
$n > r$ $\qquad$ $n < r$



$\mathrm{V}b_2$

$x^n y^m + x^n + 1 = 0$ n, r odd, m even

The curve cuts the x-axis at x = - 1. Form at infinity on the x-axis is $x^n y^m + x^n = 0$ or $y^m = -1/x^{n-r}$. Since m and n-r are even this is imaginary. Form at infinity on the y-axis is $x^m y^m + 1 = 0$.



Writing the equation in the form:

$$y^m = -\frac{1}{x^m} - x^{n-1}$$

we see that the curve is imaginary for $-1 > x > 0$. The curve is symmetrical with respect to the x-axis.

$\mathrm{V}b_3$

$x^m y^m + x^n + 1 = 0$ m, n odd, r even.

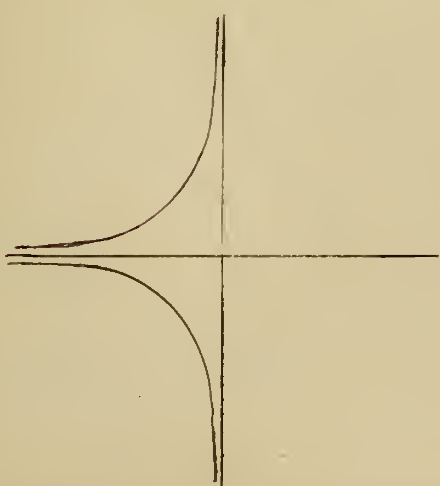
There are no finite intersections with the axes and the curve is symmetrical with respect to the origin. Form at infinity on the y-axis is given by $x^m y^m + 1 = 0$ and at infinity on the x-axis by $x^m y^m + x^n = 0$. The last form depends upon whether $n \gtrless r$

n > r

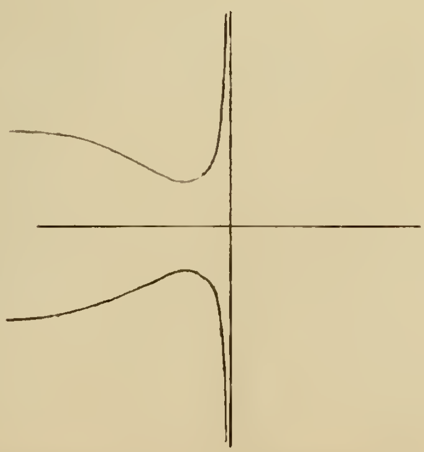
n < r

Hence we conclude that the curve has approximately the following forms:

n > r



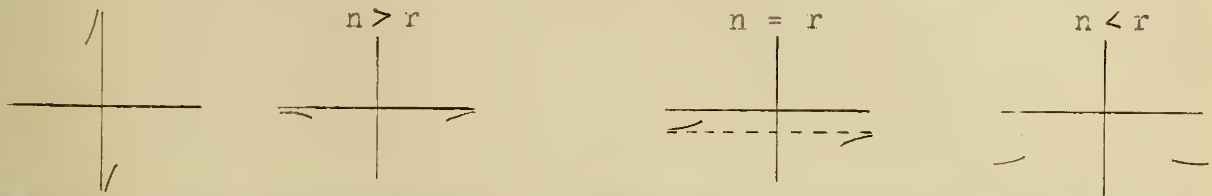
n < r



Vb_4

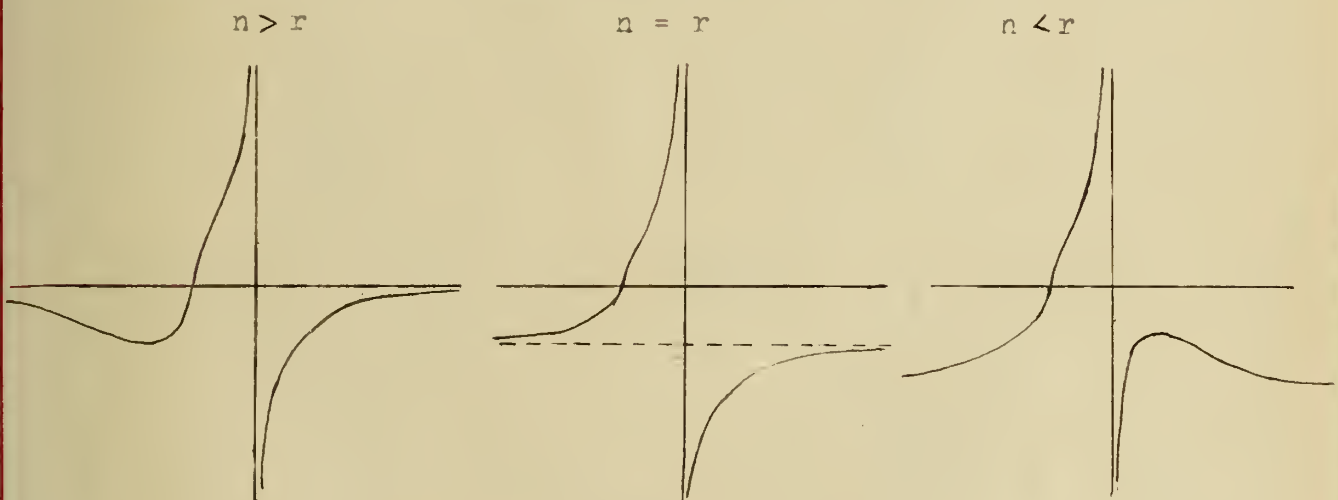
$x^n y^m + x^r + 1 = 0$ n, m, r odd

The curve intersects the x-axis at $x - 1$. The form at infinity on the y-axis is given by $x^n y^m + 1 = 0$ and at infinity on the x-axis by $x^n y^m + x^r = 0$. The latter has three forms according as $n \gtreqless r$.



When n = r the asymptote y = - 1 does not intersect the curve in any finite points.

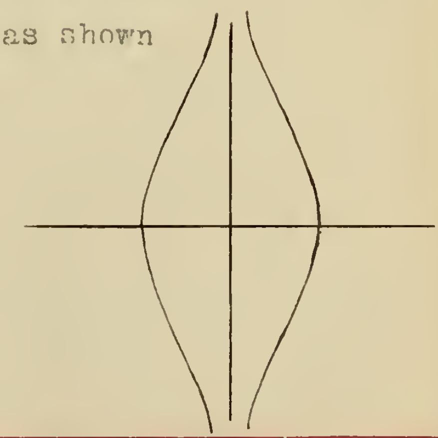
Hence we conclude that the curve has the following forms:



Va_1

$x^n y^m + x^r - 1 = 0$ n, m, r even

The intercepts are $x = \pm 1$. The form at infinity on the x-axis $x^n y^m + x^r = 0$ is imaginary. The form at infinity on the y-axis is $x^n y^m - 1 = 0$ which is as shown on the right. The curve is symmetrical with respect to the x-axis and is imaginary for $|x| > 1$ as is seen by writing the equation $y^m = \dfrac{1 - x^r}{x^n}$.



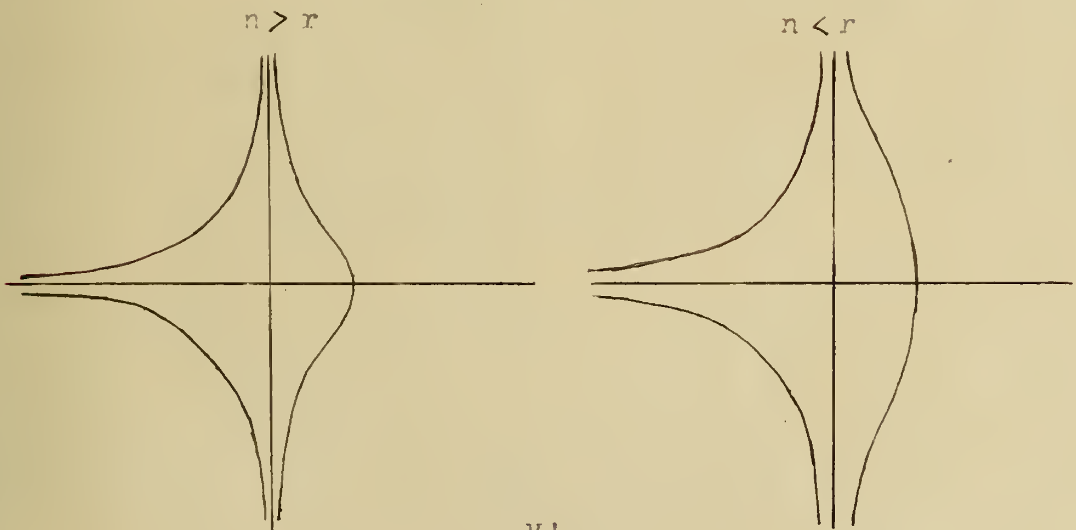
$V'a_2$

$x^n y^m + x^r - 1 = 0$ n, m even, r odd

The x-intercept is x = 1

The forms at infinity on the two axes given by $x^n y^m - 1 = 0$ and $x^m y^m + x^r = 0$ are as follows;

The curve is symmetrical with respect to the x-axis and writing the equation in the form $y^m = \dfrac{1 - x^r}{x^n}$ we see that it is imaginary for x > 1

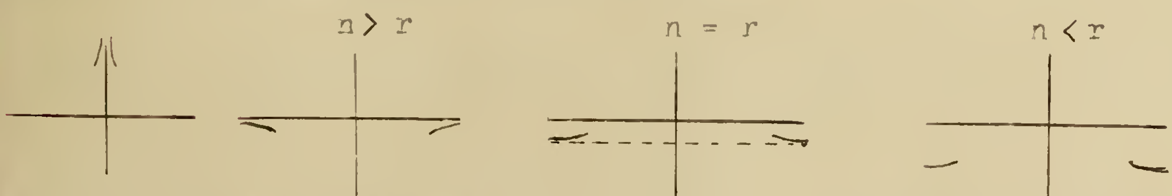


$V'a_3$

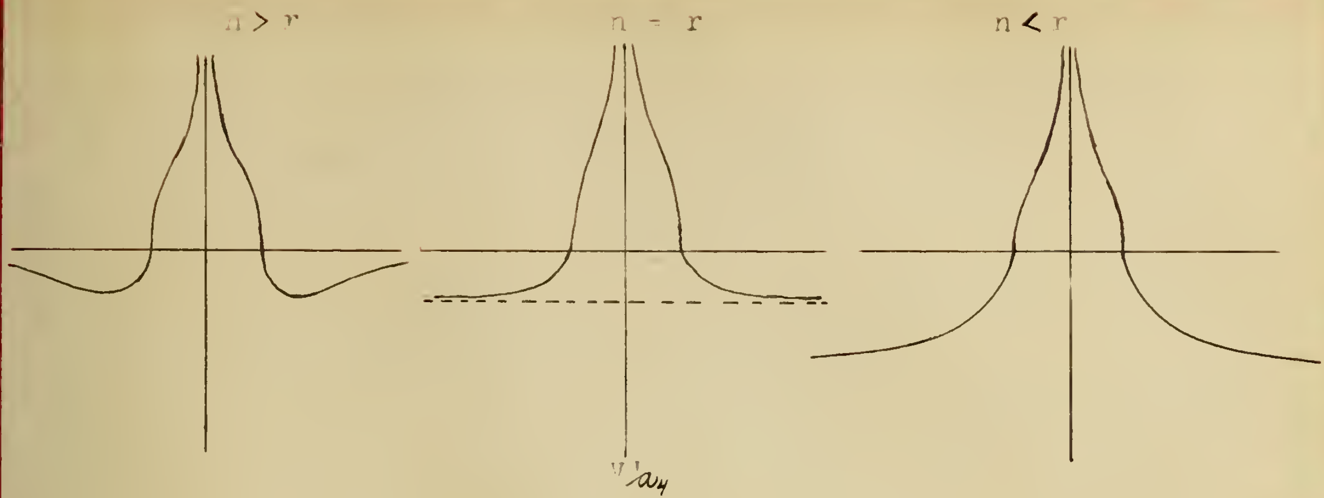
$x^n y^m + x^r - 1 = 0$ n, r even, m odd

The curve intersects the x-axis at x = ±1

The forms at infinity on the x- and y- axes given by $x^n y^m + x^r = 0$ and $x^n y^m - 1 = 0$ respectively are as follows.



For n = r the asymptote y = - 1 does not intersect the curve in any finite points. The curve is symmetrical with respect to the y-axis.



$x^n y^m + x^r - 1 = 0$ n, even, m, r odd

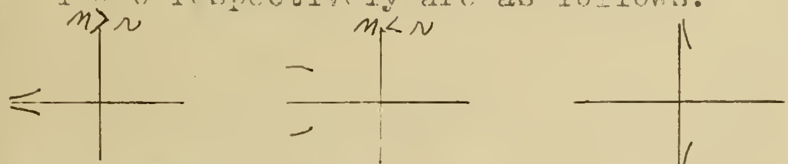
This is changed to $V'a_4$ by the transformation $x = -x$, $y = -y$

$V'b_1$

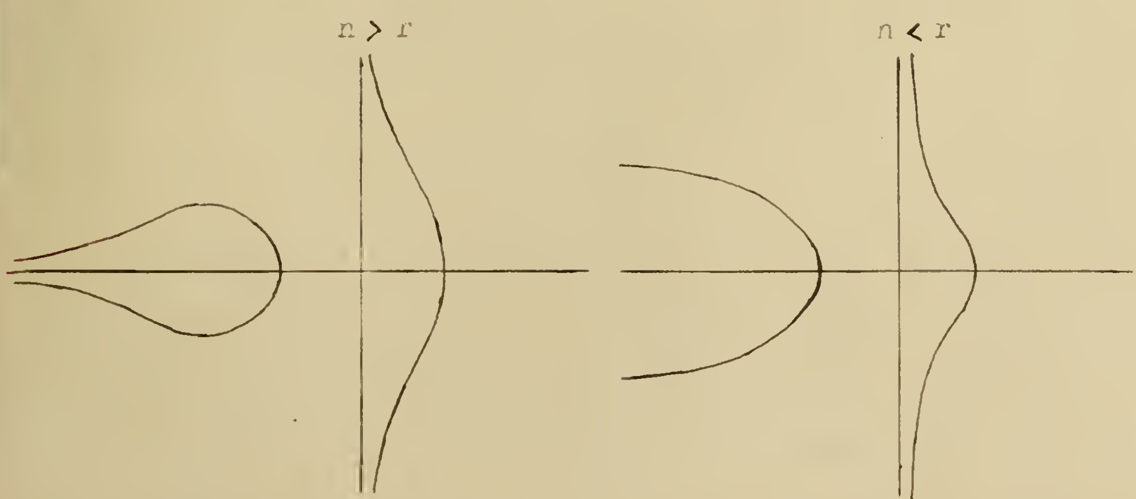
$x^n y^m + x^r - 1 = 0$ n odd, m, r even

The intercepts are $x = \pm 1$

The forms at infinity on the x- and y- axes given by $x^n y^m + x = 0$ and $x^n y^m - 1 = 0$ respectively are as follows.



The curve is symmetrical with respect to the x-axis. Writing the equation in the form $y^m = \dfrac{1-x^r}{x^n}$ we see that the curve is imaginary for $x > 1$. Hence we conclude that the curve has approximately the following forms.



V'_{b_2}

$x^n y^m + x^r - 1 = 0$ $\qquad$ n, r odd, m even

This is transformed to $V b_2$ by putting $x = -x$.

V'_{b_3}

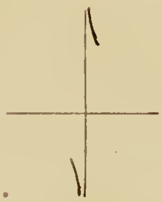
$x^n y^m + x^r - 1 = 0$ $\qquad$ n, m odd, r even

The curve intersects the x-axis at $x = \pm 1$

The forms at infinity on the x- and y-axes are given by $x^n y^m + x^r = 0$ and $x^n y^m - 1 = 0$ respectively.

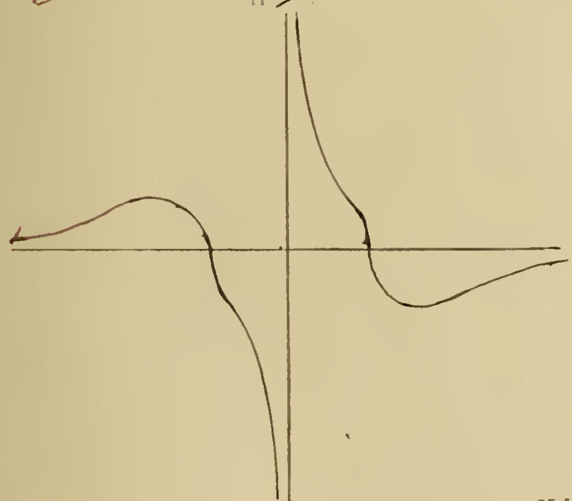
n > r

n < r

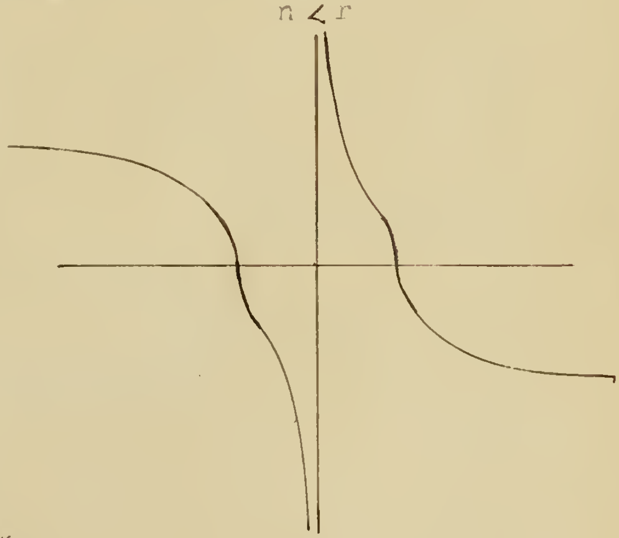


The curve is symmetrical with respect to the origin.

n > r



n < r



V'_{b_4}

$x^n y^m + x^r - 1 = 0$ $\qquad$ n, m, r odd

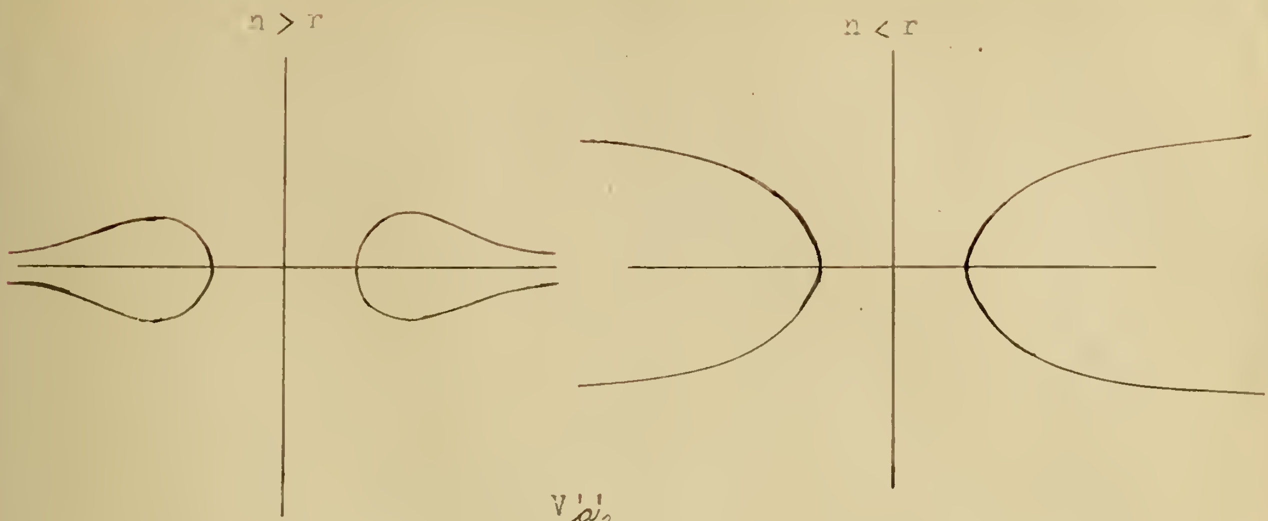
This is changed to $V b_4$ by putting $x = -x$.

