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EXISTENCE OF UNIVERSAL CONNECTIONS.*

By M. S. Narasimhan and S. Ramanan.

1. Introduction. The purpose of this paper is to prove the existence of universal connections for principal bundles with a compact Lie group as structure group. We prove (Theorem 2) that given a compact Lie group G and a positive integer n, there exist a differentiable principal G-bundle E *and* a connection γ_0 on E such that any connection on a differentiable principal G-bundle P with base of dimension $\leqq n$ can be obtained as the inverse image of the connection γ_0 by a differentiable bundle homomorphism of P into E. As is well-known, the analogous problem for bundles without connections is treated in the topology of fibre bundles [1].

It is also known that the Stiefel bundles play the role of universal bundles for the unitary groups $U(k)$. One can define in a natural way a connection on every Stiefel bundle (§ 2). We prove that these connections themselves are universal for connections in $U(k)$-bundles. A precise formulation is found in Theorem 1.

In the unitary case the problem is first solved locally by explicit construction, the crucial step being the lemma in § 3. The local solutions are then pieced up with the help of a special type of covering by coordinate cells.

In the general case, the compact Lie group G is identified with a closed subgroup of a unitary group. Starting from a universal connection for this unitary group, a universal connection for G is constructed by generalizing the usual method of construction of an invariant connection in the principal bundle associated with a Lie group and a closed subgroup ([3], p. 45).

A theorem of A. Weil ([1], p. 57) asserts that the cohomology classes of the base of a principal G-bundle obtained by substitution of the curvature form of a connection on P in the invariant polynomials of G are independent of the connection. Our result seems to explain this invariance and in fact furnishes an alternate proof in the case of compact Lie groups.

For definitions of the notions related to connections in principal bundles we refer to [1] and [3]. We use connections and connection forms interchangeably. By "differentiable" we always mean "indefinitely differentiable."

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All manifolds, bundles, bundle homomorphisms and differential forms are assumed to be differentiable. Also all manifolds that occur are paracompact.

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2. Canonical connections in Stiefel bundles. Let $\boldsymbol{C}^N$ be the N-dimensional complex number space with O as origin. The Stiefel manifold $V(N,k)$ (with $N \geqq k$) of all unitary k-frames at O may then be identified with the left coset space $U(N)/I_k \times U(N-k)$ where I_k is the unit (k,k) matrix. To every frame $(v_1,\cdots,v_k)$ with $v_i = \sum_{j=1}^{N} b_{i,j}e_j$ where (e_j) is the canonical base in $\boldsymbol{C}^N$, we associate the (N,k)-matrix $A = (a_{i,j})$ with $a_{i,j} = b_{j,i}$. Since $(v_1,\cdots,v_k)$ is orthonormal, we have $\sum_{\alpha=1}^{N} b_{j,\alpha}\bar{b}_{l,\alpha} = \delta_{j,l}$, i.e. A satisfies the condition $A^*A = I_k$ where A^* is the conjugate transpose of A. Thus $V(N,k)$ is identified with (N,k) matrices A satisfying $A^*A = I_k$. The action of $U(k)$ (resp. $U(N)$) on $V(N,k)$ to the right (resp. to the left) goes over under the above identification into multiplication of (N,k) matrices by unitary (k,k) matrices on the right (resp. by unitary (N,N) matrices on the left). Under the action of $U(k)$, $V(N,k)$ becomes a principal $U(k)$-bundle (known as the Stiefel bundle) with the Grassman manifold $G(N,k)$ of k-subspaces of $\boldsymbol{C}^N$ as base. $G(N,k)$ may again be identified with the left coset space $U(N)/U(k)\times U(N-k)$.

Let S be the (N,k) matrix-valued function on $V(N,k)$ which associates to each frame $(v_1,\cdots,v_k)$ the matrix A. Consider the (k,k) matrix-valued differential form S^*dS on $V(N,k)$. Since $S^*S = I_k$ for every frame, on differentiation we obtain $S^*dS + (dS^*)S = 0$, or again $S^*dS + (S^*dS)^* = 0$. Hence S^*dS has actually values in the Lie algebra $\mathfrak{u}(k)$ (which is the vector space of skew-Hermitian matrices) of $U(k)$.

