DUALITY SYMMETRY GROUP OF TWO DIMENSIONAL HETEROTIC STRING THEORY

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Abstract

The equations of motion of the massless sector of the two dimensional string theory, obtained by compactifying the heterotic string theory on an eight dimensional torus, is known to have an affine $o(8,24)$ symmetry algebra generating an $O(8,24)$ loop group. In this paper we study how various known discrete S- and T- duality symmetries of the theory are embedded in this loop group. This allows us to identify the generators of the discrete duality symmetry group of the two dimensional string theory.

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1 Introduction and Summary

There is mounting evidence that heterotic string theory compactified on a six dimensional torus is invariant under an $SL(2, Z)$ group of duality transformation that interchanges the strong and weak coupling limits of the theory[1]-[15]. This duality symmetry is known as S-duality, and is distinct from the target space duality or T-duality symmetry[16, 17] of the theory which generates the group $O(6, 22; Z)$. T-duality symmetry holds order by order in string perturbation theory, and hence is easy to test, whereas S-duality transformation mixes different orders in string perturbation theory and hence is not testable in perturbation theory. Nevertheless, non-trivial tests of S-duality symmetry have been designed, and so far the results of all such tests support the hypothesis that S-duality is a genuine symmetry of the full string theory. If this hypothesis turns out to be correct, then we can use this symmetry to unravel some of the non-perturbative features of string theory, since this symmetry acts non-trivially on the string coupling constant. In particular, we can calculate scattering amplitude of elementary string states at strong coupling by calculating the scattering amplitude of solitons at weak coupling.

Although string theory in four dimensions is the theory that is of most interest to us, the particular string theory in which S-duality is most manifest, namely, ten dimensional heterotic string theory compactified on a torus, does not give a phenomenologically acceptable four dimensional theory. There are of course four dimensional theories which are much better for low energy phenomenology (including models which have $SU(3) \times SU(2) \times U(1)$ as low energy gauge group and three generations of quarks and leptons[18]), but at present it seems to be difficult to understand the role of S-duality in such theories. (Some progress in this direction has been made in refs.[19]-[22].) A more fruitful approach seems to be not to restrict ourselves only to physically motivated models, but study S-duality in a more general context. One might hope that understanding S-duality in a widely different class of models may finally give us a clue as to how it manifests itself in a physically motivated string theory.

Guided by this principle, in a previous paper we had studied the manifestation of S-duality in a three dimensional string theory, obtained by compactifying ten dimensional heterotic string theory on a seven dimensional torus[23]. It was found that in three dimensions the T-duality group $O(7, 23; Z)$ and the S-duality group combine into a much bigger group

$O(8,24;Z)$. This analysis was facilitated by an earlier work of Marcus and Schwarz[24], where they had studied the symmetries of the three dimensional supergravity theory, which forms the massless sector of the three dimensional string theory, and had found it to be $O(8,24)$. When we take into account the massive string states, only a discrete subgroup $O(8,24;Z)$ of $O(8,24)$ has a chance of being an exact symmetry of the spectrum, and this was conjectured to be the symmetry of the three dimensional string theory. One support to this conjecture comes from the fact that this discrete $O(8,24;Z)$ subgroup is generated by the T-duality group $O(7,23;Z)$ of the three dimensional string theory, and the S-duality group $SL(2,Z)$ of the four dimensional theory before compactification from four to three dimensions. Thus if S-duality is a symmetry of the four dimensional theory, and if it survives compactification of one of the dimensions, then $O(8,24;Z)$ must be the symmetry group of the three dimensional theory. (Similar arguments have been used in ref.[25] to construct the full T-duality group of string theories compactied on $K3$.)

In this paper, we shall carry out a similar analysis for the two dimensional string theory, obtained by compactifying heterotic string theory on an eight dimensional torus. Again our analysis is facilitated by known results about two dimensional supergravity theory, whose classical equations of motion are known to be invariant under an $o(\widehat{8,24})$ current algebra symmetry[26]-[32]. (For a history of this subject, see refs.[30, 32].) Thus one would expect the duality symmetry group of the two dimensional string theory to be a discrete subgroup of the loop group $O(\widehat{8,24})$. (This was already conjectured by Hull and Townsend[33]). If we assume that the S-duality is a symmetry of the theory even after two of the four dimensions are compactified, then the minimal duality symmetry group of the two dimensional theory is generated by the usual T-duality transformations $O(8,24;Z)$ of the two dimensional string theory, and the S-duality transformations of the four dimensional string theory. As in the case of three dimensional string theory, the S- and T- duality transformations do not commute, and generate a much bigger group. Following the convention of Hull and Townsend[33], we shall call this the U-duality group, and denote this by G.

The main aim of this paper will be to find explicit representation of the generators of the group G. As we shall see, these will be represented by O(8,24) valued functions (possibly containing poles at the origin) of a real variable v. There are two main steps involved in this project. In section 2 we study how the various known symmetry groups of two and higher dimensional supergravity theories are embedded in the loop group $O(\widehat{8,24})$. This includes the manifest $O(8,24)$ symmetry group of the two dimensional supergravity theory that contains the T-duality group $O(8,24;Z)$ as its natural subgroup, as well as the $O(8,24)$ symmetry of the three dimensional supergravity theory that contains the S-duality group $SL(2,Z)$ as its subgroup. Thus this analysis provides an explicit embedding of the various known duality transformations in the loop group $O(\widehat{8,24})$. In section 3 we use this knowledge to construct explicitly the elements of the loop group $O(\widehat{8,24})$ (represented by $O(8,24)$ matrix valued functions of a real variable) corresponding to the generators of the duality transformation. The appendix contains some of the technical results that are needed for the analysis of section 2.

We conclude this section by stating some of the conventions that we shall be using in this paper. We use the normalization $\alpha' = 16$. We shall denote algebras by lower case letters *e.g.* $o(8,24)$, and groups by upper case letters *e.g.* $O(8,24)$. We shall use a hat to denote the affine algebra or loop group, *e.g.* $o(\widehat{8,24})$ or $O(\widehat{8,24})$. Algebras or groups without hat will denote the ordinary Lie algebras or Lie groups generated by the zero mode subalgebra of the corresponding affine algebra. Finally, in our convention, the group element g and the generators T^α are related by $g = \exp(\theta_\alpha T^\alpha)$; note the absence of i in the exponent.

2 Embedding of Various $o(8,24)$ Subalgebras in the Affine Algebra $o(\widehat{8,24})$

We start with the massless sector of the heterotic string theory in ten dimensions. This theory is described by the action

$$\begin{aligned} S &= \frac{2\pi}{(8\pi)^8}\int d^{10}z\sqrt{-G^{(10)}}e^{-\Phi^{(10)}}\Big[R^{(10)} + G^{(10)MN}\partial_M\Phi^{(10)}\partial_N\Phi^{(10)} \\ &\quad -\frac{1}{12}H^{(10)}_{MNP}H^{(10)MNP} - \frac{1}{4}F^{(10)I}_{MN}F^{(10)IMN}\Big], \end{aligned} \tag{2.1}$$

where $R^{(10)}$ denotes the scalar curvature,

$$F^{(10)I}_{MN} = \partial_M A^{(10)I}_N - \partial_N A^{(10)I}_M,$$

$$H^{(10)}_{MNP} = (\partial_M B^{(10)}_{NP} - \frac{1}{2} A^{(10)I}_M F^{(10)I}_{NP}) + \text{cyclic permutations of } M,\ N,\ P\,, \tag{2.2}$$

and $G^{(10)}_{MN}$, $B^{(10)}_{MN}$, $A^{(10)I}_M$ and $\Phi^{(10)}$ $(0 \le M \le 9)$ denote the metric, anti-symmetric tensor field, $U(1)^{16}$ gauge fields, and the dilaton field in ten dimensions. z^M are the coordinates of the ten dimensional space. The overall normalization factor multiplying the action has been chosen appropriately for later convenience. We now dimensionally reduce the theory to two dimensions by defining[34, 35],

$$\begin{aligned} z^\mu &= x^\mu, \quad 0 \le \mu \le 1 \\ z^{m+1} &= y^m, \quad 1 \le m \le 8\,. \end{aligned} \tag{2.3}$$

We shall take each of the coordinates y^m to represent a compact direction with length 8π : $y^m \equiv y^m + 8\pi$. We take various fields to be independent of the coordinates y^m and define,[1]

$$\begin{aligned}
&\widehat{G}_{mn} = G^{(10)}_{m+1,n+1}, \quad \widehat{B}_{mn} = B^{(10)}_{m+1,n+1}, \quad \widehat{A}^I_m = A^{(10)I}_{m+1}, \\
&A^{(2m-1)}_\mu = \frac{1}{2}\widehat{G}^{mn} G^{(10)}_{n+1,\mu}, \quad A^{(I+16)}_\mu = -(\frac{1}{2} A^{(10)I}_\mu - \widehat{A}^I_n A^{(2n-1)}_\mu), \\
&A^{(2m)}_\mu = \frac{1}{2} B^{(10)}_{(m+1)\mu} - \widehat{B}_{mn} A^{(2n-1)}_\mu + \frac{1}{2}\widehat{A}^I_m A^{(I+16)}_\mu, \\
&G_{\mu\nu} = G^{(10)}_{\mu\nu} - G^{(10)}_{(m+1)\mu} G^{(10)}_{(n+1)\nu} \widehat{G}^{mn}, \\
&B_{\mu\nu} = B^{(10)}_{\mu\nu} - 4\widehat{B}_{mn} A^{(2m-1)}_\mu A^{(2n-1)}_\nu - 2(A^{(2m-1)}_\mu A^{(2m)}_\nu - A^{(2m-1)}_\nu A^{(2m)}_\mu) \\
&\qquad\quad + 2\widehat{A}^I_n (A^{(2n-1)}_\mu A^{(I+16)}_\nu - A^{(2n-1)}_\nu A^{(I+16)}_\mu)\,, \\
&\Phi = \Phi^{(10)} - \frac{1}{2}\ln\det\widehat{G}, \qquad 1 \le m,n \le 8, \quad 0 \le \mu,\nu \le 1, \quad 1 \le I \le 16.
\end{aligned} \tag{2.4}$$

