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The Witt group of a real Enriques surface having real points is computed purely in terms of the topology of the real part. For a real Enriques surface without real points the level of the function field is shown to be 2, and the Witt group is computed in this case as well.

The Witt group of a real projective curve was computed by Knebusch in [**Kn**]. In [**S**], there is a computation of the structure of the Witt group of a smooth projective real algebraic surface in terms of certain birational invariants of the surface. Let X be a smooth, projective, geometrically integral surface over $\mathbb{R}$, and let s be the number of connected components (for the euclidean topology) of $X(\mathbb{R})$. Then the Witt group $W(X)$ of X is isomorphic to a direct sum of $\mathbb{Z}^s$ and a 2-primary torsion group that depends on cohomological invariants of the scheme X (see §1 A). When X is a *real rational surface*, by which we mean that $X_{\mathbb{C}} = X \times_{\mathbb{R}} \mathbb{C}$ is birational to $\mathbb{P}^2_{\mathbb{C}}$, the Witt group is completely determined by the number of connected components of $X(\mathbb{R})$. By [**S**, Th. 4.1] we then have that

$$W(X) \simeq \mathbb{Z}^s \oplus (\mathbb{Z}/2)^{s-1}$$

whenever $X(\mathbb{R}) \neq \emptyset$. If X has no real points, the natural map $W(\mathbb{R}) \to W(X)$ is surjective and $W(X) \simeq \mathbb{Z}/4$. The latter result is based on a computation of the level of the function field of X (see §1 A), which was shown to be 2, in a joint work of Parimala and the first author [**P-S**].

In general, the situation is different. For example, let X be a smooth, projective, geometrically irreducible surface over $\mathbb{R}$ with $H_1(X(\mathbb{C}), \mathbb{Z}) = 0$. The calculations of [**N**] imply that if $X(\mathbb{R})$ has $s > 0$ connected components, then

$$W(X) \simeq \mathbb{Z}^s \oplus (\mathbb{Z}/2)^{s-1+t},$$

where t is the dimension of the cokernel of the characteristic class mapping from $\mathrm{Pic}(X) \otimes \mathbb{Q}$ into the subspace $H^2(X(\mathbb{C}), \mathbb{Q})^-$ of $H^2(X(\mathbb{C}), \mathbb{Q})$ formed by the classes that are anti-invariant under the Galois action. When $X(\mathbb{R}) = \emptyset$, the level of the function field is not always 2. For example, if X is a sufficiently general smooth hypersurface of degree $2d \geq 4$ in $\mathbb{P}^3_{\mathbb{R}}$, the Picard group of complex line bundles $\mathrm{Pic}(X_{\mathbb{C}})$ is generated by the hyperplane section, so $\mathrm{Pic}(X) = \mathrm{Pic}(X_{\mathbb{C}})$. This implies that if $X(\mathbb{R}) = \emptyset$, the function field of X has level 4. On the other hand, the function field of the hypersurface given by the equation $x_0^{2d} + x_1^{2d} + x_2^{2d} + x_3^{2d} = 0$ has level 2, since it contains the function field of the quadric without real points. See also Remark 3.4.

A *real Enriques surface* X is by definition a smooth, projective, geometrically integral surface over $\mathbb{R}$ with irregularity $q(X) = 0$, and such that the canonical line bundle $\mathcal{K}_X$ is nontrivial, but $\mathcal{K}_X^{\otimes 2} \simeq \mathcal{O}_X$. As in the case of rational surfaces, the geometric genus p_g and the irregularity $q(X)$ are both zero for an Enriques surface X, but unlike the situation for rational surfaces, we have that $H^1(X(\mathbb{C}), \mathbb{Z}/2)$ is isomorphic to $\mathbb{Z}/2$. We will show that the Witt group of a real Enriques surface X with $X(\mathbb{R}) \neq \emptyset$ is completely determined by the topology of $X(\mathbb{R})$, as in the case of the Witt group of a real rational surface. It does not, however, depend exclusively on the number of connected components of the set of real points; a certain natural decomposition of $X(\mathbb{R})$ into two parts (see §1, C) comes into play, as well as the orientability and the Euler characteristic of the connected components. The full result is given by Theorem 2.6. For a real Enriques surface X without real points we show that the level of the function field is 2, as in the case of real rational surfaces, but now $W(X) \simeq (\mathbb{Z}/2)^2 \oplus \mathbb{Z}/4$ (see Theorems 3.2 and 3.3).

Let us mention the key ingredients in the computations. In addition to the results of [**S**], they include a result of Krasnov [**Kr**] on separating the real connected components of a surface by étale cohomology classes, some results of Nikulin on equivariant cohomology from [**N**], a computation — due to Mangolte and the second author [**M-vH**] — of the Brauer groups of real Enriques surfaces, and arithmetic results concerning the Galois action on the cohomology lattice, due to Nikulin (see [**N-S**]) and Degtyarev and Kharlamov [**D-K**].

The paper consists of three sections. Section 1 lists various results that are used in the subsequent sections. In Section 2, we consider the case of a real Enriques surface with real points, and the case when there is no real point is treated in Section 3.

We would like to thank V. Kharlamov for helpful discussions.

