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Stresses in Links

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An Investigation of the Stresses

... IN ...

Links with Elliptical and Oval Center-Lines

... BY ...

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THIS IS TO CERTIFY THAT THE THESIS PREPARED UNDER MY SUPERVISION BY

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IS APPROVED BY ME AS FULFILLING THIS PART OF THE REQUIREMENTS FOR THE DEGREE

OF Mechanical Engineer.

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HEAD OF DEPARTMENT OF Mechanical Engineering.

CONTENTS.

REFERENCES.

A History of the Strength and Elasticity of Materials, Todhunter and Pearson, Vol. II, Part I, pp. 422-445.

Elasticität und Festigkeit, C. von Bach, Section V.

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AN INVESTIGATION OF THE STRESSES IN LINKS WITH ELLIPTICAL AND OVAL CENTER-LINES

I. OBJECTS OF THE INVESTIGATION.

1. The investigations contained in this Thesis have been made with the following objects in view:

a. To ascertain the stresses actually induced in links of various forms, with and without restraining studs, when such links are subjected to the action of external forces.

b. To compare links of different form as regards strength, and thus to determine the form of link that will give maximum strength; in particular to compare the relative strengths of open links and stud links.

c. To derive from the results thus obtained working formulas for the loading of chains of the ordinary commercial sizes, and to compare these formulas with the formulas now in vogue.

II. PREVIOUS INVESTIGATIONS.

2. The investigation of the stresses in links was suggested to me by Bach's analysis of the stresses in a hollow cylindrical roller, "Elasticität und Festigkeit",p. 458. It was evident that the general method there employed could be used to compute the stresses in links with elliptical center-lines,and I attempted to extend the analysis to links with elliptical center-lines and make the circular center-lines a special case. Some time after I had completed this analysis, I found that that Grashof had made an analysis in his " Elasticität und Festigkeit", Arts:178-180,pp.273-277. While the method employed by Grashof seems correct,the assumptions he makes are clearly untenable,and the results he obtains are far from the truth. Grashof's treatment,however,suggested to me an idea of substituting an oval center-line of four circular arcs for the elliptical center-line.

Before proceeding further,I made a search through German Technical periodicals,in particular the Zeitschrift des Vereines Deutscher Ingenieur,with the expectation of finding a discussion of the subject that would render superfluous further efforts on my part. My search was fruitless,and I proceeded with the analysis of the cases that follow.

After I had finished the analytical work and had partly completed the computations,I accidentally discovered Pearson's elegant discussion of Winkler's memoir "Formandering und Festigkeit gekrummter

Körper inbesondere der ringe". This discussion is by Prof. Karl Pearson, and may be found in Todhunter and Pearson's, "History of the Elasticity and strength of materials; Vol 11, part 1, p.423 et sequi. To show Pearson's estimate of Winkler's work and of the method employed in the analysis, I quote from the first paragraphs of the discussion:

"This is an important memoir both from the theoretical and practical standpoint; although many of its results require correction and modification. Some of these corrections have been made in Kapitel XL (Ringformige Körper) of the authors well known treatise: "Die Lehre von der Elasticität und Festigkeit", Prag, 1867, but this treatise does not cover anything like the same area as the memoir, I propose therefore to indicate the correct analysis and compare its results with those of Winkler.

"The importance of the subject will be sufficiently grasped when I remind the reader that it is the only existing theory of the strength of the links of chains. To investigate the strength of such links by the complete theory of elasticity would involve even for the case of anchor rings an appalling investigation in toroidal and allied functions, While for the oval chain links with studs in ordinary use, any successful attempt at a general investigation seems inconceivable. We shall have the less hesitation, however in applying the Bernoulli-Eulerian theory, if we remember how close an approximation Saint-Venant's researches on flexure have shown it to be in the case of straight bars. At the same time we are certainly going to put it to the very limit of its application, namely to curved

bars in which the dimensions of the cross sections are not very small as compared with either the length or the radius of curvature of the central axis"... ... .

"Remembering that we need not assume adjacent cross sections of our link to remain undistorted, if we only suppose them to be approximately equally distorted, we can easily investigate an expression for the stretch at any point by a method akin to that which results from the Bernoulli-Eulerian theory". X

The method here referred to is that given by Bach and Grashof in connection with bodies having curved center-lines, and is the one that I have used as the basis of the following analysis.

Prof. Pearson considers only the case of the elliptical center-line and the center-line composed of two circular arcs and two straight lines. In both cases he pronounces Winkler's work incorrect and gives correct equations. The case of the center-line with four circular arcs seems to have been discussed by Winkler in his treatise instead of in the memoir .

I have not had access to the original memoir, which is published in "<u>Der Civilingenieur</u>" Bd. IV, S, 232-246. From Prof. Pearson's discussion, however, it appears that in all cases Winkler assumes the external force acting on the link to be concentrated at a point at the end of the link. Such an assumption considerably simplifies the analytical work. The results obtained from analysis based on this assumption may properly be used when the assumption is justified by the facts of the case; certainly not otherwise. In the case of chains and chain cables, the external force is the pressure between

two adjacent links. Unless the links are circular, this pressure cannot be concentrated at a single point, but is distributed over an area; in fact with links of ordinary proportions, the action between two links is that of a journal and bearing. As will be shown subsequently, this distribution reduces in a marked degree the stresses computed on the purely arbitrary assumption that the pressure is concentrated. For this reason the results obtained by the correct analysis of Prof. Pearson can hardly be used as basis for formulas giving the strength of chains. So far as I know, the discussions mentioned are the only ones concerning this subject in existence. Both are faulty; Winkler, by making an assumption not justified by fact, underestimates the strength of the link. Grashof recognizes the influence of the distribution of the load, but by incorrect reasoning arrives at results that if adopted would lead to a serious overestimation of the strength of the link.

III. GENERAL THEORY OF THE STRESSES IN BARS WITH CURVED CENTER-LINES.

3. The best statement of the fundamental theory underlying all the subsequent investigations is contained in Bach's "Elasticität und Festigkeit", section V. For the sake of completeness I give an outline of the theory. The method of presentation is substanially that of Bach. In Fig 1, BCC_1B_1 represents a part of a body of uniform cross

section. OO_1 is the center-line passing through the center of gravity of the sections. BOC and $B_1 O_1 C_1$ are two sections normal to the center-line. If the center-line is straight, the planes of the sections are parallel, but if it is curved the planes meet in the axis of curvature M, and make with each other the angle $d\varphi$. Suppose a normal force p to be uniformly distributed over the cross section $B_1 O_1 C_1$. Were the sections parallel, all fibers lying between them would have the same length, and as a consequence every fiber would be extended by the same amount; hence the action of the force p would result in a change in the distance between the sections, the parallelism remaining undisturbed. When the sections are inclined, as in Fig 1, the action is different. The force p being uniformly distributed over the section, each fiber is subjected to the same stress σ;

now since the <u>relative</u> extensions of all fibers are equal, it follows that the absolute change in the length of a fiber is proportional to the length of the fiber, or what is equivalent, to the distance of the fiber from the axis of curvature, M. Assuming, therefore, that the cross section remains plane, its plane after extension will pass through the axis M.

4. Suppose now that the section is subjected to the action of a normal force p and also to a couple whose moment may be denoted by M_b. the force p brings the section $B_1 O_1 C_1$ to the position $B_0 C_0$ and the couple of moment M_b induces a stress couple which produces further extensions of the fiber-either positive or negative-and brings the section to a new position $B_1' O_1' C_1'$. By the action of the couple, the

center of curvature of the center-line OO_1 is changed from M to M' and the radius of curvature is decreased from r to ρ. The inclination between the sections is increased from $d\phi$ to $d\phi + \Delta d\phi$.

Let ε_0 denote the relative extension of the center-line OO_1 and ε that of a fiber PP_1 at a distance η from the center-line; also let ds denote the length of the fiber OO_1; then

$$\varepsilon_0 = \frac{\Delta ds}{ds} = \frac{\overline{O_1O_1'}}{\overline{OO_1}},$$

and $$\varepsilon = \frac{\overline{P_1P_1'}}{\overline{PP_1}}.$$

Through O_1' let a line be drawn parallel to P_1C_1 cutting PP_1 in H; then

$$\overline{P_1P_1'} = \overline{P_1H} + \overline{HP_1'} = \overline{O_1O_1'} + \overline{HP_1'};$$

but $$\overline{OO_1'} = \varepsilon_0 \cdot \overline{OO_1} = \varepsilon_0\, ds = \varepsilon_0\, r\, d\phi,$$

and from the geometry of the figure,

$$\overline{HP_1'} = \overline{O_1'H} \times (d\varphi + \Delta d\varphi - d\phi) = \eta \cdot \Delta d\varphi,$$

and $$PP_1 = (r+\eta)\, d\varphi.$$

Substituting these values in the expression for ε,

$$\varepsilon = \frac{\varepsilon_0\, r\, d\varphi + \eta \cdot \Delta d\varphi}{(r+\eta)\, d\varphi} = \frac{\varepsilon_0 + \frac{\eta}{r}\frac{\Delta d\varphi}{d\varphi}}{1+\frac{\eta}{r}}.$$

Let the ratio $\frac{\Delta d\varphi}{d\varphi}$ be denoted by ω; then

$$\varepsilon = \varepsilon_0 + (\omega - \varepsilon_0)\frac{\frac{\eta}{r}}{1+\frac{\eta}{r}}. \qquad (1)$$

The normal stress corresponding to extension ε is

$$\sigma = E\varepsilon = E\left[\varepsilon_0 + (\omega-\varepsilon_0)\frac{\frac{\eta}{r}}{1+\frac{\eta}{r}}\right]. \qquad (2)$$

in which E denotes the modulus of elasticity.

Placing the stresses in equilibrium with the external forces, we obtain,

$$P = \int \sigma \, df = \int E \left[\varepsilon_0 + (\omega - \varepsilon_0) \frac{\frac{\eta}{r}}{1+\frac{\eta}{r}} \right] df \; ; \qquad (3)$$

$$M_b = \int \eta \sigma \, df = \int E \eta \left[\varepsilon_0 + (\omega - \varepsilon_0) \frac{\frac{\eta}{r}}{1+\frac{\eta}{r}} \right] df. \qquad (4)$$

Assuming the modulus E to be a constant, the equations become, respectively

$$P = E \left[\varepsilon_0 \int df + (\omega - \varepsilon_0) \int \frac{\eta}{r+\eta} \, df \right],$$

$$M_b = E \left[\varepsilon_0 \int \eta \, df + (\omega - \varepsilon_0) \int \frac{\eta^2}{r+\eta} \, df \right].$$

The center-line O O_1, Fig 1, passes through the center of gravity of each normal section; as a consequence

$$\int \eta \, df = 0.$$

Let

$$\int \frac{\eta}{r+\eta} \, df = - \varkappa f \; ;$$

then

$$\int \frac{\eta^2}{r+\eta} \, df = \int \left(\eta - r \frac{\eta}{r+\eta} \right) df = - r \int \frac{\eta}{r+\eta} \, df = \varkappa f r.$$

Introducing these values for the integrals in the preceding expressions for P and M_b,

$$P = E f \left[\varepsilon_0 - \varkappa (\omega - \varepsilon_0) \right],$$

$$M_b = E f (\omega - \varepsilon_0) \varkappa r.$$

Solving for ε_0 and ω we obtain the following important equations:

$$\left.\begin{aligned} \omega - \varepsilon_0 &= \frac{M_b}{Efr\varkappa}. \\ \varepsilon_0 &= \frac{P}{Ef} + \varkappa(\omega - \varepsilon_0) = \frac{1}{Ef}\left(P + \frac{M_b}{r}\right) \\ \omega &= \varepsilon_0 + \frac{1}{Ef}\frac{M_b}{\varkappa r} = \frac{1}{Ef}\left(P + \frac{M_b}{r} + \frac{M_b}{\varkappa r}\right). \end{aligned}\right\} \quad (5)$$

Substituting these values of ε_0 and ω in (2),

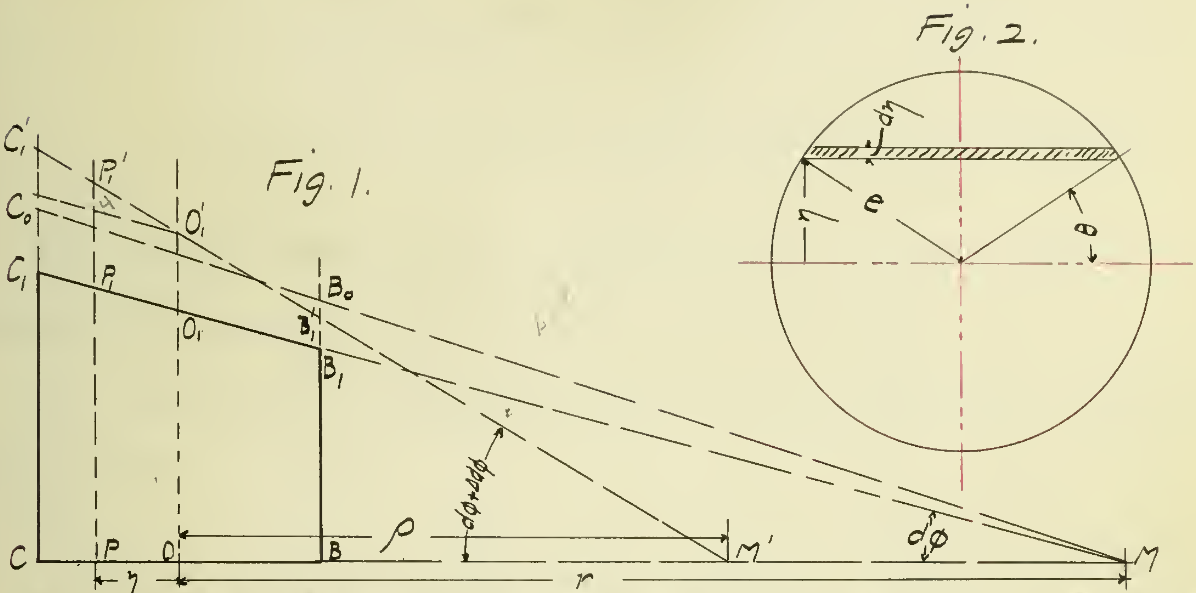
$$\sigma = \frac{1}{f}\left(P + \frac{M_b}{r} + \frac{M_b}{\varkappa r}\frac{\eta}{r+\eta}\right)$$

$$\text{or} \quad \sigma = \frac{P}{f} + \frac{M_b}{fr}\left(1 + \frac{1}{\varkappa}\frac{\eta}{r+\eta}\right). \qquad (A)$$

This formula gives the magnitude of the normal stress σ in a fiber at a distance η from the center-line, in terms of the normal force P, bending moment M, and constants depending upon the configuration. The convention of signs adopted is as follows: P is considered positive when it produces tension, negative when it produces compression; M_b is positive when it tends to increase the curvature, negative when it tends to decrease that curvature. When σ is found to be positive the stress is tensile; when found to be negative, the stress is compression.

5. The value of the function $\varkappa$ must be determined for any given form of cross section. When the section is a simple geometrical figure, as a circle or a square, an expression for $\varkappa$ may be found analytically; When the cross section is irregular in outline approximate

methods must be used. Since the bodies considered in this investigation have only circular cross sections, we shall deduce the expression for λ for that form of section only.



Let us take an elementary strip at a distance η from the center-line, (See Fig 2). The width of this strip is $d\eta$ and its length is $2e\cos\theta$, if e denotes the radius of the section; hence the area of the strip is

$$df = 2e\cos\theta\, d\eta.$$

But

$$\eta = e\sin\theta,$$

$$d\eta = e\cos\theta\, d\theta,$$

$$df = 2e^2\cos^2\theta\, d\theta;$$

hence

$$x = -\frac{1}{f}\int_{-e}^{+e}\frac{\eta}{r+\eta}\,df = -\frac{1}{\pi e^2}\int_{-\frac{\pi}{2}}^{+\frac{\pi}{2}}\frac{e\sin\theta}{r+e\sin\theta}\cdot 2e^2\cos^2\theta\, d\theta.$$

$$= -\frac{2e}{\pi}\int_{-\frac{\pi}{2}}^{+\frac{\pi}{2}}\frac{\sin\theta\cos^2\theta\, d\theta}{r+e\sin\theta}.$$

$$\frac{1}{r+e\sin\theta} = \frac{1}{r}\left(1 - \frac{e}{r}\sin\theta + \left(\frac{e}{r}\right)^2\sin^2\theta - \left(\frac{e}{r}\right)^3\sin^3\theta + \cdots\right),$$

a converging series. Therefore

$$\varkappa = -\frac{2}{\pi}\frac{e}{r}\left[\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\sin\theta\cos^2\theta\,d\theta + \frac{e^2}{r^2}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\sin^3\theta\cos^2\theta\,d\theta + \frac{e^4}{r^4}\int_{-\frac{\pi}{2}}^{+\frac{\pi}{2}}\sin^5\theta\cos^2\theta\,d\theta + \cdots\right]$$

$$+\frac{2}{\pi}\left[\frac{e^2}{r^2}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\sin^2\theta\cos^2\theta\,d\theta + \frac{e^4}{r^4}\int_{-\frac{\pi}{2}}^{+\frac{\pi}{2}}\sin^4\theta\cos^2\theta\,d\theta + \frac{e^6}{r^6}\int_{-\frac{\pi}{2}}^{+\frac{\pi}{2}}\sin^6\theta\cos^2\theta\,d\theta + \cdots\right].$$

The integrals in the first parenthesis are various powers of and with the assigned limits, each reduces to zero.

The values of the integrals in the second parenthesis are obtained as follows:

$$\int\sin^2\theta\cos^2\theta\,d\theta = \int\sin^2\theta\,d\theta - \int\sin^4\theta\,d\theta;$$

$$\int\sin^4\theta\cos^2\theta\,d\theta = \int\sin^4\theta\,d\theta - \int\sin^6\theta\,d\theta;$$

$$\int\sin^6\theta\cos^2\theta\,d\theta = \int\sin^6\theta\,d\theta - \int\sin^8\theta\,d\theta, \quad \text{and so on.}$$

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\sin^2\theta\,d\theta = -\frac{1}{2}\cos\theta\sin\theta + \frac{1}{2}\theta\Big]_{-\frac{\pi}{2}}^{\frac{\pi}{2}} = \frac{1}{2}\pi;$$

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\sin^4\theta\,d\theta = -\frac{1}{4}\cos\theta\sin^3\theta + \frac{3}{4}\int_{-\frac{\pi}{2}}^{+\frac{\pi}{2}}\sin^2\theta\,d\theta = \frac{1}{2}\cdot\frac{3}{4}\pi;$$

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\sin^6\theta\,d\theta = \frac{1}{2}\cdot\frac{3}{4}\cdot\frac{5}{6}\pi$$

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$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\sin^n\theta\,d\theta = \frac{1}{2}\cdot\frac{3}{4}\cdot\frac{5}{6}\cdots\frac{n-1}{n}\cdot\pi$$

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin^2\theta \cos^2\theta \, d\theta = \pi\left(\frac{1}{2} - \frac{1}{2}\cdot\frac{3}{4}\right);$$

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin^4 \cos^2\theta \, d\theta = \pi\left(\frac{1}{2}\cdot\frac{3}{4} - \frac{1}{2}\cdot\frac{3}{4}\cdot\frac{5}{6}\right);$$

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin^6 \cos^2\theta \, d\theta = \pi\left(\frac{1}{2}\cdot\frac{3}{4}\cdot\frac{5}{6} - \frac{1}{2}\cdot\frac{3}{4}\cdot\frac{5}{6}\cdot\frac{7}{8}\right), \text{ etc.}$$

$$\varkappa = \frac{2}{\pi}\left[\pi\frac{\epsilon^2}{r^2}\left(\frac{1}{2} - \frac{1}{2}\cdot\frac{3}{4}\right) + \pi\frac{\epsilon^2}{r^4}\left(\frac{1}{2}\cdot\frac{3}{4} - \frac{1}{2}\cdot\frac{3}{4}\cdot\frac{5}{6}\right) + \pi\frac{\epsilon^6}{r^6}\left(\frac{1}{2}\cdot\frac{3}{4}\cdot\frac{5}{6} - \frac{1}{2}\cdot\frac{3}{4}\cdot\frac{5}{6}\cdot\frac{7}{8}\right) + \cdots\right]$$

$$= \frac{1}{4}\left(\frac{\epsilon}{r}\right)^2 + \frac{3}{4}\cdot\frac{1}{6}\left(\frac{\epsilon}{r}\right)^4 + \frac{3}{4}\cdot\frac{5}{6}\cdot\frac{1}{8}\left(\frac{e}{r}\right)^6 + \cdots\cdots + \frac{3}{4}\cdot\frac{5}{6}\cdot\cdots\cdots\frac{n-1}{n}\cdot\frac{1}{2+n}\left(\frac{\epsilon}{r}\right)^n + \cdots$$

Finally

$$\left.\begin{aligned} \varkappa &= \frac{1}{4}\left(\frac{e}{r}\right)^2 + \frac{1}{8}\left(\frac{\epsilon}{r}\right)^4 + \frac{5}{64}\left(\frac{e}{r}\right)^6 + \frac{7}{128}\left(\frac{\epsilon}{r}\right)^8 + \cdots \\ &= \frac{1}{16}\left(\frac{d}{r}\right)^2 + \frac{1}{128}\left(\frac{d}{r}\right)^4 + \frac{5}{4096}\left(\frac{d}{r}\right)^6 + \cdots \end{aligned}\right\} \quad (6)$$

where d = diameter of circular section.

IV. APPLICATION OF THE THEORY TO VARIOUS FORM OF LINKS.

a. Link with Elliptical Center-Line.

6. Let the center-line of the link be an ellipse with semi-axes a and b; and let the center of the ellipse be taken as the origin and the axes as coordinate axes. Then the equation of the center-line is

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1.$$

Consider one fourth of the link, as showen in Fig:3. If 2Q is the load in the direction of the long axes, the section A will be subjected to a load Q normal to it. Suppose the link at section A to be subjected to a bending moment M, at present unknown, but to be determined from the conditions of the problem. This moment being found, the bending moment at any section may be determined, and this, with the normal force P at the section, will give the data required in finding the stress at any fiber of the section.

Assume a normal section cutting the center-line at C, and consider the part of the link between this section and section A a free body. At C let two opposite forces each equal to Q be added to the system. One of these, together with Q at section A, forms a couple whose moment is Q (b-y): the other force may be resolved into two components, one $Q\sin\phi$ normal to the section and producing tensile stress tn the fibers and the other $Q\cos\phi$ along the section and producing a shearing stress. The latter component will be neglected in the following investigation.

At the section in question, therefore,

Normal force $= P = Q\sin\phi$;

Bending moment $= M_b = M + Q(b-y)$.

