GOUWENS

The Groups of Isomorphisms of Groups of Degree Eight and of Order Less than Forty-Eight

Mathematics

A. M.

1911

THE GROUPS OF ISOMORPHISMS OF GROUPS OF DEGREE EIGHT AND OF ORDER LESS THAN FORTY-EIGHT

BY

CORNELIUS GOUWENS
B. S., Northwestern University, 1910

THESIS

Submitted in Partial Fulfillment of the Requirements for the

Degree of

MASTER OF ARTS

IN MATHEMATICS

IN

THE GRADUATE SCHOOL

OF THE

UNIVERSITY OF ILLINOIS

1911

5

1911
G74

UNIVERSITY OF ILLINOIS

THE GRADUATE SCHOOL

June 3, 1911. 190

I HEREBY RECOMMEND THAT THE THESIS PREPARED UNDER MY SUPERVISION BY

Cornelius Gouwens

ENTITLED THE GROUPS OF ISOMORPHISMS OF GROUPS OF DEGREE EIGHT AND ORDER LESS THAN FORTY EIGHT.

BE ACCEPTED AS FULFILLING THIS PART OF THE REQUIREMENTS FOR THE

DEGREE OF Master of Arts.

G. A. Miller
In Charge of Major Work

E. J. Townsend
Head of Department

Recommendation concurred in:

Committee
on
Final Examination

THE GROUPS OF ISOMORPHISMS OF GROUPS OF DEGREE EIGHT AND ORDER LESS THAN FORTY EIGHT.

Contents.

I. INTRODUCTION.

Poincaré in his article published in the Monist[1] has shown how the group concept is connected with the most ancient mathematical thought. The group idea, however, was first explicitly used by Lagrange[2] where he considers the permutations of letters and their use in the solution of equations. Vandermonde[3] also used group theory in the solution of algebraic equations.

In the latter part of the tenth century Ruffini worked on the solution of algebraic equations of order higher than four

(1) Poincaré - Monist, 9, (1898), pp. 1-43.
(2) Lagrange - Oeuvres, 3, pp. 205-421.
(3) Vandermonde -"Memoire sur la résolution des equations," Histoire de l'academie des Sciences, Paris, 1771, pp. 365-414.

by means of groups or the permutations of letters among themselves. Started by these men the final definition of groups was developed in the long stretch of a century. The following definition: "A set consisting of a finite number of substitututions such that the product of any two (identical or distinct) of the set equals a substitution of the set, is termed a group of substitutions" was given by Galois (1811-1832).

The study of abstract groups was of a later date. Jordan was one of the first to make any considerable study of these groups and their properties. Klein also dealt with groups other than substitution groups in some of his early memoirs. The honor of the first explicit statements in reference to abstract groups is, however, due to Cayley[1] with this dictum: A group is defined by means of the laws of the combinations of its symbols." The earliest explicit set[6] of postulates for abstract groups were given by Kronecker[2] and Weber[3]. Weber's definition was somewhat simplified by Burnside[4] and more explicitly by Pierpont[5] and others.

Next we come to isomorphisms. If we arrange the elements of a group in two orders and if these arrangements are made so that in them corresponding elements have the same law of combination, they are said to define an isomorphism of the group with itself.

(1) Cayley - Philosophical Magazine, Vol. 7 (1854) p. 40.
American Journal of Mathematics, Vol. 1, (1878) p. 50.
(2) Kronecker - Monatsberichte der königlich preusschen Akademie der Wissenschaften zu Berlin, 1870, p. 882.
(3) Weber - Mathematische Annalen, Vol. 20 (1882) p. 521.
(4) - Burnside - Theory of Groups of Finite Order, 1897, p. 11.
(5) Pierpont - Annals of Mathematics, Ser. 2, Vol. 2 (1900-01) p. 47.
(6) See E. V. Huntington - Transactions of the American Mathematical Society, Vol. 6 (1895) p. 181.

This is often called an automorphism, the term automorphism being due to Frobenius. For example, let us take the elements of a group G as

$$s_1\ (=1),\ s_2,\ s_3,\ - - - - -,\ s_n.$$

In general it is possible to rearrange the operators in a different way

$$s_1',\ s_2',\ - - - - - - - - -,\ s_n'.$$

but not affecting the multiplication table so that

$$s_p\ s_q = s_r$$
$$s_p'\ s_q' = s_r'$$

taking any arbitrary values for p and q. Each such arrangement represents an isomorphism. These isomorphisms are divided into two classes: the cogredient or inner and the contragredient or outer. It is called cogredient when the isomorphism is obtained by transforming G by an operator of G. All others are contragredient. All isomorphisms of a group are obtained by permuting the elements, and any one isomorphism may be regarded as an operation performed on the elements of a group. That the total of these operations form a group was explicitly stated by Holder(1) and Moore(2). Frobenius(3) showed that all automorphisms of a group can be obtained by transforming it when it is written in the regular form. Each automorphism may be represented as a substitution, and hence two

(1) Holder - Bildung zusammengesetzten Gruppen - Mathematishe Annalen, Vol. 43 (1893) p. 301.
(2) Moore - American Mathematical Society Proceedings, Vol. 1, Ser. 2 (1894-5) p. 61.
(3) Frobenius - Situngsberichte der Akademie der Wissenschaften zu Berlin, Vol. 1 (1895) p. 184.

successive automorphisms may be represented as the product of two substitutions. These substitutions must form a group.