Here $\widehat{G}^{mn}$ denotes the inverse of the matrix $\widehat{G}_{mn}$. We also define,

$$\widehat{C}_{mn} = \frac{1}{2}\widehat{A}^I_m \widehat{A}^I_n\,. \tag{2.5}$$

For every m, n $(1 \le m, n \le 8)$ we define H_{mn} to be the 2×2 matrix

$$H_{mn} = \begin{pmatrix} \widehat{G}^{mn} & \widehat{G}^{mp}(\widehat{B}_{pn} + \widehat{C}_{pn}) \\ (-\widehat{B}_{mp} + \widehat{C}_{mp})\widehat{G}^{pn} & (\widehat{G} - \widehat{B} + \widehat{C})_{mp}\widehat{G}^{pq}(\widehat{G} + \widehat{B} + \widehat{C})_{qn} \end{pmatrix}, \tag{2.6}$$

[1] The last term in the expression for $B_{\mu\nu}$ was inadvertently left out in refs.[6, 23].

and for every m, I ($1 \leq m \leq 8$, $1 \leq I \leq 16$) we define $Q_m^{(I)}$ to be the two dimensional column vector

$$Q_m^{(I)} = \begin{pmatrix} \widehat{G}^{mn}\widehat{A}_n^I \\ \\ (\widehat{G} - \widehat{B} + \widehat{C})_{mn}\widehat{G}^{np}\widehat{A}_p^I \end{pmatrix} . \tag{2.7}$$

We also define K to be a 16×16 matrix whose components are,

$$K_{IJ} = \delta_{IJ} + \widehat{A}_m^I \widehat{G}^{mn} \widehat{A}_n^J . \tag{2.8}$$

In terms of H, Q and K, we now define a 32×32 matrix M as follows:

$$M = \begin{pmatrix} H_{11} & \cdot & H_{18} & Q_1^{(1)} & \cdot & Q_1^{(16)} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ H_{81} & \cdot & H_{88} & Q_8^{(1)} & \cdot & Q_8^{(16)} \\ & & & & & \\ Q_1^{(1)T} & \cdot & Q_8^{(1)T} & & & \\ \cdot & \cdot & \cdot & & K & \\ Q_1^{(16)T} & \cdot & Q_8^{(16)T} & & & \end{pmatrix} . \tag{2.9}$$

M satisfies

$$M^T = M, \qquad MLM^T = L\,, \tag{2.10}$$

where L is a 32×32 matrix, defined as,

$$L = \begin{pmatrix} \sigma_1 & & & & \\ & \cdot & & & \\ & & \cdot & & \\ & & & \sigma_1 & \\ & & & & -I_{16} \end{pmatrix} , \qquad \sigma_1 = \begin{pmatrix} & 1 \\ 1 & \end{pmatrix} . \tag{2.11}$$

I_n denotes the $n \times n$ identity matrix. It is well known that the matrix M contains exactly the same information as the set of fields $\{\widehat{G}_{mn}, \widehat{B}_{mn}, \widehat{A}_m^I\}$.

We now use the fact that the antisymmetric tensor field $B_{\mu\nu}$ and the gauge fields $A_\mu^{(a)}$ ($1 \leq a \leq 32$) have no dynamics in two dimensions. Thus we can set them to zero. The remaining dynamical variables in two dimensions are given by the fields Φ, the matrix M, and the two dimensional metric $G_{\mu\nu}$. In terms of these fields the ten dimensional action (2.1) takes the form:

$$S = (2\pi) \int d^2x \sqrt{-G}\, e^{-\Phi}\, [R_G + G^{\mu\nu}\partial_\mu\Phi\partial_\nu\Phi + \frac{1}{8}G^{\mu\nu}Tr(\partial_\mu ML\partial_\nu ML)] . \tag{2.12}$$

This action gives rise to the following set of independent equations of motion in the conformal gauge $G_{\mu\nu} = \lambda e^{2\Phi}\eta_{\mu\nu}$:

$$\eta^{\mu\nu}\partial_\mu\partial_\nu(e^{-\Phi}) = 0\,, \tag{2.13}$$

$$\partial_\mu(e^{-\Phi}\eta^{\mu\nu}\partial_\nu M M^{-1}) = 0\,, \tag{2.14}$$

and,

$$\partial_\pm(\ln\lambda)\partial_\pm(e^{-\Phi}) = \partial_\pm^2(e^{-\Phi}) - \frac{1}{4}e^{-\Phi}\,Tr(\partial_\pm ML\partial_\pm ML)\,, \tag{2.15}$$

where $x^\pm$ are the light cone variables,

$$x^\pm = \frac{1}{\sqrt{2}}(x^0 \pm x^1)\,. \tag{2.16}$$

This system of equations has an $o(8,24)$ current algebra symmetry[26]-[32] which we shall denote by $\widehat{o(8,24)}$. It is generated by conserved charges J_n^α $(-\infty < n < \infty)$ and the central charge[29] C:

$$[J_m^\alpha, J_n^\beta] = f^{\alpha\beta\gamma}\eta_{\gamma\delta}J_{m+n}^\delta + m\eta^{\alpha\beta}\delta_{m+n,0}C\,, \tag{2.17}$$

where $f^{\alpha\beta\gamma}$ are the structure constants of $o(8,24)$, and $\eta^{\alpha\beta}$ is the metric.

Let us now discuss the action of various generators of the current algebra on the fields M, Φ and $G_{\mu\nu}$. We shall follow the approach of Breitenlohner and Maison[27] adapted for the present system. The central charge C simply causes a shift of λ (*i.e.* a scaling of $G_{\mu\nu}$) without any transformation of the fields M and Φ:

$$\delta_C G_{\mu\nu} = 2G_{\mu\nu}, \qquad \delta_C\Phi = 0, \qquad \delta_C M = 0\,, \tag{2.18}$$

where for any symmetry generator $\mathcal{O}$ and a field ϕ, $\epsilon\delta_{\mathcal{O}}\phi$ denotes the infinitesimal transformation of ϕ generated by $\mathcal{O}$, ϵ being the infinitesimal transformation parameter. J_0^α generate the usual $O(8,24)$ transformations

$$\delta_0^\alpha M = -(T^\alpha M + M(T^\alpha)^T)\,, \qquad \delta_0^\alpha\Phi = 0\,, \qquad \delta_0^\alpha G_{\mu\nu} = 0\,, \tag{2.19}$$

where T^α are the generators of $o(8,24)$ algebra satisfying

$$T^\alpha L + L(T^\alpha)^T = 0\,. \tag{2.20}$$

The finite version of this transformation is given by,

$$M \to \Omega M \Omega^T, \qquad \Phi \to \Phi, \qquad G_{\mu\nu} \to G_{\mu\nu}, \tag{2.21}$$

where Ω is an O(8,24) matrix satisfying,

$$\Omega L \Omega^T = L\,. \tag{2.22}$$

The transformations generated by J^α_n for $n \neq 0$ act on M in a nonlocal fashion and are more difficult to describe. We first note that eq.(2.13) gives

$$\rho \equiv e^{-\Phi} = \rho_+(x^+) + \rho_-(x^-)\,. \tag{2.23}$$

The next thing to note is that the equations of motion (2.14) and the Bianchi identity of M,

$$\partial_+(\partial_- M M^{-1}) - \partial_-(\partial_+ M M^{-1}) + [\partial_- M M^{-1}, \partial_+ M M^{-1}] = 0\,, \tag{2.24}$$

allow us to define a matrix $\mathcal{U}(x;v)$ satisfying[30]

$$\begin{aligned} (\partial_+ \mathcal{U}^{-1})\mathcal{U} &= \frac{t}{1+t}\partial_+ M M^{-1}\,, \\ (\partial_- \mathcal{U}^{-1})\mathcal{U} &= -\frac{t}{1-t}\partial_- M M^{-1}\,, \end{aligned} \tag{2.25}$$

$$\mathcal{U}(x; v=0) = I_{32}\,, \tag{2.26}$$

where v is an arbitrary real parameter[2], and,

$$t = \frac{\sqrt{1+4v\rho_+} - \sqrt{1-4v\rho_-}}{\sqrt{1+4v\rho_+} + \sqrt{1-4v\rho_-}}\,. \tag{2.27}$$