The following considerations determine the unknown moment M: Let ϕ denote the angle which the plane of any section makes with the X-axis, and as in Fig. 1, let $d\phi$ denote the angle between two adjacent sections. The distortion of the link by the load varies the relative positions of

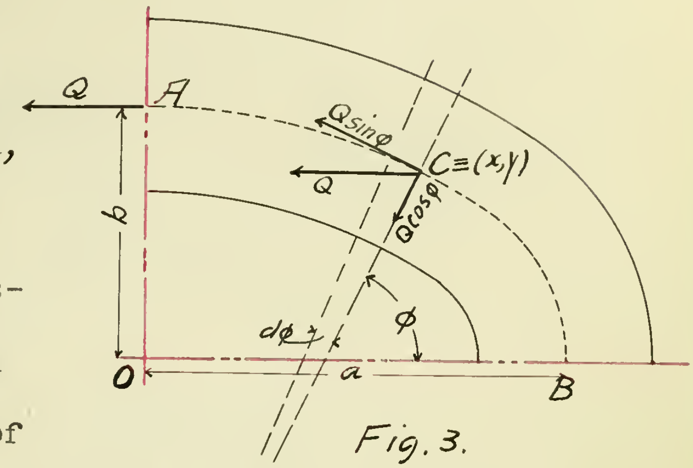


Fig. 3.

the normal sections, and in general the angle $d\phi$ is changed to $d\phi + \Delta d\phi$, the change $\Delta d\phi$ being either positive, negative, or zero. However, owing to the symmetry of the link, the sections A and B, ~~originally~~ remain at right angles, however the center-line is distorted; hence the summation of these changes of angle, $\Delta d\phi$, between these sections must be zero; that is,

$$\int_0^{\frac{\pi}{2}} \Delta d\varphi = 0, \qquad \text{or since} \quad \omega = \frac{\Delta d\varphi}{d\phi},$$

$$\int_0^{\frac{\pi}{2}} \omega \, d\varphi = 0$$

From the third of equations (5),

$$\omega = \frac{1}{Ef}\left[P + \frac{M_b}{r}\left(1 + \frac{1}{\varkappa}\right)\right];$$

hence

$$\frac{1}{Ef}\int_0^{\frac{\pi}{2}}\left[P + \frac{M_b}{r}\left(1 + \frac{1}{\varkappa}\right)\right] d\varphi = 0,$$

or

$$\int_0^{\frac{\pi}{2}}\left[Q\sin\varphi + \frac{M + Q(b-y)}{r}\left(1 + \frac{1}{\varkappa}\right)\right] d\varphi = 0$$

To integrate this expression, the variable ordinate y, radius of curvature r, and the variable x must be expressed as functions of the variable angle ϕ.

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 .$$

$$-\frac{dx}{dy} = \frac{a^2 y}{b^2 x} = \tan\varphi .$$

$$\frac{x^2}{a^2} = 1 - \frac{y^2}{b^2} = \frac{a^2 y^2}{b^4 \tan^2\varphi} .$$

$$y^2 = \frac{b^4 \tan^2\varphi}{a^2 + b^2 \tan^2\varphi} = \frac{b^4}{b^2 + a^2\cot^2\varphi} ;$$

$$y = \frac{b^2}{(b^2 + a^2\cot^2\varphi)^{\frac{1}{2}}} \qquad (a)$$

For the ellipse the radius of curvature is

$$r = \frac{(a^4 y^2 + b^4 x^2)^{\frac{3}{2}}}{a^4 b^4} .$$

Since $b^4 x^2 = a^4 y^2 \cot^2\varphi$,

$$r = \frac{[a^4 y^2 (1 + \cot^2\varphi)]^{\frac{3}{2}}}{a^4 b^4} = \frac{a^6 y^3 \csc^3\varphi}{a^4 b^4} = \frac{a^2 y^3}{b^4 \sin^3\varphi}$$

$$= \frac{a^2 b^6}{b^4 \sin^3\varphi (b^2 + a^2\cot^2\varphi)^{\frac{3}{2}}} = \frac{a^2 b^2}{(b^2\sin^2\varphi + a^2\cos^2\varphi)^{\frac{3}{2}}} \qquad (b)$$

For the circular cross section of radius e,

$$x = \frac{1}{4}\left(\frac{e}{r}\right)^2 + \frac{1}{8}\left(\frac{e}{r}\right)^4 + \frac{5}{64}\left(\frac{e}{r}\right)^6 + \cdots\cdots ,$$

and $$\frac{1}{x} = 4\left(\frac{r}{e}\right)^2 - 2 - \frac{1}{4}\left(\frac{e}{r}\right)^2 - \cdots\cdots . \qquad (c)$$

Since $\frac{e}{r}$ is small, a close approximation is

$$\frac{1}{x} = 4\left(\frac{r}{e}\right)^2 - 2.$$

Then $$1 + \frac{1}{x} = 4\left(\frac{r}{e}\right)^2 - 1,$$

and $$\frac{1}{r}\left(1 + \frac{1}{x}\right) = 4\frac{r}{e^2} - \frac{1}{r}.$$

Substituting this in the expression for $\int \omega\, d\varphi$, the integral becomes:

$$\frac{1}{Ef}\int_0^{\frac{\pi}{2}} \omega\, d\varphi = \int_0^{\frac{\pi}{2}} \left\{ Q\sin\varphi + (M + Qb)\frac{4r}{e^2} - (M + Qb)\frac{1}{r} - 4Q\frac{ry}{e^2} + \frac{Qy}{r} \right\} d\varphi.$$

Substituting the values of y and r given in equations (a) and (b),

$$\frac{1}{Ef}\int_0^{\frac{\pi}{2}} \omega\, d\varphi = Q\int_0^{\frac{\pi}{2}} \sin\varphi\, d\varphi + \frac{4a^2b^2}{e^2}(M + Qb)\int_0^{\frac{\pi}{2}} \frac{d\varphi}{(b^2\sin^2\varphi + a^2\cos^2\varphi)^{\frac{3}{2}}}$$

$$- \frac{M + Qb}{a^2 + b^2}\int_0^{\frac{\pi}{2}} (b^2\sin^2\varphi + a^2\cos^2\varphi)^{\frac{3}{2}} d\varphi - 4Q\frac{a^2b^4}{e^2}\int \frac{\sin\varphi\, d\varphi}{(b^2\sin^2\varphi + a^2\cos^2\varphi)^2}$$

$$+ \frac{Q}{a^2}\int_0^{\frac{\pi}{2}} (b^2\sin^2\varphi + a^2\cos^2\varphi)\sin\varphi\, d\varphi.$$

The integrals have the following values;

$$Q\int_0^{\frac{\pi}{2}} \sin\varphi\, d\varphi = Q \quad ; \quad \int_0^{\frac{\pi}{2}} (b^2\sin^2\varphi + a^2\cos^2\varphi)\sin\varphi\, d\varphi = \frac{1}{3}(a^2 + 2b^2);$$

$$\int_0^{\frac{\pi}{2}} \frac{d\varphi}{(b^2\sin^2\varphi + a^2\cos^2\varphi)^{\frac{3}{2}}} = \frac{\pi}{2a^3}\,\frac{1}{1-k^2}\left(1 - \frac{1}{4}k^2 - \frac{3}{64}k^4 - \frac{5}{256}k^6 - \cdots\right)$$

$$= \frac{\pi}{2ab^2}\alpha, \text{ where } k^2 = 1 - \frac{b^2}{a^2}, \text{ and } \alpha = \text{the series};$$

$$\int_0^{\frac{\pi}{2}} (b^2\sin^2\varphi + a^2\cos^2\varphi)^{\frac{3}{2}} d\phi = a^3\frac{\pi}{2}\left(1 - \frac{3}{4}k^2 + \frac{9}{64}k^4 + \frac{5}{256}k^6 + \cdots\right)$$

$$= \frac{\pi}{2}a^3\beta, \text{ where } \beta \text{ denotes the series};$$

$$\int_0^{\frac{\pi}{2}} \frac{\sin\phi\, d\phi}{(b^2\sin^2\varphi + a^2\cos^2\varphi)^2} = \frac{1}{2b^4}\left(\frac{b^2}{a^2} + \frac{b}{\sqrt{a^2-b^2}}\tan^{-1}\frac{\sqrt{a^2-b^2}}{b}\right) = \frac{1}{2b^4}\gamma.$$

Substituting now in the preceding equation,

$$\frac{1}{Ef}\int_0^{\frac{\pi}{2}} \omega \, d\varphi = 0 = Q + (M+Qb)\frac{\pi}{2}\cdot\frac{4a}{e^2}\alpha - (M+Qb)\frac{\pi}{2}\frac{a}{b^2}\beta - 2\frac{a^2}{e^2}\gamma + \frac{1}{3}Q + \frac{2}{3}\frac{b^2}{a^2}Q.$$

$$M + Qb = Qa\left[\frac{2\frac{a^2}{e^2}\gamma - \frac{2}{3}\frac{b^2}{a^2} - \frac{4}{3}}{\frac{\pi}{2}\left(\frac{4a^2}{e^2}\alpha - \frac{a^2}{b^2}\beta\right)}\right].$$

$$M = Qd\left[\frac{a}{d}\;\frac{8\frac{a^2}{d^2}\gamma - \frac{2}{3}\frac{b^2}{a^2} - \frac{4}{3}}{\frac{\pi}{2}\left(\frac{16a^2}{d^2}\alpha - \frac{a^2}{b^2}\beta\right)} - \frac{b}{d}\right]. \quad (B)$$

When the center-line of the link is a circle--the limit of the ellipse-- the factor x becomes a constant, and the radius of curvature r becomes equal to the axes a and b, that is,

$$r = a = b.$$

In this case,

$$\frac{1}{Ef}\int_0^{\frac{\pi}{2}} \omega \, d\varphi = Q\int_0^{\frac{\pi}{2}} \sin\phi \, d\varphi + \frac{1}{r}(M+Qb)\left(1+\frac{1}{x}\right)\int_0^{\frac{\pi}{2}} d\varphi - Q\left(1+\frac{1}{x}\right)\int_0^{\frac{\pi}{2}} \sin\varphi \, d\varphi = 0.$$

Integrating,

$$Q + \frac{\pi}{2}\cdot\frac{1}{r}(M+Qr)\left(1+\frac{1}{x}\right) - Q\left(1+\frac{1}{x}\right) = 0$$

$$M = Qr\left(\frac{2}{\pi(1+x)} - 1\right). \qquad (C)$$

b. Link with Center-Line of four Circular Arcs.

7. The assumption that the center-line of the link is an ellipse makes the computation of the stresses tedious because of the continuous variation of the radius of curvature and the consequent variation of the value of the function $\varkappa$. For this reason, and for another that will be given presently, it is considered advisable to substitute for the elliptical center-line one made up of four arcs of circles as shown in Fig 4. The arcs $E E'$ and $F F'$ have the points H and H' respectively as centers, and the arcs E F and $E'F'$ have C and C' as centers. H and H' are the centers of the sections of the adjacent links that fit into the link in question. This center-line coincides very nearly with the elliptical center-line, and its adoption greatly simplifies the analysis.

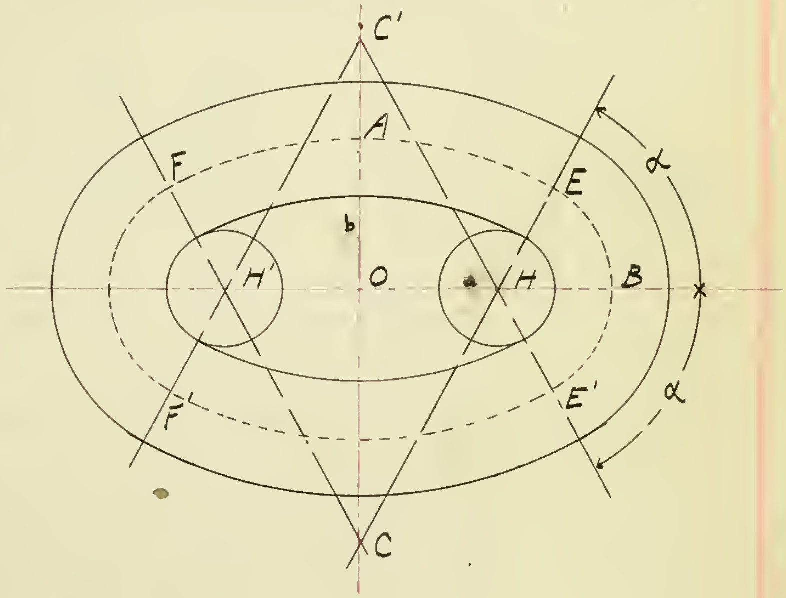


Fig. 4.

Let α denote the angle between the radius C E and the long axis of the link, and let r denote the radius C E = C A; also let the semi-axis OA be denoted by b and the semi-axis OB by a . Then from the geometry of the figure the following relations are readidly obtained.:

$$\tan\alpha = \frac{r-b}{a-d}\,;\quad \sin\alpha = \frac{r-b}{r-d}\,;\quad \cos\alpha = \frac{a-d}{r-d}\,. \qquad (7)$$

$$r = \frac{a^2+b^2-2ad}{2(b-d)}\,, \qquad (8)$$

Let it be assumed first that the pressure between two links is concentrated at a point. Denoting this pressure by 2Q, the normal force at the section A is Q. As before, let M denote the unknown bending moment at the section A.

For sections between B and E, that is, for values of ϕ lying between 0 and α

$$P = Q \sin\phi,$$
$$M_b = M + Q(b - d\sin\phi);$$

and for sections between E and A,

$$P = Q\sin\phi,$$
$$M_b = M + Qr(1 - \sin\phi).$$

The general expression for ω is

$$\omega = \frac{1}{Ef}\left(P + \frac{M_b}{r} + \frac{M_b}{\varkappa r}\right)$$

For the sections between $\phi = 0$ and $\phi = \alpha$; $r = d$; hence

$$\omega_1 = \frac{1}{Ef}\left(Q\sin\phi + \frac{M_b}{d} + \frac{M_b}{\varkappa_1 d}\right),$$

the subscript 1 being used to distinguish the ω and $\varkappa$ of this part of the link from those of the other part.

For the section lying between $\varphi = \alpha$ and $\varphi = \frac{\pi}{2}$,

$$\omega_2 = \frac{1}{Ef}\left(Q\sin\phi + \frac{M_b}{r} + \frac{M_b}{\varkappa_2 r}\right).$$

Inserting the proper values of M_b

$$Ef\,\omega_1 = \frac{M + Qb}{d}\left(1 + \frac{1}{\varkappa_1}\right) - \frac{Q}{\varkappa_1}\sin\phi.$$

$$Ef\,\omega_2 = \left(\frac{M}{r} + Q\right)\left(1 + \frac{1}{\varkappa_2}\right) - \frac{Q}{\varkappa_2}\sin\varphi.$$

The total change of inclination between the section at B and A is

$$\int_0^{\alpha}\omega_1\,d\varphi + \int_{\alpha}^{\frac{\pi}{2}}\omega_2\,d\varphi;$$

but since these sections remain at right angles, this change must be zero; hence

$$Ef\left[\int_0^{\alpha}\omega_1\,d\varphi + \int_{\alpha}^{\frac{\pi}{2}}\omega_2\,d\varphi\right] = 0,$$

or

$$0 = \frac{M+Qd}{d}\left(1+\frac{1}{x_1}\right)\int_0^{\alpha}d\phi - \frac{Q}{x_1}\int_0^{\alpha}\sin\phi\,d\phi + \left(\frac{M}{r}+Q\right)\left(1+\frac{1}{x_2}\right)\int_{\alpha}^{\frac{\pi}{2}}d\phi - \frac{Q}{x_2}\int_{\alpha}^{\frac{\pi}{2}}\sin\phi\,d\phi.$$

Integrating,

$$0 = \frac{M+Qb}{d}\left(1+\frac{1}{x_1}\right)\alpha - \frac{Q}{x_1}\left(1-\cos\alpha\right) + \left(\frac{M}{r}+Q\right)\left(1+\frac{1}{x_2}\right)\left(\frac{\pi}{2}-\alpha\right) - \frac{Q}{x_2}\cos\alpha.$$

Solving for M,

$$M = Qd\left[\frac{\frac{1}{x_1}\left(1-\cos\alpha\right) + \frac{1}{x_2}\cos\alpha - \frac{b}{d}\alpha\left(1+\frac{1}{x_1}\right) - \left(\frac{\pi}{2}-\alpha\right)\left(1+\frac{1}{x_2}\right)}{\alpha\left(1+\frac{1}{x_1}\right) + \frac{d}{r}\left(\frac{\pi}{2}-\alpha\right)\left(1+\frac{1}{x_2}\right)}\right]. \quad (D)$$

The value of M being found, the bending moment at any section may readily be obtained; and with the bending moment and normal force as data, the stress in any fiber of the section is found from formula (A).

8. The assumption that the load on the link is concentrated at one point, while rendering the analytical work easier, cannot be justified by the facts of the case. In reality the adjacent links have a considerable surface in contact, especially after some use, and the pressure between them must be distributed in some way or other over this surface. In the absence of absolute knowledge, the law of distribution must be assumed, care being exercised that the assumption made is justified by experience and commom sense.

The law of "Equal wear" gives a clue to a reasonable assumption in this case. Though strictly the parts of the links in contact are curved, the action of one link on the other may be compared to that of a journal and its bearing. As shown in Fig 5, let the arc of contact between the link and its bearing be denoted by 2α. As the links wear, the surface of contact will become that shown by the dotted line $e\ b\ e'$, which is approximately a circular arc with a radius equal to the radius of a section of the link. Evidently the wear is greatest at b and least at e and e′. Let t denote the depth of wear at a section making the angle ϕ with the axis HX, and h the depth at section B. Since the center of the arc $e\ b\ e'$ is at a distance h to the right of the center O, we have the relation

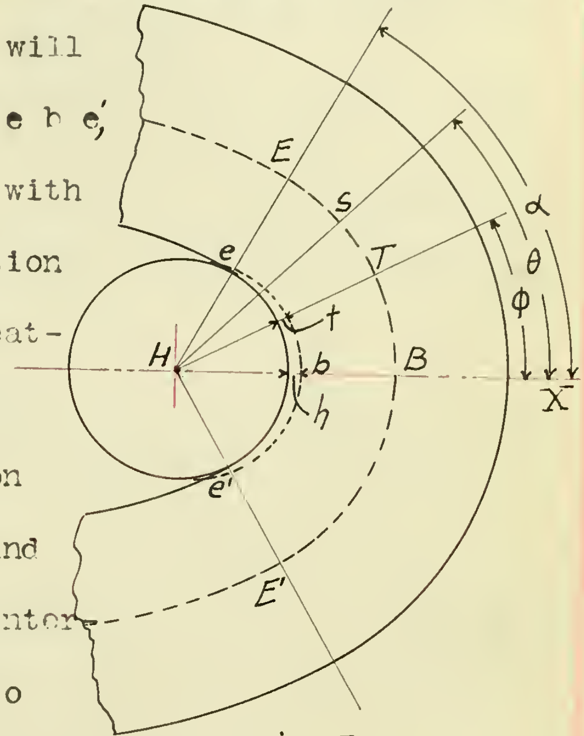


Fig. 5

$$r^2 = h^2 + (r+t)^2 - 2h(r+t)\cos\phi.$$

$$2rt + t^2 = h\left[2(r+t)\cos\phi - h\right].$$

$$\frac{t}{h} = \frac{2(r+t)\cos\phi}{2r+t} - \frac{h}{2r+t}.$$

$$\underset{h \doteq 0}{\text{Limit}}\left(\frac{t}{h}\right) = \cos\phi.$$

That is, the wear at any point of the circumference in contact is proportional the the cosine of the angle ϕ which the section at

this point makes with the long axis of the link. Now the wear is proportional to the work of friction, which is in turn proportional to the normal pressure; hence we conclude that the pressure between the links is distributed in such manner that if p denotes the intensity of pressure at the axis HB, $p \cos\phi$ will be the intensity at a point whose radius makes an angle ϕ with HB.

9. With the pressure thus distributed instead of concentrated, the sections of that portion of the link in contact with its neighbor will be subjected to a bending moment and a normal force different from those deduced for the concentrated load.

Let p denote the intensity of pressure at the section at the small end of the link, that is, the one containing the axis HB, Fig 5. Then $p \cos\phi$ will be the intensity at a section making an angle ϕ with this axis. The length of an elementary arc of the circumference in contact is $\frac{d}{2}\, d\varphi$; hence the pressure on the elementary arc is

$$p \cos\phi \frac{d}{2} d\phi .$$

The horizontal component of this force is

$$\frac{pd}{2} \cos^2\varphi \, d\phi ,$$

The sum of the horizontal component of these elementary forces must balance the external force 2Q; hence

$$\frac{pd}{2}\int_{-\alpha}^{+\alpha} \cos^2\varphi \, d\varphi = 2Q .$$

$$\text{But} \quad \int_{-\alpha}^{+\alpha} \cos^2\varphi \, d\varphi = \frac{1}{2}(\varphi + \sin\varphi \cos\varphi)\Big]_{-\alpha}^{+\alpha} = \alpha + \sin\alpha \cos\alpha = k .$$

hence $\frac{pdk}{2} = 2Q$, or $p = \frac{4Q}{kd}$, where $k = \alpha + \sin\alpha\cos\alpha$.