PROPOSITION 1. *S^*dS is a connection form on the Stiefel bundle $V(N,k)$ which is invariant under the action of $U(N)$.*

In fact, if X_ξ is a tangent vector at $\xi \in V(N,k)$ and $s \in U(k)$, we shall denote by $X_\xi s$ the image of X_ξ under the differential of the map $\xi \to \xi s$ of $V(N,k)$. Then we have

$$\begin{aligned}(S^*dS)(X_\xi s) &= S^*(\xi s)(X_\xi s)(S)\\ &= s^*S^*(\xi)(X_\xi S)s\\ &= s^{-1}(S^*dS)(X_\xi)s.\end{aligned}$$

On the other hand, if $a \in \mathfrak{u}(k)$, we identify a with a tangent vector at e and denote by ξa the image of a under the differential of the map $s \to \xi s$ of G into $V(N,k)$. Then

$$\begin{aligned}(S^*dS)(\xi a) &= S(\xi)^*(\xi a)(S)\\ &= S(\xi)^*S(\xi)a\\ &= a,\end{aligned}$$

since $S(\xi)^*S(\xi) = I_k$. Hence S^*dS is a connection form on the Stiefel bundle. Moreover, if $t \in U(N)$, the left translation of the differential form by t yields $(tS)^*d(tS) = S^*t^*tdS = S^*dS$.

Remark. This connection will hereafter be referred to as the canonical connection and will be denoted by γ_0.

The horizontal subspace for this connection at the point $\xi_0 = \begin{pmatrix} I_k \\ 0 \end{pmatrix}$ of $V(N,k)$ may be described as follows: The tangent space at ξ_0 can be identified with (N,N) skew Hermitian matrices of the type $\begin{pmatrix} P & -Q^* \\ Q & 0 \end{pmatrix}$ where P is a (k,k) skew Hermitian matrix and Q is a rectangular $(N, N-k)$ matrix. The horizontal vectors at ξ_0 for the connection γ_0 are then given by matrices of the type $\begin{pmatrix} 0 & -Q^* \\ Q & 0 \end{pmatrix}$. This description together with the invariance under the action of $U(N)$ characterises the connection γ_0 completely.

Analogous statements are true for the real Stiefel manifold $W(N,k)$ and the corresponding $O(k)$-bundle. In particular $S'dS$ (where S' is the transpose of S) is a connection form on the Stiefel bundle, the corresponding horizontal subspace at ξ_0 being given by matrices of the type $\begin{pmatrix} 0 & -Q' \\ Q & 0 \end{pmatrix}$. This is easily seen to be the orthogonal complement of the vertical subspace at ξ_0 with respect to the killing form $\operatorname{tr}(\operatorname{ad}x \operatorname{ad}y)$ on $\mathfrak{o}(k)$, the Lie algebra of $O(k)$.

3. The local problem.

LEMMA. *Let U be an open subset of $\mathbf{R}^n$ and V a relatively compact open subset whose closure is contained in U. For every differential form α of degree 1 on U with values in $\mathfrak{u}(k)$ (the space of skew-Hermitian matrices), there exist differentiable functions $\phi_1, \cdots, \phi_{m'}$ in V with values in the space $\mathfrak{M}_k(\mathbf{C})$ of (k,k) complex matrices such that*

i) $$\sum_{j=1}^{m'} \phi_j^*\phi_j = I_k, \quad \textit{and}$$

ii) $$\sum_{j=1}^{m'} \phi_j{}^* d\phi_j = \alpha,$$

where $m' = (2n+1)k^2$.

