HELLMANN

The Effect of Wave Form Upon the Transformer Losses

Electrical Engineering

M. S.

1909

THE EFFECT OF WAVE FORM UPON THE TRANSFORMER LOSSES

BY

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THESIS

Submitted in Partial Fulfillment of the Requirements for the

Degree of

MASTER OF SCIENCE

IN ELECTRICAL ENGINEERING

IN

THE GRADUATE SCHOOL

OF THE

UNIVERSITY OF ILLINOIS

1909

UNIVERSITY OF ILLINOIS

THE GRADUATE SCHOOL

May 21, 1909.

I HEREBY RECOMMEND THAT THE THESIS PREPARED UNDER MY SUPERVISION BY

Carl August Hellmann

ENTITLED THE EFFECT OF WAVE FORM UPON THE TRANSFORMER LOSSES

BE ACCEPTED AS FULFILLING THIS PART OF THE REQUIREMENTS FOR THE DEGREE OF Master of Science in Electrical Engineering

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Committee on Final Examination

REFERENCE :

See the Bulletin of the Bureau of Standards, Vol. 4, #4,

" THE EFFECT OF WAVE FORM UPON THE IRON LOSSES IN TRANSFORMERS."

I N T R O D U C T I O N.

It has been assumed in the construction of alternating current generators that the aim of the designer should be to produce a machine whose electromotive force is as nearly as possible a pure sine wave. While this is undoubtedly good practice where the generator is to supply power to a circuit composed mostly of motors or other devices having considerable inductance or capacity, yet where these are absent, there is no real need for a sine wave, and it is the purpose of this paper to show how the iron losses of transformers may be materially decreased by the use of non-sine wave forms, and to illustrate this by showing certain curves and their characteristics and effects upon the losses.

It is proposed to develop the whole paper directly from the fundamental principle of induced currents, namely that the magnitude and direction of the induced electromotive force are given by the rate of change of the inducing flux, and the Steinmetz formula, which says that the hysteresis loss is proportional to the 1.6 power of the maximum of the flux density. In the course of the paper every equation is derived from the two principles just now stated, and all the necessary steps involved in obtaining these equations are clearly shown, and diagrams and curves inserted where thought desirable. The mathematical part of the work is made as simple as possible, though the problem from its very nature necessarily involves some rather intricate equations. Several graphical solutions are offered to replace the tedious and often difficult process of solving complicated trigonometrical equations. It has also been the aim of the writer to limit the discussion to those data which are of practical use to the engineer, rather than

the mere theoretical manipulation of mathematical-physical conceptions which do not directly aid the designer in his work.

Such, then, is the purpose of this paper. It is only a very elemantary discussion of the subject, but yet, since it treats of a branch of alternating current phenomena of which very little has been written, and which is of interest to the engineer, the writer feels justified in offering it to such as may chance to read it, with his apologies.

C. A. Hellmann,

Washington, D. C. April 21, 1909.

D I S C U S S I O N .

Let us assume that we have a periodic electromotive force of any given wave form. Since the wave is periodic, it must pass cyclically through a definite given set of values and repeat this cycle indefinitely. Now let us denote the time in seconds, from a given point in one cycle to the corresponding point in the following cycle by $2T$. Then let us call the quantity $2T$, just now defined, the PERIOD of the emf. in question, and the reciprocal, $\frac{1}{2T}$, the FREQUENCY of that emf..

It is well known that any periodic quantity can be represented as made up of superposed sine curves of various amplitudes, and in various phase relations to each other, and whose frequencies are to each other as 1, 2, 3, 4, 5, ...etc.. We shall call the sine curve whose frequency corresponds to 1, the FUNDAMENTAL,and all the others the HARMONICS, the latter being ODD or EVEN according as the frequency ratio corresponding to the given harmonic, stated above, is an odd or an even number. Further, we shall use the letter m to denote the ORDER of the harmonic, the order being the ratio of the frequency of the harmonic to that of the fundamental.

Now let us digress a little and consider the periodic flux, say in the core of a transformer, which produces the emf. wave we have assumed. We shall denote by S the cross-sectional area in square centimeters of the core, and assume that the flux is at every instant uniformly distributed over this area S, and shall denote its density, or number of lines of force per square centimeter, at any instant, by ϕ, and the number of turns in the transformer coil by N, and time in seconds by t. If then, we denote the induced emf. by e, we shall have the following equation:

$$e = -\frac{d\phi}{dt}SN\cdot10^{-8} \text{ volts} \quad \ldots\ldots\ldots\ldots \quad (1)$$

This is derived directly from the definition of an induced emf., and if the quantities involved in the equation are given in C. G. S. units as here assumed, the equation necessarily gives the result in volts. Now, S and N are constants for any given apparatus, hence for simplicity, we shall denote the expression $-SN\cdot10^{-8}$ by the letter k, a constant, so that we may write:

$$e = k\frac{d\phi}{dt} \quad \ldots\ldots\ldots\ldots \quad \ldots\ldots\ldots\ldots \quad (2)$$

We thus obtain, from (2),

$$kd\phi = edt \quad \ldots\ldots\ldots\ldots \quad \ldots\ldots\ldots\ldots \quad (3)$$

Now, since we have assumed that e is periodic, it necessarily follows that e is some function of the time, t, which we shall state mathematically thus:

$$e = F'(t) \quad \ldots\ldots\ldots\ldots \quad \ldots\ldots\ldots\ldots \quad (4)$$

where $F'(t)$ is a functional symbol, and denotes the first derived function of the function $F(t)$ with respect to t. Obviously, this places no restriction on the nature of $F'(t)$, since any function is the derived function of some other function, and this notation is here used for convenience, as will appear later. Substitution of this value of e in (3) gives:

$$kd\phi = F'(t)dt \quad \ldots\ldots\ldots\ldots \quad \ldots\ldots\ldots\ldots \quad (5)$$

and integration of this, for the interval $t..t_0$ gives:

$$k\phi = \int_{t_0}^{t} F'(t)dt = F(t) - F(t_0) \quad \ldots\ldots\ldots\ldots \quad (6)$$

Now let us denote the maximum value of ϕ by the letter B and proceed to find the value of B, since the magnitude of the hysteresis loss depends upon this quantity. The calculus tells us that when a function of any variable attains a maximum or a minimum value, its first derivative with respect to that variable must be zero. This is readily seen by plotting the curves representing these two

quantities. Also, since the derivative is the "rate of change" of the flux, then obviously, when the flux is not changing, the induced emf. will be zero, while the tangent to the flux curve at the corresponding point will be parallel to the axis of abscissae, and consequently the flux either a maximum or a minimum value. Applying this criterion to the present case where, in (6) we have expressed ϕ as a function of t, we see that:

$$k\frac{d\phi}{dt} = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (7)$$

is a condition which must be fulfilled if ϕ = B.

But the first member of (7) is seen from (2) to be equal to e, hence equation (7) means that when the flux is a maximum, the emf. must be zero. From (4) we have that e equals $F'(t)$, hence from (2), (4) and (7), we see that:

$$F'(t) = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (8)$$

is a condition that ϕ be a maximum. Equation (8) is an equation of the m^{th} degree in sine and cosine of $\frac{t}{2T}$, that is, when expanded so as to be expressed in terms of the sine and cosine of the fundamental, hence it will in general, have m roots, and also in general, if m be greater than 3 the roots can be only approximated. Hence by far the simplest method is to get the oscillograph curve for e and thus get the roots by means of a curve analyzer. Having obtained these roots, and denoting one of them by t_r, the substitution of t_r in equation (6) gives:

$$k\phi_r = F(t_r) - F(t_o) \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (9)$$

Now let us see what the meaning of t_o is. In equation (6) we have integrated $F'(t)dt$ between the limits t and t_o, and in obtaining (9) we have put t_r, one of the roots of (8), in place of t in equation (6). Hence, the second member of (9) is the value of the integral of $F'(t)dt$ between the limits t_o and t_r. This integral represents

the area under the curve $e = F'(t)$ between the limits t_r and t_o, that is, the area under the emf. curve, and evidently is a maximum only when the limits of the integral are both points at which e is equal to zero, that is, t_r and t_o must both be roots of (8). This is evident from an inspection of the emf. curve shown in fig. 1 below, and is true for any periodic curve without restriction.

Now let us assume that the emf. is such as would be given by any well designed alternator, that is, that the portion of the curve above the axis bears such a relation to that below the axis that any half-cycle, above the axis, when revolved from positive to negative position about the axis, and also shifted as a whole through a distance equal to T, along the axis, will be exactly coincident with the following half-cycle, below the axis. For brevity, let us designate such a wave to be "symmetrical". Having thus limited the discussion to symmetrical waves, we thereby eliminate the harmonics of even order, since such harmonics always produce non-symmetrical waves, as will be evident from figures 2 and 3, below:

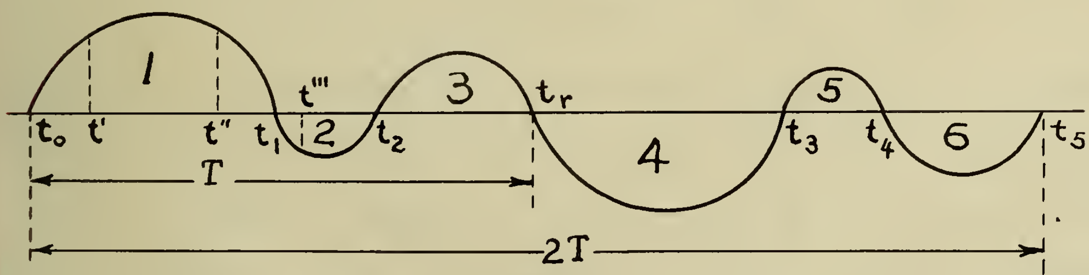

Figure 1.

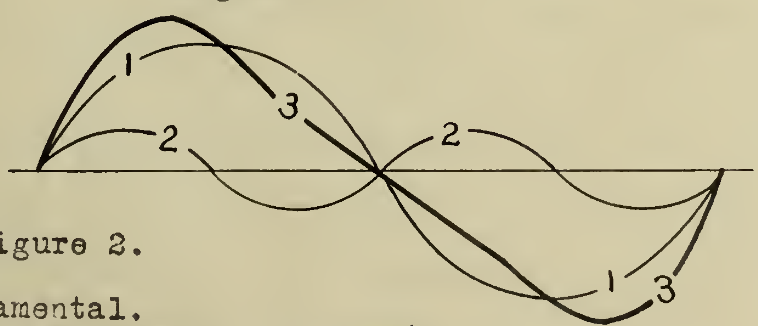

Figure 2.