The Bianchi identity,

$$\partial_+(\partial_- \mathcal{U}^{-1}\mathcal{U}) - \partial_-(\partial_+ \mathcal{U}^{-1}\mathcal{U}) + [\partial_- \mathcal{U}^{-1}\mathcal{U}, \partial_+ \mathcal{U}^{-1}\mathcal{U}] = 0\,, \tag{2.28}$$

follows as a consequence of eqs.(2.25), (2.14), (2.23), (2.24) and (2.27). Eq.(2.26) guarantees that $\mathcal{U}$ has a series expansion of the form

$$\mathcal{U} = \exp(V(x;v))\,, \tag{2.29}$$

[2] v is related to the spectral parameter w of ref.[30] through the relation $v = 1/4w$.

$$V(x;v) = \sum_{n=1}^{\infty} v^n V^{(n)}(x)\,. \tag{2.30}$$

Let us now decompose M as,

$$M = \mathcal{V}\mathcal{V}^T\,, \tag{2.31}$$

where $\mathcal{V}$ is an O(8,24) matrix. The choice of $\mathcal{V}$ is not unique, since we can multiply $\mathcal{V}$ by an arbitrary x dependent $O(8)\times O(24)$ matrix from the right without changing M. We shall refer to this as a *gauge transformation*. Let us now define,

$$\widehat{\mathcal{V}}(x;v) = \mathcal{U}(x;v)\mathcal{V}\,. \tag{2.32}$$

The above procedure gives us an algorithm for constructing $\widehat{\mathcal{V}}(x;v)$ for a given M. On the other hand, given $\widehat{\mathcal{V}}(x;v)$ which has a well defined Taylor series expansion around $v=0$, we can define $\mathcal{V}(x)$ to be $\widehat{\mathcal{V}}(x;0)$, and then construct M using eq.(2.31). We can also reconstruct $\mathcal{U}$ (and hence the potentials $V^{(n)}$) from $\widehat{\mathcal{V}}$ using eq.(2.32).

Note, however, that eqs.(2.25) and (2.26) do not determine $\mathcal{U}$ (and hence $\widehat{\mathcal{V}}$) unambiguously for a given M, since we have the freedom of multiplying $\mathcal{U}$ from the left by a v-dependent but x independent $O(8,24)$ matrix which reduces to the identity matrix at $v=0$. This introduces an ambiguity in the definition of the potentials $V^{(n)}$. Put another way, the equations of motion (2.25), (2.26), written in terms of $\widehat{\mathcal{U}}$ (or $\widehat{\mathcal{V}}$), have a trivial symmetry corresponding to left multiplication of $\widehat{\mathcal{U}}$ by a v-dependent $O(8,24)$ matrix, which tends to identity as $v\to 0$. This transformation does not affect M. Since M was the basic variable in supergravity theory, one might tend to conclude from this that the extra degrees of freedom contained in the matrix $\widehat{\mathcal{V}}$ due to this ambiguity are spurious. While this is certainly true for the classical supergravity theory, this is not true in quantum string theory. The symmetry corresponding to multiplying $\mathcal{U}$ from the left by a v-dependent O(8,24) matrix is broken in the quantum theory, and the full spectrum of string theory is not invariant under this symmetry. (One example of such a symmetry transformation would be the translation of the would be axion in four dimensions). Thus, in string theory, $\widehat{\mathcal{V}}$ is a more natural dynamical variable than M, and from now on we shall treat $\widehat{\mathcal{V}}$ as the basic variable of the theory. In particular, all the transformation laws will be given for $\widehat{\mathcal{V}}$ instead of M. (We can, of course, find the transformation laws of M from the transformation laws of $\widehat{\mathcal{V}}$, since the construction of M from $\widehat{\mathcal{V}}$ is free from any ambiguity.)[3]

We are now in a position to describe the action of the generators J^α_n on the basic variables in the theory, *e.g.* Φ, λ and $\widehat{\mathcal{V}}$. We shall first specify the transformation laws under finite elements of the loop group, specified by an $O(8,24)$ valued function $g(v)$, and then specialize to infinitesimal elements. These are given by[27, 30],

$$\widehat{\mathcal{V}}(x;v) \to \widehat{\mathcal{V}}'(x;v) = g(v)\widehat{\mathcal{V}}(x;v)H(x,t)\,, \qquad \Phi \to \Phi\,, \tag{2.33}$$

where $H(x,t)$ is a suitably chosen $O(8,24)$ matrix which makes the right hand side of eq.(2.33) finite in the $v \to 0$ limit, and satisfies,

$$H^T(x, \frac{1}{t})H(x,t) = I_{32}\,. \tag{2.34}$$

Since eq.(2.15) determines λ in terms of other variables up to an overall multiplicative constant, eq.(2.33) also determines the transformation law of λ. (A more explicit form of the transformation law of λ has been given in ref.[27] in terms of the group cocycle.) It has been argued (see, *e.g.* ref.[27, 30]) that these transformations preserve the equations of motion. In other words, if we define,

$$\mathcal{V}'(x) = \widehat{\mathcal{V}}'(x; v=0)\,, \tag{2.35}$$

$$\mathcal{U}'(x;v) = \widehat{\mathcal{V}}'(x;v)(\mathcal{V}'(x))^{-1}\,, \tag{2.36}$$

and,

$$M'(x) = \mathcal{V}'(x)(\mathcal{V}'(x))^T\,, \tag{2.37}$$

then analogs of eq.(2.25), with the variables $\mathcal{U}$, M replaced by $\mathcal{U}'$, M', are satisfied.

The infinitesimal transformation laws which follow from eq.(2.33) are,

$$\begin{aligned}\widehat{\mathcal{V}}'(x;v) &\equiv \widehat{\mathcal{V}}(x;v) + \epsilon\delta^\alpha_{-n}\widehat{\mathcal{V}}(x;v)\\ &= \Big(1 - \epsilon T^\alpha v^{-n}\Big)\widehat{\mathcal{V}}(x;v)H^\alpha_n(x;t)\,,\end{aligned}$$

[3]On the other hand, if one is only interested in the symmetries of the classical supergravity theory, then one can take M as the basic variable, and end up with a different symmetry algebra[32].

$$\delta^{\alpha}_{-n}\Phi = 0\,, \tag{2.38}$$

where $H^{\alpha}_n(x;t)$ is an appropriately chosen infinitesimal O(8,24) matrix, satisfying,

$$(H^{\alpha}_n)^T(x;\frac{1}{t})H^{\alpha}_n(x;t) = I_{32}\,. \tag{2.39}$$

This specifies the transformation laws of $\widehat{\mathcal{V}}$ under the generators J^{α}_n. For $n\le 0$ we can take $H^{\alpha}_n = I_{32}$. Thus for $n=0$, $\mathcal{V}'(x) = \mathcal{V}(x) - \epsilon T^{\alpha}\mathcal{V}(x)$. This agrees with the transformation laws given in eq.(2.19). For $n<0$, $\widehat{\mathcal{V}}'(x;v=0) = \widehat{\mathcal{V}}(x;v=0)$, which shows that the matrix M does not transform under these transformations. However, using the expansion (2.29), (2.30) of $\mathcal{U}$ we see that the potentials $V^{(m)}$ for $m\ge -n$ do transform under the generators J^{α}_{-n}.

Let us now start from the same ten dimensional theory, and carry out the dimensional reduction in two stages. We first compactify the directions 3-9 so that we get a three dimensional theory. At the next stage we compactify the second direction to get a two dimensional theory. For the first stage of dimensional reduction we introduce coordinates

$$\begin{aligned} z^{\bar\mu} &= \bar x^{\bar\mu}, \qquad 0\le\bar\mu\le 2, \\ z^{\bar m+2} &= \bar y^{\bar m}, \qquad 1\le\bar m\le 7. \end{aligned} \tag{2.40}$$

We also define new fields:

$$\begin{aligned}
&\widehat{\bar G}_{\bar m\bar n} = G^{(10)}_{\bar m+2,\bar n+2}, \quad \widehat{\bar B}_{\bar m\bar n} = B^{(10)}_{\bar m+2,\bar n+2}, \quad \widehat{\bar A}^I_{\bar m} = A^{(10)I}_{\bar m+2}, \\
&\bar A^{(2\bar m-1)}_{\bar\mu} = \frac{1}{2}\widehat{\bar G}^{\bar m\bar n} G^{(10)}_{\bar n+2,\bar\mu}, \quad \bar A^{(I+14)}_{\bar\mu} = -(\frac{1}{2}A^{(10)I}_{\bar\mu} - \widehat{\bar A}^I_{\bar n}\bar A^{(2\bar n-1)}_{\bar\mu}), \\
&\bar A^{(2\bar m)}_{\bar\mu} = \frac{1}{2}B^{(10)}_{(\bar m+2)\bar\mu} - \widehat{\bar B}_{\bar m\bar n}\bar A^{(2\bar n-1)}_{\bar\mu} + \frac{1}{2}\widehat{\bar A}^I_{\bar m}\bar A^{(I+14)}_{\bar\mu}, \\
&\bar G_{\bar\mu\bar\nu} = G^{(10)}_{\bar\mu\bar\nu} - G^{(10)}_{(\bar m+2)\bar\mu}G^{(10)}_{(\bar n+2)\bar\nu}\widehat{\bar G}^{\bar m\bar n}, \\
&\bar B_{\bar\mu\bar\nu} = B^{(10)}_{\bar\mu\bar\nu} - 4\widehat{\bar B}_{\bar m\bar n}\bar A^{(2\bar m-1)}_{\bar\mu}\bar A^{(2\bar n-1)}_{\bar\nu} - 2(\bar A^{(2\bar m-1)}_{\bar\mu}\bar A^{(2\bar m)}_{\bar\nu} - \bar A^{(2\bar m-1)}_{\bar\nu}\bar A^{(2\bar m)}_{\bar\mu}) \\
&\qquad +2\widehat{\bar A}^I_{\bar n}(\bar A^{(2\bar n-1)}_{\bar\mu}\bar A^{(I+14)}_{\bar\nu} - \bar A^{(2\bar n-1)}_{\bar\nu}\bar A^{(I+14)}_{\bar\mu})\,, \\
&\bar\Phi = \Phi^{(10)} - \frac{1}{2}\ln\det\widehat{\bar G}, \qquad 1\le\bar m,\bar n\le 7, \quad 0\le\bar\mu,\bar\nu\le 2, \quad 1\le I\le 16.
\end{aligned} \tag{2.41}$$