Note. This paper is based on our preprint with the same title that has been circulating since the end of 1996. While it was under review we found that, independently, Krasnov has obtained results very close to ours (see [**Kr2**, Th. 0.7]) using similar methods. The main difference with our paper is that he allows for some extra possibilities that do not actually occur (for instance, the case of $\mathbb{R}(X)$ having level 4 when $X(\mathbb{R})$ is empty). We are able to exclude these possiblities using calculations by Nikulin and by Degtyarev and Kharlamov.

1. Preliminary results.

This section lists various results that will be used in later computations.

A: Witt groups of real algebraic surfaces. Let X be a smooth, projective, geometrically integral surface over a field k, of characteristic $\neq 2$. Let $H^i(X)$ denote the étale cohomology groups $H^i_{ét}(X, \mu_2)$. Let $\mathcal{H}^q$ be the Zariski sheaf associated to the presheaf $U \mapsto H^q(U)$. It follows from [**B-O**, Th. 4.2] that the group $\Gamma(X, \mathcal{H}^n)$ of global sections of the sheaf $\mathcal{H}^n$ concides with the unramified cohomology group of degree n with coefficients in μ_2. We have for every n a canonical mapping

$$\varepsilon_n : H^n(X) \to \Gamma(X, \mathcal{H}^n).$$

By [**B-O**, Th. 6.1, Th. 7.7] and the Kummer exact sequence, the mappings ε_n induce isomorphisms $\Gamma(X, \mathcal{H}^1) \simeq H^1(X)$ and $\Gamma(X, \mathcal{H}^2) \simeq {}_2\mathrm{Br}(X)$, where ${}_2\,\mathrm{Br}(X)$ denotes the 2-torsion in the Brauer group of X.

Let X be a smooth, projective, geometrically integral surface over $\mathbb{R}$. We follow the notation in [**S**]. We denote by $\Gamma_t(X, \mathcal{H}^i)$ the (-1)-*torsion* subgroups [**AEJ**] i.e.,

$$\Gamma_t(X, \mathcal{H}^i) = \{\alpha \in \Gamma(X, \mathcal{H}^i) \mid \alpha \cup (-1)^l = 0 \text{ for some } l\},$$

where (-1) is the nontrivial element of $H^1(\mathbb{R}) = \mathbb{R}^*/\mathbb{R}^{*2} \simeq \mathbb{Z}/2$. Let

$$N = \mathrm{Ker}\{\Gamma_t(X, \mathcal{H}^1) \overset{\cup(-1)}{\to} \Gamma_t(X, \mathcal{H}^2)\}$$

and let j, k, l (as in [**S**]) denote the $\mathbb{Z}/2$- dimensions of $\Gamma_t(X, \mathcal{H}^1)$, $\Gamma_t(X, \mathcal{H}^2)$ and N respectively. We have:

Theorem A ([**S**, Theorem 3.1]). *Let X be a smooth projective, geometrically integral real surface such that $X(\mathbb{R}) \neq \emptyset$. Let s denote the number of connected components of $X(\mathbb{R})$ in the euclidean topology, and let j, k, l be as above. Then*

$$W(X) \simeq \mathbb{Z}^s \oplus (\mathbb{Z}/2)^m \oplus (\mathbb{Z}/4)^n,$$

where $n = j - l$ and $m = k + 2l - j$.

Now suppose that X is a geometrically integral surface over $\mathbb{R}$ without real points. Recall that the *level* of a field F is the smallest integer n such that -1 is expressible as a sum of n squares. The level of a field F is finite if and only if F has no real orderings, so $X(\mathbb{R}) = \emptyset$ implies that the level of the function field $\mathbb{R}(X)$ of X is finite. Moreover, by results of Pfister [**Pf**], the level of $\mathbb{R}(X)$ is a power of 2 and is at most 4. Further, the groups $\Gamma(X, \mathcal{H}^i)$ are purely (-1)-torsion, in other words, $\Gamma_t(X, \mathcal{H}^i) = \Gamma(X, \mathcal{H}^i)$. Let j, k and l be the dimensions of $\Gamma(X, \mathcal{H}^1)$, $\Gamma(X, \mathcal{H}^2)$ and N as before.

Theorem A1 ([**S**, Theorem 3.2]). *Let X be a smooth, projective, geometrically integral surface over $\mathbb{R}$ with $X(\mathbb{R}) = \emptyset$. Let j, k, l be as above. Then*

$$W(X) \simeq (\mathbb{Z}/2)^{m_1} \oplus (\mathbb{Z}/4)^{n_1} \oplus (\mathbb{Z}/8)^{t_1},$$

where

$$\begin{cases} t_1 = 0,\ n_1 = j - l + 1,\ m_1 = k + 2l - j - 1 & \text{if the level of } \mathbb{R}(X) \text{ is } 2, \\ t_1 = 1,\ n_1 = j - l - 1,\ m_1 = k + 2l - j & \text{if the level of } \mathbb{R}(X) \text{ is } 4. \end{cases}$$

