THE APOLAR INVARIANT OF BILINEAR FORMS.

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In this note an attempt is made to work out two ideas. Poristic conditions which express the apolarity of a quadric locus and a quadric envelope are well known in Projective Geometry. The general form of such poristic conditions for two apolar bilinear forms in digredient variables is investigated here. Secondly, certain rules of a geometrically convenient character are obtained for the calculation of the apolar invariant, and are illustrated from the properties of the Ricci Tensor in Riemannian space.

§ 1. We begin by considering the bilinear form $a_j^i x^j U_i$, where according to the tensor convention, summation over the values from 1 to n of any repeated index is implied. The variables x and U are supposed to be contragredient, so that we can take $(x^1, x^2, \cdots, x^n)$ as the homogeneous coordinates of a point x, and $(U_1, U_2, \cdots, U_n)$ as the homogeneous coordinates of a prime U in a projective space S_n of n-1 dimensions.

Consider now a point y and a prime V in S_n, and think of them together as a *point-prime* whose homogeneous coordinates are the n^2 linearly independent quantities $b_j^i = y^i V_j$. The point-prime $(y\text{V})$ may be identified with the factorisable bilinear form $b_j^i x^j U_i$. From the linear independence of the coordinates of a point-prime, it follows that any form can be expressed as the sum of factorisable forms or point-primes of the type $(y\text{V})$; the apolar invariant a_i^i of the form is then equal to the sum of the inner products $(y^i V_i)$ of each point-prime which occurs. Hence:

Theorem I. *In whatever way the form* a_j^i *is expressed as the sum of point-primes, the sum of the inner products of the point-primes is constant and equal to the apolar invariant* a_i^i *of the form.*

We may now deduce the poristic condition for the vanishing of the apolar invariant. A point-prime $(y\text{V})$ may be called *incident*, if its apolar invariant vanishes, or if the point y is incident with the prime V. Supposing the rank of a_j^i to be n, it is known from the theory of rank that if $y_{(1)}, y_{(2)}, \cdots, y_{(n)}$ be n assigned lineary independent points, we can choose n primes $V_{(i)}$, so that the form a_j^i is the sum of the n point-primes $(y_{(t)}V_{(t)})$. By considering the collineation determined by the form a_j^i it is easy to shew that we can choose the points y in an infinite number of ways, so that

$(n-1)$ of the point-primes $(y_{(t)}V_{(t)})$ are incident. If now the apolar invariant a_i^i of the form vanishes, it follows from Theorem I that the remaining point-prime (yV) must also be incident. Hence:

THEOREM II. *A form* a_j^i *with vanishing apolar invariant (or 'spur') can be expressed in an infinite number of ways as the sum of* n *incident point-primes.*

Let

$$a_j^i = \sum_{t=1}^{n} (y^i_{(t)} \, V_{(t)j}) \, ; \; y^i_{(t)} V_{(t)i} = 0 \text{ for } t = 1, 2, \cdots, n,$$

be one such expression for a_j^i. If $z_{(t)}$ be the intersection of the $(n-1)$ primes V other than $V_{(t)}$, it is immediately evident that $a_j^i z^j_{(t)} = \rho y^i_{(t)}$ for $i = 1, 2, \cdots, n$. Hence if U is any prime passing through $y_{(t)}$, $a_j^i z^j_{(t)} U_i = 0$. A point $z_{(t)}$ and a prime U which satisfy this relation may be conveniently described as *incident* with the form a_j^i. If now S_1 represent the simplex whose vertices are the points y, $\bar{S}_1$, the dual simplex constituted by the prime faces of S_1, $\bar{S}_2$ the simplex of the primes V, and S_2 the dual simplex constituted by the vertices of $\bar{S}_2$, we see that the corresponding elements of $S_1, \bar{S}_2$ are incident, while all pairs of non-corresponding elements of $\bar{S}_1, S_2$ are incident with a_j^i. Hence:

THEOREM III. *The poristic condition for the vanishing of the apolar invariant of the form* a_j^i *is the existence of a point simplex* S_1 *and a prime simplex* $\bar{S}_2$*, whose corresponding elements are mutually incident, such that the non-corresponding elements of the dual simplexes* $\bar{S}_1, S_2$ *are incident with* a_j^i.