Two years later Burnside(1) gave a proof of this theorem. He also showed that the cogredient isomorphisms are transformed into themselves by all other isomorphisms and hence form a subgroup invariant under I where I represents the group of isomorphisms.

The importance of the groups of isomorphisms was first brought into prominence by the early writings of Holder(2) and of Moore(3), who independently of each other discussed groups of isomorphisms and some of their properties. The distinction between cogredient and contragredient isomorphisms had, however, been discussed at an earlier time by Klein(4). It was but a short time after these two men had studied some of the properties of these groups of isomorphisms that Burnside published his article in the "Proceedings of the London Mathematical Society" which brought out new properties and immediately created more interest in this new subject.

We wish to mention one of these special properties. Using the general symbol

$$\begin{pmatrix} s \\ s' \end{pmatrix}$$

to define an isomorphism as

$$\begin{pmatrix} s_1, & s_2, & ----, & s_p, & -----s_n \\ s_1', & s_2', & ----, & s_p', & -----s_n' \end{pmatrix}$$

and supposing that $s_p\, s_q = s_r$

(1) Burnside - Proceedings of London Math. Soc.,Vol.27(1895-96),p.354
(2) Holder - Mathematische Annalen, Vol. 43 (1893), p. 313.
(3) Moore, E.H. - Bulletin of the American Mathematical Society, Ser. (2), Vol. 1 (1894-5), p. 61.
(4) Klein - "Vorlesungen uber das Ikosaeder" (1884).

where G equals

$$s_1(=1),\ s_2,\ \text{------}\ s_n$$

we can take two operations in I as

$$\begin{pmatrix} s \\ s_p^{-1}\ s\ s_p \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} s \\ s_q^{-1}\ s\ s_q \end{pmatrix}$$

Multiplying these we have

$$\begin{pmatrix} s \\ s_p^{-1}\ s\ s_p \end{pmatrix} \begin{pmatrix} s \\ s_q^{-1}\ s\ s_q \end{pmatrix} = \begin{pmatrix} s \\ s_p^{-1}\ s\ s_p \end{pmatrix} \begin{pmatrix} s_p^{-1}\ s\ s_p \\ s_p^{-1}\ s_q^{-1}\ s\ s_q\ s_p \end{pmatrix}$$

$$= \begin{pmatrix} s \\ s_r^{-1}\ s\ s_r \end{pmatrix}.$$

This shows that the group of cogredient isomorphisms is isomorphic with the original group G. When G contains no invariant operator we see that

$$\begin{pmatrix} s \\ s_p^{-1}\ s\ s_p \end{pmatrix}, \quad \begin{pmatrix} s \\ s_q^{-1}\ s\ s_q \end{pmatrix}$$

can be identical only when

$$s_p = s_q.$$

In that case the group of cogredient isomorphisms is of the same order as G. There is then said to be a holohedric isomorphism between the two groups. If G contains invariant operators, these operators will correspond to themselves or to each other in every isomorphism. Suppose we have h invariant operators in G. They form an invariant subgroup H, and all other operators are transformed into themselves. The group of cogredient isomorphisms is of order not greater than g/h and is said to be merihedrically isomorphic with G.

We might add here that the property that the group of cogredient isomorphisms could not be cyclical was not discovered

until 1899.[1]

II. FUNDAMENTAL THEOREMS.

A complete list of the substitution groups whose degree does not exceed eight has been given by G. A. Miller[2]. It is our object in this paper to study the groups of isomorphisms of thegroups of degree eight and of order less than forty-cight. I wish here to express my thanks to Professor G. A. Miller, under whose direction this paper was written, for the help he has given me in preparing this paper. The groups of isomorphisms of the groups of degree less than eight have already been published[3]. We wish to make use of several of the theorems given in the last article which are frequently used.

Theorem I:[6] If a group is generated by two characteristic[4] subgroups which have only the identity in common, its I is the direct product of the I's[5] of these two characteristic subgroups and its K is the direct product of their K's.

Corollary I: The I of a cyclic group of order p^{α}, p geing an odd prime, is the cyclic group of order $p^{\alpha-1}(p - 1)$.

Corollary II: The I of a cyclic group of order 2^{α}, $\alpha > 1$, is the direct product of the group of order 2 and the cyclic group of order $2^{\alpha-2}$.

