Junction of several weakly interacting quantum wires: a renormalization group study

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We study the conductance of three or more semi-infinite wires which meet at a junction. The electrons in the wires are taken to interact weakly with each other through a short-range density-density interaction, and they encounter a general scattering matrix at the junction. We derive the renormalization group equations satisfied by the S-matrix, and we identify its fixed points and their stabilities. The conductance between any pair of wires is then studied as a function of physical parameters such as temperature. We discuss the possibility of observing the effects of junctions in present day experiments, such as the four-terminal conductance of a quantum wire and crossed quantum wires.

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I. INTRODUCTION

Recent advances in the fabrication of semiconductor heterostructures have made it possible to study electronic transport in a variety of geometries. Recent studies of ballistic transport through a quantum wire (QW) have brought out the important role played by both scattering centers and the interactions between the electrons inside the QW. Theoretical studies using a renormalization group (RG) analysis show that repulsive interactions between electrons tend to increase the effective strength of the scattering as one goes to longer distance scales [1]; experimentally, this leads to a decrease in the conductance as the temperature is reduced or the wire length is increased [2]. Considerable effort has also gone into understanding the effects of (Fermi) leads [3], multiple impurities [4] and also contacts [5]. Motivated by the understanding of the effects of scattering in a one-dimensional problem, we are led to address the following question in this work: what is the effect of interactions between electrons on the conductance of more complicated geometrical structures such as three or more QWs meeting at a junction? This problem has been studied before in Ref. [6]; as explained below, our model differs from theirs in certain ways, and our results are quite different. We will show that for the case of weak interactions, the effects of a junction (characterized by an arbitrary scattering matrix S) on the conductance can be understood in great detail by using a RG technique introduced in Ref. [7]. We will also complement this with a study of the effects of certain special kinds of junctions for arbitrary electron interaction to gain a more complete picture.

The plan of the paper is as follows. In Sec. II, we will define a junction in terms of a scattering matrix, and we will provide a microscopic lattice model of a junction. In Sec. III, we will discuss an interacting theory of spinless fermions in the presence of an S-matrix at the junction, and we will enumerate some of the special S-matrices for which the theory can be bosonized. Sec. IV will contain a derivation of the RG equations for the junction S-matrix for the case of weak interactions in the wires. In Sec. V, we will study the fixed points of the RG equations and their stabilities for the case of three wires meeting at a junction. Wherever possible, we will compare our weak interaction results with the exact results available from bosonization. In Sec. VI, the results of the previous section will be used to study the conductance of a three-wire system as a function of the temperature in the vicinity of one of the fixed points. In Sec. VII, we will consider the temperature dependence of the four-terminal conductance of a quantum wire (which is often studied experimentally). In Sec. VIII, we will study the fixed points and stabilities of the RG equations of a four-wire system, and its four-terminal conductance. In Sec. IX, we will briefly discuss how to extend the previous analysis to the case of spinful fermions. We will make some concluding remarks (including a comparison of our model to that given in Ref. [6]) in Sec. X.

II. A MODEL FOR THE JUNCTION

To study the problem, we first need a model for the junction. Let us assume that N semi-infinite wires meet at a junction. The wires are parameterized by coordinates x_i, $i = 1, 2, ..., N$. The junction is the point where all the x_i are simultaneously equal to 0. We adopt the convention that each x_i increases from 0 as one goes outwards from the junction along wire i. Let us denote the incoming and outgoing one-electron wave functions on

wire i by $\psi_{Ii}(x_i)$ and $\psi_{Oi}(x_i)$ respectively (we are ignoring the spin label σ for the moment); see Fig. 1. For a given wave number $k > 0$, these wave functions are proportional to the plane waves $\exp(-ikx_i)$ and $\exp(ikx_i)$.

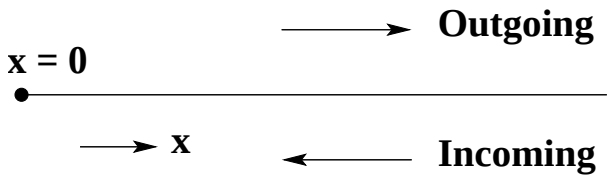

FIG. 1. Picture of a single wire showing the incoming and outgoing directions and the junction at $x = 0$.

The coefficients of the plane waves are related to each other by a $N \times N$ scattering matrix called S. Denoting the incoming and outgoing wave functions at the junction by the columns $\psi_I(0)$ and $\psi_O(0)$, we have the relation

$$\psi_O(0) \;=\; S\;\psi_I(0)\;. \tag{1}$$

Clearly, S must be unitary. (If we want the system to be invariant under time reversal, S must also be symmetric). The diagonal entries of S are the reflection amplitudes r_{ii}, while the off-diagonal entries are the transmission amplitudes t_{ij} to go from wire j to wire i.

It is useful, though not necessary, to have in mind a microscopic model of a junction with an unitary and symmetric S-matrix. A simple lattice model for this is shown in Fig. 2 for the case of three wires labeled by $i = 1, 2, 3$. The junction is the site labeled as 0, while the sites on the wires have labels going from 1 to ∞. The electrons hop from site to site with a hopping constant which is -1 on all bonds except the 3 bonds which join the sites labeled as 1 with the junction; on those three bonds, we take the hopping constants to be the real numbers $-u_i$. In addition, we have a chemical potential λ at the junction, while the chemical potential on all the other sites is 0. The momenta of the electrons go from $-\pi$ to π (taking the lattice spacing to be 1); the dispersion relation is given by $E = -2\cos k$. Since the chemical potential is zero at all sites except one, the system is at half-filling, and the Fermi points lie at $\pm k_F$ where $k_F = \pi/2$. For incoming momenta k close to k_F, we find that the entries of the S-matrix are given by

$$\begin{aligned} r_{ii} \;&=\; \frac{2u_i^2}{D} - 1\;, \\ t_{ij} \;&=\; \frac{2u_i u_j}{D}\;, \\ \text{where}\quad D \;&=\; \sum_{k=1}^{3} u_k^2 \;+\; i\lambda\;. \end{aligned} \tag{2}$$

This matrix is both unitary and symmetric, although it is not the most general possible matrix with those properties.

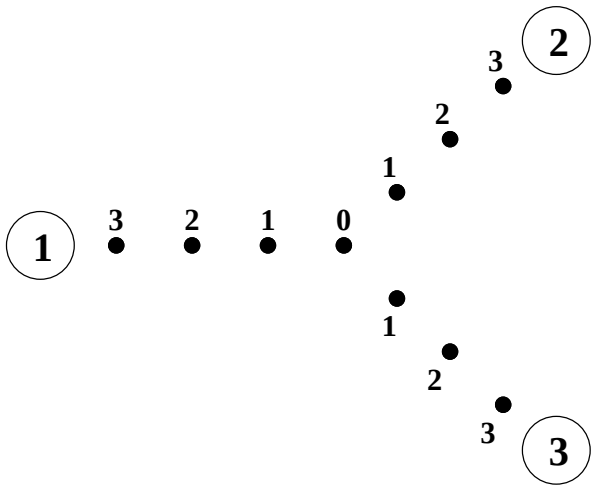

FIG. 2. Picture of the lattice model for three wires meeting at a junction.

III. TOMONAGA-LUTTINGER LIQUIDS WITHOUT SPIN

Let us now discuss the fermion fields in some more detail. We will consider a single wire for the moment, so that the label i can be dropped. Since all low-energy and long-wavelength processes are dominated by modes near the Fermi points $\pm k_F$, let us write the second-quantized field $\Psi(x)$ (which implicitly contains both fermion annihilation operators and wave functions) as

$$\Psi(x) \;=\; \Psi_I(x)\;e^{-ik_F x} \;+\; \Psi_O(x)\;e^{ik_F x}\;. \tag{3}$$

Note that the fields Ψ_I and Ψ_O defined in Eq. (3) vary slowly on the scale of the inverse Fermi momentum k_F^{-1}, since we have separated out the rapidly varying functions $\exp(\pm ik_F x)$. We will henceforth use the notation Ψ_I and Ψ_O for these slowly varying second-quantized fields, rather than the incoming and outgoing fields defined earlier. For these fields, we will only be interested in Fourier components with momenta k which satisfy $|k| << k_F$. We can then make a linear approximation for the dispersion relations which take the form $E = \pm\hbar v_F k$ for the fields Ψ_O and Ψ_I respectively, where v_F is the Fermi velocity. (For instance, for the lattice model discussed above, $v_F = 2\sin k_F$). We will also assume that the entries of the S-matrix do not vary with k in the limited range of momenta that we are interested in.