Proof. Let $f_1, \cdots, f_{k^2}$ be a set of positive definite matrices which form a base for the complex Hermitian matrices over the reals, such that $\| f_r \| = 1$ for every r ($\| f \|$ being the norm as a linear transformation). Since α has values in $\mathfrak{u}(k)$, we may write α/i in U as $\sum_{s=1}^{n} \sum_{r=1}^{k^2} \lambda_{r,s} f_r dx_s$, where $\lambda_{r,s}$ are real-valued functions and x_s the coordinate functions in $\boldsymbol{R}^n$. If $a_{r,s} = \sup_V | \lambda_{r,s} |$, we have $\lambda_{r,s} = \mu_{r,s} - \nu_{r,s}$ where

$$\mu_{r,s} = \tfrac{1}{2}\{\lambda_{r,s} + a_{r,s} + 1\} \quad \text{and}$$

$$\nu_{r,s} = \tfrac{1}{2}\{a_{r,s} - \lambda_{r,s} + 1\}$$

are both strictly positive differentiable functions. Hence we may write $\mu_{r,s} = p_{r,s}{}^2$ and $\nu_{r,s} = q_{r,s}{}^2$ where $p_{r,s}$ and $q_{r,s}$ are positive differentiable functions. Clearly one may assume that $\sum_{s=1}^{n} (\mu_{r,s} + \nu_{r,s})$ is bounded by $1/2k^2$ on V for every r, by altering the coordinate functions x_s by a constant multiple, if necessary. The matrix valued function $1/k^2 I_k - \{\sum_{s=1}^{n} (\mu_{r,s} + \nu_{r,s})\} f_r$ is then positive. For,

$$\| \sum_{s=1}^{n} (\mu_{r,s} + \nu_{r,s}) f_r \| \leqq 1/2k^2 \| f_r \| < 1/k^2.$$

Let g_r be the (unique) positive square-root of the positive matrix f_r and h_r the differentiable positive matrix-valued function satisfying

$$h_r{}^2(x) = 1/k^2 I_k - \{ \sum_{s=1}^{n} (\mu_{r,s}(x) + \nu_{r,s}(x)) \} f_r.$$

We now define $\mathfrak{M}_k(\boldsymbol{C})$-valued functions $\phi_j (1 \leqq j \leqq (2n+1)k^2)$ as follows.

For $1 \leqq j \leqq nk^2$, ϕ_j shall be the nk^2 functions $p_{r,s} e^{ix_s} \cdot g_r$ arranged in some order.

For $nk^2 + 1 \leqq j \leqq 2nk^2$, ϕ_j shall be the nk^2 functions $q_{r,s} e^{-ix_s} \cdot g_r$ arranged in some order.

For $2nk^2 + 1 \leqq j \leqq (2n+1)k^2$, ϕ_j shall be the functions h_r in some order.

We have to verify that the ϕ_j thus defined fulfil the conditions i) and ii) of the lemma. In fact,

$$\begin{aligned}\sum_{j=1}^{m'} \phi_j{}^*\phi_j &= \sum_{r,s} p_{r,s}{}^2 g_r{}^2 + \sum_{r,s} q_{r,s}{}^2 g_r{}^2 + \sum_r h_r{}^2 \\ &= \sum_{r,s} \mu_{r,s} f_r + \sum_{r,s} \nu_{r,s} f_r + I_k - \sum_{r,s} (\mu_{r,s} + \nu_{r,s}) f_r \\ &= I_k.\end{aligned}$$

On the other hand,

$$\begin{aligned}\sum_{j=1}^{m'} \phi_j{}^* d\phi_j &= \sum_{r,s} p_{r,s} e^{-ix_s}\{p_{r,s} i e^{ix_s}(dx_s) + (dp_{r,s}) e^{ix_s}\} g_r{}^2 \\ &\quad + \sum_{r,s} q_{r,s} e^{ix_s}\{- q_{r,s} \cdot i e^{-ix_s}(dx_s) + (dq_{r,s}) e^{-ix_s}\} g_r{}^2 \\ &\quad + \sum_r h_r dhr \\ &= \sum_{r,s} i \cdot p_{r,s}{}^2 g_r{}^2 dx_s + \sum_{r,s} p_{r,s} dp_{r,s} g_r{}^2 \\ &\quad + \sum_{r,s} i(- q_{r,s}{}^2) g_r{}^2 dx_s + \sum_{r,s} q_{r,s} dq_{r,s} g_r{}^2 + \sum_r h_r dh_r \\ &= \sum_{r,s} i(\mu_{r,s} - \nu_{r,s}) f_r dx_s + \tfrac{1}{2} \sum_{r,s} d(p_{r,s}{}^2 + q_{r,s}) f_r + \sum_r h_r dh_r \\ &= \alpha + \tfrac{1}{2} \sum_{r,s} d(\mu_{r,s} + \nu_{r,s}) f_r + \sum_r h_r dh_r.\end{aligned}$$