Here $\widehat{\bar{G}}^{\bar{m}\bar{n}}$ denotes the inverse of the matrix $\widehat{\bar{G}}_{\bar{m}\bar{n}}$. We also define,

$$\widehat{\bar{C}}_{\bar{m}\bar{n}} = \frac{1}{2}\widehat{\bar{A}}^I_{\bar{m}}\widehat{\bar{A}}^I_{\bar{n}}\,. \tag{2.42}$$

For every $\bar{m},\bar{n}$ $(1\le \bar{m},\bar{n}\le 7)$ we define $\bar{H}_{\bar{m}\bar{n}}$ to be the 2×2 matrix

$$\bar{H}_{\bar{m}\bar{n}} = \begin{pmatrix} \widehat{\bar{G}}^{\bar{m}\bar{n}} & \widehat{\bar{G}}^{\bar{m}\bar{p}}(\widehat{\bar{B}}_{\bar{p}\bar{n}} + \widehat{\bar{C}}_{\bar{p}\bar{n}}) \\ (-\widehat{\bar{B}}_{\bar{m}\bar{p}} + \widehat{\bar{C}}_{\bar{m}\bar{p}})\widehat{\bar{G}}^{\bar{p}\bar{n}} & (\widehat{\bar{G}} - \widehat{\bar{B}} + \widehat{\bar{C}})_{\bar{m}\bar{p}}\widehat{\bar{G}}^{\bar{p}\bar{q}}(\widehat{\bar{G}} + \widehat{\bar{B}} + \widehat{\bar{C}})_{\bar{q}\bar{n}} \end{pmatrix}, \tag{2.43}$$

and for every $\bar{m}$, I $(1\le\bar{m}\le 7,\ 1\le I\le 16)$ we define $\bar{Q}^{(I)}_{\bar{m}}$ to be the two dimensional column vector

$$\bar{Q}^{(I)}_{\bar{m}} = \begin{pmatrix} \widehat{\bar{G}}^{\bar{m}\bar{n}}\widehat{\bar{A}}^I_{\bar{n}} \\ (\widehat{\bar{G}} - \widehat{\bar{B}} + \widehat{\bar{C}})_{\bar{m}\bar{n}}\widehat{\bar{G}}^{\bar{n}\bar{p}}\widehat{\bar{A}}^I_{\bar{p}} \end{pmatrix}. \tag{2.44}$$

We also define $\bar{K}$ to be a 16×16 matrix whose components are,

$$\bar{K}_{IJ} = \delta_{IJ} + \widehat{\bar{A}}^I_{\bar{m}}\widehat{\bar{G}}^{\bar{m}\bar{n}}\widehat{\bar{A}}^J_{\bar{n}}\,. \tag{2.45}$$

In terms of $\bar{H}$, $\bar{Q}$ and $\bar{K}$, we now define a 30×30 matrix $\bar{M}$ as follows:

$$\bar{M} = \begin{pmatrix} \bar{H}_{11} & \cdot & \bar{H}_{17} & \bar{Q}^{(1)}_1 & \cdot & \bar{Q}^{(16)}_1 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \bar{H}_{71} & \cdot & \bar{H}_{77} & \bar{Q}^{(1)}_7 & \cdot & \bar{Q}^{(16)}_7 \\ & & & & & \\ \bar{Q}^{(1)T}_1 & \cdot & \bar{Q}^{(1)T}_7 & & & \\ \cdot & \cdot & \cdot & & \bar{K} & \\ \bar{Q}^{(16)T}_1 & \cdot & \bar{Q}^{(16)T}_7 & & & \end{pmatrix}. \tag{2.46}$$

$\bar{M}$ satisfies

$$\bar{M}^T = \bar{M}, \qquad \bar{M}\bar{L}\bar{M}^T = \bar{L}, \tag{2.47}$$

where $\bar{L}$ is the 30×30 matrix:

$$\bar{L} = \begin{pmatrix} \sigma_1 & & & & \\ & \cdot & & & \\ & & \cdot & & \\ & & & \sigma_1 & \\ & & & & -I_{16} \end{pmatrix}. \tag{2.48}$$

Let us now define

$$\bar{g}_{\bar{\mu}\bar{\nu}} = e^{-2\bar{\Phi}}\bar{G}_{\bar{\mu}\bar{\nu}}\,, \tag{2.49}$$

and note that the equations of motion for the gauge fields $\bar{A}^{(\bar{a})}_{\bar{\mu}}$ $(1 \le \bar{a} \le 30)$ allows us to define a set of scalar fields $\bar{\psi}^{\bar{a}}$ through the relations[23]:

$$\sqrt{-\bar{g}}e^{-2\bar{\Phi}}\bar{g}^{\bar{\mu}\bar{\mu}'}\bar{g}^{\bar{\nu}\bar{\nu}'}(\bar{M}\bar{L})_{\bar{a}\bar{b}}\bar{F}^{(\bar{b})}_{\bar{\mu}'\bar{\nu}'} = \frac{1}{2}\epsilon^{\bar{\mu}\bar{\nu}\bar{\rho}}\partial_{\bar{\rho}}\bar{\psi}^{\bar{a}}\,. \tag{2.50}$$

We now define

$$\widetilde{M} = \begin{pmatrix} e^{-2\bar{\Phi}} + \bar{\psi}^T\bar{L}\bar{M}\bar{L}\bar{\psi} + \frac{1}{4}e^{2\bar{\Phi}}(\bar{\psi}^T\bar{L}\bar{\psi})^2 & -\frac{1}{2}e^{2\bar{\Phi}}\bar{\psi}^T\bar{L}\bar{\psi} & \bar{\psi}^T\bar{L}\bar{M} + \frac{1}{2}e^{2\bar{\Phi}}\bar{\psi}^T(\bar{\psi}^T\bar{L}\bar{\psi}) \\ -\frac{1}{2}e^{2\bar{\Phi}}\bar{\psi}^T\bar{L}\bar{\psi} & e^{2\bar{\Phi}} & -e^{2\bar{\Phi}}\bar{\psi}^T \\ \bar{M}\bar{L}\bar{\psi} + \frac{1}{2}e^{2\bar{\Phi}}\bar{\psi}(\bar{\psi}^T\bar{L}\bar{\psi}) & -e^{2\bar{\Phi}}\bar{\psi} & \bar{M} + e^{2\bar{\Phi}}\bar{\psi}\bar{\psi}^T \end{pmatrix}. \tag{2.51}$$

$\widetilde{M}$ satisfies

$$\widetilde{M}^T = \widetilde{M}, \qquad \widetilde{M}^T L \widetilde{M} = L\,. \tag{2.52}$$

All information about the scalar fields $\{\bar{G}, \bar{B}, \bar{A}, \bar{\Phi}\}$, as well as the gauge fields $\{\bar{A}^{(\bar{a})}_{\bar{\mu}}\}$ (or, equivalently, the scalar fields $\bar{\psi}^{\bar{a}}$) are contained in the matrix $\widetilde{M}$. Finally, since in three dimensions the anti-symmetric tensor field $\bar{B}_{\bar{\mu}\bar{\nu}}$ has no dynamics, we can set the corresponding field strength $\bar{H}_{\bar{\mu}\bar{\nu}\bar{\rho}}$ to zero.