We now introduce a model for interactions between electrons. Let us consider a short-range density-density interaction of the form

$$H_{\rm int} \;=\; \frac{1}{2}\int\int dxdy\;\rho(x)\;V(x-y)\;\rho(y)\;, \tag{4}$$

where $V(x)$ is a real and even function of x, and the

density ρ is given in terms of the fermion field as $\rho(x) = \Psi^\dagger(x)\Psi(x)$. Using Eq. (3), we find that

$$\rho(x) \;=\; \Psi_I^\dagger \Psi_I \;+\; \Psi_O^\dagger \Psi_O \\ +\; \Psi_I^\dagger \Psi_O \; e^{i2k_F x} \;+\; \Psi_O^\dagger \Psi_I \; e^{-i2k_F x} \; . \qquad (5)$$

We can now rewrite the interaction in Eq. (4) in a simple way if $V(x)$ is so short-ranged that the arguments x and y of the two density fields can be set equal to each other wherever possible. [In doing so, we will be ignoring terms which have scaling dimension greater than 2, and are therefore irrelevant in the RG sense. We note that the assumption of a short-ranged interaction is often made in the context of the Tomonaga-Luttinger liquid description of systems of interacting fermions in one dimension.] Using the anticommutation relations between different fermion fields, we obtain

$$H_{\rm int} \;=\; g_2 \int dx \; \Psi_I^\dagger \Psi_I \Psi_O^\dagger \Psi_O \; , \qquad (6)$$

where g_2 is related to the Fourier transform of $V(x)$ as $g_2 = \tilde{V}(0) - \tilde{V}(2k_F)$. [Note that g_2 is zero if $V(x)$ is a δ-function; so $V(x)$ should have a finite range in order to have an effect.] Thus the interaction depends on a single parameter g_2 on each wire. Different wires may have different values of this parameter which we will denote by g_{2i}. For later use, we define the dimensionless constants

$$\alpha_i \;=\; \frac{g_{2i}}{2\pi\hbar v_F} \; , \qquad (7)$$

where we assume that the velocity v_F is the same on all wires.

For many problems involving a Tomonaga-Luttinger liquid, it is useful to bosonize the theory [8,9]. For spinless fermions, the bosonic theory is characterized by two quantities, namely, the velocity of the bosonic excitations v, and a dimensionless parameter K which is a measure of the interactions between the fermions. (Typically, K governs the exponents which appear in the power-law fall-offs of various correlation functions in the theory). For a model defined on the entire real line with the interaction parameter g_2 or α defined above, we find that [8]

$$v \;=\; v_F \; (1-\alpha^2)^{1/2} \; , \\ K \;=\; \Big(\frac{1-\alpha}{1+\alpha}\Big)^{1/2} \; . \qquad (8)$$

Thus $K = 1$ for noninteracting fermions, while $K < 1$ for short-range repulsive interactions. For weak interactions, we see that $v = v_F$ while $K = 1 - \alpha$ to first order in α. In this work, we will be interested in the case in which the interactions are weak and repulsive, i.e., the parameters α_i are all positive and small.

Although bosonization is a very powerful technique, it is not always possible to bosonize a system of interacting fermions with boundaries. In particular, for our system of interest, i.e., three or more semi-infinite wires meeting at a junction with some arbitrary S-matrix defined at that point, bosonization is a difficult task in general. The reason is that although one can always find linear combinations of the incoming and outgoing fermion fields which unitarily diagonalize the S-matrix, the four-fermion interactions in the bulk of the wires are generally not diagonal in terms of the same linear combinations. Conversely, the interactions in the bulk of the wires can be bosonized, but it is then generally not clear what boundary conditions should be imposed on the bosonic fields at the junction.

However, it is possible to bosonize the system easily for some special forms of the S-matrix at the junction. For the case of three wires, there seem to be only six such forms. These are as follows.

(a) Case I: Here $|r_{11}| = |r_{22}| = |r_{33}| = 1$, and all the other entries of the S-matrix are zero. This can be realized by the lattice model of Fig. 2 if we take the limit $\lambda \to \infty$. This case corresponds to the three wires being disconnected from each other. Each wire can then be bosonized by an unfolding technique described in Ref. [9].

(b) Cases II-IV: In case II, $|r_{33}| = |t_{12}| = |t_{21}| = 1$, and all the other entries of S are zero. This can be realized by our lattice model if we set $u_3 = \lambda = 0$ and $u_1 = u_2 \neq 0$. This corresponds to wire 3 being disconnected from wires 1 and 2; the latter two have perfect transmission into each other. Wire 3 can be bosonized as in case I, while wires 2 and 3 can be bosonized as a single infinite wire. Similarly, there are two other cases, called cases III and IV, which are obtained from case II by cyclically permuting the three wires. We note that cases I - IV are all invariant under time reversal, if we choose all the entries of the S-matrix to be real.

(c) Cases V-VI: In case V, $|t_{21}| = |t_{32}| = |t_{13}| = 1$, and all the other entries of S are zero. No matter how the phases of the three non-zero entries of S are chosen, this is not invariant under time reversal. (Therefore it cannot be realized by our lattice model for any choice of the parameters u_i, λ and k_F). This can be thought of as three infinite wires with chiral fields; for instance, one such wire is the incoming field along wire 1 which transmits perfectly into the outgoing field along wire 2. Finally, we have case VI obtained by time reversing case V; namely, $|t_{12}| = |t_{23}| = |t_{31}| = 1$, and all the other entries of S are zero. Cases V and VI can both be bosonized.

Before ending this section, we would like to make some remarks about the physical applicability of cases V and VI. If we think of the three wires as having finite widths, with the incoming and outgoing waves running along two different edges of each wire, then the forms of the S-matrices in cases V and VI are very similar to those describing the edge states of a quantum Hall system. However, the value of K in a quantum Hall system is fixed by

the filling fraction of the (two-dimensional) bulk of the system, not by the interaction between the edge states. (In fact, the interactions between the states at the opposite edges of a quantum Hall system are often ignored because of their spatial separation). In contrast to this, our model of the Tomonaga-Luttinger liquids in the wires and our derivation of the RG equations for the S-matrix given below both depend on the short-range interaction between the incoming and outgoing modes on each wire. Hence the results obtained by us may not be applicable to quantum Hall systems.

IV. RENORMALIZATION GROUP EQUATIONS FOR THE S-MATRIX

Rather than employ bosonization to study the case of an arbitrary S-matrix, we use an instructive and physically transparent method introduced in Ref. [7] to directly obtain RG equations for the entries of the S-matrix in the presence of electron interactions (provided that the interactions are weak). The basic idea of this method is the following. In the presence of a non-zero reflection amplitude r_{ii}, the density of noninteracting fermions in wire i has Friedel oscillations with wavenumber $2k_F$. When a weak interaction is turned on, an electron scatters from these oscillations by an amount proportional to the parameter α_i. Yue et al use this idea to derive the RG equations for an arbitrary S-matrix located at the junction of two semi-infinite wires. In the limits of both weak scattering ($r_{11} \to 0$) and strong scattering ($|r_{11}| \to 1$), their results reduce to those known from bosonization [1,9]. We will use the same idea for a junction of more than two wires. Not surprisingly, we will find that the results are much richer than those for two wires.