But since for any $x, y \in V$, $h_r{}^2(x)$ and $h_r{}^2(y)$ commute, their positive square roots $h_r(x)$ and $h_r(y)$ also commute. It readily follows that $h_r dh_r = dh_r \cdot h_r$. Hence $\frac{1}{2} d(h_r{}^2) = h_r dh_r$. Therefore, finally we have

$$\begin{aligned}\sum_{j=1}^{m'} \phi_j{}^* d\phi_j &= \alpha + \tfrac{1}{2} d\{ \sum_{r,s} (\mu_{r,s} + \nu_{r,s}) f_r + \sum_r h_r{}^2) \} \\ &= \alpha,\end{aligned}$$

since $\sum_{r,s} (\mu_{r,s} + \nu_{r,s}) f_r + \sum_r h_r{}^2 = I_k$, and the lemma is completely proved.

The problem is solved locally by the following

PROPOSITION 2. Let P be a principal $U(k)$-bundle over a manifold X of dimension $\leqq n$ and γ a connection form on P. For every relatively compact open subset W of X with $\bar{W}$ contained in a coordinate neighborhood U of X over which P is trivial, there exists a differentiable bundle map Φ of $p^{-1}(W)$ into $V(m'', k)$ such that the inverse image of the canonical connection γ_0 by Φ is γ, where $m'' = (2n + 1)k^3$ and p is the projection $P \rightarrow X$.

Proof. Let σ be a section of P over U and α the inverse image of γ by σ.

By the lemma, we can find differentiable $\mathfrak{M}_k(\boldsymbol{C})$-valued functions $\phi_1, \cdots, \phi_{m'}$ in W such that

i) $\sum_{j=1}^{m'} \phi_j^* \phi_j = I_k$, and

ii) $\sum_{j=1}^{m'} \phi_j^* d\phi_j = \alpha$, where $m' = (2n+1)k^2$.

Define a map Φ of P over W into the space of (m'', k)-matrices by setting for

$$\xi \in P, \quad \Phi(\xi) = \begin{pmatrix} \phi_1(p\xi) \\ \vdots \\ \phi_{m'}(p\xi) \end{pmatrix} \cdot s \quad \text{where } s \in U(k) \text{ is determined by } \xi = \sigma(p\xi)s.$$

Φ is easily seen to be a bundle homomorphism. We then have

$$\begin{aligned} \Phi^*\Phi(\xi) &= s^*\big(\sum_{j=1}^{m'} (\phi_i^*\phi_i)(p\xi)\big)s \\ &= s^*s, \text{ by (i)} \\ &= I_k, \text{ since } s \text{ is unitary.} \end{aligned}$$

Hence Φ maps $P|p^{-1}(W)$ actually into $V(m'', k)$. On the other hand, it is obvious that the inverse image by Φ of $\gamma_0 = S^*dS$ is given by $\Phi^*d\Phi$. But the inverse image by σ of $\Phi^*d\Phi$ is $(\Phi\circ\sigma)^*d(\Phi\circ\sigma) = \sum_{i=1}^{m'} \phi_i^* d\phi_i = \alpha$ by construction. Now γ and $\Phi^*d\Phi$ are two connections on $P|p^{-1}(W)$ such that their inverse image by the section σ are the same. Hence $\gamma = \Phi^*d\Phi$ on $p^{-1}(W)$.