We have now to find the bending moment at a section T making an angle ϕ with OX, due to the distributed pressure between the sections S and T. Take any section, as S, making the variable angle θ with OX, the angle ϕ being considered for the present a constant. Intensity of the pressure in the direction $\overset{H}{O}$ S is

$$p \cos\theta,$$

and the pressure on an infinitesimal arc of the circumference is

$$p \frac{d}{2} \cos\theta \, d\theta.$$

The component of this force perpendicular to $\overset{H}{O}$ T is

$$p \frac{d}{2} \cos\theta \sin(\theta - \phi) \, d\theta.$$

The moment of this component, whose line of action of course passes through $\overset{H}{O}$, about the point T is

$$d \times \frac{pd}{2} \cos\theta \sin(\theta - \varphi) \, d\theta = \frac{pd^2}{2} \cos\theta \sin(\theta - \phi) \, d\theta.$$

Hence

normal force at $\overset{T}{R} = \frac{pd}{2} \int_{\varphi}^{\alpha} \cos\theta \sin(\varphi - \theta) \, d\theta;$

moment at $\overset{T}{R} = \frac{pd^2}{2} \int_{\varphi}^{\alpha} \cos\theta \sin(\varphi - \theta) \, d\theta.$

These are merely the force and moment due to the distributed pressure between the sections S and T.

$$\int_{\varphi}^{\alpha} \cos\theta \sin(\theta - \varphi) \, d\theta = \cos\varphi \int_{\varphi}^{\alpha} \sin\theta \cos\theta \, d\theta - \sin\varphi \int_{\varphi}^{\alpha} \cos^2\theta \, d\theta$$

$$= \frac{\cos\phi}{2}\left(\sin^2\alpha - \sin^2\varphi\right) - \frac{\sin\varphi}{2}\left(\alpha + \sin\alpha\cos\alpha - \varphi - \sin\varphi\cos\varphi\right)$$

$$= \frac{1}{2}\left(\cos\varphi\sin^2\alpha - k\sin\varphi + \varphi\sin\varphi\right).$$

Normal force $= \frac{pd}{4}\left(\sin^2\alpha\cos\varphi - k\sin\varphi + \varphi\sin\varphi\right)$

$$= \frac{Q}{k}\left(\sin^2\alpha\cos\varphi - k\sin\varphi + \varphi\sin\varphi\right)$$

Moment $= \frac{Qd}{k}\left(\sin^2\alpha\cos\varphi - k\sin\varphi + \varphi\sin\varphi\right)$

As has been shown ,the normal force at section T due to the force Q at section A , Fig 4,is Q sin ϕ ; adding to this the normal force due to the distributed pressure,the total normal force is

$$P = Q\sin\varphi + \frac{Q}{k}\left(\sin^2\alpha\cos\varphi - k\sin\varphi + \varphi\sin\varphi\right)$$

$$= \frac{Q}{k}\left(\sin^2\alpha\cos\varphi + \varphi\sin\varphi\right).$$

The bending moment at the section T due to Q was found to be

$$M + Q\left(b - d\sin\varphi\right)$$

From this must be subtracted the moment due to the distributed pressure,the two having opposite senses. The net moment is therefore

$$M_b = M + Qb - Qd\sin\varphi - \frac{Qd}{k}\left(\sin^2\alpha\cos\varphi - k\sin\varphi + \varphi\sin\varphi\right)$$

$$= M + Qb - \frac{Qd}{k}\left(\sin^2\alpha\cos\varphi + \varphi\sin\varphi\right).$$

Substituting these values of P and M_b in the expression for ω_1,

$$\omega_1 = \frac{1}{Ef}\left[\frac{Q}{k}\left(\sin^2\alpha\cos\phi + \phi\sin\phi\right) + \frac{1}{d}\left(1+\frac{1}{x_1}\right)\left\{M+Qb-\frac{Qd}{k}\left(\sin^2\alpha\cos\varphi+\varphi\sin\varphi\right)\right\}\right]$$

Reducing

$$Ef\omega_1 = \frac{1}{d}(M+Qb)\left(1+\frac{1}{x_1}\right) - \frac{Q}{kx_1}\left(\sin^2\alpha\cos\phi + \phi\sin\varphi\right).$$

$$Ef\int_0^\alpha \omega_1\,d\varphi = \frac{1}{d}(M+Qb)\left(1+\frac{1}{x_1}\right)\int_0^\alpha d\phi - \frac{Q}{kx_1}\sin^2\alpha\int_0^\alpha\cos\varphi\,d\varphi - \frac{Q}{kx_1}\int_0^\alpha\varphi\sin\phi\,d\phi.$$

$$= \frac{\alpha}{d}(M+Qb)\left(1+\frac{1}{x_1}\right) - \frac{Q\sin^3\alpha}{kx_1} - \frac{Q\sin\alpha}{kx_1} + \frac{Q\alpha\cos\alpha}{kx_1}.$$

Adding to this the integral $\int_\alpha^{\frac{\pi}{2}}\omega_2\,d\varphi$, previously obtained,

$$0 = \frac{\alpha}{d}(M+Qb)\left(1+\frac{1}{x_1}\right) - \frac{Q}{kx_1}\left(\sin^3\alpha+\sin\alpha-\alpha\cos\alpha\right) + \left(\frac{M}{r}+Q\right)\left(1+\frac{1}{x_2}\right)\left(\frac{\pi}{2}-\alpha\right) - \frac{Q}{x_2}\cos\alpha.$$

Solving for M,

$$M = Qd\left\{\frac{\frac{1}{x_1}\left(\frac{2\sin\alpha}{k}-\cos\alpha\right)+\frac{1}{x_2}\cos\alpha-\frac{b}{d}\alpha\left(1+\frac{1}{x_1}\right)-\left(\frac{\pi}{2}-\alpha\right)\left(1+\frac{1}{x_2}\right)}{\alpha\left(1+\frac{1}{x_1}\right)+\frac{d}{r}\left(1+\frac{1}{x_2}\right)\left(\frac{\pi}{2}-\alpha\right)}\right\}. \quad (E)$$

The difference between the two expressions for M lies in the first term of the numerator, which for concentrated load is $\frac{1}{x_1}(1-\cos\alpha)$ and for distributed load $\frac{1}{x_1}\left(\frac{2\sin\alpha}{k}-\cos\alpha\right)$. For $\alpha=0$, the load must be concentrated, and the second expression should be equal to the first; thus

$$1-\cos\alpha\Big]_{\alpha=0} = 1-1 = 0\,;$$

and
$$\frac{2\sin\alpha}{k}-\cos\alpha\Big]_{\alpha=0} = \frac{2\sin\alpha}{\alpha+\sin\alpha\cos\alpha}-\cos\alpha\Big]_{\alpha=0} = \frac{2}{\frac{\alpha}{\sin\alpha}+\cos\alpha}-1\Big]_{\alpha=0}$$

$$= \frac{2}{1+1}-1 = 0.$$

For $\alpha = \frac{\pi}{2}$, $1 - \cos\alpha = 1$,

and $\frac{2\sin\alpha}{k} - \cos\alpha = \frac{2\sin\alpha}{\alpha + \sin\alpha\cos\alpha} - \cos\alpha = \frac{2}{\frac{\pi}{2}} = \frac{4}{\pi} = 1.2732$.

The difference between the values of M given by the two equations is greatest when $\alpha = \frac{\pi}{2}$ and decreases with α, becoming zero when $\alpha = 0$, that is, when the link has a circular instead of an oval center-line.

With a concentrated load, the moment for $\phi = 0$, that is, at section B, is

$$M_b = M + Qb,$$

while for a distributed load it is

$$M_b = M + Qb - \frac{Q\alpha \sin^2\alpha}{k}.$$

As will be shown, when we arrive at numerical results, the assumption of distribution results in a marked reduction in the stresses computed by the first assumption of a concentrated load.

c. Link of Lemniscate Form.

10. The equations so far deduced hold for a link in which all parts of the center-line are concave to the geometrical center of the link. The two limiting forms are the link with circular center-line and the link with the center-line made up of two semicircles and two straight sides. We now investigate the case of a link in which the sides are convex to the center - one whose center-line has somewhat the form of the lemniscate.

A link of this form is shown in Fig 6. The part BF of the quarter link is a circular arc with H as a center. The remaining part FA is a circular arc with C as a center and with a radius R. Evidently the pressure between two links is distributed over a half circumference, that is, the angle α is 90° or $\frac{\pi}{2}$.

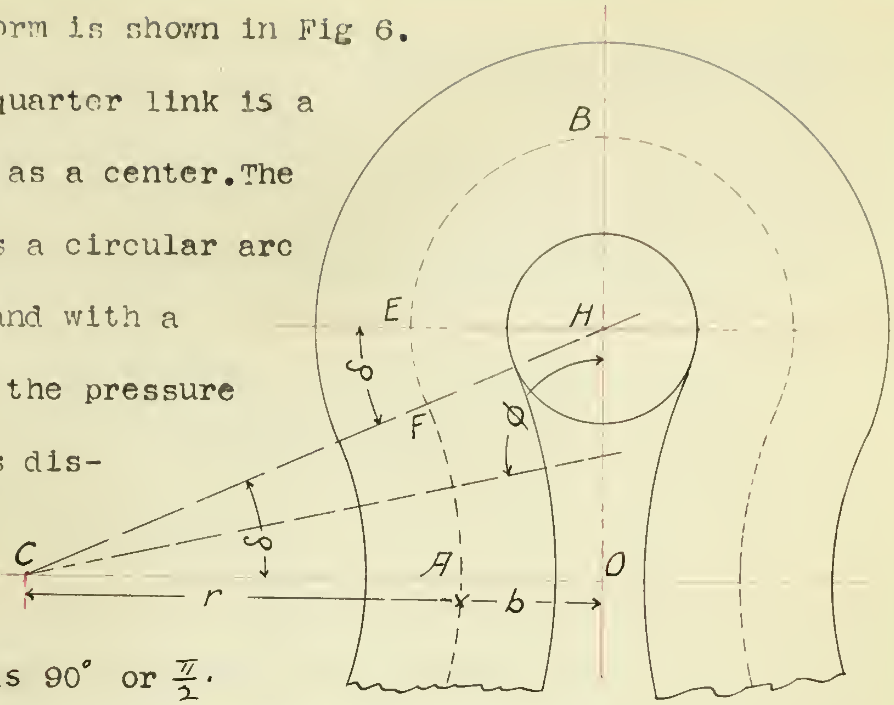


Fig. 6.

The center-line of the quarter link AB must therefore be considered in three parts:

1. The circular arc AF, of radius R, and subjected to the external force Q;

2. The arc EF, of radius d, and likewise subjected to the external force Q;

3. The arc BE, of radius d, and subjected to the force Q and to the distributed pressure of the adjacent link.

For a section of the link between B and E, the expressions previously deduced for the normal force and bending moment still hold; thus

$$P = \frac{Q}{k}\left(\sin^2\alpha \cos\varphi + \varphi \sin\varphi\right);$$

$$M_b = M + Qb - \frac{Qd}{k}\left(\sin^2\alpha \cos\varphi + \varphi \sin\varphi\right).$$

However in the present case, $\alpha = 90° = \frac{\pi}{2}$, $\sin^2\alpha = 1$, and

$K = \alpha + \sin\alpha\cos\alpha = \frac{\pi}{2}$. Substituting these values,

$$P = \frac{2Q}{\pi}(\cos\varphi + \varphi\sin\varphi);$$

$$M_b = M + Qb - \frac{2Qd}{\pi}(\cos\varphi + \varphi\sin\varphi).$$

Between E and F

$$P = Q\sin\varphi;$$

$$M_b = M + Q(b - d\sin\varphi;$$

and between F and A

$$P = Q\sin\varphi;$$

$$M_b = -M + Qr(1 - \sin\varphi)$$

The negative sign must be given to the moment M for the arc AF, since M, being assumed clockwise, tends to decrease the curvature of the arc.

For the arc BE,

$$Ef\omega_1 = \frac{M + Qb}{d}\left(1 + \frac{1}{x_1}\right) - \frac{2Q}{\pi x_1}(\cos\varphi + \varphi\sin\varphi).$$

For the arc EF

$$Ef\omega_2 = \frac{M + Qb}{d}\left(1 + \frac{1}{x_1}\right) - \frac{Q}{x_1}\sin\varphi;$$

and finally for the arc FH,

$$Ef\omega_3 = \frac{-M + Qr}{r}\left(1 + \frac{1}{x_2}\right) - \frac{Q}{x_2}\sin\varphi.$$

These are obtained by substituting the proper values of P and M in the general equation

$$\omega = \frac{1}{Ef}\left(P + \frac{M_b}{r} + \frac{M_b}{xr}\right).$$

For the arcs BE and EF, the radius of curvature is d and

$$x_1 = \frac{1}{16}\left(\frac{d}{d}\right)^2 + \frac{1}{32}\left(\frac{d}{d}\right)^4 + \cdots$$

For the arc FA, the radius is R, and

$$\varkappa_2 = \frac{1}{16}\left(\frac{d}{r}\right)^2 + \frac{1}{128}\left(\frac{d}{r}\right)^4 + \cdots$$

Since the sections at H and B remain at right angles, we have

$$\int_0^{\frac{\pi}{2}} \omega_1 \, d\varphi + \int_{\frac{\pi}{2}}^{\frac{\pi}{2}+\alpha} \omega_2 \, d\varphi + \int_{\frac{\pi}{2}+\delta}^{\frac{\pi}{2}} \omega_3 \, d\varphi = 0$$

Substituting the values of ω_1, ω_2 and ω_3, and dropping the factor Ef, the condition that the total change of inclination of the sections A and B shall be zero leads to the equation

$$\frac{M+Qb}{d}\left(1+\frac{1}{\varkappa_1}\right)\int_0^{\frac{\pi}{2}} d\varphi - \frac{2Q}{\pi\varkappa_1}\int_0^{\frac{\pi}{2}} (\cos\varphi + \varphi\sin\varphi)\, d\varphi + \frac{M+Qb}{d}\left(1+\frac{1}{\varkappa_1}\right)\int_{\frac{\pi}{2}}^{\frac{\pi}{2}+\delta} d\varphi$$

$$- \frac{Q}{\varkappa_1}\int_{\frac{\pi}{2}}^{\frac{\pi}{2}+\delta} \sin\varphi \, d\varphi - \frac{M-Qr}{r}\left(1+\frac{1}{\varkappa_2}\right)\int_{\frac{\pi}{2}+\delta}^{\frac{\pi}{2}} d\phi - \frac{Q}{\varkappa_2}\int_{\frac{\pi}{2}+\delta}^{\frac{\pi}{2}} \sin\varphi \, d\varphi = 0.$$

The integrals have the following values:

$$\int_0^{\frac{\pi}{2}} d\varphi = \frac{\pi}{2} \; ; \qquad \int_{\frac{\pi}{2}}^{\frac{\pi}{2}+\delta} d\varphi = \delta \; ;$$

$$\int_0^{\frac{\pi}{2}} \cos\varphi \, d\varphi = \sin\frac{\pi}{2} = 1 \; ;$$

$$\int_0^{\frac{\pi}{2}} \varphi\sin\varphi \, d\varphi = \sin\varphi - \varphi\cos\varphi \Big]_0^{\frac{\pi}{2}} = 1 \; ;$$

$$\int_{\frac{\pi}{2}}^{\frac{\pi}{2}+\delta} \sin\varphi \, d\varphi = -\cos\varphi \Big]_{\frac{\pi}{2}}^{\frac{\pi}{2}+\delta} = \sin\delta \; ;$$

$$\int_{\frac{\pi}{2}+\delta}^{\frac{\pi}{2}} d\varphi = -\delta \; ; \qquad \int_{\frac{\pi}{2}+\delta}^{\frac{\pi}{2}} \sin\varphi \, d\varphi = -\sin\delta.$$

Using these values, the equation becomes

$$\frac{M+Qb}{d}\left(1+\frac{1}{x_1}\right)\left(\frac{\pi}{2}+\delta\right)+\frac{M-Qr}{r}\left(1+\frac{1}{x_2}\right)\delta-\frac{Q}{x_1}\left(\frac{4}{\pi}+\sin\delta\right)+\frac{Q}{x_2}\sin\delta=0,$$

or $$(M+Qb)\left(1+\frac{1}{x_1}\right)\left(\frac{\pi}{2}+\delta\right)+\left(M\frac{d}{r}-Qd\right)\left(1+\frac{1}{x_2}\right)\delta-\frac{Qd}{x_1}\left(\frac{4}{\pi}+\sin\delta\right)+\frac{Qd}{x_2}\sin\delta=0.$$

Solving,

$$M=Qd\left\{\frac{\frac{1}{x_1}\left(\frac{4}{\pi}+\sin\delta\right)-\frac{b}{d}\left(1+\frac{1}{x_1}\right)\left(\frac{\pi}{2}+\delta\right)+\left(\frac{1}{x_2}+1\right)\delta-\frac{1}{x_2}\sin\delta}{\left(1+\frac{1}{x_1}\right)\left(\frac{\pi}{2}+\delta\right)+\frac{d}{r}\left(1+\frac{1}{x_2}\right)\delta}\right\}.\quad (F)$$

d. Link with Straight Sides.

11. A limiting case that must receive special consideration is that in which the sides of the link are straight. This case forms the boundary between the two cases just considered, and therefore equations (E) and (F) should, for this form of center-line, be identical.

This requirement furnishes a test for the correctness of the equation in question.

With the notation heretofore used, we have the following, when the sides of the link are straight;

$$\alpha=\frac{\pi}{2};\quad r=\infty;\quad \frac{1}{x_2}=16\frac{r^2}{d^2}-2-\frac{1}{16}\frac{d^2}{r^2}\Bigg]_{r=\infty}=16\frac{r^2}{d^2}\Bigg]_{r=\infty}=\infty.$$

Let the distance OH, Fig 4, be denoted by l so that $l=a-d$; further let β denote the angle OCH, whence $\beta=90^\circ-\alpha=\frac{\pi}{2}-\alpha$; then

$$\beta=\frac{\text{arc } AE}{r},$$

and when r recedes to infinity and AE becomes equal to OH = l,

$$\beta = \frac{l}{r}\Big]_{r=\infty}$$

The evaluation of the separate terms in the numerator and denominator of the second member of equation (E) proceeds as follows:

$$\frac{2}{k x_1}\sin\alpha - \frac{1}{x_1}\cos\alpha\Big]_{\alpha=\frac{\pi}{2}} = \frac{4}{x_1 \pi};$$

$$-\frac{b}{d}\alpha\left(1+\frac{1}{x_1}\right)\Big]_{\alpha=\frac{\pi}{2}} = -\frac{\pi}{2}\frac{b}{d}\left(1+\frac{1}{x_1}\right);$$

$$\frac{1}{x_2}\cos\alpha - \left(1+\frac{1}{x_2}\right)\left(\frac{\pi}{2}-\alpha\right)\Big]_{\alpha=\frac{\pi}{2}} = \frac{1}{x_2}(\sin\beta-\beta)-\beta\Big]_{\beta=0} = \frac{1}{x_2}(\sin\beta-\beta)\Big]_{\beta=0}$$

Now $\frac{1}{x_2} = \frac{16r^2}{d^2}$,

and $$\sin\beta-\beta = \left[\beta-\frac{\beta^3}{6}+\frac{\beta^5}{120}-\cdots\right]-\beta$$

$$= -\frac{\beta^3}{6}+\cdots = -\frac{l^3}{r^3} + \text{terms with higher powers of } r.$$

Hence,

$$\frac{1}{x_2}\cos\alpha - \left(1+\frac{1}{x_2}\right)\left(\frac{\pi}{2}-\alpha\right)\Big]_{\alpha=\frac{\pi}{2}} = -\frac{16r^2}{d^2}\cdot\frac{l^3}{r^3}\Big]_{r=\infty} = 0;$$

$$\alpha\left(1+\frac{1}{x_1}\right)\Big]_{\alpha=\frac{\pi}{2}} = \frac{\pi}{2}\left(1+\frac{1}{x_1}\right);$$

$$\frac{d}{r}\left(1+\frac{1}{x_2}\right)\left(\frac{\pi}{2}-\alpha\right)\Big]_{\alpha=\frac{\pi}{2}} = \frac{d}{r}\left(1+\frac{1}{x_2}\right)\beta\Big]_{\beta=0}$$

$$= \left[\frac{\beta d}{r}+\frac{\beta d}{r x_2}\right]_{\beta=0} = \frac{\beta d}{r x_2}\Big]_{\beta=0}$$

$$= \frac{16r^2}{d^2}\frac{l}{r}\frac{d}{r} = 16\frac{l}{d}.$$

Substituting the values thus found, the equation becomes

$$M = Qd\left\{\frac{\frac{4}{\pi x_1} - \frac{\pi}{2}\frac{b}{d}\left(1+\frac{1}{x_1}\right)}{\frac{\pi}{2}\left(1+\frac{1}{x_1}\right) + 16\frac{l}{d}}\right\}. \qquad (G)$$

In equation (F) we have, when the sides are straight,

$$r = \infty;\quad \frac{1}{x_2} = 16\frac{r^2}{d^2}\Big]_{r=\infty} = \infty;\quad \text{and}\quad \delta = \frac{l}{r}\Big]_{r=\infty} = 0.$$

The various terms have the following values:

$$\frac{1}{x_1}\left(\frac{4}{\pi} + \sin\delta\right)_{\delta=0} = \frac{4}{x_1\pi};$$

$$\frac{b}{d}\left(1+\frac{1}{x_1}\right)\left(\frac{\pi}{2}+\delta\right)_{\delta=0} = \frac{\pi}{2}\frac{b}{d}\left(1+\frac{1}{x_1}\right);$$

$$\left(1+\frac{1}{x_2}\right)\delta - \frac{1}{x_2}\sin\delta\Big]_{\substack{\delta=0\\ x_2=\infty}} = \frac{1}{x_2}(\delta - \sin\delta) = 0;$$

$$\left(1+\frac{1}{x_1}\right)\left(\frac{\pi}{2}+\delta\right]_{\delta=0} = \frac{\pi}{2}\left(1+\frac{1}{x_1}\right);$$

$\frac{d}{r}\left(1+\frac{1}{x_2}\right)\delta\Big]_{\delta=0} = 16\frac{l}{d}$, since δ and β have the same limiting values, and $\frac{d}{r}\left(1+\frac{1}{x_2}\right)\beta = 16\frac{l}{d}$.

Inserting these values,

$$M = \left\{\frac{\frac{4}{x_1\pi} - \frac{\pi}{2}\frac{b}{d}\left(1+\frac{1}{x_1}\right)}{\frac{\pi}{2}\left(1+\frac{1}{x_1}\right) + 16\frac{l}{d}}\right\}.$$

This equation is identical with (G) as it should be.

e. Link with Circular Center-Line.

12. One other limiting value remains to be tested. When in Fig 4 the half-axis b of the oval link is made equal to the half-axis a, the oval center-line becomes a true circle and the angle α becomes zero. If in equation (E) this value of α is inserted, the resulting value of M should be precisely that given by equation (C). Making this substitution in the second member of (E), the first and third terms of the numerator and the first term of the denominator become zero, and the equation reduces to

$$M = Qd\left[\frac{\frac{1}{x_2} - \left(1+\frac{1}{x_2}\right)\frac{\pi}{2}}{\frac{d}{r}\left(1+\frac{1}{x_2}\right)\frac{\pi}{2}}\right] = Qr\left[\frac{\frac{1}{x_2} - \left(1+\frac{1}{x_2}\right)\frac{\pi}{2}}{\left(1+\frac{1}{x_2}\right)\frac{\pi}{2}}\right]$$

$$= Qr\left[\frac{2}{\pi(1+x_2)} - 1\right],$$

which is identical with (C), since $x_2 = x$.

f. Link with Stud.

13. The links of crane chains and anchor chains are frequently provided with lateral struts to prevent the collapsing of the sides. It is the general impression that the resistance of the link is increased by the use of such a strut or stud; however, some doubt has been thrown on this conclusion by recent experiment. These experiments showed that with chains made of the same size of iron the chain with the ordinary open link withstood a greater breaking load

than the chain with the stud links. On the strength of these experiments, it is claimed that for general purposes the open link chain is preferable to the stud link chain and that the addition of the stud actually weakens the link rather than strengthens it. The discussion following will show that exactly the reverse is true; that with loads within the elastic limit the use of the stud reduces the stresses in links of the usual form by 100 % or more.