B: Separation of real connected components. Let X be a smooth, projective, geometrically integral variety over $\mathbb{R}$ of dimension d such that $X(\mathbb{R})$ has $s > 0$ connected components for the euclidean topology. Let $H^0(X(\mathbb{R}), \mathbb{Z}/2)$ be the set of continuous maps from $X(\mathbb{R})$ into $\mathbb{Z}/2$. Clearly, $H^0(X(\mathbb{R}), \mathbb{Z}/2) \simeq (\mathbb{Z}/2)^s$. For every $n \geq 0$ there is a map

$$h_n : \Gamma(X, \mathcal{H}^n) \to H^0(X(\mathbb{R}), \mathbb{Z}/2)$$

which is defined by specializing an element $\alpha \in \Gamma(X, \mathcal{H}^n)$ at a real point P to get an element $\alpha_P \in \Gamma(\text{Spec } \mathbb{R}, \mathcal{H}^n) \simeq H^n(\mathbb{R}) \simeq \mathbb{Z}/2$. For us, the importance of h_n lies in the well-known fact that

$$\Gamma_t(X, \mathcal{H}^n) = \operatorname{Ker} h_n.$$

Colliot-Thélène and Parimala have shown that the map h_n is an isomorphism if $n \geq d+1$, where d is the dimension of X (see [**CT-P**, Th. 2.3.2]). Moreover, they proved that if X is a smooth projective real surface with $H^3(X(\mathbb{C}), \mathbb{Z}/2) = 0$, the map h_2 is surjective (see [**CT-P**, Prop. 3.2.1]) and they raised the question of surjectivity of h_2 for an arbitrary surface (see [**CT-P**, Rem. 2.4.4]).

In [**N**] Nikulin applied topological methods in studying the mapping h_n; we will sketch his approach here. Consider the space $X(\mathbb{C})$ equipped with the euclidean topology and with the natural continuous action of $G = \text{Gal}(\mathbb{C}/\mathbb{R})$. The quotient map will be denoted by $\pi : X(\mathbb{C}) \to X(\mathbb{C})/G$. For any G-sheaf $\mathcal{A}$ on $X(\mathbb{C})$ we have the *equivariant cohomology groups* $H^k(X(\mathbb{C}); G, \mathcal{A})$, as defined in [**Gr**, Ch. 5]. There is a well-known identification $H^n(X) = H^n_{ét}(X, \mathbb{Z}/2) \simeq H^n(X(\mathbb{C}); G, \mathbb{Z}/2)$. Moreover, for every $n \geq 0$ there is a canonical isomorphism

$$H^n(X(\mathbb{R}); G, \mathbb{Z}/2) \simeq \bigoplus_{i=0}^{n} H^i(X(\mathbb{R}), \mathbb{Z}/2),$$

so the restriction from $X(\mathbb{C})$ to $X(\mathbb{R})$ induces a homomorphism $e_n : H^n(X(\mathbb{R}); G, \mathbb{Z}/2) \to H^0(X(\mathbb{R}), \mathbb{Z}/2)$. Nikulin observed ([**N**, Remark 1.8]) that the following diagram is commutative:

$$\begin{array}{ccc} H^n(X) & = & H^n(X(\mathbb{C}); G, \mathbb{Z}/2) \\ \varepsilon_n \downarrow & & e_n \downarrow \\ \Gamma(X, \mathcal{H}^n) & \xrightarrow{h_n} & H^0(X(\mathbb{R}), \mathbb{Z}/2). \end{array}$$

It follows that h_n is surjective if e_n is surjective. Using this fact, Nikulin showed that h_2 is surjective if $X/\mathbb{R}$ is a smooth projective geometrically integral surface with $H^3(X(\mathbb{C})/G, \mathbb{Z}/2) = 0$ (see [**N**, Th. 0.1]), a condition that is satisfied by many, but not all Enriques surfaces.

It was Krasnov who proved in [**Kr**], that the map $e_d : H^d(X(\mathbb{C}); G, \mathbb{Z}/2) \to H^0(X(\mathbb{R}), \mathbb{Z}/2)$ is surjective for any smooth projective variety of dimension d. His result is a consequence of a much more general result (see [**Kr**, Cor. 3.2], see also [**vH**, §2.3] for another proof). Again, the surjectivity of e_d implies the surjectivity of h_d, so we obtain the following result.

Theorem B ([**Kr**]). *Let X be a smooth projective geometrically integral variety over $\mathbb{R}$ of dimension d. The map*

$$h_d : \Gamma(X, \mathcal{H}^d) \to H^0(X(\mathbb{R}), \mathbb{Z}/2)$$

is surjective, so the elements of $\Gamma(X, \mathcal{H}^d)$ separate the real connected components of X.

Corollary B1. *Let X be a smooth projective geometrically integral surface over $\mathbb{R}$. We have*

$$k = \dim \Gamma_t(X, \mathcal{H}^2) = \dim {}_2\mathrm{Br}(X) - s.$$

C: Brauer groups of real Enriques surfaces. In view of Corollary B1, we need to know the Brauer groups in order to be able to apply Theorems A and A1. For real Enriques surfaces partial computations of the Brauer groups were made in [**N-S**], and for a larger class of surfaces by Nikulin in [**N**]. A complete solution for real Enriques surfaces is given in [**Kr**, Th. 4.5] and, independently, in [**M-vH**, Th. 1.3]. In order to state this result we need to introduce some more terminology concerning real Enriques surfaces.

Let X be a real Enriques surface. Then $X_{\mathbb{C}}$ admits a double covering $Y \to X_{\mathbb{C}}$ by a complex K3 surface Y. Since a K3 surface is simply connected, $Y(\mathbb{C})$ is the universal covering space of $X(\mathbb{C})$. Let τ be the involution of the covering. The complex conjugation on X can be lifted to an antiholomorphic involution of the covering space $Y(\mathbb{C})$ in two different ways, σ and $\tau\sigma$. Hence Y can be given the structure of a real variety in two different ways, which we denote by Y_1 and Y_2. We obtain a decomposition $X(\mathbb{R}) = X_1 \sqcup X_2$, where each X_i consists of the connected components of $X(\mathbb{R})$ covered by connected components of $Y_i(\mathbb{R})$. The subsets X_1 and X_2 are referred to as the *two halves* [**D-K**, §1.3] of $X(\mathbb{R})$. We can now describe the Brauer groups of real Enriques surfaces.