4. Universal connection for the unitary group.

THEOREM 1. *Let P be a principal $U(k)$-bundle over a manifold X of dimension $\leqq n$ and γ any connection form on P. Then there exists a differentiable bundle homomorphism Φ of P into the Stiefel bundle $V(m, k)$ such that γ is the inverse image by Φ of the canonical connection γ_0 on $V(m, k)$, where $m = (n+1)(2n+1)k^3$.*

Proof. We can find a covering of X by relatively compact open sets $\{V_i\}$ such that i) each $\bar{V}_i$ is contained in a coordinate cell, and ii) the V_i's can be divided into $(n+1)$ classes $\mathcal{C}_j$ in such a way that no two V_i's of the same class intersect ([2], p. 61). Let $\{W_i\}$ be a shrinking of this covering, i.e., an open covering $\{W_i\}$ such that $\bar{W}_i \subset V_i$. Let D_j $(j = 1, \cdots, n+1)$ be the union of the open sets $p^{-1}(W_i)$ where $\bar{W}_i \subset V_i$ with V_i belonging to $\mathcal{C}_j$.

The bundle is trivial over the coordinate cells and hence, by Proposition 2, one can find differentiable bundle homomorphisms Φ_i on $p^{-1}(V_i)$ into

$V((2n+1)k^3, k)$ inducing the connection γ on $p^{-1}(V_i)$. Corresponding to each D_j there exists a $\{(2n+1)k^3, k\}$-matrix-valued differentiable function Ψ on P such that Ψ coincides with Φ_i on $p^{-1}(W_i)$ for V_i in $\mathcal{B}_j$. Let then $\Psi_1, \cdots, \Psi_{n+1}$ be the $(n+1)$ functions thus constructed. Consider a differentiable partition of unity with respect to the covering $\{D_j\}$ consisting of non-negative differentiable functions ζ_i invariant under the action of the group $U(k)$ such that the support of $\zeta_i \subset D_i$ and $\sum \zeta_i^2 = 1$.

Consider now the map Φ on P defined by

$$\Phi(\xi) = \begin{pmatrix} \zeta_1(\xi)\Psi_1(\xi) \\ \vdots \\ \zeta_{n+1}(\xi)\Psi_{n+1}(\xi) \end{pmatrix} \text{for every } \xi \in P.$$

We have to prove that Φ is bundle map of P into $V(m,k)$ such that $\Phi^* d\Phi = \alpha$. But

$$\begin{aligned} \Phi^*\Phi(\xi) &= \sum_{i=1}^{n+1} \zeta_i(\xi)^2 \Psi_i^*(\xi)\Psi_i(\xi) \\ &= \sum \zeta_i(\xi)^2 \Psi_i^*(\xi)\Psi_i(\xi), \end{aligned}$$

the summation being over those i's for which $\xi \in D_i$. But on D_i, $\Psi_i^*\Psi_i = I$ and we have $(\Psi_i^*\Psi_i) = I$ for every i over which the summation extends. Hence $\Phi^*\Phi(\xi) = \sum \zeta_i(\xi)^2 I = I$, since $\sum \zeta_i^2(\xi) = 1$. Moreover

$$\begin{aligned} \Phi(\xi s) &= \begin{pmatrix} \zeta_1(\xi s)\Psi_1(\xi s) \\ \vdots \\ \zeta_{n+1}(\xi s)\Psi_{n+1}(\xi s) \end{pmatrix} = \begin{pmatrix} \zeta_1(\xi)\Psi_1(\xi)s \\ \vdots \\ \zeta_{n+1}(\xi)\Psi_{n+1}(\xi)s \end{pmatrix} \\ &= \begin{pmatrix} \zeta_1(\xi)\Psi_1(\xi) \\ \vdots \\ \zeta_{n+1}(\xi)\Psi_{n+1}(\xi) \end{pmatrix} s \\ &= \Phi(\xi)\cdot s \end{aligned}$$

for every $\xi \in P$ and $s \in U(k)$.