Let us briefly present the method of Yue et al. We first derive the form of the density oscillations in one particular wire given that there is a reflection coefficient r for waves coming in along that wire. For a momentum in the vicinity of k_F, we can write the wave function in the form

$$\psi_k(x) \;=\; e^{-i(k+k_F)x} \;+\; r\; e^{i(k+k_F)x} \;, \tag{9}$$

where $|k| << k_F$. In the ground state of the noninteracting system, the density is given by

$$< \rho(x) > \;\;=\;\; \int_{-\infty}^{0} \frac{dk}{2\pi}\; \psi_k^\star(x)\psi_k(x) \;, \tag{10}$$

where we have used the fact that only states with energy less than E_F (i.e., momenta less than k_F) are occupied, and we have extended the lower limit to $-\infty$ for convenience. (Alternatively, we can impose a cut-off at the lower limit of the form $\exp(\epsilon k)$, and take the limit $\epsilon \to 0$ at the end of the calculation). We then find that ρ has a constant piece ρ_0 (which can be eliminated by normal ordering the density operator), and a x-dependent piece given by

$$< \rho(x) > \;\; - \;\; \rho_0 \;\; = \;\; \frac{i}{4\pi x}\; (\; r^\star\; e^{-i2k_F x} - r\; e^{i2k_F x}\;) \;. \tag{11}$$

Using the expression in (5), we see that the expectation value $< \Psi_I^\dagger \Psi_I + \Psi_O^\dagger \Psi_O >$ is a constant, while

$$\begin{aligned} < \Psi_O^\dagger \Psi_I > \;\; &= \;\; \frac{ir^\star}{4\pi x} \;, \\ < \Psi_I^\dagger \Psi_O > \;\; &= \;\; -\; \frac{ir}{4\pi x} \;. \end{aligned} \tag{12}$$

Note that there is also a contribution to $\rho(x)$ from the waves transmitted from the other wires; however those are independent of x and can be absorbed in ρ_0. Thus the Friedel oscillations Eq. (11) in a given wire only arise from reflections within that wire.

Next we derive the reflection of the fermions from the Friedel oscillations, using a Hartree-Fock decomposition of the interaction in Eq. (6). The reflection is caused by the following terms in the decomposition

$$\begin{aligned} H_{\rm int} \;\; &= \;\; -\, g_2 \int_0^\infty dx\; (\;\; < \Psi_O^\dagger \Psi_I > \Psi_I^\dagger \Psi_O \\ &\qquad\qquad + \;\; < \Psi_I^\dagger \Psi_O > \Psi_O^\dagger \Psi_I \;) \;, \\ &= \;\; -\, \frac{ig_2}{4\pi} \int_0^\infty \frac{dx}{x}\; (r^\star\; \Psi_I^\dagger \Psi_O \;\; - \;\; r\; \Psi_O^\dagger \Psi_I) \;, \end{aligned} \tag{13}$$

where we have used (12) to write the second equation. Now we can derive the amplitude to go from a given incoming wave with momentum k to an outgoing wave (or vice versa) under the action of $\exp(-iH_{\rm int}t)$. The amplitude is given by

$$\begin{aligned} -i \int \frac{dk'}{2\pi}\; 2\pi\delta(E_k - E_{k'})\; &|{\rm outgoing}, k'> \\ &\times < {\rm outgoing}, k'|\; H_{\rm int}\; |{\rm incoming}, k> \\ = \;\; |{\rm outgoing}, k> \;\; &\frac{ig_2 r}{4\pi\hbar v_F} \int_0^\infty \frac{dx}{x}\; e^{-i2kx} \;, \end{aligned} \tag{14}$$

where we have used Eq. (13), the dispersion relation $E_k = \hbar v_F k$ (so that $\delta(E_k - E_{k'}) = (1/\hbar v_F)\delta(k - k')$), and the wave functions $\exp(\pm ikx)$ of the outgoing and incoming waves respectively. The integral over x in (14) is divergent at the lower end; we therefore introduce a short-distance cut-off d there. The amplitude in (14) then reduces to

$$-\, \frac{\alpha r}{2}\; \ln(kd) \tag{15}$$

plus pieces which remain finite as $kd \to 0$; we have used Eq. (7) here. Similarly, the amplitude to go from an outgoing wave to an incoming wave is given by

$$\frac{\alpha r^\star}{2}\; \ln(kd) \;. \tag{16}$$

These reflections from the Friedel oscillations can then be combined along with the S-matrix at the junction to calculate the corrections to the S-matrix. For instance, consider r_{ii}. To first order in the interaction parameters α_i, this gets corrections from the following processes. An incoming wave on wire i can either (i) turn into an outgoing wire on the same wire with the amplitude in (15) (with r replaced by r_{ii} in that expression), or (ii) get reflected from the junction with amplitude r_{ii} thereby turning into an outgoing wave, turn back into an incoming wave according to (16), then get reflected again from the junction, or (iii) transmit through the junction into wire j (with $j \neq i$) with amplitude t_{ji}, turn from an outgoing wave to an incoming wave on wire j according to (16) (with r replaced by r_{jj}), then transmit back through the junction to wire i with amplitude t_{ij}. The correction to r_{ii} is therefore

$$\begin{aligned} dr_{ii} &= - A_{ii} \ln(kd) \ , \\ \text{where} \quad A_{ii} &= - \frac{1}{2} [- \alpha_i r_{ii} + \alpha_i |r_{ii}|^2 r_{ii} \\ &\quad + \sum_{j \neq i} \alpha_j t_{ij} r^\star_{jj} t_{ji}] \ . \end{aligned} \tag{17}$$

Similarly, the transmission amplitude t_{ji} from wire i to wire j can get corrections from the following processes. The incoming wave on wire i can either (i) get reflected from the junction with amplitude r_{ii}, turn back into an incoming wave according to (16), and then transmit into wire j with amplitude t_{ji}, or (ii) transmit into to wire j first, turn into an incoming wave on wire j according to (16), then get reflected from the junction with amplitude r_{jj}, or (iii) transmit into a wire k (with $k \neq i, j$), turn into an incoming wave in wire k according to (16) (with r replaced by r_{kk}), then transmit into wire j with amplitude t_{jk}. Hence the correction to t_{ji} is

$$\begin{aligned} dt_{ji} &= - A_{ji} \ln(kd) \ , \\ \text{where} \quad A_{ji} &= - \frac{1}{2} [\alpha_i t_{ji} |r_{ii}|^2 + \alpha_j |r_{jj}|^2 t_{ji} \\ &\quad + \sum_{k \neq i,j} \alpha_k t_{jk} r^\star_{kk} t_{ki}] \ . \end{aligned} \tag{18}$$

Yue et al then derive the RG equations for the S-matrix which is now considered to be a function of a distance scale L; they show that $-\ln(kd)$ in Eqs. (17-18) can effectively be replaced by dl, where $l = \ln(L/d)$. The RG equations therefore take the from

$$\begin{aligned} \frac{dr_{ii}}{dl} &= A_{ii} \ , \\ \frac{dt_{ij}}{dl} &= A_{ij} \ , \end{aligned} \tag{19}$$

where A_{ii} and A_{ij} are given above. We can write Eqs. (19) in a simpler way. Given the matrix S and the parameters α_i (which do not flow under RG), we can define a diagonal matrix F whose entries are

$$F_{ii} = - \frac{1}{2} \alpha_i r_{ii} \ . \tag{20}$$

Then the RG equations can be written in the matrix form

$$\frac{dS}{dl} = S F^\dagger S - F \ . \tag{21}$$

This is the central result of our work. One can verify from (21) that S continues to remain unitary under the RG flow; it also remains symmetric if it begins with a symmetric form.

We note also that the form of (21) remains unchanged if S is multiplied either from the left or from the right by a diagonal unitary matrix with entries of the form

$$U_{ii} = e^{i\phi_i} \ , \tag{22}$$

where the real numbers ϕ_i are independent of the length parameter l. The fixed points discussed below will therefore also remain unchanged under such phase transformations. We will generally not distinguish between S-matrices which differ only by such transformations.

V. FIXED POINTS AND STABILITY ANALYSIS

We will now study the RG flow in some detail. We will consider the case of three wires for convenience, although much of the discussion below can be generalized to more than three wires. Let us first find the fixed points of Eq. (21). The required condition is that $SF^\dagger = FS^\dagger$, i.e., that $SF^\dagger$ is hermitian. It is easy to see that the six cases I - VI considered above are all fixed points of the RG. In addition, there is another fixed point which we will call case VII. For the physically interesting situation in which all the α_i are positive, this case is described as follows. We first define a quantity a as

$$a = \frac{1}{\sum_{i=1}^3 \alpha_i^{-1}} \ . \tag{23}$$

Then the fixed point S-matrix has the entries

$$\begin{aligned} r_{ii} &= - \frac{a}{\alpha_i} \quad \text{for all } i \ , \\ t_{ij} &= \sqrt{(1 - \frac{a}{\alpha_i})(1 - \frac{a}{\alpha_j})} \quad \text{for all } i, j \ . \end{aligned} \tag{24}$$

It is possible to obtain a family of fixed points related to the above by multiplying the various amplitudes by some phases as discussed in Eq. (22). However, we will mainly consider the above form of case VII for simplicity. Note that for the case of equal interactions in the three wires (i.e., all the α_i equal to each other), and $\lambda = 0$, the fixed point is a well-known S-matrix whose entries are $r_{ii} = -1/3$ for all i, and $t_{ij} = 2/3$ for all i, j. This is symmetric under all possible permutations of the three wires, and has the maximum transmission (in all channels simultaneously) allowed by unitarity.