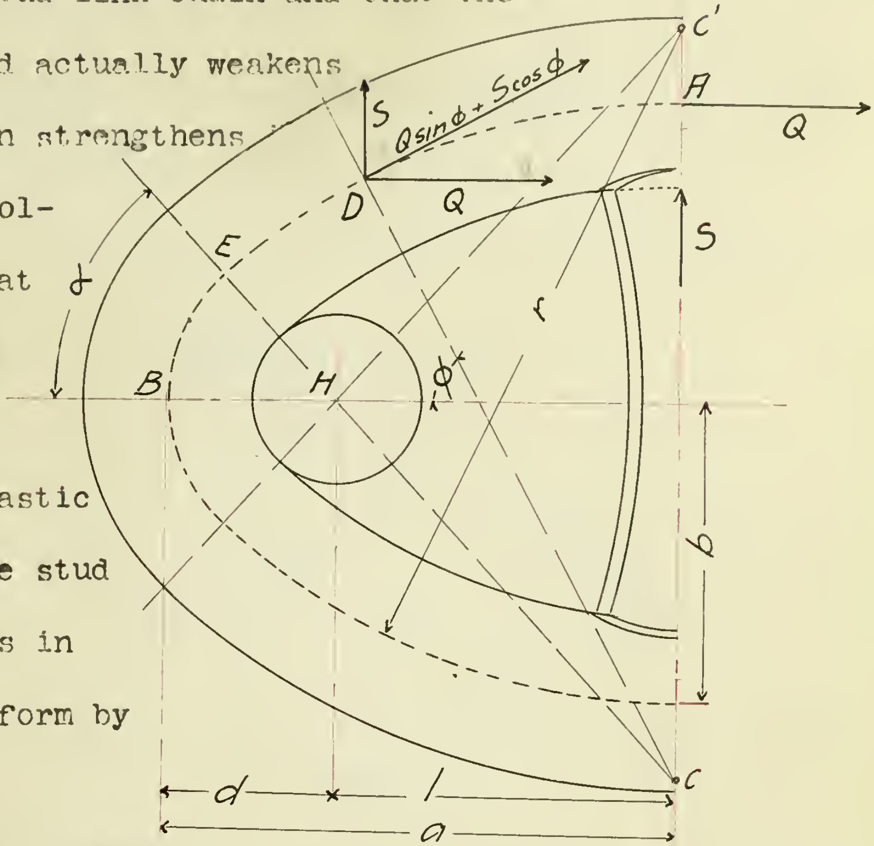


Fig. 7

14. To the system of external forces heretofore considered as acting on the link, a new force, the pressure of the stud, is added. Thus in Fig 7 there are acting at the section OA the normal force Q, the force S, the reaction of the stud, at right angle to Q, and the unknown bending moment M. At a section D making an angle ϕ with the X-axis, the normal force is evidently $Q\sin\phi + S\cos\phi$; the tangential force is $Q\cos\phi - S\sin\phi$, and the bending moment will be made up of the moment M and the moments of the forces, Q and S. There are now two unknown quantities to determine—the moment M and the force S; hence there must be two relations connecting M, Q, and S. As in the previous discussion one equation is given by the integration of the inclinations of the

the sections; thus since sections OA and OB remain at right angles,

$$\int_0^{\frac{\pi}{2}} \omega \cdot d\varphi = 0.$$

To obtain a second equation, we find the deflection of the side of the link from its original unstrained position under the action of the known system of external forces, and equate this deflection to the shortening of the stud under the action of the compressive force S.

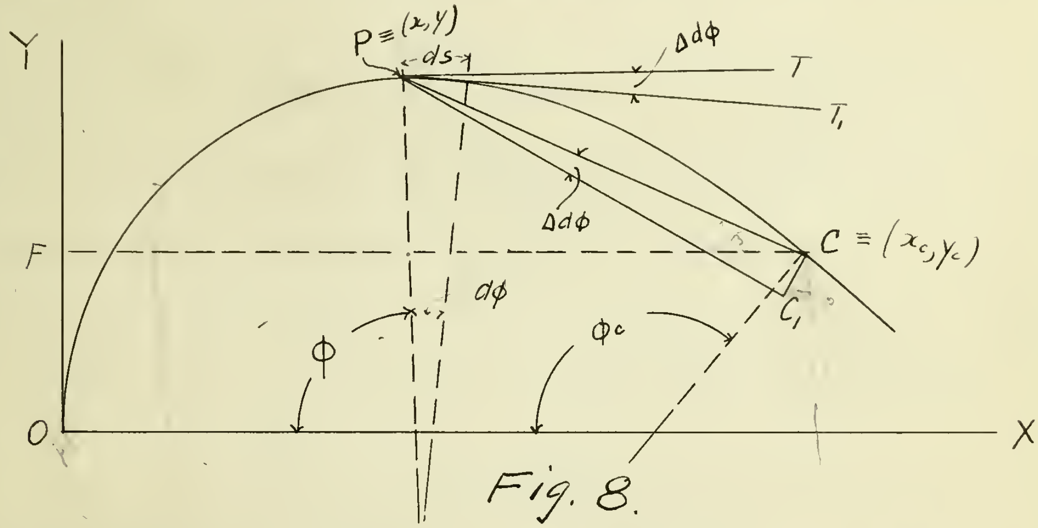


Fig. 8.

15. The change in the coordinates of the center-line of a curved link subjected to external forces may be determined as follows: Let the curve, Fig. 8, be the given center-line, P any point in it, and C the point, the change in the coordinates of which is to be found.

Let the coordinates of C be X_c and y_c, and let the coordinate increments be denoted by Δx_c and Δy_c; then the new coordinates of C will be

$$x_c + \Delta x_c$$
$$y_c + \Delta y_c$$

Let x, y be the coordinates of P. By the action of the external forces, the inclination of a normal section at P will be changed by the angle $\Delta d\phi$, and the tangent to the center-line at P will change its direction from PT to PT′. This change of angle

will evidently cause the point C to move through an arc of length $pc \cdot \Delta d\phi$ and assume the position C_1. The X-component of CC_1 is $CC_1 \sin PCF$, and the Y-component is $CC_1 \cos PCF$.

But $CC_1 = pc\,\Delta d\varphi$; $pc \sin PCF = y - y_c$, and $pc \cos PCF = x_c - x$; hence the x-component is

$$-(y - y_c) \cdot \Delta d\varphi;$$

and the Y-component is

$$-(x_c - x) \cdot \Delta d\varphi$$

In addition to the coordinate increments due to this change in the inclination of the section at P, there is an increment due to the actual lengthening (or shortening) of an element of arc ds at P. The amount of this extension is $\varepsilon_o\,ds$; hence by reason of it, the point C is carried in the direction of the axis x a distance

$$\varepsilon_o\,ds \sin\varphi = \varepsilon_o\,dx$$

and in the direction of the axis Y a distance

$$\varepsilon_o\,ds \cos\varphi = \varepsilon_o\,dy$$

Adding together the increments due to the change in direction and the change in length of the elementary arc ds,

$$d(\Delta x_c) = -(y - y_c)\,\Delta d\varphi + \varepsilon_o\,dx = y_c\,\omega\,d\varphi - y\,\omega\,d\varphi + \varepsilon_o\,dx;$$

$$d(\Delta y_c) = -(x_c - x)\,\Delta d\varphi + \varepsilon_o\,dy = -x_c\,\omega\,d\varphi + x\,\omega\,d\varphi + \varepsilon_o\,dy.$$

Summing for all the elementary arcs between C and the origin or stationary point A, at which point $\phi = 0$,

$$\Delta x_c = y_c \int_0^{\varphi_c} \omega\,d\varphi - \int_0^{\varphi_c} y\,\omega\,d\varphi + \int_0^{x_c} \varepsilon_o\,dx. \qquad (9)$$

$$\Delta y_c = -x_c \int_0^{\varphi_c} \omega\,d\varphi + \int_0^{\varphi_c} x\,\omega\,d\varphi + \int_0^{y_c} \varepsilon_o\,dy. \qquad (10)$$

16. Referring to Fig 7, it is evident that the point A of the center-line is the point whose coordinate increments are desired. The ordinate OA is the semi-axis b when the link is unstrained.

Under the action of the external forces the sides of the link approach each other, A approaches O, and the ordinate b receives the increment $-\Delta b$. The immediate problem is to determine this negative increment.

Taking the point B, Fig 7, as origin, the abscissa of the point A is the semi-axis a; this corresponds to x_c, equation (10).

For values of ϕ between 0 and α,

$$x = d(1-\cos\phi);$$
$$x_c - x = a - d(1-\cos\varphi) = a - d + d\cos\varphi = l + d\cos\varphi.$$

For values of φ between α and $\frac{\pi}{2}$,

$$x = a - r\cos\varphi;$$
$$x_c - x = a - (a - r\cos\varphi) = r\cos\varphi$$

From (10)

$$-\Delta b = \int_0^{\varphi_c} (x_c - x)\,\omega\, d\varphi - \int_0^{y_c} \mathcal{E}_0\, dy;$$

hence denoting by ω_1 and ω_2 the values of ω for the arcs of radius d and r, respectively,

$$-\Delta b = l\int_0^{\alpha} \omega_1 d\varphi + d\int_0^{\alpha} \cos\varphi\, d\varphi - \int_0^{d\sin\alpha} \mathcal{E}_{0_1}\, dy$$
$$+ r\int_{\alpha}^{\frac{\pi}{2}} \omega_2 \cos\varphi\, d\varphi - \int_{d\sin\alpha}^{b} \mathcal{E}_{0_2}\, dy. \qquad (11)$$

To obtain the values of ω_1, ω_2, $\mathcal{E}_{0_1}$, and $\mathcal{E}_{0_2}$, we must find the general expressions for the normal force and bending moment at any section.

For sections between $\phi = 0$ and $\varphi = \alpha$,

$$P = \frac{Q}{k}\left(\sin^2\alpha\cos\varphi + \varphi\sin\varphi\right) + S\cos\phi\ ;$$

$$M_b = M + Qb - \frac{Qd}{k}\left(\sin^2\alpha\cos\varphi + \varphi\sin\varphi\right) - S(l + d\cos\varphi).$$

From equations (5), remembering that within these limits, the radius of curvature of the center-line is $\underline{d}$,

$$Ef\varepsilon_{0_1} = P + \frac{M_b}{d} = \frac{M + Qb - Sl}{d}\ ;$$

$$Ef\omega_1 = Ef\varepsilon_{0_1} + \frac{M_b}{\varkappa_1 d} = \frac{M+Qb-Sl}{d}\left(1 + \frac{1}{\varkappa_1}\right) - \frac{Q}{k\varkappa_1}\left(\sin^2\alpha\cos\varphi + \varphi\sin\varphi\right) - \frac{S}{\varkappa_1}\cos\varphi.$$

For sections between $\varphi = \alpha$ and $\varphi = \frac{\pi}{2}$

$$M_b = M + Qr - Qr\sin\phi - Sr\cos\varphi\ ;$$

$$P = Q\sin\varphi + S\cos\varphi\ ;$$

$$Ef\varepsilon_{0_2} = \frac{M}{r} + Q\ ;$$

$$Ef\omega_2 = \left(\frac{M}{r} + Q\right)\left(1 + \frac{1}{\varkappa_2}\right) - \frac{1}{\varkappa_2}\left(Q\sin\varphi + S\cos\varphi\right).$$

From the condition

$$\int_0^\alpha \omega_1\,d\varphi + \int_\alpha^{\frac{\pi}{2}} \omega_2\,d\varphi = 0,$$

we obtain

$$\frac{M+Qb-Sl}{d}\left(1+\frac{1}{\varkappa_1}\right)\int_0^\alpha d\varphi - \frac{Q}{k\varkappa_1}\sin^2\alpha\int_0^\alpha \cos\varphi\,d\varphi - \frac{Q}{k\varkappa_1}\int_0^\alpha \varphi\sin\varphi\,d\varphi - \frac{S}{\varkappa_1}\int_0^\alpha \cos\varphi\,d\varphi$$

$$+\left(\frac{M}{r} + Q\right)\left(1 + \frac{1}{\varkappa_2}\right)\int_\alpha^{\frac{\pi}{2}} d\varphi - \frac{Q}{\varkappa_2}\int_\alpha^{\frac{\pi}{2}} \sin\varphi\,d\varphi - \frac{S}{\varkappa_2}\int_\alpha^{\frac{\pi}{2}} \cos\varphi\,d\varphi = 0.$$

Performing the integrations and reducing,

$$\left(M + Qb - Sl\right)\left(1+\frac{1}{\varkappa_1}\right)\alpha + \left(M\frac{d}{r} + Qd\right)\left(1+\frac{1}{\varkappa_2}\right)\left(\frac{\pi}{2} - \alpha\right) - \frac{Qd}{\varkappa_1}\left(\frac{2\sin\alpha}{k} - \cos\alpha\right)$$

$$- \frac{Sd}{\varkappa_1}\sin\alpha - \frac{Qd}{\varkappa_2}\cos\alpha - \frac{Sd}{\varkappa_2}\left(1 - \sin\alpha\right) = 0. \qquad (12)$$

Now substituting the values of $\omega_1, \omega_2, \varepsilon_{o_1}, \varepsilon_{o_2}$ in equation (11),

$$-Ef\cdot\Delta b = \frac{l}{d}(M+Qb-Sl)\left(1+\frac{1}{x_1}\right)\int_0^\alpha d\varphi - \frac{Ql}{kx_1}\sin^2\alpha\int_0^\alpha\cos\varphi\, d\varphi$$

$$-\frac{Ql}{kx_1}\int_0^\alpha \varphi\sin\varphi\, d\varphi - \frac{Sl}{x_1}\int_0^\alpha\cos\varphi\, d\varphi + (M+Qb-Sl)\left(1+\frac{1}{x_1}\right)\int_0^\alpha\cos\varphi\, d\varphi$$

$$-\frac{Qd}{kx_1}\int_0^\alpha\cos^2\varphi\, d\varphi - \frac{Qd}{kx_1}\int_0^\alpha\varphi\sin\varphi\cos\varphi\, d\varphi - \frac{Sd}{x_1}\int_0^\alpha\cos^2\varphi\, d\varphi$$

$$+(M+Qr)\left(1+\frac{1}{x_2}\right)\int_\alpha^{\frac{\pi}{2}}\cos\varphi\, d\varphi - \frac{Qr}{x_2}\int_\alpha^{\frac{\pi}{2}}\sin\varphi\cos\varphi\, d\varphi - \frac{Sr}{x_2}\int_\alpha^{\frac{\pi}{2}}\cos^2\varphi\, d\varphi$$

$$-(M+Qb-Sl)\sin\alpha - \left(\frac{M}{r}+Q\right)(b-d\sin\alpha).$$

Performing the integrations,

$$-Ef\cdot\Delta b = (M+Qb-Sl)\left(1+\frac{1}{x_1}\right)\frac{\alpha l}{d} - \frac{Ql}{kx_1}\sin^3\alpha - \frac{Ql}{kx_1}(\sin\alpha-\alpha\cos\alpha)$$

$$-\frac{Sl}{x_1}\sin\alpha + (M+Qb-Sl)\left(1+\frac{1}{x_1}\right)\sin\alpha - \frac{Qd}{2kx_1}\sin^2\alpha\,(\alpha+\sin\alpha\cos\alpha)$$

$$-\frac{Qd}{4kx_1}(2\sin^2\alpha+\sin\alpha\cos\alpha-\alpha) - \frac{Sd}{2x_1}(\alpha+\sin\alpha\cos\alpha)$$

$$+(M+Qr)\left(1+\frac{1}{x_2}\right)(1-\sin\alpha) - \frac{Qr}{2x_2}\cos^2\alpha - \frac{Sr}{2x_2}\left(\frac{\pi}{2}-\alpha-\sin\alpha\cos\alpha\right)$$

$$-(M+Qb-Sl)\sin\alpha - \left(\frac{M}{r}+Q\right)(b-d\sin\alpha). \qquad (13)$$

Let E' and f' denote, respectively, the modulus of elasticity of the material of the stud and the cross section of the stud; then since $-\Delta b$ is the amount of compression in the half length, $b-\frac{d}{2}$, of the stud, we have

$$-E'f'\cdot\Delta b = S\left(b-\frac{d}{2}\right).$$

If we denote by c the ratio $\frac{Ef}{E'f'}$; then

$$-Ef\cdot\Delta b = cS\left(b-\frac{d}{2}\right).$$

After slight reduction, equations (12) and (13) may be written

as follows:

$$\Gamma M = \Omega \cdot Sd - \Psi \cdot Qd. \qquad (14)$$

$$\Omega M = \Sigma \cdot Sd - X \cdot Qd. \qquad (15)$$

in which the coefficients of M, Sd, and Qd have the following values:

$$\Gamma = \alpha\left(1+\frac{1}{x_1}\right) + \frac{d}{r}\left(1+\frac{1}{x_2}\right)\left(\frac{\pi}{2}-\alpha\right); \qquad (16)$$

$$\Omega = \frac{l}{d}\left(1+\frac{1}{x_1}\right)\alpha - \sin\alpha\left(\frac{1}{x_2}-\frac{1}{x_1}\right) + \frac{1}{x_2}; \qquad (17)$$

$$\Psi = \frac{b}{d}\left(1+\frac{1}{x_1}\right)\alpha + \left(1+\frac{1}{x_2}\right)\left(\frac{\pi}{2}-\alpha\right) - \frac{\cos\alpha}{x_2} - \frac{1}{x_1}\left(\frac{2\sin\alpha}{k}-\cos\alpha\right); \qquad (18)$$

$$\Sigma = \frac{l}{d}\left(1+\frac{1}{x_1}\right)\left(\alpha\frac{l}{d}+\sin\alpha\right) + \frac{k}{2x_1} + \frac{1}{2x_2}\frac{r}{d}\left(\frac{\pi}{2}-k\right)$$
$$+ \frac{l}{d}\sin\alpha\left(\frac{1}{x_1}-1\right) + c\left(\frac{b}{d}-\frac{1}{2}\right); \qquad (19)$$

$$X = \frac{1}{x_1}\left(\frac{b}{d}\sin\alpha - \frac{\sin^2\alpha}{2} - \frac{1}{4}\right) - \frac{l}{d}\frac{1}{x_1}\left(2\frac{\sin\alpha}{k}-\cos\alpha\right) + \frac{\alpha}{2kx_1}$$
$$+ \frac{l}{d}\frac{b}{d}\alpha\left(1+\frac{1}{x_1}\right) + \frac{r}{d}\cdot\frac{1}{2x_2}\left(1-\sin\alpha\right)^2. \qquad (20)$$

Solving (14) and (15) for M and S,

$$M = Qd\,\frac{\Psi\Sigma - \Omega X}{\Omega^2 - \Gamma\Sigma}; \qquad (H)$$

$$S = Q\,\frac{\Omega\Psi - \Gamma X}{\Omega^2 - \Gamma\Sigma}. \qquad (I)$$

Having found from formulas (H) and (I) the values of M and S, we can readily find the normal force and bending moment at any section, and then from formula (A) the stresses in the various fibers.

If in (14) we make S = 0, we have the case of an open link; the value of M then becomes

$$M = -Qd\,\frac{\Psi}{\Gamma},$$

which is identical with formula (E), as it should be.

17. The limiting case in which the sides of the link are straight must receive special consideration. For this case,

$$\alpha = \frac{\pi}{2}; \quad \beta \left(= \frac{\pi}{2} - \alpha\right) = \frac{l}{r}\Big]_{r=\infty} = 0; \quad \frac{1}{x_2} = \frac{16r^2}{d^2}\Big]_{r=\infty} = \infty.$$

$$\Gamma = \frac{\pi}{2}\left(1 + \frac{1}{x_1}\right) + \frac{d}{r}\left(1 + \frac{16r^2}{d^2}\right)\frac{l}{r}\Big]_{r=\infty} = \frac{\pi}{2}\left(1 + \frac{1}{x_1}\right) + 16\frac{l}{d}.$$

$$\Omega = \frac{\pi}{2}\frac{l}{d}\left(1 + \frac{1}{x_1}\right) + \frac{1}{x_2}\left(1 - \sin\alpha\right)_{\alpha=\frac{\pi}{2}} + \frac{1}{x_1} = \frac{\pi}{2}\frac{l}{d}\left(1 + \frac{1}{x_1}\right) + \frac{1}{x_1} + \frac{1}{x_2}\left(1 - \cos\beta\right)\Big]_{\substack{x_2=\infty \\ \beta=0}}$$

$$= \frac{\pi}{2}\frac{l}{d}\left(1 + \frac{1}{x_1}\right) + \frac{1}{x_1} + \frac{16r^2}{d^2}\cdot\frac{1}{2}\frac{l^2}{r^2} = \frac{\pi}{2}\frac{l}{d}\left(1 + \frac{1}{x_1}\right) + \frac{1}{x_1} + 8\frac{l^2}{d^2}.$$

$$\Psi = \frac{\pi}{2}\frac{b}{d}\left(1 + \frac{1}{x_1}\right) + \frac{1}{x_2}\left(\beta - \sin\beta\right)_{\beta=0} - \frac{4}{\pi x_1} = \frac{\pi}{2}\frac{b}{d}\left(1 + \frac{1}{x_1}\right) + \frac{16r^2}{d^2}\cdot\frac{l^3}{6r^3}\Big]_{r=\infty} - \frac{4}{\pi x_1}$$

$$= \frac{\pi}{2}\frac{b}{d}\left(1 + \frac{1}{x_1}\right) - \frac{4}{\pi x_1} \quad \text{— since } \frac{16r^2}{d^2}\cdot\frac{l^3}{6r^3}\Big]_{r=\infty} = \frac{8}{3}\frac{l^3}{d^2 r}\Big]_{r=\infty} = 0.$$

$$\Sigma = \frac{l}{d}\left(\frac{\pi}{2}\frac{l}{d} + 1\right)\left(1 + \frac{1}{x_1}\right) + \frac{\pi}{4x_1} + \frac{l}{d}\left(\frac{1}{x_1} - 1\right) + c\left(\frac{b}{d} - \frac{1}{2}\right) + \frac{1}{2x_2}\frac{r}{d}\left(\frac{\pi}{2} - k\right).$$

To evaluate the last term we have

$$\frac{\pi}{2} - k = \frac{\pi}{2} - \alpha - \sin\alpha\cos\alpha = \beta - \sin\beta\cos\beta = \beta - \left(\beta - \frac{2}{3}\beta^3 + \cdots\right) = \frac{2}{3}\beta^3 + \cdots$$

and terms with higher powers of β.

$$\frac{1}{2x_2}\frac{r}{d}\left(\frac{\pi}{2} - k\right) = \frac{1}{2x_2}\cdot\frac{r}{d}\frac{2}{3}\beta^3 = \frac{8r^2}{d^2}\cdot\frac{r}{d}\cdot\frac{2}{3}\frac{l^3}{r^3} = \frac{16}{3}\frac{l^3}{d^3}; \text{ hence}$$

$$\Sigma = \frac{l}{d}\left(\frac{\pi}{2}\frac{l}{d} + 1\right)\left(1 + \frac{1}{x_1}\right) + \frac{\pi}{4x_1} + \frac{l}{d}\left(\frac{1}{x_1} - 1\right) + c\left(\frac{b}{d} - \frac{1}{2}\right) + \frac{16}{3}\frac{l^3}{d^3}.$$

In the expression for X the indeterminate term is

$$\frac{r}{d}\cdot\frac{1}{2x_2}\left(1 - \sin\alpha\right)^2.$$

Since $\left(1 - \sin\alpha\right)^2_{\alpha=\frac{\pi}{2}} = \left(1 - \cos\beta\right)^2_{\beta=0} = \left(\frac{\beta^2}{2}\right)^2_{\beta=0} = \frac{l^4}{4r^4}$

the expression is

$$\frac{r}{d}\cdot\frac{8r^2}{d^2}\cdot\frac{l^4}{4r^4}\Big]_{r=\infty} = \frac{2l^4}{d^3 r}\Big]_{r=\infty} = 0$$

hence,

$$\chi = \frac{1}{x_1}\left(\frac{b}{d} - \frac{3}{4}\right) - \frac{4}{\pi x_1}\frac{l}{d} + \frac{l}{d}\frac{b}{d}\frac{\pi}{2}\left(1 + \frac{1}{x_1}\right).$$

g. Résumé of Formulas.

