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UNBOUNDED C^*-SEMINORMS AND UNBOUNDED C^*-SPECTRAL ALGEBRAS

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Abstract. Several $*$-algebras $\mathcal{A}$ carry with them unbounded C^*-seminorms in the sense that they are C^*-seminorms defined on $*$-subalgebras. Unbounded operator representations of $\mathcal{A}$ are constructed from such unbounded C^*-seminorms and they are investigated. The notions of spectrality and stability of unbounded C^*-seminorms are defined and studied.

Keywords: *(Hereditary) spectral unbounded C^*-seminorms, unbounded $*$-representations, stable unbounded C^*-seminorms.*

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1. INTRODUCTION

Unbounded C^*-seminorms on $*$-algebras in the sense that they are C^*-seminorms defined on $*$-subalgebras have appeared in many mathematical and physical subjects (for example, locally convex $*$-algebras in [5]–[8] and [18], and the quantum field theory in [1], [14] and [32] etc.). But this systematical study has not yet done sufficiently. The main purpose of this paper is to do a systematical study of unbounded C^*-seminorms and to apply it to a study of unbounded $*$-representations and that of locally convex $*$-algebras.

The paper is organized as follows: In Section 2 we construct unbounded $*$-representations of a $*$-algebra from unbounded C^*-seminorms and investigate them. Let $\mathcal{A}$ be a $*$-algebra. Let p be a C^*-seminorm defined on $\mathcal{A}$. Every $*$-representation of the Hausdorff completion of $(\mathcal{A}, p)$ gives rise to a $*$-representation of $\mathcal{A}$ into bounded Hilbert space operators. However, there are a number of situations in which natural C^*-seminorms are defined on $*$-subalgebras of $\mathcal{A}$. Then they should lead to unbounded operator representations of $\mathcal{A}$. An *unbounded m^*-(respectively C^*-) seminorm* is a submultiplicative $*$-(respectively C^*-) seminorm p defined on a $*$-subalgebra $\mathcal{D}(p)$ of $\mathcal{A}$. Then $N_p := \{x \in \mathcal{D}(p) : p(x) = 0\}$ is a $*$-ideal of $\mathcal{D}(p)$ and $\mathfrak{N}_p := \{x \in \mathcal{D}(p) : ax \in \mathcal{D}(p), \forall a \in \mathcal{A}\}$ is a left ideal of $\mathcal{A}$.

It is shown that any faithful nondegenerate $*$-representation $\Pi_p : \mathcal{A}_p \to \mathcal{B}(\mathcal{H})$ of the C^*-algebra $\mathcal{A}_p$ obtained by the Hausdorff completion of $(\mathcal{D}(p), p)$ leads to an unbounded $*$-representation π_p of $\mathcal{A}$ such that $\|\overline{\pi_p(x)}\| \leqslant p(x)$ for all $x \in \mathcal{D}(p)$. But, π_p is not necessarily nontrivial (that is, $\mathcal{H}_{\pi_p} \neq \{0\}$), and π_p is nontrivial if and only if $\mathfrak{N}_p \not\subset N_p$. We assume that an unbounded C^*-seminorm satisfies the condition $\mathfrak{N}_p \not\subset N_p$. Then π_p is always strongly nondegenerate. Here we say that a $*$-representation π is strongly nondegenerate if there exists a left ideal $\mathcal{I}$ of $\mathcal{A}$ contained in $\mathcal{A}^{\pi}_{\flat} := \{x \in \mathcal{A} : \pi(x) \text{ is bounded}\}$, such that $[\overline{\pi(\mathcal{I})}\mathcal{H}_\pi] = \mathcal{H}_\pi$, where $[\mathcal{K}]$ denotes the closed linear span of a subset $\mathcal{K}$ of a Hilbert space. We denote by $\mathrm{Rep}(\mathcal{A}, p)$ the set of all such $*$-representations π_p of $\mathcal{A}$. In order to investigate representations in $\mathrm{Rep}(\mathcal{A}, p)$ in details, we introduce the notions of nondegenerate, finite, uniformly semifinite, semifinite and weakly semifinite unbounded C^*-seminorms, and show that if p is weakly semifinite or semifinite, then there exists a strongly nondegenerate $*$-representation π_p in $\mathrm{Rep}(\mathcal{A}, p)$ such that $\|\overline{\pi_p(x)}\| = p(x)$ for all $x \in \mathcal{D}(p)$. Such a π_p is called *well-behaved*. In Section 3 we consider the converse direction of Section 2. We construct an unbounded C^*-seminorm r_π on $\mathcal{A}$ from a strongly nondegenerate $*$-representation π of $\mathcal{A}$ and a natural well-behaved representation $\pi^N_{r_\pi}$ of $\mathcal{A}$ constructed from r_π which is the restriction of the closure $\widetilde{\pi}$ of π. Further, it is shown that if p is a weakly semifinite unbounded C^*-seminorm on $\mathcal{A}$ and π_p is any well-behaved $*$-representation, then r_{π_p} is a maximal extension of p. In Section 4 we define and characterize the notion of regular unbounded C^*-seminorms. An unbounded C^*-seminorm on a $*$-algebra $\mathcal{A}$ is *regular* if it is a restriction of the unbounded C^*-seminorm $\sup\limits_\alpha p_\alpha$ defined by a family $\{p_\alpha\}$ of C^*-seminorms on $\mathcal{A}$. It is shown that given a semifinite unbounded C^*-seminorm p on $\mathcal{A}$, p is regular if and only if there exists a well-behaved $*$-representation π_p of $\mathcal{A}$ which is a restriction of the direct sum $\bigoplus\limits_\alpha \pi_\alpha$ of bounded $*$-representations π_α of $\mathcal{A}$.

In Section 5 we construct the unbounded Gelfand-Naimark C^*-seminorm $|\cdot|_p$ on $\mathcal{A}$ from an unbounded m^*-seminorm p on $\mathcal{A}$. Yood ([33]) has investigated some aspects of bounded C^*-seminorms by re-examining the construction of Gelfand-Naimark pseudo-norm discussed in [9]. Here we extend some of Yood's results about C^*-seminorms to unbounded C^*-seminorms. In Section 6 we apply the results developed earlier to the study of spectral algebras. Following Palmer ([22]) a *spectral algebra* $\mathcal{A}$ is an algebra on which there is defined a submultiplicative seminorm p (called a *spectral seminorm*) such that $\{x \in \mathcal{A} : p(x) < 1\} \subset \mathcal{A}^{\mathrm{qr}}$(= the set of all quasi-regular elements of $\mathcal{A}$). The morale of [22] and [23] is that even though a spectral algebra need not be normable, it is rich enough to recapture the pure algebraic flavour of much of the spectral theory of Banach algebras. We call an unbounded m^*-seminorm p to be *spectral* (respectively *hereditary spectral*) if $\{x \in \mathcal{D}(p) : p(x) < 1\} \subset \mathcal{D}(p)^{\mathrm{qr}}$ (respectively $p\lceil\mathcal{B}$ is spectral for each $*$-subalgebra $\mathcal{B}$ of $\mathcal{A}$). An unbounded $*$-representation π of $\mathcal{A}$ is a *spectral $*$-representation* (respectively a *hereditary spectral $*$-representation*) if $\mathrm{Sp}_{\mathcal{A}^\pi_\flat}(x) \subset \mathrm{Sp}_{C^*(\pi)}(\overline{\pi(x)}) \bigcup \{0\}$ for all $x \in \mathcal{A}, C^*(\pi)$ being the C^*-algebra generated by $\overline{\pi(\mathcal{A}^\pi_\flat)}$ (respectively $\pi\lceil\mathcal{B}$ is spectral for each $*$-subalgebra $\mathcal{B}$ of $\mathcal{A}$). It is shown that there exists a strongly nondegenerate $*$-representation π of $\mathcal{A}$ such that $\pi_\flat := \pi\lceil \mathcal{A}_\flat^\pi$ is (hereditary) spectral if and only if there exists a maximal, weakly semifinite, (hereditary) spectral unbounded C^*-seminorm on $\mathcal{A}$. Further, we define the notion of stability of unbounded m^*- (or C^*-) seminorms and characterize it by spectral unbounded C^*-seminorms. An unbounded m^*-seminorm p on $\mathcal{A}$ is called *stable* if for any $*$-subalgebra $\mathcal{B}$ of $\mathcal{A}$, any $*$-representation π of $\mathcal{B}$ such that $\mathcal{B}\bigcap \mathcal{D}(p) \subset \mathcal{B}_\flat^\pi$ and $[\pi(\mathcal{B}\bigcap\mathcal{D}(p))\mathcal{D}(\pi)] = \mathcal{H}_\pi$ can be dilated to a $*$-representation ϱ of $\mathcal{A}$ such that $\mathcal{D}(p) \subset \mathcal{A}_\flat^\varrho$ and $[\varrho(\mathcal{D}(\varrho))\mathcal{D}(\varrho)] = \mathcal{H}_\varrho$. It is shown that a semifinite unbounded C^*-seminorm on $\mathcal{A}$ is hereditary spectral if and only if it is spectral and stable. In Section 7 we give some examples of (regular, spectral, weakly semifinite, semifinite) unbounded C^*-seminorms on special $*$-algebras (locally m-convex $*$-algebras, pro-C^*-algebras, M*-like (or C^*-like) locally convex $*$-algebras, Köthe sequence algebras, O*-algebras). Throughout this paper we assume that a $*$-algebra $\mathcal{A}$ has always an identity $\mathbb{1}$ to simplify the arguments. This assumption does not lose the generality.

2. REPRESENTATIONS INDUCED BY UNBOUNDED C^*-SEMINORMS

In this section we construct a family of $*$-representations of a $*$-algebra $\mathcal{A}$ induced by an unbounded C^*-seminorm on $\mathcal{A}$ and investigate the properties. We begin with the review of (unbounded) $*$-representations of $\mathcal{A}$. Throughout this section let $\mathcal{A}$ be a $*$-algebra with identity $\mathbb{1}$. Let $\mathcal{D}$ be a dense subspace in a Hilbert space $\mathcal{H}$ and let $\mathcal{L}^\dagger(\mathcal{D})$ denote the set of all linear operators X in $\mathcal{H}$ with the domain $\mathcal{D}$ for which $X\mathcal{D} \subset \mathcal{D}$, $\mathcal{D}(X^*) \supset \mathcal{D}$ and $X^*\mathcal{D} \subset \mathcal{D}$. Then $\mathcal{L}^\dagger(\mathcal{D})$ is a $*$-algebra under the usual operations and the involution $X \to X^\dagger := X^*\lceil\mathcal{D}$. A $*$-subalgebra of the $*$-algebra $\mathcal{L}^\dagger(\mathcal{D})$ is said to be an O**-algebra on* $\mathcal{D}$ *in* $\mathcal{H}$. A $*$*-representation* π of $\mathcal{A}$ on a Hilbert space $\mathcal{H}$ with a domain $\mathcal{D}$ is a $*$-homomorphism of $\mathcal{A}$ into $\mathcal{L}^\dagger(\mathcal{D})$ and $\pi(\mathbb{1}) = I$, and then we write $\mathcal{D}$ and $\mathcal{H}$ by $\mathcal{D}(\pi)$ and $\mathcal{H}_\pi$, respectively. Let π_1 and π_2 be $*$-representations of $\mathcal{A}$. If $\mathcal{H}_{\pi_1}$ is a closed subspace of $\mathcal{H}_{\pi_2}$ and $\pi_1(x) \subset \pi_2(x)$ for each $x \in \mathcal{A}$, then π_2 is said to be an *extension* of π_1 and denoted by $\pi_1 \subset \pi_2$. In particular, if $\pi_1 \subset \pi_2$ and $\mathcal{H}_{\pi_1} = \mathcal{H}_{\pi_2}$, then π_2 is said to be an *extension of* π_1 *in the same Hilbert space*. Let π be a $*$-representation of $\mathcal{A}$. If $\mathcal{D}(\pi)$ is complete with the graph topology t_π defined by the family of seminorms $\{\|\cdot\|_{\pi(x)} := \|\cdot\| + \|\pi(x)\cdot\| : x \in \mathcal{A}\}$, then π is said to be *closed*. It is well known that π is closed if and only if $\mathcal{D}(\pi) = \bigcap_{x\in\mathcal{A}} \mathcal{D}(\overline{\pi(x)})$. The *closure* $\widetilde{\pi}$ of π is defined by

$$\mathcal{D}(\widetilde{\pi}) = \bigcap_{x\in\mathcal{A}} \mathcal{D}(\overline{\pi(x)}) \quad \text{and} \quad \widetilde{\pi}(x)\xi = \overline{\pi(x)}\xi \quad \text{for } x \in \mathcal{A},\ \xi \in \mathcal{D}(\widetilde{\pi}).$$

Then $\widetilde{\pi}$ is the smallest closed extension of π. The *weak commutant* $\pi(\mathcal{A})'_{\mathrm{w}}$ of π is defined by

$$\pi(\mathcal{A})'_{\mathrm{w}} = \{C \in \mathcal{B}(\mathcal{H}_\pi) : C\pi(x)\xi = \pi(x^*)^*C\xi,\ \forall x \in \mathcal{A},\ \forall\xi \in \mathcal{D}(\pi)\},$$

where $\mathcal{B}(\mathcal{H}_\pi)$ is the set of all bounded linear operators on $\mathcal{H}_\pi$, and it is a weakly closed $*$-invariant subspace of $\mathcal{B}(\mathcal{H}_\pi)$, but it is not necessarily an algebra. It is known that $\pi(\mathcal{A})'_{\mathrm{w}}\mathcal{D}(\pi) \subset \mathcal{D}(\pi)$ if and only if $\pi(\mathcal{A})'_{\mathrm{w}}$ is a von Neumann algebra and $\overline{\pi(x)}$ is affiliated with the von Neumann algebra $(\pi(\mathcal{A})'_{\mathrm{w}})'$ for each $x \in \mathcal{A}$. For more details we refer to [16], [19], [26] and [29].

DEFINITION 2.1. A mapping p of a subspace $\mathcal{D}(p)$ of $\mathcal{A}$ into $\mathbb{R}^+ = [0, \infty)$ is said to be an *unbounded (semi)norm* on $\mathcal{A}$ if it is a (semi)norm on $\mathcal{D}(p)$, and p is said to be an *unbounded m^*- (respectively C^*-) (semi)norm* on $\mathcal{A}$ if $\mathcal{D}(p)$ is a $*$-subalgebra of $\mathcal{A}$ and p is a submultiplicative $*$- (respectively C^*-) (semi)norm on $\mathcal{D}(p)$.

By [31], if a seminorm p on a $*$-algebra $\mathcal{A}$ is a C^*-seminorm, that is, it satisfies the C^*-property $p(x^*x) = p(x)^2$, $\forall x \in \mathcal{A}$, then it is a m^*-seminorm on $\mathcal{A}$, that is, $p(x^*) = p(x)$ and $p(xy) \leqslant p(x)p(y)$ for $\forall x, y \in \mathcal{A}$.

Let p be an unbounded C^*-seminorm on $\mathcal{A}$. We put

$$N_p = \{x \in \mathcal{D}(p) : p(x) = 0\} \quad \text{and} \quad \mathfrak{N}_p = \{x \in \mathcal{D}(p) : ax \in \mathcal{D}(p),\ \forall a \in \mathcal{A}\}.$$

Then N_p is a $*$-ideal of $\mathcal{D}(p)$ and $\mathfrak{N}_p$ is a left ideal of $\mathcal{A}$, and the quotient $*$-algebra $\mathcal{D}(p)/N_p$ is a normed $*$-algebra with the C^*-norm $\|x + N_p\|_p := p(x)$ $(x \in \mathcal{D}(p))$. We denote by $\mathcal{A}_p$ the C^*-algebra obtained by the completion of $\mathcal{D}(p)/N_p$, and denote by $\text{Rep}(\mathcal{A}_p)$ the set of all faithful nondegenerate $*$-representations Π_p of the C^*-algebra $\mathcal{A}_p$ on Hilbert spaces $\mathcal{H}_{\Pi_p}$. It is well known that $\text{Rep}(\mathcal{A}_p) \neq \emptyset$. For each $\Pi_p \in \text{Rep}(\mathcal{A}_p)$ we can define a bounded $*$-representation π_p^0 of $\mathcal{D}(p)$ on the Hilbert space $\mathcal{H}_{\Pi_p}$ by

$$\pi_p^0(x) = \Pi_p(x + N_p), \quad x \in \mathcal{D}(p).$$

The natural question arises: Can we extend the bounded $*$-representation π_p^0 of the $*$-algebra $\mathcal{D}(p)$ to a (generally unbounded) $*$-representation of the $*$-algebra $\mathcal{A}$? We show that this question has affirmative answer.

