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Black Hole Entropy Function, Attractors and Precision Counting of Microstates

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Abstract

In these lecture notes we describe recent progress in our understanding of attractor mechanism and entropy of extremal black holes based on the entropy function formalism. We also describe precise computation of the microscopic degeneracy of a class of quarter BPS dyons in $\mathcal{N} = 4$ supersymmetric string theories, and compare the statistical entropy of these dyons, expanded in inverse powers of electric and magnetic charges, with a similar expansion of the corresponding black hole entropy. This comparison is extended to include the contribution to the entropy from multi-centered black holes as well.

Contents

1 Motivation

It is well known that low energy limit of string theory gives rise to gravity coupled to other fields. As a result these theories typically have black hole solutions. Thus string theory gives a framework for studying classical and quantum properties of black holes.

Classically black holes are solutions of Einstein's equations with special properties. They have a hypothetical surface – known as the event horizon – surrounding them such that no object inside the event horizon can escape the black hole. In quantum theory however the black hole behaves as a black body with finite temperature – known as the Hawking temperature. Consequently it emits Hawking radiation in accordance with the laws of black body radiation, and in its interaction with matter it behaves as a thermodynamic system characterized by entropy and other thermodynamic quantities. In the low curvature approximation where we ignore terms in the action with more than two derivatives, this entropy, known as the Bekenstein-Hawking entropy S_{BH}, is given by a simple expression

$$S_{BH} = A/(4G_N)\,, \tag{1.1}$$

where A is the area of the event horizon and G_N is the Newton's constant. One of the important questions is: Can we understand this entropy from statistical viewpoint i.e. as logarithm of the number of quantum states associated with the black hole?

Although we do not yet have a complete answer to this question, for a special class of black holes in string theory, – known as extremal black holes, – this question has been answered in the affirmative. These black holes have zero temperature and hence do not Hawking radiate and are usually stable. Often, but not always, extremal black holes are also invariant under certain number of supersymmetry transformations. In that case they are called BPS black holes. Due to the stability and the supersymmetry properties one has some control over the dynamics of the microscopic configuration (typically involving D-branes, fundamental strings and other solitonic objects) representing these black holes. This in turn allows us to calculate the degeneracy of such states at weak coupling where gravitational backreaction of the system can be ignored. Supersymmetry allows us to continue the result to strong coupling where gravitational backreaction becomes important and the system can be described as a black hole. In string theory one finds that for a wide class of extremal BPS black holes we have, in the limit where the size of the black hole is large [1],

$$S_{BH}(Q) = S_{stat}(Q)\,, \tag{1.2}$$

where $S_{BH}(Q)$ denotes the Bekenstein-Hawking entropy of an extremal black hole carrying a given set of charges labelled by Q, and $S_{stat}(Q)$ is defined as

$$S_{stat}(Q) = \ln d(Q)\,, \tag{1.3}$$

where $d(Q)$ is the degeneracy of BPS states in the theory carrying the same set of charges. This clearly gives a good understanding of this Bekenstein-Hawking entropy from microscopic viewpoint.

The initial comparison between S_{BH} and S_{stat} was carried out in the limit of large charges. Typically in this limit the horizon size is large so that the curvature and other field strengths at the horizon are small and hence we can calculate the entropy via eq.(1.1) without worrying about the higher derivative corrections to the effective action of string theory. On the other hand the computation of $S_{stat}(Q)$ also simplifies in this limit since the dynamics of the corresponding microscopic system is often described by a 1+1 dimensional conformal field theory (CFT) with the spatial coordinate compactified on a circle. An extremal black hole with large charges typically corresponds to a state of this CFT with large L_0 (or $\bar{L}_0$) eigenvalue and zero $\bar{L}_0$ (or L_0) eigenvalue. The degeneracy of such states can be computed via the Cardy formula in terms of the left- and right-handed central charges (c_L, c_R) of the conformal field theory and the L_0 (or $\bar{L}_0$) eigenvalue without knowing the details of the conformal field theory:

$$
\begin{aligned}
S_{stat}(Q) &\simeq 2\pi\sqrt{\frac{c_L L_0}{6}} \quad \text{for } \bar{L}_0 = 0 \\
&\simeq 2\pi\sqrt{\frac{c_R \bar{L}_0}{6}} \quad \text{for } L_0 = 0\,.
\end{aligned} \tag{1.4}
$$

The pleasant surprise is that these two completely different computations, – one for $S_{BH}(Q)$ and the other for $S_{stat}(Q)$, – give the same answer.

Given this success, it is natural to carry out this comparison to finer details. When we move away from the large charge limit, the curvature and other field strengths at the horizon are no longer negligible. Thus we must take into account the effect of higher derivative terms in the effective action on the black hole entropy. Typical example of such higher derivative terms are terms involving square and higher powers of the Riemann tensor. For a large but finite size black hole we expect the effect of these higher derivative terms at the horizon to be small but non-zero, giving rise to small modifications of the horizon geometry and consequently the black hole entropy. On the other hand for finite but large charges the statistical entropy computed from the Cardy formula will also receive corrections which are suppressed by inverse powers of charges. Thus it would be natural to ask: Does the agreement between $S_{BH}(Q)$ and $S_{stat}(Q)$ continue to hold even after taking into account the effects of higher derivative corrections on the black hole side, and deviation from the Cardy formula on the statistical side?

Due to an inherent ambiguity in defining the black hole entropy and the statistical entropy beyond the large charge limit, the issue involved is more complex than it sounds. The original Bekenstein-Hawking entropy was computed in classical general theory of relativity. However once we begin including higher derivative corrections, various string dualities can map the classical contribution to the effective action to the quantum contribution and vice versa. Thus it no longer makes sense to restrict our analysis to the classical theory. A natural choice will be to use the one particle irreducible (1PI) effective action of the theory since it respects all the duality symmetries. However since string theory has massless particles, the 1PI effective action typically gets non-local contributions if we go to sufficiently high order in the derivatives. As we shall describe in this review, while for any higher derivative theory of gravity with a local Lagrangian density there is a well defined algorithm for computing the entropy of a black hole, at present no technique is available for treating theories with non-local action. This causes a potential problem in defining the entropy of a black hole in string theory beyond the leading order. We could circumvent this problem using the Wilsonian effective action which is manifestly local, but this does not respect all the duality symmetries of the theory. There is a similar ambiguity on the statistical side as well. Since the corrections to the entropy are suppressed by inverse powers of various charges, they can be regarded as finite size corrections. However such corrections are known to depend crucially on the ensemble we choose to define the entropy. For example we could use duality invariant microcanonical or grand canonical ensembles, or use duality non-invariant mixed ensembles where we treat a subset of the charges as we would do in a microcanonical ensemble and the rest of the charges as we would do in a grand canonical ensemble [2].

We hope that by studying explicit examples where one could compute the corrections to both the statistical entropy and the black hole entropy, we may be able to resolve the above mentioned ambiguities and make a more precise formulation of the relationship between the two entropies. In order to proceed along these lines we need to open two fronts. First of all we need to learn how to take into account the effect of the higher derivative terms on the computation of black hole entropy. But we also need to know how to calculate the statistical entropy to greater accuracy. This involves precise computation of the degeneracy of states with a given set of charges. In this review we shall address both these issues. The first part of the review, dealing with the computation of black hole entropy in the presence of higher derivative terms, will be based on the entropy function formalism of [3, 4], – this in turn is an adaptation of a more general formalism for computing black hole entropy in the presence of higher derivative terms [5] to the special case of extremal black holes. The second part of the review, dealing with precision computation of statistical entropy for a class of four dimensional black holes in $\mathcal{N} = 4$ supersymmetric string theories, will follow the analysis of [6, 7, 8]. The original formula for the statistical entropy was first proposed in [9] for the special case of heterotic string theory compactified on T^6, and later extended to more general models in [10, 11, 12]. Various alternative approaches to proving these formulæ have been explored in [13, 14, 15, 16, 17]. We shall not review these different approaches, except parts of [13, 15] that will be directly relevant for our approach to the counting problem, and also part of [18] that will be useful in extracting the asymptotic behaviour of the statistical entropy for large charges.

In our analysis we shall try to maintain manifest duality invariance by using the 1PI effective action on the black hole side and microcanonical ensemble on the statistical side. As we shall see, to the extent we can compute both sides, the black hole and the statistical entropy agree even after taking into account higher derivative corrections. The analysis on the statistical side is quite clean, and we find an explicit algorithm to generate an expansion of the statistical entropy in inverse powers of charges. The analysis on the black hole side suffers from the inherent problem of having to deal with non-local terms, but we carry out our analysis by including the effect of a duality invariant set of local four derivative terms in the action, – the Gauss-Bonnet term. Although we cannot prove that the other four derivative terms (including non-local terms) do not contribute to this order, we find that in various limits where this additional contribution can be computed, the result for the black hole entropy agrees with the one computed using the Gauss-Bonnet term. We believe that the limitation due to the non-local nature of the 1PI effective action can be overcome in the future, and we shall find a systematic procedure for calculating the black hole entropy using the 1PI effective action. On the other hand it is also possible in principle that we can give up manifest duality invariance and work with Wilsonian effective action on the black hole side and a mixed ensemble on the statistical side. This approach has been advocated in [2].

The rest of the review is organised as follows. In §2 we develop the entropy function formalism for computing the entropy of extremal black holes. We include in this discussion spherically symmetric black holes in various dimensions, rotating black holes, theories with Chern-Simons terms etc., and also describe how the entropy function formalism provides a simple proof of the attractor phenomenon. §3 contains application of this formalism to the computation of entropy of a wide class of extremal black holes in a wide class of theories. These include in particular a class of quarter BPS black holes in the $\mathcal{N}=4$ supersymmetric four dimensional string theories; these are the black holes for which we later carry out a detailed comparison between the black hole entropy and the statistical entropy. In §4 we compare our approach, which holds for a general extremal black hole, with that of [19, 20, 21, 22] where for a special class of extremal black holes – with an AdS_3 factor in the near horizon geometry – a more powerful technique for computing the entropy was developed. In §5 we describe the computation of statistical entropy of a special class of black holes in $\mathcal{N}=4$ supersymmetric string theories in four dimensions, and compare the results with the black hole entropy computed in §3. We conclude in §6 with a list of open questions. We also speculate on how our degeneracy formula might be extendable to the $\mathcal{N}=2$ supersymmetric string theories. The appendices provide us with some technical results which are used mainly in the computation of the statistical entropy.