(1) Miller - Comptes Rendus, Vol. 128 (1899) p. 229.
(2) G. A. Miller - American Journal of Mathematics, Vol. 21 (1899) p. 326.
(3) G. A. Miller - Philosophical Magazine, Ser. 6, Vol. 15 (1908) p. 223.
(4) See Frobenius - Sitzungsberichte der Akademie der Wissenschaften zu Berlin, Vol. 1, (1895) p. 183.
(5) I is the symbol generally used to represent the group of isomorphisms and K to represent the holomorph.
(6) This theorem is also given in the Transactions of the American Mathematical Society, Vol. 1 (1900) p. 396.

Theorem II: The symmetric group of degree n, $n \neq 2$, or 6, is simply isomorphic with its I, and the alternating of degree n, $n \neq 3$, has the same group of isomorphisms as the symmetric group of the same degree.

Let us assume it true for (n - 1) as the degree of the symmetric group and prove it true for n. We know that the symmetric group has n conjugate subgroups of degree (n - 1). We will show that if we fix the isomorphism between any two of these subgroups, all the isomorphisms are fixed. Call these subgroups G_{n-1}, G'_{n-1}, etc. We can make G_{n-1} isomorphic with itself. Now take any transposition as <u>ah</u> where <u>a</u> is in G_{n-1} but <u>h</u> is not. We will now prove that the operator corresponding to <u>ah</u> is <u>ah</u> itself. Let us represent the isomorphic G_{n-1} by G'_{n-1} and the operator corresponding to <u>ah</u> by $(\underline{ah})'$. Now, when we transform G_{n-1} by <u>ah</u>, <u>a</u> goes into <u>h</u> in every operator, but the other letters remain fixed. also, <u>ah</u> is transformed into itself. Since G_{n-1} is a symmetric group of degree (n - 1), it has (n - 1) symmetric conjugate subgroups of degree (n - 2). If we take this subgroup omitting both <u>a</u> and <u>h</u>, we have G_{n-2} commutative with <u>ah</u>. Therefore $(ah)'$ must be commutative with G'_{n-2}. This immediately fixes $(ah)'$ and if we take as G'_{n-2} the subgroup of G'_{n-1}, omitting <u>a</u>, the transposition $(ah)'$ must be <u>ah</u> or it would not be commutative with G'_{n-2}. But we assume that the total number of isomorphisms of G_{n-1} symmetric are (n - 1)! Since there are <u>n</u> subgroups of degree (n - 1), and a G_n can have only n times the order of I of G_{n-1} for the order of its I, this order is n(n - 1)! = n! This proves it true for (n - 1) = 7 so that G is written on the letters

a, b, c, d, e, f, g

There are $\dfrac{7\cdot 6\cdot 5\cdot 4\cdot 3\cdot 2}{6}$ substitutions of the form abcdef.

There are 6 substitutions of the form ag where g is fixed. But abcdef and ag generate the group and the I cannot be of order greater than the number of ways that the generators may be chosen. This is 7! . There might, perhaps, be substitutions of the form abc.de corresponding to abcdef since both are of the same order. The second is transformed into itself only by its powers, the first by its powers and also by substitutions in fg, hence they have a different number of conjugates and so could not correspond.

This method of proof fails when $n = 6$ because there are then more subgroups of index n than the n symmetric subgroups of degree $(n - 1)$.

Theorem III: If an abelian group G which involves operators whose orders exceed 2 is extended by means of an operator of order 2 which transforms each operator of G into its inverse, then the I of this extended group is the K of G.

Corollary I: The group of isomorphisms of the dihedral group of order 2n, $n > 2$, is the holomorph of the cyclic group of order n.

"The group of isomorphisms of this dihedral group may be represented as a transitive substitution group of degree n, and it involves an invariant cyclic subgroup of order n composed of all its operators which are commutative with every operator of the cyclic subgroup of order n."(1)

Theorem IV: If a complete group has only one subgroup of index 2, the direct product formed with it and the group of order 2 is simply isomorphic with its group of isomorphisms.

(1) Miller - Lecture Notes given in year (1910-1911).

Corollary I: The direct product of the symmetric group of order n, $n \neq 6$, and the group of order 2 is simply isomorphic with its I.

Corollary II: The direct product of the metacyclic group of order $p(p - 1)$, p being any odd prime, and the group of order 2 is simply isomorphic with its I.

III. THE GROUPS WHOSE ORDER IS SIXTEEN OR LESS.

We shall now consider the groups of isomorphisms of some of these groups. In this article in the American Journal[1] all distinct groups are denoted by Greek letter while the isomorphic groups are represented by Roman letters. Since these have the same I as the groups with which they are isomorphic, we need consider only the distinct groups. The groups of isomorphisms of groups isomorphic with lower degree are known[2], all others isomorphic with groups of degree eight have I's identical with the distinct group of degree eight.

The I of the cyclic group of order eight is the four group by the theorem I, corollary II. The I of the quatemion group is the symmetric group of order 24.[3]

The group of order 15 is the direct product of two cyclic groups of order 5 and 3. Their group of isomorphisms is the direct product of the I's of their factors and so is the product of a group of order 2 and the cyclic group of crder 4.