The resulting equations of motion are derivable from an action[23]:

$$S = \frac{1}{4}\int d^3\bar{x}\sqrt{-\bar{g}}[R_{\bar{g}} + \frac{1}{8}\bar{g}^{\bar{\mu}\bar{\nu}}Tr(\partial_{\bar{\mu}}\widetilde{M}L\partial_{\bar{\nu}}\widetilde{M}L)]\,. \tag{2.53}$$

We now compactify the direction $\bar{x}^2$ on a circle of length 8π and ignore dependence of all the fields on the coordinate $\bar{x}^2$. We then get a two dimensional theory. The components $\bar{g}_{2\mu}$ act as gauge fields in this two dimensional theory, but since in two dimension gauge fields have no dynamics, we can set these fields to zero. Thus the metric $\bar{g}_{\bar{\mu}\bar{\nu}}$ has the form:

$$\bar{g} = \begin{pmatrix} g_{\mu\nu} & \\ & \bar{g}_{22} \end{pmatrix}. \tag{2.54}$$

The resulting action can be written as[36],

$$S = 2\pi\int d^2x\sqrt{-g}\sqrt{\bar{g}_{22}}\,[R_g + \frac{1}{8}g^{\mu\nu}Tr(\partial_\mu\widetilde{M}L\partial_\nu\widetilde{M}L)]\,. \tag{2.55}$$

Comparing the definition of Φ given in eq.(2.4) and the definition of $\bar{g}_{22}$ given in eqs.(2.41) and (2.49), we see that,

$$\bar{g}_{22} = e^{-2\Phi}\,. \tag{2.56}$$

We now write down the independent equations of motion in the gauge $g_{\mu\nu} = \widetilde{\lambda}\eta_{\mu\nu}$:

$$\eta^{\mu\nu}\partial_\mu\partial_\nu(e^{-\Phi}) = 0\,, \tag{2.57}$$

$$\partial_\mu(e^{-\Phi}\eta^{\mu\nu}\partial_\nu\widetilde{M}\widetilde{M}^{-1}) = 0\,, \tag{2.58}$$

and,

$$\partial_\pm(\ln\widetilde{\lambda})\partial_\pm(e^{-\Phi}) = \partial_\pm^2(e^{-\Phi}) - \frac{1}{4}e^{-\Phi}Tr(\partial_\pm\widetilde{M}L\partial_\pm\widetilde{M}L)\,. \tag{2.59}$$

These equations are identical to eqs.(2.13)-(2.15) with the replacement $M \to \widetilde{M}$, $\lambda \to \widetilde{\lambda}$. Thus we can generate another current algebra $\widehat{o(8,24)}_{(2)}$ acting on these fields, generated by the generators $\widetilde{J}^\alpha_m$:

$$[\widetilde{J}^\alpha_m, \widetilde{J}^\beta_n] = f^{\alpha\beta\gamma}\eta_{\gamma\delta}\widetilde{J}^\delta_{m+n} + m\eta^{\alpha\beta}\delta_{m+n,0}\widetilde{C}\,. \tag{2.60}$$

The subscript (2) denotes the fact that in identifying the algebra $\widehat{o(8,24)}_{(2)}$ we have given special treatment to the coordinate z^2. The action of $\widetilde{C}$, $\widetilde{J}^\alpha_m$ on various fields are defined in a manner exactly analogous to the action of C, J^α_m on the various fields. First of all, $\widetilde{C}$ generates a scaling of $g_{\mu\nu}$ leaving the fields $\widetilde{M}$ and Φ fixed:

$$\delta_{\widetilde{C}}g_{\mu\nu} = 2g_{\mu\nu}, \qquad \delta_{\widetilde{C}}\Phi = 0, \qquad \delta_{\widetilde{C}}\widetilde{M} = 0\,. \tag{2.61}$$

In order to define an action of $\widetilde{J}^\alpha_n$ on various fields, we define $\widetilde{\mathcal{U}}(x;v)$ through the equations:

$$\begin{aligned}(\partial_+\widetilde{\mathcal{U}}^{-1})\widetilde{\mathcal{U}} &= \frac{t}{1+t}\partial_+\widetilde{M}\widetilde{M}^{-1}\,,\\ (\partial_-\widetilde{\mathcal{U}}^{-1})\widetilde{\mathcal{U}} &= -\frac{t}{1-t}\partial_-\widetilde{M}\widetilde{M}^{-1}\,,\end{aligned} \tag{2.62}$$

$$\widetilde{\mathcal{U}}(x;v=0) = I_{32}\,, \tag{2.63}$$

where t has been defined in eq.(2.27). $\widetilde{\mathcal{U}}$ has a series expansion in v of the form

$$\widetilde{\mathcal{U}} = \exp(\widetilde{V}(x;v))\,, \tag{2.64}$$

$$\widetilde{V}(x;v) = \sum_{n=1}^{\infty} v^n \widetilde{V}^{(n)}(x)\,. \tag{2.65}$$

We now decompose $\widetilde{M}$ as,

$$\widetilde{M} = \widetilde{\mathcal{V}}\widetilde{\mathcal{V}}^T\,, \tag{2.66}$$

where $\widetilde{\mathcal{V}}$ is an O(8,24) matrix. We also define

$$\widehat{\widetilde{\mathcal{V}}}(x;v) = \widetilde{\mathcal{U}}(x;v)\widetilde{\mathcal{V}}\,. \tag{2.67}$$

As for the J_n^α's, the transformation laws of the field $\widehat{\widetilde{\mathcal{V}}}$ under $\widetilde{J}_n^\alpha$ are defined as follows:

$$\begin{aligned} \widehat{\widetilde{\mathcal{V}}}''(x;v) &\equiv \widehat{\widetilde{\mathcal{V}}}(x;v) + \epsilon\widetilde{\delta}_{-n}^\alpha \widehat{\widetilde{\mathcal{V}}}(x;v) \\ &= (1 - \epsilon T^\alpha v^{-n})\widehat{\widetilde{\mathcal{V}}}(x;v)\widetilde{H}_n^\alpha(x;t)\,, \end{aligned} \tag{2.68}$$

where $\widetilde{H}_n^\alpha(x;t)$ is a suitably chosen infinitesimal $O(8,24)$ matrix which makes the right hand side of eq.(2.68) finite in the $v \to 0$ limit, and satisfies,

$$(\widetilde{H}_n^\alpha)^T(x;\frac{1}{t})\widetilde{H}_n^\alpha(x;t) = I_{32}\,. \tag{2.69}$$

This specifies the transformation laws of $\widehat{\widetilde{\mathcal{V}}}$ under the generators $\widetilde{J}_n^\alpha$. The finite form of this transformation is,

$$\widehat{\widetilde{\mathcal{V}}}''(x;v) = \widetilde{g}(v)\widehat{\widetilde{\mathcal{V}}}(x;v)\widetilde{H}(x,t)\,. \tag{2.70}$$

Φ remains invariant under these transformations, and the transformation law of $\widetilde{\lambda}$ is determined using the equation of motion (2.59).

Since the symmetry algebras $o\widehat{(8,24)}$ and $o\widehat{(8,24)}_{(2)}$ are in fact the symmetry algebras of the same theory, the two algebras must be related by an automorphism. We now ask:

Question: What is the automorphism that relates the algebras $o\widehat{(8,24)}$ and $o\widehat{(8,24)}_{(2)}$?

Answer: Let T^0 be the $o(1,1)$ generator

$$T^0 = \begin{pmatrix} 1 & & \\ & -1 & \\ & & 0_{30} \end{pmatrix} . \tag{2.71}$$

We arrange the generators T^α of $o(8,24)$ such that,

$$[T^0, T^\alpha] = \lambda^\alpha T^\alpha , \tag{2.72}$$

where λ^α are numbers. Then the generators C, J_n^α of $o(\widehat{8,24})$ and the generators $\widetilde{C}$, $\widetilde{J}_n^\alpha$ of $o(\widehat{8,24})_{(2)}$ are related by the following automorphism:

$$\widetilde{C} = C, \tag{2.73}$$

$$\widetilde{J}_n^\alpha = J_{n+\lambda^\alpha}^\alpha + \delta_{\alpha,0}\, \delta_{n,0}\, C . \tag{2.74}$$

Proof: First of all, both C and $\widetilde{C}$ generate a scaling of the components $G_{\mu\nu}^{(10)}$ $(0 \le \mu, \nu \le 1)$ of the ten dimensional metric leaving all other components of the ten dimensional fields fixed. Thus eq.(2.73) is manifest.

For proving eq.(2.74) we follow the approach of ref.[27]. In appendix A we have proved the following relation between $\widehat{\mathcal{V}}$ and $\widehat{\widehat{\mathcal{V}}}$:

$$\widehat{\widehat{\mathcal{V}}}(x; v) = (v)^{-T^0} \widehat{\mathcal{V}}(x; v) h(x; t) , \tag{2.75}$$

where $h(x,t)$ is an appropriately chosen O(8,24) transformation satisfying

$$h^T(x, \frac{1}{t}) h(x,t) = I_{32} . \tag{2.76}$$

Eq.(2.75) implies that if a given element of the loop group $O(\widehat{8,24})$ acts on $\widehat{\widehat{\mathcal{V}}}$ as,

$$\widehat{\widehat{\mathcal{V}}}(x; v) \to \widetilde{g}(v) \widehat{\widehat{\mathcal{V}}}(x; v) \widetilde{H}(x,t) , \tag{2.77}$$

then it acts on $\widehat{\mathcal{V}}$ as,

$$\widehat{\mathcal{V}}(x; v) \to g(v) \widehat{\mathcal{V}}(x; v) H(x,t) , \tag{2.78}$$

where,

$$g(v) = (v)^{T^0} \widetilde{g}(v) (v)^{-T^0} . \tag{2.79}$$

The infinitesimal version of the above equation will give us the relations between J_n^α and $\widetilde{J}_n^\alpha$. From eqs.(2.38), (2.75), we get,

$$\begin{aligned}
\widehat{\widetilde{\mathcal{V}}}'(x;v) &\equiv (v)^{-T^0}\widehat{\mathcal{V}}'(x;v)h'(x,t) \\
&= (v)^{-T^0}(1-\epsilon T^\alpha v^{-n})\widehat{\mathcal{V}}(x;v)H_n^\alpha(x,t)h'(x,t) \\
&= (v)^{-T^0}(1-\epsilon T^\alpha v^{-n})(v)^{T^0}\widehat{\widetilde{\mathcal{V}}}(x;v)h^{-1}(x;t)H_n^\alpha(x,t)h'(x,t) \\
&= (1-\epsilon T^\alpha (v)^{-n-\lambda^\alpha})\widehat{\widetilde{\mathcal{V}}}(x;v)h^{-1}(x,t)H_n^\alpha(x,t)h'(x,t)\,.
\end{aligned} \tag{2.80}$$