Having found the fixed points of the RG equations, we can study their stabilities. Let us write a fixed point of the S-matrix as S_0, and a small deviation from this as the matrix ϵS_1, where ϵ is a small real parameter. Given S_0, we are interested in finding the various flow 'directions' S_1 such that Eq. (21) takes the simple form

$$\frac{d\epsilon}{dl} \;=\; \mu\epsilon \;, \tag{25}$$

where μ is a real number. The solution of this equation is $\epsilon(l) = \epsilon(0)\exp(\mu l)$, where $\epsilon(0)$ is given by the deviation of the S-matrix from S_0 at the microscopic (e.g., lattice) scale. Thus $\mu < 0$ indicates that S is stable against a perturbation in the direction of the corresponding S_1, while $\mu > 0$ indicates an instability in the direction of the corresponding S_1. We now consider the various fixed points in turn. [All the fixed points have directions in which $\mu = 0$ corresponding to the phase rotations of the S-matrix described in Eq. (22). We will ignore these directions in the following discussion.]

(a) Case I: This turns out to be stable against perturbations in all directions. There are three directions in which μ takes the values $-(\alpha_1+\alpha_2)/2$, $-(\alpha_2+\alpha_3)/2$, and $-(\alpha_1+\alpha_3)/2$ respectively. These are negative since we are assuming that the interactions in all the wires are repulsive. Note that this result agrees, to first order in the α_i, with the exact results one obtains from bosonization. The operator which tunnels a particle from wire 1 to wire 2 has the scaling dimension $(K_1+K_2)/(2K_1K_2)$. For weak interactions, this is equal to $1+(\alpha_1+\alpha_2)/2$. Under a RG flow, therefore, the coefficient of the tunneling operator satisfies Eq. (25) with $\mu = -(\alpha_1+\alpha_2)/2$.

(b) Cases II-IV: Case II has two stable directions, both with $\mu = -\alpha_3/2$ (these correspond to tunneling from wire 3 to wire 1 or wire 2), and one unstable direction with $\mu = (\alpha_1+\alpha_2)/2$ corresponding to reflection between wires 1 and 2. These results also agree, to first order in the α_i, with those obtained from bosonization. For tunneling from wire 3 to wire 1, the operator has the dimension $1/(2K_3) + (K_1 + 1/K_1)/4$; this is equal to $1+\alpha_3/2$ to first order in the α_i, and therefore gives $\mu = -\alpha_3/2$. A weak reflection between wires 1 and 2 has the dimension $(K_1+K_2)/2$ which is equal to $1-(\alpha_1+\alpha_2)/2$ to first order. This gives a flow with $\mu = (\alpha_1+\alpha_2)/2$ which goes in the direction of case I. The RG flows in cases III and IV can be worked out similarly.

(c) Cases V-VI: Case V has three unstable directions with $\mu = \alpha_1/2$, $\alpha_2/2$ and $\alpha_3/2$ respectively. The three directions give flows towards cases II, III and IV. However, if we start out with an appropriate linear combination of the three directions, we can flow directly to case I. Similarly, one can work out the flows for case VI.

(d) Case VII: The stability of flows near this fixed point appears to be difficult to study in general. However, the case of equal interactions, $\alpha_1 = \alpha_2 = \alpha_3 = \alpha$, can be studied more easily. It has one unstable direction with $\mu = \alpha$ which flows in the direction of case I (this is discussed further in Eq. (35) below), and two unstable directions with $\mu = 2\alpha/3$ which flow towards one of the four cases I - IV depending on the precise choice of the initial direction. Further, for appropriately chosen directions of the initial flow, one can go from cases V and VI to case VII. Near case VII, these correspond to two stable directions with $\mu = -\alpha$.

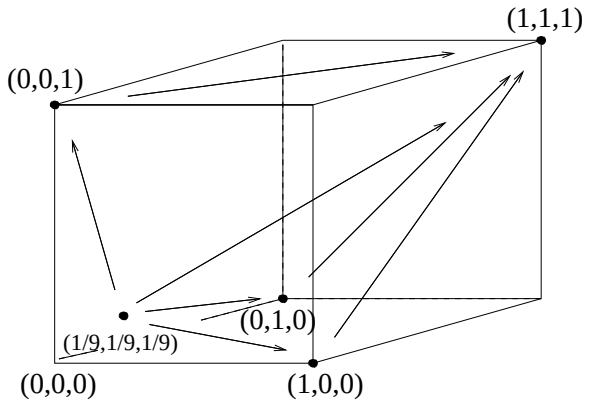

FIG. 3. Schematic diagram of the various time-reversal invariant fixed points for the 3-wire junction problem. Cases I-IV and VII are indicated as (1,1,1), (0,0,1), (0,1,0), (1,0,0) and (1/9,1/9,1/9), where the sets of three numbers denote the modulus squared of the diagonal entries of the respective S-matrices. RG flows between the various fixed points are indicated by the arrows.

Based on the above, we can state the flow diagram in the space of all S-matrices as follows. In general, case I is the most stable. Cases II, III and IV are only unstable to a flow towards case I. Cases V and VI are unstable to flows towards cases I - IV. Finally, for the case of equal interactions α_i, case VII is unstable to flow towards cases I - IV, and cases V and VI are also unstable to a flow towards case VII (if one starts out in the appropriate direction). We have verified this flow diagram numerically by starting from a number of S-matrices close to the various fixed points and letting them evolve according to Eq. (21).

We thus see that the flow diagram for the case of 3 wires (with repulsive interactions on all the wires) is much richer than in the case of two wires. In the latter case, there are only two fixed points, a stable one at $|r_{11}| = |r_{22}| = 1$ (two disconnected wires), and an unstable one at $|t_{12}| = |t_{21}| = 1$ (a perfectly transmitting wire). The RG flow simply goes from the first point to the second [1,9].

VI. CONDUCTANCE OF A THREE-WIRE SYSTEM

Having studied the S-matrix for a three-wire system as a function of the RG distance scale l, we can now discuss the conductance of this system. We will assume that the three wires, instead of being really semi-infinite, are connected to three Fermi liquid leads (with the interaction parameter being given by $K = 1$) at a large distance from the junction. We will also assume that there is only one transverse channel of spinless fermions in each wire; in this band, there is a resistance of e^2/h at the contacts between the leads and the wires [10]. Although the contacts can themselves scatter the fermions [5], we will ignore such effects here.

We take the fermions in all the leads to have the same Fermi energy E_F, and the net current on all wires to be zero in the absence of any applied voltage on the leads. Now suppose that the voltage is changed by a small amount V_i on lead i. Then the net current flowing out of wire i will satisfy the linear relationship [11,10]

$$I_i \;=\; \frac{e^2}{h} \sum_{j=1}^{3} T_{ij} V_j \;, \tag{26}$$

where the T_{ij} (for $i \neq j$) define the various transmission probabilities, and $T_{ii} + 1$ denote the reflection probabilities. These are related to the S-matrix at the junction as follows

$$\begin{aligned} T_{ij} \;&=\; |t_{ij}|^2 \quad \text{for} \quad i \neq j \;, \\ \text{and} \quad T_{ii} \;&=\; |r_{ii}|^2 \;-\; 1 \;. \end{aligned} \tag{27}$$

Since the unitarity of the S-matrix implies that

$$|r_{ii}|^2 \;+\; \sum_{j \neq i} |t_{ij}|^2 \;=\; 1 \;, \tag{28}$$

we see from (26) that the currents I_j do not change if all the voltages V_i are changed by the same amount. When a small voltage V_i is applied on lead i in addition to the Fermi energy, it increases the number of incoming fermions on that lead by an amount given by eV_i times the density of states in energy per unit length. For noninteracting spinless fermions in one dimension, the density of states in a continuum theory is given by

$$\rho(E_F) \;=\; \frac{1}{2\pi\hbar v_F} \;, \tag{29}$$

where v_F is the Fermi velocity. We assume this expression for $\rho(E_F)$ to be the same on all leads. In the absence of any scattering from the contact i or from impurities inside wire i, these fermions will travel ballistically towards the junction where they will be either reflected back or transmitted to one of the other two wires. Following that, the fermions again travel ballistically till they emerge from one of the three wires. The outgoing currents are therefore given by ev_F times the extra number of electrons coming in on wire i times the appropriate transmission coefficients on the other two wires and the reflection coefficient (subtracted from the incoming current) on wire i.