V'_{a_1}

$x^n y^m - x^r + 1 = 0$ $\qquad$ n, m, r even

Intercepts on the x-axis are $x = \pm 1$. The form at infinity on the y-axis $x^n y^m + 1 = 0$ is imaginary since n and m are even. The form at infinity on the x-axis given by $x^n y^m - x^r = 0$ is as shown on the right. The curve is symmetrical with respect to both axes. Writing the equation in the form $y^m = \frac{x^r - 1}{x^n}$ we see

that the curve is imaginary for $|x| < 1$



V''_{a_2}

$x^n y^m - x^r + 1 = 0$ n, m even, r odd

This is transformed to V_{a_2} by putting x = - x.

V''_{a_3}

$x^n y^m - x^r + 1 = 0$ n, r even, m odd

This is transformed to V'_{a_3} by putting y = - y.

$x^n y^m - x^r + 1 = 0$ V''_{a_4} n even, m, r odd

This is transformed to V_{a_4} by putting x = - x.

$x^n y^m - x^r + 1 = 0$ V''_{b_1} n odd, m, r even

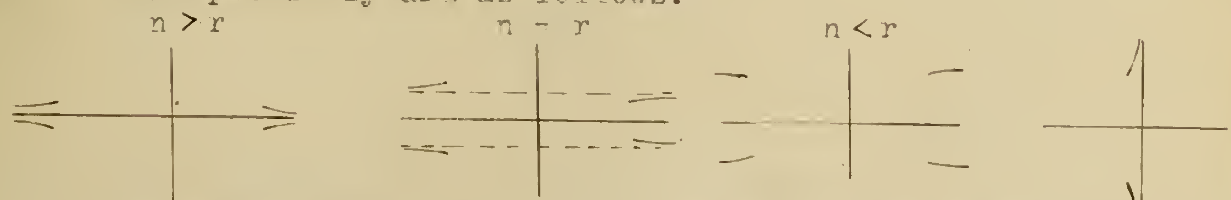
This is transformed to V'_{b_1} by putting x = - x.

V''_{b_2}

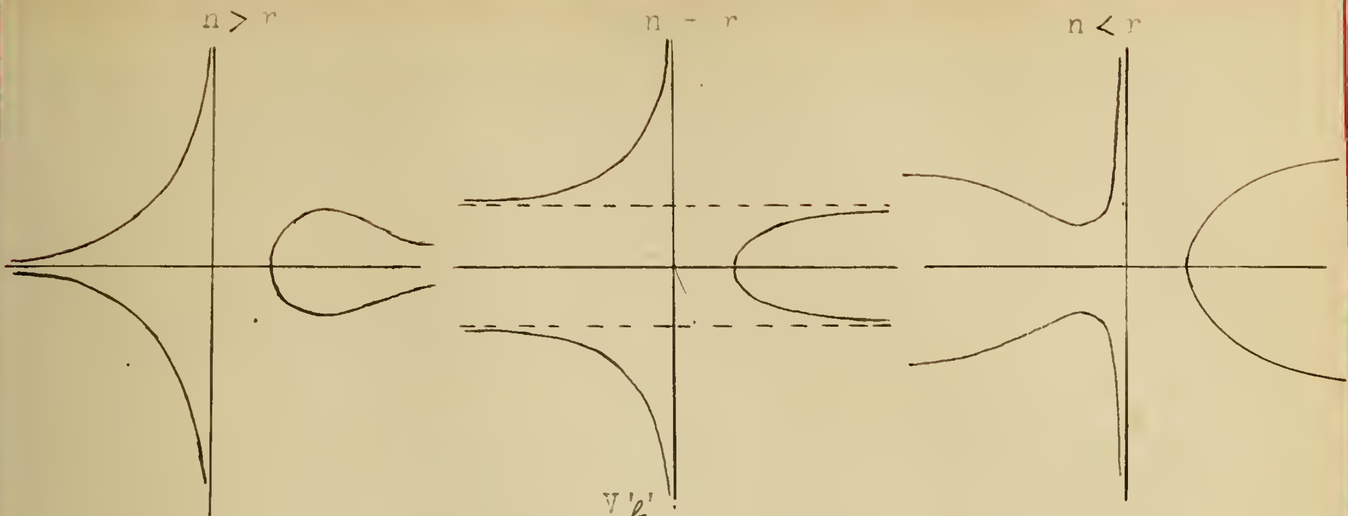
$x^n y^m - x^r + 1 = 0$ n, r odd, m even

The curve cuts the x-axis at x = 1

The forms at infinity on the x- and y-axes given by $x^n y^m - x^r = 0$ and $x^n y^m + 1 = 0$ respectively are as follows.



If n = r the asymptotes $y = \pm 1$ do not intersect the curve in any finite points. The curve is symmetrical with respect to the x-axis.



$V'b'_3$

$x^n y^m - x^r + 1 = 0$ n, m odd, r even

This is changed to Vb_3 by putting y = - y.

$V'b'_4$

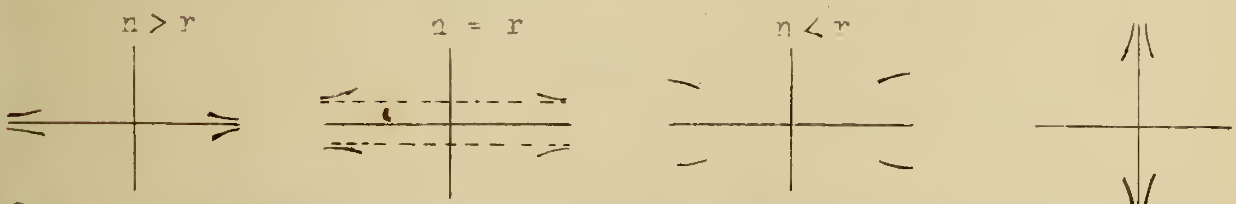
$x^n y^m - x^r + 1 = 0$ n, m, r odd

This is changed to Vb_4 by the transformation x = - x, y = - y.

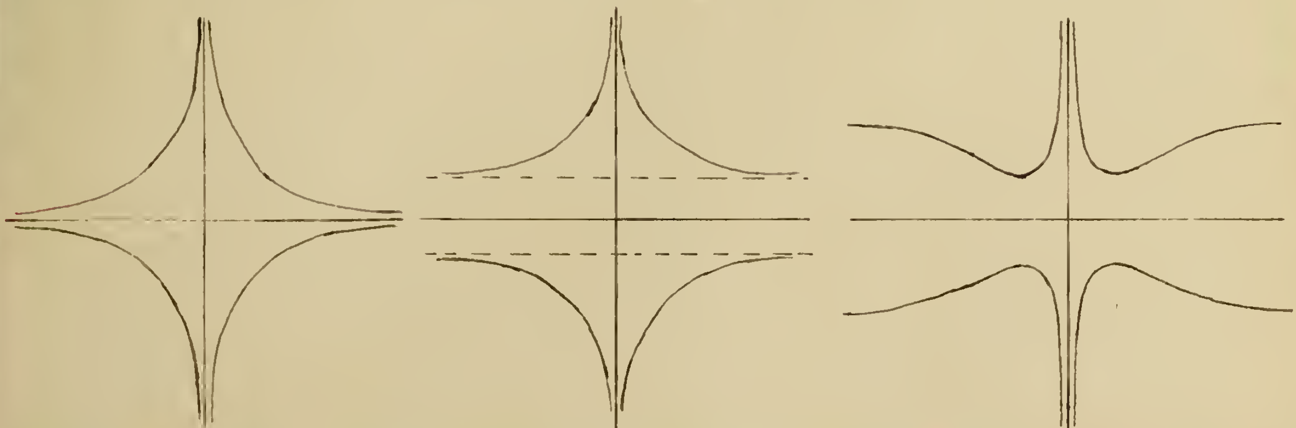
$V'a'''_1$

$x^n y^m - x^r - 1 = 0$ n, m, r even

The forms at infinity on the x- and y-axes given by $x^n y^m - x^r = 0$ and $x^n y^m - 1 = 0$ respectively are as follows:



If n = r the asymptotes y = ±1 do not intersect the curve in finite points. The curve is symmetrical with respect to both axes.



V'''_{a_2}

$x^n y^m - x^r - 1 = 0$ n, m even, r odd

This becomes V'_{a_2} by putting $x = -x$.

V'''_{a_3}

$x^n y^m - x^r - 1 = 0$ n, r even, m odd

This becomes V_{a_3} by putting $y = -y$.

V'''_{a_4}

$x^n y^m - x^r - 1 = 0$ n even, m, r odd

This becomes V_{a_4} by putting $y = -y$.

V'''_{b_1}

$x^n y^m - x^r - 1 = 0$ n, odd, m, r even

This becomes V_{b_1} by putting $x = -x$.

V'''_{b_2}

$x^n y^m - x^r - 1 = 0$ n, r odd, m even

This becomes V'_{b_2} by putting $x = -x$.

V'''_{b_3}

$x^n y^m - x^r - 1 = 0$ n, m, odd, r even

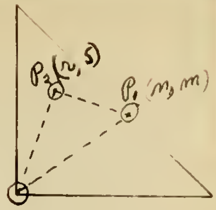
This becomes V_{b_3} by putting $y = -y$.

V'''_{b_4}

$x^n y^m - x^r - 1 = 0$ n, m, r odd

This becomes $V b_4$ by putting $y = -y$.

Case VI



$$x^n y^m \pm x^r y^s \pm 1 = 0 \qquad (n+m > r+s)$$

If the point $P_2(r, s)$ falls within the triangle OMP_1 there will be two forms at infinity on the y-axis and if it falls within the triangle ONP_1 there will be two forms at infinity on the x-axis. Since x and y are similarily involved it is evidently sufficient to consider only one of these cases. We shall therefore suppose that P_2 falls in the triangle OMP_1. Hence it will follow that $\frac{n}{r} > \frac{m}{s}$, $n > r$ and $m \gtreqless s$.

We will make the following classification according as the exponents are even or odd.

a n even, m even

1. r even, s even
2. r even, s odd
3. r odd, s even
4. r odd, s odd

b n even, m odd

1. r even, s even
2. r even, s odd
3. r odd, s even
4. r odd, s odd

c n odd, m even

1. r even, s even
2. r even, s odd
3. r odd, s even
4. r odd, s odd

d n odd, m odd

1. r even, s even
2. r even, s odd
3. r odd, s even
4. r odd, s odd

None of these curves intersect the axes in finite points.

VI a_1

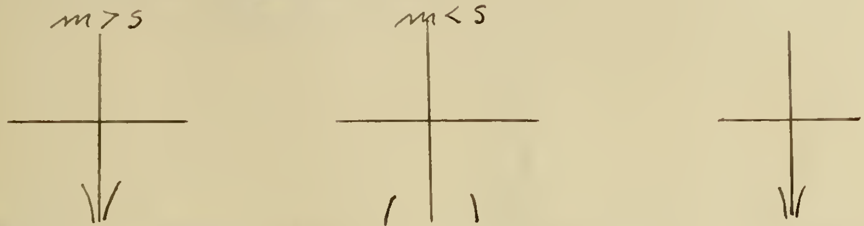
$$x^n y^m + x^r y^s + 1 = 0 \qquad n,\ m,\ r,\ s \text{ even}$$

The curve is imaginary.

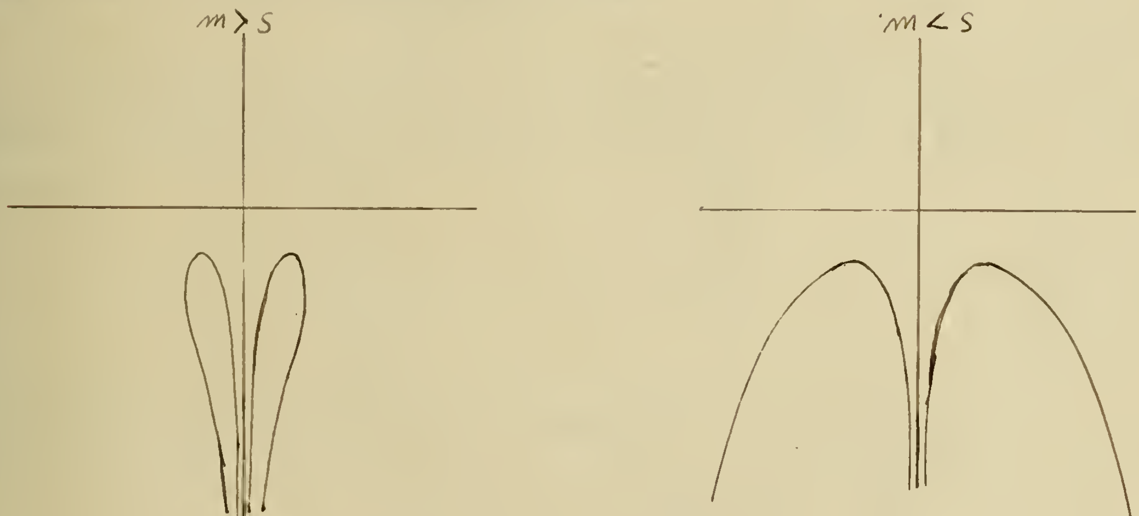
VI a_2

$x^n y^m + x^r y^s + 1 = 0$ n, m, r even, s odd

The form at infinity on the x-axis, $x^n y^m + 1 = 0$ is imaginary. The form at infinity on the y-axis given by $x^n y^m + x^r y^s = 0$ and $x^r y^s + 1 = 0$ are as shown below.



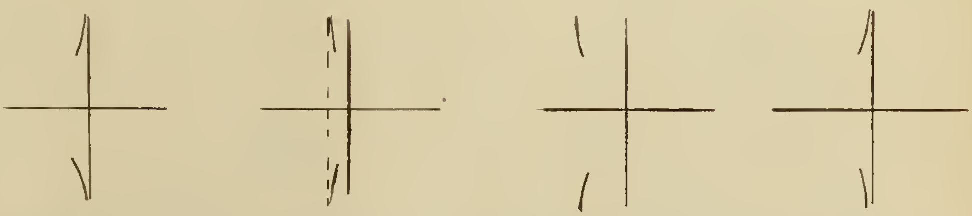
The curve cannot extend into the first or second quadrants. It is symmetrical with respect to the y-axis. Hence we conclude that the form is approximately as follows.



VI a_3

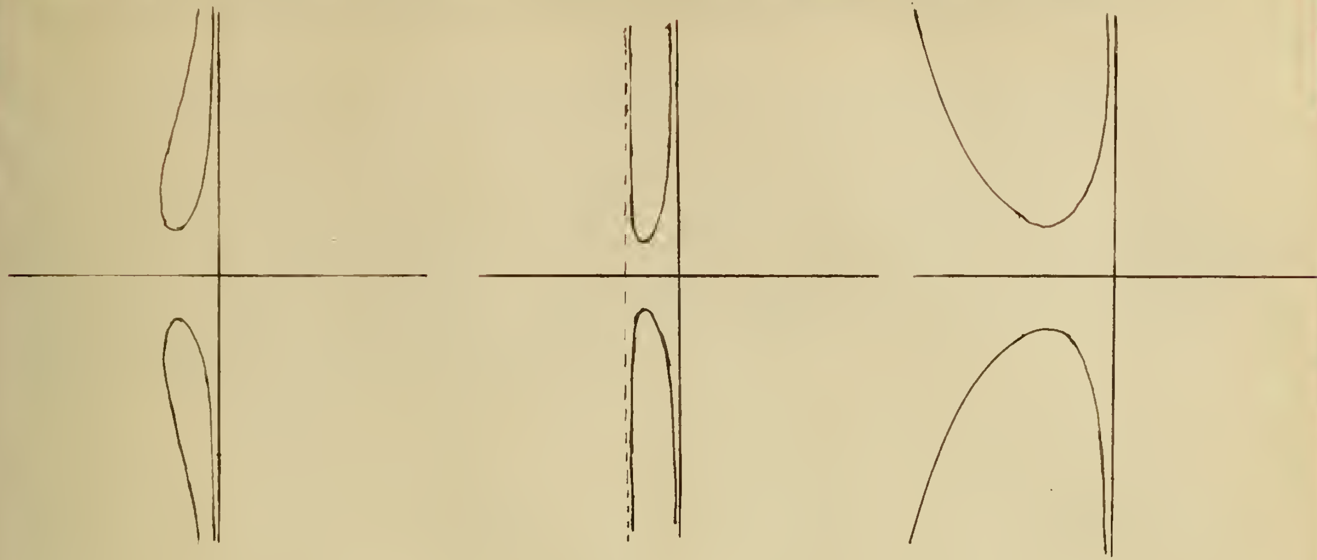
$x^m y^m + x^r y^s + 1 = 0$ n, m, s even, r odd

Form at infinity on the x-axis is imaginary as before. The forms at infinity on the y-axis given by $x^m y^m + x^r y^s = 0$ and $x^r y^s + 1 = 0$ are as follows.



If m = s the asymptote x = - 1 does not intersect the curve in any finite points. The curve is symmetrical with respect to the x-axis.

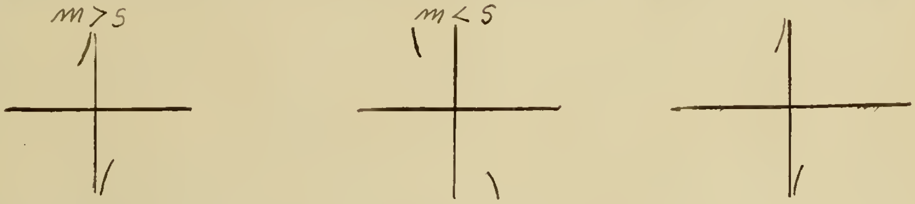
Hence we conclude that the form is approximately as follows.



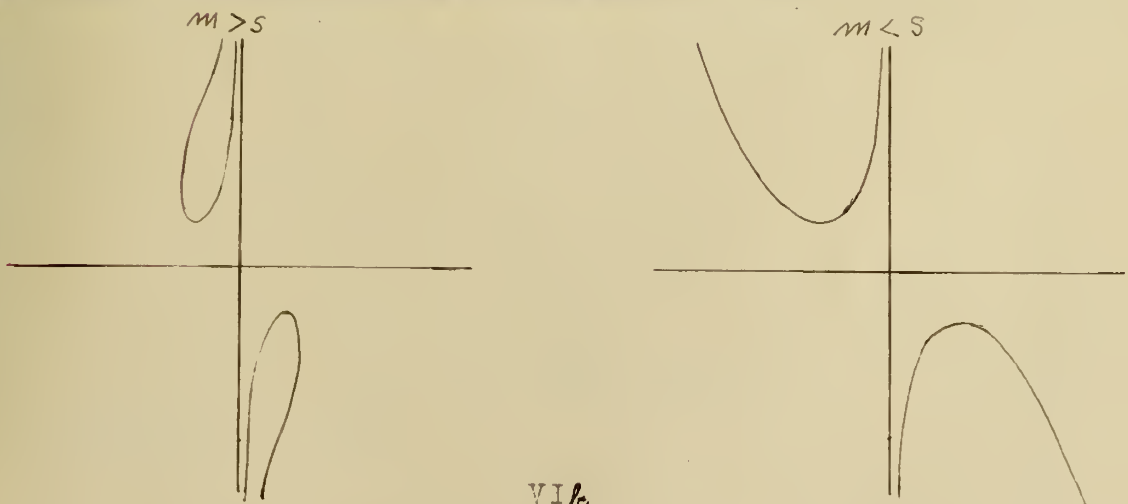
VI a_4

$x^n y^m + x^r y^s + 1 = 0$ n, m even, r, s odd

The form at infinity on the x-axis is imaginary. The forms at infinity on the y-axis given by $x^n y^m + x^r y^s = 0$ and $x^r y^s + 1 = 0$ are as follows.



The curve is symmetrical with respect to the origin.