PROPOSITION 2.2. *Let p be an unbounded C^*-seminorm on $\mathcal{A}$. For any $\Pi_p \in \text{Rep}(\mathcal{A}_p)$, there exists a $*$-representation π_p of $\mathcal{A}$ on a Hilbert space $\mathcal{H}_{\pi_p}$ such that $\|\overline{\pi_p(b)}\| \leqslant p(b)$ for each $b \in \mathcal{D}(p)$ and $\|\overline{\pi_p(x)}\| = p(x)$ for each $x \in \mathfrak{N}_p$.*

Proof. Let $\Pi_p \in \text{Rep}(\mathcal{A}_p)$. We put

$$\begin{aligned} &\mathcal{D}(\pi_p) = \text{ linear span of } \{\Pi_p(x + N_p)\xi : x \in \mathfrak{N}_p, \xi \in \mathcal{H}_{\Pi_p}\}, \\ &\pi_p(a)\Big(\sum_k \Pi_p(x_k + N_p)\xi_k\Big) = \sum_k \Pi_p(ax_k + N_p)\xi_k \quad \text{(finite sums)} \end{aligned}$$

for $a \in \mathcal{A}$, $\{x_k\} \subset \mathfrak{N}_p$ and $\{\xi_k\} \subset \mathcal{H}_{\Pi_p}$. Since

$$\begin{aligned} (\Pi_p(ax + N_p)\xi | \Pi_p(y + N_p)\eta) &= (\xi | \Pi_p((ax + N_p)^*(y + N_p))\eta) \\ &= (\xi | \Pi_p(x^*a^*y + N_p)\eta) \\ &= (\xi | \Pi_p(x^* + N_p)\Pi_p(a^*y + N_p)\eta) \\ &= (\Pi_p(x + N_p)\xi | \Pi_p(a^*y + N_p)\eta) \end{aligned}$$

for each $a \in \mathcal{A}, x, y \in \mathfrak{N}_p$ and $\xi, \eta \in \mathcal{H}_{\Pi_p}$, it follows that $\pi_p(a)$ is a well-defined linear operator on $\mathcal{D}(\pi_p)$ for each $a \in \mathcal{A}$, so that it is easily shown that π_p is a $*$-representation of $\mathcal{A}$ on the Hilbert space $\mathcal{H}_{\pi_p} := [\mathcal{D}(\pi_p)] = \overline{\mathcal{D}(\pi_p)}^{\|\cdot\|}$ (the closure of $\mathcal{D}(\pi_p)$ in $\mathcal{H}_{\Pi_p}$) with domain $\mathcal{D}(\pi_p)$. Take an arbitrary $b \in \mathcal{D}(p)$. By the definition of π_p we have $\pi_p(b) = \pi_p^0(b)\lceil\mathcal{D}(\pi_p)$, and hence

$$\|\overline{\pi_p(b)}\| \leqslant \|\Pi_p(b + N_p)\| \leqslant \|b + N_p\|_p = p(b).$$

Suppose $x \in \mathfrak{N}_p$. It is sufficient to show that $\|\overline{\pi_p(x)}\| \geqslant p(x)$. If $p(x) = 0$, then it is obvious. Suppose $p(x) \neq 0$. We put $y = x/p(x) \in \mathfrak{N}_p$. For each $\xi \in \mathcal{H}_{\Pi_p}$ with $\|\xi\| \leqslant 1$, we have

$$\|\Pi_p(y + N_p)\xi\| \leqslant \|\Pi_p(y + N_p)\| \, \|\xi\| = p(y)\|\xi\| \leqslant 1,$$

and so

$$\begin{aligned}\|\overline{\pi_p(y)}\| = \|\overline{\pi_p(y^*)}\| &\geqslant \sup\left\{\|\pi_p(y^*)\Pi_p(y + N_p)\xi\| : \xi \in \mathcal{H}_{\Pi_p} \text{ such that } \|\xi\| \leqslant 1\right\} \\ &= \sup\left\{\|\Pi_p(y^*y + N_p)\xi\| : \xi \in \mathcal{H}_{\Pi_p} \text{ such that } \|\xi\| \leqslant 1\right\} \\ &= \|\Pi_p(y^*y + N_p)\| = p(y^*y) = p(y)^2 = 1.\end{aligned}$$

Hence, we have $\|\overline{\pi_p(x)}\| \geqslant p(x)$. This completes the proof. ∎

We simply sketch the method of the construction of the $*$-representation π_p:

REMARK 2.3. Let p be an unbounded C^*-seminorm on $\mathcal{A}$. As above, we can construct a set $\{\pi_p\}$ of $*$-representations of $\mathcal{A}$ from any $\Pi_p \in \operatorname{Rep}(\mathcal{A}_p)$, but π_p is not necessarily nontrivial, that is, the case $\mathcal{H}_{\pi_p} = \{0\}$ may arise (Example 7.1, (2)). It is clear that $\mathcal{H}_{\pi_p} \neq \{0\}$ if and only if $\mathfrak{N}_p \not\subset N_p$. Hereafter we shall assume that unbounded C^*-seminorms satisfy always this condition: $\mathfrak{N}_p \not\subset N_p$.

Let p be an unbounded C^*-seminorm on $\mathcal{A}$. We denote by $\operatorname{Rep}(\mathcal{A}, p)$ the set of all $*$-representations of $\mathcal{A}$ constructed as above by $(\mathcal{A}, p)$, that is,

$$\operatorname{Rep}(\mathcal{A}, p) = \{\pi_p : \Pi_p \in \operatorname{Rep}(\mathcal{A}_p)\}.$$

DEFINITION 2.4. An unbounded m^*-seminorm q on $\mathcal{A}$ is said to be *nondegenerate* if $\mathcal{D}(q)^2$ is total in $\mathcal{D}(q)$ with respect to the seminorm q. An unbounded m^*-seminorm q on $\mathcal{A}$ is said to be *finite* if $\mathcal{D}(q) = \mathfrak{N}_q$; and q is said to be *uniformly semifinite* if there exists a net $\{u_\alpha\}$ in $\mathfrak{N}_q$ such that $u_\alpha^* = u_\alpha$ and $q(u_\alpha) \leqslant 1$ for each α and $\lim_\alpha q(xu_\alpha - x) = 0$ for each $x \in \mathcal{D}(q)$; and q is said to be *semifinite* if $\mathfrak{N}_q$ is dense in $\mathcal{D}(q)$ with respect to the seminorm q. An unbounded C^*-seminorm p on $\mathcal{A}$ is said to be *weakly semifinite* if $\operatorname{Rep}^{\mathrm{WB}}(\mathcal{A}, p) := \{\pi_p \in \operatorname{Rep}(\mathcal{A}, p) : \mathcal{H}_{\pi_p} = \mathcal{H}_{\Pi_p}\} \neq \emptyset$. An element π_p of $\operatorname{Rep}^{\mathrm{WB}}(\mathcal{A}, p)$ is said to be a *well-behaved* $*$-representation of $\mathcal{A}$ in $\operatorname{Rep}(\mathcal{A}, p)$.

Definition 2.5. A $*$-representation π of $\mathcal{A}$ is said to be *strongly nondegenerate* if there exists a left ideal $\mathcal{I}$ of $\mathcal{A}$ contained in the bounded part $\mathcal{A}_\flat^\pi := \{x \in \mathcal{A} : \overline{\pi(x)} \in \mathcal{B}(\mathcal{H}_\pi)\}$ of π such that $[\overline{\pi(\mathcal{I})}\mathcal{H}_\pi] = \mathcal{H}_\pi$.

Proposition 2.6. *Let p be an unbounded C^*-seminorm on $\mathcal{A}$ and $\pi_p \in \mathrm{Rep}(\mathcal{A}, p)$. Then the following statements hold:*

(1) $[\overline{\pi_p(\mathfrak{N}_p)}\mathcal{H}_{\pi_p}] = \mathcal{H}_{\pi_p}$*, and so π_p is strongly nondegenerate.*

(2) *Suppose $\pi_p \in \mathrm{Rep}^{\mathrm{WB}}(\mathcal{A}, p)$. Then:*

(i) $\|\overline{\pi_p(x)}\| = p(x)$, $\forall x \in \mathcal{D}(p)$;

(ii) $\pi_p(\mathcal{A})'_{\mathrm{w}} = \overline{\pi_p(\mathcal{D}(p))}'$ *and* $\pi_p(\mathcal{A})'_{\mathrm{w}}\mathcal{D}(\pi_p) \subset \mathcal{D}(\pi_p)$.

(3) π_p *satisfies the condition* (2) (i) *if and only if there exists an element* π_p^{WB} *of* $\mathrm{Rep}^{\mathrm{WB}}(\mathcal{A}, p)$ *which is a restriction of* π_p.

(4) *Suppose p is semifinite. Then $\pi_p \in \mathrm{Rep}^{\mathrm{WB}}(\mathcal{A}, p)$ and $\mathfrak{N}_p^2$ is total in $\mathcal{D}(p)$ with respect to p, and so p is nondegenerate.*

(5) *Suppose p is uniformly semifinite. Then:*

$$\mathcal{A}_\flat^{\pi_p} = \mathcal{A}_\flat^p := \{a \in \mathcal{A} : \exists k_a > 0 \text{ such that } p(ax) \leqslant k_a p(x),\ \forall x \in \mathfrak{N}_p\},$$

$$\|\overline{\pi_p(b)}\| = \sup\{p(bx) : x \in \mathfrak{N}_p \text{ and } p(x) \leqslant 1\}, \quad \forall b \in \mathcal{A}_\flat^p$$

for each $\pi_p \in \mathrm{Rep}(\mathcal{A}, p)$.

(6) *p is finite if and only if $\mathcal{D}(p)$ is a left ideal of $\mathcal{A}$.*

Proof. (1) Since the $\|\cdot\|_p$-closure $\overline{\mathfrak{N}_p\lceil N_p}^{\|\cdot\|_p}$ of $\{x + N_p : x \in \mathfrak{N}_p\}$ in $\mathcal{A}_p$ is a left ideal of the C^*-algebra $\mathcal{A}_p$, it follows that there exists a left approximate identity $\{E_\alpha\}$ in $\overline{\mathfrak{N}_p\lceil N_p}^{\|\cdot\|_p}$, so that $\lim_\alpha \|(x + N_p)E_\alpha - (x + N_p)\|_p = 0$ for each $x \in \mathfrak{N}_p$. For any α, it follows since $E_\alpha \in \overline{\mathfrak{N}_p\lceil N_p}^{\|\cdot\|_p}$ that there exists a sequence $\{e_\alpha^{(n)}\}$ in $\mathfrak{N}_p$ such that $\lim_{n\to\infty} \|(e_\alpha^{(n)} + N_p) - E_\alpha\|_p = 0$. Take an arbitrary $\eta \in [\Pi_p(\mathfrak{N}_p + N_p)\mathcal{H}_{\Pi_p}] \ominus [\pi_p(\mathfrak{N}_p)\Pi_p(\mathfrak{N}_p + N_p)\mathcal{H}_{\Pi_p}]$. Then we have

$$\begin{aligned}(\Pi_p(x + N_p)\xi|\eta) &= \lim_\alpha(\Pi_p(x + N_p)\Pi_p(E_\alpha)\xi|\eta)\\ &= \lim_\alpha \lim_{n\to\infty}(\Pi_p(x + N_p)\Pi_p(e_\alpha^{(n)} + N_p)\xi|\eta)\\ &= \lim_\alpha \lim_{n\to\infty}(\pi_p(x)\Pi_p(e_\alpha^{(n)} + N_p)\xi|\eta) = 0\end{aligned}$$

for each $x \in \mathfrak{N}_p$ and $\xi \in \mathcal{H}_{\Pi_p}$, which implies that $[\pi_p(\mathfrak{N}_p)\Pi_p(\mathfrak{N}_p + N_p)\mathcal{H}_{\Pi_p}] = [\Pi_p(\mathfrak{N}_p + N_p)\mathcal{H}_{\Pi_p}] = \mathcal{H}_{\pi_p}$. Hence π_p is strongly nondegenerate.

(2) Suppose $\pi_p \in \mathrm{Rep}^{\mathrm{WB}}(\mathcal{A}, p)$. Since $\pi_p(b) = \Pi_p(b + N_p)\lceil\mathcal{D}(\pi_p)$, $\forall b \in \mathcal{D}(p)$ and $\mathcal{H}_{\Pi_p} = \overline{\mathcal{D}(\pi_p)}^{\|\cdot\|}$, it follows that $\overline{\pi_p(b)} = \Pi_p(b + N_p)$, $\forall b \in \mathcal{D}(p)$, which implies the statement (i). The statement (ii) follows since

$$\begin{aligned}C\Pi_p(x + N_p)\xi &= \Pi_p(x + N_p)C\xi \in \mathcal{D}(\pi_p(a)),\\ \pi_p(a)C\Pi_p(x + N_p)\xi &= \pi_p(a)\Pi_p(x + N_p)C\xi = \Pi_p(ax + N_p)C\xi\\ &= C\pi_p(a)\Pi_p(x + N_p)\xi\end{aligned}$$

for each $C \in \overline{\pi_p(\mathcal{D}(p))}'$, $a \in \mathcal{A}$, $x \in \mathfrak{N}_p$ and $\xi \in \mathcal{H}_{\pi_p}$.

(3) Suppose π_p satisfies condition (i) above. We put

$$\Pi_p^{\mathrm{WB}}(b+N_p)=\overline{\pi_p(b)}, \quad b\in\mathcal{D}(p).$$

Since $\|\Pi_p^{\mathrm{WB}}(b+N_p)\|=\|\overline{\pi_p(b)}\|=p(b)=\|b+N_p\|_p$ for each $b\in\mathcal{D}(p)$, it follows from (1) that Π_p^{WB} can be extended to a faithful nondegenerate $*$-representation of the C^*-algebra $\mathcal{A}_p$ on the Hilbert space $\mathcal{H}_{\pi_p}$ and denote it by the same Π_p^{WB}. We also denote by π_p^{WB} the strongly nondegenerate $*$-representation of $\mathcal{A}$ induced by Π_p^{WB}. Since

$$\begin{aligned}\mathcal{D}(\pi_p^{\mathrm{WB}}) &= \text{ linear span of } \{\Pi_p^{\mathrm{WB}}(x+N_p)\xi : x\in\mathfrak{N}_p,\ \xi\in\mathcal{H}_{\pi_p}\}\\ &= \text{ linear span of } \{\overline{\pi_p(x)}\xi : x\in\mathfrak{N}_p,\ \xi\in\mathcal{H}_{\pi_p}\},\end{aligned}$$

it follows from (1) that $\mathcal{H}_{\pi_p^{\mathrm{WB}}}=\mathcal{H}_{\pi_p}=\mathcal{H}_{\Pi_p^{\mathrm{WB}}}$, which means that $\pi_p^{\mathrm{WB}}\in\mathrm{Rep}^{\mathrm{WB}}(\mathcal{A},p)$. The converse follows from (2) (i).

(4) Suppose p is semifinite. Since p is semifinite, it follows that $\{\Pi_p(x+N_p) : x\in\mathfrak{N}_p\}$ is uniformly dense in the C^*-algebra $\Pi_p(\mathcal{A}_p)$, which implies by the nondegenerateness of Π_p that $\mathcal{H}_{\Pi_p}=\mathcal{H}_{\pi_p}$. Hence $\pi_p\in\mathrm{Rep}^{\mathrm{WB}}(\mathcal{A},p)$. By (1) we have $\mathrm{Rep}^{\mathrm{WB}}(\mathcal{A},p)=\mathrm{Rep}(\mathcal{A},p)$. Since the C^*-algebra $\mathcal{A}_p$ has a bounded approximate identity and $\mathfrak{N}_p$ is dense in $\mathcal{D}(p)$ with respect to p, it follows that $\mathfrak{N}_p^2$ is total in $\mathcal{D}(p)$ with respect to p.