By considering the collineation $x'^i = a_j^i x^j$ associated with the form, we may express this poristic condition in an alternative way, as the existence of a simplex whose vertices are carried into points on the opposite face by the associated collineation.

§ 2. Consider next the two bilinear forms $a_{ij} x^i y^j$, $b^{ij} U_i V_j$, where we suppose that (x, U) belong to a space S_n, while (y, V) belong to a second space S'_n. To obtain a geometrical interpretation of their apolar invariant $a_{ij} b^{ij}$, take n linearly independent primes $U_{(t)}$ $(t = 1, 2, \cdots, n)$ in S_n, and let the coordinates $\xi^i_{(t)}$ of the point of intersection $\xi_{(t)}$ of the $n-1$ primes other than $U_{(t)}$ be so normalised that:

$$\mu_{(t)i} \, \xi^i_{(t)} = 1 \, ; \; t = 1, 2, \cdots, n \qquad \qquad (1)$$

Further take n points $\eta_{(t)}$ in S'_n, given by:

$$\eta^j_{(t)} = b^{ij} \, U_{(t)i}.$$

Let the primes $V_{(t)}$ of the simplex constituted by the points η be so normalised that:

$$1 = V_{(t)i}\,\eta^i{}_{(t)} = b^{ij}\,U_{(t)i}\,V_{(t)j} \qquad \ldots \qquad \ldots \qquad \ldots \quad (2)$$

Now the form a_{ij} can be expressed as a linear combination of the n^2 prime-pairs $U_{(p)}\,V_{(q)}$; but for $p \neq q$ this prime-pair is incident with b^{ij} and therefore contributes nothing to the apolar invariant. Hence if

$$a_{ij} = \sum_{p,q} \lambda_{pq}\,(U_{(p)i}\,V_{(q)j}),$$

$$a_{ij}\,b^{ij} = \sum_{p} \lambda_{pp}\,b^{ij}\,U_{(p)i}\,V_{(p)j} = \Sigma\lambda_{pp} \text{ by } (2)$$

But $\sum_{t} a_{ij}\,\xi^i{}_{(t)}\,\eta^j{}_{(t)} = \Sigma\lambda_{tt} = a_{ij}\,b^{ij}$.

Thus the apolar invariant is the sum of the values of the form $a_{ij}\,x^i\,y^j$ for the n pairs of points ξ, η, the coordinates of ξ being normalised by (1).

§ 3. To obtain a poristic condition for the apolarity of a_{ij}, b^{ij}, we observe that if they are apolar, a_{ij} can be expressed in an infinite number of ways as the sum of n products of prime-pairs $U_{(t)}V_{(t)}$ incident with b^{ij}. It follows that $a_{ij}\xi^i{}_{(t)} = \rho V_{(t)j}$, if $\xi_{(t)}$ are the vertices of the simplex constituted by the U's. Hence it $\eta_{(t)}$ are the vertices of the simplex constituted by the V's,

$$a_{ij}\,\xi^i{}_{(p)}\,\eta^j{}_{(q)} = 0 \text{ for } p \neq q.$$

Hence:

THEOREM IV. *The poristic condition for the apolarity of the forms* a_{ij}, b^{ij} *is the existence of two simplexes in the respective spaces, whose corresponding prime-pairs are incident with* b^{ij}, *and whose non-corresponding vertex-pairs are incident with* a_{ij}.

§ 4. We now examine the various special cases of Theorem IV which arise:—

Suppose first that S'_n is the dual space of S_n, so that the points of S'_n are the primes of S_n. Then Theorem IV gives a poristic condition for the apolarity of two forms $a_j{}^i$, $b_i{}^j$ in S_n. If we suppose further that $b_i{}^j$ is the identical Kronecker bilinear form $\delta_j{}^i\,x^j U_i \equiv x^1U_1 + x^2U_2 + \ldots + x^nU_n$, the poristic condition reduces to the result of Theorem III.