(1) Miller - American Journal, Vol. 21 (1899), p. 287.
(2) Miller - Philosophical Magazine Ser. 6, Vol. 15 (1908), p. 223.
(3) Miller - American Philosophical Society Proceedings, Vol. 37 (1898), p. 315.

There are nine groups of order 16. The abelian group of type (1, 1, 1, 1,) has a group of order $(2^4 - 1)(2^4 - 2)(2^4 - 2)(2^4 - 2^3) = 20160$. These factors represent the number of ways in which the generators may be chosen. The I is isomorphic with the alternating group of degree 8[1]. The abelian group of type (2,1,1) has a group of order 192 for its I. If we select a set of generators from its operators of order 4, the first generator may be chosen in 8 ways, the second in 6, and the third in 4 ways. These three generate the group. The group of isomorphisms can be written on eight letters, each letter representing one of the eight operators of order 4 in G. I has four systems of imprimitivity because the operators of order 4 in G come in pairs, each operator with its inverse. These 4 systems of imprimitivity can at most be permuted according to the symmetric group on 4 letters. This would give a group of order 384. I is therefore a subgroup of this G_{384}. We cannot have a transposition in I because the operators in our group are always permuted at least four at a time. We can therefore choose only eight operators from our head for I. Transforming these according to the symmetric group, we have our G_{192}. These eight operators in the head were positive and when transformed according to the alternating group gives 96 positive substitutions. The remaining operators of the symmetric group are negative and of order 2 or 4. The operators in I formed by transforming the head by the alternating group are of order greater than 2 and positive. We can, however, get an

(1) Burnside - Theory of Groups (1897) p. 339.
Miller - American Journal of Mathematics, Vol. 20 (1898) p.320.
E. H. Moore - Mathematische Annalen, Vol. 51 (1899) p. 417.

operator or order 2 by multiplying thegroup of order 16 by an operator of order 2 not a square. We then get an operator of order 2 in I but it is positive. We now have more than one-half of the operators in I positive, hence all are positive and our I must be the subgroup of G_{384} containing all positive substitutions.

The I of the abelian group of type (2, 2) is of order 96. The first generating operator can correspond in 12 ways and the second can then correspond in only 8 ways. This I has three systems of imprimitivity composed of the four groups, and these can at most be permuted according to the summetric group on 3 letters. But, again there can be no transposition in I, so we take only the positive substitutions from our head. It is also impossible to transform the systems of imprimitivity according to an operator of order 2. Hence our group must be the positive substitutions of a group of order 192 with three systems of imprimitivity composed of the four groups, where the systems are permuted according to a cyclic group of order 3.

The group obtained by dimitiating the product of the cyclic group of order 4 and the octic group has a group of order 32 for its I. The substitutions of G are

1	eg	abcd.efgh	abcd.ef.gh
eg.fh	fh	abcd.ehgf	abcd.eh.fg
ac.bd	ac.bd.eg	adcb.efgh	adcb.ef.gh
ig.fh.ac.bd	ac.bd.fh.	adcb.ehgf	adcb.eh.fg

The first generating operator can be made to correspond in 8 ways and the second in four ways. These generate G. The commutator is of order 2 and always corresponds to itself in every isomorphism and so the two systems of imprimitivity must be the four-groups. These are permuted according to a group of order 2. Lettering the operators of order 4 by the letters A, B, C, D, E, F,

G, H, respectively, the head of I is

1	1
AB.CD	EF.GH
AC.BD	EG.FH
AD.BC	EH.FG

We find that an operator t = AE.BF.CH.DG. permutes the systems. The group (H, t) is therefore of order 32, with 19 operators of order 2, identity, and 12 operators of order 4 with three distinct squares. It is the group numbered 11 of crder 32 and degree 8 by G. A. Miller[1].

A second dimitiation of the same two groups is the group

1	efgh	abcd.eg	abcd.ef.gh
ac.bd	ac.bd.efgh	adcb.eg	adcb.ef.gh
eg.fh	ehgf	abcd.fh	abcd.eh.fg
ac.bd.eg.fh	ac.bd.ehgf	adcb.fh	adcb.eh.fg

The I is a group of order 32 and degree 8. We have two subgroups of order 4 in G which are invariant, the others are no-invariant, and our I is written on letters representing these invariant operators. In selecting the generators the first may chosen in 8 ways and the second in 4 ways. I has two systems of imprimitivity the octic groups. They must be taken in a 2 : 2 isomorphism, since the commutator is of order 2 while the operators of order 2,of which the commutator is a subgroup, form an invariant subgroup and so permutations in one system of imprimitivity must correspond to identical permutations in the other system. These two systems can be permuted cyclically among themselves by an operator of order 2. The I is therefore number 12 as given by Miller[1]. These include all of the intransitive groups. There are only four distinct transitive groups of order 16 and degree 8 not isomorphic with other groups.

(1) G. A. Miller - American Journal of Mathematics, Vol. 21 (1899) p. 332.