Comparing this with eq.(2.68) we see that this is the transformation generated by $\widetilde{J}^\alpha_{-n-\lambda^\alpha}$. Thus we get

$$\delta^\alpha_{-n}\widehat{\mathcal{V}} = \widetilde{\delta}^\alpha_{-n-\lambda^\alpha}\widehat{\mathcal{V}}\,. \tag{2.81}$$

This proves eq.(2.74) acting on the field $\widehat{\mathcal{V}}$ (and hence $\widehat{\widetilde{\mathcal{V}}}$). Note that the central charge term appearing on the right hand side of eq.(2.74) does not show up here, since it does not act on $\widehat{\mathcal{V}}$.

To see the presence of the central charge term in eq.(2.74), we need to compare the action of J_0^0 and $\widetilde{J}_0^0$ on the component $G^{(10)}_{\mu\nu}$ of the ten dimensional metric. This can be easily done, since we know their action on the fields M, Φ and $G_{\mu\nu}$ ($\widetilde{M}$, Φ, $g_{\mu\nu}$). The non-trivial transformations are,

$$\begin{aligned}
\delta_0^0 G^{(10)}_{22} &= 2G^{(10)}_{22}, \qquad \delta_0^0 G^{(10)}_{2,\bar m+2} = G^{(10)}_{2,\bar m+2}, \qquad \delta_0^0 B^{(10)}_{2,\bar m+2} = B^{(10)}_{2,\bar m+2}, \\
\delta_0^0 A_2^{(10)I} &= A_2^{(10)I}, \qquad \delta_0^0\Phi^{(10)} = 1, \qquad \delta_0^0 G^{(10)}_{\mu\nu} = 0, \\
& \qquad 1\le \bar m\le 7, \qquad 1\le I\le 16, \qquad 0\le \mu,\nu\le 1\,,
\end{aligned} \tag{2.82}$$

$$\begin{aligned}
\widetilde{\delta}_0^0 G^{(10)}_{22} &= 2G^{(10)}_{22}, \qquad \widetilde{\delta}_0^0 G^{(10)}_{2,\bar m+2} = G^{(10)}_{2,\bar m+2}, \qquad \widetilde{\delta}_0^0 B^{(10)}_{2,\bar m+2} = B^{(10)}_{2,\bar m+2}, \\
\widetilde{\delta}_0^0 A_2^{(10)I} &= A_2^{(10)I}, \qquad \widetilde{\delta}_0^0\Phi^{(10)} = 1, \qquad \widetilde{\delta}_0^0 G^{(10)}_{\mu\nu} = 2G^{(10)}_{\mu\nu}.
\end{aligned} \tag{2.83}$$

Thus the difference between these two transformations is simply a scaling of $G^{(10)}_{\mu\nu}$. The result can be written in terms of the four dimensional fields as,

$$\widetilde{\delta}_0^0 M - \delta_0^0 M = 0, \qquad \widetilde{\delta}_0^0\Phi - \delta_0^0\Phi = 0,$$

$$\widetilde{\delta}_0^0 G_{\mu\nu} - \delta_0^0 G_{\mu\nu} = 2G_{\mu\nu}, \qquad 0 \le \mu, \nu \le 1\,. \tag{2.84}$$

Comparing this with eq.(2.18) we see that

$$\widetilde{J}_0^0 - J_0^0 = C\,. \tag{2.85}$$

This agrees with eq.(2.74) for $n = 0$, $\alpha = 0$.

3 Discrete Duality Symmetry Group of the Two Dimensional String Theory

In the previous section we have seen that the equations of motion of the massless fields in the two dimensional string theory have an $O(\widehat{8,24})$ loop group symmetry. The question that we shall be addressing in this section is: what (discrete) subgroup of this infinite dimensional group is a symmetry of the full string theory? First of all, a discrete subgroup $O(8, 24; Z)$ is already known to be a symmetry of the full string theory. This is the T-duality group of the two dimensional theory. Besides this, if we regard the two dimensional string theory to be the result of compactification of a four dimensional theory, then the four dimensional theory is expected to be invariant under an S-duality symmetry group $SL(2, Z)$. This can also be identified as a subgroup of the loop group $O(\widehat{8,24})$. If we believe that compactification does not destroy S-duality invariance of a theory, then this $SL(2, Z)$ subgroup of the loop group must also be a symmetry of the full two dimensional string theory. Since the S-duality group and the T-duality group in two dimensions do not commute, they will generate a much bigger subgroup of $O(\widehat{8,24})$. This, we believe, will be the minimal duality symmetry group of the two dimensional string theory.

At this point, we should note that there are many (precisely 28) different ways in which a two dimensional string theory may be regarded as a compactified four dimensional theory, giving rise to many different S-duality groups. These different choices will correspond to the choice of two out of eight coordinates $z^2, \ldots z^9$ that we use to label the non-compact directions of the four dimensional theory. However, these different choices are permuted by the T-duality group, and hence the final duality group generated by the T-duality group $O(8,24;Z)$ and the S-duality group $SL(2,Z)$ will be independent of the initial choice of the $SL(2,Z)$ group.

Thus, in order to describe the duality group of the two dimensional theory, we should first determine how the S- and the T-duality groups are embedded in the loop group $O\widehat{(8,24)}$. This is done by specifying the $O(8,24)$ group valued function $g(v)$ for each generator of the S- and the T- duality groups. This representation, however, is not faithful, in that a given $g(v)$ will represent a whole one dimensional orbit in the $O\widehat{(8,24)}$ group which differ from each other by the action of the central charge. We shall later discuss how this ambiguity gets resolved.

We shall first study how the O(8,24;Z) target space duality transformations are embedded in $O\widehat{(8,24)}$. This is easy, since this O(8,24;Z) is a subgroup of the $O(8,24)$ group that is generated by the generators J_0^α. If U is an $O(8,24;Z)$ matrix, its action on $\widehat{\mathcal{V}}$ is given by,

$$\widehat{\mathcal{V}} \to U\widehat{\mathcal{V}} . \tag{3.1}$$

Comparing with eq.(2.33) we see that this corresponds to a loop group element,

$$g(v; U \in O(8,24;Z)) = U . \tag{3.2}$$

At this stage, let us briefly recall the way we characterize the $O(8,24;Z)$ matrices U. A general $O(8,24)$ group element acts naturally on a 32 dimensional vector space. We define a lattice Λ_{32} in this vector space, spanned by vectors of the form:

$$\begin{pmatrix} \vec{m} \\ \vec{\alpha} \end{pmatrix} , \tag{3.3}$$

where $\vec{m}$ is a 16 dimensional vector with integer entries, and $\vec{\alpha}$ is a 16 dimensional vector in the $E_8 \times E_8$ root lattice. The O(8,24;Z) subgroup of O(8,24) is defined to be the group of O(8,24) matrices that preserve the lattice Λ_{32}.

Let us now see how the S-duality transformations fit inside the loop group $O\widehat{(8,24)}$. We shall consider the S-duality group of the four dimensional theory obtained by compactifying the directions $z^4 - z^9$. This S-duality group has a natural embedding in the group $O(8,24)_{(2)}$ generated by $\widetilde{J}_0^\alpha$. If $\widetilde{U}$ is an $O(8,24;Z)_{(2)}$ matrix, then its action on $\widehat{\widehat{\mathcal{V}}}$ is given by,

$$\widehat{\widehat{\mathcal{V}}} \to \widetilde{U}\widehat{\widehat{\mathcal{V}}} . \tag{3.4}$$

Comparing with (2.70) we see that this corresponds to the loop group element

$$\widetilde{g}(v;\widetilde{U} \in O(8,24;Z)_{(2)}) = \widetilde{U}\,. \tag{3.5}$$

Eq.(2.79) now gives,

$$g(v;\widetilde{U} \in O(8,24;Z)_{(2)}) = (v)^{T^0}\widetilde{U}(v)^{-T^0}\,. \tag{3.6}$$