We must caution the reader that this is not a complete review of attractor mechanism or computation of black hole and statistical entropy. Instead it deals with only a very specific approach to computing entropy of extremal black holes using the entropy function formalism, and computation of the statistical entropy of a special class of dyons in a special class of theories. It does not even contain all aspects of the entropy function formalism or the computation of the statistical entropy in $\mathcal{N} = 4$ supersymmetric string theories. However we have tried to make the review self-contained in the sense that for the material that does get covered in the review, one does not always have to go back and consult the original literature in order to follow the material. As a result the review has become somewhat long; however we hope that this has not significantly increased the time needed to go through the rest of the review.

2 Black Hole Entropy Function and the Attractor Mechanism

In this section we shall develop a general method for computing the entropy of extremal black holes in a theory of gravity with higher derivative corrections. The first question we need to address is: How do we define extremal black holes in a general higher derivative theory of gravity? For this we shall take the clue from usual theories of gravity with two derivative actions, – namely study the properties of extremal black holes in these theories and then identify certain universal features which can be adopted as the definition of extremal black holes in a more general class of theories with higher derivative terms.

2.1 Definition of extremal black holes

We begin our analysis with the Reissner-Nordstrom solution describing a spherically symmetric charged black hole in the usual Einstein-Maxwell theory in four dimensions. This theory is described by the action

$$\mathcal{S} = \int d^4x \sqrt{-\det g}\, \mathcal{L}, \qquad \mathcal{L} = \frac{1}{16\pi G_N} R - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} \,. \tag{2.1.1}$$

We shall be using the following notations for the Christoffel symbol and Riemann tensors:

$$\begin{aligned}
\Gamma^\mu_{\nu\rho} &= \frac{1}{2} g^{\mu\sigma} \left(\partial_\nu g_{\sigma\rho} + \partial_\rho g_{\sigma\nu} - \partial_\sigma g_{\nu\rho}\right) \\
R^\mu{}_{\nu\rho\sigma} &= \partial_\rho \Gamma^\mu_{\nu\sigma} - \partial_\sigma \Gamma^\mu_{\nu\rho} + \Gamma^\mu_{\tau\rho}\Gamma^\tau_{\nu\sigma} - \Gamma^\mu_{\tau\sigma}\Gamma^\tau_{\nu\rho} \\
R_{\nu\sigma} &= R^\mu{}_{\nu\mu\sigma}, \qquad R = g^{\nu\sigma} R_{\nu\sigma} \,.
\end{aligned} \tag{2.1.2}$$

The Reissner-Nordstrom solution in this theory is given by

$$
\begin{aligned}
ds^2 &= -(1-a/\rho)(1-b/\rho)d\tau^2 + \frac{d\rho^2}{(1-a/\rho)(1-b/\rho)} + \rho^2(d\theta^2 + \sin^2\theta d\phi^2)\,, \\
F_{\rho\tau} &= \frac{q}{4\pi\rho^2}, \qquad F_{\theta\phi} = \frac{p}{4\pi}\sin\theta\,,
\end{aligned}
\tag{2.1.3}
$$

where ρ, θ, ϕ and τ are the space-time coordinates, a and b are two constants determined from the relation

$$
a + b = 2G_N M, \qquad ab = \frac{G_N}{4\pi}(q^2 + p^2)\,,
\tag{2.1.4}
$$

and q, p and M denote the electric and magnetic charges and the mass of the black hole respectively. If we take $a > b$ then the inner and the outer horizon of the black hole are at $r = b$ and at $r = a$ respectively. The extremal limit corresponds to choosing

$$
M^2 = \frac{1}{4\pi G_N}(q^2 + p^2)\,,
\tag{2.1.5}
$$

so that we have

$$
a = b = \sqrt{\frac{G_N}{4\pi}(q^2 + p^2)}\,.
\tag{2.1.6}
$$

We now define

$$
t = \lambda\,\tau/a^2, \qquad r = \lambda^{-1}(\rho - a),
\tag{2.1.7}
$$

where λ is an arbitrary constant, and rewrite the extremal solution in this new coordinate system. This gives

$$
\begin{aligned}
ds^2 &= -\frac{r^2 a^4}{(a+\lambda\,r)^2}dt^2 + \frac{(a+\lambda\,r)^2}{r^2}dr^2 + (a+\lambda\,r)^2(d\theta^2 + \sin^2\theta d\phi^2)\,, \\
F_{rt} &= \frac{qa^2}{4\pi(a+\lambda r)^2}, \qquad F_{\theta\phi} = \frac{p}{4\pi}\sin\theta\,.
\end{aligned}
\tag{2.1.8}
$$

Finally we take the 'near horizon' limit $\lambda \to 0$. In this limit the solution takes the form:

$$
\begin{aligned}
ds^2 &= a^2\left(-r^2 dt^2 + \frac{dr^2}{r^2}\right) + a^2(d\theta^2 + \sin^2\theta d\phi^2)\,, \\
F_{rt} &= \frac{q}{4\pi}, \qquad F_{\theta\phi} = \frac{p}{4\pi}\sin\theta\,.
\end{aligned}
\tag{2.1.9}
$$

The entropy of the black hole, obtained by dividing the area of the horizon by $4G_N$, is

$$
S_{BH} = \frac{1}{4}(q^2 + p^2)\,.
\tag{2.1.10}
$$

The field configuration given in (2.1.9) has the following features:

1. In the limit $\lambda \to 0$ keeping r fixed, the original coordinate ρ approaches a. Thus (2.1.9) describes the field configuration of the black hole near the horizon.

2. Since for any λ (2.1.8) describes an exact classical solution, in the $\lambda \to 0$ limit also we have an exact classical solution for all finite r, not just for small r. Indeed, we could have obtained (2.1.9) by directly solving the equations of motion of the Einstein-Maxwell theory without any reference to black holes.

3. The space-time described by (2.1.9) splits into a product of two spaces. One of them, labelled by (θ, ϕ) describes an ordinary two dimensional sphere S^2. The other, labelled by (r, t), describes a two dimensional space-time known as AdS_2. This is a solution of two dimensional Einstein gravity with negative cosmological constant.

4. The background described in (2.1.9) has an $SO(3)$ isometry acting on the sphere S^2. This reflects the spherical symmetry of the original black hole and is present even in the full black hole solution. The background also has an SO(2,1) isometry acting on the AdS_2 space that was not present in the full black hole solution. This isometry is generated by

$$L_1 = \partial_t, \quad L_0 = t\partial_t - r\partial_r, \quad L_{-1} = \frac{1}{2}\left(\frac{1}{r^2} + t^2\right)\partial_t - t\, r\, \partial_r\,. \qquad (2.1.11)$$

   Not only the metric, but also the gauge field strengths given in (2.1.9) can be shown to be invariant under the $SO(2,1) \times SO(3)$ transformation.

It turns out that all known extremal spherically symmetric black holes in four dimensions with non-singular horizon have near horizon geometry $AdS_2 \times S^2$ and an associated isometry $SO(2,1) \times SO(3)$.

Consider now the effect of adding higher derivative terms in the action. In general it is quite difficult to find the full black hole solution after taking into account these higher derivative terms. However it is natural to postulate that the symmetries of the near horizon geometry will not be destroyed by these higher derivative terms. This suggests the following postulate:

In any generally covariant theory of gravity coupled to matter fields, the near horizon geometry of a spherically symmetric extremal black hole in four dimensions has $SO(2,1) \times SO(3)$ isometry.

We shall take this as the definition of spherically symmetric extremal black holes in four dimensions. Although we arrived at this definition by analyzing extremal black holes in theories with only two derivative terms in the action, and possible small modification of the solution due to higher derivative terms, we shall extend the definition to include even black holes with large curvature at the horizon so that the higher derivative terms are as important as the two derivative terms.[1]

This analysis can be generalized to study spherically symmetric extremal black holes in other dimensions, as well as rotating extremal black holes in four and other dimensions. In every known example one finds that the near horizon geometry of extremal black holes has an enhanced $SO(2,1)$ isometry that is not present in the full black holes solution. The full isometry of the near horizon geometry is then the product of the $SO(2,1)$ isometry and rotational isometry of the full black hole solution. For example

1. The near horizon geometry of an extremal spherically symmetric black hole in D dimensions has $SO(2,1) \times SO(D-1)$ isometry.

2. The near horizon geometry of an extremal rotating black hole in four dimensions has $SO(2,1) \times U(1)$ isometry.

We shall take these as definitions of the corresponding extremal black holes even after inclusion of higher derivative terms.[2]

Based on these postulates we shall now develop a general procedure for computing the entropy of extremal black holes. Our discussion will follow [19, 4, 24].

2.2 Spherically symmetric black holes in $D = 4$

Let us consider a four dimensional theory of gravity coupled to a set of abelian gauge fields $A^{(i)}_\mu$ and neutral scalar fields $\{\phi_s\}$. Let $\sqrt{-\det g}\,\mathcal{L}$ be the lagrangian density, expressed as a function of the metric $g_{\mu\nu}$, the scalar fields $\{\phi_s\}$, the gauge field strengths $F^{(i)}_{\mu\nu}$, and covariant derivatives of these fields. We have not included any antisymmetric rank two tensor field in our list of fields since such fields can always be dualized to a scalar field. When written in terms of the anti-symmetric tensor field, the definition of the field strength often contains gauge and Lorentz Chern-Simons terms. However when written in terms of the dual scalar fields there are no Chern-Simons type term in the action. Hence reparametrization and gauge invariance of the action implies that $\mathcal{L}$ is manifestly reparametrization invariant and gauge invariant under the usual transformation laws of various fields. Thus $\mathcal{L}$ must be constructed from the scalar fields ϕ_s, gauge field strengths $F^{(i)}_{\mu\nu} \equiv \partial_\mu A^{(i)}_\nu - \partial_\nu A^{(i)}_\mu$, the inverse metric $g^{\mu\nu}$, the Riemann tensor $R_{\mu\nu\rho\sigma}$ and covariant derivatives of these fields.

[1] Such black holes are called small black holes and will be the subject of discussion in §3.3 and §4.4.
[2] In four and five dimensions these postulates have recently been proved in [23].