(5) It is clear that $\mathcal{A}_\flat^{\pi_p}\subset\mathcal{A}_\flat^p$ without the assumption of the uniform semifiniteness of p. Suppose p is uniformly semifinite. Then we show the converse inclusion. Let $\{u_\alpha\}$ be in Definition 2.4. Take an arbitrary $a\in\mathcal{A}_\flat^p$, $\{x_k\}\subset\mathfrak{N}_p$ and $\{\xi_k\}\subset\mathcal{H}_{\Pi_p}$. Since

$$\begin{aligned}&\|\pi_p(a)\Pi_p(u_\alpha x_k+N_p)\xi_k-\pi_p(a)\Pi_p(x_k+N_p)\xi_k\|=\|\Pi_p(a(u_\alpha x_k-x_k)+N_p)\xi_k\|\\ &\qquad\leqslant k_a p(u_\alpha x_k-x_k)\|\xi_k\|=k_a p(x_k^*u_\alpha-x_k^*)\|\xi_k\|\xrightarrow[\alpha]{}0,\end{aligned}$$

it follows that

$$\begin{aligned}&\Big\|\pi_p(a)\sum_k\Pi_p(x_k+N_p)\xi_k\Big\|=\lim_\alpha\Big\|\pi_p(a)\sum_k\Pi_p(u_\alpha x_k+N_p)\xi_k\Big\|\\ &\quad=\lim_\alpha\Big\|\pi_p(au_\alpha)\sum_k\Pi_p(x_k+N_p)\xi_k\Big\|\leqslant\varlimsup_\alpha\|\overline{\pi_p(au_\alpha)}\|\,\Big\|\sum_k\Pi_p(x_k+N_p)\xi_k\Big\|\\ &\quad=\varlimsup_\alpha p(au_\alpha)\Big\|\sum_k\Pi_p(x_k+N_p)\xi_k\Big\|\leqslant k_a\Big\|\sum_k\Pi_p(x_k+N_p)\xi_k\Big\|,\end{aligned}$$

which implies $a\in\mathcal{A}_\flat^{\pi_p}$. Hence we have $\mathcal{A}_\flat^p=\mathcal{A}_\flat^{\pi_p}$.

(6) This is trivial. This completes the proof. ∎

3. UNBOUNDED C^*-SEMINORMS DEFINED BY $*$-REPRESENTATIONS

In Section 2 we constructed a family $\mathrm{Rep}(\mathcal{A},p)$ (respectively $\mathrm{Rep}^{\mathrm{WB}}(\mathcal{A},p)$) of strongly nondegenerate $*$-representations of $\mathcal{A}$ from an (respectively weakly semifinite) unbounded C^*-seminorm p on $\mathcal{A}$. Conversely we shall construct an unbounded C^*-seminorm r_π on $\mathcal{A}$ from a strongly nondegenerate $*$-representation π of $\mathcal{A}$ and the natural representation $\pi^N_{r_\pi}$ of $\mathcal{A}$ constructed from r_π, and investigate the relation between π and $\pi^N_{r_\pi}$. Let π be a strongly nondegenerate $*$-representation of $\mathcal{A}$ on a Hilbert space $\mathcal{H}_\pi$. We put

$$\mathcal{A}^\pi_\flat = \{x \in \mathcal{A} : \overline{\pi(x)} \in \mathcal{B}(\mathcal{H}_\pi)\} \quad \text{and} \quad \pi_\flat(x) = \overline{\pi(x)}, \quad x \in \mathcal{A}^\pi_\flat.$$

Then $\mathcal{A}^\pi_\flat$ is a $*$-subalgebra of $\mathcal{A}$ with the identity $\mathbb{1}$ and $\pi_\flat$ is a bounded $*$-representation of $\mathcal{A}^\pi_\flat$ on $\mathcal{H}_\pi$. We denote by $C^*(\pi)$ the C^*-algebra generated by $\pi_\flat(\mathcal{A}^\pi_\flat)$. We now define an unbounded C^*-seminorm r_π on $\mathcal{A}$ as follows:

$$\mathcal{D}(r_\pi) = \mathcal{A}^\pi_\flat \quad \text{and} \quad r_\pi(x) = \|\pi_\flat(x)\|, \quad x \in \mathcal{D}(r_\pi).$$

Then r_π satisfies the condition $\mathfrak{N}_{r_\pi} \not\subset N_{r_\pi}$. In fact, this follows since $\mathcal{I} \subset \mathfrak{N}_{r_\pi}$, where $\mathcal{I}$ is a left ideal of $\mathcal{A}$ contained in $\mathcal{A}^\pi_\flat$ such that $[\pi(\mathcal{I})\mathcal{D}(\pi)] = \mathcal{H}_\pi$. Here we put

$$\Pi(x + N_{r_\pi}) = \pi_\flat(x), \quad x \in \mathcal{A}^\pi_\flat.$$

Since $\|\Pi(x + N_{r_\pi})\| = r_\pi(x) = \|x + N_{r_\pi}\|_{r_\pi}$ for each $x \in \mathcal{A}^\pi_\flat$, it follows that Π can be extended to a faithful $*$-representation $\Pi^N_{r_\pi}$ of $\mathcal{A}_{r_\pi}$ on the Hilbert space $\mathcal{H}_\pi$. The $*$-representation $\pi^N_{r_\pi}$ of $\mathcal{A}$ defined by $\Pi^N_{r_\pi}$ as above is called the *natural representation* of $\mathcal{A}$ induced by π. Since $\mathcal{H}_{\Pi^N_{r_\pi}} = \mathcal{H}_\pi$, it follows that $\mathcal{H}_{\pi^N_{r_\pi}}$ is a closed subspace of $\mathcal{H}_\pi$. We simply sketch the above method of the construction of $\pi^N_{r_\pi}$:

We have the following results for the relation between π and $\pi^N_{r_\pi}$:

PROPOSITION 3.1. *Let π be a $*$-representation of $\mathcal{A}$. Suppose that π is strongly nondegenerate, that is, there exists a left ideal $\mathcal{I}$ of $\mathcal{A}$ contained in $\mathcal{A}^\pi_b$ such that $[\pi(\mathcal{I})\mathcal{D}(\pi)] = \mathcal{H}_\pi$. Then $\pi^N_{r_\pi} \in \mathrm{Rep}^{\mathrm{WB}}(\mathcal{A}, r_\pi)$ and $\pi^N_{r_\pi} \subset \widetilde{\pi}$. Furthermore, if $\pi(\mathcal{I})\mathcal{D}(\pi)$ is total in $\mathcal{D}(\pi)$ with respect to the graph topology t_π, then $\widetilde{\pi^N_{r_\pi}} = \widetilde{\pi}$.*

Proof. Since

$$\begin{aligned} \mathcal{D}(\pi^N_{r_\pi}) &= \text{ linear span of } \{\Pi^N_{r_\pi}(x+N_{r_\pi})\xi : x \in \mathfrak{N}_{r_\pi},\, \xi \in \mathcal{H}_\pi\} \\ &= \text{ linear span of } \{\overline{\pi(x)}\xi : x \in \mathfrak{N}_{r_\pi},\, \xi \in \mathcal{H}_\pi\}, \end{aligned} \tag{3.1}$$

it follows that

$$\begin{aligned} (\pi(a)^*\eta | \Pi^N_{r_\pi}(x+N_{r_\pi})\xi) &= (\pi(a)^*\eta|\overline{\pi(x)}\xi) = (\pi(x)^*\pi(a)^*\eta|\xi) \\ &= (\pi(ax)^*\eta|\xi) = (\eta|\overline{\pi(ax)}\xi) = (\eta|\pi^N_{r_\pi}(a)\Pi^N_{r_\pi}(x+N_{r_\pi})\xi) \end{aligned}$$

for each $a \in \mathcal{A}, \eta \in \mathcal{D}(\pi(a)^*), x \in \mathfrak{N}_{r_\pi}$ and $\xi \in \mathcal{H}_\pi$, which implies $\Pi^N_{r_\pi}(x+N_{r_\pi})\xi \in \mathcal{D}(\overline{\pi(a)})$ and $\overline{\pi(a)}\Pi^N_{r_\pi}(x+N_{r_\pi})\xi = \pi^N_{r_\pi}(a)\Pi^N_{r_\pi}(x+N_{r_\pi})\xi$. Hence, $\mathcal{D}(\pi^N_{r_\pi}) \subset \mathcal{D}(\widetilde{\pi})$ and $\widetilde{\pi}\lceil\mathcal{D}(\pi^N_{r_\pi}) = \pi^N_{r_\pi}$.

Since π is strongly nondegenerate and $\mathcal{A}^\pi_\flat = \mathcal{D}(r_\pi)$, it follows that $[\overline{\pi(\mathfrak{N}_{r_\pi})}\mathcal{H}_\pi] = \mathcal{H}_\pi$, which implies by (3.1) that $\mathcal{H}_{\pi^N_{r_\pi}} = \mathcal{H}_\pi = \mathcal{H}_{\Pi^N_{r_\pi}}$, so that $\pi^N_{r_\pi} \in \mathrm{Rep}^{\mathrm{WB}}(\mathcal{A}, r_\pi)$. Suppose that $\pi(\mathcal{I})\mathcal{D}(\pi)$ is total in $\mathcal{D}(\pi)[t_\pi]$. Then it follows from (3.1) that $\widetilde{\pi^N_{r_\pi}} = \widetilde{\pi}$. This complete the proof. ∎

By Proposition 2.6 and Proposition 3.1 we have the following diagram:

And we have the following

COROLLARY 3.2. *The following statements are equivalent:*

(i) *There exists an unbounded C^*-seminorm p on $\mathcal{A}$ such that $\mathfrak{N}_p \not\subset N_p$.*
(ii) *There exists a strongly nondegenerate $*$-representation of $\mathcal{A}$.*
(iii) *There exists a well-behaved $*$-representation of $\mathcal{A}$.*

Next we investigate the relations between unbounded C^*-seminorms p and r_{π_p} and the $*$-representations π_p and $\pi^N_{r_{\pi_p}}$. We first define an order relation among unbounded seminorms as follows:

DEFINITION 3.3. Let p and q be unbounded seminorms on $\mathcal{A}$. We say that p is an *extension* of q (or q is a *restriction* of p) if $\mathcal{D}(q) \subset \mathcal{D}(p)$ and $q(x) = p(x)$ for each $x \in \mathcal{D}(q)$, and then denote by $q \subset p$.

We denote by $\mathrm{C^*N}(\mathcal{A})$ the set of all unbounded C^*-seminorms p on $\mathcal{A}$ such that $\mathfrak{N}_p \not\subset N_p$. Then $\mathrm{C^*N}(\mathcal{A})$ is a partially ordered set with the order $\subset$. For any $p \in \mathrm{C^*N}(\mathcal{A})$ we put

$$\mathrm{C^*N}(p) = \{q \in \mathrm{C^*N}(\mathcal{A}) : p \subset q\}.$$

Then it follows from Zorn's lemma that $\mathrm{C^*N}(p)$ has a maximal element. We show that if p is weakly semifinite then r_{π_p} is a maximal element of $\mathrm{C^*N}(p)$.

PROPOSITION 3.4. *Suppose p is a weakly semifinite unbounded C^*-seminorm on $\mathcal{A}$ and $\pi_p \in \mathrm{Rep}^{\mathrm{WB}}(\mathcal{A}, p)$. Then r_{π_p} is a maximal element of* $\mathrm{C^*N}(p)$ *and $r_{\pi_p} = r_{\pi'_p}$ for each $\pi_p, \pi'_p \in \mathrm{Rep}^{\mathrm{WB}}(\mathcal{A}, p)$.*

Proof. We show that r_{π_p} is a maximal element of $\mathrm{C^*N}(p)$. Take an arbitrary $r \in \mathrm{C^*N}(r_{\pi_p})$. By Proposition 2.6 we have $p \subset r_{\pi_p} \subset r$, and so it follows that the linear map: $x + N_p \in \mathcal{D}(p)/N_p \longmapsto x + N_r \in \mathcal{D}(p)/N_r$ is a bijection and isometry, so that $\mathcal{A}_p$ is regarded as a closed $*$-subalgebra of the C^*-algebra $\mathcal{A}_r$. By the stability of C^*-algebras ([11], Proposition 2.10.2) there exists a $*$-representation Π_r of $\mathcal{A}_r$ such that $\Pi_p \subset \Pi_r$. Then we can construct in the same way as the proof of Proposition 2.6 the $*$-representation π_r of $\mathcal{A}$ induced by Π_r which is an extention of π_p, which implies that $\pi_p(a)$ is bounded and

$$\|\overline{\pi_p(a)}\| \leqslant \|\overline{\pi_r(a)}\| \leqslant r(a), \quad \forall a \in \mathcal{D}(r). \tag{3.2}$$

Hence we have

$$\mathcal{D}(r) \subset \mathcal{D}(r_{\pi_p}). \tag{3.3}$$

On the other hand, since $r_{\pi_p} \subset r$, we have $r = r_{\pi_p}$. We next show that $r_{\pi_p} = r_{\pi'_p}$ for each $\pi_p, \pi'_p \in \mathrm{Rep}^{\mathrm{WB}}(\mathcal{A}, p)$. Since $p \subset r := r_{\pi'_p}$, it follows from (3.2) and (3.3) that $\mathcal{D}(r_{\pi'_p}) = \mathcal{D}(r) \subset \mathcal{D}(r_{\pi_p})$ and $r_{\pi_p}(x) = \|\overline{\pi_p(x)}\| \leqslant r(x) = r_{\pi'_p}(x)$ for each $x \in \mathcal{D}(r) = \mathcal{D}(r_{\pi'_p})$ Similarly we have that $\mathcal{D}(r_{\pi_p}) \subset \mathcal{D}(r_{\pi'_p})$ and $r_{\pi'_p}(x) \leqslant r_{\pi_p}(x)$ for each $x \in \mathcal{D}(r_{\pi_p})$. Hence, $r_{\pi_p} = r_{\pi'_p}$. This completes the proof. ∎

By Proposition 3.1 and Proposition 3.4 we have the following

COROLLARY 3.5. *Suppose π is a strongly nondegenerate $*$-representation of $\mathcal{A}$. Then r_π is maximal.*

For the relation of $*$-representations π_p and $\pi^N_{r_{\pi_p}}$ we have the following

PROPOSITION 3.6. *Suppose p is a weakly semifinite unbounded C^*-seminorm on $\mathcal{A}$ and $\pi_p \in \mathrm{Rep}^{\mathrm{WB}}(\mathcal{A}, p)$. Then $\pi_p \subset \pi^N_{r_{\pi_p}}$ and $\widetilde{\pi^N_{r_{\pi_p}}} = \widetilde{\pi_p}$.*

Proof. It follows from the definition of $\pi^N_{r_{\pi_p}}$ that $\mathcal{H}_{\pi_p} = \mathcal{H}_{\Pi^N_{r_{\pi_p}}}$ and since $\mathfrak{N}_p \subset \mathfrak{N}_{r_{\pi_p}} \subset \mathcal{A}^{\pi_p}_{\flat}$ and

$$\Pi_p(x + N_p)\xi = \overline{\pi_p(x)}\xi = \Pi^N_{r_{\pi_p}}(x + N_{r_{\pi_p}})\xi$$

for each $x \in \mathfrak{N}_p$ and $\xi \in \mathcal{H}_{\pi_p}$, we have $\mathcal{D}(\pi_p) \subset \mathcal{D}(\pi^N_{r_{\pi_p}})$. Furthermore, since

$$\pi_p(a)\Pi_p(x + N_p)\xi = \overline{\pi_p(ax)}\xi = \pi^N_{r_{\pi_p}}(a)\Pi^N_{r_{\pi_p}}(x + N_{r_{\pi_p}})\xi = \pi^N_{r_{\pi_p}}(a)\Pi(x + N_p)\xi$$

for each $a \in \mathcal{A}, x \in \mathfrak{N}_p$ and $\xi \in \mathcal{H}_{\pi_p}$, it follows that $\pi_p = \pi^N_{r_{\pi_p}} \lceil \mathcal{D}(\pi_p)$. On the other hand, we have $\mathcal{D}(\pi^N_{r_{\pi_p}}) \subset \mathcal{D}(\widetilde{\pi_p})$ by Proposition 3.1. Therefore it follows that $\mathcal{H}_{\pi_p} = \mathcal{H}_{\pi^N_{r_{\pi_p}}}$, $\pi_p \subset \pi^N_{r_{\pi_p}}$ and $\widetilde{\pi_p} = \widetilde{\pi^N_{r_{\pi_p}}}$. This completes the proof. ∎

4. REGULAR UNBOUNDED C^*-SEMINORMS

In this section we define and characterize the notion of regular unbounded C^*-seminorms on $*$-algebras. We first prepare an unbounded C^*-seminorm $\sup\limits_\alpha p_\alpha$ constructed by a family $\{p_\alpha\}$ of unbounded C^*-seminorms on $\mathcal{A}$ and the notion of direct sum of $*$-representations of $\mathcal{A}$. Let $\{p_\alpha\}$ be a family of unbounded C^*-seminorms on $\mathcal{A}$. We put

$$\mathcal{D}(\sup_\alpha p_\alpha) = \big\{x \in \bigcap_\alpha \mathcal{D}(p_\alpha) : \sup_\alpha p_\alpha(x) < \infty\big\},$$
$$(\sup_\alpha p_\alpha)(x) = \sup_\alpha p_\alpha(x), \quad x \in \mathcal{D}(\sup_\alpha p_\alpha).$$

Then $\sup\limits_\alpha p_\alpha$ is an unbounded C^*-seminorm on $\mathcal{A}$, and it is an unbounded C^*-norm if and only if $p_\alpha(x) = 0$, $\forall\alpha$ implies $x = 0$.