1 Fundamental.
2 Second Harmonic. (in phase)
3 Resultant.

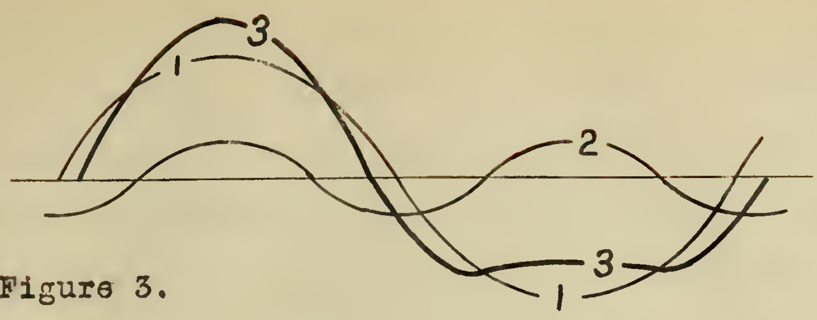

Figure 3.

1 Fundamental.
2 Second Harmonic. (90° out of phase)
3 Resultant.

Figures 2 and 3 show waves which are not "symmetrical".

With curves which are symmetrical, we have the additional relation that the interval from t_r to t_o is equal to T or in symbols:

$$t_r - t_o = T \quad \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \quad (10)$$

to guide us in choosing the proper values of t to get the value of B. This does not except in very special cases, hold for curves which contain the even harmonics, as will be at once apparent upon drawing the diagram.

Figure 1 represents one cycle of a curve which is very much distorted by harmonics, yet is symmetrical, that is, it contains only the fundamental and the odd harmonics. The loops 1, 2, 3, etc., are seen to be alternately positive and negative, the complete cycle comprising six loops, three positive, and three negative. Now this curve is simply a specific one of the infinite number of forms the general equation (4) may assume, that is, this curve represents graphically one case of $e = F'(t)$. Hence the area under the curve between any two values of t represents the value of the integral $F(t)$, for the corresponding interval. Obviously, from the diagram, this integral is greater between, for instance, t_o and t_1 than between any other two of the values in the series, t_o, t', t'', t_1, and t''', since it is at once apparent that the best values of t are

those which give as much as possible of the positive area of the loop and deduct the least possible by including negative parts of the same. It will also be seen from the figure that since the positive loops during a complete cycle are exactly equal in area to the negative loops, the value of the integral, starting from any point whatever, to the end of a complete cycle of values, is the same as that at the starting point, and hence that the maximum value of the integral will be obtained by choosing a half cycle so as to include the largest positive loops, when it follows necessarily that the same interval will include the smallest negative loops, since these are arranged in rotation, by the conditions of the problem, as will be seen on inspection of any such symmetrical curve.

The AVERAGE ordinate of any curve may be obtained by finding the area under the curve, that is the area enclosed between two ordinates, the curve, and the axis of abscissae, and dividing this area by the distance between the two ordinates. This amonnts to saying that the average ordinate is the distance that a straight line, parallel to the axis, would be from the said axis in order that the area between the line, the axis, and two ordinates the same distance apart as those used above, equal the area of the given curve as above found. Since we have limited this discussion to symmetrical curves, it follows at once from the definition of such curves, as before stated, that the area under the curve represented by equation (4) will be zero if the distance between the two limiting ordinates is equal to 2T or any multiple thereof, and as previously stated, will be a maximum for one particular interval whose length is equal to the period, T. Let us denote the abscissae corresponding to the two ordinates of this interval, by t_0 and t_0+T. Then, from equation (6), putting these values as the limits of the integration, we

obtain:

$$k\phi_{max.} \ (=kB) = F(t_0+T) - F(t_0) \ \ldots\ldots\ldots\ldots\ldots \qquad (11)$$

thus showing that the maximum flux density is a constant times the maximum area of the curve. Now, from the definition of the average ordinate of a curve stated before, we see that the average emf., E_a, is the second member of (11)divided by the constant, T,hence we have shown that the value of B is directly proportional to the average value of e, for a half cycle chosen as above, and in symbols we have:

$$B = \frac{TE_a}{k} = k_1 \cdot E_a \ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \qquad (12)$$

where k_1 is a constant.

The effective value of an alternating current wave is defined as the square root of the average square of the ordinate of the curve, and is obtained by first finding the integral of the square of $F'(t)$ between the limits t_0 and t_0+T, as was done in the case of finding the area, with $F'(t)$ itself, then dividing the result by T, and taking the square root of the quotient, thus:

$$E = \sqrt{\frac{\int_{t_0}^{T+t_0} [F'(t)]^2 dt}{T}} \ \ldots\ldots\ldots\ldots \qquad (13)$$

where E is the effective value of the emf. e.

The ratio of the effective value, E to the average value, E_a, is the FORM-FACTOR, and will be denoted by the letter <u>f</u>, so that:

$$f = \frac{E}{E_a} \ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \qquad (14)$$

and from (12) and (14) we obtain that:

$$fB = k_1 \cdot E \ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \qquad (15)$$

thus showing that the effective emf. is proportional to the product of the form factor and the maximum of the flux density.

The IRON-LOSSES of the transformer are of two kinds, those due to EDDY-CURRENTS, and those due to HYSTERESIS, so that we may write:

$$W = W_e + W_h \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (16)$$

where W denotes the total iron loss, and W_e and W_h are the eddy current loss and the hysteresis loss respectively.

Of these two losses, the eddy current losses are very easily dealt with, and will be at once considered. Eddy currents are produced by the same flux which produces the emf. e,and, since they flow in the iron itself,it may be assumed that the resistance and the impedance of their paths remain constant, and hence that the power lost is,by the well known principle, proportional to the square of the effective value of the eddy current or the eddy emf. to which this is proportional. Now, since the same flux produces this and e, it follows at once that if E be kept constant, the effective eddy emf. will also remain the same, and hence the eddy current loss for any emf. whose frequency and effective value remain constant, will also remain constant, no matter what is the shape of the wave.

The hysteresis losses depend upon the frequency and the maximum, B, of the flux density. In curves of the type shown in figure 1, there are several maxima, corresponding to the points t_1, t_r, t_4, etc., and hence in curves which are so irregular as to have more than one zero value of emf. per half cycle, it is necessary to choose the greatest of the corresponding flux maxima. The usual method of obtaining the loss is to assume the empirical relation that the loss is proportional to the 1.6 power of the maximum flux density, but of course this does not even pretend to take account of the additional loss due to the intermediate maxima and minima in the curve. But it is not worth our while to concern ourselves much with curves which are so irregular as to have several maxima of the corresponding flux curves, since such emfs. contain an

abnormal percentage of the harmonics and are not such as would be often found in practice, and in case they were, the error resulting from the application of the formula would be comparatively small, as will be seen from the two figures below, figs. 4 and 5, which show the emf. curve, and the corresponding flux curve and the hysteresis loop.

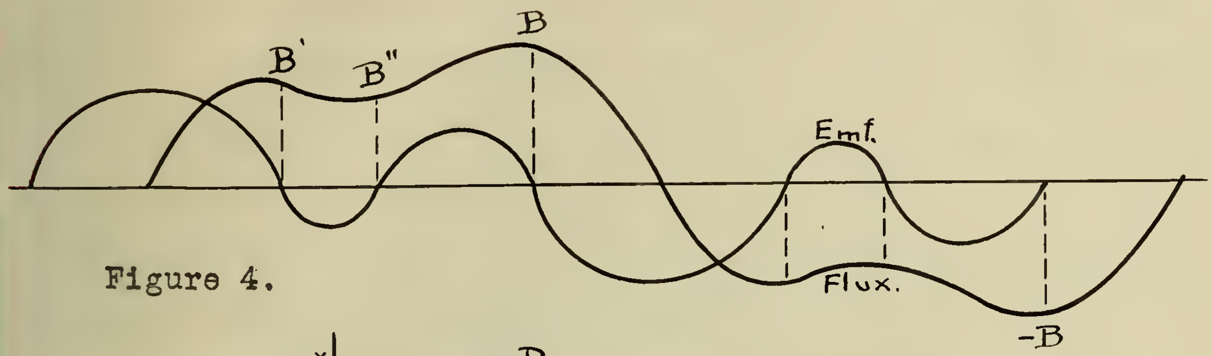

Figure 4.

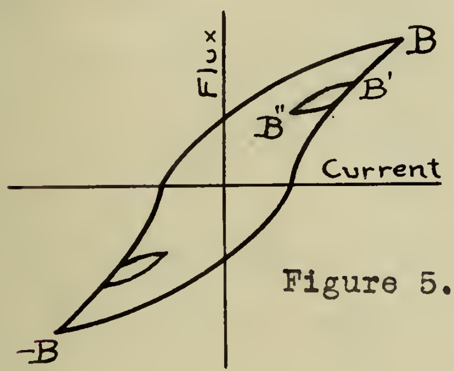

Figure 5.

Figure 4 shows the emf. and the flux curves, and Figure 5 shows the hysteresis loop corresponding thereto.

Figure 4 shows an unusually irregular emf. curve, and the corresponding curve of flux. It will be observed that the flux irregularities are not anywhere nearly so marked as the irregularities in the emf. curve. This, as will be more fully explained later, is due to the fact that the flux curve corresponding to the m^{th} harmonic of the emf. curve has the relative amplitude of only $\frac{h}{m}$ as compared with the amplitude of the fundamental of the flux curve considered as unity, instead of the corresponding ratio h of the emf. curve. Hence it is seen that the additional loops in the hysteresis loop shown in figure 5, produced by the small range of flux between B' and B" will be of negligible area in comparison with the large loop

and that their omission will introduce only a very small error.