VI b_1

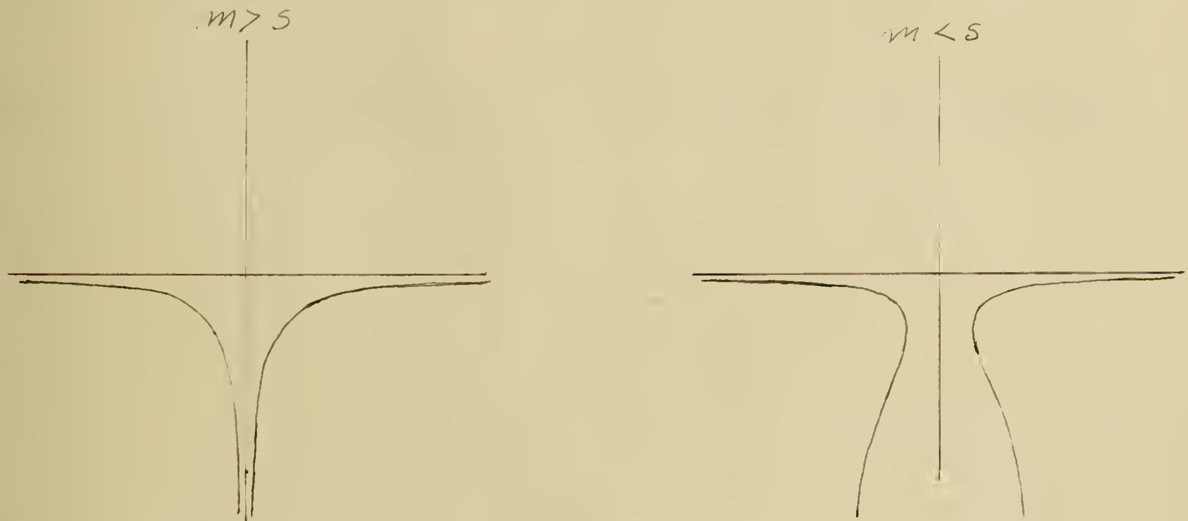
$x^n y^m + x^r y^s + 1 = 0$ n, r, s even, m odd

The form at infinity on the x-axis, is given by $x^n y^m + 1 = 0$

Of the two forms at infinity on the y-axis, $x^r y^s + 1 = 0$ and $x^n y^m + x^r y^s = 0$

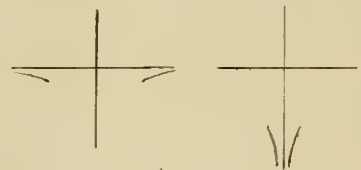
the first is imaginary and the second depends upon whether $m \gtrless s$.

The curve is symmetrical with respect to the y-axis and lies entirely in the third and fourth quadrants. Hence we conclude that the form is approximately as follows.

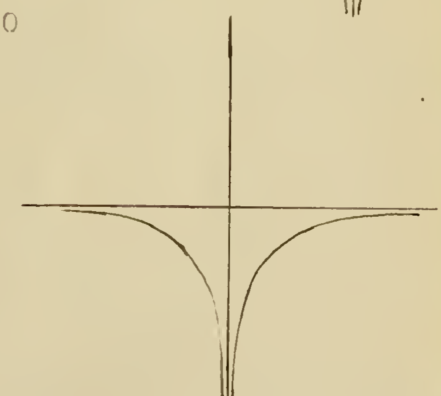


VI b_2

$x^n y^m + x^r y^s + 1 = 0$ n, r even, m, s odd



The form at infinity on the x-axis is the same as in VI b_1 above. Of the two forms at infinity on the y-axis, $x^n y^m + x^r y^s$ and $x^r y^s + 1 = 0$ the first is imaginary and the second is as shown on the right. The curve is symmetrical with respect to the y-axis. Hence we conclude that it has approximately the accompanying form.



VI b_3

$x^n y^m + x^r y^s + 1 = 0$ n, s even, m, r odd

The form at infinity on the x-axis is the same as in VI b_1 above. The forms at infinity on the y-axis given by $x^n y^m + x^r y^s = 0$ and

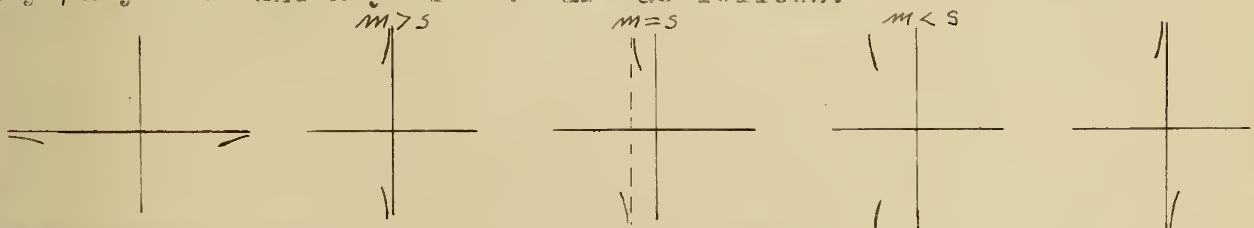
$x^r y^s + 1 = 0$ are as shown below.

We conclude that the curve has approximately the following forms.

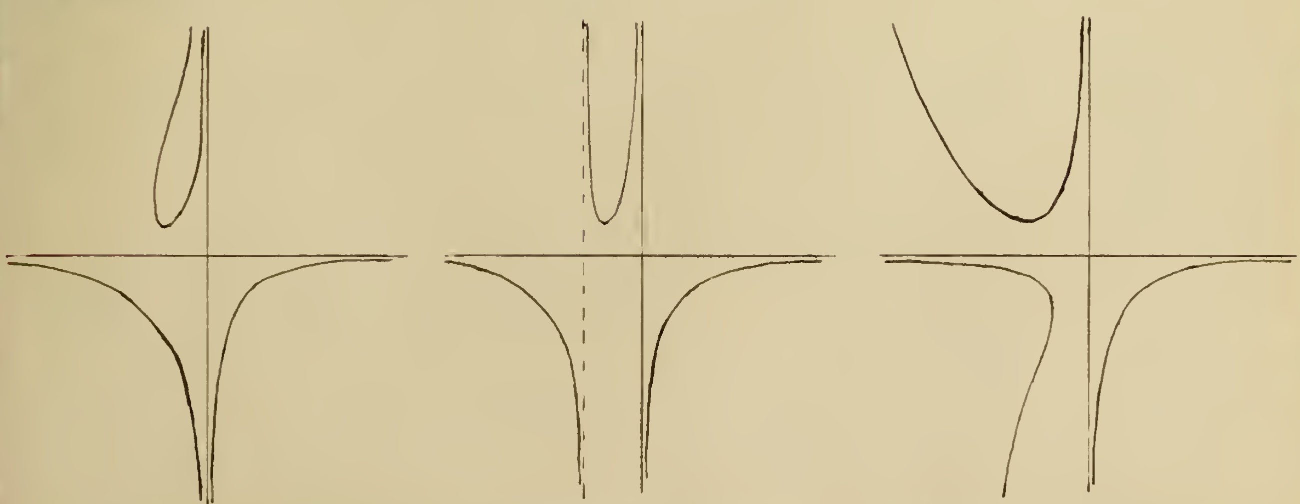
VI b_4

$x^n y^m + x^r y^s + 1 = 0$ n, even, m, r, s odd

The forms at infinity on the two axes given by $x^n y^m + 1 = 0$, $x^n y^m + x^r y^s = 0$ and $x^r y^s$ 1 = 0 are as follows.



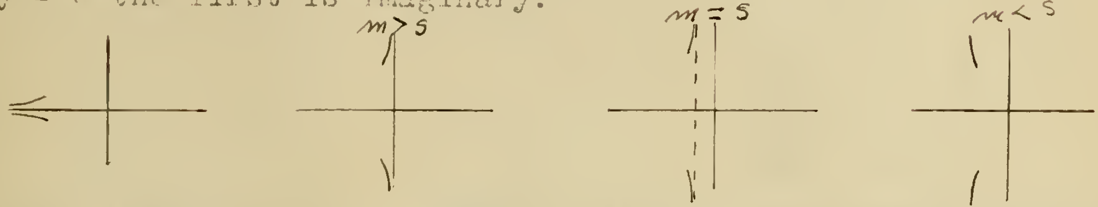
If m = s the asymptote x = - 1 does not intersect the curve in finite points.



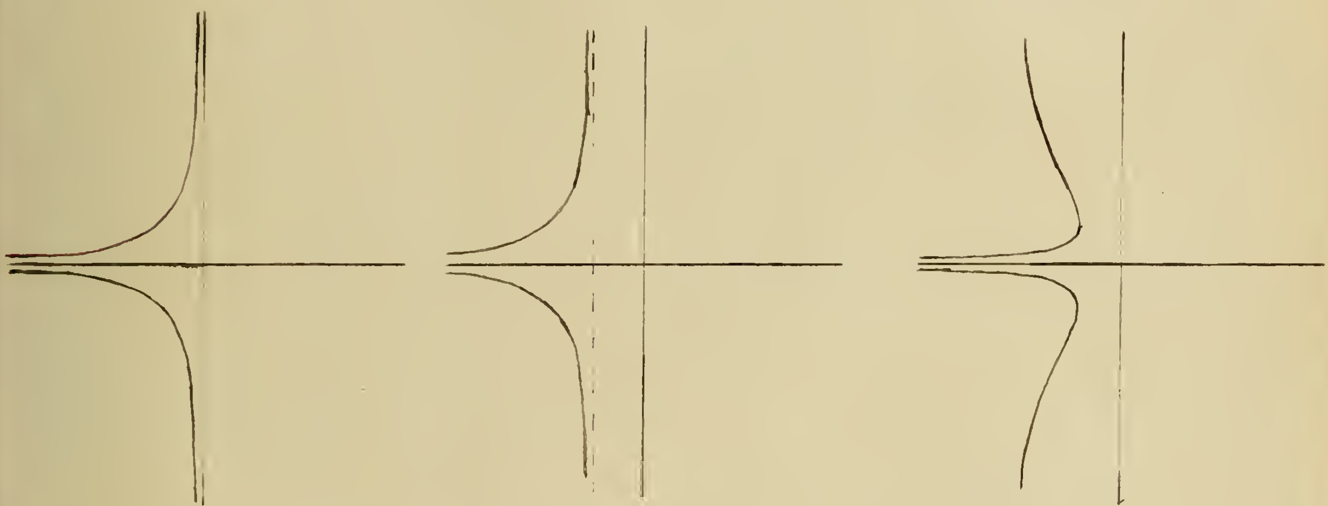
VI e_1

$x^n y^m + x^r y^s + 1 = 0$ n odd, m, r, s even

The form at infinity on the x-axis is given by $x^n y^m + 1 = 0$. Of the two forms at infinity on the y-axis $x^r y^s + 1 = 0$ and $x^n y^m + x^r y^s = 0$ the first is imaginary.



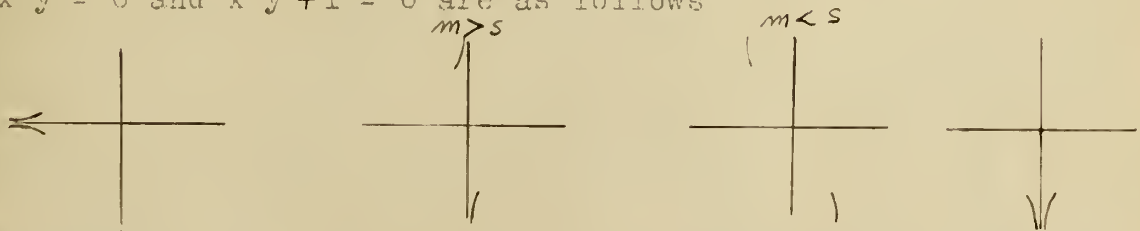
If m = s the asymptote x = - 1 has no finite intersections with the curve. The curve is symmetrical with respect to the x-axis.



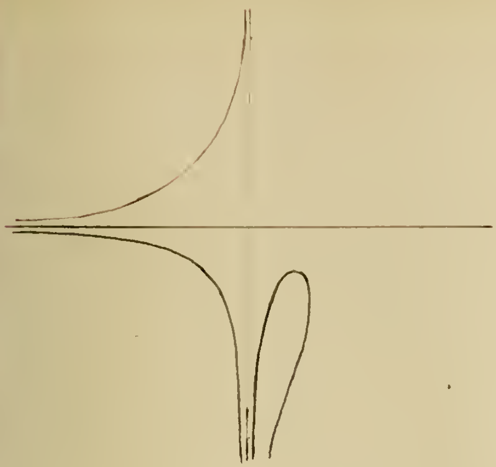
VI e_2

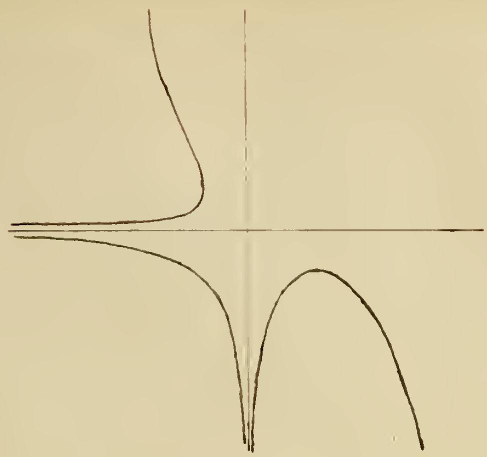
$x^n y^m + x^r y^s + 1 = 0$ n, s odd, m, r even

The forms at infinity on the two axes given by $x^n y^m + 1 = 0$, $x^n y^m + x^r y^s = 0$ and $x^r y^s + 1 = 0$ are as follows



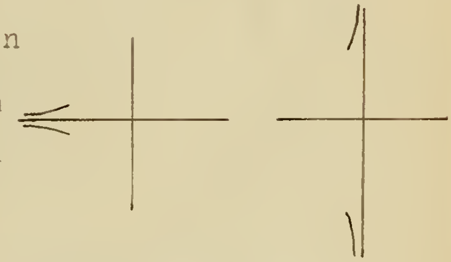
We conclude that the curve has approximately the following forms. (See figures on next page)



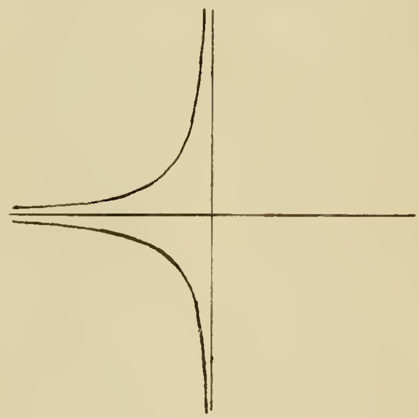


VI e_3

$x\ y \quad x\ y \quad 1 = 0$ $\qquad$ n, r odd, m, s even



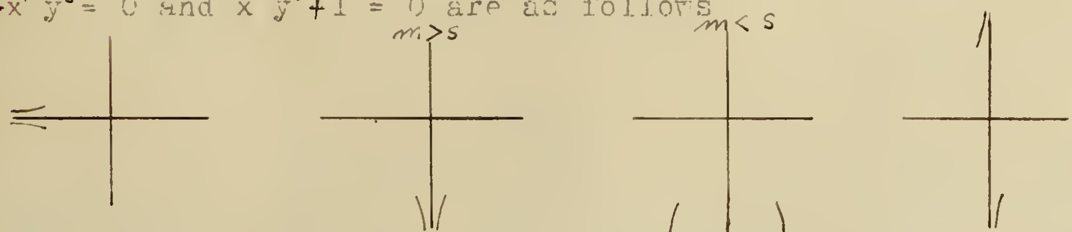
The form at infinity on the y-axis given by $x^m y^m + x^n y^s = 0$ is imaginary. The other form given by $x^n y^s + 1 = 0$ and the form at infinity on the x-axis given by $x^m y^m + 1 = 0$ are as shown on the right. The curve is symmetrical with respect to the x-axis. Hence we conclude that it has approximately the form indicated.



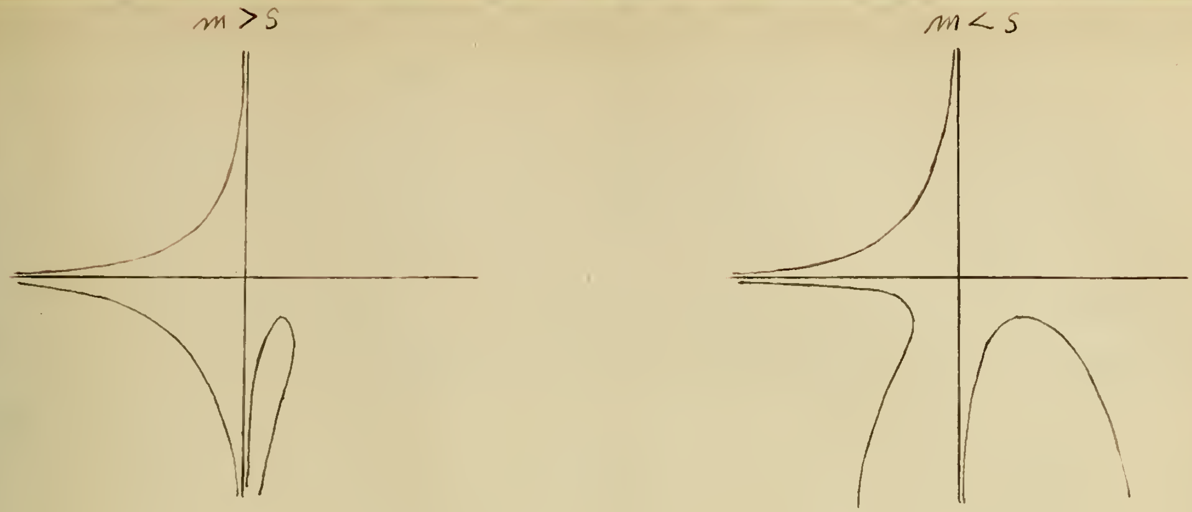
VI e_4

$x^m y^m + x^n y^s + 1 = 0$ $\qquad$ n, r, s odd, m even

The forms at infinity on the two axes, given by $x^m y^m + 1 = 0$, $x^m y^m + x^n y^s = 0$ and $x^n y^s + 1 = 0$ are as follows



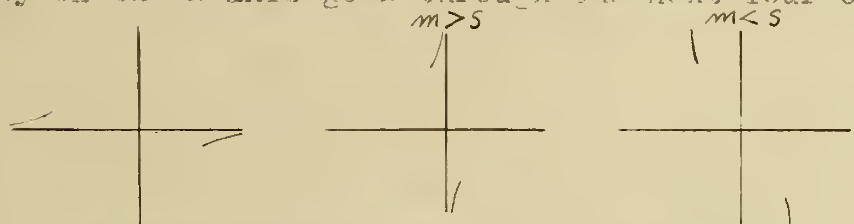
Hence we conclude that the curve has approximately the following forms. (See next page)



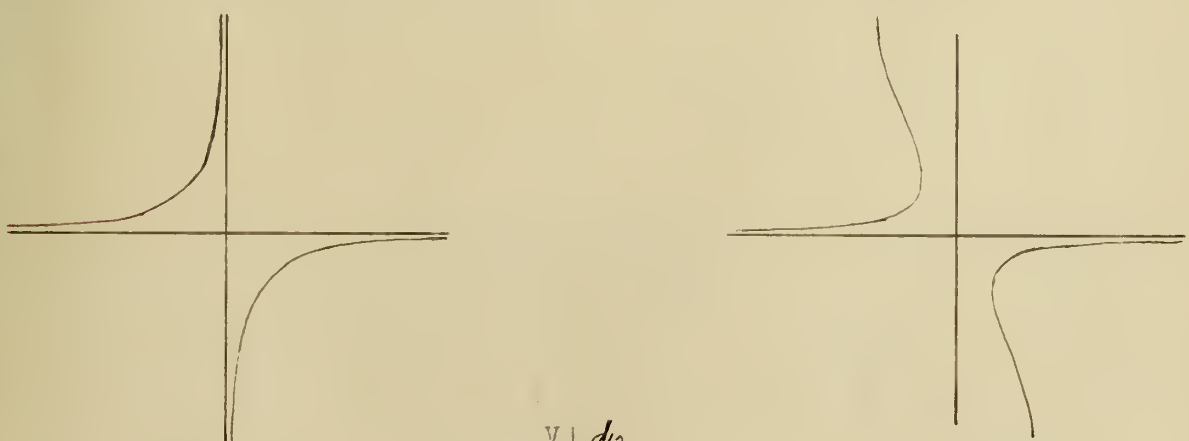
VI d_1

$x^n y^m + x^r y^s + 1 = 0$ n, m odd, r, s even

The form at infinity on the y-axis given by $x^r y^s + 1 = 0$ is imaginary. The other given by $x^n y^m + x^r y^s = 0$ and the form at infinity on the x-axis given by $x^n y^m + 1 = 0$ are as shown below. The same form at infinity on the x-axis goes through the next four cases.