DEFINITION 4.1. An unbounded C^*-(semi)norm p on $\mathcal{A}$ is said to be *regular* if $p \subset \sup\limits_\alpha p_\alpha$, where $\{p_\alpha\}$ is a family of C^*-seminorms on $\mathcal{A}$.

Let $\{\pi_\alpha\}$ be a family of $*$-representations of $\mathcal{A}$. We put

$$\mathcal{D}\Big(\bigoplus_\alpha \pi_\alpha\Big) = \Big\{\xi = (\xi_\alpha) \in \bigoplus_\alpha \mathcal{H}_{\pi_\alpha} : \xi_\alpha \in \mathcal{D}(\pi_\alpha),\ \forall\alpha \ \text{and} \sum_\alpha \|\pi_\alpha(a)\xi_\alpha\|^2 < \infty,\ \forall a \in \mathcal{A}\Big\},$$
$$\Big(\bigoplus_\alpha \pi_\alpha\Big)(a)(\xi_\alpha) = (\pi_\alpha(a)\xi_\alpha), \quad a \in \mathcal{A},\ (\xi_\alpha) \in \mathcal{D}\Big(\bigoplus_\alpha \pi_\alpha\Big).$$

Then $\bigoplus\limits_\alpha \pi_\alpha$ is a $*$-representation of $\mathcal{A}$ on $\bigoplus\limits_\alpha \mathcal{H}_{\pi_\alpha}$ such that

$$x \in \mathcal{A}_\flat^{\oplus_\alpha \pi_\alpha} \ \text{iff}\ \pi_\alpha(x) \text{ is bounded } \forall\alpha, \quad \text{and} \quad \sup_\alpha \|\overline{\pi_\alpha(x)}\| < \infty.$$

DEFINITION 4.2. A $*$-representation π of $\mathcal{A}$ is said to be *weakly bounded* if $\pi \subset \bigoplus\limits_\alpha \pi_\alpha$ as the same Hilbert space, where $\{\pi_\alpha\}$ is a family of bounded $*$-representations of $\mathcal{A}$.

LEMMA 4.3. *Let p be an unbounded C^*-seminorm on $\mathcal{A}$. Suppose $p \subset \sup\limits_\alpha p_\alpha$ for a net $\{p_\alpha\}$ of weakly semifinite unbounded C^*-seminorms on $\mathcal{A}$, and further $\mathfrak{N}_p$ is dense in $\mathcal{D}(p_\alpha)$ with respect to $\{p_\alpha\}$. Then p is weakly semifinite, and for any π_{p_α} of $\mathrm{Rep}^{\mathrm{WB}}(\mathcal{A}, p_\alpha)$ $\forall\alpha$, there exists an element π_p of $\mathrm{Rep}^{\mathrm{WB}}(\mathcal{A}, p)$ such that $\pi_p \subset \bigoplus\limits_\alpha \pi_{p_\alpha}$.*

Proof. We put

$$\Pi_p(x + N_p)(\xi_\alpha) = (\Pi_{p_\alpha}(x + N_{p_\alpha})\xi_\alpha), \quad x \in \mathcal{D}(p),\ (\xi_\alpha) \in \bigoplus_\alpha \mathcal{H}_{\pi_{p_\alpha}}.$$

Since

$$\|\Pi_p(x + N_p)\| = \sup_\alpha \|\Pi_{p_\alpha}(x + N_{p_\alpha})\| = \sup_\alpha p_\alpha(x) = p(x)$$

for each $x \in \mathcal{D}(p)$, it follows that Π_p can be extended to a faithful $*$-representation of $\mathcal{A}_p$ on $\bigoplus_\alpha \mathcal{H}_{\pi_{p_\alpha}}$. We denote π_p the $*$-representation of $\mathcal{A}$ induced by Π_p. Then we have

$$\begin{cases} \mathcal{D}(\pi_p) = \text{ linear span of } \{\Pi_p(x+N_p)\xi : x \in \mathfrak{N}_p,\, \xi \in \mathcal{H}_{\pi_p}\} \\ \qquad\quad = \text{ linear span of } \big\{(\overline{\pi_{p_\alpha}(x)}\xi_\alpha) : x \in \mathfrak{N}_p,\, \xi = (\xi_\alpha) \in \bigoplus_\alpha \mathcal{H}_{\pi_{p_\alpha}}\big\}, \\ \pi_p(a)(\overline{\pi_{p_\alpha}(x)}\xi_\alpha) = (\overline{\pi_{p_\alpha}(ax)}\xi_\alpha). \end{cases}$$

We show that p is weakly semifinite, that is, $\mathcal{D}(\pi_p)$ is dense in $\bigoplus_\alpha \mathcal{H}_{\pi_{p_\alpha}}$. Take an arbitrary $\xi = (\xi_\alpha) \in \bigoplus_\alpha \mathcal{H}_{\pi_{p_\alpha}} \ominus \overline{\mathcal{D}(\pi_p)}$. Take an arbitrary α. For any $\eta_\alpha \in \mathcal{H}_{\pi_{p_\alpha}}$ we have

$$(\overline{\pi_{p_\alpha}(x)}\eta_\alpha|\xi_\alpha) = (\delta_{\alpha\beta}\overline{\pi_{p_\beta}(x)}\eta_\beta|\xi) = 0 \tag{4.1}$$

for each $x \in \mathfrak{N}_p$. Since $\mathfrak{N}_p$ is dense in $\mathcal{D}(p_\alpha)$ with respect to p_α, it follows that $\overline{\pi_{p_\alpha}(\mathfrak{N}_p)}\mathcal{H}_{\pi_{p_\alpha}}$ is total in $\overline{\pi_{p_\alpha}(\mathcal{D}(p_\alpha))}\mathcal{H}_{\pi_{p_\alpha}}$, and further it follows from the weak semifiniteness of p_α that $\overline{\pi_{p_\alpha}(\mathcal{D}(p_\alpha))}\mathcal{H}_{\pi_{p_\alpha}}$ is total in $\mathcal{H}_{\pi_{p_\alpha}}$. Hence, $\overline{\pi_{p_\alpha}(\mathfrak{N}_p)}\mathcal{H}_{\pi_{p_\alpha}}$ is total in $\mathcal{H}_{\pi_{p_\alpha}}$, and so by (4.1) $\xi_\alpha = 0$. Hence, $\xi = 0$. Thus, $\mathcal{D}(\pi_p)$ is dense in $\bigoplus_\alpha \mathcal{H}_{\pi_{p_\alpha}}$. By the definition of π_p we have $\pi_p \subset \bigoplus_\alpha \pi_{p_\alpha}$. This completes the proof. ∎

By Lemma 4.3 we have the following

PROPOSITION 4.4. *Let p be an unbounded C^*-seminorm on $\mathcal{A}$. Suppose p is regular, that is, $p \subset \sup_\alpha p_\alpha$ for some net $\{p_\alpha\}$ of C^*-seminorms on $\mathcal{A}$, and further $\mathfrak{N}_p$ is dense in $\mathcal{A}$ with respect to $\{p_\alpha\}$. Then there exists an element π_p of* $\mathrm{Rep}^{\mathrm{WB}}(\mathcal{A}, p)$ *which is weakly bounded. Conversely suppose $\pi_p \in \mathrm{Rep}^{\mathrm{WB}}(\mathcal{A}, p)$ and it is weakly bounded. Then p is regular.*

In Section 7 we shall give several examples of regular unbounded C^*-(semi) norms.

5. UNBOUNDED GELFAND-NAIMARK C^*-SEMINORMS

In this section we construct and characterize an unbounded Gelfand-Naimark C^*-seminorm $|\cdot|_p$ from an unbounded m^*-seminorm p on a $*$-algebra $\mathcal{A}$. An unbounded m^*-seminorm p on $\mathcal{A}$ is said to be *representable* if there exists a non-zero nondegenerate bounded $*$-representation π of $\mathcal{D}(p)$ such that $\|\pi(x)\| \leqslant p(x)$ for each $x \in \mathcal{D}(p)$. Every unbounded C^*-seminorm on $\mathcal{A}$ is representable, but an unbounded m^*-seminorm is not necessarily representable (see Section 37, Example 16 in [9]). Let p be a representable unbounded m^*-seminorm on $\mathcal{A}$ and $\mathrm{Rep}(p)$ the set of all nondegenerate bounded $*$-representations π of $\mathcal{D}(p)$ on $\mathcal{H}_\pi$ such that $\|\pi(x)\| \leqslant k_\pi p(x)$, $\forall x \in \mathcal{D}(p)$ for some constant k_π. Let $\pi \in \mathrm{Rep}(p)$. It is easily shown that $\|\pi(x)\| \leqslant p(x)$ for each $x \in \mathcal{D}(p)$, and so we can define an unbounded C^*-seminorm $|\cdot|_p$ on $\mathcal{A}$ by

$$\mathcal{D}(|\cdot|_p) = \mathcal{D}(p) \quad \text{and} \quad |x|_p = \sup_{\pi \in \mathrm{Rep}(p)} \|\pi(x)\|, \quad x \in \mathcal{D}(p)$$

and call it the *unbounded Gelfand-Naimark C^*-seminorm* of the unbounded m^*-seminorm p. To investigate the unbounded Gelfand-Naimark C^*-seminorm $|\cdot|_p$, we prepare another order $\leqslant$ on $\mathrm{C^*N}(p)$ as follows: $r_1 \leqslant r_2$ iff $\mathcal{D}(r_2) \subset \mathcal{D}(r_1)$ and $r_1(x) \leqslant r_2(x)$, $\forall x \in \mathcal{D}(r_2)$.

Proposition 5.1. *Let p be a representable unbounded m^*-seminorm on a $*$-algebra $\mathcal{A}$. Then the following statements hold:*

(i) *$|\cdot|_p$ is the largest element of* $(\mathrm{C^*N}(p), \leqslant)$.

(ii) *If p is semifinite, then $|\cdot|_p$ is semifinite.*

(iii) *Suppose $\mathfrak{N}_p$ is dense in $\mathcal{D}(p)$ with respect to the set $\{r_\pi : \pi \in \mathrm{Rep}(p)\}$ of seminorms r_π. Then $|\cdot|_p$ is weakly semifinite and there exists a $*$-representation π_p of $\mathcal{A}$ such that $\|\pi_p(x)\| = |x|_p$ for each $x \in \mathcal{D}(p)$.*

(iv) *Suppose p is an unbounded C^*-seminorm on $\mathcal{A}$. Then $|\cdot|_p = p$.*

Proof. (i) Let r be any unbounded C^*-seminorm on $\mathcal{A}$ such that $r \leqslant p$. For any $\Pi_r \in \mathrm{Rep}(A_r)$ we define a bounded $*$-representation π_r^0 of $\mathcal{D}(r)$ by

$$\pi_r^0(x) = \Pi_r(x + N_r), \qquad x \in \mathcal{D}(r).$$

Then since $\mathcal{D}(p) \subset \mathcal{D}(r)$, it follows that $\pi_r^0 \lceil \mathcal{D}(p)$ is a bounded $*$-repesentation of $\mathcal{D}(p)$ and $\|\pi_r^0(x)\| = r(x) \leqslant p(x)$ for each $x \in \mathcal{D}(p)$, which implies $\pi_r^0 \lceil \mathcal{D}(p) \in \mathrm{Rep}(p)$. Hence it follows that $r(x) \leqslant |x|_p$ for each $x \in \mathcal{D}(p)$.

(ii) This follows since $\mathcal{D}(|\cdot|_p) = \mathcal{D}(p)$, $\mathfrak{N}_{|\cdot|_p} = \mathfrak{N}_p$ and $|x|_p \leqslant p(x)$, $\forall x \in \mathcal{D}(p)$.

(iii) We put

$$\Pi_p(x + N_{|\cdot|_p}) = \Big(\bigoplus_{\pi \in \mathrm{Rep}(p)} \pi\Big)(x), \quad x \in \mathcal{D}(p).$$

Then Π_p can be extended to a faithful nondegenerate $*$-representation of the C^*-algebra $\mathcal{A}_{|\cdot|_p}$ on $\bigoplus_{\pi \in \mathrm{Rep}(p)} \mathcal{H}_\pi$ and denote it by the same Π_p. Here we denote by π_p the $*$-representation of $\mathcal{A}$ defined by Π_p, that is,

$$\begin{aligned}
\mathcal{D}(\pi_p) &= \text{ linear span of } \{\Pi_p(x + N_{|\cdot|_p})(\xi_\pi) : x \in \mathfrak{N}_p,\, \xi_\pi \in \mathcal{H}_\pi\} \\
&= \text{ linear span of } \{(\pi(x)\xi_\pi) : x \in \mathfrak{N}_p,\, \xi_\pi \in \mathcal{H}_\pi\},
\end{aligned}$$
$$\pi_p(a)(\pi(x)\xi_\pi) = (\pi(ax)\xi_\pi), \quad a \in \mathcal{A},\, x \in \mathfrak{N}_p,\, \xi_\pi \in \mathcal{H}_\pi.$$

Since $\mathfrak{N}_p$ is dense in $\mathcal{D}(p)$ with respect to r_π $(\pi \in \mathrm{Rep}(p))$ and any π is nondegenerate, it follows that $\mathcal{D}(\pi_p)$ is dense in $\bigoplus_\pi \mathcal{H}_\pi$, which implies that $|\cdot|_p$ is weakly semifinite. Hence, it follows from Proposition 2.6 that $|x|_p = \|\pi_p(x)\|$ for each $x \in \mathcal{D}(p)$.

(iv) Suppose p is an unbounded C^*-seminorm on $\mathcal{A}$. Take an arbitrary $\Pi_p \in \mathrm{Rep}(\mathcal{A}_p)$. We put

$$\pi_p^0(x) = \Pi_p(x + N_p), \quad x \in \mathcal{D}(p).$$

Then it follows that $\pi_p^0 \in \mathrm{Rep}(p)$ and $\|\pi_p^0(x)\| = p(x)$ for each $x \in \mathcal{D}(p)$, which implies $|\cdot|_p = p$. This completes the proof. ∎

We next characterize the unbounded Gelfand-Naimark C^*-seminorm $|\cdot|_p$ of a representable unbounded m^*-seminorm p extending some main results in [33] about C^*-seminorms on $*$-algebras with identity to unbounded C^*-seminorms on $*$-algebras without identity. A positive linear functional f on $\mathcal{A}$ is said to be *representable* if there exists a constant $\gamma > 0$ such that $|f(x)|^2 \leqslant \gamma f(x^*x)$ for all $x \in \mathcal{A}$.