The group denoted by

$$\left((abcd)\ cyc\ (efgh)\ cyc \right) pos\ \ (ae.bf.cg.dh)$$

has the following substitutions:

1	abcd.efgh	ae.bf.cg.dh	afch.bgde
ac.bd	adcb.efgh	agcd.bhdf	ah.cf.be.dg
eg.fh	abcd.ehgf	ecga.fdhb	af.bg.ch.de
ac.bd.eg.fh	adcb.ehgf	ag.ce.bh.df	ahcf.bedg

This has one invariant cyclic group of order 4 and an invariant quaternion group. The I of the quaternion group is the symmetric group of order 24. The two operators of order 4 in the invariant cyclic group can correspond in two ways. Hence the I of the group is a group of degree 8, the direct product of the symmetric group of order 24 written on six letters and a group of order 2.

The group denoted by

$$\left((abcd)\ cyc\ (efgh)\ cyc \right) pos\ \ afbgchde$$

has the substitutions:

1	abcd.efgh	afbgchde	agdfcebh
ac.bd	adcb.efgh	abdgcfbe	aebfcgdh
eg.fh	abcd.ehgf	echbgafd	agbhcedf
ac.bd.eg.fh	adcb.ehgf	ahbecfdg	aedhcgbf

This has a cyclic group of order 8 for a head and a tail where each operator transforms the head into its fifth power. Again lettering the operators of order eight by A, B, C, D, E, F, G, H, we see that I must have 4 systems of imprimitivity of order 2 as

1	1	1	1
AB	CD	EF	GH

where the square of A is the inverse of the square of B, and similarly for the other three sets. We can have no transposition in I, if, say, A is replaced by B, F must be replaced by E, as is seen by noting the position of the squares of the operators of order 8. We then see that an operator I permuting the systems is AE.BF.CG.DH. This gives a group of order 16 with 11 operators of

order 2, and four of order 4 all having the same square. The I is a two to two isomorphism of two octic groups written so

1	1
ab.ef	cd.gh
afbe	chdg
aebf	cgdh
ab	cd
ef	gh
af.be	ch.dg
ae.bf	cg.dh

Of the two remaining transitive groups of order 16 formed by extending an isomorphism of two octic groups by means of operators of order 8, the first is denoted by

$$(abcd.efgh)8_2\ (aebfcgdh)$$

Its substitutions are

1	aebfcgdh
abcd.efgh	afdechbg
ac.bd.eg.fh	agbhcedf
adcb.ehgf	ahdgcfbe
ac.ef.gh	ag.ce.fb.hd
ab.cd.fh	af.be.ch.dg
bd.eh.fg	bh.df.ea.eg
ad.bf.cf	ah.de.bg.ef

and the commutator is

1
abcd.efgh
ac.bd.eg.fh
adcb.ehgf

The generators can be shosen in 8 x 4 ways. The I is a group of order 32 and degree 8, representing the eight no-invariant operators of order 2. It has two systems of imprimitivity, and from the form of the commutator we see that they must be the four groups. They are transformed according to an operator as AEDHCGBF. The group is the dihedral group of order 16 and its I is the holomorph of the cyclic group of order 8 according to the theorem.

Its substitutions are:

1	AEDHCGBF
AB.CD	AF.BEDG.CH
AC.BD	AGBHCEDF
AD.BC	AHCF.DE.BG
EF.GH	EA.FDHB.GC
AB.CDEFGH	AFDGCHBE
AC.BD.EF.GH	AGCE.BHDF
AD.BC.EF.GH	AHBGCFDE
EG.FH	EBFCGDHA
AB.CD.EG.FH	AFCH.BE.DG
AC.BD.EG.FH	AGDFCEBH
AD.BC.EG.FH	AH.DEBG.FC
EH.FG	ECGA.HD.FB
AB.CD.EH.FG	AFBECHDG
AC.BD.EH.FG	AG.CE.BHDF
AD.BC.EH.FG	AHDECFBG

The substitutions of the other group are:

1	afbgchde
abcd.efgh	agdfcebh
ac.bd.eg.fh	ahbecfdg
adcb.ehgf	aedhcgbf
ac.ef.gh	ahcf.ebgd
ab.cd.fh	agce.bfdh
bd.eh.fg	bedg.hafc
ad.bc.eg	aecg.dfbh

This is a group with a cyclic group of order 8 for its head and a tail which transforms the head into its third power. In this group the I must be represented on eight letters including the four operators of order 8 and either the 4 operators of order 4 or the 4 non-invariant operators of order 2. This group has the same commutator subgroup as the preceding one. Taking the cyclic group of order 8 as the head of G, we have the I of the head as the four-group. The next four operators all transform the head in the same way; hence the group of isomorphisms keeping the head fixed is the cyclic group of order 4. The group of isomorphisms of the whole group is therefore the direct product of the cyclic group of order 4 and the four-group.

IV. THE GROUPS OF ORDER EIGHTEEN, TWENTY FOUR, AND THIRTY.

The only distinct abstract group of order 18 is the abelian group of type (p, p, q). It is the direct product of two cyclic groups of order 3 and a group of order 2. It has therefore two characteristic subgroups and its I is the product of the I's of these subgroups. The I is the transitive group of degree 8 and order 48(1) which has operators of order 8 which is the I of the group of order 9 of type (1, 1).