The curve is symmetrical with respect to the origin.

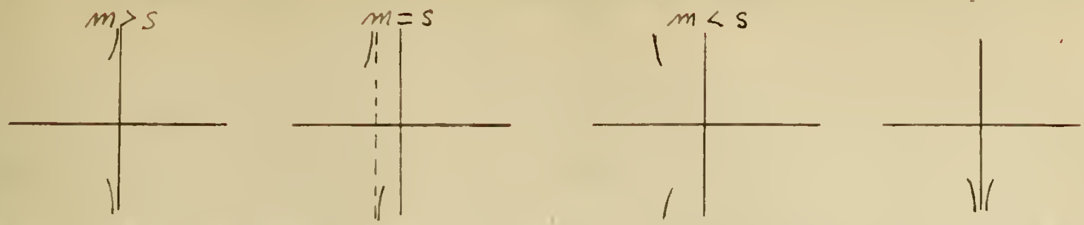


VI d_2

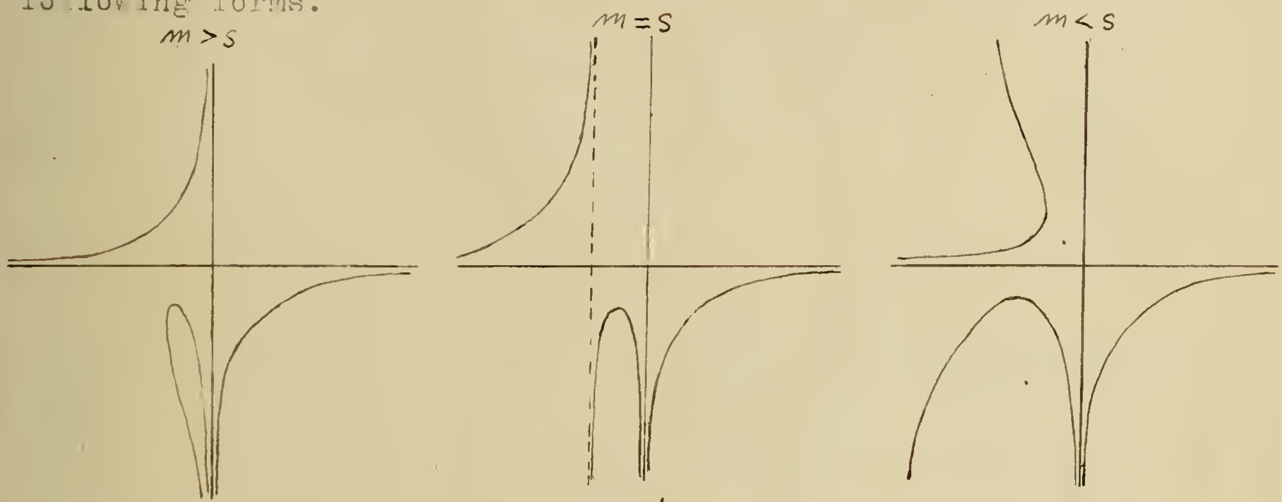
$x^n y^m + x^r y^s + 1 = 0$ n, m, s odd, r even

The form at infinity on the x-axis is the same as in VI d_1 above. The forms at infinity on the y-axis are given by $x^n y^m + x^r y^s = 0$ and

$x^N y^s + 1 = 0$. The first divides into three cases according as $m \gtreqless s$.



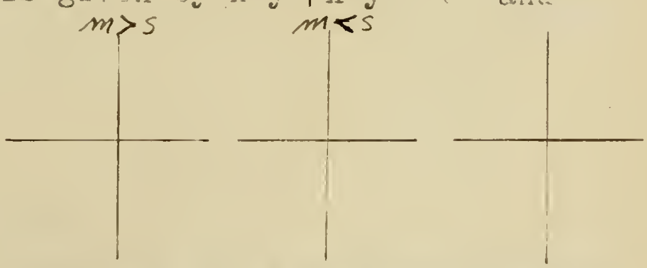
If m = s the asymptote x = - 1 has no finite intersections with the curve. Hence we conclude that the curve has approximately the following forms.



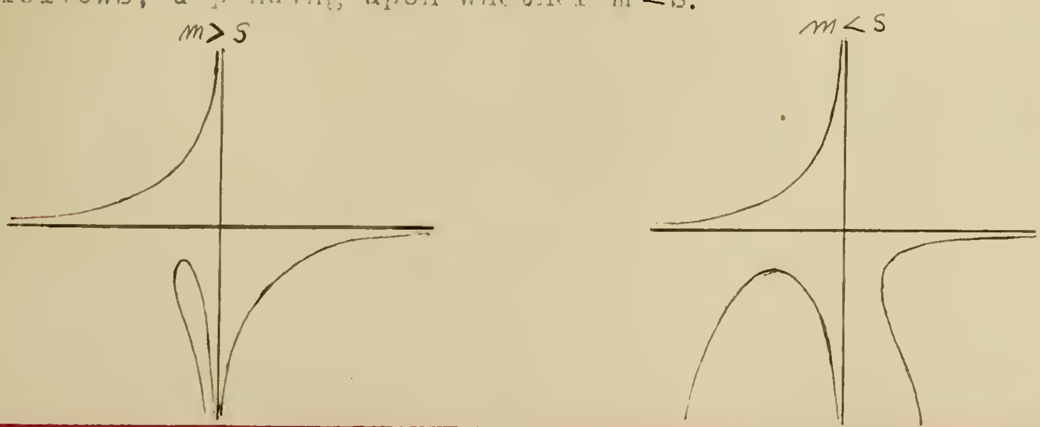
VI d_3

$x^m y^m + x^N y^s + 1 = 0$ n, m, r odd, s even

Form at infinity on the x-axis is the same as in VI d_1 above. The form at infinity on the y-axis given by $x^m y^m + x^N y^s = 0$ and $x^N y^s + 1 = 0$ are as shown below.

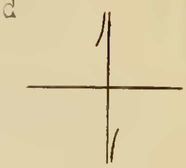


Hence we conclude that the curve has two forms approximately as follows, depending upon whether $m \gtrless s$.

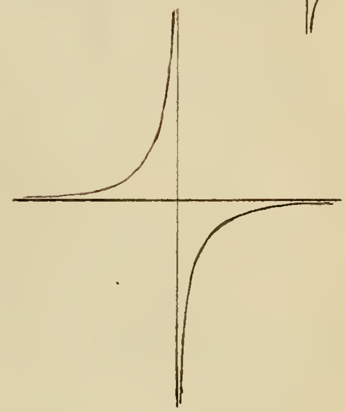


VI d_4

$$x^n y^m + x^r y^s + 1 = 0 \qquad n,\ m,\ r,\ s \text{ odd}$$



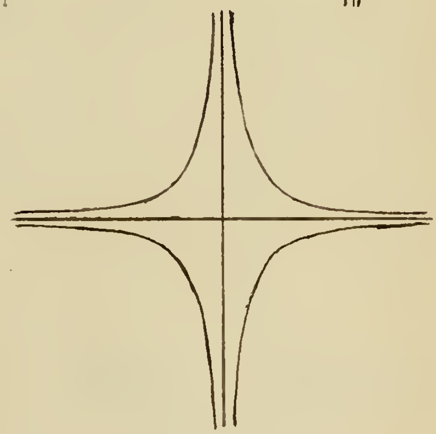
The form at infinity on the y-axis given by $x^n y^m + x^r y^s = 0$ is imaginary. The other, given by $x^r y^s + 1 = 0$ is as shown on the right. The form at infinity on the x-axis is the same as above in VId_1. The curve is symmetrical with respect to the origin. Hence we conclude that it has approximately the form indicated.



VI' a_1

$$x^n y^m + x^r y^s - 1 = 0 \qquad n,\ m,\ r, s \text{ even}$$

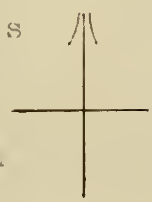
The form at infinity on the y-axis given by $x^n y^m + x^r y^s = 0$ is imaginary. The other form $x^r y^s - 1 = 0$ is shown on the right. The form at infinity on the x-axis, $x^n y^m - 1 = 0$ is also shown on the right and this form goes through the next four cases. The curve is symmetrical with respect to both axes. We conclude that its form is approximately as indicated on the right.

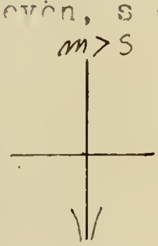


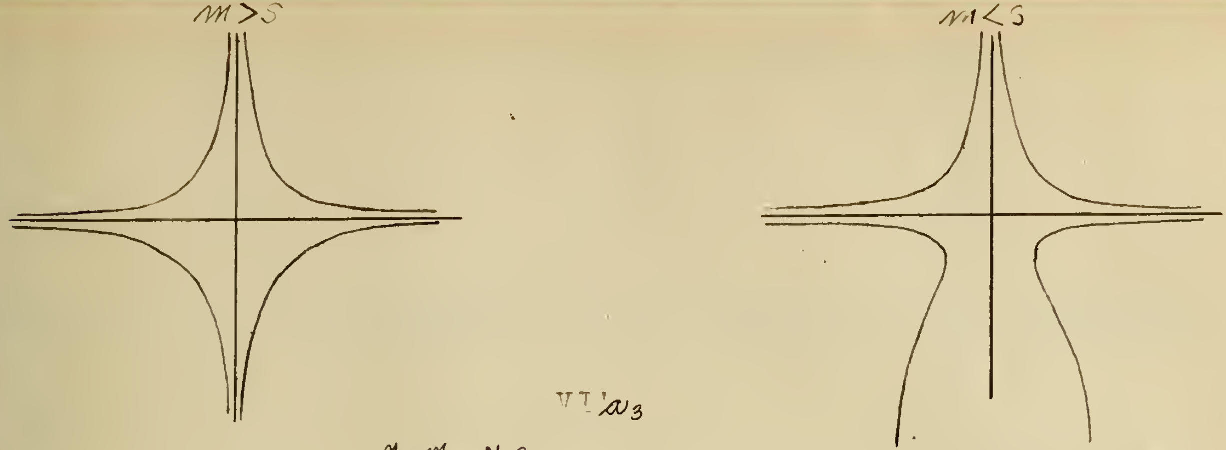
VI' a_2

$$x^n y^m + x^r y^s - 1 = 0 \qquad n,\ m,\ r \text{ even},\ s \text{ odd}$$

The two forms at infinity on the y-axis $x^n y^m + x^r y^s = 0$ and $x^r y^s - 1 = 0$ are as shown on the right. The curve is symmetrical with respect to the y-axis.





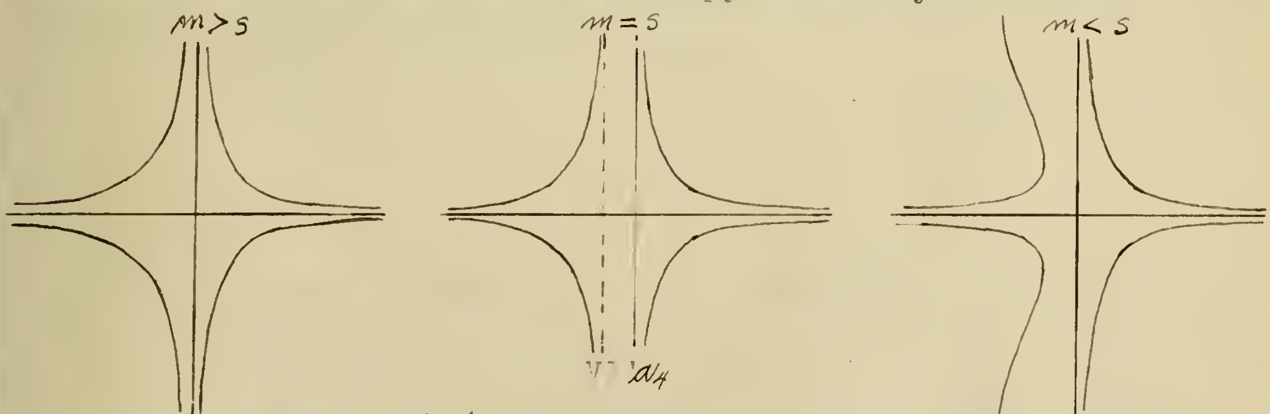


VI'a₃

$x^n y^m + x^r y^s - 1 = 0$ n, m, s even, r odd

The forms at infinity on the y-axis $x^n y^m + x^r y^s = 0$ and $x^r y^s - 1 = 0$ are as shown below. At infinity on the x-axis the form is the same as in VIa, above.

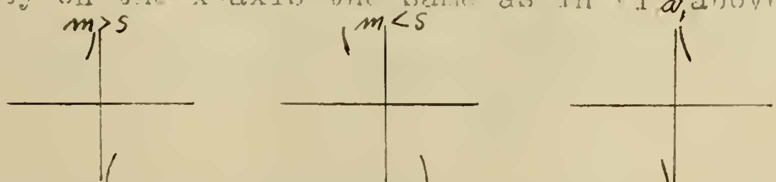
If m = s the asymptote x = - 1 does not intersect the curve in finite points. The curve is symmetrical with respect to the x-axis. Hence we conclude that the form is approximately as follows.



VI'a₄

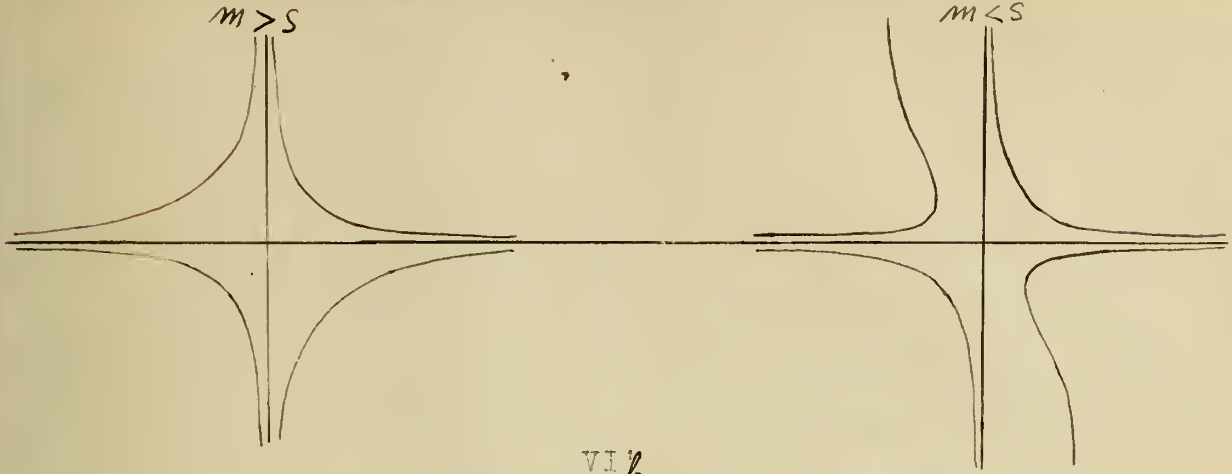
$x^n y^m + x^r y^s - 1 = 0$ n, m even, r, s odd

The forms at infinity on the y-axis are as indicated below, at infinity on the x-axis the same as in VI'a, above.



The curve is symmetrical with respect to the origin. We conclude

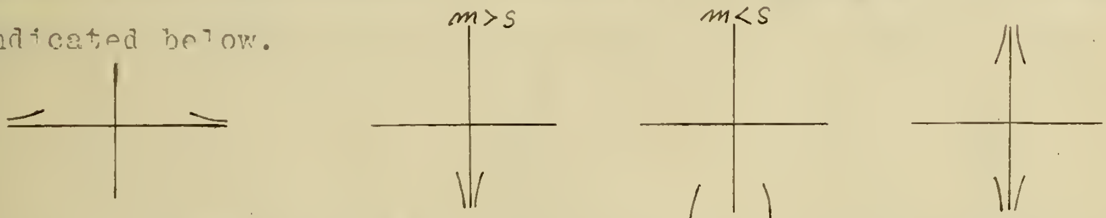
that the curve has approximately the following forms.



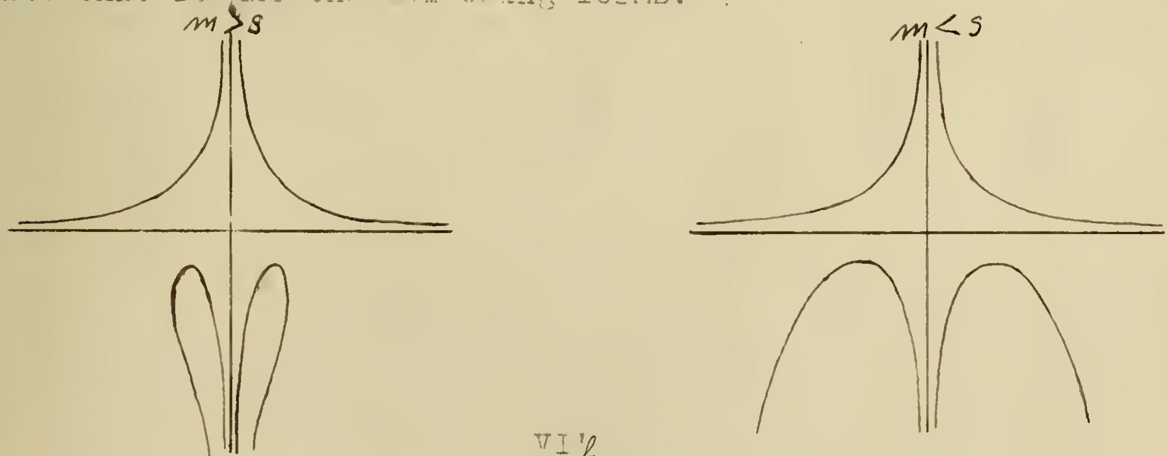
VI'b_1

$x^n y^m + x^r y^s - 1 = 0$ n, r, s even, m odd

The form at infinity on the x-axis is $x^n y^m - 1 = 0$ and this goes through the next four cases. This together with the forms at infinity on the y-axis given by $x^n y^m + x^r y^s = 0$ and $x^r y^s - 1 = 0$ are as indicated below.



The curve is symmetrical with respect to the origin. Hence we conclude that it has the following forms.



VI'b_2

$x^n y^m + x^r y^s - 1 = 0$ n, r even, m, s odd

This becomes the same as VIb_2 by putting $y = -y$.

VI'b_3

$x^n y^m + x^r y^s - 1 = 0$ n, s even, m, r odd

This becomes VIb_3 by putting $x = -x$, $y = -y$.

VI'b_4

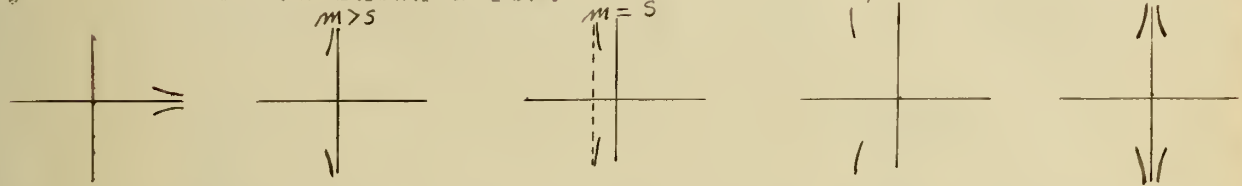
$$x^n y^m + x^r y^s - 1 = 0 \qquad n, \text{ even}, m, r, s \text{ odd}$$

This becomes VI b_4 by putting $y = - y$.