We consider a spherically symmetric extremal black hole solution with $SO(2,1)\times SO(3)$ invariant near horizon geometry. The most general field configuration consistent with this isometry is of the form:

$$\begin{aligned}
&ds^2 \equiv g_{\mu\nu}dx^\mu dx^\nu = v_1\left(-r^2dt^2+\frac{dr^2}{r^2}\right)+v_2\left(d\theta^2+\sin^2\theta d\phi^2\right)\\
&\phi_s = u_s\\
&F^{(i)}_{rt} = e_i, \qquad F^{(i)}_{\theta\phi} = \frac{p_i}{4\pi}\sin\theta\,,
\end{aligned} \tag{2.2.1}$$

where v_1, v_2, $\{u_s\}$, $\{e_i\}$ and $\{p_i\}$ are constants. For this background the nonvanishing components of the Riemann tensor are:

$$\begin{aligned}
R_{\alpha\beta\gamma\delta} &= -v_1^{-1}(g_{\alpha\gamma}g_{\beta\delta}-g_{\alpha\delta}g_{\beta\gamma})\,, \qquad \alpha,\beta,\gamma,\delta = r,t\,,\\
R_{mnpq} &= v_2^{-1}\,(g_{mp}g_{nq}-g_{mq}g_{np})\,, \qquad m,n,p,q=\theta,\phi\,.
\end{aligned} \tag{2.2.2}$$

It follows from the general form of the background that the covariant derivatives of the scalar fields ϕ_s, the gauge field strengths $F^{(i)}_{\mu\nu}$ and the Riemann tensor $R_{\mu\nu\rho\sigma}$ all vanish for the near horizon geometry. By the general symmetry consideration it follows that the contribution to the equation of motion from any term in $\mathcal{L}$ that involves covariant derivatives of the gauge field strengths, scalars or the Riemann tensor vanish identically for this background and we can restrict our attention to only those terms which do not involve covariant derivatives of these fields.

Let us denote by $f(\vec{u},\vec{v},\vec{e},\vec{p})$ the Lagrangian density $\sqrt{-\det g}\,\mathcal{L}$ evaluated for the near horizon geometry (2.2.1) and integrated over the angular coordinates:

$$f(\vec{u},\vec{v},\vec{e},\vec{p}) = \int d\theta\, d\phi\,\sqrt{-\det g}\,\mathcal{L}\,. \tag{2.2.3}$$

The scalar and the metric field equations in the near horizon geometry correspond to extremizing f with respect to the variables $\vec{u}$ and $\vec{v}$:

$$\frac{\partial f}{\partial u_s}=0, \qquad \frac{\partial f}{\partial v_i}=0\,. \tag{2.2.4}$$

Furthermore since u_s, v_1 and v_2 describe the most general $SO(2,1)\times SO(3)$ invariant scalar and metric deformations, these are the only independent components of the equations of motion of scalar fields and the metric.

On the other hand the non-trivial components of the gauge field equations and the Bianchi identities for the full black hole solution takes the form:

$$\partial_r\left(\frac{\delta\,\mathcal{S}}{\delta F^{(i)}_{rt}}\right)=0, \qquad \partial_r F^{(i)}_{\theta\phi}=0\ , \tag{2.2.5}$$

where $\mathcal{S} = \int d^4x\sqrt{-\det g}\mathcal{L}$ is the action. These equations are of course automatically satisfied by the near horizon background (2.2.1), but we can extract more information from them. From (2.2.5) it follows that

$$\int d\theta d\phi \, \frac{\delta \mathcal{S}}{\delta F^{(i)}_{rt}} = a_i, \quad \int d\theta d\phi \, F^{(i)}_{\theta\phi} = b_i \,, \tag{2.2.6}$$

where a_i and b_i are r independent constants. Evaluating these integrals on the near horizon geometry (2.2.1) gives

$$a_i = \frac{\partial f}{\partial e_i}, \qquad b_i = p_i \,. \tag{2.2.7}$$

On the other hand if we evaluate the integrals in (2.2.6) at asymptotic infinity, then a_i and b_i are just the integrals of electric and magnetic flux at infinity, and hence can be identified with the electric and magnetic charges respectively. From this it follows that the constants p_i appearing in (2.2.1) correspond to magnetic charges of the black hole, and

$$\frac{\partial f}{\partial e_i} = q_i \tag{2.2.8}$$

where q_i denote the electric charges carried by the black hole.

For fixed $\vec{p}$ and $\vec{q}$, (2.2.4) and (2.2.8) give a set of equations which are equal in number to the number of unknowns $\vec{u}$, $\vec{v}$ and $\vec{e}$. In a generic case we may be able to solve these equations completely to determine the background in terms of only the electric and the magnetic charges $\vec{q}$ and $\vec{p}$. [3] This is consistent with the attractor mechanism for supersymmetric background which says that the near horizon configuration of a black hole depends only on the electric and magnetic charges carried by the black hole and not on the asymptotic values of these scalar fields. We shall elaborate on this in §2.3.

Let us define

$$\mathcal{E}(\vec{u}, \vec{v}, \vec{e}, \vec{q}, \vec{p}) \equiv 2\,\pi(e_i\, q_i - f(\vec{u}, \vec{v}, \vec{e}, \vec{p})) \tag{2.2.9}$$

The equations (2.2.4), (2.2.8) determining $\vec{u}$, $\vec{v}$ and $\vec{e}$ are then given by:

$$\frac{\partial \mathcal{E}}{\partial u_s} = 0, \quad \frac{\partial \mathcal{E}}{\partial v_1} = 0\,, \quad \frac{\partial \mathcal{E}}{\partial v_2} = 0\,, \quad \frac{\partial \mathcal{E}}{\partial e_i} = 0\,. \tag{2.2.10}$$

Thus all the near horizon parameters may be determined by extremizing a single function $\mathcal{E}$.

[3] The situation in string theory is not completely generic. For example in $\mathcal{N} = 2$ supersymmetric string theories there is no coupling of the hypermultiplet scalars to the vector multiplet fields or the curvature tensor to lowest order in α', and hence in this approximation the function f does not depend on the hypermultiplet scalars. Thus the equations (2.2.4), (2.2.8) do not fix the values of the hypermultiplet scalars in this approximation.

We shall now turn to the analysis of the entropy associated with this black hole [3]. A general formula for the entropy in the presence of higher derivative terms has been given in [5, 25, 26, 27].[4] For a spherically symmetric black hole this formula takes the form

$$S_{BH} = -8\pi \int_H d\theta\, d\phi\, \frac{\delta \mathcal{S}}{\delta R_{rtrt}} \sqrt{-g_{rr}\, g_{tt}}\,, \tag{2.2.11}$$

where H denotes the horizon of the black hole. In computing $\delta\mathcal{S}/\delta R_{\mu\nu\rho\sigma}$ in (2.2.11) we need to

1. express the action $\mathcal{S}$ in terms of symmetrized covariant derivatives of fields by replacing anti-symmetric combinations of covariant derivatives in terms of the Riemann tensor, and then

2. treat $R_{\mu\nu\rho\sigma}$ as independent variables.

This formula simplifies enormously here since the covariant derivatives of all the tensors vanish, and as a result

$$\frac{\delta \mathcal{S}}{\delta R_{rtrt}} = \sqrt{-\det g}\frac{\partial \mathcal{L}}{\partial R_{rtrt}}\,, \tag{2.2.12}$$

where in the expression for $\mathcal{L}$ we need to keep only those terms which do not involve explicit covariant derivatives, and $\partial\mathcal{L}/\partial R_{\mu\nu\rho\sigma}$ is defined through the equation

$$\delta\, \mathcal{L} = \frac{\partial \mathcal{L}}{\partial R_{\mu\nu\rho\sigma}}\, \delta\, R_{\mu\nu\rho\sigma}\,. \tag{2.2.13}$$

In computing $\delta\mathcal{L}$ we need to treat the components of the Riemann tensor as independent variables not related to the metric. Substituting (2.2.12) into (2.2.11) we get simple formula for the entropy

$$S_{BH} = 8\pi\, \frac{\partial \mathcal{L}}{\partial R_{rtrt}}\, g_{rr}\, g_{tt}\, A_H = -8\pi\, v_1^2\, \frac{\partial \mathcal{L}}{\partial R_{rtrt}}\, A_H\,, \tag{2.2.14}$$

where A_H is the area of the event horizon.

In order to express this in terms of the function f defined in (2.2.3), let us denote by $f_\lambda(\vec{u}, \vec{v}, \vec{e}, \vec{p})$ an expression similar to the right hand side of (2.2.3) except that each factor of R_{rtrt} in the expression of $\mathcal{L}$ is multiplied by a factor of λ. Then we have the relation:

$$\left.\frac{\partial f_\lambda(\vec{u}, \vec{v}, \vec{e}, \vec{p})}{\partial \lambda}\right|_{\lambda=1} = \int d\theta\, d\phi\, \sqrt{-\det g}\, R_{\alpha\beta\gamma\delta}\, \frac{\partial \mathcal{L}}{\partial R_{\alpha\beta\gamma\delta}}\,, \tag{2.2.15}$$

[4] This formula for the entropy has been derived for regular black holes with bifurcate event horizon and not for extremal black holes. We are defining the entropy of extremal black holes as the entropy of a non-extremal black hole in the extremal limit. This allows us to use Wald's formula.

where the repeated indices $\alpha, \beta, \gamma, \delta$ are summed over the coordinates r and t. Now since by symmetry consideration $(\partial\mathcal{L}/\partial R_{\alpha\beta\gamma\delta})$ is proportional to $(g^{\alpha\gamma}g^{\beta\delta} - g^{\alpha\delta}g^{\beta\gamma})$, we have

$$\frac{\partial\mathcal{L}}{\partial R_{\alpha\beta\gamma\delta}} = -v_1^2\,(g^{\alpha\gamma}g^{\beta\delta} - g^{\alpha\delta}g^{\beta\gamma})\,\frac{\partial\mathcal{L}}{\partial R_{rtrt}}\,. \tag{2.2.16}$$

The constant of proportionality has been fixed by taking $(\alpha\beta\gamma\delta) = (rtrt)$. Using (2.2.2) and (2.2.16), and that for a spherically symmetric background $\partial\mathcal{L}/\partial R_{rtrt}$ is independent of the (θ, ϕ) coordinates, we can rewrite (2.2.15) as

$$\frac{\partial f_\lambda(\vec{u}, \vec{v}, \vec{e}, \vec{p})}{\partial\lambda}\bigg|_{\lambda=1} = 4\,v_1^2\,\frac{\partial\mathcal{L}}{\partial R_{rtrt}}\,A_H\,. \tag{2.2.17}$$