18. For convenience of reference we collect the various formulas for the bending moment M,

(1) Link with elliptical center-line, load assumed as concentrated:

$$M = Qd\left\{\frac{a}{d}\,\frac{8\frac{a^2}{d^2}\gamma - \frac{2}{3}\frac{b^2}{a^2} - \frac{4}{3}}{\frac{\pi}{2}\left(16\frac{a^2}{d^2}\alpha - \frac{a^2}{b^2}\beta\right.} - \frac{b}{d}\right\}, \quad (B)$$

in which

$$\alpha = 1 - \frac{1}{4}k^2 - \frac{3}{64}k^4 - \frac{5}{256}k^6 - \dots$$

$$\beta = 1 - \frac{3}{4}k^2 + \frac{9}{64}k^4 + \frac{5}{256}k^6 + \dots$$

$$\gamma = \frac{b^2}{a^2} + \frac{b}{\sqrt{a^2-b^2}}\tan^{-1}\frac{\sqrt{a^2-b^2}}{b}$$

and $k^2 = 1 - \frac{b^2}{a^2}$.

(2) Link with circular center-line of radius r:

$$M = Qr\left\{\frac{2}{\pi(1+x)} - 1\right\}. \qquad (C)$$

(3) Link with center-line of four circular arcs, two of radius d and two of radius r, load assumed concentrated:

$$M = Qd\left\{\frac{\frac{1}{x_1}(1-\cos\alpha) + \frac{1}{x_2}\cos\alpha - \frac{b}{d}\alpha\left(1+\frac{1}{x_1}\right) - \left(1+\frac{1}{x_2}\right)\left(\frac{\pi}{2}-\alpha\right)}{\alpha\left(1+\frac{1}{x_1}\right) + \frac{d}{r}\left(1+\frac{1}{x_2}\right)\left(\frac{\pi}{2}-\alpha\right)}\right\}. \quad (D)$$

(4) Link as described under (3) but with load assumed as distributed:

$$M = Qd\left\{\frac{\frac{1}{x_1}\left(\frac{2\sin\alpha}{\kappa} - \cos\alpha\right) + \frac{1}{x_2}\cos\alpha - \frac{b}{d}\alpha\left(1+\frac{1}{x_1}\right) - \left(\frac{\pi}{2}-\alpha\right)\left(1+\frac{1}{x_2}\right)}{\alpha\left(1+\frac{1}{x_1}\right) + \frac{d}{r}\left(1+\frac{1}{x_2}\right)\left(\frac{\pi}{2}-\alpha\right)}\right\}. \quad (E)$$

(5) Link of four circular arcs but of lemniscate form; load distributed:

$$M = Qd\left\{\frac{\frac{1}{x_1}\left(\frac{4}{\pi}+\sin\delta\right) - \frac{b}{d}\left(1+\frac{1}{x_1}\right)\left(\frac{\pi}{2}+\delta\right) + \left(1+\frac{1}{x_2}\right)\delta - \frac{1}{x_2}\sin\delta}{\left(\frac{\pi}{2}+\delta\right)\left(1+\frac{1}{x_1}\right) + \frac{d}{r}\delta\left(1+\frac{1}{x_2}\right)}\right\}, \quad (F)$$

(6) Link composed of two circular arcs and two straight lines; load distributed:

$$M = Qd\left\{\frac{\frac{4}{x_1\pi} - \frac{\pi}{2}\frac{b}{d}\left(1+\frac{1}{x_1}\right)}{\frac{\pi}{2}\left(1+\frac{1}{x_1}\right) + 16\frac{l}{d}}\right\}. \quad (G)$$

(7) Link of four circular arcs with stud:

$$M = \frac{\Psi\Sigma - \Omega X}{\Omega^2 - \Gamma\Sigma}, \quad (H)$$

$$S = \frac{\Omega\Psi - \Gamma X}{\Omega^2 - \Gamma\Sigma}. \quad (I)$$

in which Γ, Ω, Ψ, and Σ are given by equations (16), (17), (18), and (19).

V. COMPUTATION OF STRESSES.

a. Stresses in Link of Length 6 d

19. Proportions of links- The proportions of chain links in ordinary use vary somewhat. The semi-axis a of the center-line may be as small as 1.8 d, or as large as 4.5 d; while the semi-axis may vary between 12 d and 2.25 d. For anchor chains with studs, Bach gives the proportions

$$a = 2.5\,d,$$
$$b = 1.3\,d.$$

To show the influence of the breadth of the link, I have as a first example, assumed a = 2.5 d (that is, total length = 6 d) and have varied the semi-axis b from d to 2.5 d. With b = d, the sides of the link are straight, and with b = 2.5 d, the center-line becomes a true circle.

20. Computation of normal forces and bending moments.- The first step in the computation is the determination of the radius r of the arcs that form the sides and of the angle α and its functions.

Taking the ratio $\frac{b}{d}$ as the independent variable, the values given in table 1 are readily found. Formula (7) is used to compute $\sin\alpha$ and $\cos\alpha$, (8) to compute $\frac{r}{d}$, and the formula

$$\frac{1}{x_2} = 4\left(\frac{r}{e}\right)^2 - 2 - \frac{1}{4}\left(\frac{e}{r}\right)^2 = 16\left(\frac{r}{d}\right)^2 - 2 - \frac{1}{16}\left(\frac{d}{r}\right)^2,$$

which may be obtained by taking the reciprocal of the series of formula (6), is used to compute the function $\frac{1}{x_2}$ for different values of $\frac{r}{d}$. The value of $\frac{1}{x_1}$ is $16\left(\frac{d}{d}\right)^2 - 2 - \frac{1}{16}\left(\frac{d}{d}\right)^2 = 13.93$

The substitution of the numerical values of these functions in

the expression for Γ, Ω, Σ, etc, equations (16) to (20) gives the numerical values of these coefficients; and the substitutions of these last values in formulas (H) and (I) gives the numerical values of the moment M and the stress S in the stud. The results, as obtained, are shown in table II.

The value of $c = \frac{Ef}{E'f'}$ in the expression for Σ is assumed to be 4 ; this holds for a cast iron stud with a sectional area one-half the area of the link section. The values of M′ for the open link given in the last column are obtained from the formula

$$M = -\frac{\Psi}{\Gamma} Qd;$$

thus for $\frac{b}{d} = 1$, $M = -\frac{5.7178}{47.452} Qd = -.1205 Qd$, and so on.

It will be shown later that in the case of the open link, the maximum stresses occur at sections A and B, that is, at the side and end of the link; that with the stud link there are in each quadrant two sections of minimum stress and that maximum stresses occur at sections A and B and at a third section lying between them. The stresses at sections A and B are therefore of prime importance, and to these we now turn our attention.

It has been shown that for sections between $\phi = 0$ and $\varphi = \alpha$,

$$P = \frac{Q}{k}(\sin^2\alpha \cos\varphi + \varphi \sin\varphi) + S\cos\varphi;$$

$$M_b = M + Qb - \frac{Qd}{k}(\sin^2\alpha\cos\varphi + \varphi\sin\varphi) - S(l + d\cos\varphi).$$

At section B, $\phi = 0$: hence

$$P = Q\frac{\sin^2\alpha}{k} + S;$$

$$M_b = M + Qb - Qd\frac{\sin^2\alpha}{k} - S(l+d)$$

$$= M + Qd\left(\frac{b}{d} - \frac{\sin^2\alpha}{k}\right) - Sa.$$

At section A,

$$P = Q\ ;$$
$$M_b = M.$$

The expressions as here given, apply to the stud link; for the open link we have merely to substitute for M the moment M′ for the open link and to make S zero. The numerical values of the normal force and bending moment at section A and B for both stud and open links are given in table III.

21. Stresses in Sections A and B.-It is evident from formula (A) that the fibers of the cross section for which η is greatest will be subjected to the greatest intensity of stress. For a circular section of diameter d the maximum values of η are $+\frac{d}{2}$ at the outermost fiber and $-\frac{d}{2}$ at the fiber nearest the center of curvature; and to find the absolute maximum intensity of stress at any section of the link, we need only to consider the stresses in these two fibers.

At section B the radius of curvature r is d, and $\frac{1}{x_1}$ is 13.93.

At the outermost fiber, $\frac{\eta}{r+\eta} = \frac{\frac{d}{2}}{d+\frac{d}{2}} = \frac{1}{3}$;

and at the innermost fiber $\frac{\eta}{r+\eta} = \frac{-\frac{d}{2}}{d-\frac{d}{2}} = -1$;

hence $1 + \frac{1}{x_1}\frac{\eta}{r+\eta}$ becomes

$$1 + \tfrac{1}{3} \times 13.93 = 5.6433, \text{ at outer fiber;}$$
$$1 - 1 \times 13.93 = -12.93, \text{ at inner fiber.}$$

Formula (A) reduces to

$$\sigma = \frac{P}{f} + 5.6433\frac{M_b}{fd}, \text{ at outer fiber,}$$

$$\text{and} \quad \sigma = \frac{P}{f} - 12.93\frac{M_b}{fd}, \text{ at inner fiber.}$$

Substituting in these equations the numerical values of p and M, as given in columns 4 and 5, 8 and 9, table 111, we obtain the results given in table V for section B.

The computation of the stresses at section A is complicated by the fact that for different values of b, we have different values of r and x_2. The various steps in the calculation and the numerical results are shown in table IV.

Let the expression $\frac{d}{r}\left(1 + \frac{1}{x_2}\frac{\eta}{r+\eta}\right)$ be denoted by K; then formula (A) may be written

$$\sigma = \frac{P}{f} + \frac{KM_b}{f} = \frac{Q}{f} + \frac{KM_b}{f},$$

since at Section A, $P = Q$.

The values of M for stud and open link, respectively are given in table III, and the corresponding values of K are given in the last two columns of table IV. The substitution of the values of K and M in the equation just given leads to the results given in table V for section A.

22. Link of Lemniscate Form. In table VI are given the data and results for the link with sides convex to the center. For the value $\frac{b}{a} = \frac{1}{2}$, the sides touch; for $\frac{b}{a} = 1$, the sides are straight. The values given in the table are taken from tables III, IV, and V. The value of the bending moment M is obtained from formula (F), the values of the variables δ, r, x_2, etc. in this formula being readily found from the geometry of the configuration. The moment at section B is obtained by making $\phi = 0$ in the general expression

$$M_b = M + Qb - \frac{2Qd}{\pi}(\cos\varphi + \phi\sin\phi);$$

hence M_b at section $B = M + Qb - \frac{2Qd}{\pi} = M + Qd\left(\frac{b}{d} - \frac{2}{\pi}\right).$

The normal force at section A is Q ; that at section B is $\frac{2}{\pi}Q$.

The normal force and bending moment at section A and B being given, the stresses in the extreme fibers at these sections, as given in table VI, are readily found from the general formula (A)

It is evident that with a link of this form the stud will have little influence on the stresses. When the sides are straight, the moment M at the sides is small and there is little tendency for the sides to approach each other; and, as shown by table VI, when $\frac{b}{d} = .8$, or less, the moment M becomes positive and the sides tend to recede from each other. For this reason it has been deemed unnecessary to consider the case of a link of this form with a restraining stud.

23. Concentrated Loads. If the pressure between two links is assumed to be concentrated at a single point we have the results shown in table VII. The bending moment M is computed from formula (D), the functions α, $\sin\alpha$, $\cos\alpha$, $\frac{1}{x_2}$, etc, having the values given in table I; the bending moment at section B is

$$M_b = M + Qb.$$

At section A the moment force is Q, and at section B it is 0.

The moment and normal force at each section being given, the method of computating of the fiber stresses is precisely the same as in the case of the open link, distributed load.

24. Curves of Bending Moments. The bending moments at section A under the different assumed conditions are shown graphically by the curves of sheet I. These curves were obtained by plotting the values of M in tables III, VI and VII. The ratio $\frac{b}{d}$ is taken as the abscissa and the bending moment at section A, as the ordinate.

It will be observed that the curve for the open link, distributed load is continuous after the ratio $\frac{b}{d}$ becomes less than 1, and the link has the lemniscate form. This curve crosses the zero line for $\frac{b}{d} = .83$; hence for this value of $\frac{b}{d}$, the moment at section A disappears and the section is subjected to the direct tension Q .

Regarding the sense of the moment, the negative sign of M for the open link shows that the moment tends to decrease the curvature at A. In the case of the stud link, on the other hand, the moment has the positive sign and therefore tends to increase the curvature at section A. Regarding the magnitude of the moment, it appears that with the same value of b, the numerical value of M is greater for the open link than for the stud link between the limits $\frac{b}{d}$ = 1 and $\frac{b}{d} = 2.5$. A peculiar and unexpected fact is that the minimum value of M for the stud link occurs for $\frac{b}{d}$ = 1.2, about. After passing this minimum, the value of M increases uniformly with $\frac{b}{d}$. In the case of the open link there is no minimum value of M in the algebraic sense. The numerical value of M increases from 0 at $\frac{b}{d} = .83$ to .97 Qd for $\frac{b}{d} = 2.5$. The assumption of a concentrated load increases the moment M, as shown by the curves. Naturally this increase is greatest with the smaller values of b and the corresponding

larger values of the angle α . As $\frac{b}{d}$ increases, the angle α decreases in value and the moments grow nearly equal: finally when $\frac{b}{d} = 2.5$, the center-line of the link is a circle, the angle α is zero, the load is concentrated, and the two values of M are the same.

The bending moment at section B are shown in the curves of sheet II . At this section the moment is positive in every case and therefore tends to increase the curvature at the end of the link, As at section A, the moment is very greatly decreased by the use of the stud ; but even in the case of the stud link, the moment increases very rapidly as b increases, much more so than the moment at section A. The influence of the assumption of a concentrated load is very marked. With a link of the ordinary width, $b = 1.3d$, the moment with a concentratd load is more than double that with a distributed load, As before, this difference decreases as the link becomes more nearly circular.

25. Curves showing Stresses in Sections A and B. The curves of sheet III shows the fiber stresses at sections A and B, the unit being $\frac{Q}{f}$, the stress in a straight bar of cross sections f and subjected to a load Q.

From $\frac{b}{d} = 1$ to $\frac{b}{d} = 1.3$ the tension in the outer fiber at both sections A and B is about the same, and approximately $1.5 \frac{Q}{f}$.

For larger values of $\frac{b}{d}$ the tension in the outer fiber of section B exceeds that in the outer fiber of section A, the difference increasing rapidly as $\frac{b}{d}$ increases. The stress in the inner fiber of section A is small and changes from tension to compression for $\frac{b}{d} = 1.93$ .The compression in the inner fiber of section B ,

though small for small values of $\frac{b}{d}$, increases very rapidly when $\frac{b}{d}$ exceeds 1.5. The stresses at section A and B of the open link with distributed load are shown graphically in sheet IV.

For values of $\frac{b}{d}$ greater than 1, the tension in the inner fiber of section A is greater than that in the outer fiber of section B, though the difference is not marked. The compression in the inner fiber of section B is, however largely in excess of that in the outer fiber of section A, and increases very rapidly as the width b is made larger. A comparison of sheets III and IV shows that for $\frac{b}{d}$ greater than 1, the stresses in sections A and B are *much* greater in the open link than in the stud link. It is interesting to note the stresses for values of $\frac{b}{d}$ less than 1, that is, for links of the lemniscate form. At section B, the tension in the outer fiber gradually diminishes with $\frac{b}{d}$ becoming about 1.2 $\frac{Q}{d}$ for $\frac{b}{d} = .5$; likewise the compression in the inner fiber diminishes to about $.85\frac{Q}{d}$ for $\frac{b}{d} = .5$. The stress in the outer fiber of section A is practically 0 for $\frac{b}{d} = 1$, and increases as b grows smaller until it reaches the value 2. 5 5 $\frac{Q}{d}$ when $\frac{b}{d} = .5$. The tension in the inner fiber grows smaller with b and becomes 0 when $\frac{b}{d} = .68$; for smaller values of $\frac{b}{d}$, this fiber is in compression. When $\frac{b}{d} = .83$, the stress in the inner and outer fibers of section A is the same and is a tension of magnitude $\frac{Q}{f}$. As was shown in sheet I, the bending moment at section A disappears for this value of $\frac{b}{d}$, and the section is acted upon by the normal force Q alone.

36. Stresses in Intermediate Sections.- While sections A and B are of prime importance, it is instructive and interesting to find the stresses in intermediate cross sections and to observe the law of variation of stress from section to section. With the length of link we have assumed in the previous computations, we shall find that in the case of the open link, the maximum absolute stress occurs in either section A or section B; but in the case of the stud link, it will be found that the maximum tensile stress occurs at some intermediate section.

For the purpose of exhibiting the stress throughout the link, let us take a link with the proportions b $= 1.5d$, $a = 2.5d$.

Referring to the preceeding tables, we find in connection with this link the following values:

$$r = 3.5d; \qquad \frac{1}{x_2} = 194;$$

$$\alpha = .9273 \text{ radian}; \qquad M = .0696\,Qd;$$

$$\sin\alpha = .8; \qquad M' = -.4467\,Qd;$$

$$\cos\alpha = .6; \qquad S = .3625\,Q;$$

$$k = \alpha + \sin\alpha\cos\alpha = 1.4073; \qquad \frac{1}{k} = .71058.$$

We will consider first the case of the stud link. The general equations for the normal force and bending moment are:

$$\varphi = 0 \text{ to } \varphi = \alpha \quad \begin{cases} P = \dfrac{Q}{k}(\sin^2\alpha\cos\varphi + \varphi\sin\varphi) + S\cos\varphi; \\ M_b = M + Qb - \dfrac{Qd}{k}(\sin^2\alpha\cos\varphi + \varphi\sin\varphi) - S(l + d\cos\varphi). \end{cases}$$

$$\varphi = \alpha \text{ to } \varphi = \frac{\pi}{2} \quad \begin{cases} P = Q\sin\varphi + S\cos\varphi; \\ M_b = M + Qr(1 - \sin\varphi) - Sr\cos\varphi. \end{cases}$$

We now assume values of ϕ as $0°$, $10°$, $20°$, ... $90°$ and compute for each value of ϕ the value of p and M_b at that section; these being obtained, we find the stresses in the outer and inner fibers or in any desired intermediate fibers by means of formula (A). The results obtained are shown in table VIII.

The bending moment M_b, it will be observed, is positive for value of ϕ between 0 and α, negative between $60°$ and $80°$, and is again positive at $90°$, that is, at section A. There are therefore two sections at which M_b passes through the value 0; one lies between α and $60°$ the other between $80°$ and $90°$. To determine these sections, we have the equation

$$M_b = 0 = M + Qr(1 - \sin\varphi) - Sr\cos\varphi,$$

from which

$$Q(1 - \sin\varphi) - S\cos\varphi = -\frac{M}{r},$$

$$\sin\varphi + \frac{S}{Q}\cos\varphi = 1 + \frac{M}{Qr},$$

$$\sin\varphi + .3625\cos\varphi = 1.0199.$$

Solving,

$$\cos\varphi = \begin{cases} .05974; \\ .59371. \end{cases}$$

$$\varphi = \begin{cases} 53° \ 34\tfrac{3}{4}'; \\ 86° \ 34\tfrac{1}{2}'. \end{cases}$$

The sections corresponding to these values of ϕ are subject to a uniformly distributed stress whose magnitude is

$$\frac{P}{f} = \frac{Q}{f}\sin\varphi + \frac{S}{f}\cos\varphi = 1.0199\frac{Q}{f}.$$

The stress in the outer fiber is positive, that is, tensile for

$\phi = 0$, and decreases in magnitude as ϕ increases until between 60° and 70° it changes sign and becomes negative or compressive. Between 70° and 80° the stress again changes sign, becomes tensile, and increases as ϕ approaches 90°. The stress in the inner fiber is compressive for $\phi = 0$ and decreases in absolute magnitude as ϕ increases from 0° to 40°. Between 40° and 50° the stress changes sign, becomes tensile and remains tensile as ϕ approaches 90°.

The maximum value of this stress occurs when ϕ is about 70°; as shown by the table this stress is the greatest that occurs any section of the link. To find the exact location of this section of maximum tensile stress, we proceed as follows:

From formula (A),

$$f_6 = P + \frac{M_b}{r}\left(1 + \frac{1}{x}\frac{\eta}{r+\eta}\right)$$

$$= P + \frac{M_b}{r}\left(1 - \frac{1}{x}\frac{d}{2r-d}\right),$$

since for the inner fiber, $\eta = -\frac{1}{2}d$.

The section in question lies between $\phi = \alpha$ and $\varphi = \frac{\pi}{2}$; hence

$$P = Q\sin\varphi + S\cos\varphi$$

$$M_b = M + Qr - r(Q\sin\varphi + S\cos\varphi)$$

$$P + \frac{M_b}{r} = \frac{M}{r} + Q$$

$$f_6 = P + \frac{M_b}{r} - \frac{M_b}{r}\frac{1}{x_2}\frac{d}{2r-d} = Q + \frac{M}{r} - h\frac{M_b}{r},$$

where h denotes the constant $\frac{1}{x_2}\frac{d}{2r-d}$.

$$f_6 = Q + \frac{M}{r} - h\left(Q + \frac{M}{r}\right) + h(Q\sin\varphi + S\cos\varphi)$$

$$= m + h(Q\sin\varphi + S\cos\varphi),$$

where m is another constant.

$$\frac{d\sigma}{d\varphi} = \frac{h}{f}(Q\cos\varphi - S\sin\varphi)$$

For σ a maximum,

$$Q\cos\varphi - S\sin\varphi = 0,$$

$$\frac{\cos\varphi}{\sin\varphi} = \cot\varphi = \frac{S}{Q} = .3625,$$

$$\varphi = 70^\circ\ 4.5'.$$

If desired, the values of φ for which the stress in the outer and inner fibers is zero, may readily be computed. For the outer fiber:

$$f\sigma = 0 = P + \frac{M_b}{r}\left(1 + \frac{1}{\varkappa_2}\,\frac{d}{2r+d}\right) = P + 25.25\,\frac{M_b}{r}.$$

Substituting $\quad P = Q\sin\varphi + S\cos\varphi,$

and $\quad M_b = M + Qr(1-\cos\varphi) - Sr\cos\varphi,$

we obtain after reduction

$$\sin\varphi + .3625\cos\varphi = 1.06196,$$

from which

$$\cos\varphi = \begin{cases} .39363 \\ .28687 \end{cases}$$

$$\varphi = \begin{cases} 66^\circ\ 49'\ 10'' \\ 73^\circ\ 19'\ 45''. \end{cases}$$

The location of the neutral line is obtained as follows: For this line the stress is zero: hence

$$f\sigma = 0 = P + \frac{M_b}{d}\left(1 + \frac{1}{\varkappa_1}\,\frac{\eta}{d+\eta}\right); \qquad \Big|_{\varphi=0}^{\varphi=\alpha}$$

and $$f\sigma = 0 = P + \frac{M_b}{r}\left(1 + \frac{1}{\varkappa_2}\,\frac{\eta}{r+\eta}\right). \qquad \Big|_{\varphi=\alpha}^{\varphi=\frac{\pi}{2}}$$

From the first of these equations,

$$\frac{\eta}{d+\eta} = -\left(\frac{Pd}{M_b}+1\right)\varkappa_1 = -\varkappa_1\,\frac{Pd+M_b}{M_b}$$

$$\frac{d+\eta}{\eta} = -\frac{M_b}{\varkappa_1(Pd+M_b)} = -\frac{13.93\,M_b}{Pd+M_b}$$

$$\frac{d}{\eta} = -\frac{13.93\,M_b}{Pd+M_b} - 1 = -\frac{14.93\,M_b+Pd}{M_b+Pd}$$

$$\frac{\eta}{d} = -\frac{M_b+Pd}{14.93\,M_b+Pd} = -\frac{\frac{M_b}{d}+P}{14.93\,\frac{M_b}{d}+P},$$

for values of φ between 0 and α.