We can now compute the conductance by setting, say, wire 3 to be the potential probe, i.e., $I_3 = 0$ [11]. Then, using the set of equations (26)-(28) given above, the corresponding three-terminal relations are found to be

$$G_{12,13} = \frac{I_1}{V_1 - V_3} = \frac{e^2}{h} \left(T_{12} + T_{13} + \frac{T_{12}T_{13}}{T_{32}}\right) , \tag{30}$$

$$G_{12,23} = \frac{I_1}{V_2 - V_3} = \frac{e^2}{h} \left(T_{12} + T_{32} + \frac{T_{12}T_{32}}{T_{13}}\right) , \tag{31}$$

where $I_1 = -I_2$, and the two-terminal conductance is given by

$$G_{12,12} = \frac{I_1}{V_1 - V_2} = \frac{e^2}{h} \left(T_{12} + \frac{T_{13}T_{32}}{T_{13} + T_{32}}\right) . \tag{32}$$

In the above conductance expressions, we have employed the standard convention for specifying the current (first pair of indices) and voltage (second pair of indices) leads. It is worth noting the incoherence introduced in $G_{12,12}$ through the non-zero transmissions of carriers T_{13} and T_{32} into the additional arm (here, wire 3). The conductances given in Eqs. (31) and (32) will flow under RG following Eq. (21). Let us begin with some S-matrix at a microscopic distance scale d (such as the spacing between the sites in a lattice model). The RG flow in (21) is valid till the logarithmic length scale reaches a physical long-distance cut-off. The appropriate cut-off in this problem is the smaller of the scales $\ln(L_i/d)$ (where L_i is the length of wire i from the junction to its lead) and $\ln(L_T/d)$, where

$$L_T \;=\; \frac{\hbar v_F}{k_B T} \;, \tag{33}$$

with T being the temperature [5]. For simplicity, let us consider the case of high temperature where L_T is smaller than all the wire lengths L_i, but larger than the microscopic length d. Then the RG flow has to be stopped at the scale $l_T = \ln(L_T/d)$ since there is no phase coherence at distance scales larger than this. Now let us suppose that at the microscopic level, the S-matrix deviates slightly from a fixed point S_0 by an amount $\epsilon(0)S_1$, where S_1 is an unstable direction with $\mu > 0$. Then at the scale l_T, the deviation is given by

$$dS(l_T) \;=\; \left(\frac{L_T}{d}\right)^{\mu} \epsilon(0) S_1 \;. \tag{34}$$

We thus see that the deviations from S_0 will grow as $1/T^{\mu}$ as the temperature decreases. Of course, this is only true as long as the deviation is not too large, since Eq. (25)

is only valid to first order in ϵ. These power-law dependences of the conductances on the temperature should be observable experimentally if a three-wire system can be fabricated.

As a specific example, consider case VII in which S_0 has $r_{ii} = -1/3$ and $t_{ij} = 2/3$. If all the interactions are equal, with $\alpha_i = \alpha$, we saw above that this is unstable to a perturbation towards case I (three disconnected wires) with $\mu = \alpha$. The small deviation which takes case VII towards case I is given by

$$dS = \begin{pmatrix} i4\epsilon - 3\epsilon^2 & i\epsilon - 15\epsilon^2/2 & i\epsilon - 15\epsilon^2/2 \\ i\epsilon - 15\epsilon^2/2 & i4\epsilon - 3\epsilon^2 & i\epsilon - 15\epsilon^2/2 \\ i\epsilon - 15\epsilon^2/2 & i\epsilon - 15\epsilon^2/2 & i4\epsilon - 3\epsilon^2 \end{pmatrix} \quad (35)$$

to second order in the real parameter ϵ. We have gone up to second order so as to calculate the correction to T_{ij} which only begins at that order. Namely, $t_{ij} = 2/3 + i\epsilon - 15\epsilon^2/2$, which gives $T_{ij} = 4/9 - 9\epsilon^2$. Since $\mu = \alpha$, the deviation of T_{ij} from 4/9 will grow as $\epsilon^2 \sim 1/T^{2\alpha}$ as the temperature is reduced. For example, the two-terminal conductance in this case will be

$$G^{VII}_{12,12} = \frac{e^2}{h}(\frac{2}{3} - \frac{27}{2}c_1 T^{-2\alpha}) \ , \quad (36)$$

where c_1 is some constant, while the three-terminal conductances for this case are identical and are given by

$$G^{VII}_{12,13} = \frac{e^2}{h}(\frac{4}{3} - 27c_1 T^{-2\alpha}) \simeq 2G^{VII}_{12,12} \quad (37)$$

Thus the power-law dependence on T can provide information on the strength of the interaction α.

VII. FOUR-TERMINAL CONDUCTANCE OF A QUANTUM WIRE

We consider here the case of the four-terminal conductance of a quantum wire of finite length L measured at high temperatures such that the thermal length L_T $(= \hbar v_F/k_B T)$ $> l$.

The quantum wire is connected to two reservoirs 1 and 2 which act as current probes. In addition, the quantum wire is also weakly coupled to two voltage probes 3 and 4 via identical barriers with tunneling amplitudes $\delta \ll 1$. We consider the current and voltage probes to be semi-infinite, two-dimensional electron gas (2DEG) Fermi liquid reservoirs; these can be modeled as Tomonaga-Luttinger leads with interaction parameter $K_L = 1$, i.e., in our case of very weak interactions, $\alpha_L = 0$. The quantum wire is itself modeled as a TLL with weak repulsive interactions characterized by a parameter α_W. We can now see that this case is akin to that of a system of two 3-wire junctions with a common arm (of finite length L and with $L \ll L_T$). Further, one arm in each of the two 3-wire junctions is coupled to the other two through a weak tunneling amplitude; this case

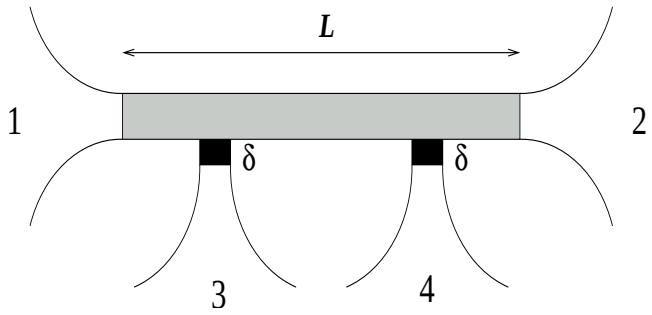

FIG. 4. Schematic diagram of a quantum wire of length L (grey shaded region) connected to the two current probes (1 and 2) and two voltage probes (3 and 4). The voltage probes are very weakly coupled to the quantum wire via tunneling barriers of amplitude $\delta \ll 1$ (black shaded regions).

thus falls somewhere between the fixed point I and any one of the fixed points II-IV discussed earlier.

Following the analysis of Ref. [11], we can write the four-terminal conductance of this system as

$$G_{12,34} = \frac{e^2}{h} T_{12} \frac{(T_{31} + T_{32})(T_{41} + T_{42})}{T_{31}T_{42} - T_{32}T_{41}} \quad (38)$$

where T_{ij} stands, as usual, for the transmission from lead i to lead j. As transmission from lead 1 to lead 2 can take place through paths which never cross any of the two tunneling barriers, we can write (to lowest order in δ) $T_{12} = T^{(0)}_{12}$. Transmission from lead 3 to lead 1 can take place, to lowest order in δ, through a path that crosses one tunneling barrier; this gives $T_{31} = \delta T^{(1)}_{31}$, where $T^{(1)}_{31}$ is a positive number of order 1. Similarly, even the simplest path from lead 3 to lead 4 needs the crossing of two barriers, giving $T_{34} = \delta^2 T^{(2)}_{34}$. Thus, keeping only terms till order δ^2, we can write $G_{12,34}$ as

$$G_{12,34} = \frac{e^2}{h} T^{(0)}_{12} \frac{(T^{(1)}_{31} + T^{(1)}_{32})(T^{(1)}_{41} + T^{(1)}_{42})}{T^{(1)}_{31}T^{(1)}_{42} - T^{(1)}_{32}T^{(1)}_{41}} \quad (39)$$

The four-terminal resistance $R_{12,34} = 1/G_{12,34}$ lies in the range $-h/(e^2\ T^{(0)}_{12}) \leq R_{12,34} \leq h/(e^2\ T^{(0)}_{12})$. The RG flow of the tunneling barriers δ will take place as given earlier, but with a parameter μ which is dependent on the interaction parameter of the quantum wire α_W (note that $\alpha_L = 0$ for all the four probes). Now, as we have identical barriers connecting two identical voltage probes 3 and 4 to the quantum wire, the various wire-voltage probe transmissions, such as $T^{(1)}_{31}, T^{(1)}_{32}, T^{(1)}_{41}$ and $T^{(1)}_{42}$, will have identical power-law dependences on the temperature. Thus, in the expression (39) given above, the only temperature dependence of $G_{12,34}$ will come from the RG flow of the transmission $T^{(0)}_{12}$, since all the temperature dependences coming from the wire-voltage probe transmissions will cancel out. Further, as the two-terminal conductance $G_{12,12} \sim (e^2/h)\ T^{(0)}_{12}$ to lowest order in δ,

we can see that the temperature dependences of $G_{12,12}$ and $G_{12,34}$ are identical!