Substituting this into (2.2.14) gives

$$S_{BH} = -2\pi\;\frac{\partial f_\lambda(\vec{u}, \vec{v}, \vec{e}, \vec{p})}{\partial\lambda}\bigg|_{\lambda=1}\,. \tag{2.2.18}$$

We shall now try to express the right hand side of (2.2.18) in terms of derivatives of f with respect to the variables $\vec{u}$, $\vec{v}$, $\vec{e}$ and $\vec{p}$. For this let us focus on the v_1 dependence of f_λ. Since the expression for $\mathcal{L}$ is invariant under reparametrization of the r, t coordinates, every factor of R_{rtrt} in the expression for f_λ must appear in the combination $\lambda\, g^{rr}g^{tt}R_{rtrt} = -\lambda v_1^{-1}$, every factor of $F^{(i)}_{rt}$ must appear in the combination $\sqrt{-g^{rr}g^{tt}}F^{(i)}_{rt} = e_i v_1^{-1}$, and every factor of $F^{(i)}_{\theta\phi} = p_i/4\pi$ and $\phi_s = u_s$ must appear without any accompanying power of v_1. The contribution from all terms which involve covariant derivatives of $F^{(i)}_{\mu\nu}$, $R_{\mu\nu\rho\sigma}$ or ϕ_s vanish; hence there is no further factor of v_1 coming from contraction of the metric with these derivative operators. The only other v_1 dependence of $f_\lambda(\vec{u}, \vec{v}, \vec{e}, \vec{p})$ is through the overall multiplicative factor of $\sqrt{-\det g} \propto v_1$. Thus $f_\lambda(\vec{u}, \vec{v}, \vec{e}, \vec{p})$ must be of the form

$$f_\lambda(\vec{u}, \vec{v}, \vec{e}, \vec{p}) = v_1 g(\vec{u}, v_2, \vec{p}, \lambda v_1^{-1}, \vec{e}v_1^{-1})\,, \tag{2.2.19}$$

for some function g. This gives

$$\lambda\frac{\partial f_\lambda(\vec{u}, \vec{v}, \vec{e}, \vec{p})}{\partial\lambda} + v_1\frac{\partial f_\lambda(\vec{u}, \vec{v}, \vec{e}, \vec{p})}{\partial v_1} + e_i\frac{\partial f_\lambda(\vec{u}, \vec{v}, \vec{e}, \vec{p})}{\partial e_i} - f_\lambda(\vec{u}, \vec{v}, \vec{e}, \vec{p}) = 0\,. \tag{2.2.20}$$

Setting $\lambda = 1$ in (2.2.20), using the equation of motion of v_1 as given in (2.2.4), and substituting the result into eq.(2.2.18) we get

$$S_{BH} = 2\pi\left(e_i\,\frac{\partial f}{\partial e_i} - f\right)\,. \tag{2.2.21}$$

This together with (2.2.8) shows that $S_{BH}(\vec{q}, \vec{p})/2\pi$ may be regarded as the Legendre transform of the function $f(\vec{u}, \vec{v}, \vec{e}, \vec{p})$ with respect to the variables e_i after eliminating $\vec{u}$ and $\vec{v}$ through their equations of motion (2.2.4). Using (2.2.9) we can also express (2.2.21) as

$$S_{BH} = \mathcal{E}(\vec{u}, \vec{v}, \vec{e}, \vec{q}, \vec{p})\,, \tag{2.2.22}$$

at the extremum (2.2.10). This suggests that we call the function $\mathcal{E}$ the entropy function [3].

We can take a slightly different viewpoint in which we define the entropy function $\mathcal{E}$ as a function of $\vec{u}$, $\vec{v}$, $\vec{q}$ and $\vec{p}$ after eliminating the electric field variables $\vec{e}$ by the $\partial f/\partial e_i = q_i$ condition. In this form the entropy function given in (2.2.9) will just be 2π times the Legendre transform of the function f with respect to the variables $\{e_i\}$. We shall continue to use the same symbol $\mathcal{E}$ for both entropy functions since the second definition is obtained from the first simply by extremizing the latter with respect to $\{e_i\}$.

Given an action, the entropy function formalism reduces the problem of computing the entropy of an extremal black hole into the problem of solving a set of algebraic equations. We shall illustrate this by applying this formalism to extremal Reissner-Nordstrom black hole in the Maxwell-Einstein theory described by the action (2.1.1). The most general $SO(2,1)\times SO(3)$ invariant background in this theory is given by

$$\begin{aligned} ds^2 &= v_1\left(-r^2dt^2+\frac{dr^2}{r^2}\right)+v_2\left(d\theta^2+\sin^2\theta d\phi^2\right) \\ F_{rt} &= e, \qquad F_{\theta\phi}=p\sin\theta/4\pi\,. \end{aligned} \tag{2.2.23}$$

Using (2.1.1), (2.2.2) we get

$$\begin{aligned} f(v_1,v_2,e,p) &\equiv \int d\theta d\phi\,\sqrt{-\det g}\,\mathcal{L} \\ &= 4\pi\,v_1v_2\left[\frac{1}{16\pi G_N}\left(-\frac{2}{v_1}+\frac{2}{v_2}\right)+\frac{1}{2}\,v_1^{-2}e^2-\frac{1}{2}v_2^{-2}\left(\frac{p}{4\pi}\right)^2\right]. \end{aligned} \tag{2.2.24}$$

This in turn gives

$$\begin{aligned} \mathcal{E}(v_1,v_2,e,q,p) &\equiv 2\pi(q\,e-f) \\ &= 2\pi\left[q\,e-\frac{1}{4G_N}(2v_1-2v_2)-2\pi\;v_2\,v_1^{-1}\,e^2+2\pi\,v_1\,v_2^{-1}\left(\frac{p}{4\pi}\right)^2\right]. \end{aligned} \tag{2.2.25}$$

It is easy to verify that $\mathcal{E}$ has an extremum at

$$v_1=v_2=G_N\,\frac{q^2+p^2}{4\pi}\,,\qquad e=\frac{q}{4\pi}\,. \tag{2.2.26}$$

Substituting this into the expression for $\mathcal{E}$ we get

$$S_{BH}\equiv\mathcal{E}=\frac{1}{4}(q^2+p^2)\,. \tag{2.2.27}$$

Eqs.(2.2.26) reproduces (2.1.6), (2.1.9) and (2.2.27) reproduces (2.1.10).

Finally we note that although the entropy function formalism developed in this section gives a simple method for computing the entropy of an extremal black hole if such a solution exists, our analysis does not tell us if the full black hole solution, interpolating between $AdS_2 \times S^2$ near horizon geometry and the asymptotically flat Minkowski space, really exists. For a general two derivative theory this issue has been addressed in [28] where it was shown that such a solution exists provided the matrix of second derivatives of the entropy function with respect to the scalar field values at the horizon is positive definite at the extremum of the entropy function. Whether there is a generalization of this result in higher derivative theories is still an open question.

2.3 Attractor, field redefinition and duality transformation

In this section we shall discuss some important consequences of the results derived in §2.2.

1. Since the construction of the function $\mathcal{E}$ involves knowledge of only the Lagrangian density, the functional form of $\mathcal{E}$ is independent of asymptotic values of the moduli scalar fields, – scalar fields which have no potential in flat space-time and hence can take arbitrary constant values asymptotically. Thus if the extremization equations (2.2.10) determine all the parameters $\vec{u}$, $\vec{v}$, $\vec{e}$ uniquely then the value of $\mathcal{E}$ at the extremum and hence the entropy S_{BH} is completely independent of the asymptotic values of the moduli fields. If on the other hand the function $\mathcal{E}$ has flat directions then only some combinations of the parameters $\vec{u}$, $\vec{v}$, $\vec{e}$ are determined by extremizing $\mathcal{E}$, and the rest may depend on the asymptotic values of the moduli fields. However since $\mathcal{E}$ is independent of the flat directions, it depends only on the combination of parameters which are fixed by the extremization equations. As a result the value of $\mathcal{E}$ at the extremum is still independent of the asymptotic moduli. This shows that the entropy of the black hole is independent of the asymptotic values of the moduli fields irrespective of whether or not $\mathcal{E}$ has flat directions. This is a generalization of the usual attractor mechanism for black holes in supergravity theories [29, 30, 31].

   This result in particular implies that the entropy of an extremal black hole does not change as we change the asymptotic value of the string coupling constant from a sufficiently large value where the black hole description is good to a sufficiently small value where the microscopic description is expected to be valid. This fact has been used to argue that under certain conditions the statistical entropy of the system, computed at weak string coupling, should match the black hole entropy even for non-supersymmetric extremal black holes [32].

2. An arbitrary field redefinition of the metric and the scalar fields will induce a redefinition of the parameters $\vec{u}$, $\vec{v}$, and hence the functional form of $\mathcal{E}$ will change.[5] However, since the value of $\mathcal{E}$ at the extremum is invariant under non-singular field redefinition, the entropy is unchanged under a redefinition of the metric and other scalar fields. To see this more explicitly, let us consider a reparametrization of $\vec{u}$ and $\vec{v}$ of the form:

$$\widehat{u}_s = g_s(\vec{u}, \vec{v}, \vec{e}, \vec{p}), \quad \widehat{v}_i = h_i(\vec{u}, \vec{v}, \vec{e}, \vec{p}) \,, \tag{2.3.1}$$

for some functions $\{g_s\}, \{h_i\}$. Then it follows from eqs.(2.2.3), (2.2.9) that the new entropy function $\widehat{\mathcal{E}}(\vec{\widehat{u}}, \vec{\widehat{v}}, \vec{e}, \vec{q}, \vec{p})$ is given by:

$$\widehat{\mathcal{E}}(\vec{\widehat{u}}, \vec{\widehat{v}}, \vec{e}, \vec{q}, \vec{p}) = \mathcal{E}(\vec{u}, \vec{v}, \vec{e}, \vec{q}, \vec{p}) \,. \tag{2.3.2}$$

It is now easy to see that eqs.(2.2.10) are equivalent to:

$$\frac{\partial \widehat{\mathcal{E}}}{\partial \widehat{u}_s} = 0, \quad \frac{\partial \widehat{\mathcal{E}}}{\partial \widehat{v}_1} = 0 \,, \quad \frac{\partial \widehat{\mathcal{E}}}{\partial \widehat{v}_2} = 0 \,, \quad \frac{\partial \widehat{\mathcal{E}}}{\partial e_i} = 0 \,. \tag{2.3.3}$$

Thus the value of $\widehat{\mathcal{E}}$ evaluated at this extremum is equal to the value of $\mathcal{E}$ evaluated at the extremum (2.2.10), showing that the entropy of an extremal black hole remains unchanged under field redefinition. This result of course is a consequence of the field redefinition invariance of Wald's entropy formula as discussed in [25].