Thus far we have considered the emf. merely as $F'(t)$, and have made little use of the fact that the curve may be represented as a composite sine function, as previously stated. Let us now take the case where:

$$e = A[\sin.pt + h_3\sin.(3pt + \theta_3) + h_5\sin.(5pt + \theta_5) + etc]. \quad (17)$$

In the above equation, $\underline{A}$ is the amplitude of the fundamental of the emf., $\underline{p}$ is the angular velocity, $\underline{h_m}$ is the ratio of the amplitude of the m^{th} harmonic to that of the fundamental, and $\underline{\theta_m}$ is the phase angle of the m^{th} harmonic.

By substituting the value of e from (2) in (17) above, for e, and then multiplying both members by pdt, we obtain:

$$pkd\phi = Ap[\sin.(pt)dt + h_3\sin.(3pt + \theta_3)dt....etc] \quad ... (18)$$

Integration of equation (18) gives us:

$$(-pk\phi =)-k'\phi = A[\cos.(pt) + \frac{h_3}{3}\cos.(3pt + \theta_3) ...etc] \quad .... (19)$$

Here the coefficient of the m^{th} harmonic of the flux curve is seen to be $\frac{h_m}{m}$, that is, only one-m^{th} of that of the corresponding emf. harmonic. On putting $m = 3$, and $h = .333$, and $\theta = 0$, we obtain the curves shown in Figure 6, below:

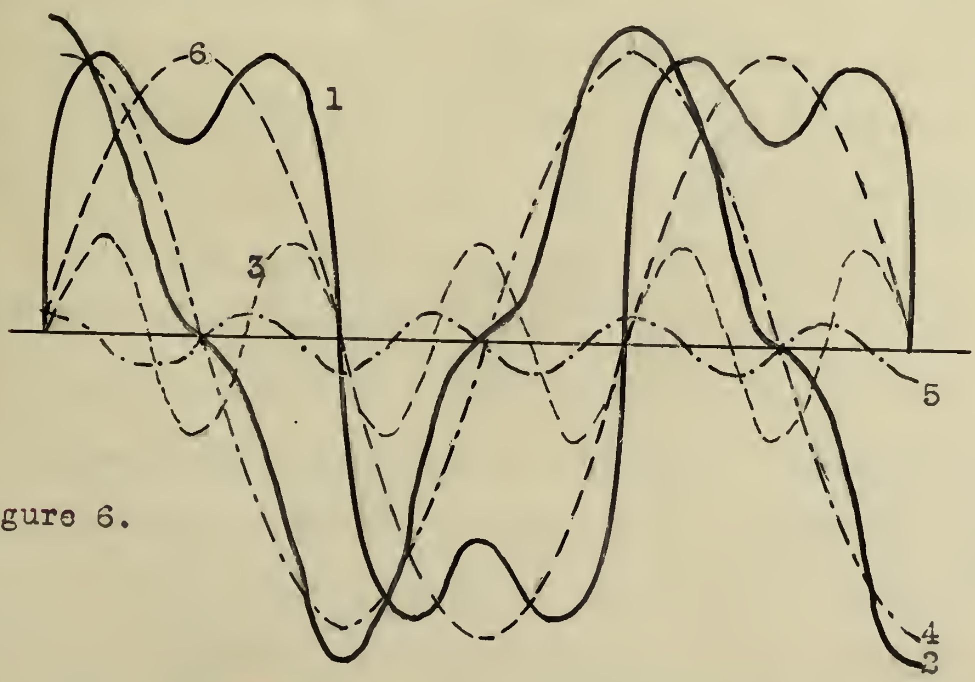

Figure 6.

In figure 6, the full lines 1 and 2 represent the emf. and the flux curves respectively, and the ----- lines show the fundamental, 6, and the triple harmonic, 3, of the emf., while the -·-·- lines, show the corresponding components, 4 and 5, of the flux . Here it will be observed that while the curve 4 is represented as having the same amplitude as the curve 6, the curve 5 on the same scale has only one third the amplitude of its corresponding emf. curve, 3. It will also be noticed that while the terms in equation (18) are all sines, the corresponding terms in (19) are all cosines, hence the corresponding points of emf. and flux are 90° out of phase, as shown in the figure.Thus it is seen that in general , because of this phase difference the flux curve corresponding to a flat top emf. will be peaked, and vice versa. This results from the fact that a displacement of each component through 90 of its degrees will change their phase relations with respect to each other by 180°, hence the maxima will fall where the minima were before, thus changing the nature of the result as stated.

On a previous page the statement is made that in order to produce more than one maximum per cycle in the flux, the emf. curve must be very irregular. Let us see how great an amplitude of the m^{th} harmonic must be introduced into the emf. curve to produce such secondary maxima. Obviuosly, ϕ will be a maximum only when e equals zero. Hence if we take the value of e to be:

$$e = A(\sin.pt + h_m \sin.(mpt + \theta_m)) \qquad \ldots\ldots\ldots\ldots \qquad (20)$$

we must obtain the values of t to make this equal to zero by equating the second member to zero and solving for t. However, as stated before, this equation is of the m^{th} degree in sine and cosine of pt, and while in a few special cases the equation may be easily solved algebraically, probably the simplest method is the following graph-

ical solution. Let us first take the special case where θ_m is 180°. The curves corresponding to this are shown in Figure 7, below:

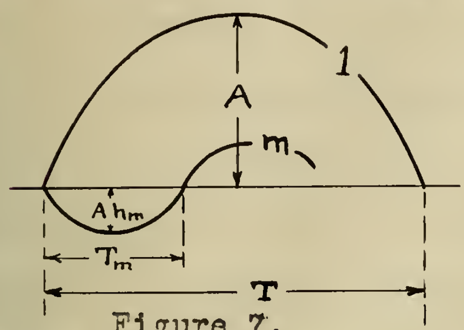

Figure 7.

In this figure, 1 is the fundamental, whose amplitude is A, m is the m^{th} harmonic, of amplitude Ah_m, and T and T_m are the periods of the fundamental and of the harmonic respectively.

If now, 1 and m represent the two components of the emf., of the first and m^{th} order, respectively, as shown, we shall obtain a certain resultant e, and it will be zero when either of the following relations is true:

First, if the corresponding ordinates of 1 and m are simultaneously equal to zero.

Second, if the corresponding ordinates of 1 and m are equal in length, but of opposite signs.

The first case gives what may be considered the normal zero of the emf., while the second gives the zero values corresponding to the secondary maxima or minima, (if any) of the flux curve.

The slopes of the two components will be equal and opposite where t is zero, if the two curves are similar, that is, if A is to Ah_m as T is to T_m. But, $T = mT_m$, hence it follows that in order that the two curves be similar, A and Ah_m must also be in the same ratio, that is, $A = mAh_m$. This gives us the condition that h_m must equal $\frac{1}{m}$. In case this relation holds, the two curves will have equal and opposite slopes at t = 0, and , since the slope of the harmonic at this point is decreasing m times as rapidly as that of the fundamental, the resultant will always have the same sign as the fundamental. Hence it follows at once that if the value of h_m be $\frac{1}{m}$ or less there will be no secondary zero values in the emf. curve, but if h_m exceed this

limiting value, the resultant will first fall below the axis due to the larger ordinates of the harmonic, and then recross the axis as the ordinates of the fundamental increase, thus introducing additional zero values into the emf. curve. The above discussion shows that where there is only one harmonic present, and it is 180° out of phase with the fundamental, its amplitude must be $\frac{1}{m}$ or less, if there are to be no secondary maxima in the flux curve.

Next we shall consider the case where the angle θ_m is equal to zero. Substitution of this value of θ_m in equation (20) gives :

$$e = A(\sin.pt + h_m \cdot \sin.mpt) \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (21)$$

Now let us plot the curves representing this as done in figure 8:

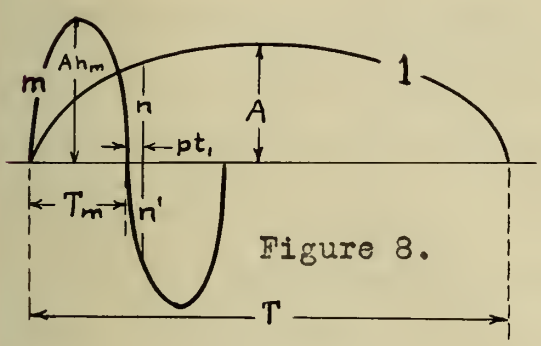

Figure 8.

This figure shows the fundamental, 1, and the m^{th} harmonic, the symbols used being the same as those of Fig. 7, with the addition of n and n', which represent corresponding ordinates of the two component curves.

Now, if the ratio h_m be so chosen that e is equal to zero at some point inside the half-cycle, T, of the fundamental, shown, then at that point the two component curves must have equal and opposite ordinates. Now revolve the harmonic from its present "positive" position to the corresponding "negative" position, about the axis of abscissae. Then the ordinate n' will fall upon the ordinate n, and if this ordinate is to be that corresponding to the zero value of e, the two ordinates will also be of the same length, and hence coincide exactly. This is shown in figure 9, below:

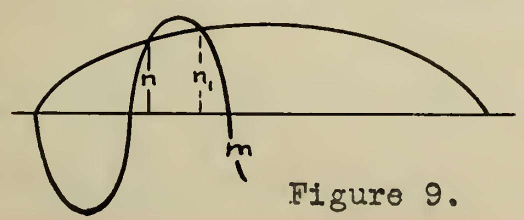

Figure 9.

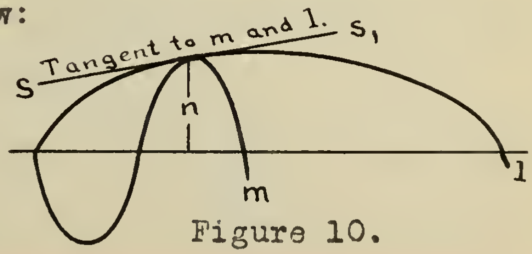

Figure 10.

Figure 9 shows the harmonic revolved as stated, and it will be seen from the figure that there will in some cases be two crossing points of the two curves, each of which corresponds to a zero value of e.