Likewise from the second equation,

$$\frac{\eta}{r} = -\frac{\frac{M_b}{r}+P}{\left(1+\frac{1}{\varkappa_2}\right)\frac{M_b}{r}+P}$$

$$= -\frac{\frac{M_b}{r}+P}{195\,\frac{M_b}{r}+P}, \text{ in the present case.}$$

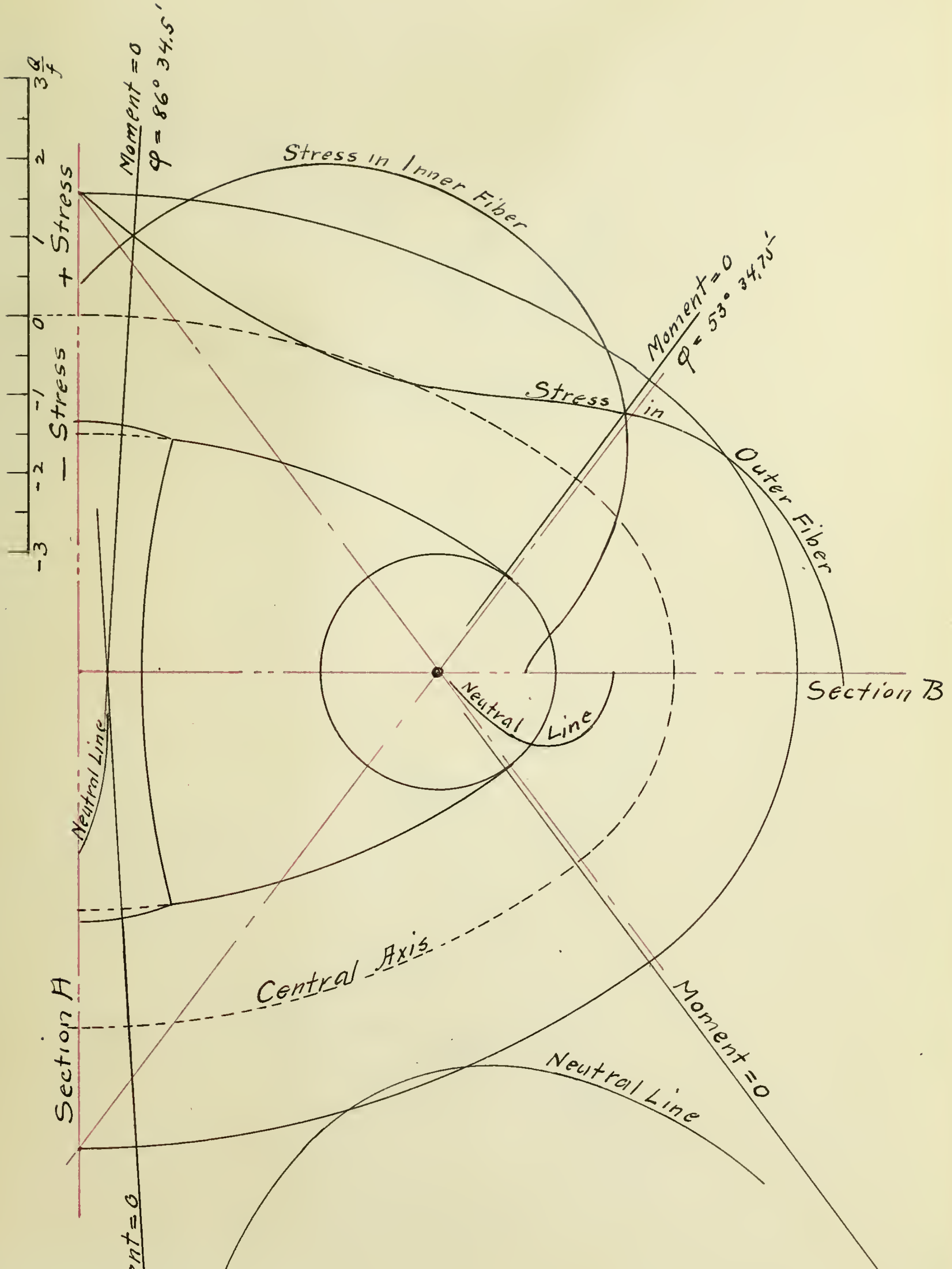
$$\frac{\eta}{d} = \frac{\eta}{r}\cdot\frac{r}{d} = -\frac{\frac{M_b}{d}+P\frac{r}{d}}{195\,\frac{M_b}{d}\cdot\frac{d}{r}+P}$$

$$= -\frac{\frac{M_b}{d}+3.5\,P}{55.7\,\frac{M_b}{d}+P}, \text{ in the present case.}$$

Fig 9 shows graphically the variation of the stresses in the intermediate sections of a stud link. The two curves in the upper quadrant are obtained by lying off radically the intensities of stress in the inner and outer fibers of any section, using the center line as a base, Tensile stresses are measured outwards, compressive stresses towards the center of curvature. The scale adopted is shown in the figure. In the lower quadrant is shown the location of the neutral line. Taking the whole link, there are eight sections at which the bending moment disappears- two in each quadrant; hence the neutral line is a curve with eight branches, each branch having radii through two sections of zero bending moment as asymptotes.

27. The curves of the stresses in the intermediate sections of the open link are shown in Fig 10. The figure is self-explaining and hardly needs comment. There are only four sections of zero bending moment and in consequence the neutral line has four branches, of the form shown in the lower quadrant. The data from which the figure is drawn are readily obtained by methods already explained, and are given in table IX .

28. Stress at a Section at which the Link has a Sudden Change of Curvature.- In connection with the results given in tables VIII and IX, there is one difficulty to be noted. Referring to Fig 4, it is seen that the sections E, E′, F and F′ separate parts of the link having quite different radii of curvature; therefore, at these sections there is a sudden change of curvature. Now ordinary static conditions require that the change in the normal force and likewise the change in the bending moment, as we pass from one side of the section to the



+ Stress
− Stress
Moment = 0
φ = 86° 34.5'
Stress in Inner Fiber
Moment = 0
φ = 53° 34.75'
Stress in Outer Fiber
Section B
Neutral Line
Neutral Line
Section A
Central Axis
Moment = 0
Neutral Line
Moment = 0

Fig. 9.

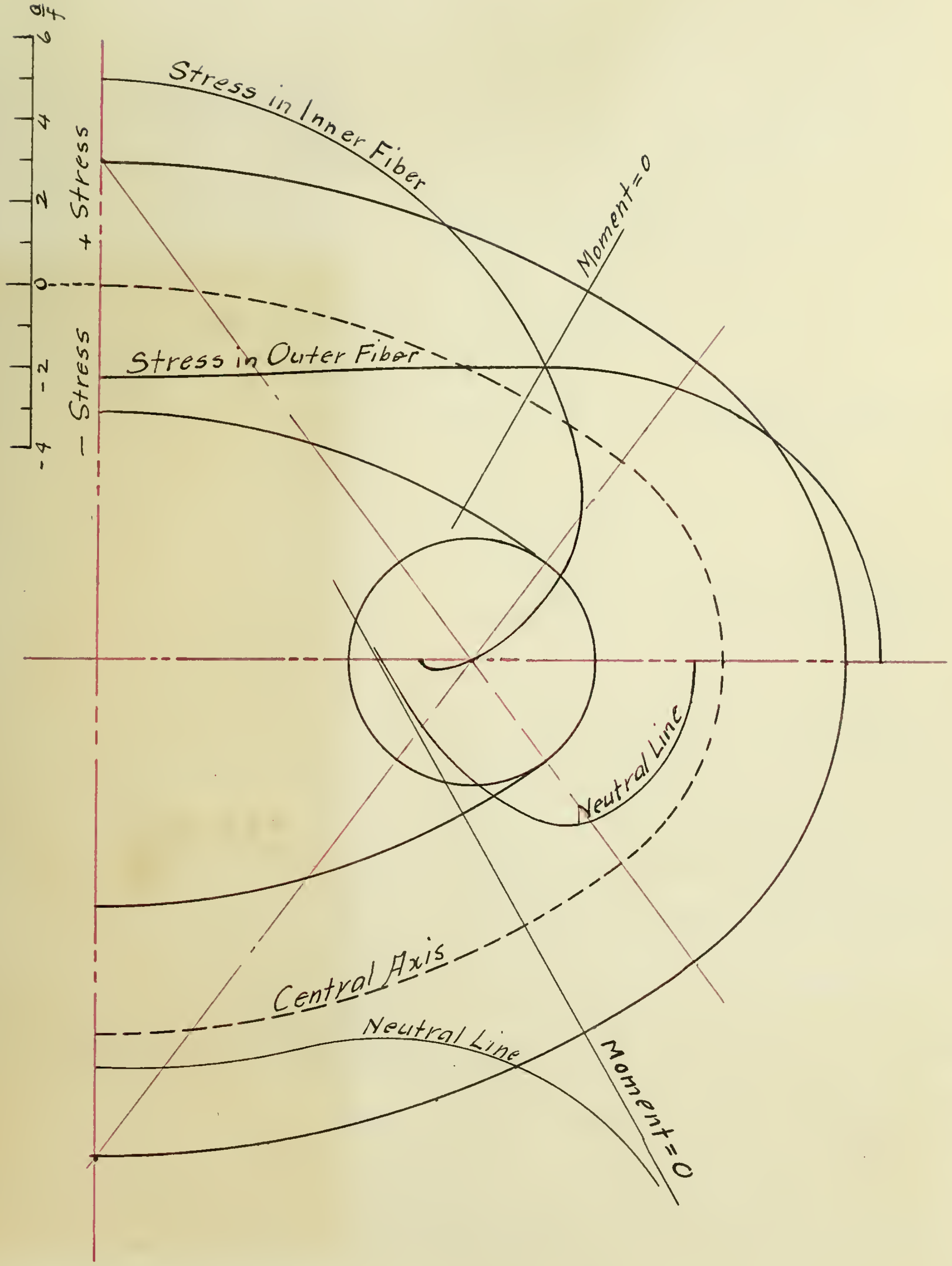

6
4
2
0
-2
-4
+ Stress
− Stress
Stress in Inner Fiber
Moment = 0
Stress in Outer Fiber
Neutral Line
Central Axis
Neutral Line
Moment = 0

Fig. 10.

other, shall be continuous; that is, arrive at the same normal force and moment whether we approach the section E from along the arc AE or along the arc BE. But the arc BE has the radius of curvature BH while the arc AE has the greater radius CE = r; furthermore, the function x has different values for the two arcs. If now we consider the section E as belonging to the point BF, the stress is given by the expression

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over such sections seems on that hypothesis (the Bernoulli-Eulerian) to be arbitrary, but it probably may be safely taken equal to the mean of the tractions on either side. I do not think this peculiarity invalidates the solution for sections at small distances from those of discontinuity." Doubtless the actual stress at such a section can be determined by an extension of the method employed to derive formula (A). Since, however, the stress at the section in question is of relatively small importance, I have not attempted to derive a formula for it.

other, shall be continuous; that is, arrive at the same normal force and moment whether we approach the section E from along the arc AE or along the arc BE. But the arc BE has the radius of curvature BH while the arc AE has the greater radius CE = r; furthermore, the function x has different values for the two arcs. If now we consider the section E as belonging to the point BE, the stress is given by the expression

$$\sigma = \frac{P}{f} + \frac{M_b}{d}\left(1 + \frac{1}{x_1}\frac{\eta}{d+\eta}\right);$$

but if we consider it as belonging to the part AE, the stress is given by

$$\sigma = \frac{P}{f} + \frac{M_b}{r}\left(1 + \frac{1}{x_2}\frac{\eta}{r+\eta}\right).$$

These stresses are different, the magnitude of the difference being shown in tables VIII and IX by the two values for $\varphi = \alpha$. This result is manifestly absurd, and is due to the fact that in the derivation of formula (A) it is tacitly assumed that the variation of curvature is continuous. Prof. Pearson says: "The exact distribution of the stress over such sections seems on that hypothesis (the Bernoulli-Eulerian) to be arbitrary, but it probably may be safely taken equal to the mean of the tractions on either side. I do not think this peculiarity invalidates the solution for sections at small distances from those of discontinuity." Doubtless the actual stress at such a section can be determined by an extension of the method employed to derive formula (A). Since, however, the stress at the section in question is of relatively small importance, I have not attempted to derive a formula for it.

29. Maximum Tensile Stresses in Stud Links.- It was shown in Art. 26 that the tensile stress in the inner fiber of a stud link has a maximum value for a value of ϕ given by the equation

$$\cot\varphi = \frac{S}{Q}.$$

It is necessary to investigate the magnitude of this stress for different values of $\frac{b}{d}$ and compare it with the tensile stress in the outer fiber at section A .

since $S = Q\cot\varphi$,

the normal force and moment at this section are

$$P = Q\sin\varphi + S\cos\varphi = Q\sin\varphi + Q\cos\varphi\cot\varphi = \frac{Q}{\sin\varphi};$$

$$M_b = M + Qr - Qr\cos\varphi - Sr\cos\varphi = M - (P-Q)r.$$

The values of the moment, normal force, and resulting tensile stress are given in table X. The stresses are shown by the dash line curve, sheet III. Comparing this curve with the other curves, it appears that for values of $\frac{b}{d}$ less than 2, this stress is the absolute maximum tensile stress in the link, and that for values of $\frac{b}{d}$ less than about 1.6, it is the maximum stress, tensile or compressive.

b. Stresses in Link of Width 3.5 d

30. To determine the influence of length, I have chosen a constant width 3.5 d ($b = 1.25 d$) for the link, and have varied the length from 4 d to 8 d ($a = 1.5 d$ to $a = 3.5 d$).

The details of the computation need not be given; the principal data and the results obtained are exhibited in table XI.

The results are shown graphically by the curves of sheets V and VI. Inspection of sheet V shows that the bending moment at section A has its greatest numerical value for $\frac{a}{d} = 1.5$, that is, for the shortest link; this moment decreases as the length of the link is taken greater, and becomes zero when the link becomes infinitely long. The moment at section B has likewise a maximum value for $\frac{a}{d} = 1.5$; it reaches a minimum when $\frac{a}{d} = 2.25$, about, and then increases, though very slowly, as the length of the link is increased. The rise in the moment for values of $\frac{a}{d}$ less than 2 is accounted for by the rapid decrease in the normal force as the link becomes shorter; this is shown by the curve of the normal force at section B. The drop in the normal force is due, of course, to the decrease in the angle α. If the length is made infinite, the normal force has the value $\frac{2}{\pi}Q = .6366\,Q$, and the moment at B the value $(1.25-.6366)\,Qd = .6134\,Qd$; these are, therefore, the limits that the normal force and moment at B cannot pass.

The variation of the stress at sections A and B as the length is varied is shown by the curves in sheet VI. For values of $\frac{a}{d}$ less than 3.5, the tensile stress in the inner fiber of section A exceeds that in the outer fiber of section B by an appreciable amount. However, the first mentioned stress decreases, while the other increases, as the length of the link in increased; and for all values of $\frac{a}{d}$ greater than 3.5 the tensile stress at section B will exceed that at section A. The compressive stress in the outer fiber of

section A is small, and decreases as we lengthen the link. That in the inner fiber of section B is numerically the greatest of the four stresses; it is a minimum for values of $\frac{a}{d}$ lying between 2 and 2.5, and slowly increases as the link is lengthened.

It is evident from this investigation that within reasonable limits the length of the link exerts comparatively little influence on the maximum stresses. With the width chosen, the most favorable value of a is 3.5 d, if only tension is considered, and about 2.5 d if we base the resistance of the link upon the absolute maximum stress, which is the compressive stress in section B.

VI. DISCUSSION OF RESULTS.

a. Influence Of the Form Of the Link.

31. The inferences to be drawn from the results of the preceding analyses, as regard the form of the link, may be stated in few words.

The breadth of the link has a marked influence upon the stresses produced by a given load. As shown by the curves of sheet IV, the wider the link, the greater the maximum stresses. This conclusion might have been predicted, as it is evident that the wider the link the greater the bending action.

The introduction of the stud practically doubles the strength of the link, provided the load is never great enough to induce stresses beyond the elastic limit of the material. It has been the general opinion of engineers that the stud link chain is stronger than the open link chain; however the experiments of Committee D of the United States board appointed to test iron steel and other metals (See executive Document No 98, House of Representatives, Forty fifth

Congress, Second session) seem to indicate that the stud actually weakens the chain, causing it to rupture at a load lower than that required to break an open link chain. At first sight these experiments seem to disprove the results given in the preceeding pages; however, in this case, fact and theory are easily reconciled. It is quite easy to understand that while the stud link is much stronger than the open link, ~~when~~ provided the elastic limit is not reached, the former may rupture with a smaller load than the latter. In the first place, the collapse of the sides of the open link after the elastic limit is passed decreases the effective width of the link, and thus decreases the bending moments and stresses. If the iron of which the link is constructed is ductile, the link may collapse until the sides are nearly parallel, and the stresses are lower than in the stud link, the sides of which are prevented from collapsing by the stud. Thus the actual distorsion of the open link gives it a form of greater strength, which is not the case with the stud link.]

Again, as will be shown presently the collapse of the sides causes the links to nip each other, and this nipping action still further reduces the stresses.

[On the whole, it seems probable near the point of rupture the stresses in the open link are less than those in the stud link; but this fact cannot be made to prove anything regarding the stresses within the elastic limit, and there can be no doubt that for ordinary working loads the chain made of stud links is materially stronger than one made of open links.]

The length of the link has comparatively little influence upon the strength of the link.

The strongest link is one with sides convex to the center (Lemniscate form) with a semi-axis b = .7 d, about; that is, the breadth of the link is about 2.4 d.

In any link there is in each quadrant at least one section at which the bending moment is zero. At this section the stress is minimum, and here the link should be welded. The end of the link is one of the dangerous points; hence the link should never be welded at the end.

b. The Distribution of the Pressure between the Links.

32. It has been shown very clearly in the preceding analyses that considerable importance attaches to the question of the distribution of the pressure between the adjacent links. As has been shown, the worst possible case so far as concerns the strength of the link is that in which this pressure is concentrated at a point or along a line, as would be the case were a knife edge employed. In rare instances- as in weighing machinery- knife edges may be used, and the load may thus be concentrated. On the other hand, links of chains fit each other to some extent, and the pressure must be more or less distributed. In the analysis, I have assumed that the action is of a journal and bearing and that the intensity of pressure varies as the cosine of a certain angle. This assumption I believe to be near the truth in the case of chains that have been some used so that the links have worn to a bearing and provided the load is not great

enough to appreciably distort the link. Under different circumstances, however, the cosine law, so to speak, may not correctly represent the distribution of pressure. In the first place the adjacent links may not fit like a journal and bearing. If the link is rather wide, the pressure may be nearly concentrated, and as a result, the link will be weaker than a link that fulfills the assumed law. On the other hand the links, if rather narrow, may wedge at their small ends so that the pressure is concentrated at two points at some distance from the ends.

In this case the stresses will be less than if the action is that of a journal and bearing. Again the link distorts when subjected to a heavy load and the sides approach each other. This action causes the sides of each link to pinch or nip the adjacent link, and there will result a new distribution of pressure that evidently will not be in accordance with the cosine law.

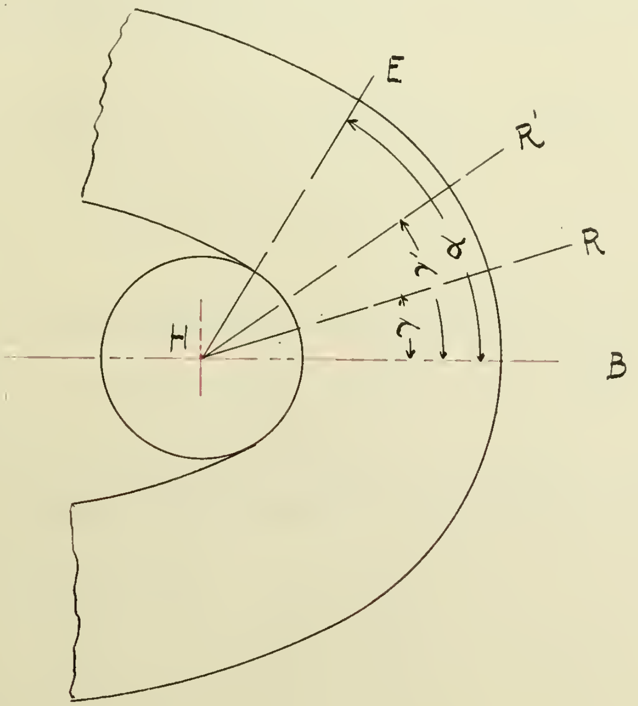


Fig. 11.

Suppose there are two links in contact as shown in Fig 11, and

that the cosine law of distribution holds good. The resultant of the pressure on one side of the axis HB has the direction HR and makes an angle γ with HB. Now if the sides of the link are made to approach each other, it is evident that there will be a pinching or nipping action set up, and because of the resistance of the link to compression, the intensity of pressure will be increased near the section HE and diminished near the section HB, On the whole, therefore, the pinching of the links will cause the resultant HR to assume a new position HR' making the greater angle γ' with the axis HB.

It is almost self-evident that the increase in the angle reduces the stresses in the link. To show the extent of the reduction I have chosen a somewhat extreme case. Let the semi-axis of the center-line of the link be $a = 2.5d$ and $b = 1.5d$. The normal force at section B we have found to be $.4548Q$ (See table III). This normal force is the V-component of the resultant R, and the H-component is Q ; hence

$$\tan\gamma = .4548Q \div Q = .4548;$$
$$\gamma = 24^\circ\ 27'.$$

Suppose now that the resultant is shifted by the pinching action between the links so that the new angle γ' is 45° ; then

$$R' = Q\sqrt{2}$$

and the normal force at the section B is equal to Q .