The group of order 24 is the one often termed the non-twelve G_{24}. It contains the quaternion group invariantly. Its substitutions are:

1	abc.def	acb.dfe
ad.fc.eb.gh	aecdbf.gh	afdce.gh
afdc.egbh	ad.fcgebh	aeg.dbh
acdf.ehbg	ceh.bgf	abgdeh.cf
abde.fgch	achdfg.be	bfg.ech
aedb.fhcg	afh.dcg	ad.efhbcg
agdh.fecb	age.dhb	agfdhc.be
ahdg.fbce	ahedgb.cf	ahf.dgc

This is a group with one invariant operator of order 2, hence we know that its group of cogredient isomorphisms is the quotient group with respect to this invariant subgroup. This is the alternating group of order 12. The group of isomorphisms of the quaternian group is the symmetric group of order 24 and its holomorph is a group of order 192, the direct product of the quaternion group and its I. To every operator of order 3 in I corresponds a group of order 24 in K. These operators of order 3 are conjugate. Hence these groups of order 24 are conjugate. These groups of order 24 are isomorphic with the non-twelve G_{24} which we are studying. The tetrahedral group written on 6 letters

(1) G. A. Miller - Philosophical Magazine, Series 6, Vol.15 (1908) p.228.

transforms the quaternion group into itself but the operators of order 3 in G are not permuted. We wish to find an operator which permutes these substitutions of order 3. We can write G isomorphic so and permute the operators of order 3 as shown.

1	1
ad.fc.eb.gh	ad.fc.eb.gh
afdc.egbh	acdf.ehbg
acdf.ehbg	afdc.egbh
abde.fgch	agdh.fecb
aedb.fhcg	ahdg.fbce
agdh.fecb	abde.fgch
ahdg.fbce	aedb.fhcg
aeg.dfh	bgf.ceh

This gives an operator of order 2, but we see that it permutes the operators of the quaternion group as well as the operators of order 3. It must therefore be an operator of the I of the quaternion group. And since we saw that it was impossible to permute the operators of order 3 in G without also permuting the operators of the quaternion group, we conclude that the I of G must also be the symmetric group of degree 4.

The group of order 30 which is the direct product of the cyclic group of order 5 and the symmetric group of order 6 has the direct product of the I's of these factors for its I. This is the direct product of the symmetric group of order 6 and the cyclic group of order 4 because of the theorems. In a cyclic group of prime order p, the group of isomorphisms is the cyclic group of order (p - 1)(1). The intransitive group which is the direct product of the dihedral group of order 10 and a cyclic group of order 3, has for its I the direct product of the holomorph of the cyclic group of order 5 and a group of order 2. The last group of order 30 is obtained by dimitiating the dihedral group with

(1) Miller - Trans. of American Math. Soc., Vol. IV (1903) p. 158.

the symmetric group of order 6. This group has 15 operators of order 2 and its generators may be chosen at most in 120 ways. Its I is written on 15 letters representing the 15 operators of order 2. Holomorphisms of the group are easily found which furnish the generating operators of I. One operator of order 15 is

AMJBNFCOGDKHELI

where the capital letters represent operators as shown:

1	be.cd.fg	A
abcde	ae.bd.fg	B
acebd	ad.bc.fg	C
adbec	ac.de.fg	D
aedcb	ab.ce.fg	E
fgh	be.cd.gh	F
abcde.fgh	ae.bd.gh	G
acebd.fgh	ad.bc.gh	H
adbec.fgh	ac.de.gh	I
aedcb.fgh	ab.ce.gh	J
fhg	be.cd.fh	K
abcde.fhg	ae.bd.fh	L
acebd.fhg	ad.bc.fh	M
adbec.fhg	ac.de.fh	N
aedcb.fhg	ab.ce.fh	O

An operator of order 10 is

ADBEC.FNGOHKILJM

and an operator of order 4 is

BCED.GHIJ.LMNO

The cube of the operator of order 15 is the square of the operator of order 10. It must therefore have three systems of imprimitivity of degree 5, because the head of G is abelian and the tail transforms the head into its inverse; hence I is the holomorph of the cyclic group according to the theorem.

V. THE GROUPS OF ORDER THIRTY TWO AND THIRTY SIX

There are 10 groups of order 32 not isomorphic with each other or with those of lower degrees. The group which is the direct product of the octic group and the abelian group of type (1,1) and order 4 has an abelian head, the direct product of the cyclic group of order 4 and the four group. The operators of the tail transforms the head into its inverse, hence by our theorem, the I is the holomorph of our head. The I of the head is found under the groups of order 16, and is a positive group of order 192. Hence I is the direct product of this group of order 192 and the head of order 16.

If we multiply the octic group by the cyclic group of order 4 instead of by the four group, we have a group with 20 operators of order 4. Three of the remaining operators of order 2 are invariant.