If, now we decrease the amplitude of the harmonic, we finally arrive at the state of affairs shown in figure 10, where the two crossing points coincide, giving only one value (or, to be mathematically exact, two coincident values) of t to make e equal to zero. Obviously at this point the two curves have a common tangent, ss_1, and also the common ordinate,n. If now we should decrease h_m still further, the curves would no longer intersect, that is, there would then be no secondary zero values in the emf. and hence no secondary maxima in the corresponding flux curve. Hence we see that the value of h_m that makes the harmonic tangent to the fundamental when revolved as in figure 10, is the limiting value, and if h_m is equal to or less than this value, there will be no secondary maxima in the flux curve.

From the above condition we may now derive the following relations between the constants of the equations of the two components of the emf. curve:

First, The two ordinates in figure 10 are equal, hence the corresponding two ordinates of the curves before revolving the harmonic about the axis must be equal but of opposite signs.

Second, The slope of the tangent of the curves of figure 10 at the point in question is the same in both components, hence the slopes of the two curves before the revolving must be equal but of opposite signs.

Now, denote by pt_1 the time from the end of the first period of the harmonic to the time where e is zero, due to the state of affairs

shown in figure 8. At this point the ordinate of the fundamental will be $A\sin.(\frac{T}{m} + pt_1)$, and that of the harmonic will be, at the same point, $Ah_m\sin.m(\frac{T}{m} + pt_1)$, hence from the first of the two conditions given above, we may write:

$$A\sin.(\frac{T}{m} + pt_1) = -Ah_m\sin.m(\frac{T}{m} + pt_1) = -Ah_m\sin.(T + mpt_1) \quad \ldots\ldots \quad (22)$$

But remembering that T is 180°, that is, π, we may rewrite the above equation (22) as follows, since $\sin(x + \pi) = -\sin.x$:

$$\sin.(\frac{\pi}{m} + pt_1) = h_m\sin.(mpt_1) \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (23)$$

From the second relation above, we may get another equation as follows The slope of the fundamental at this point is the value of $\frac{d\sin.pt}{dt}A$, and that of the harmonic is the value of $\frac{d\sin.mpt}{dt}Ah_m$. When t is equal to $(t_1+\frac{T}{pm})$ these become $Ap\cos.pt, = Ap\cos.(\frac{T}{m} + pt_1)$ and $mAph_m\cos.(T + mpt_1)$ Now, these are to be equal and of opposite signs, hence, since T is equal to π, and making use of the relation that $\cos.(x + \pi) = -\cos.x$, and substituting these values, we obtain:

$$\cos.(\frac{\pi}{m} + pt_1) = mh_m\cos.(mpt_1) \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (24)$$

This gives us two independent equations in the two unknown quantities t_1 and h_m, hence we may solve for these. Divide the members of (23) by the corresponding members of (24), and remembering that $\frac{\sin.x}{\cos.x}$ is $\tan.x$, we obtain:

$$\tan.(\frac{\pi}{m} + pt_1) = \frac{1}{m}\tan.(mpt_1) \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (25)$$

The solution of this gives us pt_1, and then from (23) we obtain h_m upon substitution of this value of pt_1. Probably the most expeditious way to solve equation (25) is by means of the diagram shown as figure 11, on the following page. It would, of course, be possible to obtain an algebraic solution of this equation, but upon attempting to solve such an equation where m is greater than 3, the expansion of $\tan.(mpt_1)$ in terms of $\tan.(pt_1)$ becomes very lengthy and of the m^{th} degree, hence the graphical solution is preferable.

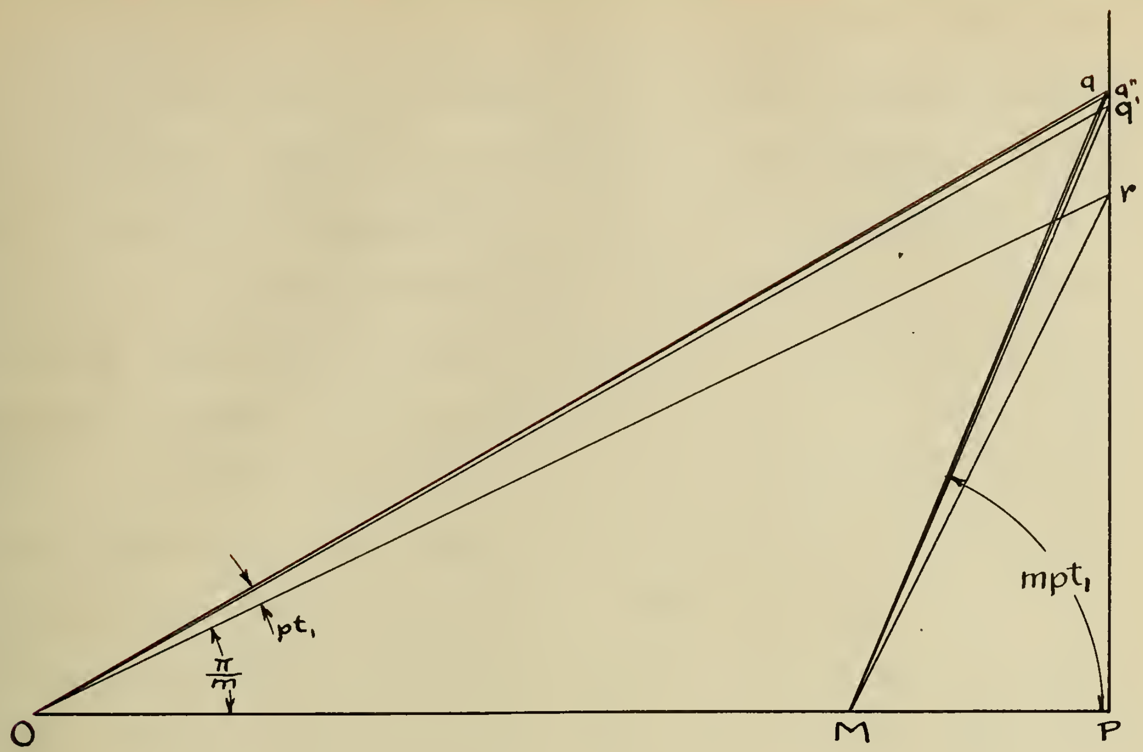

Figure 11.

Let OP be the unit of measure. Then OP = 1. Now, lay off the point M so that MP is equal to $\frac{1}{m}$. From O draw the line Or so that the angle rOP is equal to $\frac{\pi}{m}$, and connect rM, r being the point in which the line Or meets the perpendicular rP, to OP. Now if some point q is taken on Pq, we see from the diagram and from the definition of a tangent that if q be connected to O and M by the lines qO and qM, we obtain the relation that $\tan.qMP = \frac{qP}{MP}$, and $\tan.qOP = qP$, $(= \frac{qP}{OP} = \frac{qP}{1})$ But since OP is m times MP, it follows that inversely, tan.qMP is m times tan.qOP. If now we choose q so that the angle qMP is m times as large as the angle qOr, we may let these angles represent mpt_1 and pt_1 respectively, since in that case from the diagram we obtain the relation that:

$$\tan\left(\frac{\pi}{m} + pt_1\right) = \frac{1}{m}\tan(mpt_1) \qquad \text{(25')}$$

which is the same as equation (25) on the preceding page, hence we see that the diagram is a true representation of equation (25), and may therefore be used in solving that equation graphically. A very close

approximation to the root may be obtained as follows: Lay off the line OP say ten inches in length, and draw the perpendicular at P. Then lay off the angle $\frac{\pi}{m}$, as indicated on the diagram. This can easily be done by laying off Or equal to ten inches times the tangent of $\frac{\pi}{m}$ on qP, and then connecting Or. Next draw rM, M being laid off so that MP equals ten inches divided by m. Now divide the angle rMP into m equal parts, and draw q'Or equal to one of these m parts, then connect q'M, divide the angle q'Mr into m equal parts, add one of these to the angle q'OP, giving us q", connect this to M, and repeat the process as often as desired to obtain accuracy. As a rule, two applications will be sufficient to get the angle mpt_1 to within a few minutes. (Note, that when m is 3, mpt_1 will be 90°, hence in this case the construction is not very satisfactory, though none the less true. Here the easier plan is the trying by inspection to see if mpt_1 be 90°, and this will easily be seen to be the case.) Now, having the angle mpt_1, its tangent is next found by dividing qP by MP, then from trigonometrical tables, mpt_1 and pt_1 are found, and by substituting these values in equation (23) we get a linear equation with h_m as the only unknown quantity, and can at once solve for this, the solution giving the limiting value for the amplitude of the harmonic to avoid secondary maxima in the corresponding flux curve.

Having considered the above two special cases, let us pass to the general case where θ_m is not necessarily either zero or 180°. By the same reasoning as was applied in the discussion of figures 8,9, and 10, we find that in this case similar relations hold, as follows: Assume that:

$$e = A[\sin.pt + h_m\sin.(mpt + \theta_m)] \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (20)$$

The slope of the fundamental , where pt is equal to $(\theta_m + \frac{T}{m} + pt_1)$ is: $Ap\cos.(\theta_m + \frac{T}{m} + pt_1)$, and the slope of the m^{th} harmonic at the same

point is $Amph_m\cos.m(\frac{T}{m} + pt_1)$, or, since T is equal to π, the last expression may be written: $-Amph_m\cos.(mpt_1)$. Now, on equating these slopes with the sign of one changed, as done in the similar case for $\theta_m = 0$, in equation (24), we obtain:

$$\cos.(\theta_m + \frac{\pi}{m} + pt_1) = mh_m\cos.(mpt_1) \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (26)$$

as one of the conditions, and since the ordinates of the fundamental and the harmonic at this point are $A\sin.(\theta_m + \frac{T}{m} + pt_1)$ and $Ah_m\sin.m(pt_1 + \frac{T}{m})$ respectively, upon equating with the sign of one changed,putting π in place of T, and simplifying, exactly as was done in the preceding case, we obtain the other condition that:

$$\sin.(\theta_m + \frac{\pi}{m} + pt_1) = h_m\sin.(mpt_1) \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (27)$$

hence, dividing the members of (27) by the corresponding members of (26), we obtain the equation:

$$\tan.(\theta_m + \frac{\pi}{m} + pt_1) = \frac{1}{m}\tan.(mpt_1) \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (28)$$

This equation may obviously be plotted similarly to equation (25), by use of the diagram shown in figure 11, by putting $(\frac{\pi}{m} + \theta_m)$ in the place of the angle $\frac{\pi}{m}$ there plotted. Due regard must of course be paid to the sign of the angle θ_m, which may be either positive or negative so that the resulting value may be greater or less than $\frac{\pi}{m}$.