3. As is well known, Lagrangian density is not invariant under an electric-magnetic duality transformation. However, the function $\mathcal{E}$, being Legendre transformation of the Lagrangian density with respect to the electric field variables, is invariant under an electric-magnetic duality transformation. In other words, if instead of the original Lagrangian density $\mathcal{L}$, we use an equivalent dual Lagrangian density $\widetilde{\mathcal{L}}$ where some of the gauge fields have been dualized, and construct a new entropy function $\widetilde{\mathcal{E}}(\vec{u}, \vec{v}, \vec{q}, \vec{p})$ from this new Lagrangian density, then $\mathcal{E}$ and $\widetilde{\mathcal{E}}$ will be related to each other by exchange of the appropriate q_i's and p_i's.

2.4 Spherically symmetric black holes for arbitrary D

The analysis can be generalized to higher dimensional theories as follows. We begin with a D-dimensional field theory of metric, various p-form gauge fields and neutral scalars with lagrangian

[5] A redefinition of gauge fields preserving the gauge transformation laws requires adding to the gauge field a gauge invariant vector field constructed out of other fields and their covariant derivatives. Since in the $AdS_2 \times S^2$ background all such vector fields vanish, the parameters labeling the gauge field strengths are not redefined.

density $\mathcal{L}$. In this section we shall assume that the neither the definition of the field strengths associated with the p-form gauge fields, nor the Lagrangian density has any Chern-Simons terms. Thus the Lagrangian density will be manifestly invariant under general coordinate transformation and gauge transformation of the p-form gauge field $B_{\mu_1\cdots\mu_p}$ of the form

$$\delta B_{\mu_1\cdots\mu_p} = \partial_{[\mu_1}\Lambda_{\mu_2\cdots\mu_p]}\,. \tag{2.4.1}$$

The cases where either the Lagrangian density or the definition of a field strength has a Chern-Simons term will be dealt with separately in §2.6.

In D space-time dimensions the near horizon geometry of a spherically symmetric extremal black hole solution has $SO(2,1)\times SO(D-1)$ isometry. This forces the metric to have the form $AdS_2\times S^{D-2}$. The relevant fields which can take non-trivial expectation values near the horizon are scalars $\{\phi_s\}$, metric $g_{\mu\nu}$, gauge fields $A^{(i)}_\mu$, $(D-3)$-form gauge fields $B^{(a)}_{\mu_1\ldots\mu_{D-3}}$, 2-form gauge fields $C^{(m)}_{\mu\nu}$ and $(D-2)$-form gauge fields $D^{(I)}_{\mu_1\cdots\mu_{D-2}}$. If

$$H^{(a)}_{\mu_1\ldots\mu_{D-2}} = \partial_{[\mu_1}B^{(a)}_{\mu_2\cdots\mu_{D-2}]}\,, \tag{2.4.2}$$

denotes the field strength associated with the field $B^{(a)}$, then the general background consistent with the $SO(2,1)\times SO(D-1)$ symmetry of the background geometry is of the form:

$$\begin{aligned}
&ds^2 \equiv g_{\mu\nu}dx^\mu dx^\nu = v_1\left(-r^2dt^2+\frac{dr^2}{r^2}\right)+v_2\, d\Omega^2_{D-2}\\
&\phi_s = u_s, \quad C^{(m)}_{rt} = w_m, \quad D^{(I)}_{l_1\cdots l_{D-2}} = z_I\,\epsilon_{l_1\cdots l_{D-2}}\sqrt{\det h^{(D-2)}}\,/\Omega_{D-2}\\
&F^{(i)}_{rt} = e_i, \qquad H^{(a)}_{l_1\cdots l_{D-2}} = p_a\,\epsilon_{l_1\cdots l_{D-2}}\sqrt{\det h^{(D-2)}}\,/\Omega_{D-2}\,,
\end{aligned} \tag{2.4.3}$$

where v_1, v_2, $\{u_s\}$, $\{w_m\}$, $\{z_I\}$, $\{e_i\}$ and $\{p_a\}$ are constants parametrizing the background, $d\Omega^2_{D-2}\equiv h^{(D-2)}_{ll'}dx^ldx^{l'}$ denotes the line element on the unit $(D-2)$-sphere, Ω_{D-2} denotes the volume of the unit $(D-2)$-sphere, x^{l_i} with $2\le l_i\le (D-1)$ are coordinates along this sphere and ϵ denotes the totally anti-symmetric symbol with $\epsilon_{2\ldots(D-1)}=1$. Any other k-form field for k different from 1, 2, $D-3$ or $D-2$ will vanish in this background since the only $SO(2,1)\times SO(D-2)$ invariant forms on $AdS_2\times S^{D-2}$ are the 2-form corresponding to the volume form on AdS_2 and the $(D-2)$-form corresponding to the volume form on S^{D-2}. Even among the ones given above, the constant $C^{(m)}_{rt}$ background proportional to w_m can be removed by gauge transformation.

We now define

$$f(\vec u,\vec v,\vec e,\vec p) = \int dx^2\cdots dx^{D-1}\sqrt{-\det g}\,\mathcal{L}\,, \tag{2.4.4}$$

as in (2.2.3). Note that since the lagrangian density depends on the various k-form fields only through their field strength, f is independent of $\vec{w}$ and $\vec{z}$.[6] Analysis identical to that for $D = 4$ now tells us that the constants p_a represent magnetic type charges carried by the black hole, and the equations which determine the values of $\vec{u}$, $\vec{v}$ and $\vec{e}$ are

$$\frac{\partial f}{\partial u_s} = 0, \qquad \frac{\partial f}{\partial v_i} = 0\,, \qquad \frac{\partial f}{\partial e_i} = q_i\,, \tag{2.4.5}$$

where q_i denote the electric charges carried by the black hole. $\vec{w}$ and $\vec{z}$ remain undermined. Also using (2.2.14) which is valid for spherically symmetric black holes in any dimension, we can show that the entropy of the black hole is given by 2π times the Legendre transform of f:

$$S_{BH} = 2\pi \left(e_i \frac{\partial f}{\partial e_i} - f \right)\,. \tag{2.4.6}$$

as in (2.2.21).

As in the case of four dimensional black holes, we can define

$$\mathcal{E}(\vec{u}, \vec{v}, \vec{e}, \vec{q}, \vec{p}) \equiv 2\,\pi(e_i\, q_i - f(\vec{u}, \vec{v}, \vec{e}, \vec{p})) \tag{2.4.7}$$

The equations (2.4.5) determining $\vec{u}$, $\vec{v}$ and $\vec{e}$ are then given by:

$$\frac{\partial \mathcal{E}}{\partial u_s} = 0, \quad \frac{\partial \mathcal{E}}{\partial v_1} = 0\,, \quad \frac{\partial \mathcal{E}}{\partial v_2} = 0\,, \quad \frac{\partial \mathcal{E}}{\partial e_i} = 0\,. \tag{2.4.8}$$

Furthermore (2.4.6) shows that the entropy associated with the black hole is given by:

$$S_{BH} = \mathcal{E}(\vec{u}, \vec{v}, \vec{e}, \vec{q}, \vec{p})\,, \tag{2.4.9}$$

at the extremum (2.4.8).

A useful viewpoint that treats extremal black holes in all dimensions in one go is to regard the S^{D-2} part of the near horizon geometry as a compact space and treat the effective field theory governing the dynamics of the near horizon geometry as two dimensional. The effective Lagrangian density of this two dimensional theory spanned by r and t is given by

$$\sqrt{-\det h}\, \mathcal{L}^{(2)} = \int_{S^{D-2}} \sqrt{-\det g}\, \mathcal{L}\,, \tag{2.4.10}$$

where $g_{\mu\nu}$ and $\mathcal{L}$ denote the original D-dimensional metric and Lagrangian density, and $h_{\alpha\beta}$ and $\mathcal{L}^{(2)}$ denote the two dimensional metric and two dimensional Lagrangian density obtained via dimensional

[6] As we shall see in §2.6, this situation will change once we allow Chern-Simons terms.

reduction. Only non-trivial degrees of freedom in this two dimensional theory are the metric, gauge fields and scalars coming from the dimensional reduction of various D-dimensional fields on S^{D-2}. The magnetic charges p_a labeling the flux of the $(D-2)$-form field strengths through S^{D-2} appear as parameters in this two dimensional theory. The most general near horizon configuration consistent with $SO(2,1)$ isometry of AdS_2 will have an AdS_2 metric, constant two dimensional gauge field strengths and constant scalars. By regarding this as the near horizon geometry of a two dimensional extremal black hole, we can write the Wald's formula for the entropy as

$$S_{BH} = -8\pi \, \frac{\delta \mathcal{S}}{\delta R^{(2)}_{rtrt}} \sqrt{-h_{rr}\, h_{tt}} \Big|_{\rm Horizon} \,, \tag{2.4.11}$$

where $\mathcal{S} \equiv \int dr dt \sqrt{-\det h} \, \mathcal{L}^{(2)}$ now is to be regarded as a two dimensional action. For extremal black holes all covariant derivatives of scalar fields and field strengths vanish, and we have the analog of (2.2.14)

$$S_{BH} = 8\pi \, \frac{\partial \mathcal{L}^{(2)}}{\partial R^{(2)}_{rtrt}} \, h_{rr}\, h_{tt} \Big|_{\rm Horizon} \,. \tag{2.4.12}$$

If we now define

$$f = \sqrt{-\det h} \, \mathcal{L}^{(2)} \Big|_{\rm Horizon} \tag{2.4.13}$$

and

$$\mathcal{E} = 2\pi (q_i e_i - f) \,, \tag{2.4.14}$$

then one can show, following the same procedure as in §2.2, that the parameters labeling the near horizon background are obtained by extremizing $\mathcal{E}$ with respect to these various parameters, and furthermore that the entropy itself is given by the function $\mathcal{E}$ evaluated at its extremum.

The arguments of §2.3 can now be used to prove attractor behaviour of these black holes. We shall see in §2.5 that the two dimensional viewpoint provides a useful tool for proving the attractor behaviour of extremal rotating black holes as well.

2.5 Rotating black holes in $D = 4$

In this section we shall describe the construction of the entropy function for rotating black holes in four dimensions following the analysis performed in [24]. The results can be easily generalized to higher dimensions. Early work on attractor mechanism for rotating black holes has been carried out in [33, 34, 35].