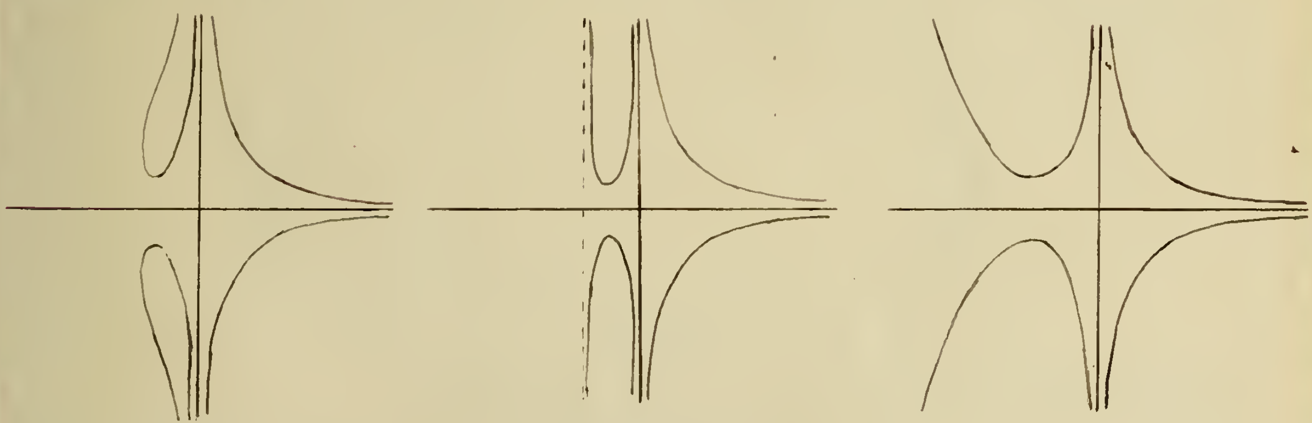
VI'c_1

$$x^n y^m + x^r y^s - 1 = 0 \qquad n, \text{ odd}, m, r, s \text{ even}$$

The form at infinity on the x-axis is given by $x^n y^m - 1 = 0$ and this form is the same for the next four cases. This together with the forms at infinity on the y-axis given by $x^n y^m + x^r y^s = 0$ and $x^r y^s - 1 = 0$ are as shown below.



The curve is symmetrical with respect to the x-axis. If $m = s$ the asymptote $x = -1$ does not intersect the curve in finite points.



VI'c_2

$$x^n y^m + x^r y^s - 1 = 0 \qquad n, s \text{ odd}, m, r \text{ even}$$

This is transformed to VI c_2 by putting $x = - x$, $y = - y$.

VI'c_3

$$x^n y^m + x^r y^s - 1 = 0 \qquad n, r \text{ odd}, m, s \text{ even}$$

This is transformed to VI c_3 by putting $x = - x$.

VI'c_4

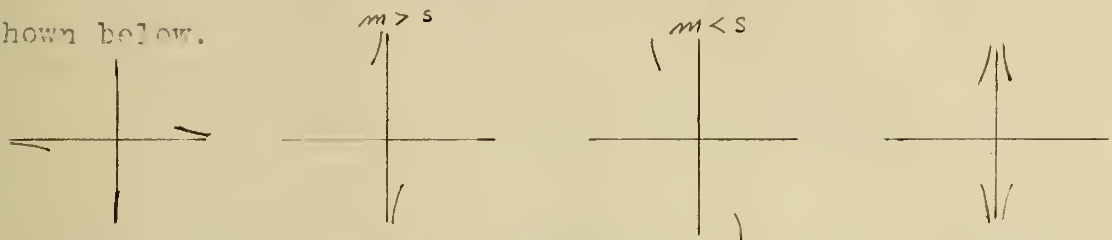
$$x^n y^m + x^r y^s - 1 = 0 \qquad n, r, s \text{ odd}, m \text{ even}$$

This is transformed to VI c_4 by putting $x = - x$.

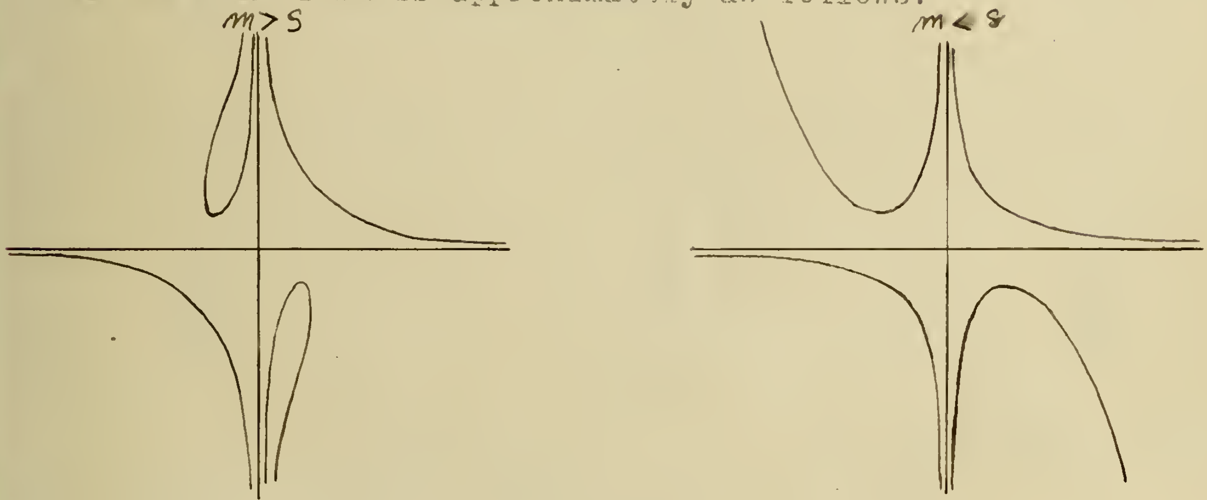
VI'd_1

$x^n y^m \; x^N y^s - 1 = 0$ n, m odd, r, s even

The form at infinity on the x-axis is given by $x^n y^m - 1 = 0$ and it is the same for the next four cases. This and the forms at infinity on the y-axis given by $x^n y^m + x^N y^s = 0$ and $x^N y^s - 1 = 0$ are as shown below.



The curve is symmetrical with respect to the origin. Hence we conclude that the form is approximately as follows.



VI'd_2

$x^n y^m + x^N y^s - 1 = 0$ n, m, s odd r even

This becomes VId_2 by the transformation $y = - y$.

VI'd_3

$x^n y^m + x^N y^s - 1 = 0$ n, m, r odd, s even

This becomes VId_3 by the transformation $x = - x$.

VI'd_4

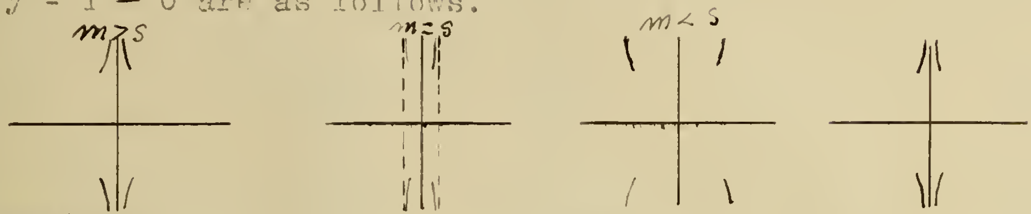
$x^n y^m + x^N y^s - 1 = 0$ n, m, r, s odd

This becomes VId_4 by the transformation $x = - x$.

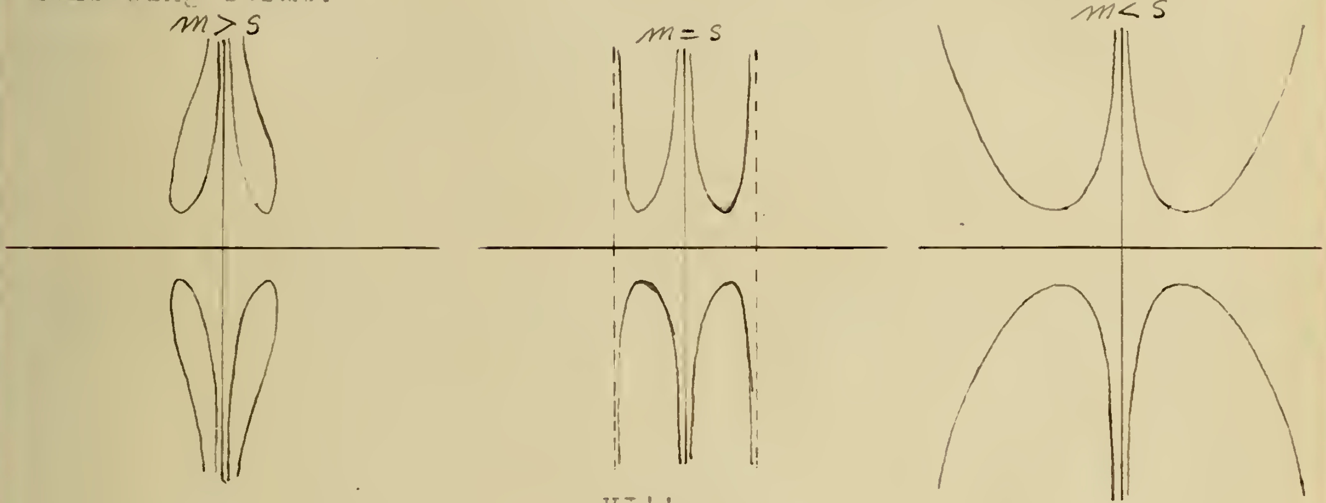
VI''d_1

$x y - x y + 1 = 0$ n, m, r, s even

The form at infinity on the x-axis given by $x^n y^m + 1 = 0$ is imaginary. The forms at infinity on the y-axis given by $x^n y^m - x^r y^s = 0$ and $x^r y^s - 1 = 0$ are as follows.



The asymptotes $x = \pm 1$ do not intersect the curve in finite points when $m = s$. Hence we conclude that the curve has approximately the following forms.



VI'a_2

$$x^n y^m - x^r y^s + 1 = 0 \qquad n,\ m,\ r \text{ even},\ s \text{ odd}$$

This becomes VIa_2 by the transformation $y = -y$.

VI'a_3

$$x^n y^m - x^r y^s + 1 = 0 \qquad n,\ m,\ s \text{ even},\ r \text{ odd}$$

This becomes VIa_3 by the transformation $x = -x$

VI'a_4

$$x^n y^m - x^r y^s + 1 = 0 \qquad n,\ m \text{ even},\ r,\ s \text{ odd}$$

This becomes VIa_4 by the transformation $x = -x$

VI'b_1

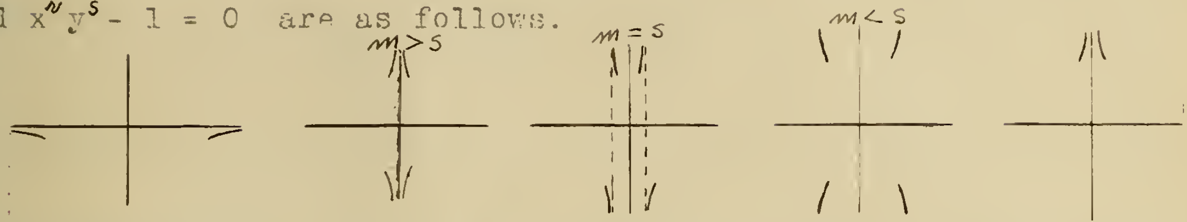
$$x^n y^m - x^r y^s + 1 = 0 \qquad n,\ r,\ s \text{ even},\ m \text{ odd}$$

This becomes VIb_1 by the transformation $y = -y$

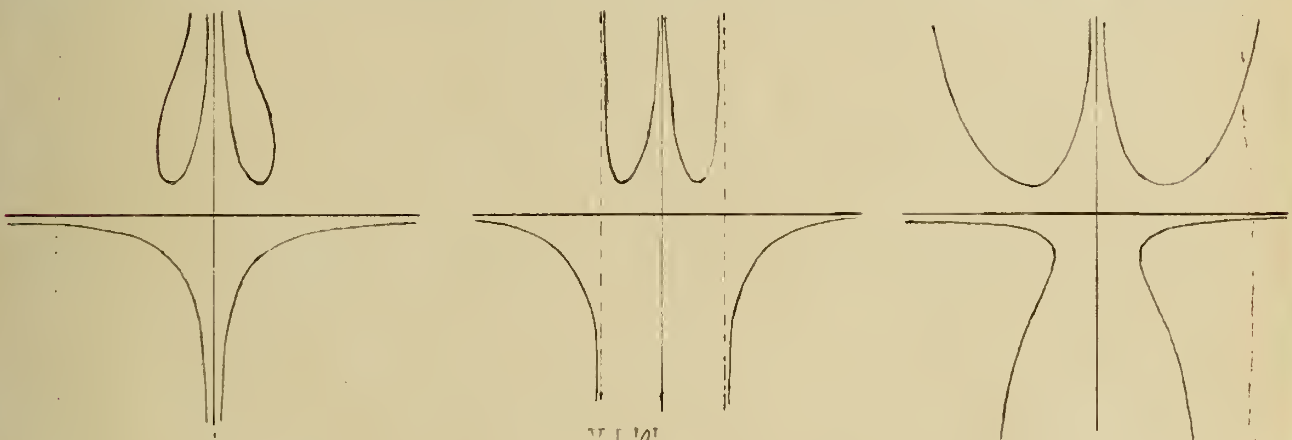
VI'b'_2

$x^n y^m - x^r y^s + 1 = 0$ n, r even, m, s odd

The forms at infinity on the axes given by $x^n y^m + 1 = 0$, $x^n y^m - x^r y^s = 0$ and $x^r y^s - 1 = 0$ are as follows.



The asymptotes $x = \pm 1$ do not intersect the curve in finite points when m = s. The curve is symmetrical with respect to the y-axis. Hence we conclude that it has approximately the following forms.



VI'b'_3

$x^n y^m - x^r y^s + 1 = 0$ n, s even, m, r odd

This becomes VIb_3 by putting x = - x.

VI'b'_4

$x^n y^m - x^r y^s + 1 = 0$ n, even, m, r, s odd

This becomes VIb_4 by putting x = - x.

VI'c'_1

$x^n y^m - x^r y^s + 1 = 0$ n, odd, m, r, s even

This becomes VI'c_1 by putting x = - x.

VI'c'_2

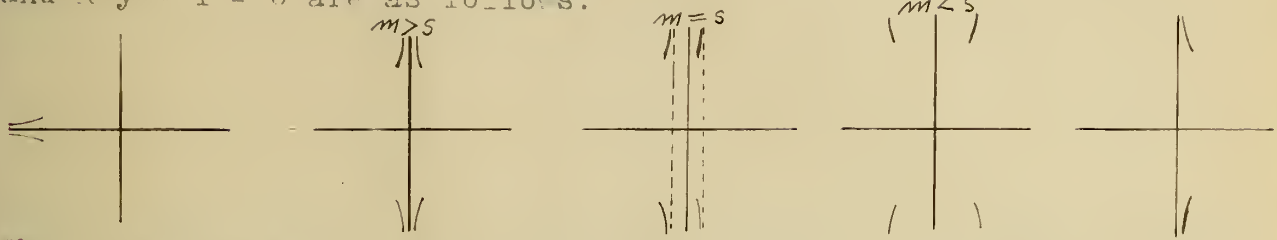
$x^n y^m - x^r y^s + 1 = 0$ n, s odd, m, r even

This becomes VIc_2 by putting y = -y.

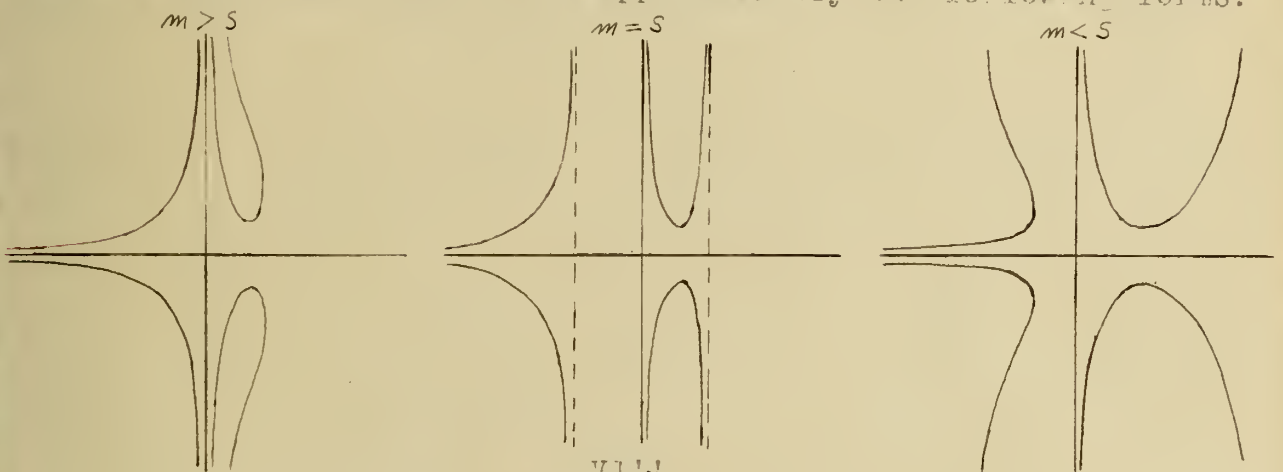
$\mathrm{VI}''c_3$

$x^n y^m - x^r y^s + 1 = 0$ m, s even, n, r odd

The form at infinity on the axes given by $x^n y^m + 1 = 0$, $x^n y^m - x^r y^s = 0$ and $x^r y^s - 1 = 0$ are as follows.



The asymptotes $x = \pm 1$ do not intersect the curve in finite points when m = s. The curve is symmetrical with respect to the x-axis. Hence we conclude that it has approximately the following forms.



$\mathrm{VI}''c_4$

$x^n y^m - x^r y^s + 1 = 0$ n, r, s odd, m even

This is changed to $\mathrm{VI}c_4$ by the transformation $y = -y$.

$\mathrm{VI}''d_1$

$x^n y^m - x^r y^s + 1 = 0$ n, m odd, r, s even

This is changed to $\mathrm{VI}'d_1$ by the transformation $x = -x$.

$\mathrm{VI}''d_2$

$x^n y^m - x^r y^s + 1 = 0$ n, m, s odd, r even

This is changed to $\mathrm{VI}d_2$ by the transformation $x = -x$, $y = -y$.

$\mathrm{VI}''d_3$

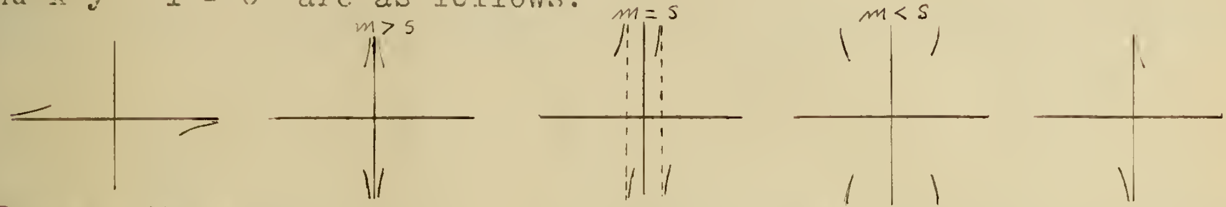
$x^n y^m - x^r y^s + 1 = 0$ n, m, r odd, s even

This is changed to $\mathrm{VI}d_3$ by the transformation $x = -x$, $y = -y$.