For sections between $\phi = 0$ and $\varphi = \alpha$

$$\text{Normal force} = Q\sin\varphi + R'\sin(\gamma' - \varphi)$$
$$\text{Moment} = M + Q(b - d\sin\varphi) - R'd\sin(\gamma' - \phi)$$

For sections between $\phi = \alpha$ and $\varphi = \frac{\pi}{2}$,

$$\text{Normal force} = Q \sin\phi,$$
$$\text{Moment} = Qr(1-\sin\varphi) + M.$$

$$Ef\omega_1 = \frac{Qb+M}{d}\left(1+\frac{1}{x_1}\right) - \frac{Q\sin\varphi}{x_1} - \frac{R'\sin(\gamma'-\varphi)}{x_1}$$

$$Ef\omega_2 = \left(\frac{M}{r}+Q\right)\left(1+\frac{1}{x_2}\right) - \frac{Q}{x_2}\sin\varphi$$

The condition

$$\int_0^\alpha \omega_1 \, d\varphi + \int_\alpha^{\frac{\pi}{2}} \omega_2 \, d\varphi$$

leads to the equation

$$0 = \alpha\left(\frac{Qb+M}{d}\right)\left(1+\frac{1}{x_1}\right) - \frac{Q}{x_1}(1-\cos\alpha) - \frac{R'}{x_1}\left[\cos(\gamma'-\alpha)-\cos\gamma'\right]$$
$$+\left(\frac{M}{r}+Q\right)\left(1+\frac{1}{x_2}\right)\left(\frac{\pi}{2}-\alpha\right) - \frac{Q}{x_2}\cos\alpha.$$

The insertion of the numerical values gives

$$M = -.3764\,Qd.$$

The bending moment at section B is therefore

$$M + Qb - Q\sqrt{2}\,d\sin 45^\circ = Qd\,(-.3764 + 1.5 - 1)$$
$$= .1236\,Qd$$

The stresses obtained by substituting these moments in equation (A) are as follows:

Outer fiber,	Section A	$-1.7155\,\frac{Q}{f}$
Inner "	" "	$+4.3696\,\frac{Q}{f}$
Outer "	Section B	$+1.6975\,\frac{Q}{f}$
Inner "	" "	$-.5981\,\frac{Q}{f}$

Comparing these with the stresses in table V, we see that there is a reduction, especially at section B. The shifting of the resultant has reduced the heavy compression in the inner fiber($-7.2838\frac{Q}{f}$) to the low value $-.5981\frac{Q}{f}$. It is, of course, evident that the wedging of the links must greatly relieve the heavy compression at section B; and as shown by this example, it reduces, though to a less degree, the other stresses.

This case is no doubt extreme, as it is unlikely that the angle γ could reach a value as great as $45°$ with a link of the assumed proportions; still it is clear that an increase of γ , however small, results in a diminution of the stresses produced by a given load.

This fact explains in some measure the high breaking load of the open link as compared with the stud link. When the open link is subjected to the heavy load imposed by the testing machine, the ends of the links are wedged so tightly that the chain becomes rigid and relative motion between the links is almost impossible. This severe wedging action must relieve to a great extent the large compression at section B, as well as the tensile stresses at both sections A and B. When the stud link is pulled in the testing machine, this wedging action is almost entirely prevented by the stud. ¶ The points presented in this article may be assumed up as follows:

(1). The maximum stresses possible are these given in table VII for a concentrated load.

(2). The stresses given in table V are substantially correct for ordinary chains in which the links have worn to a bearing.

(3). If adjacent links have but a small surface in contact, so that

the resultant R, Fig 11, makes an angle with the axis HB smaller than γ , the stresses will be between those given in table V and VII.

(4). When the open link is subjected to a load that causes its sides to collapse, the resultant R, Fig 11, will make an angle with HB greater than γ, and the stresses will be less than those of table V.

VII. FORMULAS FOR THE LOADING OF CHAINS.

33. Unwin, Elements of Machine Design, Part 1, P.438, gives the following formulas.

$$P = 9d^2 \text{ for studded link chain;}$$
$$= 6d^2 \text{ for unstudded close link chain.}$$

He says further: "For much used chain, subject frequently to the maximum load, it is better to limit the stress to 3 $\frac{1}{4}$ tons per sq in. Then

$$P = 5d^2 \text{ tons.}"$$

In these formulas, P denotes the load in tons, and d the diameter in inches of the iron from which the chain is made.

Unwin says that Towne limits the loads in ordinary crane chains to

$$P = 3.3 d^2 \text{ tons}$$

but quotes the following table from Townes "Treatise on Cranes".

Diameter of iron	$\frac{3}{16}$	$\frac{1}{4}$	$\frac{9}{32}$	$\frac{5}{16}$	$\frac{3}{8}$	$\frac{7}{16}$	$\frac{1}{2}$	$\frac{9}{16}$	$\frac{5}{8}$	$\frac{11}{16}$	$\frac{13}{16}$
Load on chain, tons	.06	.25	.5	.75	1	1.5	2	2.5	3	4	5

This table seems to be obtained from the formula

$$P = 8d^2 \text{ tons.}$$

Weisbach gives the formulas (Kents Pocket Book, p, 339)

$$P = 17800\ d^2, \quad \text{stud link}$$

$$P = 13350\ d^2, \quad \text{open link.}$$

In these formulas P denotes the load in pounds.

Bach, in his " Maschinenelemente", p.513, gives for chains with open links

$P = 1000\ d^2$ for new chains, maximum load seldom applied.

$P = 800\ d^2$ for much used chain.

P and d are taken in Kilograms and centimeters, respectively.

Using pounds and inches as the units, the formulas become

$$P = 13750\ d^2 ;$$

$$P = 11000\ d^2 .$$

For a stud link chain, Bach increases the safe load 20 per cent.

If we write the formula for the safe load

$$P = k\ d^2,$$

the values of k given by the authorites quoted are as follows, P being taken in pounds:

	Open link	Stud link
Unwin	13440 11200	20160 17800
Weisbach	13350	
Bach	13750 11000	16500 13200

34. These formulas seem to be based entirely upon the ultimate strength of the chain when tested to destruction; Thus the safe load is made a definite fraction of the proof load, which in turn is a definite fraction of average breaking load. The more rational pro-

cedure is to employ the maximum stress produced by a given load in a link of given form as a basis for the determination of the safe load.

Let S denote the maximum permissible intensity of stress;

P " the safe load;

m " a constant, which multiplied by $\frac{Q}{f}$ gives the maximum fiber stress in the link.

Then, when the chain is subjected to its maximum load, we have

$$m \frac{Q}{f} = S ;$$

$$P = 2Q = \frac{2fS}{m} = \frac{\pi}{2} \frac{S}{m} d^2 .$$

Referring to the curves, sheets III and IV, we find that for a stud link of the ordinary proportion ($a = 2.5d$; $b = 1.3d$), the maximum stress is about $2\frac{Q}{f}$, as shown by the dotted curve. With an open link of the same proportions, the maximum tensile stress is a little less than $4\frac{Q}{f}$ and the maximum compressive stress, about $5\frac{Q}{f}$. Open links however are, however, usually a little shorter, the semi-axis a being as low as 1.8 d. As shown by sheet VI, this shortening somewhat increases the stress. We are therefore justified is assuming the value $4\frac{Q}{f}$ for the maximum tensile stress for all open links of ordinary proportions; further it seems proper to base the safe load on this tensile stress rather than upon the greater compressive stress at section B, because, as we have shown, this compressive stress will be materially reduced by the nipping action between adjacent links.

Judging from the maximum permissible stresses used in machine construction in general, the value of S should not exceed 15000 pounds per square inch. We have then

$$P = \frac{\pi}{2}\,\frac{15000}{2}\,d^2 = 11780\ d^2, \quad \text{for stud link;}$$

$$P = \frac{\pi}{2}\,\frac{15000}{4}\,d^2 = 5890\ d^2, \quad \text{for open link.}$$

Or in round numbers,

$$P = 12000\ d^2, \quad \text{for stud link;}$$

$$= 6000\ d^2, \quad \text{for open link.}$$

If the link have straight sides, the value of m as shown by sheet IV is about 2 ; and for a link of lemnscate form with b = 73 d, the values of m is 1.5 ; hence the safe loads for these links are in round numbers

$$P = \frac{\pi}{2}\,\frac{15000}{2}\,d^2 = 12000\ d^2 \text{ for link with straight sides}$$

$$P = \frac{\pi}{2}\,\frac{15000}{1.5} = 16000\ d^2 \text{ for link, lemnscate form.}$$

These values of the coefficient of d^2 are much smaller than those given by Unwin, Bach, and Weisbach; it is evident, therefore, that when a chain is subjected to the maximum load permitted by the formulas in current use, the intensity of stress is considerably above 15000 pounds per sq. in, and may exceed the elastic limit. Doubtless the frequent failure of crane chains may be ascribed to this fact.

TABLE I.

$\frac{b}{d}$	$\frac{r}{d}$	sin α	cos α	α (radians)	$\frac{1}{x_2}$
1	∞	1	0	1.5708	∞
$1\frac{1}{4}$	$\frac{45}{8}$	$\frac{35}{17}$	$\frac{12}{37}$	1.2405	504.25
$1\frac{1}{2}$	$\frac{7}{2}$	$8/10$	$6/10$	.9273	194.00
$1\frac{3}{4}$	$\frac{23}{8}$	$6/10$	$8/10$	.6435	130.24
2	$\frac{21}{8}$	$\frac{5}{13}$	$\frac{12}{13}$	.3948	108.24
$2\frac{1}{4}$	$\frac{101}{40}$	$\frac{11}{61}$	$\frac{60}{61}$	.1813	100.
$2\frac{1}{2}$	$\frac{5}{2}$	0	1	.0000	98.

TABLE IV.

$\frac{b}{d}$	$\eta/_{r+\eta}$		$\frac{1}{x_2}\ \eta/_{r+\eta}$		$\frac{d}{r}\left(1+\frac{1}{x_2}\frac{\eta}{r+\eta}\right)$	
	Outer Fiber	Inner Fiber	Outer Fiber	Inner Fiber	Outer Fiber	Inner Fiber
1	0	0	∞	∞	8.	−8.
$1\frac{1}{4}$	$\frac{4}{49}$	$-\frac{4}{41}$	41.1633	−49.195	7.4959	−8.5680
$1\frac{1}{2}$	$\frac{1}{8}$	$-\frac{1}{6}$	24.2500	−32.333	7.2143	−8.9523
$1\frac{3}{4}$	$\frac{4}{27}$	$-\frac{4}{79}$	19.2950	−27.419	7.0591	−9.1892
2	$\frac{4}{25}$	$-\frac{4}{17}$	17.3184	−25.468	6.9784	−9.3214
$2\frac{1}{4}$	$\frac{20}{121}$	$-\frac{20}{81}$	16.5290	−24.691	6.9422	−9.3826
$2\frac{1}{2}$	$\frac{1}{6}$	$-\frac{1}{4}$	16.3333	−24.500	6.9333	−9.4000

Values of Various Functions

Length of Link, 6d.

Table II.

$\frac{b}{d}$	Γ	Ω	Ψ	Σ	X	M	S	M' (Open Link)
1	47.4520	67.1081	5.7178	125.4977	12.0578	+.0631 Qd	.1298 Q	−.1205 Qd
$1\frac{1}{4}$	48.1897	68.2146	13.9423	128.3350	27.3957	.0585 "	.2457 "	−.2893 "
$1\frac{1}{2}$	49.6543	70.7247	22.1795	133.5760	43.9436	.0696 "	.3625 "	−.4467 "
$1\frac{3}{4}$	51.9343	74.8612	30.5854	143.2530	60.7597	.0912 "	.4718 "	−.5891 "
2	54.8193	80.8267	38.9015	157.2191	78.6745	.1160 "	.5600 "	−.7096 "
$2\frac{1}{4}$	58.2863	88.5400	47.7417	176.1545	98.8188	.1398 "	.6312 "	−.8191 "
$2\frac{1}{2}$	62.2064	98.0000	57.5092	200.423	122.5000	.1672 "	.6929 "	−.9245 "

Values of Coefficients in Equations (H) and (I),
and of
Moments M and M'.

Stud Link, Length, 6d.

Table III.

$\frac{b}{d}$	Link with Stud				Open Link			
	Section A		Section B		Section A		Section B	
	Normal Force	Moment	Normal Force	Moment	Normal Force	Moment	Normal Force	Moment
1	Q	+.0631 Qd	.7664 Q	+.1020 Qd	Q	−.1205 Qd	.6366 Q	+.2429 Qd
$1\frac{1}{4}$	"	.0585 "	.8240 "	.1160 "	"	−.2893 "	.5783 "	.3824 "
$1\frac{1}{2}$	"	.0696 "	.8173 "	.2085 "	"	−.4467 "	.4548 "	.5985 "
$1\frac{3}{4}$	"	.0912 "	.7922 "	.3408 "	"	−.5891 "	.3204 "	.8407 "
2	"	.1160 "	.7573 "	.5187 "	"	−.7096 "	.1973 "	1.0931 "
$2\frac{1}{4}$	"	.1398 "	.7219 "	.7211 "	"	−.8191 "	.0907 "	1.3402 "
$2\frac{1}{2}$	"	.1672 "	.6929 "	.9350 "	"	−.9245 "	.0000 "	1.5755 "

Values of P and M_b

at

Sections A and B.

Length of Link, 6d.

Table V.

$\frac{b}{d}$	Stud Link				Open Link			
	Section A.		Section B		Section A		Section B	
	Outer Fiber	Inner Fiber	Outer Fiber	Inner Fiber	Outer Fiber	Inner Fiber	Outer Fiber	Inner Fiber
1	$+1.5048\frac{Q}{f}$	$+.4952\frac{Q}{f}$	$+1.3430\frac{Q}{f}$	$-.5525\frac{Q}{f}$	$+0.0360\frac{Q}{f}$	$+1.9640\frac{Q}{f}$	$+2.0074\frac{Q}{f}$	$-2.5041\frac{Q}{f}$
$1\frac{1}{4}$	1.4385 "	.4988 "	1.4786 "	-.6759 "	-1.1686 "	3.4787 "	2.8264 "	-4.3661 "
$1\frac{1}{2}$	1.5021 "	.3769 "	1.9959 "	-1.8766 "	-2.2236 "	4.9990 "	3.8323 "	-7.2838 "
$1\frac{3}{4}$	1.6438 "	.1619 "	2.7154 "	-3.6143 "	-3.1585 "	6.4104 "	5.0647 "	-10.5499 "
2	1.8095 "	-.0813 "	3.6845 "	-5.9495 "	-3.9519 "	7.6146 "	6.3660 "	-13.9365
$2\frac{1}{4}$	1.9713 "	-.3117 "	4.7913 "	-8.6019 "	-4.6864 "	8.6853 "	7.6539 "	-17.4195
$2\frac{1}{2}$	2.1593 "	-.5717 "			-5.4099 "	9.6903 "		

Stresses at Sections A and B.

Length of Link, 6d.

TABLE VI.

Stresses in Link of Lemniscate Form – Length, 6d.

$\frac{b}{d}$	$\frac{r}{d}$	$\frac{1}{\varkappa_2}$	M Moment at Section A	Moment at Section B	Stresses – Section A		Stresses – Section B	
					Outer Fiber	Inner Fiber	Outer Fiber	Inner Fiber
$\frac{1}{2}$	$\frac{3}{2}$	33.97	+.2431 Qd	+.1065 Qd	+2.5384 $\frac{Q}{f}$	−1.5906 $\frac{Q}{f}$	+1.2376 $\frac{Q}{f}$	−.7404 $\frac{Q}{f}$
$\frac{5}{8}$	$\frac{35}{16}$	74.55	+.1468 "	.1352 "				
$\frac{3}{4}$	$\frac{29}{8}$	208.	+.0536 "	.1670 "	+1.3876 "	+.5227 "	+1.5790 "	−1.5227 "
$\frac{7}{8}$	$\frac{129}{16}$	1038.	−.0354.	.2030 "				
1	∞	∞	−.1205 "	.2429 "	+.0360 "	+1.9640 "	+2.0074 "	−2.5041 "

TABLE VII.

Stresses in Open Link, Length 6d: Concentrated Load.

$\frac{b}{d}$	Moment at Section A	Moment at Section B	Stresses – Section A		Stresses – Section B	
			Outer Fiber	Inner Fiber	Outer Fiber	Inner Fiber
1	−.2007 Qd	+.7993 Qd	−0.6056 $\frac{Q}{f}$	+2.6056 $\frac{Q}{f}$	+4.5107 $\frac{Q}{f}$	−10.335 $\frac{Q}{f}$
$1\frac{1}{4}$	−.3546 "	.8954 "	−1.6580 "	4.0382 "	5.0530 "	−11.576 "
$1\frac{1}{2}$	−.4851 "	1.0149 "	−2.4997 "	5.3429 "	5.7274 .	−13.123 "
$1\frac{3}{4}$	−.6072 "	1.1428 "	−3.2863 "	6.5797 "	6.4492 "	−14.776 "
2	−.7162 "	1.2838 "	−3.9980 "	7.6760 "	7.2449 "	−16.600 "
$2\frac{1}{4}$	−.8202 "	1.4298 "	−4.6940 "	8.6956 "	8.0688 "	−18.487 "
$2\frac{1}{2}$	−.9245 "	1.5755 "	−5.4099 "	9.6903 "		

TABLE VIII.

Stresses in Sections of Stud Link

Length 6d ; Width 4d.

ϕ	P	M_b	Stresses: Outer Fiber	Stresses: Inner Fiber	Stresses: Central Axis	η/d for Neutral Line
0°	.8173 Q	+.2085 Qd	$+1.9959\frac{Q}{f}$	$-1.8766\frac{Q}{f}$	$+1.0258\frac{Q}{f}$	−.2610
10°	.8264 "	.1994 "	1.9517 "	1.7518 "	"	−.2697
20°	.8528 "	.1730 "	1.8291 "	1.3841 "	"	−.2986
30°	.8938 "	.1320 "	1.6387 "	.8130 "	"	−.3581
40°	.9520 "	.0738 "	1.3685 "	.0022 "	"	−.4995
50°	1.0047 "	.0211 "	1.1237 "	+ .7319 "	"	−.7773
α	1.0175 "	.0083 "	1.0643 " 1.0744 "	+ .9102 " + .9432 "	$1.0258\frac{Q}{f}$ 1.0199 "	+.8987
60°	1.0473 "	−.0960 "	.3547 "	1.9067 "	"	+.8300
70°	1.0637 "	−.1534 "	−.0430 "	2.4370 "	"	+.4770
80°	1.0477 "	−.0974 "	+.3450 "	1.9197 "	"	+.8130
90°	1.0000 "	+.0696 "	+1.5021 "	.3769 "	"	−.7320

TABLE IX.

Stresses in Sections of Open Link

Length 6d : Width 4d.

ϕ	P	M_b	Stresses			y/d for Neutral Line
			Outer Fiber	Inner Fiber	Central Axis	
0°	.4548Q	+.5985Qd	$+3.8323\frac{Q}{f}$	$-7.2838\frac{Q}{f}$	$+1.0533\frac{Q}{f}$	−.1122
10°	.4697 "	.5836 "	3.7631 "	−7.0762 "	"	−.1145
20°	.5122 "	.5411 "	3.5658 "	−6.4842 "	"	−.1223
30°	.5799 "	.4734 "	3.2509 "	−5.5364 "	"	−.1377
45°	.7162 "	.3371 "	2.6186 "	−3.6425 "	"	−.1832
α	.8000 "	.2533 "	2.2294 "	−2.4752 "	"	−.2300
	"	"	2.6274 "	−1.4676 "	$+.8724\frac{Q}{f}$	−.2050
60°	.866 "	.02230 "	1.0269 "	+.6663 "	"	−1.4480
70°	.9397 "	−.2357 "	−.6610 "	+3.0502 "	"	+.2504
80°	.9848 "	−.3935 "	−1.8552 "	+4.5089 "	"	+.1458
90°	1.0000 "	−.4467 "	−2.2236 "	+4.9990 "	"	+.1278

TABLE X.

Maximum Tensile Stress in Stud Link

Length = 6d.

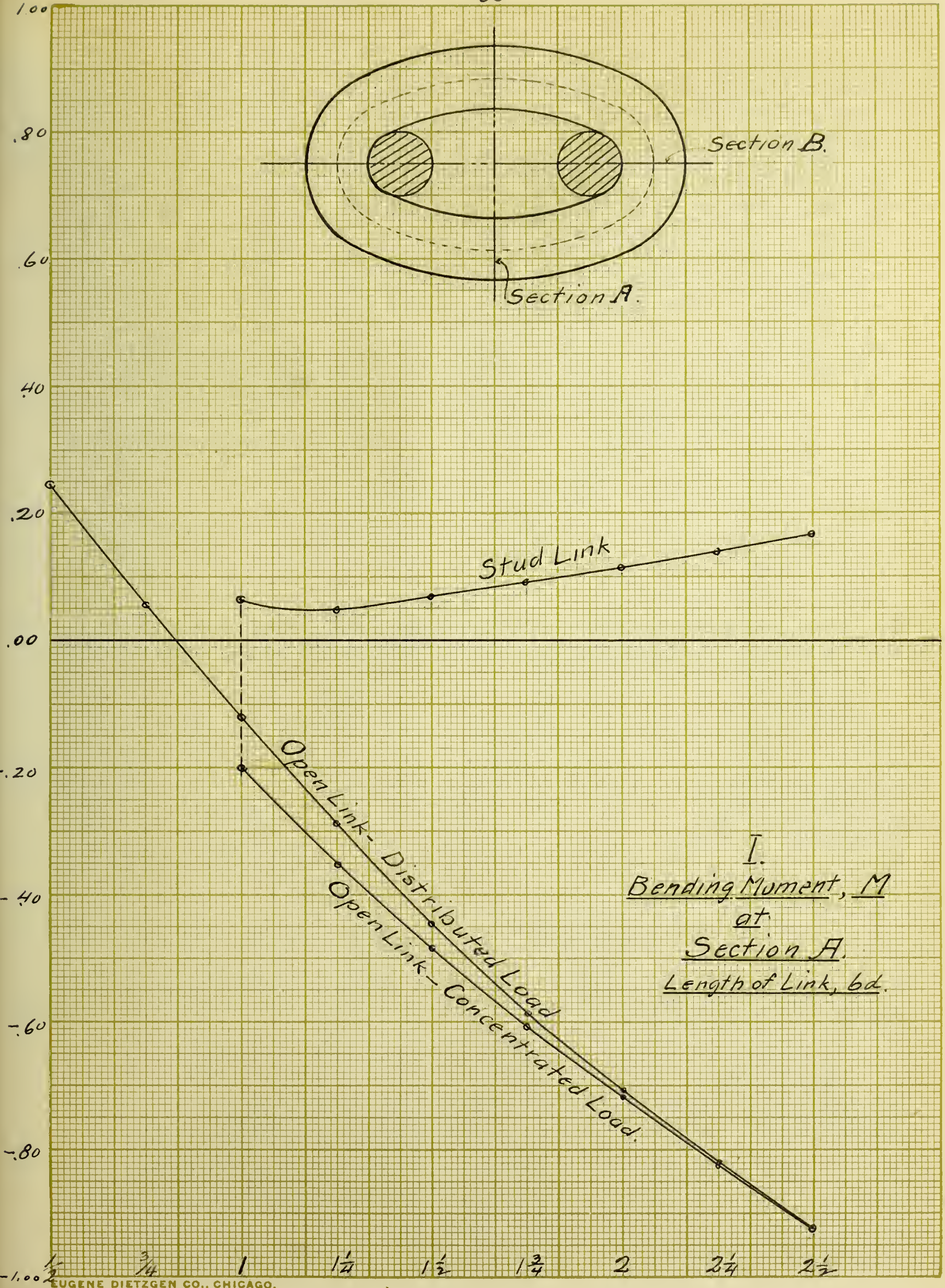
$\frac{b}{d}$	$\frac{r}{d}$	P	M	M_b	$\frac{d}{r}\left(1+\frac{1}{x_2}\frac{\eta}{r+\eta}\right)$	Stress in Inner Fiber	Angle φ of Section of Maximum Stress
$1\frac{1}{4}$	$\frac{45}{8}$	1.0298 Q	.0585 Qd	−.1088 Qd	− 8.5680	1.9619 $\frac{Q}{f}$	76° 11′ 45″
$1\frac{1}{2}$	$\frac{7}{2}$	1.0637 "	.0696 "	−.1534 "	− 8.9523	2.4370 "	70° 4′ 30″
$1\frac{3}{4}$	$\frac{23}{8}$	1.10572 "	.0912 "	−.2127 "	− 9.1892	3.0603 "	64° 44′ 30″
2	$\frac{21}{8}$	1.14613 "	.1160 "	−.2676 "	− 9.3214	3.6405 "	60° 45′ 4″
$2\frac{1}{4}$	$\frac{101}{40}$	1.1827 "	.1398 "	−.3215 "	− 9.3826	4.1992 "	57° 44′ 23″
$2\frac{1}{2}$	$\frac{5}{2}$	1.2146 "	.1672 "	−.3693 "	− 9.4000	4.6860 "	55° 16′ 55″