As in §2.2 we begin by considering a general four dimensional theory of gravity coupled to a set of abelian gauge fields $A_\mu^{(i)}$ and neutral scalar fields $\{\phi_s\}$ with action

$$\mathcal{S} = \int d^4x \, \sqrt{-\det g} \, \mathcal{L} \, , \tag{2.5.1}$$

where $\sqrt{-\det g}\, \mathcal{L}$ is the lagrangian density, expressed as a function of the metric $g_{\mu\nu}$, the Riemann tensor $R_{\mu\nu\rho\sigma}$, the scalar fields $\{\phi_s\}$, the gauge field strengths $F^{(i)}_{\mu\nu} = \partial_\mu A^{(i)}_\nu - \partial_\nu A^{(i)}_\mu$, and covariant derivatives of these fields. In general $\mathcal{L}$ will contain terms with more than two derivatives. We now define a rotating extremal black hole solution to be one whose near horizon geometry has the symmetries of $AdS_2 \times S^1$, – this holds for known extremal rotating black hole solutions [36,24] and has recently been proven in [23]. The most general field configuration consistent with the $SO(2,1)\times U(1)$ symmetry of $AdS_2 \times S^1$ is of the form:[7]

$$\begin{aligned}
& ds^2 \equiv g_{\mu\nu} dx^\mu dx^\nu = v_1(\theta) \left(-r^2 dt^2 + \frac{dr^2}{r^2} \right) + \beta^2 \, d\theta^2 + \beta^2 \, v_2(\theta) (d\phi + \alpha r dt)^2 \\
& \phi_s = u_s(\theta) \\
& \frac{1}{2} \, F^{(i)}_{\mu\nu} dx^\mu \wedge dx^\nu = (e_i + \alpha b_i(\theta)) dr \wedge dt + \partial_\theta b_i(\theta) d\theta \wedge (d\phi + \alpha r dt) \, ,
\end{aligned} \tag{2.5.2}$$

where α, β and e_i are constants, and v_1, v_2, u_s and b_i are functions of θ. Here ϕ is a periodic coordinate with period 2π and θ takes value in the range $0 \le \theta \le \pi$. The SO(2,1) isometry of AdS_2 is generated by the Killing vectors [36]:

$$L_1 = \partial_t, \qquad L_0 = t\partial_t - r\partial_r, \qquad L_{-1} = (1/2)(1/r^2 + t^2)\partial_t - (tr)\partial_r - (\alpha/r)\partial_\phi \, . \tag{2.5.3}$$

A simple way to see the $SO(2,1) \times U(1)$ symmetry of the configuration (2.5.2) is as follows. The $U(1)$ transformation acts as a translation of ϕ and is clearly a symmetry of this configuration. In order to see the SO(2,1) symmetry of this background we regard ϕ as a compact direction and interprete this as a theory in three dimensions labelled by coordinates $\{x^m\} \equiv (r, \theta, t)$ with metric $\hat{g}_{mn}$, gauge fields $a^{(i)}_m$ and a_m and scalar fields ψ and χ_i defined through the relations

$$\begin{aligned}
ds^2 &= \hat{g}_{mn} \, dx^m dx^n + \psi (d\phi + a_m dx^m)^2 \\
A^{(i)}_\mu dx^\mu &= a^{(i)}_m dx^m + \chi_i (d\phi + a_m dx^m) \, .
\end{aligned} \tag{2.5.4}$$

Besides these we also have scalar fields ϕ_s descending down from four dimensions. If we denote by $f^{(i)}_{mn}$ and f_{mn} the field strengths associated with the three dimensional gauge fields $a^{(i)}_m$ and a_m respectively,

[7] Our convention for α differs from the one used in [24] by a minus sign. With this new convention the variable J conjugate to α will represent angular momentum in the standard convention.

then the background (2.5.2) can be interpreted as the following three dimensional background:

$$\begin{aligned}
&\widehat{ds}^2 \equiv \hat{g}_{mn} dx^m dx^n = v_1(\theta) \left(-r^2 dt^2 + \frac{dr^2}{r^2} \right) + \beta^2 \, d\theta^2 \\
&\phi_s = u_s(\theta), \qquad \psi = \beta^2 \, v_2(\theta), \quad \chi_i = b_i(\theta) \, , \\
&\frac{1}{2} \, f^{(i)}_{mn} dx^m \wedge dx^n = e_i \, dr \wedge dt, \qquad \frac{1}{2} \, f_{mn} dx^m \wedge dx^n = \alpha \, dr \wedge dt \, .
\end{aligned} \tag{2.5.5}$$

The (r, t) coordinates now describe an AdS_2 space and this background is manifestly $SO(2, 1)$ invariant. In this description the Killing vectors take the standard form

$$L_1 = \partial_t, \qquad L_0 = t\partial_t - r\partial_r, \qquad L_{-1} = (1/2)(1/r^2 + t^2)\partial_t - (tr)\partial_r \, . \tag{2.5.6}$$

Let us now return to the four dimensional viewpoint. For the configuration given in (2.5.2) the magnetic charge associated with the ith gauge field is given by

$$p_i = \int d\theta d\phi F^{(i)}_{\theta\phi} = 2\pi(b_i(\pi) - b_i(0)) \, . \tag{2.5.7}$$

Since an additive constant in b_i can be absorbed into the parameters e_i, we can set $b_i(0) = -p_i/4\pi$. This, together with (2.5.7), now gives

$$b_i(0) = -\frac{p_i}{4\pi}, \qquad b_i(\pi) = \frac{p_i}{4\pi} \, . \tag{2.5.8}$$

We shall assume that the deformed horizon, labelled by the coordinates θ and ϕ, is a smooth deformation of the sphere. In particular there should be no conical defects near $\theta = 0, \pi$. We shall further assume that the gauge field strengths and scalar fields are also smooth at $\theta = 0$, π. This requires

$$\begin{aligned}
v_2(\theta) &= \theta^2 + \mathcal{O}(\theta^4) \quad \text{for } \theta \simeq 0 \\
&= (\pi - \theta)^2 + \mathcal{O}((\pi - \theta)^4) \quad \text{for } \theta \simeq \pi \, ,
\end{aligned} \tag{2.5.9}$$

$$\begin{aligned}
b_i(\theta) &= -\frac{p_i}{4\pi} + \mathcal{O}(\theta^2) \quad \text{for } \theta \simeq 0 \\
&= \frac{p_i}{4\pi} + \mathcal{O}((\pi - \theta)^2) \quad \text{for } \theta \simeq \pi \, ,
\end{aligned} \tag{2.5.10}$$

$$\begin{aligned}
u_s(\theta) &= u_s(0) + \mathcal{O}(\theta^2) \quad \text{for } \theta \simeq 0 \\
&= u_s(\pi) + \mathcal{O}((\pi - \theta)^2) \quad \text{for } \theta \simeq \pi \, .
\end{aligned} \tag{2.5.11}$$

Note that the smoothness of the background requires the Taylor series expansion around $\theta = 0, \pi$ to contain only even powers of θ and $(\pi - \theta)$ respectively. This can be seen by expressing the solutions near $\theta = 0$ or $\theta = \pi$ using the local Cartesian coordinates $(x, y) = (\sin\theta\cos\phi, \sin\theta\sin\phi)$ and requiring the solution to be non-singular and invariant under ϕ translation in this coordinate system.

Eq.(2.5.5) and hence (2.5.2) describes the most general field configuration consistent with the $SO(2,1) \times U(1)$ symmetry. Thus in order to derive the equations of motion we can evaluate the action on this background and then extremize the resulting expression with respect to the parameters labeling the background (2.5.2). The only exception to this are the parameters e_i and α labeling the field strengths. From the three dimensional viewpoint we see that the background (2.5.5) automatically satisfies the equations of motion of the gauge fields $a_m^{(i)}$ and a_m. Thus the variation of the action with respect to the parameters e_i and α need not vanish, – instead they give the corresponding conserved electric charges q_i and the angular momentum J (which can be regarded as the electric charge associated with the three dimensional gauge field a_m.)

To implement this procedure we define:

$$f[\alpha, \beta, \vec{e}, v_1(\theta), v_2(\theta), \vec{u}(\theta), \vec{b}(\theta)] = \int d\theta d\phi \sqrt{-\det g}\, \mathcal{L}\,. \tag{2.5.12}$$

Note that f is a function of α, β, e_i and a functional of $v_1(\theta)$, $v_2(\theta)$, $u_s(\theta)$ and $b_i(\theta)$. The equations of motion now correspond to

$$\frac{\partial f}{\partial \alpha} = J, \quad \frac{\partial f}{\partial \beta} = 0, \quad \frac{\partial f}{\partial e_i} = q_i\,, \quad \frac{\delta f}{\delta v_1(\theta)} = 0\,, \quad \frac{\delta f}{\delta v_2(\theta)} = 0, \quad \frac{\delta f}{\delta u_s(\theta)} = 0, \quad \frac{\delta f}{\delta b_i(\theta)} = 0\,. \tag{2.5.13}$$

Equivalently, if we define:

$$\mathcal{E}[J, \vec{q}, \alpha, \beta, \vec{e}, v_1(\theta), v_2(\theta), \vec{u}(\theta), \vec{b}(\theta)] = 2\pi\left(J\alpha + \vec{q}\cdot\vec{e} - f[\alpha, \beta, \vec{e}, v_1(\theta), v_2(\theta), \vec{u}(\theta), \vec{b}(\theta)]\right)\,, \tag{2.5.14}$$

then the equations of motion take the form:

$$\frac{\partial \mathcal{E}}{\partial \alpha} = 0, \quad \frac{\partial \mathcal{E}}{\partial \beta} = 0, \quad \frac{\partial \mathcal{E}}{\partial e_i} = 0\,, \quad \frac{\delta \mathcal{E}}{\delta v_1(\theta)} = 0\,, \quad \frac{\delta \mathcal{E}}{\delta v_2(\theta)} = 0, \quad \frac{\delta \mathcal{E}}{\delta u_s(\theta)} = 0, \quad \frac{\delta \mathcal{E}}{\delta b_i(\theta)} = 0\,. \tag{2.5.15}$$