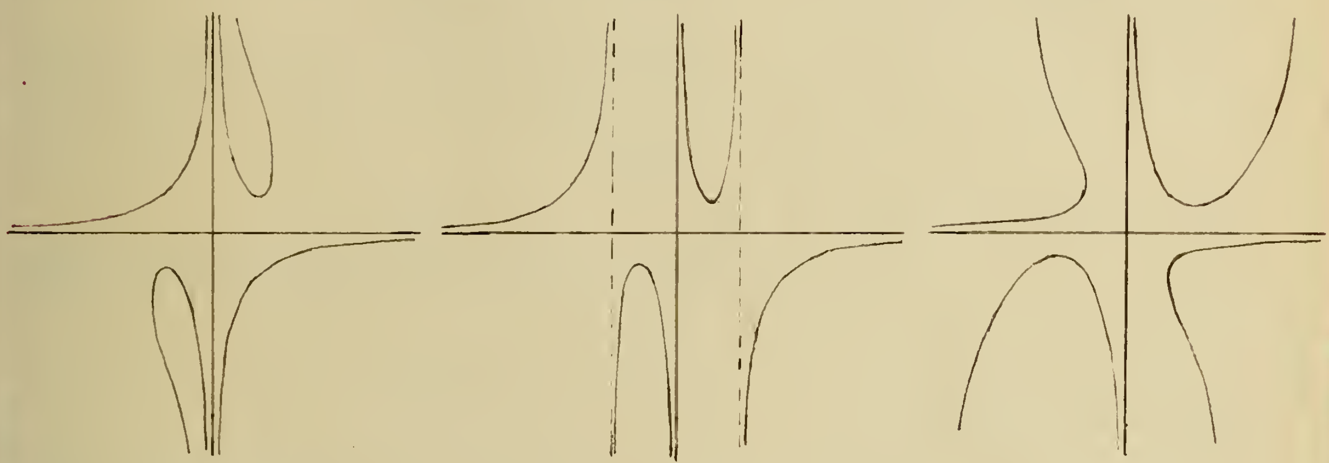
VI''_{d_4}

$x^n y^m - x^r y^s + 1 = 0$ n, m, r, s odd

The forms at infinity on the axes given by $x^n y^m + 1 = 0$, $x^n y^m - x^r y^s = 0$ and $x^r y^s - 1 = 0$ are as follows.



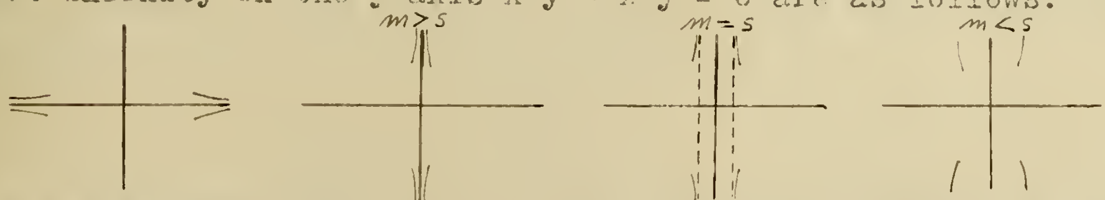
If m = s the asymptotes $x = \pm 1$ do not intersect the curve in finite points. The curve is symmetrical with respect to the origin.



VI'''_{a_1}

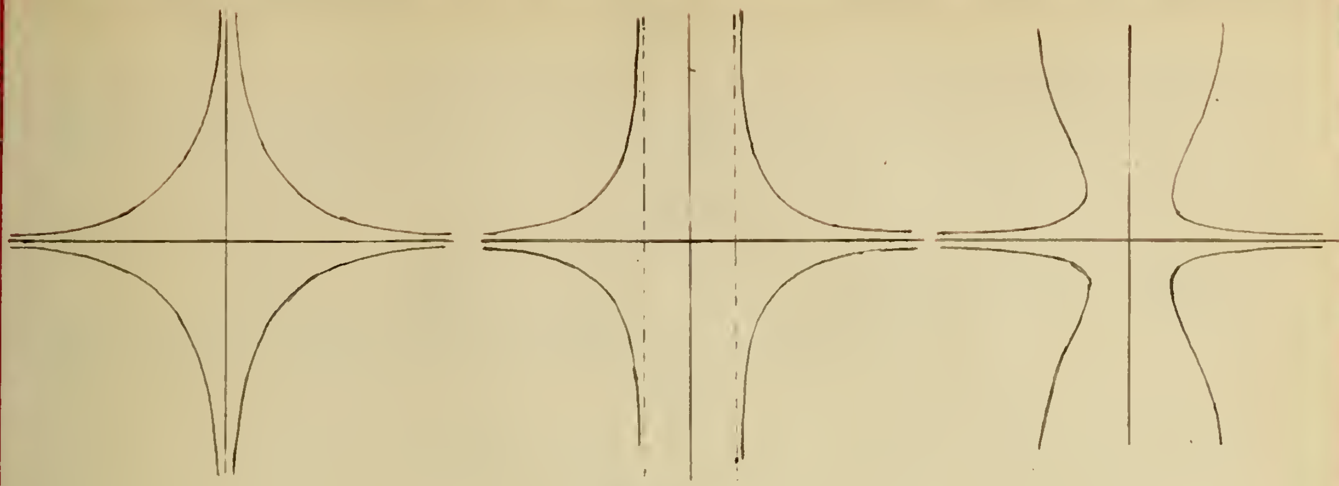
$x^n y^m - x^r y^s - 1 = 0$ n, m, r, s even

The form at infinity on the y-axis given by $x^r y^s + 1 = 0$ is imaginary. The form at infinity on the x-axis $x^n y^m - 1 = 0$ and the other form at infinity on the y-axis $x^n y^m - x^r y^s = 0$ are as follows.



If m = r the asymptotes $x = \pm 1$ do not intersect the curve in finite points. The curve is symmetrical with respect to both axes.

Hence we conclude that it has approximately the following forms.



VI'''_{a_2}

$$x^n y^m - x^r y^s - 1 = 0 \qquad n,\ m,\ r \text{ even},\ s \text{ odd}$$

This becomes VI'_{a_2} by the transformation $y = -y$.

VI'''_{a_3}

$$x\ y - x\ y - 1 = 0 \qquad n,\ m,\ s \text{ even},\ r \text{ odd}$$

This becomes VI'_{a_3} by the transformation $x = -x$.

VI'''_{a_4}

$$x^n y^m - x^r y^s - 1 = 0 \qquad n,\ m \text{ even},\ r,\ s \text{ odd}$$

This becomes VI'_{a_4} by the transformation $x = -x$.

VI'''_{b_1}

$$x^n y^m - x^r y^s - 1 = 0 \qquad n,\ r,\ s \text{ even},\ m \text{ odd}$$

This becomes VI_{b_1} by the transformation $y = -y$.

VI'''_{b_2}

$$x^n y^m - x^r y^s - 1 = 0 \qquad n,\ r \text{ even},\ m,\ s \text{ odd}$$

This becomes VI'_{b_2} by the transformation $y = -y$

VI'''_{b_3}

$$x^n y^m - x^r y^s - 1 = 0 \qquad n,\ s \text{ even},\ m,\ r \text{ odd}$$

This becomes VI_{b_3} by the transformation $y = -y$.

VI''_{b_4}

$$x^n y^m - x^r y^s - 1 = 0 \qquad n, \text{ even},\ m,\ r,\ s \text{ odd}$$

This becomes VI_{b_4} by the transformation $x = -x$.

VI''''_{c_1}

$x^n y^m - x^r y^s - 1 = 0$ n, odd, m, r, s even

This becomes VI_{c_1} by the transformation x = - x.

VI''''_{c_2}

$x^n y^m - x^r y^s - 1 = 0$ n, s odd m, r even

This becomes VI_{c_2} by the transformation x = - x.

VI''''_{c_3}

$x^n y^m - x^r y^s - 1 = 0$ n, r odd, m, s even

This becomes VI''_{c_3} by the transformation x = - x.

VI''''_{c_4}

$x^n y^m - x^r y^s - 1 = 0$ n, r, s odd, m even

This becomes VI'_{c_4} by the transformation y = - y.

VI''''_{d_1}

$x^n y^m - x^r y^s - 1 = 0$ n, m odd, r, s even

This becomes VI_{d_1} by the transformation x = - x.

VI''''_{d_2}

$x^n y^m - x^r y^s - 1 = 0$ n, m, s odd, r even

This becomes VI_{d_2} by the transformation x = - x.

VI''''_{d_3}

$x^n y^m - x^r y^s - 1 = 0$ n, m, r odd, s even

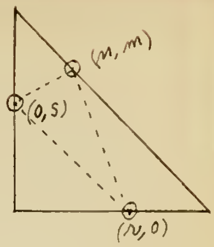
This becomes VI_{d_3} by the transformation y = - y.

VI''''_{d_4}

$x^n y^m - x^r y^s - 1 = 0$ n, m, r, s odd

This becomes VI''_{d_4} by the transformation x = - x.

Case VII



$$x^n y^m \pm x^r \pm y^s = 0 \qquad s < n+m > r$$

We will divide this into the following sub-cases.

a n, m even
- 1: r even, s even
- 2: r even, s odd
- 3: r odd, s even
- 4: r odd, s odd

b n even, m odd
- 1: r even, s even
- 2: r even, s odd
- 3: r odd, s even
- 4: r odd, s odd

c n odd, m even
- 1: r even, s even
- 2: r even, s odd
- 3: r odd, s even
- 4: r odd, s odd

d n, m odd
- 1: r even, s even
- 2: r even, s odd
- 3: r odd, s even
- 4: r odd, s odd

These sixteen cese can be reduced to six however, by the following transformations after the equations have been made homogeneous by the introduction of a new variable z.

VIIa_3	becomes	VIIa_2	by the transformation	x = y, y = x.
VIIb_2	" "	VIIb_1	" " " " "	y = z, z = y.
VIIb_4	" "	VIIb_1	" " " " "	x = y, y = z, z = x
VIIc_1	" "	VIIb_1	" " " " "	x = y, y = x.
VIIc_2	" "	VIIb_3	" " " " "	y = z, z = y.
VIIc_3	" "	VIIb_1	" " " " "	x = z, y = x, z = y
VIIc_4	" "	VIIb_1	" " " " "	x = z, z = x.
VIId_1	" "	VIIa_2	" " " " "	x = z, z = x.
VIId_2	" "	VIIa_4	" " " " "	y = z, z = y.
VIId_3	" "	VIIa_4	" " " " "	x = z, z = x.

Hence we have only to consider the sub-cases a_1, a_2, a_4, b_1, b_3 and d_4 with all the combinations of sign.

VII a_1

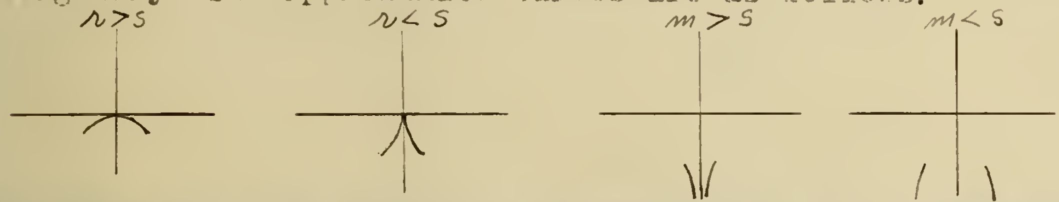
$x^n y^m + x^r + y^s = 0$ n, m, r, s even

The curve is imaginary but has one real point, the origin.

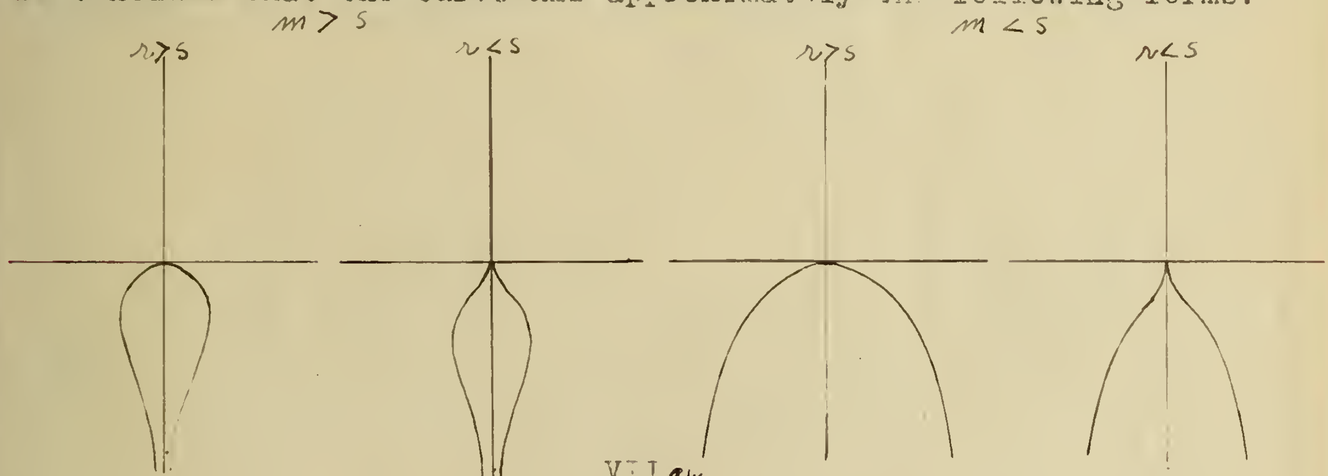
VII a_2

$x^n y^m + x^r + y^s = 0$ n, m, r even, s odd

The form at the origin is given by $x^r + y^s = 0$, at infinity on the y-axis by $x^n y^m + y^s = 0$. Each of these take two forms according as $r \gtrless s$ and $m \gtrless s$. The form at infinity on the x-axis given by $x^n y^m + x^r = 0$ is imaginary. The approximate curves are as follows.



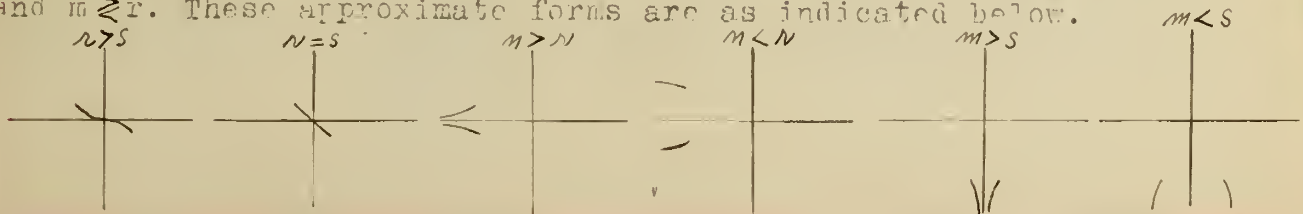
We conclude that the curve has approximately the following forms.



VII a_4

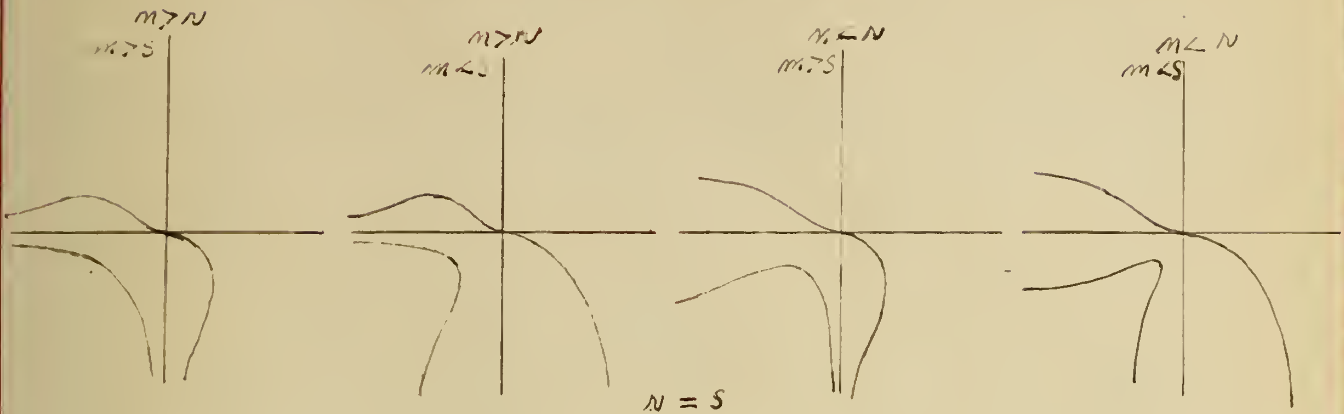
$x^n y^m + x^r + y^s = 0$ n, m even, r, s odd

The form at the origin is given by $x^r + y^s = 0$. This takes three forms according as $r \gtreqless s$. We will omit the case which $r < s$ for this can be transformed to the one in which $r > s$ by an interchange of x and y. The forms at infinity on the x-and y-axes given by $x^n y^m + x^r = 0$ and $x^n y^m + y = 0$ respectively each take two forms according as $n \gtrless r$ and $m \gtrless r$. These approximate forms are as indicated below.

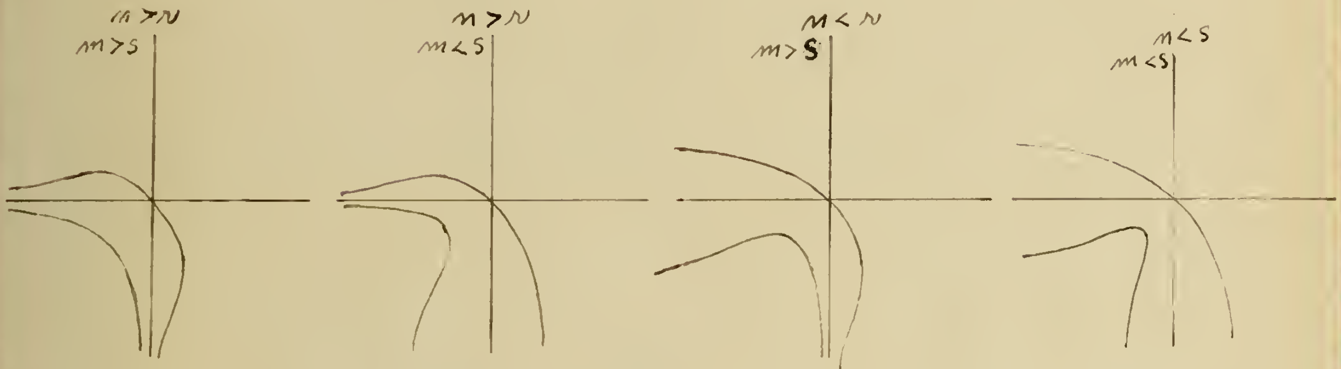


Hence we conclude that the curve has approximately the following forms.