The substitutions are

1	efgh
ac.bd	ac.bd.efgh
eg.fh	ehgf
ac.bd.eg.fh	ac.bd.ehgf
ac	ac.efgh
bd	bd.efgh
ac.eg.fh	ac.ehgf
bd.eg.fh	bd.ehgf
ab.cd	ab.cd.efgh
ad.bc.	ad.bc.efgh
ab.cd.eg.fh	ab.cd.ehgf
ad.bc.eg.fh	ad.bc.ehgf
adcb	adcb.efgh
abcd	abcd.efgh
adcb.eg.fh	adcb.ehgf
abcd.eg.fh	abcd.ehgf

Four of the operators of order 4 are invariant, eight are conjugate in sets of 4 each and eight are non invariant. These eight non invariant operators generate half of the group. In selecting the generators the first may be chosen in 8 ways and

and the second in four ways. The I of this head is a group found in our groups of order 16 and degree 8 which was the second dimin-ition of the octic group and the cyclic group. As the remaining generators we take one of the conjugate operators of order 4 not included in this head. This generating operator may be chosen in 4 ways, and in every isomorphism, with the head fixed, can be per-muted according to the four group. Hence I of the group is an in-transitive group on 12 letters the direct product of a four group and a group of order 32, the I of the he d.

Three groups are formed by dimitiating two octic groups. The substitutions are given by Cayley(1). The group

1	1
ac	eg
bd	fh
ac.bd	eg.fh
abcd	ef.gh
adcb	ehgf
abcd	efgh
ad.bc	eh.fg

has 12 operators of order 4 and 19 of order 2, of these three are squares of operators of order 4 and are invariant. The generating operators can be chosen in 12 x 8 x 4 384 ways. I is a group of degree 12 and order 384. The operators of order 4 have three dis-tinct squares, hence I may have 3 systems of imprimitivity and these are the octic groups. These systems are not independent however. If the operators of one system are permuted some opera-tors of the other system are also permuted. The I is therefore an isomorphism between 3 octic groups of which the letters are respec-tively abcd, efgh, ijkl. These three systems are permuted accord-ing to the substitutions

(1) Cayley - Quarterly Journal of Mathematics Vol 25 (1890-91)p137

ajh.fbk.cig.dle
af.bh.ce.dg
af.bh.ce.dg.il.jk

The head of the group of isomorphisms is

1	1	1
ac.bd	eg.fh	ik.jl
ab.cd	ef.gh	ij.kl
ad.bc.	eh.fg	il.jk
abcd	efgh	ijkl
adcb	ehgf	ilkj
ac	eg	ik
bd	fh	jl .

The group

1	1
abcd	efgh
ac.bd	eg.fh
adcb	ehgf
ab.cd	ef.gh
bd	fh
ad.bc	eh.fg
ac	eg

has the holomorph of the head as is I for the head is abelian and the tail transforms the head into its inverse. The I of the head we found to be a group of order 96 hence K is the product of this group of order 96 and our head of order 16.

The group which remains has the substitutions

1	1
abcd	ef.gh
ac.bd	eg.fh
adcb	eh.fg
ac	eg
ab.cd	efgh
bd	fh
ad.bc	ehgf

The 16 operators of order 4 form 2 sets which are not permutable in any isomorphism. One set consists of two invariants subgroups of order 4 and two conjugate groups of order 4. The second set has 4 conjugate subgroups. Keeping the second set fixed the I of the first set is the product of octic group and the

symmetric group of order 24. When this set is fixed the tail can be chosen in 8 ways. Hence the I is of order 1536 the direct product of the octic group, the symmetric group of order 24 and the regular abelian group of order 8 and type (1,1,1).

These include all the intransitive groups of this order. The first transitive group is the direct product of two cycles groups of order 4 and an operator of order 2 which permutes the cycles cyclically. This group includes 8 operators of the order 8 and the commutator subgroup is the group

1
abcd.ehgf
ac.bd.eg.fh
efgh.adcb

where G is

1	abcd	acbd	adcb
abcd.ehgf	acbd.ehgf	adcb.ehgf	ehgf
ac.bd.eg.fh	adcb.eg.fh	eg.fh	abcd.eg.fh
efgh.adcb	efgh	abcd.efgh	ac.bd.efgh
afbgchde	afch.bgde	afdechbg	af.bg.ch.de
agbhcedf	agce.bhdf	agdfcebh	ag.bh.ce.df
ahbecfdg	ahcf.bcdg	ahdgcfbe	ah.be.cf.dg
ebfcgdha	ecga.bfdh	edhcgbfa	ae.bf.cg.dh

The generators of this group are an operator of order 8 and an operator of order 4 whose square is not in the cyclic group generated by the operator of order 8. These generators can be chosen in 8 x 8 or 64 ways. The I is a group of order 64 and degree 8, with two systems of imprimitivity the octic groups and these are transformed according to an operator of order 2 which permutes.the systems cyclically. We cannot have a transposition in the head hence we take the positive substitutions of this group as I.