Suppose next that S'_n is identical with S_n and consider the case when the forms a_{ij}, b^{ij} are symmetric or skew-symmetric. Since a symmetric form is identically apolar to a skew-symmetric one, we need only consider the cases in which both forms are symmetric or both skew-symmetric. For the symmetric case, a_{ij} and b^{ij} represent respectively a quadric locus and quadric envelope; when the forms are apolar, an instance of the poristic simplexes of Theorem IV is furnished by the self-polar simplexes of b^{ij} which are inscribed in a_{ij}. These are not, however, the general type of poristic simplex-pairs. The general pair of poristic simplexes of Theorem IV will now consist of a pair of mutually polar simplexes of a_{ij} whose corresponding primes are conjugate in regard to b^{ij}.

For the skew-symmetric case a_{ij} will represent a linear line-complex, and b^{ij} a linear S_{n-2}-complex. For n odd these complexes are necessarily singular. We may accordingly suppose n to be even, and the complexes non-singular. If the complexes are apolar, it is known that we can choose in an infinite number of ways a set of $\frac{n}{2}$ lines $l_1\, l_2 \cdots\cdot l_{\frac{n}{2}}$ which belong to the complex a_{ij} and form a conjugate set w.r.t. the complex b^{ij}. If $a_r\, b_r$ are two points on l_r ($r = 1, 2, \cdots$), the two simplexes $(a_1 b_1 a_2 b_2 \cdots)$, $(b_1 a_1 b_2 a_2 \cdots)$ are instances of the poristic simplex-pairs of Theorem IV. These are specialised. The general pair of poristic simplexes of Theorem IV consists of a pair of simplexes S_1, S_2 (which must necessarily be each inscribed and circumscribed to the other), the joins of whose non-corresponding vertices belong to the complex a_{ij}, and the intersections of whose corresponding faces belong to the complex b^{ij}.

§ 5. We may examine the interpretation of the apolar invariant in § 2 for the special case in which S'_n is identical with S_n and the forms are symmetric or skew-symmetric. For either case it is clear that the apolar invariant can be exhibited as $\Sigma\, a_{ij}\, x^i_{(t)}\, y^j_{(t)}$ where x's and y's are vertices of two mutually polar simplexes of b^{ij} whose coordinates are normalised as in § 2. Suppose both the forms are symmetric, and identify the two simplexes into a self-polar simplex $(x_{(t)})$ of b^{ij}. It is clear that the normalisation of § 2 implies that $b_{ij}\, x^i_{(t)}\, x^j_{(t)} = 1$; $t = 1, 2, \cdots, n$, where b_{ij} is the minor of b^{ij} in $| b^{ij} |$ divided by the determinant. Hence:

THEOREM V. *The value of the apolar invariant of two symmetric bilinear forms* a_{ij}, b^{ij} *is equal to the sum of the values of* a_{ij} *for the vertices of any self-polar simplex of* b^{ij}, *provided the coordinates of the vertices are normalised w.r.t. the reciprocal tensor of* b^{ij}.

By similar reasoning we obtain the corresponding property for the skew-symmetric case, namely:

THEOREM VI. *If* n *is even, the value of the apolar invariant of the non-singular skew-symmetric forms* a_{ij}, b^{ij} *is the sum* $\sum_{t=1}^{n/2} a_{ij}\, l_{(t)}{}^{ij}$, *where* $l_{(t)}$ *are a set of mutually conjugate lines of the complex* b^{ij}, *with coordinates normalised w.r.t. the reciprocal tensor of* b^{ij}.

§ 6. We shall now apply Theorem V to the interpretation of the Ricci Tensor in a Riemannian space of n dimensions, with the coordinates

$x_1, x_2, \cdots, x_n$, and the metrical ground-form $g_{ik}\, dx^i\, dx^k$. The Riemann-Christoffel Tensor $R_{ij,kl}$ is symmetric w.r.t. the two pairs (ij), (kl), and skew-symmetric w.r.t. each of the pairs (ij) and (kl) ; the Gaussian curvature at $x_1, x_2, \cdots, x_n$ of the planar direction defined by two vectors dx, δx has the value :

$$\frac{R_{ij,kl}\, dx^i\, \delta x^j\, dx^k\, \delta x^l}{(g_{ik} g_{jl} - g_{il} g_{jk}) dx^i\, \delta x^j\, dx^k\, \delta x^l}$$