$n > s$



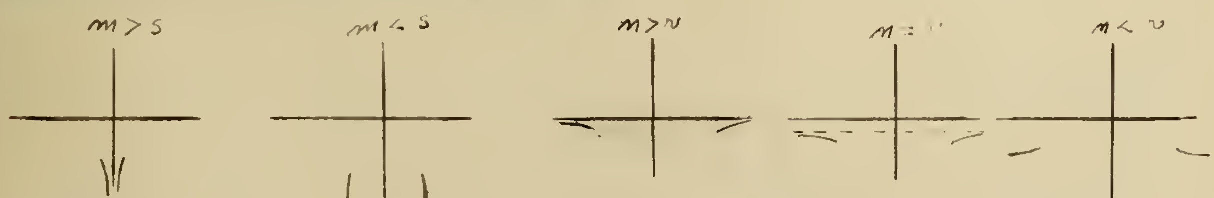
$n = s$



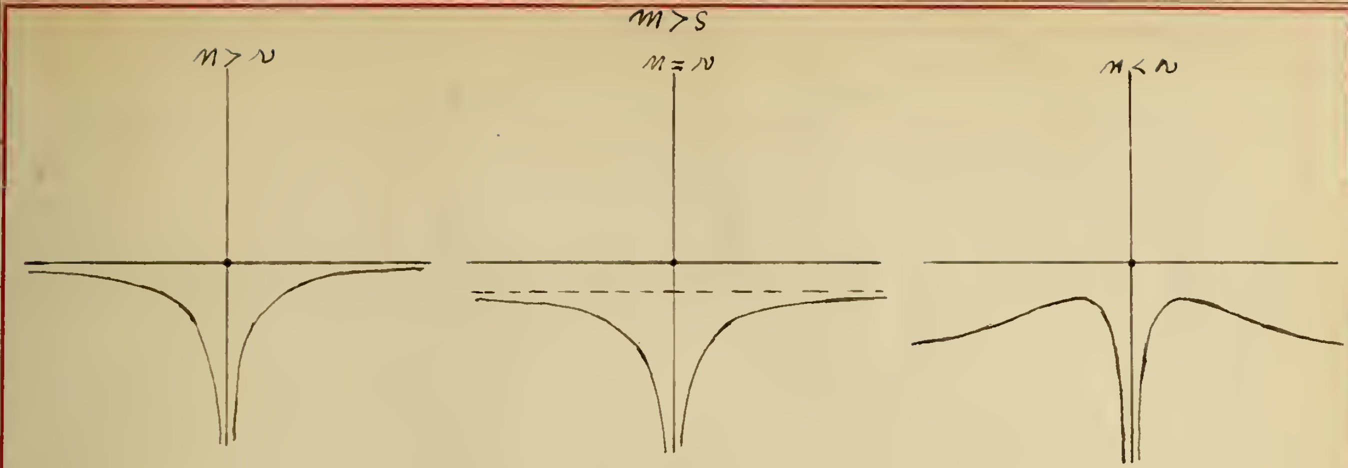
VII b_1

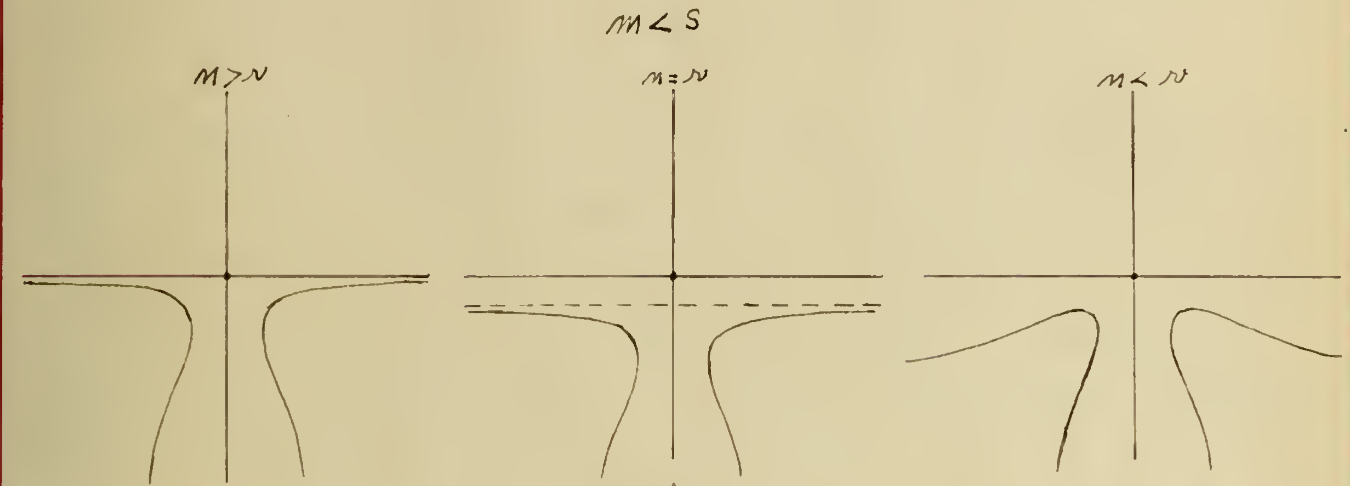
$x^n y^m + x^r + y^s = 0$ n, r, s even, m odd

The form at the origin given by $x^r + y^s = 0$ is imaginary . Hence since the origin is on the curve it is a conjugate point. The form at infinity on the y-axis given by $x^n y^m + y^s = 0$ has two forms according as $m \gtrless s$ and the form at infinity on the x-axis given by $x^n y^m + x^r = 0$ takes three forms according as $n \gtreqless r$.



If n = r the asymptote y = - 1 does not intersect the curve in finite points. The curve is symmetrical with respect to the y-axis. Hence we conclude that the curve has approximately the following forms.

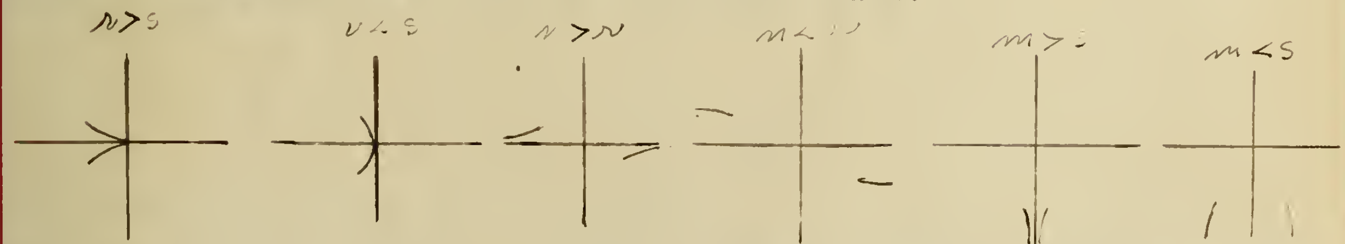




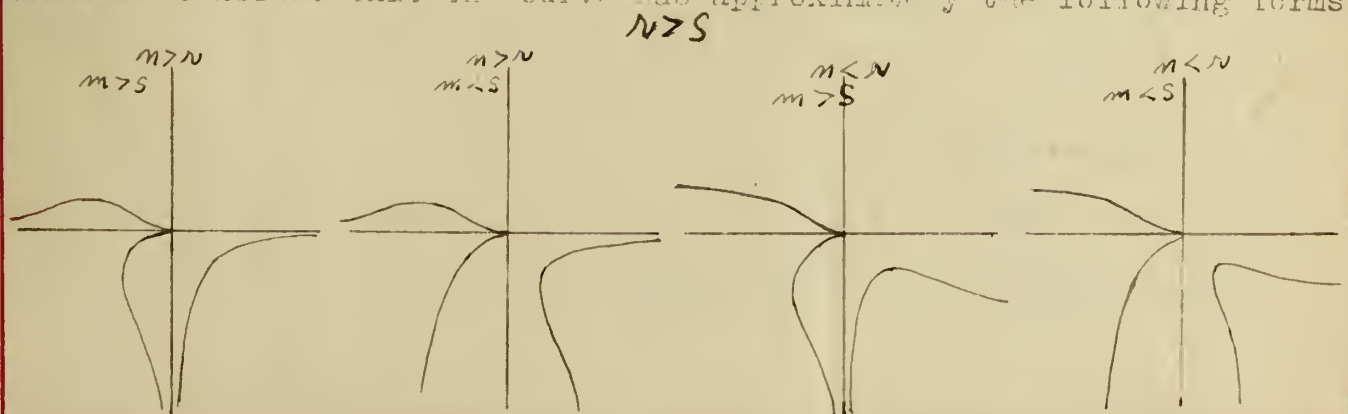
VIIb$_3$

$x^n y^m + x^r + y^s = 0$ n, s even, m, r odd

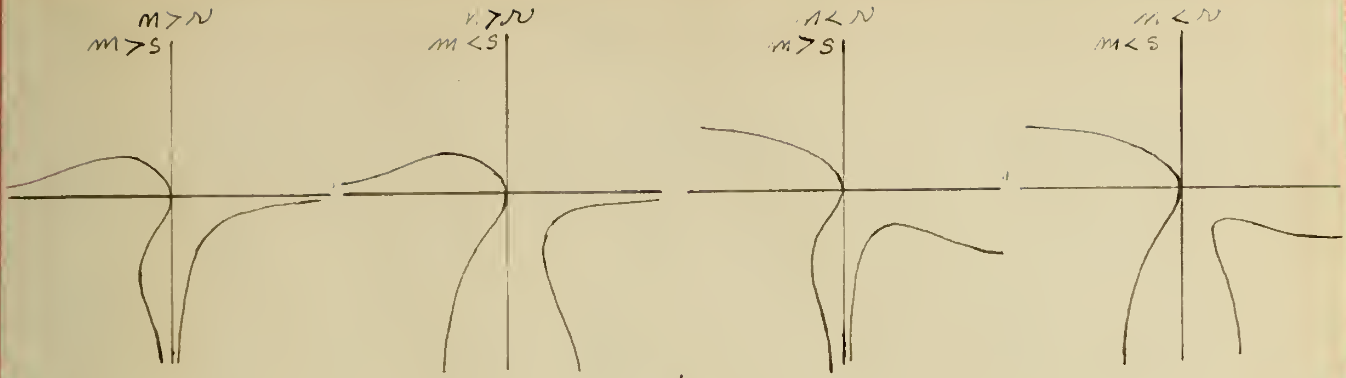
The form at the origin is given by $x^r + y^s = 0$, at infinity on the x-axis by $x^n y^m + x^r = 0$ and at infinity on the y-axis by $x^n y^m + y^s = 0$. Each of these take two forms as indicated below.



Hence we conclude that the curve has approximately the following forms.



$r < s$



VII d_4

$x^n y^m + x^r + y^s = 0$ n, m, r, s odd

The form at the origin $x^r + y^s = 0$ depends upon whether $r \gtreqless s$. The case in which $r < s$ can be transformed into the case in which $r > s$ by an interchange of x and y. The form at infinity on the x-axis is given by $x^n y^m + x^r = 0$ and on the y-axis by $x^n y^m + y^s = 0$. Each of these take three forms depending upon the relative value of the exponents. By an interchange of x and y the three cases

(1) $m = s = r$, $n > r$

(2) $m < s$, $r = s$, $n > r$

(3) $m < s$, $r = s$, $n = r$

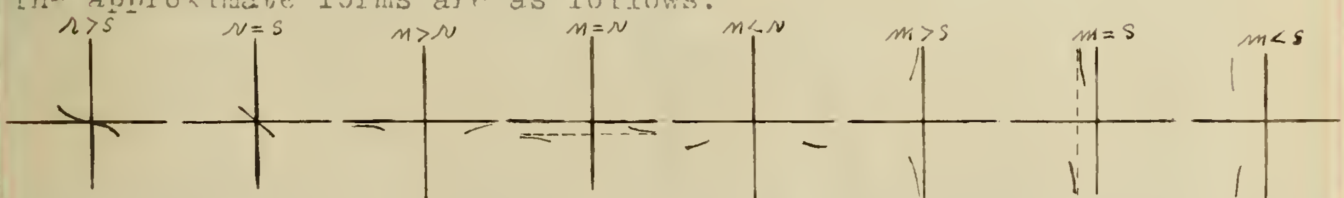
are transformed respectively into the three cases

(1') $n = r = s$, $m > s$

(2') $n < r$, $r = s$, $m > s$

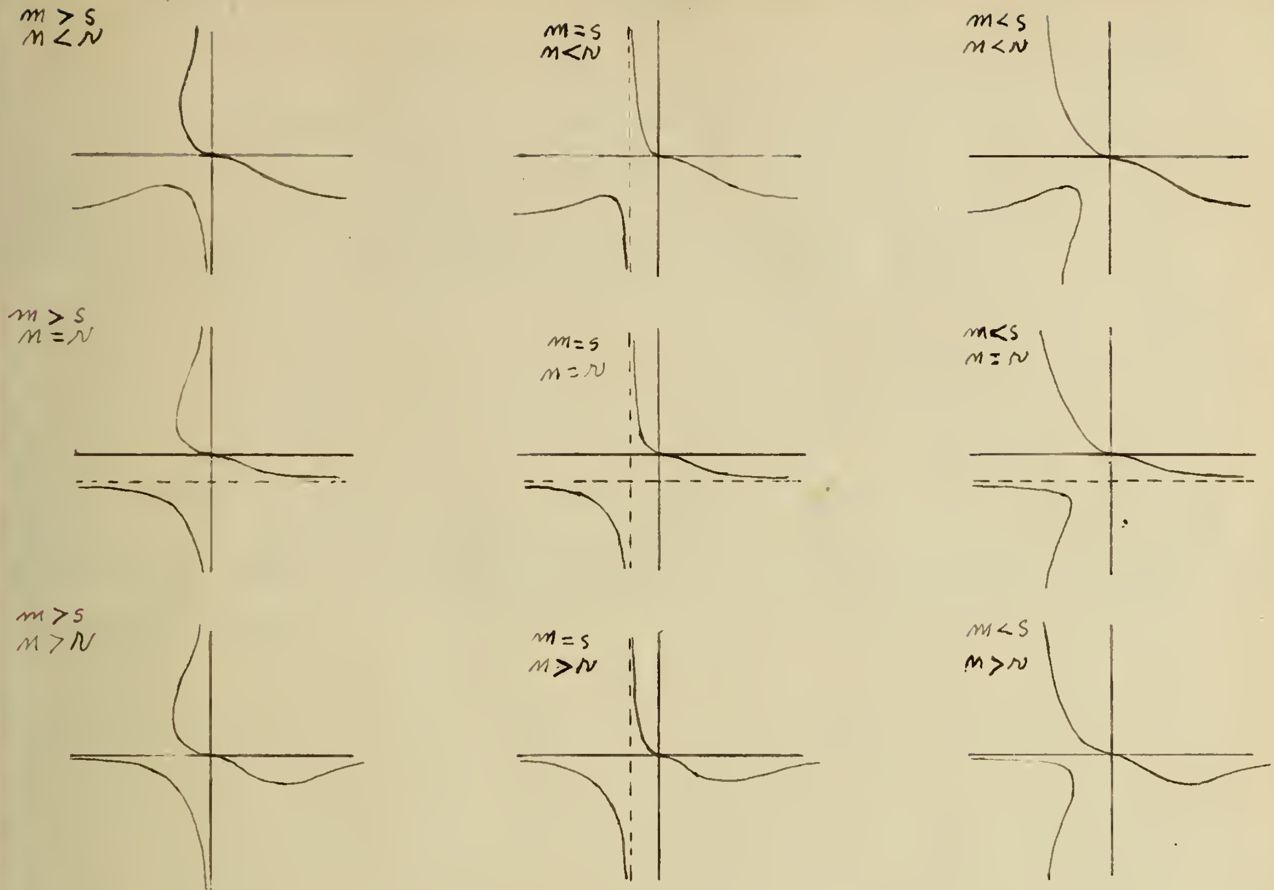
(3') $n < r$, $r = s$, $m = s$

The approximate forms are as follows.



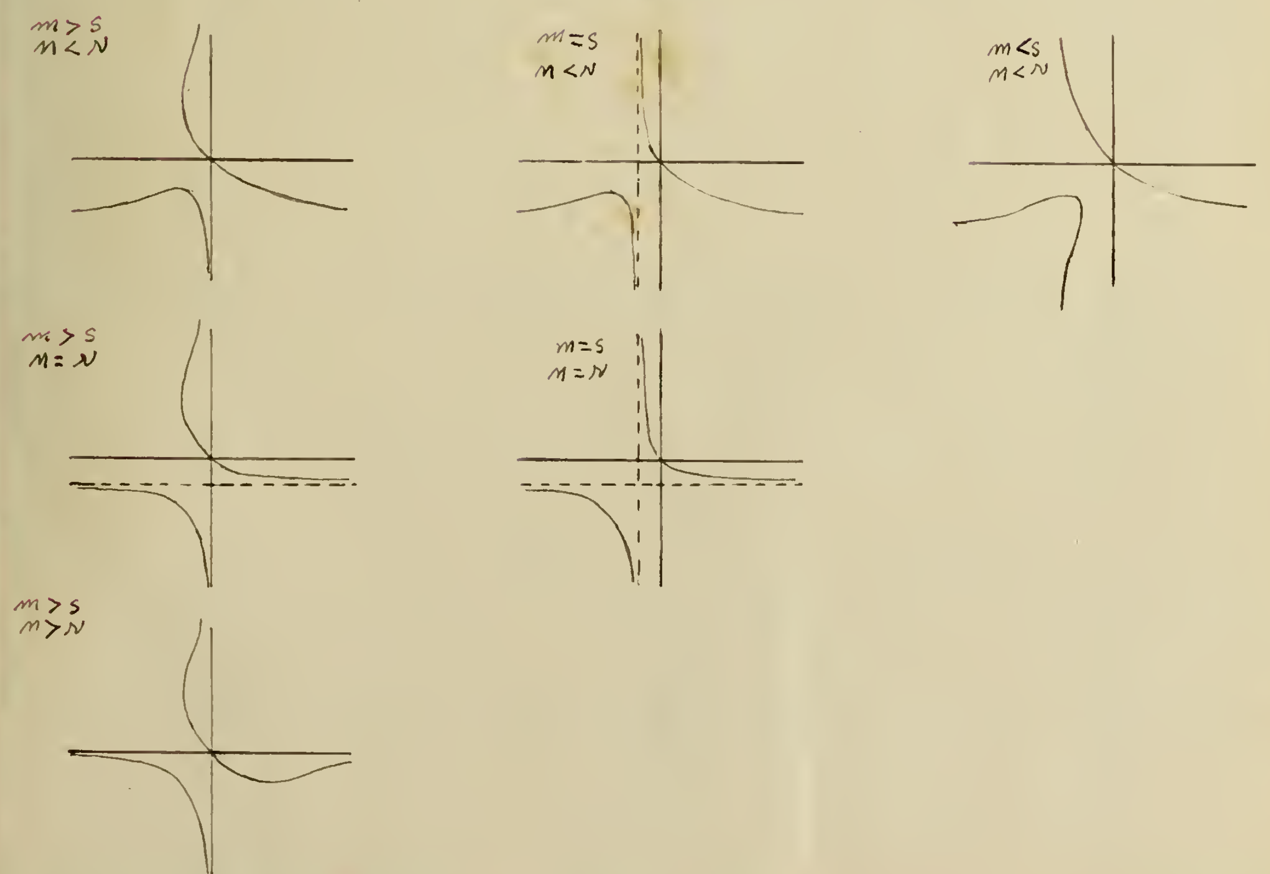
The asymptotes $y = -1$ and $x = -1$ do not intersect the curve in finite points in the cases in which $n = r$ and $m = s$ respectively. Hence, we conclude that the curve has approximately the following forms.