Now let t_o be the value of t at the starting point of a cycle as previously defined.Then, since the average value of the emf. whose equation is equation (17),is the integral of edt from t_o to $t_o + T$ divided by T, we have that:

$$E_a = A\int_{t_o}^{t_o+\pi}[\sin.ptdt + \Sigma\, h_m\sin.(mpt + \theta_m)dt] \div \pi \quad \ldots\ldots\ldots \quad (29)$$

which becomes on performing the integration as indicated,

$$E_a = \frac{2A}{\pi}[\cos.(pt_o) + \Sigma\frac{h_m}{m}\cos.(mpt_o + \theta_m)] \quad \ldots\ldots\ldots\ldots\ldots \quad (30)$$

where Σ indicates the sum of all terms of the same general form as the term following this symbol, and is to be so understood wherever used in this paper.

The value of t_o may be found, when only one harmonic is present, from equation (17), by substituting zero for e and t_o for t and solving the resulting equation for t_o. It is possible to obtain a literal solution of this general equation, but a simpler plan is to plot the equation, and thus solve it graphically, which may be done as follows. The equation is:

$$\sin . pt_o + h_m \sin . (mpt_o + \theta_m) = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (31)$$

Draw a circle of radius OP equal to the unit of length, and another circle concentric therewith, whose radius is OQ, where OP $\div$ OQ = h_m. (See figure 12.) Now, using O as the origin and OP as the initial line, of a system of polar coordinates, lay off the angle POS = θ_m. Next, lay off the points R and R' such that the angle SOR is m times the angle POR'. If R and R' have been so chosén that sin.POR' is equal to $-h_m$sin.POR, the angle POR' will be the desired angle, pt_o. Draw the lines M and M', perpendicular to OP from R and R' respectively: then M is the sine of POS + SOR, (POS being here a negative angle, as shown by the order in which its sides are designated) and M' is the sine of the angle POR', these relations following at once from the definition of a sine, since the radius of the circle is unity by construction. Here M' is positive, and M is negative, hence if h_mM is numerically equal to M', we would have M' + h_mM = 0, or, sin.POR' + $h_m$sin.POR = 0. Now, if we produce R'O to L, on the outer circle, and draw thence a parallel to OP, and this line passes through R, then R and R' are the desired points. If we draw the perpendiculars LV and L'V', we obtain the two similar triangles, LVO and L'V'O, hence LV is to L'V' as LO is to L'O, that is, as OQ is to OP, which by the construction is as 1 is to h_m. But VL is equal to M, and V'L' is equal to -M', hence, since V'L' = h_mVL, by substitution, we obtain that -M' = h_mM, that is, sin.POR' = $-h_m$sin.POR, and

if we call SOR, $(mpt + \theta_m)$, and consequently, POR', pt, then in the particular case where R and R' are so chosen that the line connecting L and R is parallel to OP, we have the equation:

$$h_m \sin.(mpt + \theta_m) + \sin.pt = 0 \text{ ....} \qquad (31')$$

hence, here t must equal t_o, giving the same equation as equation (31), showing that the diagram is a true representation of (31), and may be used in the solution of that equation. The method employed is to take a set of values to locate R and R', and see whether the line LR corresponding thereto is parallel to OP, and then changing the positions of R and R' as may be required to make this line parallel or nearly so, and repeating this as often as may be desired to find a sufficiently accurate value of pt_o. Usually two trials, if some judgment is used in regard to the probable location of these points, will suffice to get the value of pt_o to within a few minutes, and this is sufficiently accurate in consideration of the practical conditions of the problem. The diagram shown as figure 13 shows the lines actually used in solving the case where m = 3, $\theta_3 = -67^o$, and $h_3 = .667$. The line OP should be ten inches or more in length, to insure accuracy and ease in solving.

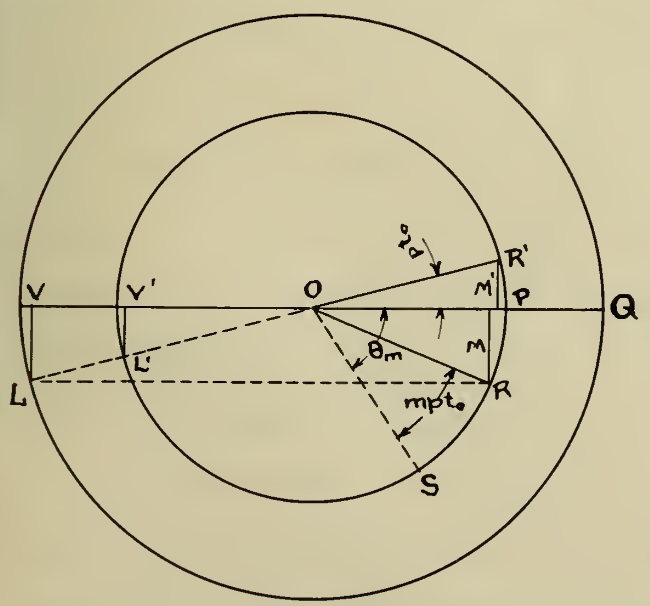

Figure 12.

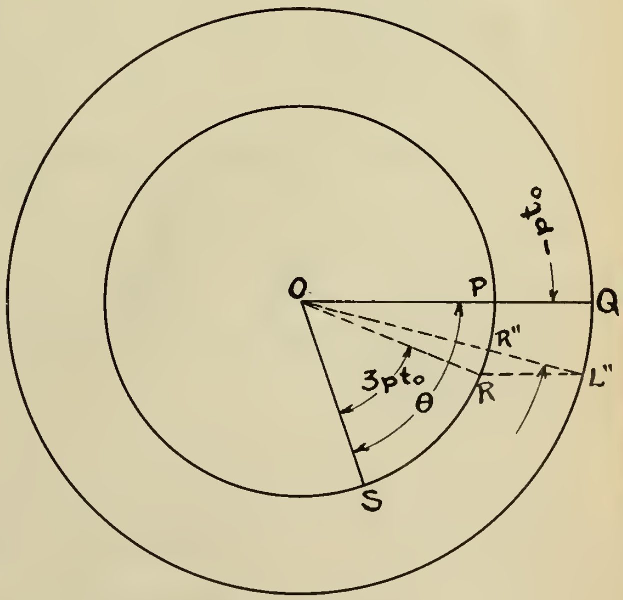

Figure 13.

The angle θ_3 is first laid off, by its sine if less than 45°, and by its cosine if greater. Now take R' so that POR' equals, say, one-fourth of θ_3. This will be found to give too large a value for t, hence we decrease POR', repeat the construction, and continue until the desired degree of accuracy is attained. In this case the second trial gives $14^\circ 50'$ as the value of pt_o. It will be seen that this is close enough for most purposes, for if this were the true value, then from the equation, we should have : $-\sin.14^\circ 50'$ equal to .667 times the sine of $(3\cdot 14^\circ 50' - 67^\circ)$ or $\sin.(-22^\circ 30')$. These sines are found to be -.256 and -.382 respectively, and .667 times -.382 is -.255, hence the agreement is to within .4 of one percent in this case. Obviously, also, it is not at all necessary in practice to draw all the lines shown in figure 12, most of these being introduced there only for use in proving that the diagram is a true representation of the equation. A little consideration will show in any particular case just which lines are needed, and the construction can thus be made very simple.

Having thus found pt_o, we can at once find the value of the average emf., E_a, by substituting this value of pt_o in equation (30). It will be observed that as θ_m varies from 0 to $\pm\pi$, the value of pt_o will vary also, from zero to its maximum, and back to zero, being zero at both extremes, and a maximum somewhere between these. If θ_m is equal to zero, the average emf. is: $E_a = \frac{2A}{\pi}(1 + \frac{h_m}{m})$, and if θ_m is equal to $\pm\pi$, the average emf. is: $E_a = \frac{2A}{\pi}(1 - \frac{h_m}{m})$, and for intermediate values of θ_m the value of the average emf. will also be intermediate these two extremes.

To get the effective value of the emf., we take the general equation, (17), which expresses e in terms of the sum of a series of sine functions, and square both members of that equation, thus:

$$e^2 = A^2[\sin^2 pt + \Sigma h_m^2 \sin^2(mpt + \theta_m) + 2\Sigma h_a h_b \sin.(apt + \theta_a)\sin.(bpt+\theta_b)] \quad \text{.....................(32)}$$

This is the square of e, hence the mean square, will be the average of the above value of e^2, and this mean square is found by integrating $e^2 dt$ from zero to π, and dividing the result by π, the length of the interval. That is,

$$E^2 = \frac{A^2}{\pi}\int_0^\pi e^2 dt = \frac{A^2}{2}(1 + \Sigma h_m^2) \quad \text{......................} \quad (33)$$

whence we obtain:

$$E = \frac{A}{\sqrt{2}}\sqrt{1 + \Sigma h_m^2} \quad \text{..................................} \quad (34)$$

The above equations result because each of the terms of the form $h_m^2 \sin^2(X)\, dt$ gives upon integration between the limits, the value πh_m^2, while all those consisting of the products of two sines will cancel, and give zero as their integral. The angle θ_m does not appear in equation (34) above, thus showing that the effective emf. is independent of the phase relations of the various components of the emf..