Using eq.(3.6), we can construct the loop group element $g(v)$ corresponding to an S-duality transformation. A generic $SL(2;Z)$ transformation is represented by an $O(8,24;Z)_{(2)}$ matrix of the form:[4]

$$\widetilde{U}(p,q,r,s) = \begin{pmatrix} p & 0 & 0 & -q & \\ 0 & s & r & 0 & \\ 0 & q & p & 0 & \\ -r & 0 & 0 & s & \\ & & & & I_{28} \end{pmatrix}, \tag{3.7}$$

where p, q, r, s are integers satisfying

$$ps - qr = 1\,. \tag{3.8}$$

This discrete group is generated by the following two $SL(2,Z)$ transformations:

$$\begin{aligned} \mathcal{T}_{(23)} \quad &: \quad \widetilde{U} = \begin{pmatrix} 1 & 0 & 0 & -1 & \\ 0 & 1 & 0 & 0 & \\ 0 & 1 & 1 & 0 & \\ 0 & 0 & 0 & 1 & \\ & & & & I_{28} \end{pmatrix}, \\ \mathcal{S}_{(23)} \quad &: \quad \widetilde{U} = \begin{pmatrix} 0 & 0 & 0 & -1 & \\ 0 & 0 & -1 & 0 & \\ 0 & 1 & 0 & 0 & \\ 1 & 0 & 0 & 0 & \\ & & & & I_{28} \end{pmatrix}, \end{aligned} \tag{3.9}$$

[4]This is obtained by appropriate rearrangements of the rows and columns in the expression given in ref.[23].

where the subscript (23) is a reminder of the fact that the directions z^2 and z^3 are given special treatment in this case. Eq.(3.6) now shows that

$$g(v;\mathcal{T}_{(23)}) = \begin{pmatrix} 1 & 0 & 0 & -v & \\ 0 & 1 & 0 & 0 & \\ 0 & v & 1 & 0 & \\ 0 & 0 & 0 & 1 & \\ & & & & I_{28} \end{pmatrix}, \tag{3.10}$$

and,

$$g(v;\mathcal{S}_{(23)}) = \begin{pmatrix} 0 & 0 & 0 & -v & \\ 0 & 0 & -v^{-1} & 0 & \\ 0 & v & 0 & 0 & \\ v^{-1} & 0 & 0 & 0 & \\ & & & & I_{28} \end{pmatrix}. \tag{3.11}$$

Let us define by G the full discrete subgroup of $O(\widehat{8,24})$, generated by $g(v;\mathcal{T}_{(23)})$, $g(v;\mathcal{S}_{(23)})$, and the T-duality group elements $g(v;U\in O(8,24;Z))$ defined in eqs.(3.10), (3.11) and (3.2) respectively. This is the minimal duality symmetry group of the two dimensional string theory. As already mentioned, the generators of the other S-duality groups can be obtained as combinations of $g(v;\mathcal{T}_{(23)})$, $g(v;\mathcal{S}_{(23)})$, and $g(v;U\in O(8,24;Z))$. In general, the S-duality generators $g(v;\mathcal{T}_{(ij)})$ and $g(v;\mathcal{S}_{(ij)})$ are obtained from $g(v;\mathcal{T}_{(23)})$ and $g(v;\mathcal{S}_{(23)})$ by the following rearrangement of the rows and columns:

$$(1,2)\to(2i-3,2i-2), \qquad (3,4)\to(2j-3,2j-2)\,. \tag{3.12}$$

In order to get some further insight into the structure of the group G, let us define $O(8,\widehat{24};Z)$ to be the following subgroup of $O(\widehat{8,24})$. $g(v)\in O(8,\widehat{24};Z)$ if,

$$g(v)Lg(v)^T = L\,, \tag{3.13}$$

and $g(v)$ has an expansion of the form:

$$g(v) = \sum_{n=-\infty}^{\infty} v^{-n} g_n\,, \tag{3.14}$$

where each g_n, acting on a vector in the lattice Λ_{32}, gives another vector in Λ_{32}. From eqs.(3.10), (3.11) and (3.2) we can easily verify that each of the generators of the duality group G is an element of $O(8,\widehat{24};Z)$. Hence G itself must be a subgroup of $O(8,\widehat{24};Z)$. G, however, does not contain all the elements of $O(8,\widehat{24};Z)$. In particular, it does not contain the element represented by the $O(8,24)$ valued function $(v)^{T^0}$. This can be seen from the fact that for all the generators of G, $g(v=1)g^{-1}(v=-1)$ represents an $O(8,24)$ matrix which can be continuously deformed to the identity element. This is not the case for the matrix $(-1)^{T^0}$. It is still tempting to conjecture[33] that the full duality symmetry group of the theory is $O(8,\widehat{24};Z)$, but justifying this will require further work. In particular, we need to study if $O(8,\widehat{24};Z)$ elements like $(v)^{T^0}$ generate duality symmetries of the two dimensional string theory.

Finally we turn to the complications that arise due to the presence of the central charge. As we have already pointed out, a given $O(8,24)$ valued function $g(v)$ actually represents to a one parameter family of $O(\widehat{8,24})$ group elements, related by the action of the central charge. Thus one needs to understand how for a given $g(v)$, representing an element of the group G, one reconstructs a specific element of the $O(\widehat{8,24})$ group. This can be done by representing $g(v)$ as a product of $g(v;\mathcal{T}_{(23)})$, $g(v;\mathcal{S}_{(23)})$, and the T-duality group elements $g(v;U\in O(8,24;Z))$. For each of these elements, we know the transformation laws of various fields, including the two dimensional metric. Thus once we express an element of G as a product of these transformations, we know how that element of G acts on various fields. This determines the specific element of $O(\widehat{8,24})$ that a given G transformation corresponds to.

There of course remains the question as to whether two different ways of expressing a given $g(v)$ as products of $g(v;\mathcal{T}_{(23)})$, $g(v;\mathcal{S}_{(23)})$, and $g(v;U\in O(8,24;Z))$ may lead to two different transformation laws of the two dimensional metric. If this is the case, then, by following one series of transformations by the inverse of the other, we can get an element of the duality group which would correspond to scaling the two dimensional metric by some fixed amount without affecting any other field. If this had been a symmetry of the full string theory, it would have been valid order by order in perturbation theory, since this transformation does not act on the string coupling constant. In other words, this should have been part of the T-duality transformation of the theory. We know, however, that the T-duality group does not contain any such element. This, in turn, implies that our initial assumption must be wrong, and two different ways of expressing a specific matrix representing an element of G must lead to identical transformations of the two dimensional metric. Thus, although the representation that we have chosen is not a faithful representation of the continuous group $\widehat{O(8,24)}$, it provides us with a faithful representation of the discrete duality symmetry group G of the two dimensional string theory.

4 Conclusion

In this paper we have analysed the structure of the discrete duality symmetry group of the two dimensional string theory, obtained by compactifying the heterotic string theory on an eight dimensional torus. The minimal duality group G, generated by the known S- and T-duality transformations of the theory, turns out to be a discrete subgroup of the loop group $\widehat{O(8,24)}$. The generators of this discrete group are represented by the $O(8,24)$ valued functions given in eqs.(3.10), (3.11) and (3.2). This discrete group is a subgroup of $O(\widehat{8,24};Z)$, $-$ the group of $O(8,24)$ valued functions of a variable v satisfying the criteria that if we expand the matrix valued function in positive and negative powers of v, then each term in this power series expansion, acting on an element of the even, self-dual, Lorentzian lattice Λ_{32} defined in eq.(3.3), will give another element of Λ_{32}. At present it remains an open question whether the theory is invariant under the full $O(\widehat{8,24};Z)$ group.

A Relation between $\widehat{\mathcal{V}}$ and $\widehat{\widehat{\mathcal{V}}}$

In this appendix we prove eq.(2.75). For this, let us define,

$$\widehat{\mathcal{V}}_1 \equiv (v)^{-T^0}\widehat{\mathcal{V}}(x;v)h(x,t)\,. \tag{A.1}$$

Thus we need to prove that $\widehat{\widehat{\mathcal{V}}} = \widehat{\mathcal{V}}_1$. First we note from eq.(2.33) and (A.1) that $\widehat{\mathcal{V}}_1$ and $\widehat{\mathcal{V}}$ are related by a finite $\widehat{O(8,24)}$ loop group transformation. Since this transformation is a symmetry of the equations of motion, $\widehat{\mathcal{V}}_1$ and $\widehat{\mathcal{V}}$ satisfy the same equations of motion. In other words, we have,

$$\widehat{\mathcal{V}}_1 = \widehat{\mathcal{U}}_1(x;v)\mathcal{V}_1(x)\,, \tag{A.2}$$

$$\mathcal{V}_1(x) \equiv \widehat{\mathcal{V}}_1(x; v=0)\,, \tag{A.3}$$

with $\widehat{\mathcal{U}}_1$ satisfying the dynamical equations:

$$\begin{aligned}(\partial_+\mathcal{U}_1^{-1})\mathcal{U}_1 &= \frac{t}{1+t}\partial_+ M_1 (M_1)^{-1}\,,\\ (\partial_-\mathcal{U}_1^{-1})\mathcal{U}_1 &= -\frac{t}{1-t}\partial_- M_1 (M_1)^{-1}\,,\end{aligned} \tag{A.4}$$

where,

$$M_1 = \mathcal{V}_1\mathcal{V}_1^T\,. \tag{A.5}$$

We furthermore note that up to a gauge transformation, and the ambiguity mentioned in the text, $\widehat{\mathcal{V}}_1$ is determined in terms of M_1 in the same way that $\widehat{\mathcal{V}}$ ($\widehat{\widehat{\mathcal{V}}}$) is determined in terms of M ($\widetilde{M}$). Comparing the above equations with eqs.(2.62), (2.63), (2.66) and (2.67) we see that $\widehat{\mathcal{V}}_1$ satisfies the defining equations of $\widehat{\widehat{\mathcal{V}}}$, provided,

$$M_1 = \widetilde{M}\,. \tag{A.6}$$

Thus the proof of eq.(2.75) boils down to proving eq.(A.6).[5]

We shall carry out the proof of eq.(A.6) in three stages. In the first stage, we shall construct $\widehat{\mathcal{V}}$ to required order in v using eqs.(2.25), (2.26), (2.31) and (2.32). From this we construct $\widehat{\mathcal{V}}_1$ using eq.(A.1). Finally we shall construct M_1 using eq.(A.5) and compare with the expression for $\widetilde{M}$ given in eq.(2.51).