Theorem C ([**Kr**, Theorem 4.5], [**M-vH**, Theorem 1.3]). *Let X be a real Enriques surface. Let s be the number of connected components of $X(\mathbb{R})$. If $X(\mathbb{R}) \neq \emptyset$ is non-orientable, then*

$$\mathrm{Br}(X) \simeq (\mathbb{Z}/2)^{2s-1}.$$

If $X(\mathbb{R}) \neq \emptyset$ is orientable, then

$$\mathrm{Br}(X) \simeq \begin{cases} (\mathbb{Z}/2)^{2s-2} \oplus (\mathbb{Z}/4) & \textit{if both halves are non-empty,} \\ (\mathbb{Z}/2)^{2s} & \textit{if one half is empty.} \end{cases}$$

If $X(\mathbb{R}) = \emptyset$, then

$$\mathrm{Br}(X) \simeq \mathbb{Z}/2.$$

Later we will also need information about the natural mapping $\mathrm{Br}(X) \to \mathrm{Br}(X_{\mathbb{C}}) \simeq \mathbb{Z}/2$. It can be checked using [**M-vH**, Lemmas 5.7, 5.8, 5.9] that the image of this mapping is as follows:

$$\mathrm{Im}\{\mathrm{Br}(X) \to \mathrm{Br}(X_{\mathbb{C}})\} \simeq \begin{cases} \mathbb{Z}/2 & \text{if both halves are non-empty or } X(\mathbb{R}) \text{ has a connected component of odd Euler characteristic,} \\ 0 & \text{otherwise.} \end{cases}$$

2. Witt groups of real Enriques surfaces having real points.

As in (§1 A), let j, k, l denote respectively the $\mathbb{Z}/2$-dimensions of $\Gamma_t(X, \mathcal{H}^1)$, $\Gamma_t(X, \mathcal{H}^2)$ and N. We first compute the invariants j, k, l for a real Enriques surface. Recall (§1 C) that the real part $X(\mathbb{R})$ decomposes into two halves $X_1 \sqcup X_2$.

Proposition 2.1. *Let X be a real Enriques surface with $X(\mathbb{R}) \neq \emptyset$. We have*

$$j = \dim\ \Gamma_t(X, \mathcal{H}^1) = \begin{cases} 0 & \textit{if both halves of } X(\mathbb{R}) \textit{ are non-empty,} \\ 1 & \textit{if precisely one of the halves is empty.} \end{cases}$$

Proof. There is an exact sequence

$$0 \to H^1(X(\mathbb{C})/G, \mathbb{Z}/2) \to H^1(X(\mathbb{C}), \mathbb{Z}/2) \xrightarrow{h_1} H^0(X(\mathbb{R}), \mathbb{Z}/2),$$

which is a special case of [**Gr**, (5.2.8)] (see also [**N**, §1]). Since $\Gamma_t(X, \mathcal{H}^1)$ is the kernel of h_1, we deduce that $j = \dim\ \Gamma_t(X, \mathcal{H}^1) = \dim\ H^1(X(\mathbb{C})/G, \mathbb{Z}/2)$. By [**N**, Corollary 0.2] we have

$$H_1(X(\mathbb{C})/G, \mathbb{Z}/2) = \begin{cases} 0 & \text{if both halves of } X(\mathbb{R}) \text{ are non-empty} \\ \mathbb{Z}/2 & \text{if precisely one of the halves is empty.} \end{cases}$$

Since $X(\mathbb{C})/G$ is a topological manifold, the conclusion follows from Poincaré duality. ☐

We will need the following three lemmas to compute in Proposition 2.5 the dimension of N, the kernel of the mapping $\Gamma_t(X, \mathcal{H}^1) \overset{\cup(-1)}{\to} \Gamma_t(X, \mathcal{H}^2)$.

Lemma 2.2. *Let X be a real Enriques surface with $X(\mathbb{R}) \neq \emptyset$. Let N be as above. There is a canonical isomorphism between N and the kernel of the mapping*

$$H^1(G, {}_2\mathrm{Pic}\, X_{\mathbb{C}}) \to H^1(G, \mathrm{Pic}\, X_{\mathbb{C}})$$

induced by the inclusion ${}_2\,\mathrm{Pic}\, X_{\mathbb{C}} \hookrightarrow \mathrm{Pic}\, X_{\mathbb{C}}$.