Let $\mathcal{F}_p$ be the set of all p-continuous representable positive linear functionals f on $\mathcal{D}(p)$ such that $|f(x)|^2 \leqslant f(x^*x)$ for each $x \in \mathcal{D}(p)$. Then we have the following

Proposition 5.2. *Let p be a representable unbounded m^*-seminorms on $\mathcal{A}$. Then*

$$\mathcal{D}(p) = \{x \in \mathcal{D}(p) : \sup_{f\in\mathcal{F}_p} f(x^*x) < \infty\},$$

$$|x|_p = \sup_{f\in\mathcal{F}_p} f(x^*x)^{1/2}, \quad x \in \mathcal{D}(p).$$

Proof. Take an arbitrary $f \in \mathcal{F}_p$. Since f is p-continuous, there exists a constant $M_f > 0$ such that $|f(x)| \leqslant M_f p(x)$, $\forall x \in \mathcal{D}(p)$, which implies

$$|f(x)|^2 \leqslant f(x^*x) \leqslant M_f p(x^*x) \leqslant M_f p(x)^2$$

for each $x \in \mathcal{D}(p)$. Repeating this, we have

$$|f(x)| \leqslant M_f^{1/n} p(x), \quad \forall x \in \mathcal{D}(p),\ \forall n \in \mathbb{N}.$$

Hence we have

$$|f(x)| \leqslant p(x), \quad \forall x \in \mathcal{D}(p). \tag{5.1}$$

For any $y \in \mathcal{D}(p)$ with $f(y^*y) = 1$ we define a positive linear functional on $\mathcal{D}(p)$ by

$$f_y(x) = f(y^*xy), \quad x \in \mathcal{D}(p).$$

Then we have

$$|f_y(x)|^2 = |f(y^*xy)|^2 \leqslant f(y^*y)f(y^*x^*xy) = f_y(x^*x)$$

and by (5.1)

$$|f_y(x)| \leqslant p(y)^2 p(x)$$

for each $x \in \mathcal{D}(p)$. Hence we have

$$f_y \in \mathcal{F}_p \text{ for each } y \in \mathcal{D}(p) \text{ with } f(y^*y) = 1. \tag{5.2}$$

Here we put

$$\begin{cases} \mathcal{D}(r_{\mathcal{F}_p}) = \{x \in \mathcal{D}(p) : \sup_{f\in\mathcal{F}_p} f(x^*x) < \infty\} \\ r_{\mathcal{F}_p}(x) = \sup_{f\in\mathcal{F}_p} f(x^*x)^{1/2}, \quad x \in \mathcal{D}(r_{\mathcal{F}_p}). \end{cases}$$

By (5.1) we have

$$\mathcal{D}(r_{\mathcal{F}_p}) = \mathcal{D}(p) \quad \text{and} \quad r_{\mathcal{F}_p}(x) \leqslant p(x), \quad \forall x \in \mathcal{D}(p). \tag{5.3}$$

Let $(\pi_f, \lambda_f, \mathcal{H}_f)$ be the GNS-construction for f. We show

$$\begin{cases} \mathcal{D}(p) = \{x \in \mathcal{D}(p) : \sup\limits_{f \in \mathcal{F}_p} \|\overline{\pi_f(x)}\| < \infty\} \\ r_{\mathcal{F}_p}(x) = \sup\limits_{f \in \mathcal{F}_p} \|\pi_f(x)\|, \quad x \in \mathcal{D}(p). \end{cases}$$

In fact, take an arbitrary $x \in \mathcal{D}(p)$. By (5.2) we have, for any $y \in \mathcal{D}(p)$ with $f(y^*y) = 1$,

$$\|\pi_f(x)\lambda_f(y)\|^2 = f_y(x^*x) \leqslant r_{\mathcal{F}}(x)^2$$

for each $x \in \mathcal{D}(p)$, which implies that $\pi_f(x)$ is bounded and $\|\overline{\pi_f(x)}\| \leqslant r_{\mathcal{F}}(x)$ for each $x \in \mathcal{D}(p)$. Hence we have

$$\sup_{f \in \mathcal{F}_p} \|\overline{\pi_f(x)}\| \leqslant r_{\mathcal{F}_p}(x), \quad \forall x \in \mathcal{D}(p).$$

Since $|f(x)| \leqslant f(x^*x)^{1/2} = \|\lambda_f(x)\|, x \in \mathcal{D}(p)$, it follows from the Riesz theorem that there exists an element ξ_f of $\mathcal{H}_f$ such that $\|\xi_f\| \leqslant 1$ and $f(x) = (\lambda_f(x)|\xi_f)$ for all $x \in \mathcal{D}(p)$, which implies by the boundedness of $\pi_f(x)$ that $\lambda_f(x) = \overline{\pi_f(x)}\xi_f$ and

$$|f(x^*x)|^{1/2} = \|\overline{\pi_f(x)}\xi_f\| \leqslant \|\overline{\pi_f(x)}\|, \quad \forall x \in \mathcal{D}(p).$$

Hence

$$r_{\mathcal{F}_p}(x) \leqslant \sup_{f \in \mathcal{F}_p} \|\overline{\pi_f(x)}\|, \quad \forall x \in \mathcal{D}(p).$$

Thus we have

$$r_{\mathcal{F}_p}(x) = \sup_{f \in \mathcal{F}_p} \|\overline{\pi_f(x)}\|, \quad x \in \mathcal{D}(p),$$

which implies that $r_{\mathcal{F}_p}$ is an unbounded C^*-seminorm on $\mathcal{A}$ such that $\mathcal{D}(r_{\mathcal{F}_p}) = \mathcal{D}(p)$ and $r_{\mathcal{F}_p}(x) \leqslant |x|_p$ for each $x \in \mathcal{D}(p)$. On the other hands, take arbitrary $\pi \in \mathrm{Rep}(p)$ and $\xi \in \mathcal{H}_\pi$ such that $\|\xi\| = 1$. Then the positive linear functional f_ξ on $\mathcal{D}(p)$ defined by $f_\xi(x) = (\pi(x)\xi|\xi)$, $x \in \mathcal{D}(p)$ belongs to $\mathcal{F}_p$, and so

$$\|\pi(x)\| = \sup_{\|\xi\|=1} f_\xi(x^*x)^{1/2} \leqslant r_{\mathcal{F}_p}(x), \quad x \in \mathcal{D}(p).$$

Hence, we have

$$|x|_p \leqslant r_{\mathcal{F}_p}(x), \quad \forall x \in \mathcal{D}(p).$$

Thus we have $|\cdot|_p = r_{\mathcal{F}_p}$. This completes the proof. ∎

6. SPECTRAL $*$-REPRESENTATIONS AND SPECTRAL UNBOUNDED C^*-SEMINORMS

In this section we define the notion of (hereditary) spectrality of unbounded C^*-seminorms and further define the notion of stable unbounded C^*-seminorms and investigate the relation of spectrality and stability of unbounded C^*-seminorms.

Let $\mathcal{B}$ be a $*$-subalgebra of a $*$-algebra $\mathcal{A}$ with identity $\mathbb{1}$ and the $*$-algebra $\mathcal{B}_{\mathbf{1}}$ obtained by adjoining the identity $\mathbb{1}$ to $\mathcal{B}$ when $\mathcal{B}$ does not have the identity. We denote by $\mathcal{B}^{\mathrm{qr}}$ the set of all *quasi-regular* elements x of $\mathcal{B}$, that is, $\mathbb{1} - x$ is invertible in $\mathcal{B}_{\mathbf{1}}$. We have the spectrum $\mathrm{Sp}_{\mathcal{B}}(x)$ and the spectral radius $r_{\mathcal{B}}(x)$ of $x \in \mathcal{B}$ as follows:

$$\mathrm{Sp}_{\mathcal{B}}(x) = \{\lambda \in \mathbb{C} : \nexists (\lambda\mathbb{1} - x)^{-1} \text{ in } \mathcal{B}_{\mathbf{1}}\} \quad \text{and} \quad r_{\mathcal{B}}(x) = \sup\{|\lambda| : \lambda \in \mathrm{Sp}_{\mathcal{B}}(x)\}.$$

By Theorem 3.1 of [21] we have the following

LEMMA 6.1. *Let p be an unbounded m^*-seminorm on $\mathcal{A}$. Then the following statements are equivalent:*

(i) $\{x \in \mathcal{D}(p) : p(x) < 1\} \subset \mathcal{D}(p)^{\mathrm{qr}}$.
(ii) $r_{\mathcal{D}(p)}(x) \leqslant p(x)$ *for each* $x \in \mathcal{D}(p)$.
(iii) $r_{\mathcal{D}(p)}(x) = \lim_{n\to\infty} p(x^n)^{1/n}$ *for each* $x \in \mathcal{D}(p)$.

In particular, if p is an unbounded C^*-seminorm on $\mathcal{A}$, then the conditions (i) $\sim$ (iii) *are equivalent to*

(iv) $r_{\mathcal{D}(p)}(x) = p(x)$ *for each* $x \in \mathcal{D}(p)$ *with* $x^*x = xx^*$.

We remark that the equivalence of (i) and (ii) in Lemma 6.1 holds for a general unbounded seminorm p.

DEFINITION 6.2. An unbounded m^*- (or C^*-) seminorm p on a $*$-algebra $\mathcal{A}$ is said to be *spectral* if it satisfies one of equivalent conditions (i) $\sim$ (iii) in Lemma 6.1.

Here we need a new notion of hereditary spectral unbounded m^*- (or C^*-) seminorms which plays an important rule in this section.

DEFINITION 6.3. An unbounded m^*- (or C^*-) seminorm p on $\mathcal{A}$ is said to be *hereditary spectral* if for any $*$-subalgebra $\mathcal{B}$ of $\mathcal{A}$ the restriction $p\lceil\mathcal{B}$ of p to $\mathcal{B}$ is spectral.

The hereditary spectrality of unbounded m^*- (or C^*-) seminorms implies the spectrality, but the converse does not hold in general. For example, if $\mathcal{A}$ is a C^*-algebra, there is a spectral m^*-seminorm on $\mathcal{A}$ which is not hereditary spectral ([23]). According to Palmer ([22] and [23]), a *spectral algebra* $\mathcal{A}$ is an algebra on which there is defined a spectral seminorm with $\mathcal{D}(p) = \mathcal{A}$. A spectral algebra need not be normable, however it is rich enough to admit a satisfactory spectral theory like Banach algebras. A C^*-spectral (hereditary C^*-spectral) algebra which is a $*$-algebra with a spectral (hereditary spectral) C^*-seminorm has been studied in [8]. C^*-spectral (hereditary C^*-spectral) algebras appear to be potential enough to recapture much of the algebraic theory of C^*-algebras. They also help to clarify the notion of local algebras that arises in non-commutative geometry, in particular, smooth structure in C^*-algebras ([10] and [11]). Here we define and characterize unbounded C^*-spectral algebras and unbounded hereditary C^*-spectral algebras.

DEFINITION 6.4. An unbounded C^*-spectral algebra is a $*$-algebra admitting a spectral unbounded C^*-seminorm. An unbounded hereditary C^*-spectral algebra is a $*$-algebra $\mathcal{A}$ admitting a hereditary spectral unbounded C^*-seminorm on $\mathcal{A}$.

We define the notion of (hereditary) spectral $*$-representations and characterize unbounded (hereditary) C^*-spectral algebras by the existence of (hereditary) spectral strongly nondegenerate $*$-representations.

DEFINITION 6.5. Let π be a $*$-representation of $\mathcal{A}$ and $x \in \mathcal{A}$. We define a *spectrum of the closed operator* $\overline{\pi(x)}$ in $C^*(\pi)$ as follows:

$$\mathrm{Sp}_{C^*(\pi)}(\overline{\pi(x)}) = \{\lambda \in \mathbb{C} : (\lambda I - \overline{\pi(x)})^{-1} \text{ does not exist in } C^*(\pi)\}.$$

If $\mathrm{Sp}_{\mathcal{A}_\flat^\pi}(x) := \{\lambda \in \mathbb{C} : (\lambda \mathbb{1} - x)^{-1} \text{ does not exist in } \mathcal{A}_\flat^\pi\} \subset \mathrm{Sp}_{C^*(\pi)}(\overline{\pi(x)}) \cup \{0\}$, $\forall x \in \mathcal{A}$, then π is said to be *spectral.* If for any $*$-subalgebra $\mathcal{B}$ of $\mathcal{A}$ the restriction $\pi\lceil\mathcal{B}$ of π to $\mathcal{B}$ is a spectral $*$-representation of $\mathcal{B}$, then π is said to be *hereditary spectral.*

Let π be a $*$-representation of $\mathcal{A}$. It is easily shown that

$$\mathrm{Sp}_{C^*(\pi)}(\overline{\pi(x)}) \cup \{0\} \subset \mathrm{Sp}_{\overline{\pi(\mathcal{A}_\flat^\pi)}}(\overline{\pi(x)}) \subset \mathrm{Sp}_{\mathcal{A}_\flat^\pi}(x), \quad \forall x \in \mathcal{A}. \tag{6.1}$$

We first characterize the spectrality of bounded $*$-repesentation $\pi_\flat$ of the $*$-algebra $\mathcal{A}_\flat^\pi$.

LEMMA 6.6. *Let π be a $*$-representation of $\mathcal{A}$. Consider the following statements:*

(i) *π is spectral;*

(ii) *$\pi_\flat$ is spectral, that is,* $\mathrm{Sp}_{\mathcal{A}_\flat^\pi}(x) \subset \mathrm{Sp}_{C^*(\pi)}(\overline{\pi(x)}) \cup \{0\}$, $\forall x \in \mathcal{A}_\flat^\pi$;

(iii) *r_π is spectral;*

(iv) $\mathrm{Sp}_{\mathcal{A}_\flat^\pi}(x) = \mathrm{Sp}_{\overline{\pi(\mathcal{A}_\flat^\pi)}}(\overline{\pi(x)})$, $\forall x \in \mathcal{A}_\flat^\pi$ *and the normed $*$-algebra $\overline{\pi(\mathcal{A}_\flat^\pi)}$ with norm r_π is a Q-algebra, that is, $\overline{\pi(\mathcal{A}_\flat^\pi)}^{\mathrm{qr}}$ is open;*

(v) $\mathrm{Sp}_{\mathcal{A}_\flat^\pi}(x) = \mathrm{Sp}_{\overline{\pi(\mathcal{A}_\flat^\pi)}}(\overline{\pi(x)})$, $\forall x \in \mathcal{A}$.

Then the implications (i) $\Rightarrow$ (ii) $\Leftrightarrow$ (iii) $\Leftrightarrow$ (iv) $\Rightarrow$ (v) *hold.*

Proof. (i) $\Rightarrow$ (ii) This is trivial. (ii) $\Rightarrow$ (iii) Suppose $\pi_\flat$ is spectral. Take an arbitrary $x \in \mathcal{A}_\flat^\pi$ with $r_\pi(x) < 1$. Since $\|\overline{\pi(x)}\| < 1, \overline{\pi(x)}$ is quasi-regular in the C^*-algebra $C^*(\pi)$, and so $1 \notin \mathrm{Sp}_{C^*(\pi)}(\overline{\pi(x)})$. Since $\pi_\flat$ is spectral, we have $1 \notin \mathrm{Sp}_{\mathcal{A}_\flat^\pi}(x)$, and so $x \in (\mathcal{A}_\flat^\pi)^{\mathrm{qr}}$. Therefore it follows from Lemma 6.1 that r_π is spectral.

(iii) $\Rightarrow$ (ii) Suppose r_π is spectral. Take arbitrary $x \in \mathcal{A}_\flat^\pi$ and $\lambda \neq 0 \in \mathbb{C}$ such that $(\lambda I - \overline{\pi(x)})^{-1} \in C^*(\pi)$. Since $C^*(\pi) = \overline{\pi(\mathcal{A}_\flat^\pi)}^{\|\cdot\|}$, there exists an element $y \in \mathcal{A}_\flat^\pi$ such that $r_\pi\big(\frac{x}{\lambda} + y - \frac{xy}{\lambda}\big) = \big\|I - \big(I - \overline{\pi\big(\frac{1}{\lambda}x\big)}\big)(I - \overline{\pi(y)})\big\| < 1$ and $r_\pi\big(\frac{x}{\lambda} + y - \frac{yx}{\lambda}\big) = \big\|I - (I - \overline{\pi(y)})\big(I - \overline{\pi\big(\frac{1}{\lambda}x\big)}\big)\big\| < 1$.

Since r_π is spectral, it follows from Lemma 6.1 that $\frac{x}{\lambda} + y - \frac{xy}{\lambda} = \mathbb{1} - \big(\mathbb{1} - \frac{1}{\lambda}x\big)(\mathbb{1} - y)$, $\frac{x}{\lambda} + y - \frac{yx}{\lambda} = \mathbb{1} - (\mathbb{1} - y)\big(\mathbb{1} - \frac{1}{\lambda}x\big)$ are contained in $(\mathcal{A}_\flat^\pi)^{\mathrm{qr}}$, and so

$\big(\mathbb{1}-\frac{1}{\lambda}x\big)(\mathbb{1}-y)$ and $(\mathbb{1}-y)\big(\mathbb{1}-\frac{1}{\lambda}x\big)$ are invertible in $\mathcal{A}^\pi_\flat$. Hence, $\mathbb{1}-\frac{1}{\lambda}x$ is invertible in $\mathcal{A}^\pi_\flat$, and so $\lambda \notin \mathrm{Sp}_{\mathcal{A}^\pi_\flat}(x)$.

(ii) $\Rightarrow$ (iv) It follows from (6.1) and the assumption (ii) that

$$\mathrm{Sp}_{C^*(\pi)}(\overline{\pi(x)}) \cup \{0\} = \mathrm{Sp}_{\overline{\pi(\mathcal{A}^\pi_\flat)}}(\overline{\pi(x)}) = \mathrm{Sp}_{\mathcal{A}^\pi_\flat}(x), \quad \forall x \in \mathcal{A}^\pi_\flat.$$

Further, it follows from Proposition 2 of [4] that

$$\begin{aligned}&\mathrm{Sp}_{C^*(\pi)}(\overline{\pi(x)}) \cup \{0\} = \mathrm{Sp}_{\overline{\pi(\mathcal{A}^\pi_\flat)}}(\overline{\pi(x)}), \quad \forall x \in \mathcal{A}^\pi_\flat \\ &\text{if and only if } \overline{\pi(\mathcal{A}^\pi_\flat)} \text{ is a } Q\text{-algebra.}\end{aligned} \tag{6.2}$$

Hence, the statement (iv) holds.

(iv) $\Rightarrow$ (ii) This follows from (6.2) and the assumption (iv).