We begin with the construction of $\widehat{\mathcal{V}}$. This involves two steps, construction of an O(8,24) matrix $\mathcal{V}$ satisfying eq.(2.31), and the construction of $\mathcal{U}$ from eqs.(2.25), (2.26). For this purpose it will be convenient to use an expression for the matrix M in terms of the fields introduced in eqs.(2.41)-

[5]Note that the construction of $\widehat{\widehat{\mathcal{V}}}$ from $\widetilde{M}$ using eqs.(2.62), (2.63), (2.66) and (2.67) suffers from ambiguities similar to those appearing in the construction of $\widehat{\mathcal{V}}$ from M. Thus even if we prove eq.(A.6), it will only imply that $\widehat{\widehat{\mathcal{V}}}$ is equal to $\widehat{\mathcal{V}}_1$ given in eq.(A.1) up to this ambiguity (and gauge transformations). But we shall take the stand that once eq.(A.6) is proved, we can take eq.(2.75) to be the definition of $\widehat{\widehat{\mathcal{V}}}$. What this means is that given $\widehat{\mathcal{V}}$, eq.(2.75) gives a procedure for constructing $\widehat{\widehat{\mathcal{V}}}$ satisfying eqs.(2.62), (2.63), (2.66) and (2.67).

(2.46). It is given by,

$$M = \begin{pmatrix} (\bar{G}_{22})^{-1} & -\frac{1}{2}(\bar{G}_{22})^{-1}\bar{\chi}^T\bar{L}\bar{\chi} & -(\bar{G}_{22})^{-1}\bar{\chi}^T \\ -\frac{1}{2}(\bar{G}_{22})^{-1}\bar{\chi}^T\bar{L}\bar{\chi} & \begin{matrix}\bar{G}_{22}+\bar{\chi}^T\bar{L}\bar{M}\bar{L}\bar{\chi} \\ +\frac{1}{4}(\bar{G}_{22})^{-1}(\bar{\chi}^T\bar{L}\bar{\chi})^2\end{matrix} & \begin{matrix}\bar{\chi}^T\bar{L}\bar{M}+\frac{1}{2}(\bar{G}_{22})^{-1} \\ \bar{\chi}^T(\bar{\chi}^T\bar{L}\bar{\chi})\end{matrix} \\ -(\bar{G}_{22})^{-1}\bar{\chi} & \bar{M}\bar{L}\bar{\chi}+\frac{1}{2}(\bar{G}_{22})^{-1}\bar{\chi}(\bar{\chi}^T\bar{L}\bar{\chi}) & \bar{M}+(\bar{G}_{22})^{-1}\bar{\chi}\bar{\chi}^T \end{pmatrix}, \tag{A.7}$$

where,

$$\bar{\chi} = 2\begin{pmatrix} \bar{A}_2^{(1)} \\ \cdot \\ \cdot \\ \bar{A}_2^{(30)} \end{pmatrix}. \tag{A.8}$$

It is a straightforward algebraic exercise to check that eq.(A.7) agrees with the expression for M given in eq.(2.9). A convenient $\mathcal{V}$ satisfying (2.31) can now be found using the expression for M given in eq.(A.7). We take,

$$\mathcal{V} = \begin{pmatrix} \mathcal{V}_{11} & 0 & 0 & \cdot & 0 \\ \mathcal{V}_{21} & \mathcal{V}_{22} & \mathcal{V}_{23} & \cdot & \mathcal{V}_{2,32} \\ \mathcal{V}_{31} & 0 & & & \\ \cdot & \cdot & & \bar{\mathcal{V}} & \\ \mathcal{V}_{32,1} & 0 & & & \end{pmatrix}, \tag{A.9}$$

where

$$\mathcal{V}_{11} = (\bar{G}_{22})^{-\frac{1}{2}}, \qquad \mathcal{V}_{22} = (\bar{G}_{22})^{\frac{1}{2}}, \qquad \mathcal{V}_{21} = -\frac{1}{2}(\bar{G}_{22})^{-\frac{1}{2}}\bar{\chi}^T\bar{L}\bar{\chi},$$
$$\begin{pmatrix} \mathcal{V}_{31} \\ \cdot \\ \cdot \\ \mathcal{V}_{32,1} \end{pmatrix} = -(\bar{G}_{22})^{-1/2}\bar{\chi}, \qquad \begin{pmatrix} \mathcal{V}_{23} \\ \cdot \\ \cdot \\ \mathcal{V}_{2,32} \end{pmatrix} = \bar{\mathcal{V}}^T\bar{L}\bar{\chi}, \tag{A.10}$$

and $\bar{\mathcal{V}}$ is an $O(7,23)$ matrix, satisfying,

$$\bar{\mathcal{V}}^T\bar{L}\bar{\mathcal{V}} = \bar{L}, \qquad \bar{\mathcal{V}}\bar{\mathcal{V}}^T = \bar{M}\,. \tag{A.11}$$

The construction of $\mathcal{U}$ proceeds as follows. We look for a solution of eqs.(2.25) of the form (2.29), (2.30). Comparing terms of order v on both sides of eq.(2.25) we get,

$$\epsilon^{\mu\nu}\partial_\nu(V^{(1)}) = \rho\eta^{\mu\nu}(\partial_\nu M M^{-1})\,, \tag{A.12}$$

Using the definition (2.50) for the vector $\bar{\psi}$, it is a straightforward algebraic exercise to verify the following relations:

$$\bar{L}\bar{\psi} = \begin{pmatrix} V^{(1)}_{13} \\ \cdot \\ \cdot \\ V^{(1)}_{1,32} \end{pmatrix}, \qquad -\bar{\psi} = \begin{pmatrix} V^{(1)}_{32} \\ \cdot \\ \cdot \\ V^{(1)}_{32,2} \end{pmatrix}. \tag{A.13}$$

Similar equations for $V^{(n)}$ for $n > 1$ may be written down, but we shall not require them. The only other information we shall need is that since $V^{(n)}$ are the generators of the $o(8,24)$ algebra, we have

$$V^{(n)}_{12} = 0. \tag{A.14}$$

We can now proceed to compute $\widehat{\mathcal{V}}_1$ using eq.(A.1). From eq.(2.27) we get,

$$v = t(\rho(1+t^2) + 2t\widetilde{\rho})^{-1}, \tag{A.15}$$

where $\widetilde{\rho} \equiv \rho_- - \rho_+$. Thus

$$(v)^{-T^0} = \begin{pmatrix} t^{-1}(\rho(1+t^2) + 2t\widetilde{\rho}) & & \\ & t(\rho(1+t^2)+2t\widetilde{\rho})^{-1} & \\ & & I_{30} \end{pmatrix}. \tag{A.16}$$

The construction of $\widehat{\mathcal{V}}_1$ also requires the knowledge of the $O(8,24)$ matrix $h(x,t)$ satisfying (2.76). Here,

$$h(x,t) = \begin{pmatrix} t & & \\ & t^{-1} & \\ & & I_{30} \end{pmatrix}. \tag{A.17}$$

With this choice of $h(x;t)$ the $\widehat{\mathcal{V}}_1$ defined in eq.(A.1) has a finite $v \to 0$ limit. In fact, using the relation between the components of $V^{(1)}$ and $\bar{\psi}$ mentioned above, one gets

$$\begin{aligned} \mathcal{V}_1(x) &\equiv \widehat{\mathcal{V}}_1(x; v=0) \\ &= \begin{pmatrix} \rho\mathcal{V}_{11} & -\frac{1}{2}\rho^{-1}\mathcal{V}_{22}\bar{\psi}^T\bar{L}\bar{\psi} & & \bar{\psi}^T\bar{L}\bar{\mathcal{V}} & \\ 0 & \rho^{-1}\mathcal{V}_{22} & 0 & \cdot & 0 \\ 0 & & & & \\ \cdot & -\rho^{-1}\mathcal{V}_{22}\bar{\psi} & & \bar{\mathcal{V}} & \\ 0 & & & & \end{pmatrix}. \end{aligned} \tag{A.18}$$

Using eqs.(2.51), (A.11) and the relations

$$\rho\mathcal{V}_{11} = e^{-\bar{\Phi}}, \qquad \rho^{-1}\mathcal{V}_{22} = e^{\bar{\Phi}}, \tag{A.19}$$

one can easily verify that

$$M_1 \equiv \mathcal{V}_1\mathcal{V}_1^T = \widetilde{M}\,. \tag{A.20}$$

This, in turn, proves eq.(2.75).

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