Proof. Since $\Gamma(X, \mathcal{H}^1) \simeq H^1(X)$, and $\Gamma(X, \mathcal{H}^2) \simeq {}_2\,\mathrm{Br}(X)$, we have that N is isomorphic to the kernel of the composite mapping

$$\begin{array}{ccccc} \Gamma(X, \mathcal{H}^1) \simeq & H^1(X) & \overset{\cup(-1)}{\to} & H^2(X) & \\ & & \searrow & \downarrow \varepsilon_1 & \\ & & & \Gamma(X, \mathcal{H}^2) & \simeq {}_2\mathrm{Br}(X). \end{array}$$

Consider the following exact sequence of étale sheaves on X

$$0 \to \mu_2 \to \pi_* \mu_2 \to \mu_2 \to 0,$$

where $\pi : X_{\mathbb{C}} \to X$ is the natural map. The boundary map $H^n(X) \to H^{n+1}(X)$ in the associated long exact sequence is cup-product with the class of (-1). Hence, from the fact that $H^1(X) \simeq \mathbb{R}^*/\mathbb{R}^{*2} \oplus {}_2\mathrm{Pic}\, X \simeq \mathbb{Z}/2 \oplus \mathbb{Z}/2$ and $H^1(X_{\mathbb{C}}) \simeq \mathbb{Z}/2$, we deduce that cup-product with the class of (-1) induces an isomorphism

$$H^1(X) \xrightarrow{\sim} \mathrm{Ker}\{H^2(X) \to H^2(X_{\mathbb{C}})\}.$$

From the Hochschild-Serre spectral sequences for the sheaves μ_2 and $\mathbb{G}_m$, we obtain the following commutative diagram with exact rows

(6)

$$\begin{array}{ccccccccc} 0 & \to & H^2(\mathbb{R}) & \to & \mathrm{Ker}\{H^2(X) \to H^2(X_{\mathbb{C}})\} & \to & H^1(G, H^1(X_{\mathbb{C}})) & \to & 0 \\ & & \downarrow i' & & \downarrow i'' & & \downarrow i''' & & \\ 0 & \to & \mathrm{Br}(\mathbb{R}) & \to & \mathrm{Ker}\{\mathrm{Br}(X) \to \mathrm{Br}(X_{\mathbb{C}})\} & \to & H^1(G, \mathrm{Pic}\ X_{\mathbb{C}}) & \to & 0. \end{array}$$

Observe that the exactness of the rows on the left and on the right follows from the condition $X(\mathbb{R}) \neq \emptyset$. It follows the remarks above that N is isomorphic to the kernel of i''. Since i' is an isomorphism, the Snake Lemma then implies that

$$N \simeq \mathrm{Ker}\, i'' \simeq \mathrm{Ker}\, i'''.$$

From the Kummer exact sequence we see that $H^1(X_{\mathbb{C}})$ is isomorphic to the 2-torsion group ${}_2\mathrm{Pic}X_{\mathbb{C}}$, so the kernel of i''' is isomorphic to the kernel of the natural mapping $H^1(G, {}_2\mathrm{Pic}\, X_{\mathbb{C}}) \to H^1(G, \mathrm{Pic}\, X_{\mathbb{C}})$. □

Recall, that ${}_2\mathrm{Pic}\, X_{\mathbb{C}}$ is isomorphic to $\mathbb{Z}/2$, and generated by the canonical class. In particular, if $X(\mathbb{R}) \neq \emptyset$, then $l \leq 1$.

Lemma 2.3 (Degtyarev-Kharlamov)**.** *Let X be a real Enriques surface, such that precisely one half of $X(\mathbb{R})$ is empty. Let $d \geq 0$ be the integer such that*

$$\dim H^*(X(\mathbb{R}), \mathbb{Z}/2) = \dim H^*(X(\mathbb{C}), \mathbb{Z}/2) - 2d = 16 - 2d,$$

let $H^2 = H^2(X(\mathbb{C}), \mathbb{Z})/\mathrm{Tors}$ *and let* $a = \mathrm{rank}\ H^2 - \dim(H^2/2H^2)^G$. *Then*

$$d - a = \begin{cases} 0 & \text{if } X(\mathbb{R}) \text{ has a connected component of} \\ & \text{odd Euler characteristic,} \\ 2 & \text{if } X(\mathbb{R}) \text{ is non-orientable and all components have} \\ & \text{even Euler characteristic,} \\ 4 & \text{if } X(\mathbb{R}) \text{ is orientable.} \end{cases}$$

Proof. The first two cases are covered by [**D-K1**, Prop. 6.1] (since $a = \dim \mathcal{D}^-$ in the notation of that paper). The case when $X(\mathbb{R})$ is orientable is also due to Degtyarev and Kharlamov (communicated by Kharlamov; to appear in [**DIK**]). □

Lemma 2.4. *With notations as above, we have*

$$\dim\ H^1(G, \mathrm{Pic}\, X_{\mathbb{C}})$$
$$= \begin{cases} 2s-3 & \text{if } X(\mathbb{R}) \text{ is non-orientable, precisely one half is empty} \\ & \text{and there is a component of odd Euler characteristic,} \\ 2s-2 & \text{if } X(\mathbb{R}) \text{ is non-orientable, precisely one half is empty} \\ & \text{and all components have even Euler characteristic,} \\ 2s-1 & \text{if } X(\mathbb{R}) \text{ is orientable and} \\ & \text{precisely one half is empty.} \end{cases}$$

Proof. Using the isomorphism $H^1(G, \mathrm{Pic}\, X_{\mathbb{C}}) \simeq H^1(G, H^2(X(\mathbb{C}), \mathbb{Z}(1)))$ this is easily computed from the spectral sequence

$$H^p(G, H^q(X(\mathbb{C}), \mathbb{Z}(1))) \Longrightarrow H^{p+q}(X(\mathbb{C}); G, \mathbb{Z}(1))$$

as determined in [**M-vH**, §5] (or [**Kr**, §4]), and the fact that $H^{2k+1}(X(\mathbb{C}); G, \mathbb{Z}(1)) \simeq \mathbb{Z}/2^{2s}$ when k is greater than the dimension of X. □