In a recent experiment [12], Picciotto *et al.* measured both the two-terminal resistance $R_{12,12}$ $(= 1/G_{12,12})$ as well as the four-terminal resistance $R_{12,34}$ of a cleaved-edge overgrowth quantum wire in a GaAs-AlGaAs heterojunction using two weakly coupled voltage probes. They found that while $R_{12,12}$ is quantized in integer multiples of $h/2e^2$, $R_{12,34}$ fluctuated above and below zero and finally vanished as the gate voltage was made less negative. $R_{12,34}$ fluctuates about zero because its value depends critically on the invasive nature of the probes (i.e., on the precise values of the transmissions $T_{31}^{(1)}, T_{32}^{(1)}, T_{41}^{(1)}$ and $T_{42}^{(1)}$), and the fact that $-h/(e^2 T_{12}^{(0)}) \leq R_{12,34} \leq h/(e^2 T_{12}^{(0)})$. The average value of $R_{12,34}$ vanishes due to the fact that the intrinsic resistance of a quantum wire without any defects or impurities comes from its connections to the 2DEG reservoirs, i.e., the so-called *contact resistances* [12]. Our prediction of the identical power-law variations of $G_{12,12}$ and $G_{12,34}$ with temperature can also be tested in such an experiment by taking measurements of the two conductances at various temperatures but at a fixed value of the gate voltage (this holds the values of the various transmissions T_{ij} fixed at the microscopic level, and their observed values will vary with the temperature through the RG equations).

VIII. ANALYSIS OF A FOUR-WIRE SYSTEM

We can carry out a similar analysis of the fixed points and the conductance for a system of four wires meeting at a junction. In this section, we will assume for simplicity that the interaction parameters $\alpha_i = \alpha$ are equal on all the wires.

Let us first consider the fixed points of the RG equations (21) for a four-wire system. To begin with, one can readily identify $4! = 24$ fixed points which are natural generalizations of the $3! = 6$ fixed points (cases I-VI) that we found above for the three-wire system. These fixed points correspond to all the possible ways in which each row (or column) of the S-matrix at the junction has only one non-zero entry whose modulus is equal to 1. We thus have the following possibilities.

(a) The simplest case is one in which all the four wires are disconnected from each other. The S-matrix is then diagonal, with all the diagonal entries having unit modulus.

(b) There are six cases in which two of the wires are disconnected from all the others, while the remaining two wires transmit perfectly into each other.

(c) There are three cases in which pairs of wires (say, 1,2 and 3,4) transmit perfectly into each other.

(d) There are eight cases in which one wire (say, wire 4) is disconnected from the other three, while the other three wires (1, 2 and 3) are connected to each other cyclically as in Cases V and VI for the three-wire system.

(e) There are six cases in which the four wires transmit perfectly into each other in a cyclical way, such as, 1 into 2, 2 into 3, 3 into 4, and 4 into 1.

We note that the 10 cases given in (a-c) are invariant under time reversal if we choose all the entries of the S-matrix to be real; these 10 cases allow bosonization to be done. The 14 cases in (d-e) necessarily violate time-reversal invariance; they can also be bosonized.

In addition to the 24 cases given above, there are 5 more fixed points of the RG equations. Four of these correspond to situations in which one of the wires (say, 4) is disconnected from the other three, while the other three wires (1, 2 and 3) have the completely symmetric and maximally transmitting S-matrix of the form given in Case VII above. The fifth case corresponds to the case in which the four wires have a completely symmetric and maximally transmitting S-matrix; the diagonal and off-diagonal entires of this matrix are given by $-1/2$ and $1/2$ respectively.

We thus have a total of 29 fixed points for a four-wire system in contrast to 7 fixed points for the three-wire system. In addition to these 29 cases, we will now see that the four-wire system has some new classes of fixed points which do not exist for systems with less than four wires. Namely, there exist *two-parameter families* of fixed points in the four-wire system. In contrast to these, the fixed points of the two- and three-wire systems are all isolated points, i.e., they have no variable parameters (apart from some trivial phases).

Although we have not studied all the two-parameter families of fixed points in the four-wire system, we can exhibit some of these families explicitly. Two examples are given by

$$S = \begin{pmatrix} 0 & x_1 & iy_1 & 0 \\ x_2 & 0 & 0 & iy_2 \\ iy_2 & 0 & 0 & x_2 \\ 0 & iy_1 & x_1 & 0 \end{pmatrix} , \tag{40}$$

where x_i and y_i are four real numbers satisfying the constraints $x_1^2 + y_1^2 = x_2^2 + y_2^2 = 1$, and

$$S = \begin{pmatrix} 0 & x & -y & -z \\ x & 0 & -z & y \\ -y & z & 0 & x \\ z & y & x & 0 \end{pmatrix} , \tag{41}$$

where x, y and z are three real numbers satisfying the constraint $x^2 + y^2 + z^2 = 1$. [It is easy to see that these are fixed points of Eq. (21) since the diagonal matrix F is equal to zero for these families]. Note that these two families have some members in common which are obtained by setting $x_1 = x_2 = x$, $y_1 = y_2 = y$ and $z = 0$, and then performing some phase transformations. Further, these families include some of the fixed points given earlier as special cases.

The two-parameter families are fixed points of the RG equations (21) which are only valid to first order in the interaction parameter α. Do they remain fixed points if we go to higher orders in α? One way to answer this question is to use the technique of bosonization. As remarked earlier, it does not seem possible to bosonize an interacting fermionic theory for all possible S-matrices. Fortunately, the two-parameter families described above contain some special points at which bosonization can be done. For instance, consider the S-matrix

$$S \;=\; \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \;, \tag{42}$$

which corresponds to wires 1,2 (and wires 3,4) transmitting perfectly into each other. We can bosonize this system; for equal interaction strengths on all the wires, the bosonic theory will have the same parameter K for all wires. We then turn on small perturbations corresponding to either y_1, y_2 in the family given in (40), or y, z in the family given in (41). These correspond to hopping at the junction between wire 1 (or 2) and wire 3 (or 4). All these hopping operators have the scaling dimension $(K + 1/K)/2$ which is necessarily larger than 1; hence they are irrelevant, and the perturbed S-matrices will therefore flow back to Eq. (42) under RG. For weak interactions with $K = 1 - \alpha$, we see that the scaling dimension differs from 1 only at order α^2 and higher, which explains why these small perturbations look like fixed points at order α.

We therefore conclude that the two-parameter families given above are generally *not* fixed points of the exact (i.e., to all orders in the interaction strengths) RG equations. Although we have shown this only in the vicinity of some bosonizable points, it is plausible that this statement will also be true for most other members of the families. However, this does not rule out the possibility that there may be non-trivial and isolated members of these families which are fixed points of the exact RG equations. Let us present a plausible example of such a non-trivial fixed point. We consider a one-parameter family of S-matrices of the form

$$S \;=\; \begin{pmatrix} 0 & x & iy & 0 \\ x & 0 & 0 & iy \\ iy & 0 & 0 & x \\ 0 & iy & x & 0 \end{pmatrix} \;, \tag{43}$$

where $x^2 + y^2 = 1$ and $0 \le x, y \le 1$. The two end-points of this family given by $(x, y) = (1, 0)$ and $(0, 1)$ are bosonizable because they consist of pairs of perfectly transmitting wires (1,2 and 3,4 at the first point, and 1,3 and 2,4 at the second point) which transmit perfectly into each other. Within this one-parameter family, the bosonization approach discussed above shows that both the end-points are stable, since small perturbations from them (corresponding to turning on $y_1 = y_2$ in Eq. (40)) are irrelevant. The simplest possibility therefore is that there is one unstable fixed point which lies between the two end-points; since the interaction strengths in all the wires are equal, this fixed point is likely to be at the half-way point given by $x = y = 1/\sqrt{2}$. However, we are unable to directly verify that this is an unstable fixed point of the exact RG equations since this point does not seem to be bosonizable.