We have shown before that $fB = k_1 E$, that is, that the form factor times the maximum of the flux is a constant times the effective emf., hence this will enable us to derive the conditions that give the best wave form when we have at our disposal a fundamental and various harmonics of odd orders. Let us take the case where we have only one harmonic, and determine the best working conditions. It has been shown previously that B is proportional to the average emf., hence it follows that the smaller we can make the average emf., without introducing secondary flux maxima, and without decreasing the effective emf., the smaller will be the resultant value of B, and consequently of the hysteresis losses. We have also shown that there are certain limiting values for h_m depending upon the values of m

and of θ_m. Now consider the case where θ_m is equal to 180°, and h_m is equal to its limiting value for that case, namely, $\frac{1}{m}$. Here E_a is equal to $\frac{2A}{\pi}(1 - \frac{h_m}{m})$, which becomes, upon the substitution of $\frac{1}{m}$ for h_m, $\frac{2A}{\pi}(\frac{m^2 - 1}{m^2})$. The effective emf. is: $E = \frac{A}{\sqrt{2}}\sqrt{1 + h_m^2}$, and this becomes on substitution, $E = \frac{A}{m\sqrt{2}}\sqrt{m^2 + 1}$, and hence the form factor which is the quotient of these two values, is:

$$f = \frac{E}{E_a} = \frac{\pi}{2\sqrt{2}} \cdot \frac{m\sqrt{m^2 + 1}}{m^2 - 1} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (35)$$

The constant coefficient of the last member of this equation is the well known value, 1.1107, which is the form factor of a pure sine curve, and consequently the remaining factor of the last member represents the ratio of the form factor of the given emf. to that in the case of a sine wave, and hence its reciprocal is the factor by which we must multiply E, and consequently E_a, to reduce the curve to one whose amplitudes are such that the effective emf. is the same as that of the sine wave. Hence, if we call the maximum of the flux in the sine wave case B_o, and the maximum in the other case B, these will be to each other in the same ratio as the two values of E_a, that is, the above reciprocal. Hence we have changed B in the ratio given, namely, $\frac{(m^2 - 1)}{m\sqrt{m^2 + 1}}$. The numerator of the ratio is always less than m^2 while the denominator is on the contrary, always greater than m^2, but as m increases, it is evident that the limiting ratio of the two very rapidly approaches unity. This shows that for the lower harmonics the change in B will be appreciable, but that it becomes negligible for the higher harmonics, and the ratio also shows that in all cases the change in B is a decrease. (That is, where θ_m is 180° and there are no secondary maxima in the flux.) Assuming, now that the hysteresis loss varies as the 1.6 power of the value of B, and calculating the values of the ratio of B to B_o from the expression given above, then the 1.6 power of that ratio, which thus represents the

corresponding hysteresis loss ratio,and tabulating these, we obtain the following table:

Table #1.(θ_m = 180°)

m	B/B_o	$(B/B_o)^{1.6}$	% decrease in loss.
3	.844	.761	23.9
5	.942	.912	8.8
7	.970	.955	4.5
9	.982	.970	3.0
11	.988	.982	1.8
13	.993	.987	1.3
15	.996	.990	1.0

From the above table we see, first, that if m be greater than 5, the decrease in the hysteresis loss is too small to offset the disadvantages incident to having harmonics in the emf. wave where the circuits contain any inductive or capacity loads, and where resonance is likely to result, so as to greatly augment one of the harmonics and thus abnormally distort the wave; and second, that if the third or the fifth harmonic be introduced, at θ_m = 180°, and h_m equal to one-third or one-fifth, respectively, the hysteresis loss may be reduced considerably. It must be remembered however that the eddy current loss remains the same, hence the actual percent decrease in the iron loss is not as great as would appear at first sight from the above table. Suppose, for example, that in the case of a sine wave the two kinds of loss were equal; then, taking the first set of values in the table, where the hysteresis is decreased about 24%, the eddy current loss would still remain the same as before, hence here the actual decrease in the total iron loss is only one-half of 24%,

that is, 12%.

Passing now to the case where θ_m is equal to zero, we have again, $pt_o = 0$, hence the value of E_a is, from equation (30), $\frac{2A}{\pi}(1 + \frac{h_m}{m})$, the value of E being $\frac{A}{\sqrt{2}}(1 + h_m^2)^{\frac{1}{2}}$, as before. Here h_m cannot be readily expressed as a simple function of m, hence we have no general expression as before, and must consider each case separately, for the harmonics, say, from the third to the fifteenth, assuming that in each case we have the maximum permissible value of the ratio h_m, and these data are tabulated in the following table:

Table #2. ($\theta_m = 0^\circ$)

m	h_m	$(1+\frac{h_m}{m})$	$(1+h_m^2)^{\frac{1}{2}}$	(B/B_o)	$(B/B_o)^{1.6}$	Decrease in Hyst. in %.
3	1.000	1.333	1.414	.943	.911	8.9
5	.800	1.160	1.280	.907	.852	14.8
7	.613	1.088	1.175	.925	.883	11.7
9	.490	1.055	1.112	.950	.920	8.0
11	.407	1.037	1.081	.959	.935	6.5
13	.347	1.027	1.058	.972	.955	4.5
15	.302	1.020	1.045	.976	.962	3.8

In these cases the emf. waves are all such that there are secondary zero values intermediate the "normal" zero points, hence in general, these curves where θ_m is zero are not desirable, since the decrease in the hysteresis loss is not of sufficient importance to offset the abnormality of the wave form, and since we can obtain a more regular wave by using the data given in table #1, where θ_m is 180°. Figures 14, 15, and 16, below, show the general shape of the first three cases given in the above table, (#2) and serve to illustrate the secondary zero points referred to above:

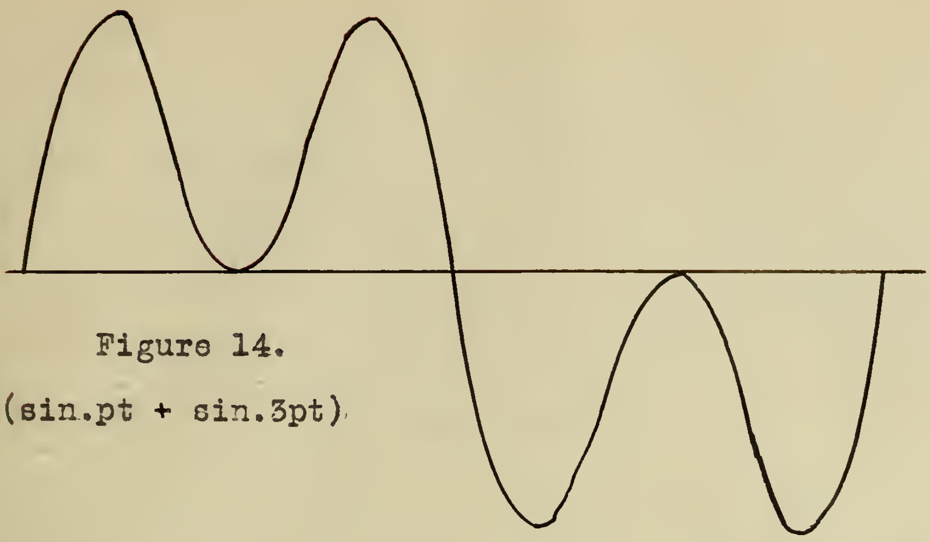

Figure 14.

$e = A(\sin.pt + \sin.3pt)$

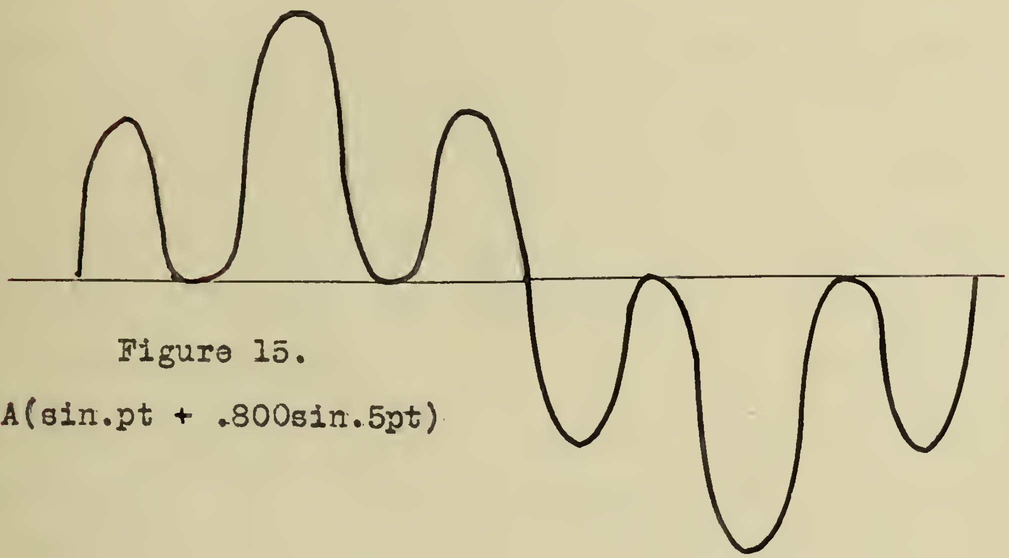

Figure 15.

$e = A(\sin.pt + .800\sin.5pt)$

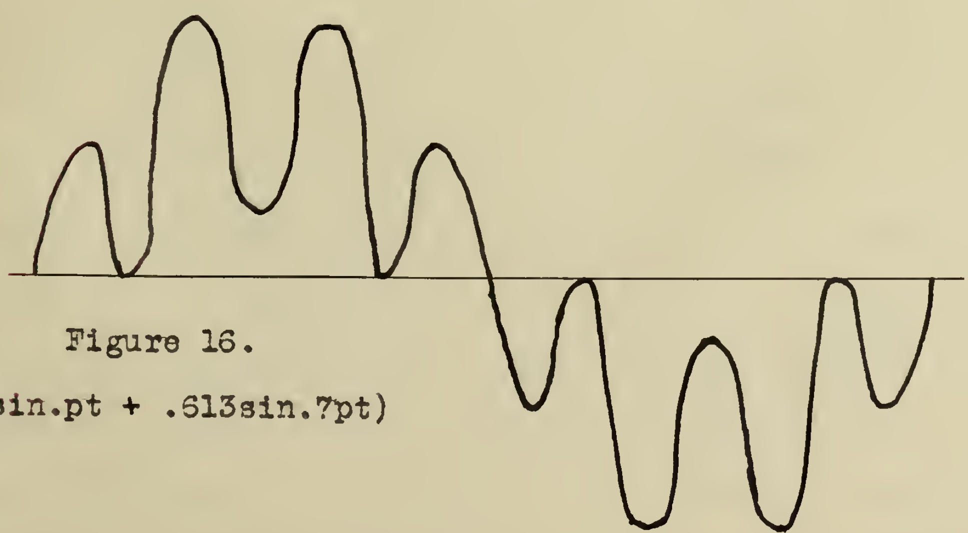

Figure 16.