Here the denominator represents the square of the area of the planar element $(dx, \delta x)$. The geometrical interpretation of the more general form $R_{ij,kl}\, dx^i\, \delta x^j\, d'x^i\, \delta' x^l$ has been given by Peré.*

The Ricci Tensor R_{jk} is defined by $R_{jk} = g^{il}\, R_{ij,kl}$. We can regard $R_{jk}\, dx^j\, \delta x^k$ for any particular dx, δx as the apolar invariant of $R_{ij,kl}\, d'x^i\, dx^j\, \delta x^k\, \delta' x^l$ and g^{il}, and therefore interpret its value by Theorem V. Thus $R_{jk}\, \lambda^j\, \lambda^k$ is equal to the sum of the products of (area)^2 and Gaussian curvature of the n planar vectors obtained by joining λ to a set of n mutually orthogonal unit vectors $\lambda_{(h)}$. Hence :

THEOREM VII. *If the unit vector* λ *at* $(x_1, \cdots, x_n)$ *makes angles* ϕ_h *with the unit orthogonal vectors* $\lambda_{(h)}$ $(h = 1, 2 \cdots, n)$, *and if* K_h *is the Gaussian curvature of the direction* $(\lambda \lambda_{(h)})$

$$R_{jk}\, \lambda^j\, \lambda^k = \sum_{h=1}^{n} K_h \sin^2 \phi_h .$$

It follows that the right side has the same value for all sets of orthogonal vectors $\lambda_{(h)}$. If we identify one of the vectors $\lambda_{(h)}$ with λ, we get Ricci's original interpretation,† namely :

$$R_{jk}\, \lambda_{(1)}{}^j\, \lambda_{(1)}{}^k = K_2 + K_3 + \cdots + K_n .$$

For the interpretation of the more general form $R_{jk}\, \lambda^j\, \mu^k$, we would have to use Peré's formula (*loc. cit.*).

§ 7. The scalar curvature R at $(x_1, x_2, \cdots, x_n)$ is defined by :

$$R = R_{jk}\, g^{jk} ;$$

That is, R is the apolar invariant of Ricci tensor and the metrical tensor. Hence by a second application of Theorem V, we have :

THEOREM VIII. *If* $\lambda_{(h)}$, $\mu_{(h)}$ $(h = 1, 2, \cdots, n)$ *be two mutually orthogonal sets of vectors,* ϕ_{hk} = *angle* $\lambda_{(h)} \wedge \mu_{(k)}$, K_{hk} = *Gaussian curvature of the planar element defined by* $\lambda_{(h)}$ *and* $\mu_{(k)}$, $\sum_{h,k} K_{hk} \sin^2 \phi_{hk}$ *is independent of the choice of the vectors and equal to the scalar curvature* R.

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* Levi-Civita, *Absolute Differential Calculus*, p. 193.

† Eisenhart, *Riemannian Geometry*, p. 113.

In particular if we identify the two sets of orthogonal vectors, we see that the sum of the Gaussian curvatures of the $\frac{n\,(n-1)}{2}$ planar directions defined by an orthogonal ennuple of vectors is constant and equal to the scalar curvature R.

If the scalar curvature R vanishes, then the forms g^{ij}, R_{ij} are apolar; since we suppose the metrical form to be definite, it follows that R_{ij} is indefinite, and therefore there exist real directions λ for which $R_{ij}\,\lambda^i\,\lambda^j = 0$. From the usual interpretation of apolarity, there follows the existence of sets of n mutually orthogonal vectors, the Gaussian curvatures K_{hk} corresponding to which satisfy the n relations:

$$\sum_h K_{hk} = 0 \ (h \neq k).$$

Or by Theorem IV there exist two sets of n mutually orthogonal vectors $\lambda_{(h)}\ \mu_{(k)}$ such that $\lambda_{(h)}\ \mu_{(k)}$ are orthogonal for $h \neq k$, while $\lambda_{(h)}$, $\mu_{(h)}$ are conjugate directions w.r.t. the Ricci tensor ($h = 1, 2, \cdots, n$).