N > S



m > s
n < N
m = s
n < N
m < s
n < N
m > s
n = N
m = s
n = N
m < s
n = N
m > s
n > N
m = s
n > N
m < s
n > N

N = S

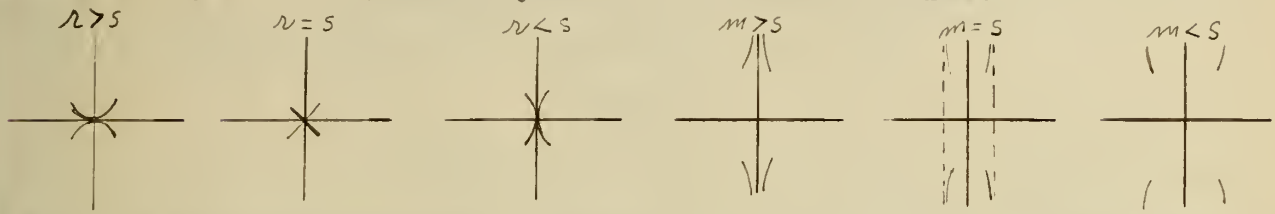


m > s
n < N
m = s
n < N
m < s
n < N
m > s
n = N
m = s
n = N
m > s
n > N

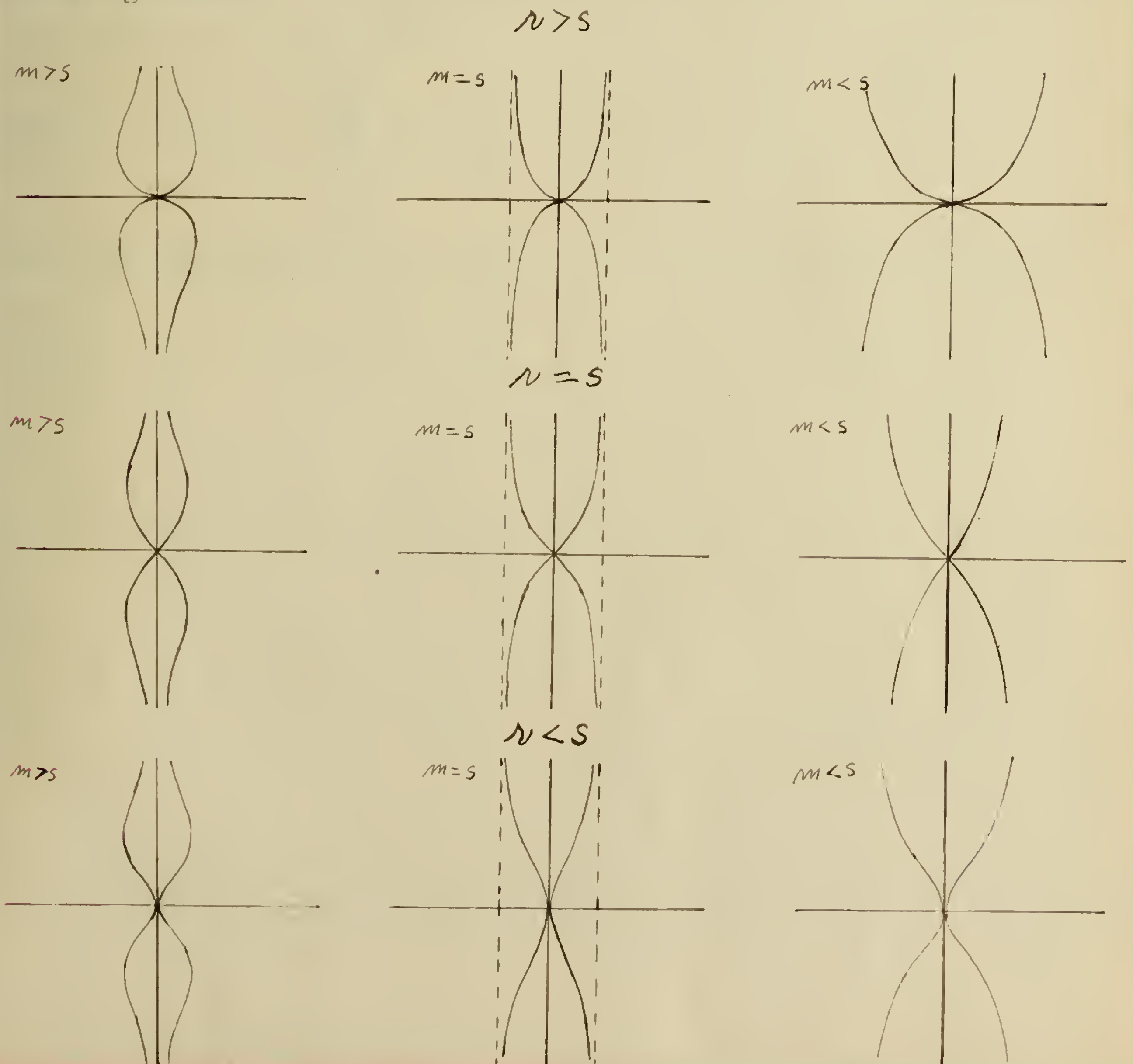
VII'a,

$$x^n y^m + x^r - y^s = 0 \qquad n,\ m,\ r,\ s \text{ even}$$

The form at infinity on the x-axis $x^n y^m + x^r = 0$ is imaginary. The forms at the origin and at infinity on the y-axis given by $x^r - y^s = 0$ and $x^n y^m - y^s = 0$ respectively are as indicated below.



If m = s the asymptotes x = ±1 do not intersect the curve in any finite points. Hence we conclude that the curve has approximately the following forms.



VII'$_{a_2}$

$$x^n y^m + x^r - y^s = 0 \qquad n,\ m,\ r \text{ even},\ s \text{ odd}$$

This is transformed to VII$_{a_2}$ by putting $y = -y$.

VII'$_{a_4}$

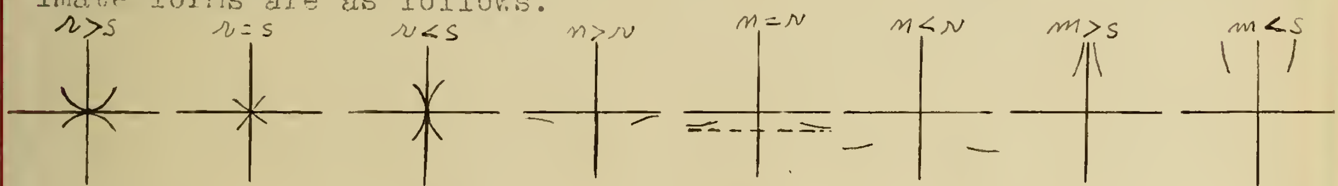
$$x^n y^m + x^r - y^s = 0 \qquad n,\ m \text{ even},\ r,\ s \text{ odd}$$

This is transformed to VII by putting $y = -y$.

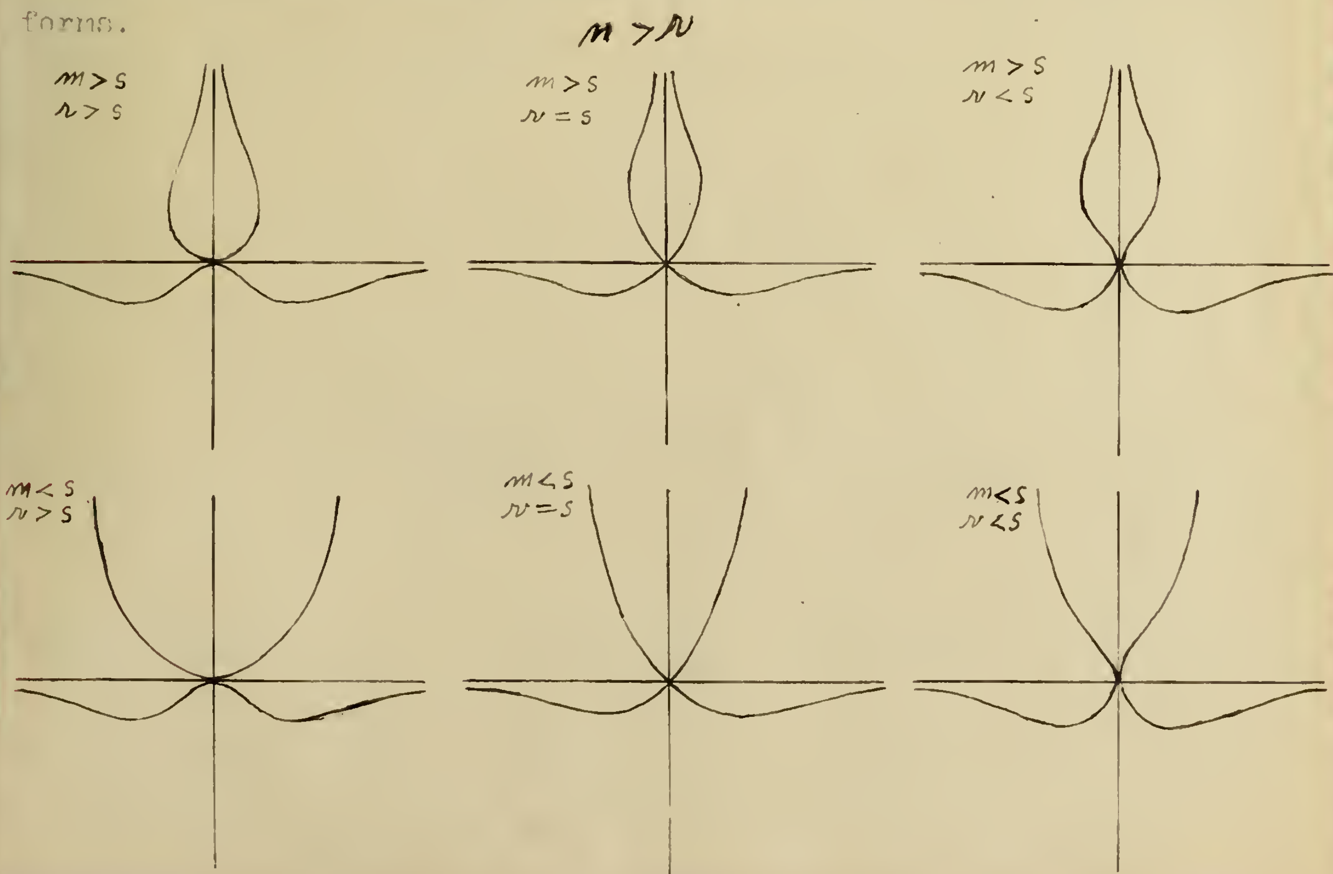
VII'$_{b_1}$

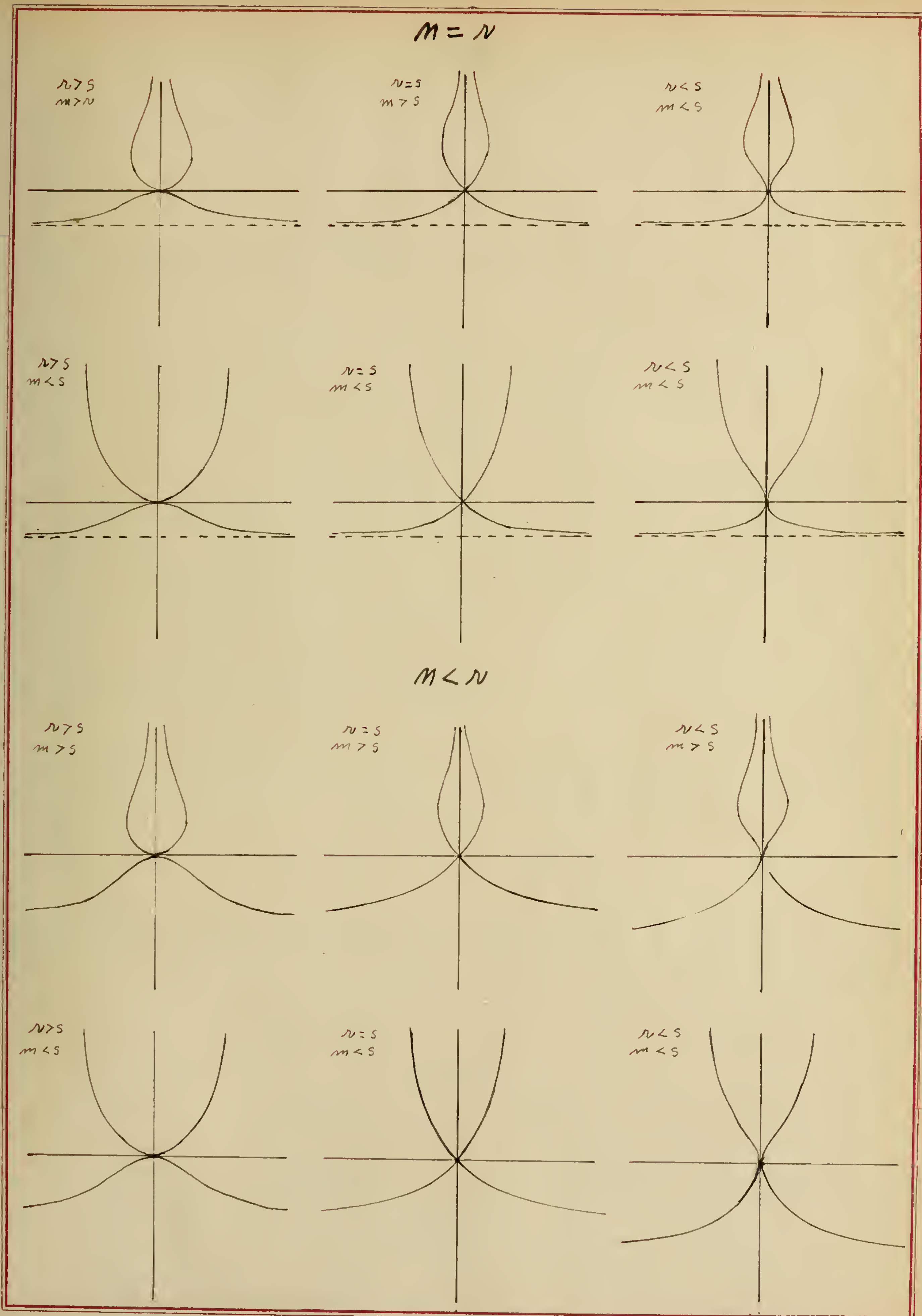
$$x^n y^m + x^r - y^s = 0 \qquad n,\ r,\ s \text{ even},\ m \text{ odd}$$

There are three forms at the origin depending upon whether $r \gtreqless s$, three at infinity on the x-axis depending upon whether $n \gtreqless r$ and two at infinity on the y-axis depending upon whether $m \gtrless s$. These approximate forms are as follows.



Hence we conclude that the curve has approximately the following forms.




m = n
n > s
m > n
n = s
m > s
n < s
m < s
n > s
m < s
n = s
m < s
n < s
m < s
m < n
n > s
m > s
n = s
m > s
n < s
m > s
n > s
m < s
n = s
m < s
n < s
m < s

VII'b_3

$$x^n y^m + x^r - y^s = 0 \qquad n,\ s \text{ even},\ m,\ r \text{ odd}$$

This is transformed to VIIb_3 by putting x = - x, y = - y.

VII'd_4

$$x^n y^m + x^r - y^s = 0 \qquad n,\ m,\ r,\ s \text{ odd}$$

This is transformed to VIId_4 by putting x = - x.

VII''a_1

$$x^n y^m - x^r + y^s = 0 \qquad n,\ m,\ r,\ s \text{ even}$$

This is transformed to VII'a_1 by an interchange of x and y.

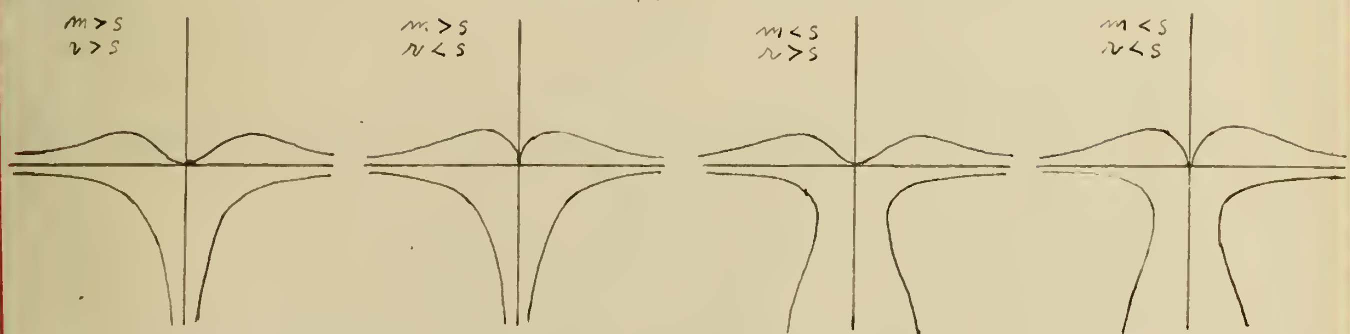
VII''a_2

$$x^n y^m - x^r + y^s = 0 \qquad n,\ m,\ r \text{ even},\ s \text{ odd}$$

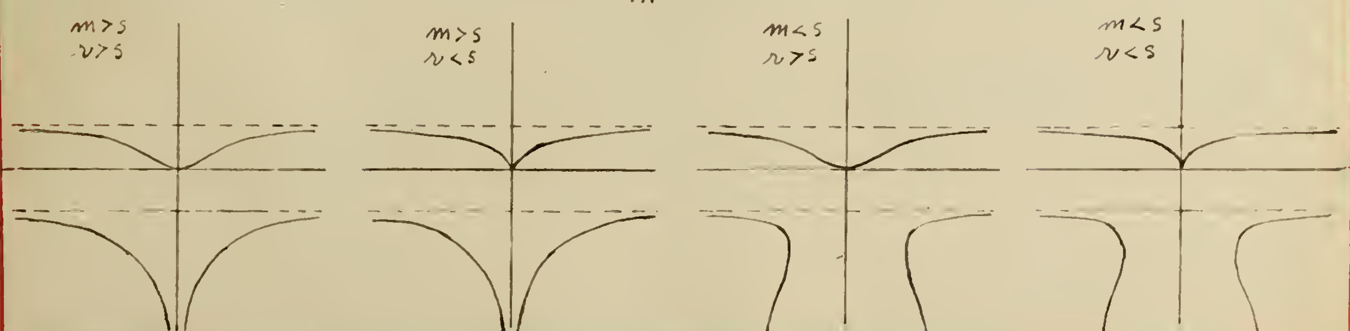
The form at the origin given by $x^r - y^s = 0$, the form at infinity on the x-axis given by $x^n y^m - x^r = 0$ and form at infinity on the y-axis given by $x^n y^m + y^s = 0$ are as follows.

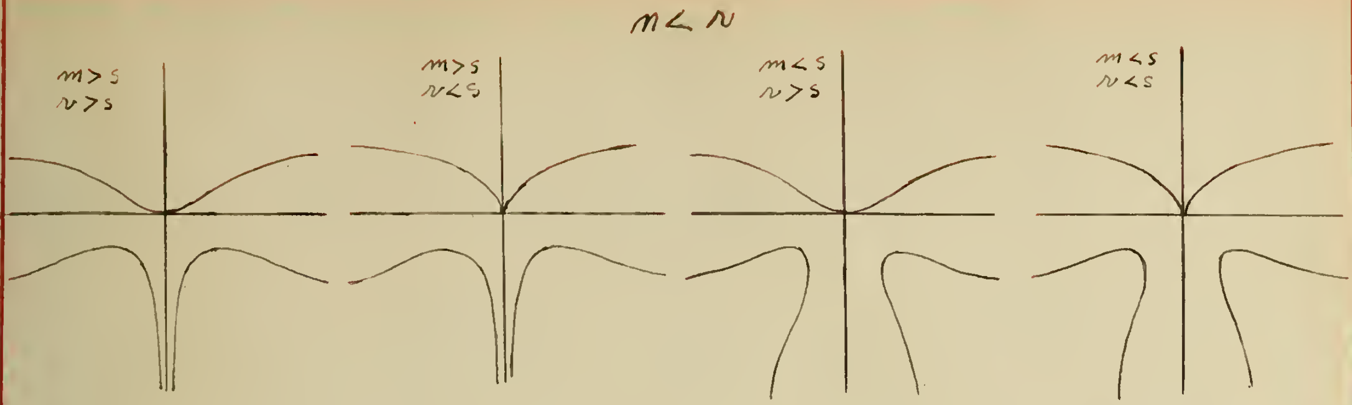
Hence we conclude that the curve has approximately the following forms.

$m > r$



$m = r$





VII'_{a_4}

$x^n y^m - x^r + y^s = 0$ n, m even, r, s odd

This becomes VII_{a_4} by the transformation x = - x.

VII'_{b_1}

$x^n y^m - x^r + y^s = 0$ n, r, s even, m odd

This becomes VII_{b_1} by the transformation y = - y.

VII'_{b_3}

$x^n y^m - x^r + y^s = 0$ n, s even, m, r odd

This is transformed to VII_{b_3} by putting x = - x.

VII'_{d_4}

$x^n y^m - x^r + y^s = 0$ n, m, r, s odd

This becomes VII_{d_4} by the transformation y = - y.

$VII'_{a_1}{}'$

$x^n y^m - x^r - y^s = 0$ n, m, r, s even

This becomes VII'_{a_1} by the transformation x = z; z = x.

$VII''_{a_2}{}'$

$x^n y^m - x^r - y^s = 0$ n, m, r even, s odd

This becomes VII''_{a_2} by the transformation y = - y.

VII'''_{a_4}

$x^n y^m - x^r - y^s = 0$ n, m even, r, s odd

This becomes VII_{a_4} by the transformation x = - x, y = - y.

$$VII'''_{b_1}$$

$$x^n y^m - x^r - y^s = 0 \qquad n,\ r,\ s \text{ even},\ m \text{ odd}$$

This becomes VII_{b_1} by the transformation $y = -y$.

$$VII'''_{b_3}$$

$$x^n y^m - x^r - y^s = 0 \qquad n,\ s \text{ even},\ m,\ r \text{ odd}$$

This becomes VII_{b_3} by the transformation $y = -y$.

$$VII'''_{d_4}$$

$$x^n y^m - x^r - y^s = 0 \qquad n,\ m,\ r,\ s \text{ odd}$$

This becomes VII_{d_4} by the transformation $x = -x$, $y = -y$.