$e = A(\sin.pt + .613\sin.7pt)$

Having shown that the best results may usually be derived by the use of the third harmonic, we shall therefore plot a few curves showing the waves corresponding to the limiting values of h_m for the third harmonic when θ_m varies from zero to 180°, and finding the proper value of A to use in each case so that the resulting value of E is the same in every one. Since the value of E is proportional to the expression $(1 + h_m^2)^{\frac{1}{2}}$, then if we make A equal to the reciprocal of this, obviously the value of E will be the same in every case. Then, for a pure sine wave, that is, $h_m = 0$, we see that A will be unity from the above condition, while in all other cases the value of A will be $\frac{1}{(1 + h_m^2)^{\frac{1}{2}}}$. Then in the sine wave the value of E will be, from equation (34), A divided by the square root of 2, which is $^1/_{1.414}$, or 0.707, and this is also the value of E in all the cases, since we have so chosen A in each case as to make E the same.

The value of E_a, however, will vary for the various cases, and is found for each case from the general equation (30). For the sine wave E_a will be $\frac{2A}{\pi}$, or 0.636, the ratio of E to this being the normal form factor, 1.11 as found previously for the sine wave. We shall plot the curves with 30° intervals in the values of θ_m, and the following table gives the preliminary data and also some values obtained from the curves, and shows the value of the average emf. ; the value of pt_0 ; the ratio of the average emf to that of the pure sine curve; and the ratio of the hysteresis loss to that of a sine curve, this being the 1.6 power of the ratio of the average emf. to the sine wave average emf., 0.636, since B is proportional to E_a. Figures 17 to 24 inclusive show the curves plotted from the data in table #3, these being all drawn to the same scale, so that they show the relative shape and size of the various waves correctly and thus permit of an accurate representation of the results of various combinations.

Table #3. (Third harmonic, θ_m varied.)

θ_m	Pure sine	0°	30°	60°	90°	120°	150°	180°
h_m	(0)	1.000	.98	.93	.85	.73	.57	.333
A	1.000	.707	.714	.733	.760	.808	.867	.950
E_a	.636	.600	.601	.592	.572	.555	.568	.537
$E_a/_{.636}$	1.000	.943	.944	.930	.900	.873	.893	.845
$('')^{1.6}$	1.000	.911	.913	.890	.842	.807	.830	.761
pt_o	0°	0°	-7.5°	-15°	-22°	-27°	-30°	0°

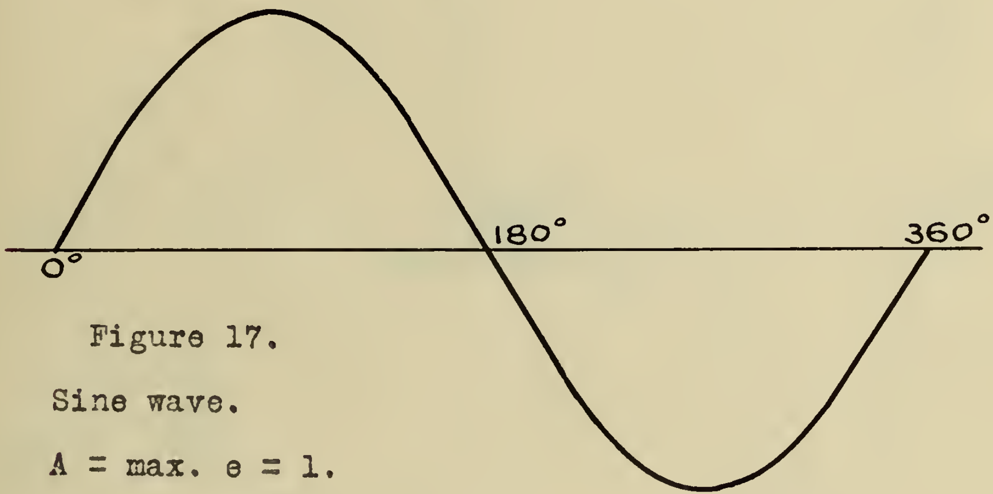

Figure 17.

Sine wave.

A = max. e = 1.

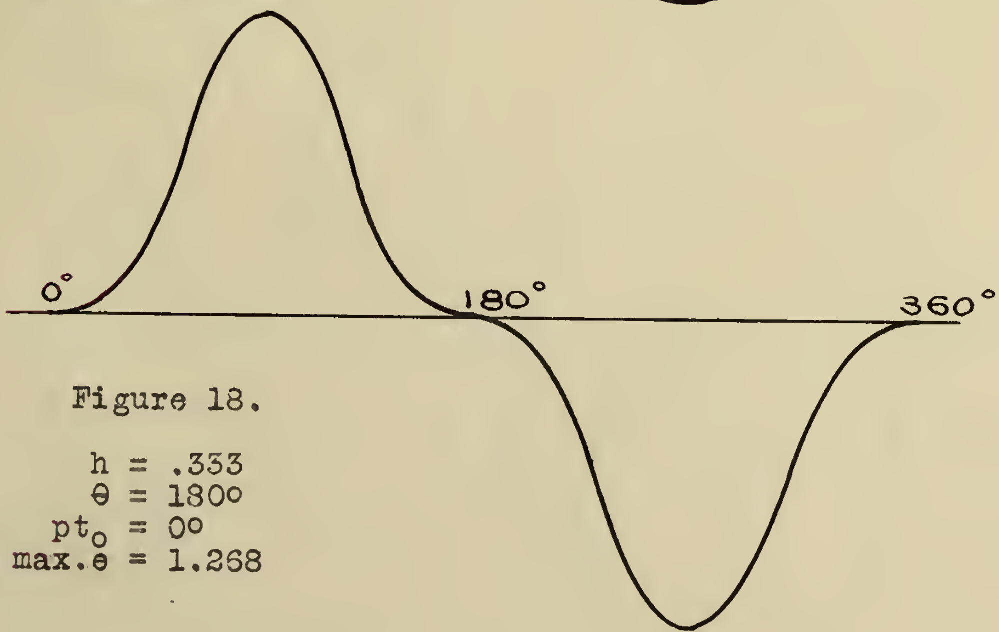

Figure 18.

$h = .333$
$\theta = 180°$
$pt_o = 0°$
$max.e = 1.268$

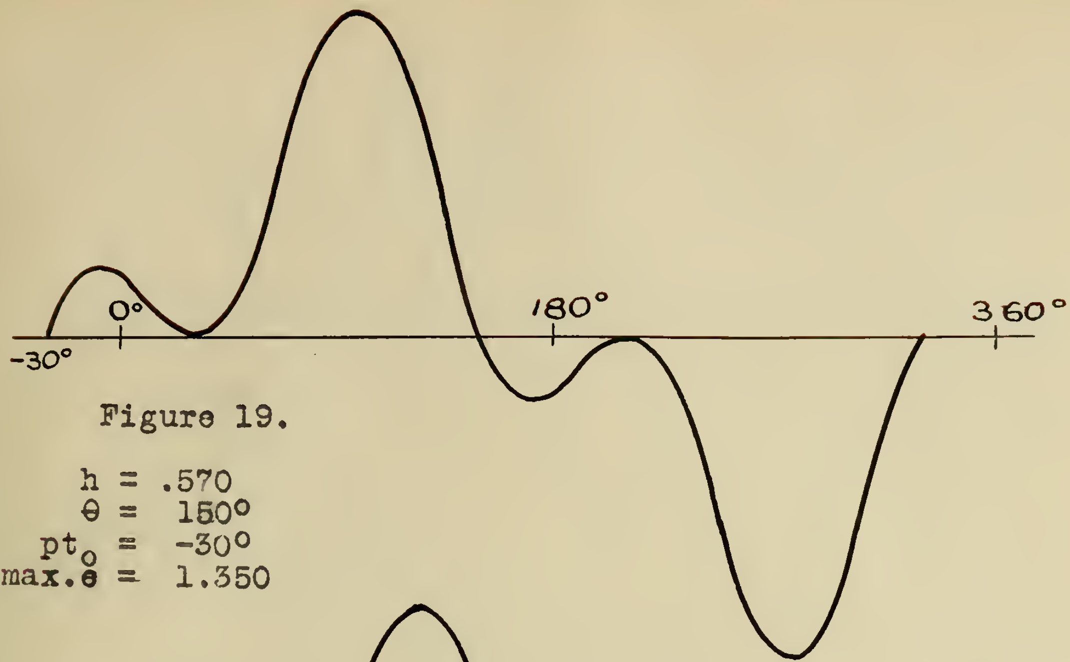

Figure 19.

$h = .570$
$\theta = 150^{\circ}$
$pt_o = -30^{\circ}$
$\max.e = 1.350$

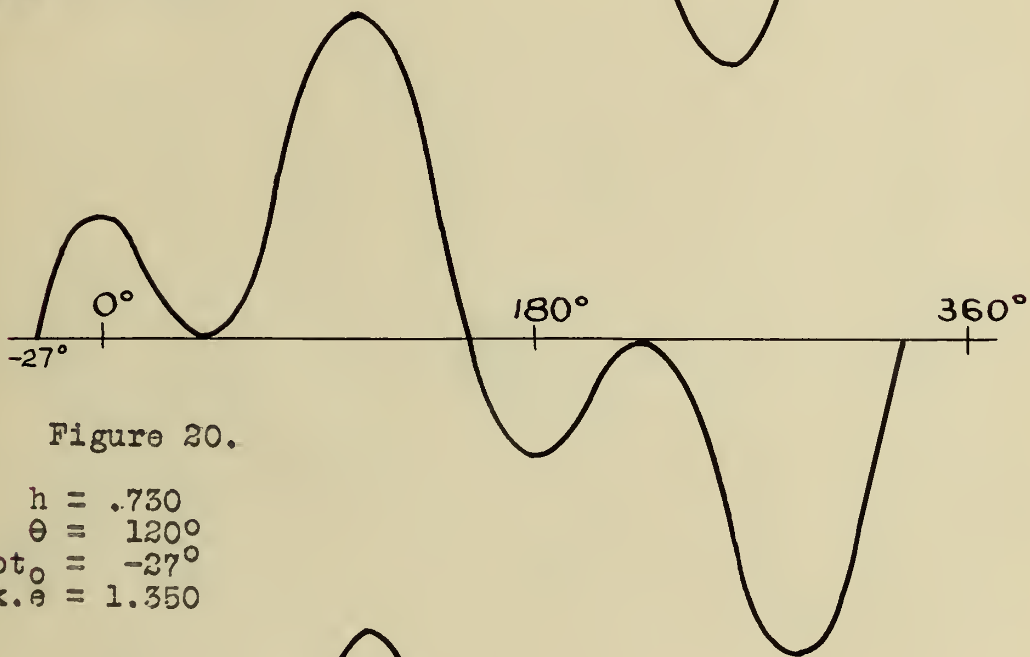

Figure 20.