(ii) $\Rightarrow$ (v) Take arbitrary $x \in \mathcal{A}$ and $\lambda \neq 0 \in \mathbb{C}$ such that $(\lambda I - \overline{\pi(x)})^{-1} \in \overline{\pi(\mathcal{A}^\pi_\flat)}$. Then there exists an element y of $\mathcal{A}^\pi_\flat$ such that $\big(I - \pi(y)\big)\big(I - \pi(\frac{x}{\lambda})\big) = \big(I - \pi(\frac{x}{\lambda})\big)(I - \pi(y)) = I$, and so $\pi\big(\frac{x}{\lambda} + y - \frac{yx}{\lambda}\big) = \pi\big(\frac{x}{\lambda} + y - \frac{xy}{\lambda}\big) = 0$. Hence, $1 \notin \mathrm{Sp}_{\overline{\pi(\mathcal{A}^\pi_\flat)}}\big(\overline{\pi\big(\frac{x}{\lambda} + y - \frac{yx}{\lambda}\big)}\big) \cup \mathrm{Sp}_{\overline{\pi(\mathcal{A}^\pi_\flat)}}\big(\overline{\pi\big(\frac{x}{\lambda} + y - \frac{xy}{\lambda}\big)}\big)$. Since $\mathrm{Sp}_{\overline{\pi(\mathcal{A}^\pi_\flat)}}(\overline{\pi(a)}) \subset \mathrm{Sp}_{C^*(\pi)}(\overline{\pi(a)})$ for each $a \in \mathcal{A}^\pi_\flat$, it follows from (ii) that $1 \notin \mathrm{Sp}_{\mathcal{A}^\pi_\flat}\big(\frac{x}{\lambda} + y - \frac{yx}{\lambda}\big)$ and $1 \notin \mathrm{Sp}_{\mathcal{A}^\pi_\flat}\big(\frac{x}{\lambda} + y - \frac{xy}{\lambda}\big)$, and so there exist elements z_1 and z_2 of $\mathcal{A}^\pi_\flat$ such that $(\mathbb{1} - z_1)(\mathbb{1} - y)\big(\mathbb{1} - \frac{x}{\lambda}\big) = \mathbb{1}$ and $\big(\mathbb{1} - \frac{x}{\lambda}\big)(\mathbb{1} - y)(\mathbb{1} - z_2) = \mathbb{1}$. Hence we have $\frac{x}{\lambda} \in (\mathcal{A}^\pi_\flat)^{\mathrm{qr}}$ and so $\lambda \notin \mathrm{Sp}_{\mathcal{A}^\pi_\flat}(x)$. This completes the proof. ∎

LEMMA 6.7. *Let $\mathcal{A}$ be a $*$-representation of $\mathcal{A}$. Then the following statements are equivalent:*

(i) *$\pi_\flat$ is hereditary spectral;*

(ii) *r_π is a hereditary spectral unbounded C^*-seminorm on $\mathcal{A}$.*

Proof. This is proved similarly to the proof of (ii) $\Leftrightarrow$ (iii) in Lemma 6.6. ∎

THEOREM 6.8. *The following statements are equivalent:*

(i) *There exists a strongly nondegenerate $*$-representation π of $\mathcal{A}$ such that $\pi_\flat$ is (hereditary) spectral.*

(ii) *There exists a maximal, weakly semifinite, (hereditary) spectral unbounded C^*-seminorm on $\mathcal{A}$.*

Proof. (i) $\Rightarrow$ (ii) Let π be a strongly nondegenerate $*$-representation of $\mathcal{A}$ such that $\pi_\flat$ is (hereditary) spectral. By Proposition 3.1 and Corollary 3.5, r_π is a maximal, weakly semifinite unbounded C^*-seminorm on $\mathcal{A}$. Further, it follows from Lemmas 6.6 and 6.7 that r_π is (hereditary) spectral.

(ii) $\Rightarrow$ (i) Let p be a maximal, weakly semifinite, (hereditary) spectral unbounded C^*-seminorm on $\mathcal{A}$. Then there exists an element π of $\mathrm{Rep}^{\mathrm{WB}}(\mathcal{A}, p)$ such that $p = r_\pi$. By Proposition 2.6 (1), π is strongly nondegenerate. Further, it follows from Lemmas 6.6 and 6.7 that π is (hereditary) spectral. This completes the proof. ∎

We next generalize the following property (stability) of C^*-algebras ([12], Proposition 2.10.2) to general $*$-algebras, and characterize it by the hereditary spectrality of unbounded C^*-seminorms.

Let $\mathcal{A}$ be a C^*-algebra and $\mathcal{B}$ any closed $*$-subalgebra of $\mathcal{A}$. For any $*$-representation π of $\mathcal{B}$ on a Hilbert space $\mathcal{H}_\pi$ there exists a $*$-representation $\widehat{\pi}$ of $\mathcal{A}$ on a Hilbert space $\mathcal{H}_{\hat{\pi}}$ such that $\mathcal{H}_{\hat{\pi}} \supset \mathcal{H}_\pi$ as a closed subspace and $\pi(x) = \widehat{\pi}(x)\lceil\mathcal{H}_\pi$ for each $x \in \mathcal{B}$.

DEFINITION 6.9. An unbounded m^*-(or C^*-)seminorms p is said to be *stable* if for any $*$-subalgebra $\mathcal{B}$ of $\mathcal{A}$ and any $*$-representation π of $\mathcal{B}$ such that $\mathcal{B}\cap\mathcal{D}(p) \subset \mathcal{B}^\pi_\flat$ and $[\pi(\mathcal{B} \cap \mathcal{D}(p))\mathcal{D}(\pi)] = \mathcal{H}_\pi$ there exists a $*$-representation ϱ of $\mathcal{A}$ such that $\mathcal{D}(p) \subset \mathcal{A}^\varrho_\flat$, $[\varrho(\mathcal{D}(p))\mathcal{D}(\varrho)] = \mathcal{H}_\varrho$, $\mathcal{H}_\varrho$ contains $\mathcal{H}_\pi$ as a closed subspace and $\overline{\pi(x)} = \overline{\varrho(x)}\lceil\mathcal{H}_\pi$ for each $x \in \mathcal{B} \cap \mathcal{D}(p)$.

The following is one of main results of the paper.

THEOREM 6.10. *Let $\mathcal{A}$ be a $*$-algebra and p a semifinite unbounded C^*-seminorm on $\mathcal{A}$. Then the following statements are equivalent:*
(i) *p is hereditary spectral;*
(ii) *p is spectral and stable.*

Proof. (i) $\Rightarrow$ (ii) Let $\mathcal{B}$ be a $*$-subalgebra of $\mathcal{A}$ and let π be a $*$-representation of $\mathcal{B}$ such that $\mathcal{B} \cap \mathcal{D}(p) \subset \mathcal{B}^\pi_\flat$ and $[\overline{\pi(\mathcal{B} \cap \mathcal{D}(p))}\mathcal{H}_\pi] = \mathcal{H}_\pi$. Since p is hereditary spectral, it follows that

$$\overline{\lim_{n\to\infty}} \|\overline{\pi(x)}^n\|^{\frac{1}{n}} = r_{C^*(\pi)}(\overline{\pi(x)}) \leqslant r_{\overline{\pi(\mathcal{B}\cap\mathcal{D}(p))}}(\overline{\pi(x)}) = r_{\mathcal{B}\cap\mathcal{D}(p)}(x) \leqslant p(x)$$

for each $x \in \mathcal{B}\cap\mathcal{D}(p)$, which implies that $\|\overline{\pi(h)}\| \leqslant p(h)$ for each $h^* = h \in \mathcal{B}\cap\mathcal{D}(p)$. Then, for any $x \in \mathcal{B} \cap \mathcal{D}(p)$ we have

$$\|\overline{\pi(x)}\|^2 = \|\overline{\pi(x^*x)}\| \leqslant p(x^*x) = p(x)^2,$$

and so

$$\|\overline{\pi(x)}\| \leqslant p(x) \quad \text{for each } x \in \mathcal{B} \cap \mathcal{D}(p). \tag{6.3}$$

By the semifiniteness of p we have $\mathrm{Rep}^{\mathrm{WB}}(\mathcal{A}, p) \neq \phi$. Let $\pi_p \in \mathrm{Rep}^{\mathrm{WB}}(\mathcal{A}, p)$ and put

$$\widetilde{\varrho}_0(\overline{\pi_p(x)}) = \overline{\pi(x)}, \quad x \in \mathcal{B} \cap \mathcal{D}(p).$$

It follows from Proposition 2.6 and (6.3) that

$$\|\widetilde{\varrho}_0(\overline{\pi_p(x)})\| \leqslant p(x) = \|\overline{\pi_p(x)}\| \tag{6.4}$$

for each $x \in \mathcal{B} \cap \mathcal{D}(p)$, and hence $\widetilde{\varrho}_0$ can be extended to a $*$-representation of the C^*-algebra $\overline{\pi_p(\mathcal{B} \cap \mathcal{D}(p))}^{\|\cdot\|}$ on $\mathcal{H}_\pi$ and it is denoted by the same $\widetilde{\varrho}_0$. By the stability of C^*-algebras there exists a Hilbert space $\mathcal{H}_{\tilde{\varrho}}$ containing $\mathcal{H}_\pi$ as a closed subspace and a $*$-representation $\widetilde{\varrho}$ of the C^*-algebra $\overline{\pi_p(\mathcal{D}(p))}^{\|\cdot\|}$ on $\mathcal{H}_{\tilde{\varrho}}$ such that $\widetilde{\varrho}(A)\lceil\mathcal{H}_\pi = \widetilde{\varrho}_0(A)$ for each $A \in \overline{\pi_p(\mathcal{B} \cap \mathcal{D}(p))}^{\|\cdot\|}$. We here put

$$\begin{cases} \mathcal{D}(\varrho) = \text{ linear span of } \{\widetilde{\varrho}(\overline{\pi_p(x)})\xi : x \in \mathfrak{N}_p,\ \xi \in \mathcal{H}_{\tilde{\varrho}}\}, \\ \varrho(a)\widetilde{\varrho}(\overline{\pi_p(x)})\xi = \widetilde{\varrho}(\overline{\pi_p(ax)})\xi \quad \text{for } a \in \mathcal{A},\ x \in \mathfrak{N}_p,\ \xi \in \mathcal{H}_{\tilde{\varrho}}. \end{cases}$$

Then it is easily shown that ϱ is a $*$-representation of $\mathcal{A}$ on $\mathcal{D}(\varrho)$ in $\mathcal{H}_\varrho := \overline{\mathcal{D}(\varrho)}$. Since p is semifinite, it follows that $\mathcal{H}_\varrho = [\widetilde{\varrho}(\overline{\pi_p(\mathcal{D}(p))})\mathcal{H}_{\tilde{\varrho}}]$, so that

$$\mathcal{H}_\pi = [\overline{\pi(\mathcal{B} \cap \mathcal{D}(p))}\mathcal{H}_\pi] = [\widetilde{\varrho}_0(\overline{\pi_p(\mathcal{B} \cap \mathcal{D}(p))})\mathcal{H}_\pi] = [\widetilde{\varrho}(\overline{\pi_p(\mathcal{B} \cap \mathcal{D}(p))})\mathcal{H}_\pi] \subset \mathcal{H}_\varrho.$$

By the definition of ϱ we have $\mathcal{D}(p) \subset \mathcal{A}_\flat^\varrho$ and $\overline{\varrho(x)}\lceil\mathcal{H}_\pi = \widetilde{\varrho}(\overline{\pi_p(x)})\lceil\mathcal{H}_\pi = \widetilde{\varrho}_0(\overline{\pi_p(x)}) = \overline{\pi(x)}$ for each $x \in \mathcal{B} \cap \mathcal{D}(p)$. Further, since p is semifinite, it follows from Proposition 2.6 (4) that $[\overline{\varrho(\mathcal{D}(p))}\mathcal{H}_\varrho] = \mathcal{H}_\varrho$. Thus we have that p is stable.

(ii) $\Rightarrow$ (i) Let $\pi_p \in \mathrm{Rep}^{\mathrm{WB}}(\mathcal{A}, p)$ and $\mathcal{B}$ be any $*$-subalgebra of $\mathcal{A}$. We first show that

$$\mathrm{Sp}_{\mathcal{B}\cap\mathcal{D}(p)}(b) \cap \mathbb{R} \subset \mathrm{Sp}_{\overline{\pi_p(\mathcal{B}\cap\mathcal{D}(p))}^{\|\cdot\|}}(\overline{\pi_p(b)}) \cup \{0\} \tag{6.5}$$

for each $b^* = b \in \mathcal{B}\cap\mathcal{D}(p)$. Let $b^* = b \in \mathcal{B}\cap\mathcal{D}(p)$ and $0 \neq \lambda \in \mathrm{Sp}_{\mathcal{B}\cap\mathcal{D}(p)}(b)\cap\mathbb{R}$. Let $\mathcal{C}$ be the $*$-subalgebra of $\mathcal{B}\cap\mathcal{D}(p)$ generated by b. Then $\mathcal{C}\big(\frac{1}{\lambda}b - \mathbb{1}\big)$ is a proper modular $*$-ideal of $\mathcal{C}$ with modular identity $u := \frac{1}{\lambda}b$. Hence there exists a maximal modular $*$-ideal $\mathfrak{M}$ of $\mathcal{C}$ containing $\mathcal{C}\big(\frac{1}{\lambda}b - \mathbb{1}\big)$. Then the quotient algebra $\mathcal{C}/\mathfrak{M}$ is isomorphic to $\mathbb{C}$. In fact, since $u^k - u \in \mathfrak{M}$ for all $k \in \mathbb{N}$, it follows that $x + \mathfrak{M} = \sum_k \alpha_k \lambda^k u + \mathfrak{M}$ for any $x = \sum_k \alpha_k b^k \in \mathcal{C}$. Thus $\mathcal{C}/\mathfrak{M} = \{\alpha u + \mathfrak{M} : \alpha \in \mathbb{C}\}$, and $\tau : \alpha u + \mathfrak{M} \to \alpha$ gives a $*$-isomorphism of $\mathcal{C}/\mathfrak{M}$ onto $\mathbb{C}$. Let $\imath : \mathcal{C} \to \mathcal{C}/\mathfrak{M}$, $\imath(x) = x + \mathfrak{M}$. Let $\pi = \tau \circ \imath$; thus, $\pi\Big(\sum_k \alpha_k b^k\Big) = \sum_k \alpha_k \lambda^k$. Then π is a 1-dimensional $*$-representation of $\mathcal{C}$ such that $\pi(b) = \lambda$. By the stability of p there exists a $*$-representation ϱ of $\mathcal{A}$ such that

$$\mathcal{A}_\flat^\varrho \supset \mathcal{D}(p), [\overline{\varrho(\mathcal{D}(p))}\mathcal{H}_\varrho] = \mathcal{H}_\varrho \quad \text{and} \quad \varrho(b)\lceil\mathbb{C} = \pi(b) = \lambda. \tag{6.6}$$

Since p is spectral and (6.6), we have

$$\|\overline{\varrho(h)}\| = r_{C^*(\varrho)}(\overline{\varrho(h)}) \leqslant r_{\mathcal{D}(p)}(h) \leqslant p(h)$$

for each $h^* = h \in \mathcal{D}(p)$, which implies

$$\|\overline{\varrho(x)}\|^2 = \|\overline{\varrho(x^*x)}\| \leqslant p(x^*x) \leqslant p(x)^2$$

for each $x \in \mathcal{D}(p)$. Hence it follows from Proposition 2.6 that

$$\|\overline{\varrho(x)}\| \leqslant p(x) = \|\overline{\pi_p(x)}\| \tag{6.7}$$

for each $x \in \mathcal{D}(p)$. Hence, $\overline{\pi_p(x)} \mapsto \overline{\varrho(x)}$ can be extended to a $*$-representation of the C^*-algebra $\overline{\pi_p(\mathcal{B} \cap \mathcal{D}(p))}^{\|\cdot\|}$, which implies by (6.6) that

$$\lambda = \pi(b) \in \mathrm{Sp}_{\overline{\pi_p(\mathcal{B}\cap\mathcal{D}(p))}^{\|\cdot\|}}(\overline{\pi_p(b)}).$$

We next show

$$\mathrm{Sp}_{\mathcal{B}\cap\mathcal{D}(p)}(x) \subset \{\lambda \in \mathbb{C} : |\lambda| < p(x)\}, \quad \forall x \in \mathcal{B} \cap \mathcal{D}(p). \tag{6.8}$$

Let $x \in \mathcal{B} \cap \mathcal{D}(p)$ and $|\lambda| > p(x) = \|\overline{\pi_p(x)}\|$. Then $(\lambda I - \overline{\pi_p(x)})^*(\lambda I - \overline{\pi_p(x)})$ is invertible in $\big(\overline{\pi_p(\mathcal{B} \cap \mathcal{D}(p))}^{\|\cdot\|}\big)_I$, and so

$$|\lambda|^2 \notin \mathrm{Sp}_{\overline{\pi_p(\mathcal{B}\cap\mathcal{D}(p))}^{\|\cdot\|}}(\overline{\pi_p(\lambda x^* + \overline{\lambda} x - x^* x)}).$$

Hence it follows from (6.5) that $|\lambda|^2 \notin \mathrm{Sp}_{\mathcal{B}\cap\mathcal{D}(p)}(\lambda x^* + \overline{\lambda} x - x^* x)$, which implies $(\lambda \mathbb{1} - x)^*(\lambda \mathbb{1} - x)$ is invertible in $(\mathcal{B} \cap \mathcal{D}(p))_{\mathbf{1}}$. Similarly, $(\lambda \mathbb{1} - x)(\lambda \mathbb{1} - x)^*$ is invertible in $(\mathcal{B} \cap \mathcal{D}(p))_{\mathbf{1}}$. Thus, we have $\lambda \notin \mathrm{Sp}_{\mathcal{B}\cap\mathcal{D}(p)}(x)$. It follows from (6.8) that $r_{\mathcal{B}\cap\mathcal{D}(p)}(x) \leqslant p(x)$ for each $x \in \mathcal{B} \cap \mathcal{D}(p)$, which means that p is hereditary spectral. This completes the proof. ∎

REMARK 6.11. As seen in the proof of Theorem 6.10, the implication (ii) $\Rightarrow$ (i) in Theorem 6.10 holds under the assumption of weak semifiniteness of the unbounded C^*-seminorm p instead of that of the semifiniteness.