Proposition 2.5. *Let X be a real Enriques surface and let* $l = \dim\ N$.

i) *If both halves of $X(\mathbb{R})$ are nonempty, then*

$$l = 0.$$

ii) *If precisely one half of $X(\mathbb{R})$ is empty, then*

$$l = \begin{cases} 0 & \text{if } X(\mathbb{R}) \text{ contains a connected component} \\ & \text{of odd Euler characteristic,} \\ 1 & \text{otherwise.} \end{cases}$$

Proof. i) If both halves of $X(\mathbb{R})$ are non-empty, then by Proposition 2.1 we have $j = 0$, hence $l = 0$.

ii) This case will be proven using Lemma 2.2 and Lemma 2.3. In order to determine the mapping

$$\psi : H^1(G, {}_2\mathrm{Pic}\, X_{\mathbb{C}}) \to H^1(G, \mathrm{Pic}\, X_{\mathbb{C}})$$

we will use the long exact sequence of Galois cohomology associated to the following short exact sequence.

$$0 \to {}_2\operatorname{Pic} X_{\mathbb{C}} \to \operatorname{Pic} X_{\mathbb{C}} \to \operatorname{Pic} X_{\mathbb{C}}/\mathrm{Tors} \to 0. \tag{7}$$

We then compare the dimension of $H^1(G, \operatorname{Pic} X_{\mathbb{C}})$, as given by Lemma 2.4, with the dimension of $H^1(G, \operatorname{Pic} X_{\mathbb{C}}/\mathrm{Tors})$ which equals $r - a$, where $r = \dim_{\mathbb{Q}} H^2(X(\mathbb{C}), \mathbb{Q})^G$ and a as in Lemma 2.3.

Let $\chi(X(\mathbb{R}))$ be the Euler characteristic of $X(\mathbb{R})$. Then

$$\dim H^*(X(\mathbb{R}), \mathbb{Z}/2) + \chi(X(\mathbb{R})) = 4s,$$

and we have the well-known relation

$$\chi(X(\mathbb{R})) = 2r - 8 \tag{8}$$

(cf. [**N-S**, p. 124]). Hence with d as in Lemma 2.3, we get

$$r = 2s - 4 + d,$$

and the results of Degtyarev and Kharlamov then give

$$r - a = \begin{cases} 2s-4 & \text{if } X(\mathbb{R}) \text{ has a connected component of} \\ & \text{odd Euler characteristic,} \\ 2s-2 & \text{if } X(\mathbb{R}) \text{ is non-orientable and all components have} \\ & \text{even Euler characteristic,} \\ 2s & \text{if } X(\mathbb{R}) \text{ is orientable.} \end{cases}$$

We now compare this information with the information on $\dim H^1(G, \operatorname{Pic} X_{\mathbb{C}})$ given by Lemma 2.4. Using the fact that ${}_2\operatorname{Pic} X \simeq \mathbb{Z}/2$, the short exact sequence (7) gives the following long exact sequence in Galois cohomology

$$\cdots \to H^1(G, \mathbb{Z}/2) \xrightarrow{\psi} H^1(G, \operatorname{Pic} X_{\mathbb{C}}) \to H^1(G, \operatorname{Pic} X_{\mathbb{C}}/\mathrm{Tors}) \to H^2(G, \mathbb{Z}/2) \to \cdots.$$

We see that the mapping ψ is injective if $X(\mathbb{R})$ contains a connected component of odd Euler characteristic, since then $\dim H^1(G, \operatorname{Pic} X_{\mathbb{C}}) = 2s - 3$, and $\dim H^1(G, \operatorname{Pic} X_{\mathbb{C}}/\mathrm{Tors}) = 2s - 4$. On the other hand, if $X(\mathbb{R})$ is orientable, the mapping ψ is zero since $\dim H^1(G, \operatorname{Pic} X_{\mathbb{C}}) = 2s - 1$, and $\dim H^1(G, \operatorname{Pic} X_{\mathbb{C}}/\mathrm{Tors}) = 2s$.

If $X(\mathbb{R})$ is non-orientable, but does not have any connected components of odd Euler characteristic, then $\dim H^1(G, \operatorname{Pic} X_{\mathbb{C}}) = 2s - 2 = \dim H^1(G, \operatorname{Pic} X_{\mathbb{C}}/\mathrm{Tors})$. Hence the long exact sequence alone is not sufficient to decide whether the map ψ is injective. However, we know that the first Stiefel-Whitney class of the real part of the canonical line bundle $\mathcal{K}$ coincides with the first Stiefel-Whitney class $w_1(X(\mathbb{R})) \in H^1(X(\mathbb{R}), \mathbb{Z}/2)$ of the tangent bundle of $X(\mathbb{R})$. Since $X(\mathbb{R})$ is non-orientable, the class $w_1(X(\mathbb{R}))$ is non-trivial, so $\mathcal{K}$ does not map to zero under the canonical mapping

$$\operatorname{Pic} X \to H^1(X(\mathbb{R}), \mathbb{Z}/2).$$

This means that the class of $\mathcal{K}$ is not of the form $(1+\theta)\mathcal{D}$ for some $\mathcal{D} \in \operatorname{Pic} X_{\mathbb{C}}$. Since the class of $\mathcal{K}$ generates the torsion in $\operatorname{Pic} X_{\mathbb{C}}$, this implies that $H^2(G, \mathbb{Z}/2) \to H^2(G, \operatorname{Pic} X_{\mathbb{C}})$ is injective. Thus the long exact sequence splits, giving an exact sequence

$$\cdots \to H^1(G, \mathbb{Z}/2) \xrightarrow{\psi} H^1(G, \operatorname{Pic} X_{\mathbb{C}}) \to H^1(G, \operatorname{Pic} X_{\mathbb{C}}/\mathrm{Tors}) \to 0.$$