Finally,

$$\begin{aligned} \Phi^* d\Phi &= \sum_{i=1}^{n+1} \zeta_i \Psi_i^* (d\zeta_i \Psi_i + \zeta_i d\Psi_i) \\ &= \sum_{i=1}^{n+1} \Psi_i^*\Psi_i \zeta_i d\zeta_i + \sum_{i=1}^{n+1} \zeta_i^2 \Psi_i^* d\Psi_i. \end{aligned}$$

As before, for a $\xi \in P$, the summation needs to be taken only over those i's for which $\xi \in D_i$. In every such D_i, however, $\Psi_i^* \Psi_i = I$ and $\Psi_i^* d\Psi_i = \alpha$. Hence $\Phi^* d\Phi = \sum \zeta_i d\zeta_i I + (\sum \zeta_i^2)\alpha = \alpha$ since $\sum \zeta_i d\zeta_i = \frac{1}{2} d(\sum \zeta_i^2) = 0$.

5. Universal connections for compact Lie groups. Let G_2 be a closed subgroup of a Lie group G_1, and $\mathfrak{g}_2$ and $\mathfrak{g}_1$ their Lie algebras. The group G_2 acts on $\mathfrak{g}_1$ by the adjoint operations and $\mathfrak{g}_2$ is invariant under this representation. Suppose $\mathfrak{m}$ is a subspace of $\mathfrak{g}_1$ invariant under the action of G_2 which is supplementary to $\mathfrak{g}_2$. (Such a space $\mathfrak{m}$ exists if G_2 is compact or semi-simple.) Let P be a principal bundle with group G_1 and ω_1 a connection on P. P is fibred by G_2 into a principal bundle with group G_2.

The direct sum decomposition $\mathfrak{g}_2 \oplus \mathfrak{m}$ of $\mathfrak{g}_1$ gives rise to a projection π of $\mathfrak{g}_1$ onto $\mathfrak{g}_2$ which commutes with the action of G_2 (i. e. $\pi \circ ads = ads \circ \pi$ for every $s \in G_2$). We define a differential form ω_2 on P by setting $\omega_2 = \pi \circ \omega_1$. It is easy to see that ω_2 is a connection on P for the fibration by G_2. In fact, (with the notations of § 2),

$$\begin{aligned} \omega_2(X_\xi s) &= (\pi \cdot \omega_1)(X_\xi s) \\ &= \pi \cdot \omega_1(X_\xi s) \\ &= \pi \cdot \operatorname{ad}s\, \omega_1(X_\xi) \\ &= \operatorname{ad}s \cdot \pi\omega_1(X_\xi) \\ &= s^{-1}\omega_2(X_\xi)s \end{aligned}$$

for every vector X_ξ at $\xi \in P$ and $s \in G_2$.

On the other hand, for every $\xi \in P$ and $a \in \mathfrak{g}_2$, we have

$$\begin{aligned} \omega_2(\xi a) &= (\pi \cdot \omega_1)(\xi a) \\ &= \pi \cdot \omega_1(\xi a) \\ &= \pi(a) \\ &= a, \text{ since } \pi \text{ is a projection.} \end{aligned}$$

THEOREM 2. *Let G be a compact Lie group and n a positive integer. There exist a principal G-bundle B and a connection form γ_1 on B such that for every principal G-bundle P with base of dimension $\leqq n$ and any connection form γ on P, one can find a bundle homomorphism f of P into B such that the inverse image of γ_1 by f is γ.*

Proof. G can be identified with a closed subgroup of a unitary group $U(k)$. Let γ_0 be a universal connection for $U(k)$ (for the dimension n) on a principal $U(k)$-bundle B, whose existence has been proved in Theorem 1. G acts on B and makes of it a principal G-bundle. One can define a connection γ_1 on this bundle by setting $\gamma_1 = \pi \circ \gamma_0$ where π is a projection of $\mathfrak{u}(k)$ onto the Lie algebra $\mathfrak{g}$ of G, as explained above. For any principal G-bundle P, with base of dimension $\leqq n$, let P' be the corresponding principal $U(k)$-bundle obtained by enlarging the group G. Then there is a natural inclusion $i\colon P \to P'$ such that $i(\xi s) = i(\xi)(s)$ for $\xi \in P$ and $s \in G$.