These equations are subject to the boundary conditions (2.5.9), (2.5.10), (2.5.11). For formal arguments it will be useful to express the various functions of θ appearing here by expanding them as a linear combination of appropriate basis states which make the constraints (2.5.9), (2.5.10) manifest, and then varying $\mathcal{E}$ with respect to the coefficients appearing in this expansion. The natural functions in terms of which we can expand an arbitrary ϕ-independent function on a sphere are the Legendre polynomials $P_l(\cos\theta)$. We take

$$v_1(\theta) = \sum_{l=0}^{\infty} \widetilde{v}_1(l)\, P_l(\cos\theta)\,, \quad v_2(\theta) = \sin^2\theta + \sin^4\theta \sum_{l=0}^{\infty} \widetilde{v}_2(l)\, P_l(\cos\theta)\,,$$

$$u_s(\theta) = \sum_{l=0}^{\infty} \widetilde{u}_s(l)\, P_l(\cos\theta)\,, \quad b_i(\theta) = -\frac{p_i}{4\pi}\cos\theta + \sin^2\theta \sum_{l=0}^{\infty} \widetilde{b}_i(l)\, P_l(\cos\theta)\,. \tag{2.5.16}$$

This expansion explicitly implements the constraints (2.5.9), (2.5.10) and (2.5.11). Substituting this into (2.5.14) gives $\mathcal{E}$ as a function of J, q_i, α, β, e_i, $\widetilde{v}_1(l)$, $\widetilde{v}_2(l)$, $\widetilde{u}_s(l)$ and $\widetilde{b}_i(l)$. Thus the equations (2.5.15) may now be reexpressed as

$$\frac{\partial\mathcal{E}}{\partial\alpha} = 0, \quad \frac{\partial\mathcal{E}}{\partial\beta} = 0, \quad \frac{\partial\mathcal{E}}{\partial e_i} = 0\,, \quad \frac{\partial\mathcal{E}}{\partial\widetilde{v}_1(l)} = 0\,, \quad \frac{\partial\mathcal{E}}{\partial\widetilde{v}_2(l)} = 0, \quad \frac{\partial\mathcal{E}}{\partial\widetilde{u}_s(l)} = 0, \quad \frac{\partial\mathcal{E}}{\partial\widetilde{b}_i(l)} = 0\,. \tag{2.5.17}$$

Let us now turn to the analysis of the entropy associated with this black hole. For this it will be most convenient to regard this configuration as a two dimensional extremal black hole by regarding the θ and ϕ directions as compact. In this interpretation the zero mode of the metric $\widehat{g}_{\alpha\beta}$ given in (2.5.5), with $\alpha, \beta = r, t$, is interpreted as the two dimensional metric $h_{\alpha\beta}$:

$$h_{\alpha\beta} = \frac{1}{2}\int_0^{\pi} d\theta\, \sin\theta\, \widehat{g}_{\alpha\beta}\,, \tag{2.5.18}$$

whereas all the non-zero modes of $\widehat{g}_{\alpha\beta}$ are interpreted as massive symmetric rank two tensor fields. This gives

$$h_{\alpha\beta}dx^\alpha dx^\beta = v_1(-r^2dt^2 + dr^2/r^2)\,, \qquad v_1 \equiv \widetilde{v}_1(0)\,. \tag{2.5.19}$$

Thus the near horizon configuration, regarded from two dimensions, involves AdS_2 metric, accompanied by background electric fields $f^{(i)}_{\alpha\beta}$ and $f_{\alpha\beta}$, a set of massless and massive scalar fields with vacuum expectation values $\widetilde{u}_s(l)$, $\widetilde{v}_2(l)$, $\widetilde{b}_i(l)$, and a set of massive symmetric rank two tensor fields with vacuum expectations values $\widetilde{v}_1(l)h_{\alpha\beta}/\widetilde{v}_1(0)$. According to the Wald formula [5, 25, 26, 27], the entropy of this black hole is given by:

$$S_{BH} = -8\pi\, \frac{\delta\mathcal{S}^{(2)}}{\delta R^{(2)}_{rtrt}}\, \sqrt{-h_{rr}\, h_{tt}}\,, \tag{2.5.20}$$

where $R^{(2)}_{\alpha\beta\gamma\delta}$ is the two dimensional Riemann tensor associated with the metric $h_{\alpha\beta}$, and $\mathcal{S}^{(2)}$ is the general coordinate invariant action of this two dimensional field theory. We now note that for this two dimensional configuration that we have, the electric field strengths $f^{(i)}_{\alpha\beta}$ and $f_{\alpha\beta}$ are proportional to the volume form on AdS_2, the scalar fields are constants and the tensor fields are proportional to the AdS_2 metric. Thus the covariant derivatives of all gauge and generally covariant tensors which one can construct out of these two dimensional fields vanish. In this case (2.5.20) simplifies to:

$$S_{BH} = -8\pi\sqrt{-\det h}\, \frac{\partial\mathcal{L}^{(2)}}{\partial R^{(2)}_{rtrt}}\, \sqrt{-h_{rr}\, h_{tt}} \tag{2.5.21}$$

where $\sqrt{-\det h}\,\mathcal{L}^{(2)}$ is the two dimensional Lagrangian density, related to the four dimensional Lagrangian density via the formula:

$$\sqrt{-\det h}\,\mathcal{L}^{(2)} = \int d\theta d\phi \sqrt{-\det g}\,\mathcal{L}\,. \tag{2.5.22}$$

Also while computing (2.5.21) we set to zero all terms in $\mathcal{L}^{(2)}$ which involve covariant derivatives of the Riemann tensor, gauge field strengths, scalars and the massive tensor fields.

We can now proceed in a manner identical to that in §2.2, §2.4 to show that the right hand side of (2.5.21) is the entropy function at its extremum. First of all from (2.5.19) it follows that

$$R^{(2)}_{rtrt} = v_1 = \sqrt{-h_{rr}h_{tt}}\,. \tag{2.5.23}$$

Using this we can express (2.5.21) as

$$S_{BH} = -8\pi\,\sqrt{-\det h}\frac{\partial \mathcal{L}^{(2)}}{\partial R^{(2)}_{rtrt}} R^{(2)}_{rtrt}\,. \tag{2.5.24}$$

Let us denote by $\mathcal{L}^{(2)}_\lambda$ a deformation of $\mathcal{L}^{(2)}$ in which we replace all factors of $R^{(2)}_{\alpha\beta\gamma\delta}$ for $\alpha,\beta,\gamma,\delta = r,t$ by $\lambda R^{(2)}_{\alpha\beta\gamma\delta}$, and define

$$f^{(2)}_\lambda \equiv \sqrt{-\det h}\,\mathcal{L}^{(2)}_\lambda\,, \tag{2.5.25}$$

evaluated on the near horizon geometry. Then

$$\lambda\frac{\partial f^{(2)}_\lambda}{\partial\lambda} = \sqrt{-\det h}\,R^{(2)}_{\alpha\beta\gamma\delta}\frac{\partial \mathcal{L}^{(2)}}{\delta R^{(2)}_{\alpha\beta\gamma\delta}} = 4\,\sqrt{-\det h}\,R^{(2)}_{rtrt}\frac{\partial \mathcal{L}^{(2)}}{\partial R^{(2)}_{rtrt}}\,. \tag{2.5.26}$$

Using this (2.5.24) may be rewritten as

$$S_{BH} = -2\pi\lambda\frac{\partial f^{(2)}_\lambda}{\partial\lambda}\Big|_{\lambda=1}\,. \tag{2.5.27}$$

Next we consider the effect of the scaling

$$\lambda\to s\lambda, \quad e_i\to se_i, \quad \alpha\to s\alpha, \quad \widetilde{v}_1(l)\to s\widetilde{v}_1(l) \quad \text{for} \quad 0\le l<\infty\,, \tag{2.5.28}$$

under which $\lambda\,R^{(2)}_{\alpha\beta\gamma\delta}\to s^2\,\lambda\,R^{(2)}_{\alpha\beta\gamma\delta}$. Since $\mathcal{L}^{(2)}$ does not involve any explicit covariant derivatives, all indices of $h^{\alpha\beta}$ must contract with the indices in $f^{(i)}_{\alpha\beta}$, $f_{\alpha\beta}$, $R^{(2)}_{\alpha\beta\gamma\delta}$ or the indices of the massive rank two symmetric tensor fields whose near horizon values are proportional to the parameters $\widetilde{v}_1(l)$. From this and the definition of the parameters e_i, $\widetilde{v}_1(l)$, and α it follows that $\mathcal{L}^{(2)}_\lambda$ remains invariant under this scaling, and hence $f_\lambda^{(2)}$ transforms to $s f_\lambda^{(2)}$, with the overall factor of s coming from the $\sqrt{-\det h}$ factor in the definition of $f_\lambda^{(2)}$. Thus we have:

$$\lambda \frac{\partial f_\lambda^{(2)}}{\partial \lambda} + e_i \frac{\partial f_\lambda^{(2)}}{\partial e_i} + \alpha \frac{\partial f_\lambda^{(2)}}{\partial \alpha} + \sum_{l=0}^{\infty} \widetilde{v}_1(l) \frac{\partial f_\lambda^{(2)}}{\partial \widetilde{v}_1(l)} = f_\lambda^{(2)} \,. \tag{2.5.29}$$

Now it follows from (2.5.12), (2.5.22) and (2.5.25) that

$$f[\alpha, \beta, \vec{e}, v_1(\theta), v_2(\theta), \vec{u}(\theta), \vec{b}(\theta)] = f_{\lambda=1}^{(2)} \,. \tag{2.5.30}$$

Thus the extremization equations (2.5.13) implies that

$$\frac{\partial f_\lambda^{(2)}}{\partial e_i} = q_i, \quad \frac{\partial f_\lambda^{(2)}}{\partial \alpha} = J, \quad \frac{\partial f_\lambda^{(2)}}{\partial \widetilde{v}_1(l)} = 0 \,, \qquad \text{at } \lambda = 1 \,. \tag{2.5.31}$$

Hence setting $\lambda = 1$ in (2.5.29) we get

$$\lambda \frac{\partial f_\lambda^{(2)}}{\partial \lambda}\Big|_{\lambda=1} = -e_i q_i - J\alpha + f_{\lambda=1}^{(2)} = -e_i q_i - J\alpha + f[\alpha, \beta, \vec{e}, v_1(\theta), v_2(\theta), \vec{u}(\theta), \vec{b}(\theta)] \,. \tag{2.5.32}$$

Eqs.(2.5.27) and the definition (2.5.14) of the entropy function now gives

$$S_{BH} = \mathcal{E} \tag{2.5.33}$$

at its extremum.