To summarize, we see that the pattern of fixed points and RG flows for a four-wire system is immensely more complicated than those of two- and three-wire systems. We do not have a complete classification of the fixed points for a four-wire system. Some families of S-matrices which appear to be fixed points at first order in the interaction strengths turn out not to be fixed points at higher orders.

We now turn to a discussion of the temperature dependences of the conductance corrections. Our arguments will be very similar to those given for a three-wire system at the end of Sec. VI. We consider the vicinity of one particular fixed point of the four-wire system, namely, the completely symmetric and maximally transmitting S-matrix. Let us perturb this in a completely symmetric way, so that the entries of the S-matrix are given by

$$\begin{aligned} r_{ii} \;&=\; -\,\frac{1}{2} \;+\; i3\epsilon \;-\; 3\epsilon^2 \;, \\ t_{ij} \;&=\; \frac{1}{2} \;+\; i\epsilon \;-\; 5\epsilon^2 \;, \end{aligned} \tag{44}$$

to second order in the small real number ϵ. [This perturbation will eventually lead to the situation in which all the four wires are disconnected from each other]. Using Eqs. (21), we find that the perturbation initially grows as in Eq. (34) with $\mu = \alpha$. The arguments presented in Sec. VI therefore imply that at high temperature, the transmission probabilities $T_{ij} = 1/4 - 4\epsilon^2$ vary with temperature as

$$T_{ij} \;=\; \frac{1}{4} \;-\; c_2\, T^{-2\alpha} \;, \tag{45}$$

where c_2 is some constant.

We can also compute the four-terminal conductances of this system by following the arguments of Ref. [11] and those given in Sections VI and VII. For a set of four probes $\{m\ n\ k\ l\}$ (which will be a permutation of $\{1\ 2\ 3\ 4\}$), we can write the relation between the currents $I_m = -I_n = I_1$, $I_k = -I_l = I_2$ and the voltages $V_1 = (\mu_m - \mu_n)/e$, $V_2 = (\mu_k - \mu_l)/e$ (where μ_i denotes the chemical potential of the $i^{\rm th}$ probe) as

$$\begin{pmatrix} I_1 \\ I_2 \end{pmatrix} \;=\; \frac{e^2}{h} \begin{pmatrix} \alpha_{11} & -\alpha_{12} \\ \alpha_{21} & \alpha_{22} \end{pmatrix} \begin{pmatrix} V_1 \\ V_2 \end{pmatrix} \;, \tag{46}$$

where

$$\begin{aligned}
\alpha_{11} &= [(1-T_{11})P - (T_{14}+T_{12})(T_{41}+T_{21})]/P \ , \\
\alpha_{12} &= (T_{12}T_{34} - T_{14}T_{32})/P \ , \\
\alpha_{21} &= (T_{21}T_{43} - T_{41}T_{23})/P \ , \\
\alpha_{22} &= [(1-T_{22})P - (T_{21}+T_{23})(T_{32}+T_{12})]/P \ , \\
P &= T_{12}+T_{14}+T_{32}+T_{34} = T_{21}+T_{41}+T_{23}+T_{43} \ .
\end{aligned} \tag{47}$$

The general expression for the four-terminal resistance $R_{mn,kl} = 1/G_{mn,kl}$ (which has six permutations) can then be written as

$$R_{mn,kl} \ = \ \frac{h}{e^2}\frac{T_{km}T_{ln} - T_{kn}T_{lm}}{D} \ , \tag{48}$$

where $D = (\alpha_{11}\alpha_{22} - \alpha_{12}\alpha_{21})/P$. From here, we can easily work out the four-terminal resistances for the case of the completely symmetric and maximally transmitting S-matrix (using Eq. (44)). In this case however, as $\alpha_{12} = \alpha_{21} = 0$, all four-terminal resistances will simply give $R_{mn,kl} = 0$. This result is interesting for the following reason: while all the four-terminal resistances $R_{mn,kl}$ vanish if the system is exactly at the fully symmetric fixed point, the above suggests that they continue to stay zero as long as interactions are weak and the RG flows of the various elements of the S-matrix take place in a symmetric fashion. This means that in this case, the various $R_{mn,kl}$ will continue to be zero even as the temperature is varied.

The other interesting (and experimentally relevant) case for which conductances can be computed is that of two crossed, perfectly transmitting quantum wires which are connected via the tunneling of electrons at one point. This point is characterized by the S-matrix given earlier in Eq. (42). We have already seen that the hopping between the two wires is an irrelevant process. Further, we can treat any small reflection in either of the two perfectly transmitting wires perturbatively; from the work of Kane and Fisher [1], it is known that such perturbations are relevant and will grow so as to cut the wires (i.e., they flow under RG towards the perfectly reflecting stable fixed point characterized by an S-matrix equal to unity). Thus, there is nothing new to be found in the computation of the conductances in this case.

Finally, we would like to mention the work of Komnik and Egger on crossed quantum wires [13]. In addition to the hopping operators considered above, they study the effects of a density-density interaction between the two wires at the point where they cross; they show that such an interaction can have a non-trivial effect if the interactions in the wires are sufficiently strong. However, such strong interactions are beyond the purview of our analysis; for the case of weak interactions considered here, such interactions are irrelevant.

IX. TOMONAGA-LUTTINGER LIQUIDS WITH SPIN

It is not difficult to extend all the results above to the case of interacting fermions with spin. Let us first discuss the form of the interactions. We again begin with a short-range interaction as in Eq. (4) where the density is now a sum of the form

$$\rho \ = \ \Psi_\uparrow^\dagger \Psi_\uparrow \ + \ \Psi_\downarrow^\dagger \Psi_\downarrow \ . \tag{49}$$

The second-quantized fields $\Psi_\uparrow$ and $\Psi_\downarrow$ have expansions near the Fermi points of the form given in Eq. (3). (We assume that there is no magnetic field, so that spin-$\uparrow$ and spin-$\downarrow$ electrons have the same Fermi energy). Following the arguments leading up to Eq. (6), we can show that

$$\begin{aligned}
H_{int} = & \\
\int dx \sum_{\sigma,\sigma'=\uparrow,\downarrow} & [g_1 \Psi_{I\sigma}^\dagger \Psi_{O\sigma'}^\dagger \Psi_{I\sigma'} \Psi_{O\sigma} + g_2 \Psi_{I\sigma}^\dagger \Psi_{O\sigma'}^\dagger \Psi_{O\sigma'} \Psi_{I\sigma} \\
& + \frac{1}{2} g_4 (\Psi_{I\sigma}^\dagger \Psi_{I\sigma'}^\dagger \Psi_{I\sigma'} \Psi_{I\sigma} + \Psi_{O\sigma}^\dagger \Psi_{O\sigma'}^\dagger \Psi_{O\sigma'} \Psi_{O\sigma})],
\end{aligned} \tag{50}$$

where

$$\begin{aligned}
g_1 \ &= \ \tilde{V}(2k_F) \ , \\
\text{and} \quad g_2 \ &= \ g_4 \ = \ \tilde{V}(0) \ .
\end{aligned} \tag{51}$$

Yue et al show that the backscattering interaction governed by g_1 leads to a logarithmic renormalization of the interaction parameters g_1 and g_2 [7]; we will ignore that effect here since it plays no role to first order in the g_i. We can also ignore the effects of the g_4 term; it renormalizes the velocity, but it does not contribute to the reflection from the Friedel oscillations which is what leads to the RG flow of the S-matrix.