Moreover, if γ is a connection form on P, one can define a natural connection γ' on P' such that the inverse image of γ' by i is γ ([3], p. 35). Let Φ be a bundle map of P into B such that the inverse image of γ_0 by Φ is γ'. We define a bundle map f of P into B fibred by G by setting $f = \Phi \cdot i$. The inverse image of γ_0 by $f = \Phi \cdot i$ is obviously γ. But since γ has values in $\mathfrak{g}$ and π is identity on $\mathfrak{g}$, we have $\pi \cdot \gamma = \gamma$ and hence the inverse image of γ_1 by f is γ.

6. Remarks.

i) We show how A. Weil's theorem on connections can be deduced from our results, at least when G is compact. Let ω_1, ω_2 be two connections on a principal G-bundle P with base X of dimension $\leqq n-1$. Consider the bundle $P \times I' \to X \times I'$ where I' is the open interval $(-\epsilon, 1+\epsilon)$, $\epsilon > 0$. Let α_1, α_2 be inverse images of ω_1, ω_2 respectively under the projection $P \times I' \to P$. The differential form $\alpha = t\alpha_1 + (1-t)\alpha_2$ where t is the projection $P \times I' \to I'$, is easily seen to be a connection on $P \times I'$. Let B be a principal G-bundle over a manifold M and γ_1 a universal connection on B for dimension $\leqq n$. It follows that there exists a differentiable family F_t of differentiable bundle maps of P into B such that the inverse image of γ_1 by F_t is $t\omega_1 + (1-t)\omega_2$. If f_t are the corresponding maps of X into M, then f_0 and f_1 are obviously homotopic. On the other hand, if K_1 and K_2 are the curvature forms of ω_1, ω_2 respectively, the 'substitution' of K_1, K_2 in each polynomial over $\mathfrak{g}$ invariant under the adjoint representation of G yields closed differential forms β_1, β_2 on X. Then β_1 and β_2 are the inverse images under f_0 and f_1 of the form on M obtained by substituting the curvature form K of the universal connection in the same polynomial. Since f_0 and f_1 are homotopic, it follows that β_1 and β_2 define the same cohmology class on the base, a characteristic class of the bundle.

ii) Our method gives a universal connection for the orthogonal group $O(k)$ in particular. But the connection was defined in the complex Stiefel manifold fibred by $O(k)$ instead of the more usual real Stiefel bundle. We have already remarked (§ 2) that if the points of the real Stiefel manifold $W(N, k)$ are represented by (N, k)-matrices A satisfying $A'A = I_k$ (A' being the transpose of A), a connection can be defined in a canonical way on the real Stiefel bundle with the corresponding connection form $A'dA$. On the other hand, the complex Stiefel manifold $V(N,k)$ may be imbedded into the real Stiefel manifold $W(2N,k)$ by associating to each (N,k) matrix A, the $(2N,k)$ real matrix $\tilde{A} = \begin{pmatrix} \mathrm{Rl}\, A \\ \mathrm{Im}\, A \end{pmatrix}$ where $\mathrm{Rl}\, A$, $\mathrm{Im}\, A$ are the real and imaginary parts of A. It is easy to see that if $A^*A = I_k$, we have $\tilde{A}'\tilde{A} = I_k$. Moreover, for every $s \in O(k) \subset U(k)$

$$(\widetilde{As}) = \begin{pmatrix} \mathrm{Rl}\,(As) \\ \mathrm{Im}\,(As) \end{pmatrix} = \begin{pmatrix} (\mathrm{Rl}\, A)s \\ (\mathrm{Im}\, A)s \end{pmatrix} = \tilde{A} \cdot s.$$

Hence the map $A \rightarrow \tilde{A}$ is a bundle map of the complex Stiefel manifold $V(N,k)$ fibred by $O(k)$, into the real Stiefel bundle. The connection form on $V(N,k)$ induced by this map is $\tilde{A}'d\tilde{A}$, but this is the same as the real part of A^*dA. If we take for the projection π of § 4, the map of $\mathfrak{u}(k)$ onto the Lie algebra $\mathfrak{o}(k)$ of $O(k)$ defined as the assignment of the real part to each skew Hermitian matrix, then the corresponding connection γ_1 is the same as the real part of A^*dA. In other words, the canonical connection in the real Stiefel bundle is universal for $O(k)$-bundles.

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