Now the equality between $\dim H^1(G, \operatorname{Pic} X_{\mathbb{C}}/\mathrm{Tors})$ and $\dim H^1(G, \operatorname{Pic} X_{\mathbb{C}})$ implies that the mapping ψ is zero. □

Theorem 2.6. *Let $X/\mathbb{R}$ be a real Enriques surface such that $X(\mathbb{R}) \neq \emptyset$. Let s denote the number of connected components of $X(\mathbb{R})$. If both halves are non-empty, then*

$$W(X) \simeq \mathbb{Z}^s \oplus (\mathbb{Z}/2)^{s-1}.$$

If one of the halves is empty, then

$$W(X) \simeq \begin{cases} \mathbb{Z}^s \oplus (\mathbb{Z}/2)^{s-2} \oplus \mathbb{Z}/4 & \textit{if } X(\mathbb{R}) \textit{ has a connected component of odd Euler characteristic,} \\ \mathbb{Z}^s \oplus (\mathbb{Z}/2)^{s} & \textit{if } X(\mathbb{R}) \textit{ is non-orientable and all components have even Euler characteristic,} \\ \mathbb{Z}^s \oplus (\mathbb{Z}/2)^{s+1} & \textit{if } X(\mathbb{R}) \textit{ is orientable.} \end{cases}$$

Proof. We apply Theorem A. The invariant j has been computed in Proposition 2.1, we have that

$$k = \begin{cases} s & \text{if } X(\mathbb{R}) \text{ is orientable and one half is empty,} \\ s-1 & \text{otherwise,} \end{cases}$$

by Theorems B and C, and the invariant l has been computed in Proposition 2.5. □

3. Witt groups of real Enriques surfaces without real points.

We now consider the case when $X(\mathbb{R}) = \emptyset$. We start by proving that in this case the level of the function field is 2. In order to do this, we study the Galois module structure of $\operatorname{Pic} X_{\mathbb{C}}$, and compute the dimension of $H^1(G, \operatorname{Pic} X_{\mathbb{C}})$. As before, let H^2 denote the lattice $H^2(X_{\mathbb{C}}, \mathbb{Z})/\mathrm{Tors}$. We will consider the triplet of invariants (r, a, δ) associated to the G-action on H^2 (see [**N-S**, §3.3]). In fact, r and a were already introduced in the previous section. The definition of the invariant δ will not be recalled here; since $X(\mathbb{R}) = \emptyset$ we have that $\delta = 0$ (see [**N-S**, (3.3.5)]). Further, by Equation (8) we have $2r = 8$, so $r = 4$. Considering all the possibilities for the triplets

(r, a, δ) of invariants of the G-action on the lattice H^2 as listed in [**N-S**, (3.3.10)], we see that $(r, a, \delta) = (4, 2, 0)$. This is crucial in the proof of the following result.

Proposition 3.1. *Let X be a real Enriques surface with $X(\mathbb{R}) = \emptyset$. Then $H^1(G, \operatorname{Pic} X_{\mathbb{C}}) \simeq \mathbb{Z}/2$, and the mapping $H^1(G, {}_2\operatorname{Pic} X_{\mathbb{C}}) \to H^1(G, \operatorname{Pic} X_{\mathbb{C}})$ is zero.*

Proof. First let us establish that $H^1(G, \operatorname{Pic} X_{\mathbb{C}})$ is either $\mathbb{Z}/2$ or zero. This follows from the exact sequence

$$\begin{aligned} (9) \quad 0 \to \operatorname{Pic} X \to (\operatorname{Pic} X_{\mathbb{C}})^G \to \operatorname{Br}(\mathbb{R}) \to \operatorname{Ker}\{&\operatorname{Br}(X) \\ &\to \operatorname{Br}(X_{\mathbb{C}})\} \to H^1(G, \operatorname{Pic} X_{\mathbb{C}}) \to 0 \end{aligned}$$

obtained from the Hochschild-Serre spectral sequence for the étale sheaf $\mathbb{G}_m$ on X. Using the long exact cohomology sequence associated to (7), and the fact that $\dim H^1(G, \operatorname{Pic} X_{\mathbb{C}}/\mathrm{Tors}) = r - a = 2$, we see that $H^1(G, \operatorname{Pic} X_{\mathbb{C}}) \simeq \mathbb{Z}/2$ and the image of $H^1(G, {}_2\operatorname{Pic} X_{\mathbb{C}})$ in $H^1(G, \operatorname{Pic} X_{\mathbb{C}})$ is zero. □

Theorem 3.2. *Let X be a real Enriques surface such that $X(\mathbb{R}) = \emptyset$. Then the level of $\mathbb{R}(X)$ is 2.*