The remaining groups are a set of six transitive groups formed by making a 2:2 isomorphism between two octic groups and then extending this head by an operator permuting the systems.

The group

1	1
ac.bd	eg.fh
abcd	efgh
adcb	ehgf
ac	eg
bd	fh
ab.cd	ef.gh
ad.bc	eh.fg

ae.bf.cg.dh

has the group (1, ac.bd.eg.fh) for its commutator subgroup. The group contains two conjugate quaternion groups and since all the operators of order 4 are contained in these the I can be written on 12 letters composed of two groups of order 24 isomorphic with symmetric group on 4 letters, and an operator permuting the cycles cyclically.

The group

1	1
ac.bd	eg.fh
abcd	eg
adcb	fh
ac	ehgf
bd	efgh
ab.cd	ef.gh
ad.bc	eh.fg

aebf.cgdh

has 20 operators of order 4. The commutator is a group of order 8 composed only of operators of order 2, and is isomorphic with the abelian group of order 8 and type (1,1,1).

The operators of order 4 are divided into three sets. One set of four operators have as a square an invariant operator of

order 2. Eight operators of order 4 have two conjugate squares of degree 4 and eight have another set of two conjugate squares of degree 8. One of these sets of 8 operators of order 4 generate a group. We can have no automorphism between the operators of one set and those of another set. Hence our generating operators may be selected in 8 x 4 = 32 ways. The I has two and also 4 systems of imprimitivity. If taken in two systems they must be the four-group. These systems may be interchanged cyclically, hence we have our group of order 32.

Of the remaining groups the one having operators of order 8 is

1	1
ac.bd	eg.fh
abcd	efgh
adcb	ehgf
ac	eh.fg
bd	ef.gh

aebfcgdh

Its commutator is a cyclic group of order 4 as

1
abcd.efgh
ac.bd.eg.fh
adcb.ehgf

G has eight operators of order 8. Two operators of order 8 and an operator of order 4 which is not contained in the cyclic groups generated by the operators of order 8 generate the group. The first operator of order 8 may be chosen in eight ways, the second in 4. The remaining generator can be selected in four ways. The I is therefore an intransitive group. The I of the head of G is a group of order 32 and degree 8 is the direct product of two four-groups and an operator permuting the systems

cyclically. The I of G is the direct product of this group of order 32 and another four group.

The last group of order 32 is the group

1	1
ac.bd	eg.fh
abcd	ef.gh
adcb	eh.gf
ac	eg
bd	fh
ab.cd	efgh
ad.bc	ehgf

ae.bf.cg.dh

This has 20 operators of order 4 and degree 8 as

1	ac.eg
ac.bd	bd.eg
eg.fh	ac.fh
ac.bd.eg.fh	bd.fh

Of these 20 operators of order 4, 4 have an invariant operator of order 2 for its square and so cannot correspond to any of the others in any automorphism. The eight operators of order 4 in the head generate half of the group. The I of this head which is isomorphic with the first dimitiation of the octic group of order 32. The I of G is therefore the direct product of this group of order 32 and a four-group representing the possible permutations of the four operators whose squares are invariant. These last four are also generators of our group.

We have two groups of order 36 and degree 8. The first is the product of the group of order 2 and the positive substitutions of the direct product of two symmetric groups of order 6. This has 8 operators of order 6, 8 of order 3, and 19 or order 2. Writing the group so

1	dh
abc	abc.dh
acb	acb.dh
efg	efg.dh
abc.efg	abc.efg.dh
acb.efg	acb.efg.dh
egf	egf.dh
abc.egf	abc.egf.dh
acb.egf	acb.egf.dh
ab.ef	ab.ef.dh
ab.fg	ab.fg.dh.
ab.eg	ab.eg.dh
bc.ef	bc.ef.dh
bc.fg	bc.fg.dh
bc.eg	bc.eg.dh
ac.ef	ac.ef.dh
ac.fg	ac.fg.dh
ac.eg	ac.eg.dh

we see that the I of the head is the group of order 48 which is the I of the group of order 9 of type (1,1). When the head is fixed the generating operator of the tail may be chosen in 18 ways, hence the order of I is 48 x 18 or 856. It is written on 18 letters representing the noninvariant operators of order 2. The other group is written

1	1	gh
abc	def	
acb	dfe	

ad.bc.cf

The I of G is the product of the I's of the characteristic subgroups. The I of the head is a group of order 48 with 2 systems of imprimitivity transformed according to the symmetric group of order 6. The operator gh is invariant and characteristic, hence the I of G is the I of the head.

VI. BIBLIOGRAPHY

The references in the foot-notes are all references to articles.

The following books have also been consulted:

(1) Pascal's Repertorium Vol. 1 (1900)

(2) Encyclopedie des Sciences Mathematiques Tome 1 Vol. 1 Fascicule 1 P. 532

(3) Weber's Algebra, Ziverte Auflage (1908)

(4) Serrets Algebra, Sixieme Edition (1910)

(5) Burnside, Theory of Groups (1897)

(6) Easton, Constructive Development of Group Theory (1902)