We consider the case of unbounded m^*-seminorms.

PROPOSITION 6.12. *Let p be a semifinite representable unbounded m^*-seminorm on a $*$-algebra $\mathcal{A}$ and $|\cdot|_p$ the unbounded Gelfand-Naimark C^*-seminorm of p. Then the following statements are equivalent:*

(i) *$|\cdot|_p$ is hereditary spectral;*
(ii) *$|\cdot|_p$ is spectral and stable;*
(iii) *p is spectral and stable.*

If this is true, then p is hereditary spectral.

Proof. Since $\mathcal{D}(p) = \mathcal{D}(|\cdot|_p)$ and $|\cdot|_p \leqslant p$ on $\mathcal{D}(p)$, it follows that $|\cdot|_p$ is semifinite, and p is stable if and only if $|\cdot|_p$ is stable, which implies by Theorem 6.10 that the statements (i) and (ii) are equivalent, and the implication (ii) $\Rightarrow$ (iii) holds. We show the implication (iii) $\Rightarrow$ (ii). Since $|\cdot|_p$ is a semifinite unbounded C^*-seminorm on $\mathcal{A}$, there exists a $*$-representation π_p of $\mathcal{A}$ such that $\|\overline{\pi_p(x)}\| = |x|_p$ for each $x \in \mathcal{D}(|\cdot|_p) = \mathcal{D}(p)$. It is shown similarly to the proof of (ii) $\Rightarrow$ (i) in Theorem 6.10 that $|\cdot|_p$ is spectral. Here we note simply the proof. Take arbitrary $h^* = h \in \mathcal{D}(p)$ and $\lambda \neq 0 \in \mathrm{Sp}_{\mathcal{D}(p)}(h) \cap \mathbb{R}$. By the stablity of p there exists a $*$-representation ϱ of $\mathcal{A}$ such that $\mathcal{A}_\flat^\varrho \supset \mathcal{D}(p)$, $[\varrho(\mathcal{D}(p))\mathcal{H}_\varrho] = \mathcal{H}_\varrho$ and $\varrho(h)\lceil\mathbb{C} = \lambda$. Further, it follows from the spectrality of p that $\|\overline{\varrho(x)}\| \leqslant p(x)$ for each $x \in \mathcal{D}(p)$, which implies that $\varrho\lceil\mathcal{D}(p) \in \mathrm{Rep}(p)$. Hence we have

$$\|\overline{\varrho(x)}\| \leqslant |x|_p = \|\overline{\pi_p(x)}\|, \qquad \forall x \in \mathcal{D}(p),$$

which implies $\lambda \in \mathrm{Sp}_{\overline{\pi_p(\mathcal{D}(p))}^{\|\cdot\|}}(\overline{\pi_p(h)})$. Hence we have

$$\mathrm{Sp}_{\mathcal{D}(p)}(h) \cap \mathbb{R} \subset \mathrm{Sp}_{\overline{\pi_p(\mathcal{D}(p))}^{\|\cdot\|}}(\overline{\pi_p(h)}) \cup \{0\},$$

which implies

$$\mathrm{Sp}_{\mathcal{D}(p)}(x) \subset \{\lambda \in \mathbb{C} : |\lambda| < |x|_p\}, \quad \forall x \in \mathcal{D}(p).$$

Hence it follows that $r_{\mathcal{D}(p)}(x) \leqslant |x|_p$ for each $x \in \mathcal{D}(p)$. Thus, $|\cdot|_p$ is spectral. This completes the proof. ∎

The implication (iii) $\Rightarrow$ (i) in Proposition 6.13 holds under a weaker assumption than that of semifiniteness of p as follows:

COROLLARY 6.13. *Suppose p is a spectral, stable, representable unbounded m^*-seminorm on $\mathcal{A}$ such that $\mathfrak{N}_p$ is dense in $\mathcal{D}(p)$ with respect to any r_π ($\pi \in \mathrm{Rep}(p)$). Then $|\cdot|_p$ is hereditary spectral and $\mathcal{A}$ is an unbounded hereditary C^*-spectral algebra.*

Proof. By Proposition 5.1, $|\cdot|_p$ is weakly semifinite and there exists a $*$-representation π_p of $\mathcal{A}$ such that $\|\overline{\pi_p(x)}\| = |x|_p$ for each $x \in \mathcal{D}(p)$. Hence it is shown in the same way as the proof (iii) $\Rightarrow$ (ii) in Proposition 6.12 that $|\cdot|_p$ is spectral, which implies by Proposition 6.12 that $|\cdot|_p$ is hereditary spectral. ∎

7. EXAMPLES

We give some examples of unbounded C^*-seminorms on $*$-algebras.

EXAMPLE 7.1. A locally convex $*$-algebra is a $*$-algebra which is also a Hausdorff locally convex space such that the multiplication is separately continuous and the involution is continuous. Let $\mathcal{A}$ be a locally convex $*$-algebra with identity $\mathbb{1}$. We denote by $\mathfrak{B}$ the collection of closed, bounded absolutely convex subsets $\mathcal{B}$ of $\mathcal{A}$ satisfying $\mathbb{1} \in \mathcal{B}$ and $\mathcal{B}^2 \subset \mathcal{B}$. For every $\mathcal{B} \in \mathfrak{B}$, the linear span $\mathcal{A}[\mathcal{B}]$ of $\mathcal{B}$ forms a normed algebra equipped with the Minkowski functional $\|\cdot\|_{\mathcal{B}}$ of $\mathcal{B}$. If $\mathcal{A}[\mathcal{B}]$ is complete for every $\mathcal{B} \in \mathfrak{B}$, then $\mathcal{A}$ is said to be *pseudo-complete*. If $\mathcal{A}$ is sequentially complete, then it is pseudo-complete. An element x of $\mathcal{A}$ is *bounded* if $\{(\lambda x)^n : n \in \mathbb{N}\}$ is bounded for some $\lambda \in \mathbb{C}$, and denote by $\mathcal{A}_0$ the set of all bounded elements of $\mathcal{A}$. G.R. Allan ([2]) and P.G. Dixon ([13]) defined the notion of GB*-algebra which is a generalization of C^*-algebra. A pseudo-complete locally convex $*$-algebra $\mathcal{A}$ is said to be a GB*-*algebra over* $\mathcal{B}_0$ if $\mathcal{B}_0$ is the greatest member in $\mathfrak{B}^* := \{\mathcal{B} \in \mathfrak{B}^* : \mathcal{B}^* = \mathcal{B}\}$ and $(\mathbb{1} + x^*x)^{-1} \in \mathcal{A}[\mathcal{B}_0]$ for every $x \in \mathcal{A}$. Then $\mathcal{A}[\mathcal{B}_0]$ is a C^*-algebra with the C^*-norm $\|\cdot\|_{\mathcal{B}_0}$. We put

$$\mathcal{D}(p_{\mathrm{GB}^*}) = \mathcal{A}[\mathcal{B}_0] \quad \text{and} \quad p_{\mathrm{GB}^*}(x) = \|x\|_{\mathcal{B}_0}, \quad x \in \mathcal{A}[\mathcal{B}_0].$$

Then p_{GB^*} is a spectral unbounded C^*-norm on $\mathcal{A}$. Hence every GB*-algebra is an unbounded C^*-spectral algebra. We consider the following questions:

(1.) When does p_{GB^*} satisfy the condition $\mathfrak{N}_{p_{\mathrm{GB}^*}} \not\subset N_{p_{\mathrm{GB}^*}}$ (equivalently $\mathfrak{N}_{p_{\mathrm{GB}^*}} \neq \{0\}$)?

(2.) When is p_{GB^*} semifinite or weakly semifinite?

(3.) When does there exist a family $\{p_\lambda\}_{\lambda\in\Lambda}$ of seminorms determining the topology such that $p_{\mathrm{GB}^*} = \sup\limits_{\lambda\in\Lambda} p_\lambda$?

Let $\mathfrak{M}$ be a left ideal of a GB*-algebra $\mathcal{A}$ contained in $\mathcal{A}[\mathcal{B}_0]$. Suppose $\mathfrak{M}$ is dense in the C^*-algebra $\mathcal{A}[\mathcal{B}_0]$. By standard C^*-algebra theory, $\mathfrak{M}$ contains a bounded approximate identity $\{u_\alpha\}$ for the C^*-algebra $\mathcal{A}[\mathcal{B}_0], u_\alpha^* = u_\alpha, \|u_\alpha\|_{\mathcal{B}_0} \leqslant 1$ for all α. By the proof of Theorem 3.6 in [5] (see also [24], Proposition 3.11 for a particular case), $\{u_\alpha\}$ is a bounded approximate identity for $\mathcal{A}$. Since $\mathfrak{M} \subset \mathfrak{N}_{p_{\mathrm{GB}^*}}$, it follows that p_{GB^*} is uniformly semifinite. Let π be any $*$-representation of $\mathcal{A}$ having $\mathcal{A}_\flat^\pi = \mathcal{A}[\mathcal{B}_0]$. Let $r_\pi(x) = \|\pi(x)\|$ for $x \in \mathcal{D}(r_\pi) = \mathcal{A}_\flat^\pi$. Since $\mathfrak{M} \subset \mathfrak{N}_{r_\pi}$, it follows from Proposition 3.1 that $\widetilde{\pi^N_{r_\pi}} = \widetilde{\pi}$. Here we consider the cases of pro-C^*-algebras and C^*-like locally convex $*$-algebras which are important in GB*-algebras.

(1) A complete locally convex $*$-algebra $\mathcal{A}[\tau]$ is said to be a pro-C^*-algebra ([24]) if the topology τ is determined by a direct family $\{p_\lambda\}_{\lambda\in\Lambda}$ of C^*-seminorms. Then $\mathcal{A}$ is a GB*-algebra over $\mathcal{B}_0 = \mathcal{U}(\sup_{\lambda\in\Lambda} p_\lambda) := \{x \in \mathcal{A} : \sup_{\lambda\in\Lambda} p_\lambda(x) \leqslant 1\}$ with $p_{\mathrm{GB}^*} = \sup_{\lambda\in\Lambda} p_\lambda$

(a) Let X be a locally compact non-compact Hausdorff-space and $\mathcal{A} = C(X)$ is a locally convex $*$-algebra of all complex-valued continuous functions on X with the compact open topology. The compact open topology is defined by a family $\{p_M : M \text{ is a compact subset of } X\} : p_M(f) = \sup_{x\in M} |f(x)|$, $f \in C(X)$. Then $\mathcal{A}$ is a pro-C^*-algebra and $\mathcal{A}[\mathcal{B}_0]$ equals the C^*-algebra $(C_\flat(X), \|\cdot\|_\infty)$ of all bounded continuous functions on X. Since $C_c(X) := \{f \in C_\flat(X) : \mathrm{supp} f \text{ is compact}\} \subset \mathfrak{N}_{p_{\mathrm{GB}^*}}$, it follows that $\mathfrak{N}_{p_{\mathrm{GB}^*}}$ is dense in $\mathcal{D}(p_{\mathrm{GB}^*})$ with respect to the compact open topology, but p_{GB^*} is not semifinite in general. For example, when $X = \mathbb{R}$, p_{GB^*} is maximal and weakly semifinite, but not semifinite.

(b) Let X be a σ-finite measure space and $\mathcal{A} = L^\infty_{\mathrm{loc}}(X)$ is a locally convex $*$-algebra of all measurable functions which are essentially bounded on every set of finite measure equipped with the topology defined by the family of C^*-seminorms $\{\|\cdot\|_A : \|f\|_A = \operatorname{ess\,sup}_{x\in A} |f(x)|$, where $A \subset X$ is any set of finite measure$\}$. Then $\mathcal{A}$ is a pro-C^*-algebra and a GB*-algebra having $\mathcal{A}[\mathcal{B}_0] = L^\infty(X)$ and $p_{\mathrm{GB}^*}(f) = \|f\|_\infty := \sup_A \|f\|_A$, $f \in L^\infty(X)$. Since

$$L^\infty_{\mathrm{c}}(X) := \{f \in L^\infty_{\mathrm{loc}}(X) : \mathrm{supp}\, f \text{ is contained in some set of finite measure}\} \subset \mathfrak{N}_{p_{\mathrm{GB}^*}},$$

it follows that $\mathfrak{N}_{p_{\mathrm{GB}^*}}$ is dense in $\mathcal{D}(p_{\mathrm{GB}^*})$ with respect to the locally convex topology and p_{GB^*} is maximal and weakly semifinite.

(c) Let $\mathcal{B}$ be a C^*-algebra without identity. Let $K_{\mathcal{B}}$ be the Pedersen ideal of $\mathcal{B}$, $M(\mathcal{B})$ be the C^*-algebra of all multipliers of $\mathcal{B}$, and $\mathcal{A} = \Gamma(K_{\mathcal{B}})$ be the $*$-algebra of all multipliers of $K_{\mathcal{B}}$ ([15] and [25]). Let p be any C^*-seminorm on $\mathcal{B}$. Then p can be regarded as an unbounded C^*-seminorm on $\mathcal{A}$ with $\mathcal{D}(p) = \mathcal{B}$. Since $K_{\mathcal{B}}$ is a $*$-ideal of $\mathcal{A}$ and it is dense in $\mathcal{B}$, it follows that $K_{\mathcal{B}} \subset \mathfrak{N}_p$ and p is uniformly semifinite. In fact, $\mathcal{A}$ is a pro-C^*-algebra with appropriate topology.

(2) A complete locally convex $*$-algebra $\mathcal{A}[\tau]$ is said to be *C^*-like* if there exists a C^*-like family $\{p_\lambda\}_{\lambda\in\Lambda}$ of seminorms determining the topology τ such that $\mathcal{D}\big(\sup_{\lambda\in\Lambda} p_\lambda\big) := \big\{x \in \mathcal{A} : \sup_{\lambda\in\Lambda} p_\lambda(x) < \infty\big\}$ is τ-dense in $\mathcal{A}$. Here we say that $\{p_\lambda\}_{\lambda\in\Lambda}$ is *C^*-like* if for any $\lambda \in \Lambda$ there exists $\lambda' \in \Lambda$ such that $p_\lambda(xy) \leqslant p_{\lambda'}(x)p_{\lambda'}(y)$, $p_\lambda(x^*) = p_\lambda(x)$ and $p_\lambda(x)^2 \leqslant p_{\lambda'}(x^*x)$ for each $x, y \in \mathcal{A}$. It follows from ([18], Theorem 2.1) that $\mathcal{A}$ is a GB*-algebra over $\mathcal{B}_0 = \mathcal{U}(\sup_{\lambda\in\Lambda} p_\lambda)$ with $p_{\mathrm{GB}^*} = \sup_{\lambda\in\Lambda} p_\lambda$. Let $\mathcal{A} = L^\omega[0,1] := \bigcap_{1\leqslant p<\infty} L^p[0,1]$ be the Arens GB*-algebra equipped with the topology defined by the family of L^p-norms ([3]). Then $\mathcal{A}$ is a C^*-like locally convex $*$-algebra with the C^*-like family $\{\|\cdot\|_p : 1 \leqslant p < \infty\}$ of seminorms, and

$\mathcal{A}[\mathcal{B}_0] = L^\infty[0,1]$ and $p_{\mathrm{GB}^*} = \sup_{1 \leqslant p < \infty} \| \cdot \|_p$. But, $L^\omega[0,1]$ is not a pro-C^*-algebra and $\mathfrak{N}_{p_{\mathrm{GB}^*}} = \{0\}$. Here is a non-commutative analogue of this ([17]). Let $\mathcal{M}_0$ be a von Neumann algebra with a faithful normal tracial state φ. Let $L^p(\mathcal{M}_0, \varphi)(1 \leqslant p \leqslant \infty)$ be the Segal L^p-space ([30]). Then $L^p(\mathcal{M}_0, \varphi)$ is a Banach space of closed operators in $\mathcal{H}$ affiliated with $\mathcal{M}_0$ with L^p-norm $\|X\|_p := \varphi(|X|^p)^{1/p}$. For $1 \leqslant r \leqslant p$, $L^\infty(\mathcal{M}_0, \varphi) = \mathcal{M}_0 \subset L^p(\mathcal{M}_0, \varphi) \subset L^r(\mathcal{M}_0, \varphi) \subset L^1(\mathcal{M}_0, \varphi)$. By using non-commutative Hölder's inequality it follows that $L^\omega(\mathcal{M}_0, \varphi) := \bigcap_{1 \leqslant p < \infty} L^p(\mathcal{M}_0, \varphi)$ is a $*$-algebra with identity and with strong operators : $\overline{X+Y}, \overline{\lambda X}, \overline{XY}$ and operator adjoint as the involution. Let τ_ω be the topology on $L^\omega(\mathcal{M}_0, \varphi)$ defined by the C^*-like family $\Gamma = \{\| \cdot \|_p : 1 \leqslant p < \infty\}$. Then $L^\omega(\mathcal{M}_0, \varphi)$ is a C^*-like locally convex $*$-algebra with $p_{\mathrm{GB}^*}(X) = \sup_{n \in \mathbb{N}} \|X\|_n = \|X\|_\infty$ (operator-norm).