Proof. By §1 C we have that $\operatorname{Ker}\{\operatorname{Br}(X) \to \operatorname{Br}(X_{\mathbb{C}})\} \simeq \mathbb{Z}/2$. Since $H^1(G, \operatorname{Pic} X_{\mathbb{C}}) \simeq \mathbb{Z}/2$ as well, by Proposition 3.1, the exact sequence (9) gives a short exact sequence

$$(10) \qquad 0 \to \operatorname{Pic} X \to (\operatorname{Pic} X_{\mathbb{C}})^G \to \operatorname{Br}(\mathbb{R}) \to 0.$$

This implies that the natural map $\operatorname{Br}(\mathbb{R}) \to \operatorname{Br}(X)$ is zero. Since $\operatorname{Br}(X) \subseteq \operatorname{Br}(\mathbb{R}(X))$, this is equivalent to saying that the quaternion algebra $(-1, -1)$ splits over $\mathbb{R}(X)$. Equivalently, the norm form $\langle 1, 1, 1, 1 \rangle$ is isotropic over $\mathbb{R}(X)$ and hence -1 is a sum of two squares in $\mathbb{R}(X)$. □

Theorem 3.3. *Let X be a real Enriques surface such that $X(\mathbb{R}) = \emptyset$. Then*

$$W(X) \simeq (\mathbb{Z}/2)^2 \oplus \mathbb{Z}/4.$$

Proof. We will determine the invariants j, k, l and apply Theorem A1. Since $\Gamma(X, \mathcal{H}^1) \simeq H^1(X)$, the Kummer exact sequence gives that $\Gamma(X, \mathcal{H}^1) \simeq \mathbb{R}^*/\mathbb{R}^{*^2} \oplus {}_2\operatorname{Pic} X \simeq \mathbb{Z}/2 \oplus \mathbb{Z}/2$, so $j = 2$. By Theorem C we have that $\Gamma(X, \mathcal{H}^2) \simeq \mathbb{Z}/2$, so $k = 1$. Finally, we prove that $l = 2$ using the following commutative diagram with exact rows, which is quite similar to diagram (6) in the proof of Lemma 2.2.

$$(11) \quad \begin{array}{ccccccccc} 0 & \to & H^2(\mathbb{R}) & \to & \operatorname{Ker}\{H^2(X) \to H^2(X_{\mathbb{C}})\} & \to & H^1(G, H^1(X_{\mathbb{C}})) & \to & 0 \\ & & \downarrow i' & & \downarrow i'' & & \downarrow i''' & & \\ \operatorname{Pic} X_{\mathbb{C}} & \to & \operatorname{Br}(\mathbb{R}) & \xrightarrow{\phi} & \operatorname{Ker}\{\operatorname{Br}(X) \to \operatorname{Br}(X_{\mathbb{C}})\} & \to & H^1(G, \operatorname{Pic}\ X_{\mathbb{C}}) & \to & 0. \end{array}$$

Even though $X(\mathbb{R}) = \emptyset$, the arguments in the proof of Lemma 2.2 used to show that N is isomorphic to the kernel of the mapping i'' are still valid. We do need an extra argument in order to establish the exactness of the upper row of diagram (6). Consider the following morphism of exact sequences, derived from the Hochschild–Serre spectral sequence.

$$
\begin{array}{ccccccccc}
0 & \to & H^1(\mathbb{R}) & \to & H^1(X) & \xrightarrow{e} & H^1(X_{\mathbb{C}})^G & \xrightarrow{\partial} & H^2(\mathbb{R}) \\
 & & \downarrow \cup(-1) & & \downarrow \cup(-1) & & \downarrow \cup(-1) & & \downarrow \cup(-1) \\
H^1(X_{\mathbb{C}})^G & \xrightarrow{\partial} & H^2(\mathbb{R}) & \to & \operatorname{Ker}\{H^2(X) \to H^2(X_{\mathbb{C}})\} & \xrightarrow{e'} & H^1(G, H^1(X_{\mathbb{C}})) & \xrightarrow{\partial'} & H^3(\mathbb{R}).
\end{array}
$$

Since $H^1(X) \simeq \mathbb{Z}/2 \oplus \mathbb{Z}/2$, we have that the mapping e is surjective, hence ∂ is zero. All vertical mappings in the above diagram are surjective (in fact, even isomorphisms), which implies that ∂' is zero as well, hence the upper row of diagram (11) is exact.

In order to finish the computation of l, observe that the mapping ϕ is zero by Theorem 3.2, so a diagram chase gives that

$$\dim \operatorname{Ker} i'' = \dim \operatorname{Ker} i''' + 1.$$

It follows from Proposition 3.1 and the observation made at the end of the proof of Lemma 2.2 that

$$\operatorname{Ker} i''' \simeq H^1(G, {}_2\mathrm{Pic}\, X_{\mathbb{C}}) \simeq \mathbb{Z}/2.$$

As a result we obtain that $l = \dim \operatorname{Ker} i'' = 2$. $\square$

Remark 3.4. Let X be a (geometrically integral) K3 surface over $\mathbb{R}$ with $X(\mathbb{R}) = \emptyset$. By Pfister's result, the level of $\mathbb{R}(X)$ is either 2 or 4. Since a smooth projective hypersurface of degree 4 is a K3 surface, we know from the introduction that both cases actually occur. Using the result on Enriques surfaces we obtain new examples of K3 surfaces whose function field has level 2. Indeed, it follows from Theorem 3.2 that, the function field of any K3 surface covering a real Enriques surface without real points has level 2.

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