If there is a non-zero reflection amplitude r on wire i, then there will again be Friedel oscillations given by Eqs. (11-12) for both spin-$\uparrow$ and spin-$\downarrow$ electrons. Then the interactions will lead to scattering of incoming electrons to outgoing electrons (and vice versa); this is given by the following Hartree-Fock decomposition of (50),

$$\begin{aligned}
H_{\rm int} \ = & \\
- \ \frac{i(g_2 - 2g_1)}{4\pi} \int_0^\infty dx \ [\ & r^\star \ (\Psi_{I\uparrow}^\dagger \Psi_{O\uparrow} \ + \ \Psi_{I\downarrow}^\dagger \Psi_{O\downarrow}) \\
& - \ r \ (\Psi_{O\uparrow}^\dagger \Psi_{I\uparrow} \ + \ \Psi_{O\downarrow}^\dagger \Psi_{I\downarrow}) \ ] \ .
\end{aligned} \tag{52}$$

(This may be compared with Eq. (13) for spinless fermions). We see from Eq. (52) that the spin-$\uparrow$ and spin-$\downarrow$ electrons have decoupled from each other in this approximation. Hence the RG analysis given above for spinless fermions will go through similarly here. The only difference is that the interaction parameter α is now given by

$$\alpha \;=\; \frac{\tilde{V}(0) - 2\tilde{V}(2k_F)}{2\pi\hbar v_F} \;, \qquad (53)$$

instead of $\alpha = [\tilde{V}(0) - \tilde{V}(2k_F)]/(2\pi\hbar v_F)$ in the spinless case. We thus see that to first order in the interaction, the analysis remains essentially the same for spinless and spinful fermions. Finally, the conductances have factors of $2e^2/h$ for spinful fermions in place of e^2/h for spinless fermions but have similar temperature power-laws dependent on the interaction parameter α defined above. In fact, it should be possible to detect such power-laws in existing 3-arm and 4-arm quantum wire systems built by the voltage-gate patterning on the 2DEG in GaAs heterojunctions [14,15]. While the early experiments with such systems focussed on carrier transport in the presence of an external magnetic field and the effects of geometry [14], measuring the two-terminal, three-terminal and four-terminal conductances for fixed values of the various gate voltages but at different temperatures should again reveal identical power-law variations as discussed earlier for spinless fermions. In fact, similar studies using the technique developed by Shepard *etal.* [15] for directly measuring the transmission matrix elements of such junctions should be able to show the temperature power-law variations of the various transmission probabilities.

X. CONCLUSION

In this work, we have derived the RG equations for a general S-matrix at the junction of several quantum wires, and we have discussed the consequences of these equations for the conductances across the system. The RG flows are a result of interactions in the wires; there is no flow if the interaction parameters α_i are all zero. Our results differ considerably from those of Ref. [6] who find RG flows even in the absence of interactions in the wires. This difference seems to be due to their model of the junction; they have a spin-1/2 degree of freedom sitting there which interacts with the electrons on the wires. This gives rise to a nontrivial interacting model of the Kondo type even if there are no interactions in the wires. Their flow diagram is therefore quite different from ours. Further, they only consider the case where both the S-matrix and the interactions are symmetric under all possible permutations of the wires; however they are able to use bosonization to study the case of an arbitrary interaction strength.

Our work can clearly be generalized to the case of more than four wires meeting at a junction. The RG flow diagram will rapidly get more complicated as the number of wires increases. Physically, we expect the cases of three and four wires to be the easiest to study; these two cases arise in the experiments discussed earlier [14,15] as well as in the cases of Y-branched carbon nanotubes [16] and crossed carbon nanotubes [17].

Besides the restriction to weak interactions in the wires, our work has the limitation that we have assumed linear relations between the incoming and outgoing fermion fields. In principle, other interesting things can happen at a junction. For instance, there may be Andreev reflection in which a fermion striking the junction from one wire is reflected back as a hole while two fermions are transmitted into some of the other wires [6]. Even more complicated things may occur for the case of spinful fermions. Some of these phenomena can be expressed as boundary conditions at the junction in the bosonic language, but not in the fermionic language. We expect that such bosonic boundary conditions will require a method of analysis which is very different from the one which we have used to study the fermion S-matrix in this paper.

Finally, it remains a challenging problem to see if some of the non-trivial fixed points that we have found (such as case VII for the three-wire case) can be bosonized for arbitrary interaction strengths. Bosonizing such points would lead to a much more complete picture of the RG flows besides increasing our understanding of conformal field theories with boundaries.

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[1] C. L. Kane and M. P. A. Fisher, Phys. Rev. B **46**, 15233 (1992).

[2] S. Tarucha, T. Honda, and T. Saku, Sol. St. Comm. **94**, 413 (1995); A. Yacoby, H. L. Stormer, N. S. Wingreen, L. N. Pfeiffer, K. W. Baldwin, and K. W. West, Phys. Rev. Lett. **77**, 4612 (1996); C. -T. Liang, M. Pepper,, M. Y. Simmons, C. G. Smith, and D. A. Ritchie, Phys. Rev. B **61**, 9952 (2000); B. E. Kane, G. R. Facer, A. S. Dzurak, N. E. Lumpkin, R. G. Clark, L. N. Pfeiffer, and K. W. West, App. Phys. Lett. **72**, 3506 (1998); D. J. Reilly, G. R. Facer, A. S. Dzurak, B. E. Kane, R. G. Clark, P. J. Stiles, J. L. O'Brien, N. E. Lumpkin, L. N. Pfeiffer, and K. W. West, Phys. Rev. B **63**, 121311(R) (2001).

[3] I. Safi and H. J. Schulz, Phys. Rev. B **52**, 17040 (1995); D. L. Maslov and M. Stone, *ibid* **52**, 5539 (1995); V. V. Ponomarenko, *ibid* **52**, 8666 (1995).

[4] D. L. Maslov, Phys. Rev. B, **52**, 14386 (1995); A. Furusaki and N. Nagaosa, *ibid* **54**, 5239 (1996); I. Safi and H. J. Schulz, *ibid* **59**, 3040 (1999).

[5] S. Lal, S. Rao and D. Sen, Phys. Rev. Lett. **87**, 026801

(2001), Phys. Rev. B **65**, 195304 (2002), and cond-mat/0104402.
[6] C. Nayak, M. P. A. Fisher, A. W. W. Ludwig, and H. H. Lin, Phys. Rev. B **59**, 15694 (1999).
[7] D. Yue, L. I. Glazman, and K. A. Matveev, Phys. Rev. B **49**, 1966 (1994).
[8] For a review, see S. Rao and D. Sen, cond-mat/0005492.
[9] A. O. Gogolin, A. A. Nersesyan and A. M. Tsvelik, *Bosonization and Strongly Correlated Systems* (Cambridge University Press, Cambridge, 1998).
[10] M. Büttiker, Y. Imry, and R. Landauer, Phys. Rev. B **31**, 6207 (1985); S. Datta, *Electronic transport in mesoscopic systems* (Cambridge University Press, Cambridge, 1995); *Transport phenomenon in mesoscopic systems*, edited by H. Fukuyama and T. Ando (Springer Verlag, Berlin, 1992); Y. Imry, *Introduction to Mesoscopic Physics* (Oxford University Press, New York, 1997).
[11] M. Büttiker, Phys. Rev. Lett. **57**, 1761 (1986); IBM J. Res. Dev. **32**, 317 (1988).
[12] R. de Picciotto, H. L. Stormer, L. N. Pfeiffer, K. W. Baldwin, and K. W. West, Nature **411**, 51 (2001).
[13] A. Komnik and R. Egger, Phys. Rev. Lett. **80**, 2881 (1998) and Eur. Phys. J. B **19**, 271 (2001).
[14] G. Timp, R. E. Behringer, E. H. Westerwick, and J. E. Cunningham, in *Quantum Coherence in Mesoscopic Systems*, edited by B. Kramer (Plenum Press, 1991) pages 113-151 and references therein; C. B. J. Ford, S. Washburn, M. Büttiker, C. M. Knoedler, and J. M. Hong, Phys. Rev. Lett. **62**, 2724 (1989).
[15] K. L. Shepard, M. L. Roukes, and B. P. Van der Gaag, Phys. Rev. Lett. **68**, 2660 (1992).
[16] C. Papadopoulos, A. Rakitin, J. Li, A. S. Vedeneev, and J. M. Xu, Phys. Rev. Lett. **85**, 3476 (2000).
[17] J. Kim, K. Kang, J.-O. Lee, K.-H. Yoo, J.-R. Kim, J. W. Park, H. M. So, and J.-J. Kim, cond-mat/0005083.