EXAMPLE 7.2. We consider Köthe sequence spaces and convolution algebras.

(1) Let ω denote the set of all sequences of complex numbers. Let $\mathcal{P}$ be a set of positive sequences $a = \{a_n\}$ in ω satisfying

(i) $\forall \{a_n\}, \{b_n\} \in \mathcal{P}$, $\exists \{c_n\} \in \mathcal{P}$; $a_n \leqslant c_n$, $b_n \leqslant c_n$, $n \in \mathbb{N}$;

(ii) $a_n > 0$, $\forall n \in \mathbb{N}$ for $\forall \{a_n\} \in \mathcal{P}$;

(iii) $a_{n+1} \leqslant a_n$, $\forall n \in \mathbb{N}$ for $\forall \{a_n\} \in \mathcal{P}$;

(iv) ${}^\forall \{a_n\} \in \mathcal{P}$, $\exists \{d_n\} \in \mathcal{P}$; $a_n \leqslant d_n^2$, $\forall n \in \mathbb{N}$.

Let $1 \leqslant q < \infty$. The Köthe sequence space $\ell^q(\mathcal{P})$ is defined as

$$\ell^q(\mathcal{P}) = \Big\{ x = \{x_n\} \in \omega : p_a^q(x) := \Big(\sum_n |x_n|^q a_n^q \Big)^{1/q} = \|xa\|_q < \infty, \forall a \in \mathcal{P} \Big\}.$$

$\ell^q(\mathcal{P})$ is a complete locally convex $*$-algebra (pointwise operations, complex conjugation) with respect to the topology $\tau_{\mathcal{P}}^q$ defined by seminorms $\{p_a^q : a \in \mathcal{P}\}$ ([6]). It is clear that $\mathcal{P} \subset \ell^\infty$ and $\ell^q(\mathcal{P})$ contains ℓ^q as a dense $*$-subalgebra. Further, it follows from (iv) that for any $a \in \mathcal{P}$, $p_a^q(xy) \leqslant p_d^q(x) p_d^q(y)$ and $p_a^q(x^*) = p_a^q(x)$ for each $x, y \in \ell^q(\mathcal{P})$, which implies that $\sup_{a \in \mathcal{P}} p_a^q$ is a spectral unbounded m^*-norm on $\ell^q(\mathcal{P})$. Let $q = \infty$. Then

$$\ell^\infty(\mathcal{P}) := \big\{ x = \{x_n\} \in \omega : p_a^\infty(x) = \|xa\|_\infty < \infty, \forall a \in \mathcal{P} \big\}$$

is a C^*-like locally convex $*$-algebra with the C^*-like direct family $\{p_a^\infty : a \in \mathcal{P}\}$ of seminorms. Hence $\sup_{a \in \mathcal{P}} p_a^\infty$ is a spectral unbounded C^*-norm on $\ell^\infty(\mathcal{P})$.

Further, suppose

(v) $\|a\|_\infty \leqslant 1$ for $\forall a \in \mathcal{P}$.

Then since $\mathcal{D}(\sup_{a \in \mathcal{P}} p_a^q) \supset \ell^q$ and

$$\mathfrak{N}_{\sup_{a \in \mathcal{P}} p_a^q} \supset \mathcal{F} := \{x = \{x_n\} \in \omega : x_n = 0 \text{ except for finite many } n\},$$

it follows that $\sup_{a \in \mathcal{P}} p_a^q$ is semifinite. Similarly, $\sup_{a \in \mathcal{P}} p_a^\infty$ is semifinite. Here is an important special case. Let

$$s = \{x = \big\{x_n\big\} \in \omega : \{n^k x_n\} \in \ell^\infty, \forall k \in \mathbb{N}\}$$

be the $*$-algebra consisting of all rapidly decreasing sequences. Then

$$\mathcal{P} := \Big\{\{|x_n|\} : \{x_n\} \in s,\ \sup_n |x_n| \leqslant 1 \text{ and } |x_{n+1}| \leqslant |x_n|,\ \forall n \in \mathbb{N}\Big\}$$

satisfies the condition (i)-(v). Then we have

$$\begin{aligned}
\ell^1(\mathcal{P}) &= \{x = \{x_n\} \in \omega : \{x_n y_n\} \in \ell^1,\ \forall y = \{y_n\} \in \mathcal{P}\} \\
&= s' \text{ (the set of all tempered sequences)} \\
&= \{x \in \omega : \sup_n |x_n| n^{-m} < \infty \text{ for some } m \in \mathbb{N}\}, \\
\mathcal{D}(\sup_{y\in\mathcal{P}} p_y^1) &= \{x \in s' : \sup_{y\in\mathcal{P}} \|xy\|_1 < \infty\}, \\
(\sup_{y\in\mathcal{P}} p_y^1)(x) &= \sup_{y\in\mathcal{P}} \|xy\|_1, \quad x \in \mathcal{D}(\sup_{y\in\mathcal{P}} p_y^1)
\end{aligned}$$

and $\sup_{y\in\mathcal{P}} p_y^1$ is a semifinite spectral unbounded m^*-norm on s'.

We can define the following unbounded m^*-norms p^q and p^q_∞ on $\ell^q(\mathcal{P})$ by

$$\begin{aligned}
&\mathcal{D}(p^q) = \ell^q(\mathcal{P}) \cap \ell^q = \ell^q \quad \text{and} \quad p^q(x) = \|x\|_q, \quad x \in \mathcal{D}(p^q); \\
&\mathcal{D}(p^q_\infty) = \ell^q(\mathcal{P}) \cap \ell^\infty \quad \text{and} \quad p^q_\infty(x) = \|x\|_\infty, \quad x \in \mathcal{D}(p^q_\infty).
\end{aligned}$$

Since $(\ell^q, \|\cdot\|_q)$ is a Banach $*$-algebra and $\mathfrak{N}_{p^q}$ contains a dense subspace $\mathcal{F}$ in ℓ^q, it follows that p^q is a semifinite spectral unbounded m^*-norm on $\ell^q(\mathcal{P})$, and p^q_∞ is the unbounded Gelfand-Naimark C^*-norm defined by the unbounded m^*-norm p^q, and it is semifinite.

(2) The above (1) can be used to model certain convolution algebra as illustrated below. Let $\triangle = \{z \in \mathbb{C} : |z| < 1\}, H(\triangle)$ be the nuclear Fréchet space of all functions holomorphic on $\triangle$. $H(\triangle)$ is a $*$-algebra with involution $f^*(z) = \overline{f(\overline{z})}$ and Hadamard product $(f * g)(x) = \frac{1}{2\pi \mathrm{i}} \int f(z) g(xz^{-1}) z^{-1}\, \mathrm{d}z$, $|x| < r < 1$. The function $e(z) = (1-z)^{-1}$ is the identity of $H(\triangle)$. The algebra $H(\triangle)$ is $*$-isomorphic to $\ell^1(\mathcal{P})$ with $\mathcal{P} = \{\{r^n\}_{n=0}^\infty : 0 < r < 1\}$ via the isomorphism $\psi : H(\triangle) \to \ell^1(\mathcal{P})$, $\psi(f) = \Big\{\frac{f^{(n)}(0)}{n!}\Big\}_{n=0}^\infty$. It follows that $a^q(f) = \sup_{0<r<1} \Big[\sum_n \Big|\frac{f^{(n)}(0)}{n!} r^n\Big|^q\Big]^{\frac{1}{q}}$ $(1 \leqslant q \leqslant \infty)$ defines a semifinite unbounded norm on $H(\triangle)$. Let $T = \{z \in \mathbb{C} : |z| = 1\}$ be the unit circle. The Fréchet space $C^\infty(T)$ of C^∞-functions on T with the topology τ defined by the seminorms $p_n(f) = \sum_{k=0}^n \frac{1}{k!} \sup_{t\in T} |f^{(k)}(t)|$ is a convolution $*$-algebra with involuiton $f^*(z) = \overline{f(\overline{z})}$. $C^\infty(T)$ is isomorphic to the sequence algebra $s(\mathbb{Z}) := \big\{x = \{x_n\}_{-\infty}^\infty : \{|n|^k x_n\}_{-\infty}^\infty \in \ell^\infty, \forall k \in \mathbb{N}\big\}$. The dual of $C^\infty(T)$ is the commutative convolution algebra $\mathcal{D}(T)$ of all distributions on T, the identity being the Dirac delta δ and the involution being $u \to u^*$, $< u^*, f >= \overline{\langle u, f^* \rangle}$ $(f \in C^\infty(T))$. Let $u \to \widehat{u}$, $\widehat{u}(n) = \langle u, \exp(-\mathrm{i}nt)\rangle$ $(n \in \mathbb{Z})$ be the Fourier-Schwarz transform that map $\mathcal{D}(T)$-$*$-isomorphically onto the $*$-algebra $s'(\mathbb{Z}) = \big\{a = \{a_n\}_{-\infty}^\infty : a_n = \mathrm{O}(|n|^m)$ for some m depending on $a\big\}$ having pointwise operations and complex conjugation as the involution. Under this map, the $*$-subalgebra $\mathrm{PM}(T)$ (*pseudo measures on* T) of $\mathcal{D}(T)$ is mapped onto $\ell^\infty(\mathbb{Z})$. By (1) we can define a semifinite spectral unbounded m^*-norm on $\mathcal{D}(T)$ and a semifinite spectral unbounded C^*-norm on $\mathrm{PM}(T)$. In fact, $\mathcal{D}(T)$ is a sequentially

complete GB*-algebra with sequentially jointly continuous multiplication and having bounded part $\mathcal{A}[\mathcal{B}_0] = \mathrm{PM}(T)$. For the unbounded C^*-norm p_{GB^*}, we have $\mathcal{D}(p_{\mathrm{GB}^*}) = \mathrm{PM}(T)$ and $p_{\mathrm{GB}^*}(x) = \sup_{n\in\mathbb{Z}} |\widehat{x}(n)| = \|\widehat{x}\|_\infty$. Further, by (12.6.2, p.74) in [15] $C^\infty(T)$ is an ideal of $\mathcal{D}(T)$ and so $C^\infty(T) \subset \mathfrak{N}_{p_{\mathrm{GB}^*}}$.

EXAMPLE 7.3. We consider unbounded C^*-norms on O*-algebras. We put

$$\mathcal{D}(p_b) = \mathcal{M}_b := \{X \in \mathcal{M} : \overline{X} \text{ is bounded }\}$$

and $p_b(X) = \|\overline{X}\|$, $X \in \mathcal{D}(p_b)$. Then p_b is an unbounded C^*-norm on $\mathcal{M}$.

(1) Let $\{\mathcal{M}_\lambda\}_{\lambda\in\Lambda}$ be a family of bounded $*$-algebras $\mathcal{M}_\lambda$ on Hilbert spaces $\mathcal{H}_\lambda$ with identity operator and $\prod_{\lambda\in\Lambda} \mathcal{M}_\lambda$ be the product of $\{\mathcal{M}_\lambda\}_{\lambda\in\Lambda}$. We put

$$\mathcal{D}\Big(\prod_{\lambda\in\Lambda}\mathcal{M}_\lambda\Big) = \Big\{(\xi_\lambda) \in \bigoplus_{\lambda\in\Lambda}\mathcal{H}_\lambda : \sum_{\lambda\in\Lambda}\|X_\lambda\xi_\lambda\|^2 < \infty, \forall (X_\lambda) \in \prod_{\lambda\in\Lambda}\mathcal{M}_\lambda\Big\},$$

$$(X_\lambda)(\xi_\lambda) = (X_\lambda\xi_\lambda), \quad (X_\lambda) \in \prod_{\lambda\in\Lambda}\mathcal{M}_\lambda, \quad (\xi_\lambda) \in \mathcal{D}\Big(\prod_{\lambda\in\Lambda}\mathcal{M}_\lambda\Big).$$

Then $\prod_{\lambda\in\Lambda}\mathcal{M}_\lambda$ is an O*-algebra on $\mathcal{D}(\prod_{\lambda\in\Lambda}\mathcal{M}_\lambda)$ in $\bigoplus_{\lambda\in\Lambda}\mathcal{H}_\lambda$. A $*$-subalgebra of such an O*-algebra is said to be *weakly bounded*. Let $\mathcal{M}$ be a weakly bounded O*-algebra, that is, a $*$-subalgebra of the O*-algebra $\prod_{\lambda\in\Lambda}\mathcal{M}_\lambda$. Then

$$\mathcal{D}(p_\flat) = \big\{(X_\lambda) \in \mathcal{M} : \sup_\lambda \|X_\lambda\| < \infty\big\},$$

$$p_\flat((X_\lambda)) = \sup_\lambda \|X_\lambda\|, \quad (X_\lambda) \in \mathcal{D}(p_\flat).$$

Suppose that $\mathcal{M}$ contains the family $\{E_\lambda\}_{\lambda\in\Lambda}$ of the projection E_λ of $\bigoplus_{\lambda\in\Lambda}\mathcal{H}_\lambda$ onto $\mathcal{H}_\lambda$, in particular, $\mathcal{M} = \prod_{\lambda\in\Lambda}\mathcal{M}_\lambda$. Then $p_\flat$ is a maximal, regular and semifinite unbounded C^*-norm on $\mathcal{M}$. Schmüdgen ([28]) has given necessary and sufficient conditions under which a closed O*-algebra is weakly bounded.

(2) Let $\mathcal{M}$ be an O*-algebra on $\mathcal{D}$ in $\mathcal{H}$. Suppose $\mathcal{M} \supset \{\xi_n \otimes \overline{\xi_n} : n \in \mathbb{N}\}$, where $\{\xi_n\}$ is an orthonormal basis in $\mathcal{H}$ contained in $\mathcal{D}$. Then $p_\flat$ is a maximal and weakly semifinite unbounded C^*-norm on $\mathcal{M}$.

(3) Let $\mathcal{M}_0$ be the O*-algebra on the Schwartz space $\mathcal{S}(\mathbb{R})$ generated by the momentum operator P and the position operator Q. Then $\mathcal{D}(p_\flat) = \mathbb{C}I$ and $\mathfrak{N}_{p_\flat} = \{0\}$. Let $\mathcal{M}$ be an O*-algebra on $\mathcal{S}(\mathbb{R})$ generated by $\mathcal{M}_0$ and $\{f_n \otimes \overline{f_n} : n = 0, 1, \ldots\}$, where $\{f_n\}$ is an orthonormal basis in $L^2(\mathbb{R})$ consisting of the normalized Hermite functions. Then it follows that $\mathfrak{N}_{p_\flat}$ equals the $*$-algebra generated by $\{A(f_n \otimes \overline{f_n}) : A \in \mathcal{M}_0, n = 0, 1, \ldots\}$, so that $p_\flat$ is a maximal and weakly semifinite unbounded C^*-norm on $\mathcal{M}$.

We intend to study unbounded m^*-(or C^*-)seminorms on *locally convex $*$-algebras*. In particular, it seems important to define and study the notions of topologically (hereditary) C^*-spectral algebras, topologically (hereditary) spectral $*$-representations and topological stability in case of locally convex $*$-algebras.

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