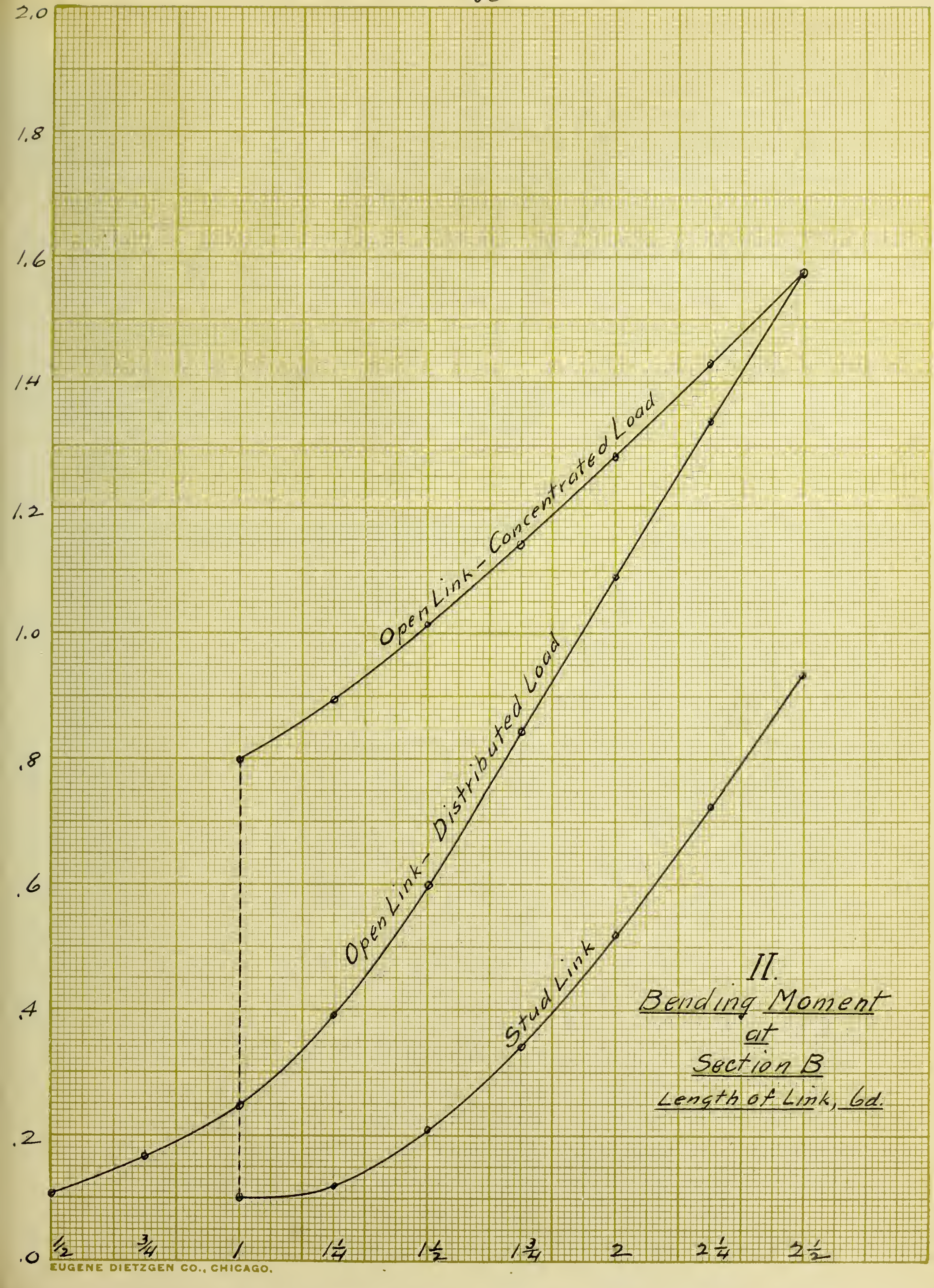
TABLE XI.

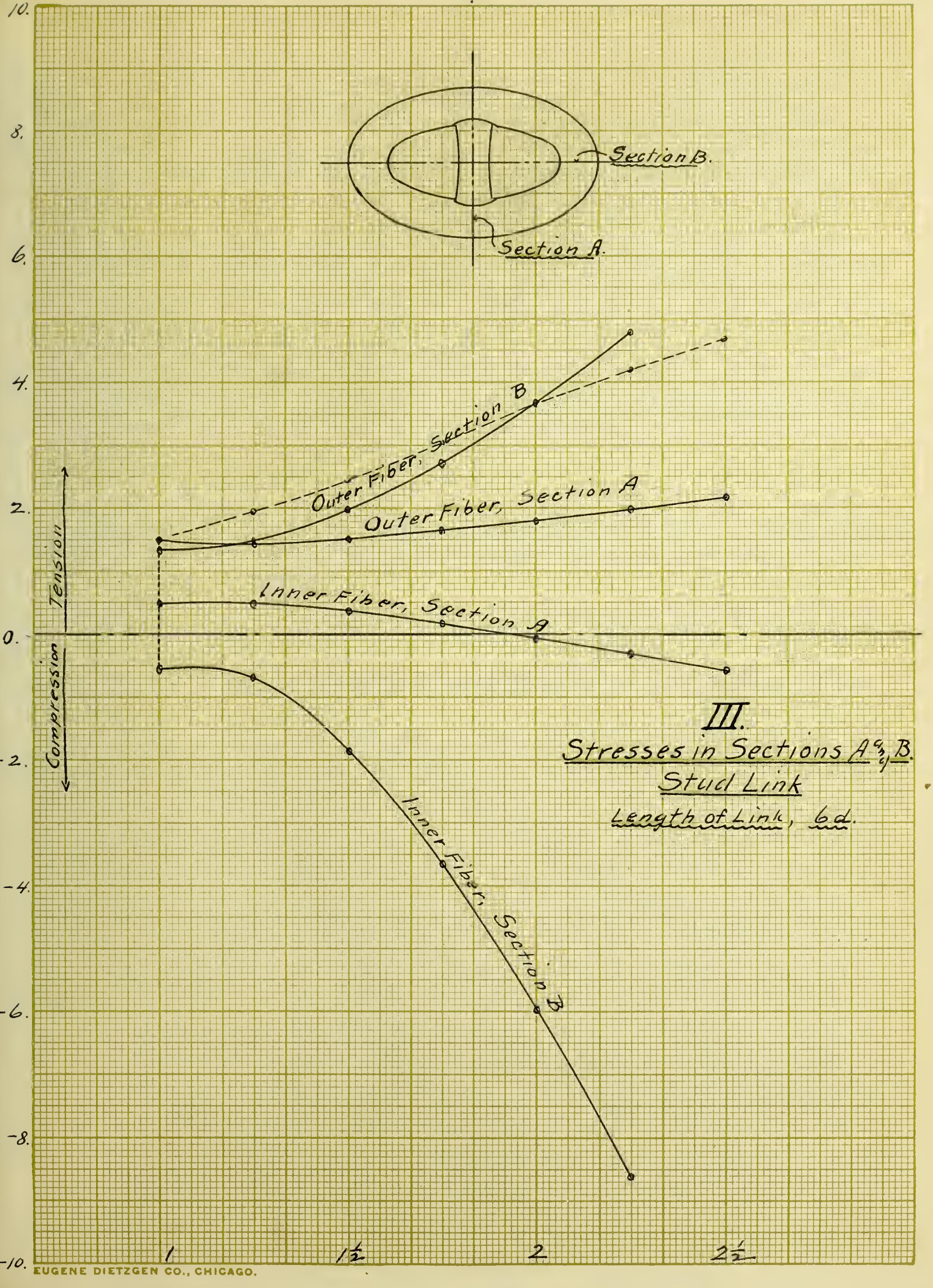
Moments and Stresses at Sections A and B.

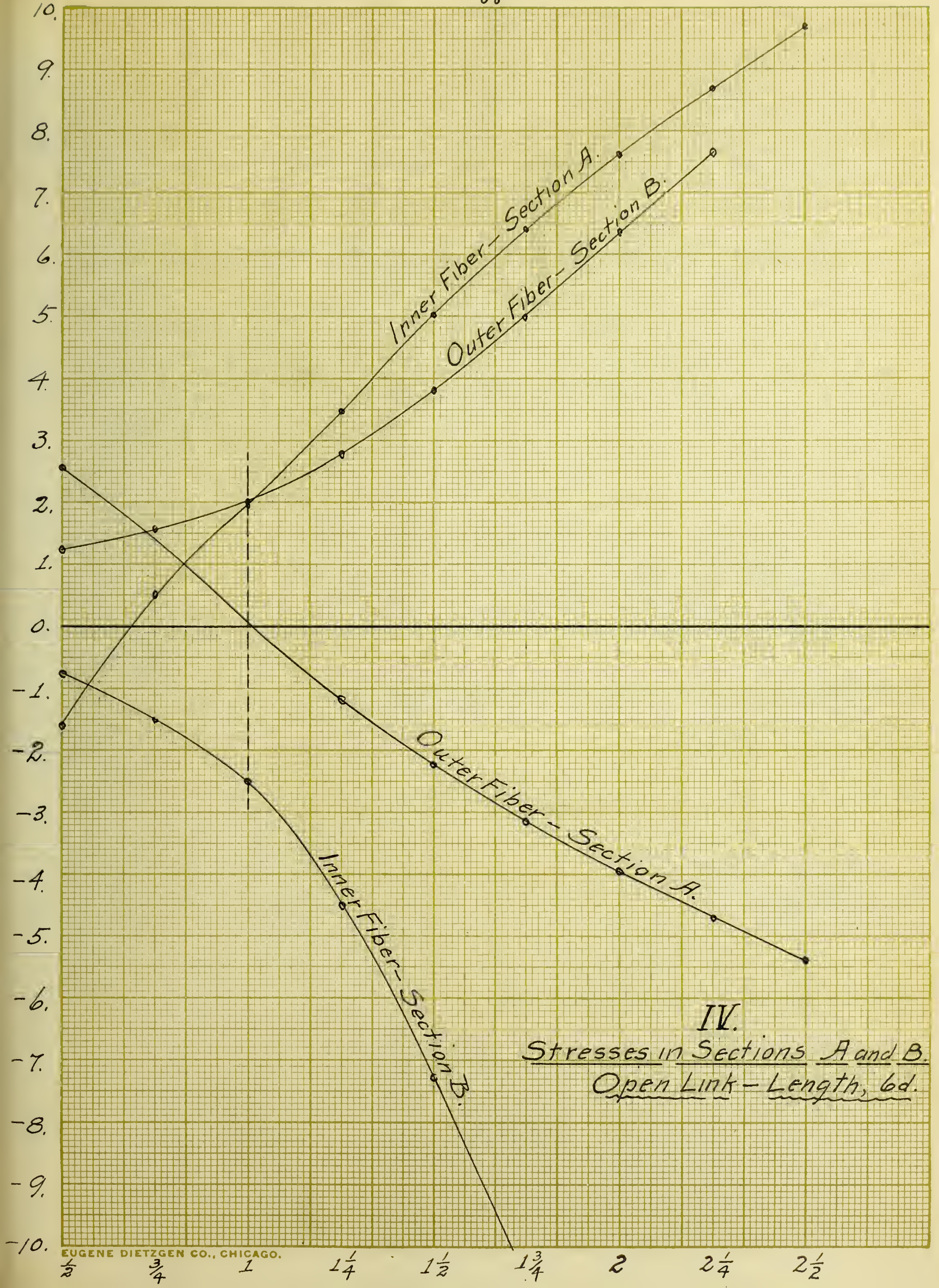
Open Link, Width 3.5d.

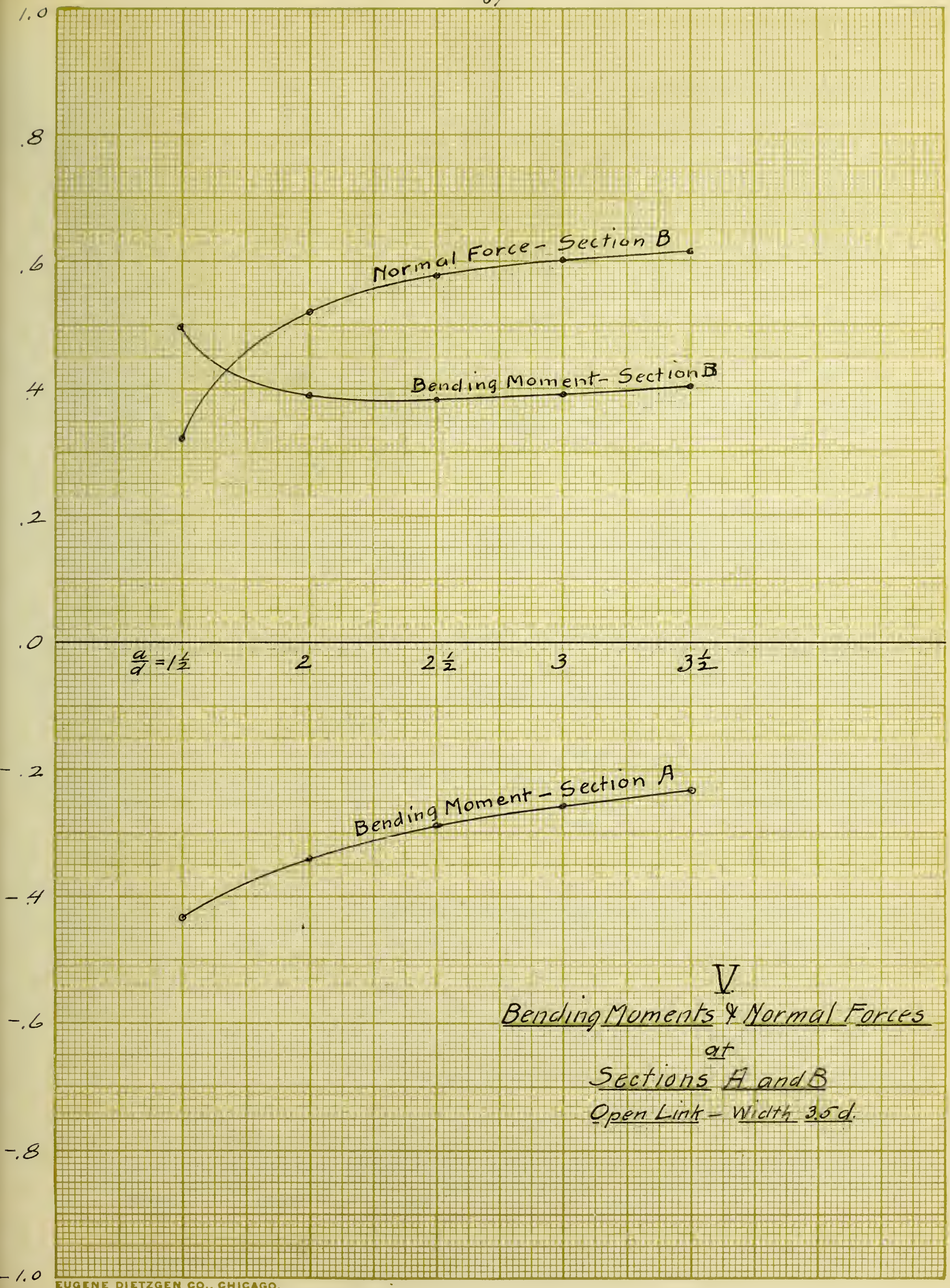
$\frac{a}{d}$	$\frac{r}{d}$	$\frac{1}{x_2}$	Moment at Section A	Moment at Section B	Normal Force at Section B	Stresses, Section A		Stresses, Section B	
						Outer Fiber	Inner Fiber	Outer Fiber	Inner Fiber
$1\frac{1}{2}$	$\frac{13}{8}$	40.244	−.4321 Qd	+.4974 Qd	.3205 Q	−1.7774 $\frac{Q}{f}$	+5.4902 $\frac{Q}{f}$	+3.1275 $\frac{Q}{f}$	−6.1109 $\frac{Q}{f}$
2	$\frac{25}{8}$	154.25	−.3406 "	+.3890 "	.5204 "	−1.4279 "	4.0933 "	2.7156 "	−4.5094 "
$2\frac{1}{2}$	$\frac{45}{8}$	504.25	−.2893 "	+.3824 "	.5783 "	−1.1686 "	3.4787 "	2.7363 "	−4.3661 "
3	$\frac{73}{8}$	1330.25	−.2577 "	+.3904 "	.6019 "	−.9798 "	3.1496 "	2.8050 "	−4.4460 "
$3\frac{1}{2}$	$\frac{109}{8}$	2968.25	−.2333 "	+.4030 "	.6137 "	−.8162 "	2.9191 "	2.8880 "	−4.5971 "


1.00
.80
.60
.40
.20
.00
-.20
-.40
-.60
-.80
-1.00
Section B.
Section A.
Stud Link
Open Link - Distributed Load
Open Link - Concentrated Load.
I.
Bending Moment, M
at
Section A.
Length of Link, 6d.
1/2
3/4
1
1 1/4
1 1/2
1 3/4
2
2 1/4
2 1/2


2.0
1.8
1.6
1.4
1.2
1.0
.8
.6
.4
.2
.0
1/2
3/4
1
1 1/4
1 1/2
1 3/4
2
2 1/4
2 1/2
Open Link – Concentrated Load
Open Link – Distributed Load
Stud Link
II.
Bending Moment
at
Section B
Length of Link, 6d.
EUGENE DIETZGEN CO., CHICAGO.


10.
8.
6.
4.
2.
0.
-2.
-4.
-6.
-8.
-10.
Section B.
Section A.
Outer Fiber, Section B
Outer Fiber, Section A
Inner Fiber, Section A
Inner Fiber, Section B
Tension
Compression
1
1½
2
2½
III.
Stresses in Sections A & B.
Stud Link
Length of Link, 6d.

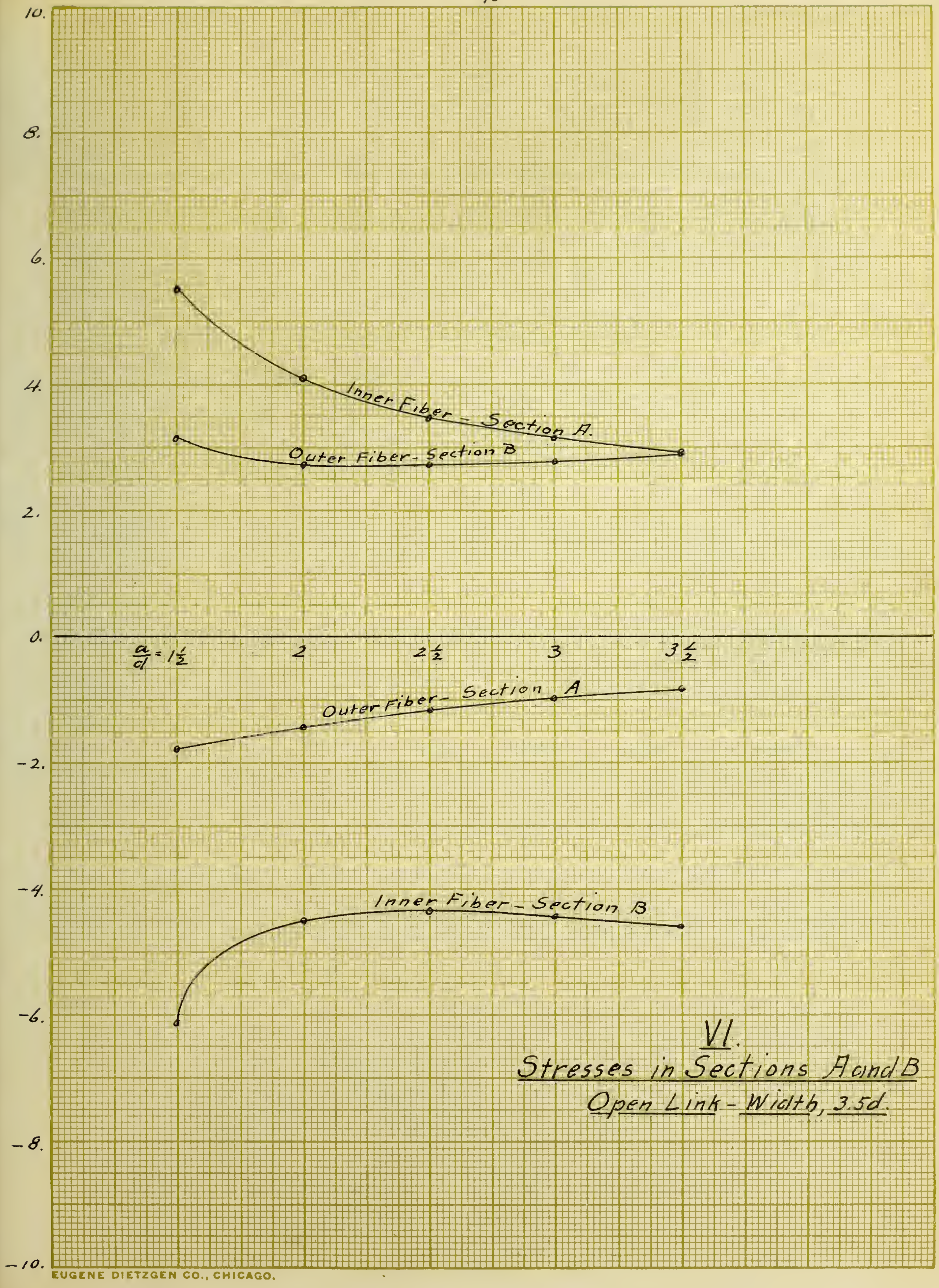

10.
9.
8.
7.
6.
5.
4.
3.
2.
1.
0.
-1.
-2.
-3.
-4.
-5.
-6.
-7.
-8.
-9.
-10.
1/2
3/4
1
1 1/4
1 1/2
1 3/4
2
2 1/4
2 1/2
Inner Fiber – Section A.
Outer Fiber – Section B.
Outer Fiber – Section A.
Inner Fiber – Section B.
IV.
Stresses in Sections A and B.
Open Link – Length, 6d.
EUGENE DIETZGEN CO., CHICAGO.



V

Bending Moments & Normal Forces at Sections A and B

Open Link – Width 3.5d.



VI.

Stresses in Sections A and B

Open Link - Width, 3.5d.