$h = .730$
$\theta = 120^{\circ}$
$pt_o = -27^{\circ}$
$\max.e = 1.350$

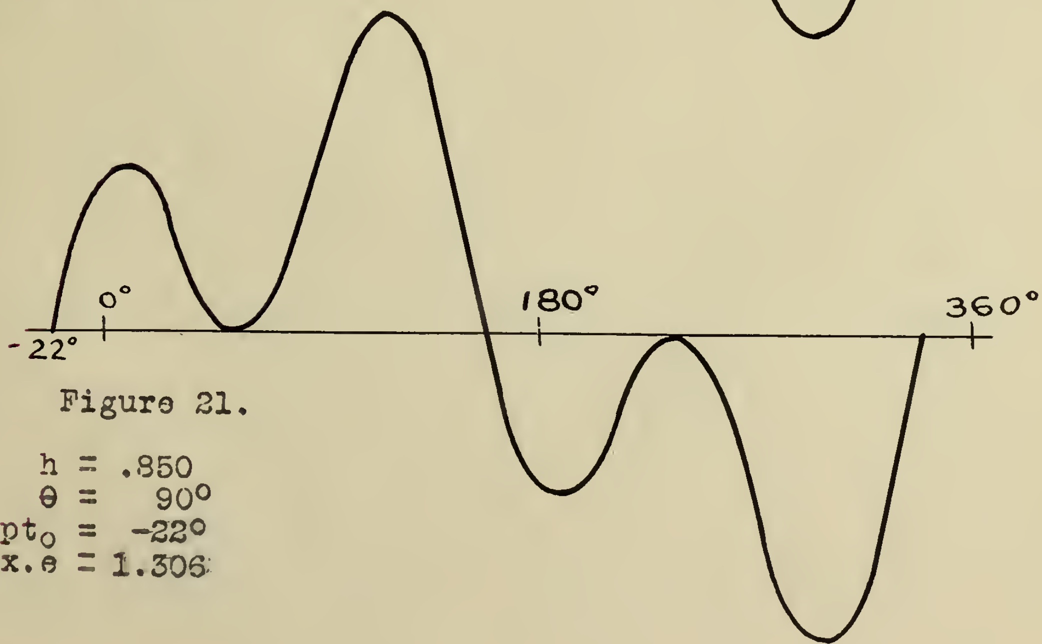

Figure 21.

$h = .850$
$\theta = 90^{\circ}$
$pt_o = -22^{\circ}$
$\max.e = 1.306$

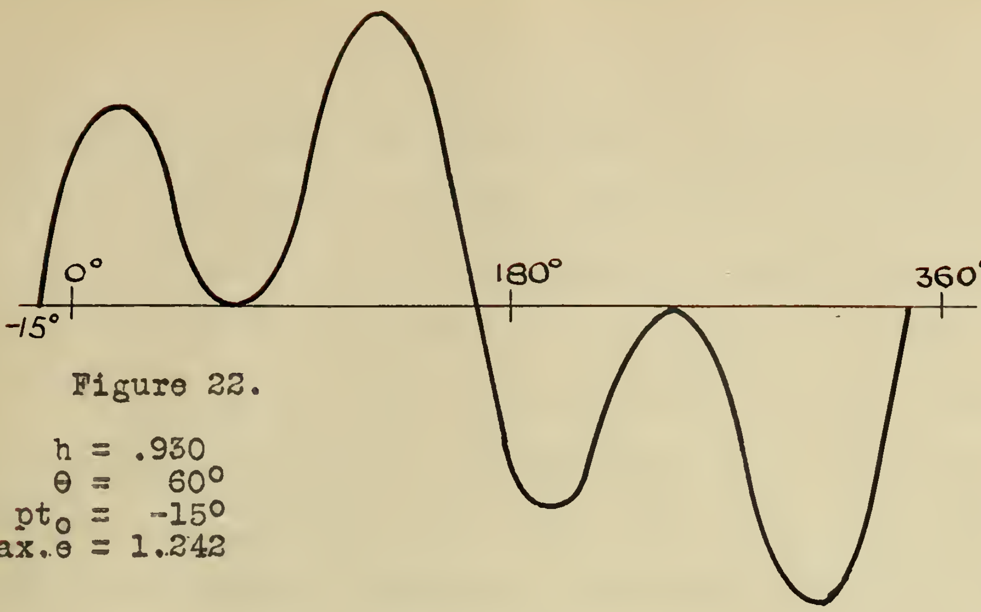

Figure 22.

$h = .930$
$\theta = 60^{\circ}$
$pt_{o} = -15^{\circ}$
$max.e = 1.242$

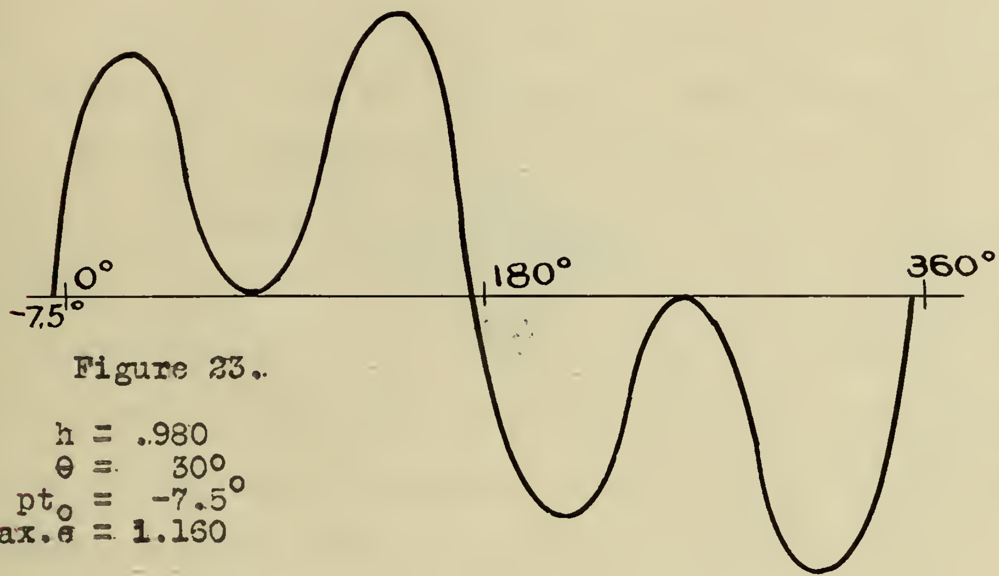

Figure 23.

$h = .980$
$\theta = 30^{\circ}$
$pt_{o} = -7.5^{\circ}$
$max.e = 1.160$

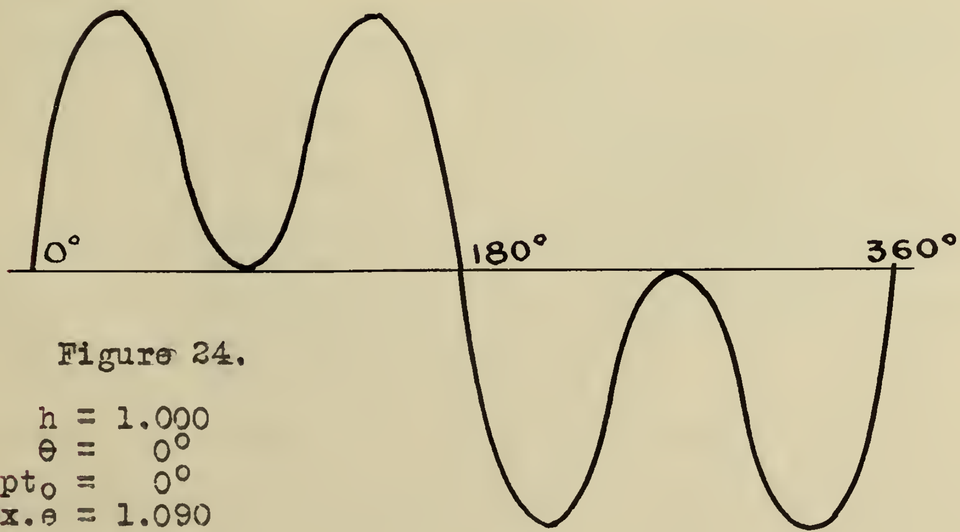

Figure 24.

$h = 1.000$
$\theta = 0^{\circ}$
$pt_{o} = 0^{\circ}$
$max.e = 1.090$

CONCLUSIONS.

From the preceding curves abd discussion, we may derive the following conclusions: While there are some cases where it is essential that the emf. wave be as nearly as possible a pure sine wave, yet in many cases it will be found desirable to introduce a certain small amount of the third or the fifth harmonic into the wave, preferably at the phase angle 180° with the fundamental, so as to get the greatest possible decrease, in the hysteresis loss, and at the same time produce the least possible distortion consistent therewith. When the amplitude of the harmonic is within the limits stated in the discussion, it has been shown that the hysteresis loss is proportional to the 1.6 power of the reciprocal of the form factor of the wave, the effective emf. being assumed the same, hence if we can, by any means whatever, obtain the wave form of a given emf. it is necessary only to determine the form factor in order to find the relative loss.

The curves shown on the three preceding pages give the maximum values of the emfs. , since these may be of interest in the case of extremely high voltages, where it may be desirable to keep the maximum as low as possible.

In three phase circuits it is necessary to avoid the use of the third harmonic, since it would in the case of a Y connected circuit, not appear in the line voltage, while in the delta connected armature the phase relations are such that a large short circuit circulating current may result in the armature, thus wasting power. Here the fifth harmonic might be used advantageously.

The foregoing discussion it is thought, is sufficient to cover the essential features of the problem, and to enable the designer of alternating current apparatus to use the formulas advantageously.