The arguments of §2.3 can now be used to prove attractor behaviour of these black holes, i.e. the black hole entropy depends only on the charges $\{q_i, p_i\}$ and the angular momentum J but not on any other asymptotic data.

For practical computations it is often useful to work with the functions $v_i(\theta)$, $u_s(\theta)$ and $b_i(\theta)$ instead of their mode decompositions given in (2.5.16). In this case the extremization of the entropy functional $\mathcal{E}$ with respect to these functions, as described in eqs.(2.5.15), would give rise to a set of ordinary differential equations in θ for these functions, and the entropy is obtained by evaluating the entropy function at a solution of these equations. In order to carry out this procedure we need to carefully keep track of all the boundary terms that arise in the expression for the entropy function. This has been discussed in detail in [24] where we have also illustrated the general method by applying it to a specific class of rotating black holes in string theory studied in [37, 38, 39, 40].

Finally, one interesting question that arises for rotating black holes is whether the horizon can have non-spherical topology, *e.g.* the topology of a torus. Although in two derivative theories the horizon of a four dimensional black hole is known to have spherical topology, once higher derivative terms are added to the action there may be other possibilities. Our analysis can be easily generalized to the case where the horizon has the topology of a torus rather than a sphere. All we need to do is to take the θ coordinate to be a periodic variable with period 2π and expand the various functions in the basis of periodic functions of θ. However if the near horizon geometry is invariant under both ϕ and θ translations, then in the expression for L_{-1} given in (2.5.3) we could add a term of the form $-(\gamma/r)\partial_\theta$, and the entropy could have an additional dependence on the charge conjugate to the variable γ. This represents the Noether charge associated with θ translation, but does not correspond to a physical charge from the point of view of the asymptotic observer since the full solution is not invariant under θ translation. These are commonly known as dipole charges. The entropy function method can also be used to compute entropies of higher dimensional black holes with non-spherical horizons where such solutions are known to exist even for two derivative action [41, 42].

2.6 Dealing with Chern-Simons terms

The analysis in the previous sections relies on several important assumptions about the structure of the Lagrangian density. In particular we have assumed that

1. The lagrangian density depends on the metric in a manifestly covariant manner, namely the only dependence on the metric comes via the metric, Riemann tensor and covariant derivatives of various tensor fields, but does not have any explicit dependence on spin connections and Christoffel symbols.

2. For any p-form gauge field B present in the theory, the covariant field strength has the form $H = dB$ so that H satisfies the Bianchi identity $dH = 0$. If this is not the case, then a field configuration of the form given in (2.4.3) will not automatically satisfy the Bianchi identity, and we shall get additional constraints on the parameters labeling the near horizon background by requiring that H satisfies the Bianchi identity.

3. The dependence on the gauge fields $A^{(i)}_\mu$ and more generally on the $(D-3)$-form gauge fields $B^{(a)}$ appears through their field strengths. Otherwise we shall encounter the following problems:

   (a) If the lagrangian density had any explicit dependence on the gauge fields then the gauge field equations of motion would not take the form given in (2.2.5).

   (b) While making the ansatz (2.4.3) we have taken the gauge field strengths $F^{(i)}_{rt}$ and $H^{(a)}_{l_1\cdots l_{D-2}}$ to be invariant under the symmetries of $AdS_2 \times S^{D-2}$, but the gauge fields themselves do not carry the symmetry. As a result the Lagrangian density evaluated in the background will be invariant under the symmetries of $AdS_2 \times S^{D-2}$ only if it does not involve explicitly the gauge fields and is a function only of the field strengths. Otherwise the extremization of the entropy function may not give the complete set of independent equations of motion.

The type of Lagrangian densities which appear in low energy effective action of string theory often violates one or more of these conditions. In particular the Lagrangian density often involves Chern-Simons forms, which are totally antisymmetric tensors $\Omega_{\mu_1\ldots\mu_n}$ which depend on one or more lower rank gauge fields or spin connetions / Christoffel symbols rather than just on the field strengths. As a result Ω is not invariant under the gauge transformation associated with these lower rank gauge fields. Nevertheless Ω has the special property that the variation of Ω under various gauge transformations are exact forms:

$$\delta\Omega_{\mu_1\ldots\mu_n} = \partial_{[\mu_1}\chi_{\mu_2\ldots\mu_n]}\,, \tag{2.6.1}$$

for some quantity χ. As a result $\partial_{[\mu_1}\Omega_{\mu_2\ldots\mu_{n+1}]}$ is a covariant tensor.

A simple example of such a Chern-Simons term is as follows. Suppose the theory contains a p-form gauge field $B^{(1)}_{\mu_1\ldots\mu_p}$ and a q-form gauge field $B^{(2)}_{\mu_1\ldots\mu_q}$, with associated gauge transformations of the form

$$\delta B^{(1)}_{\mu_1\ldots\mu_p} = \partial_{[\mu_1}\Lambda^{(1)}_{\mu_2\ldots\mu_p]} \qquad \delta B^{(2)}_{\mu_1\ldots\mu_q} = \partial_{[\mu_1}\Lambda^{(2)}_{\mu_2\ldots\mu_q]}\,. \tag{2.6.2}$$

Then the $(p+q+1)$-form

$$\Omega_{\mu_1\ldots\mu_{p+q+1}} = B^{(1)}_{[\mu_1\ldots\mu_p}\,\partial_{\mu_{p+1}}B^{(2)}_{\mu_{p+2}\ldots\mu_{p+q+1}]} \tag{2.6.3}$$

transforms by a total derivative of the form (2.6.1) under the gauge transformation induced by $\Lambda^{(1)}$. Thus $\Omega_{\mu_1\ldots\mu_{p+q+1}}$ defined in (2.6.3) is a Chern-Simons $(p+q+1)$-form.

The Chern-Simons terms could appear in the expression for the lagrangian density in two different ways:

1. The action itself may contain a Chern-Simons term of the form

$$\int d^D x\, \epsilon^{\mu_1\ldots\mu_D}\,\Omega_{\mu_1\ldots\mu_D}\,. \tag{2.6.4}$$

   Since $\delta\Omega$ is a total derivative, an action of this form is gauge invariant up to boundary terms. In the presence of such a term the Lagrangian density may fail to satisfy our assumptions on three counts. First of all since the Lagrangian density is not gauge invariant, it may not be invariant under the symmetries of $AdS_2 \times S^{D-2}$ when evaluated in the near horizon background

(2.4.3). Second, since the lagrangian density may now depend explicitly on the gauge fields and not just their field strengths, the equations of motion of the gauge fields may no longer be of the form (2.2.5). Finally, if the Chern-Simons form explicitly involves Christoffel symbol or spin connection, then even Wald's formula for the entropy is not directly applicable.[8]

2. In some theories the gauge invariant field strength associated with an antisymmetric tensor field $B_{\mu_1\ldots\mu_{n-1}}$ is given by

$$H_{\mu_1\ldots\mu_n} = \partial_{[\mu_1} B_{\mu_2\ldots\mu_n]} + \Omega_{\mu_1\ldots\mu_n} \tag{2.6.5}$$

for some Chern-Simons n-form Ω constructed out of lower dimensional gauge fields and spin connection. Under the gauge transformation (2.6.1), $B_{\mu_1\ldots\mu_{n-1}}$ is assigned the transformation

$$\delta B_{\mu_1\ldots\mu_{n-1}} = -\chi_{\mu_1\ldots\mu_{n-1}} \, , \tag{2.6.6}$$

so that

$$\delta H_{\mu_1\ldots\mu_n} = 0 \, . \tag{2.6.7}$$

A typical example of such a term is the 3-form field strength associated with the NS sector 2-form gauge field of heterotic string theory. The definition of the three form field strength involves both gauge and Lorentz Chern-Simons 3-forms. In such cases the Lagrangian density, being a function of $H_{\mu_1\ldots\mu_n}$, is invariant under the gauge transformation (2.6.1), (2.6.6). Nevertheless, since the definition of $H_{\mu_1\ldots\mu_n}$ involves various lower rank gauge fields and not just their field strengths, the presence of such terms in the Lagrangian density violates our assumptions on three counts. First the Bianchi identity of $H_{\mu_1\ldots\mu_n}$, being of the form

$$\partial_{[\mu_1} H_{\mu_2\ldots\mu_{n+1}]} = \partial_{[\mu_1} \Omega_{\mu_2\ldots\mu_{n+1}]} \, , \tag{2.6.8}$$

now could give additional constraints on the near horizon parameters besides the ones obtained by entropy function extremization condition. Second since the definition of $H_{\mu_1\ldots\mu_n}$ involves explicitly lower rank gauge fields, the equations of motion of the gauge fields may no longer be of the form (2.2.5). Finally, if the Chern-Simons form explicitly involves Christoffel symbol or spin connection, then Wald's formula for the entropy is not directly applicable.

In order to deal with these Chern-Simons terms we shall proceed in two steps. First we shall show that the second type of Chern-Simons terms described above, where it appears in the definition of a field strength, can be transformed to the first type. This will involve generalizing the analysis in [44].

[8] For some recent work on application of Wald's formula in the presence of Chern-Simons term see [43].

We shall then describe a general procedure for dealing with the first type of Chern-Simons term [45]. In carrying out these manipulations we shall need to add total derivative terms to the Lagrangian density. Since addition of such term do not affect the equations of motion we expect that the entropy computed from the new Lagrangian density will continue to describe entropy of extremal black holes in the original theory.

The first step is carried out as follows. Suppose in a theory in D dimensions we have an $(n-1)$-form field B whose field strength H contains a Chern-Simons term as in (2.6.5), but $\mathcal{L}$ depends on B only through its field strength H. We now introduce a new $(D-n-1)$-form field $\mathcal{B}_{\mu_1\ldots\mu_{D-n-1}}$ and define its field strength to be

$$\mathcal{H}_{\mu_1\ldots\mu_{D-n}} = \partial_{[\mu_1}\mathcal{B}_{\mu_2\ldots\mu_{D-n}]} \,. \tag{2.6.9}$$

$\mathcal{H}_{\mu_1\ldots\mu_{D-n}}$ is invariant under a gauge transformation

$$\delta\mathcal{B}_{\mu_1\ldots\mu_{D-n-1}} = \partial_{[\mu_1}\Lambda_{\mu_2\ldots\mu_{D-n-1}]} \,. \tag{2.6.10}$$

We now consider a new Lagrangian density

$$\sqrt{-\det g}\,\widetilde{\mathcal{L}} = \sqrt{-\det g}\,\mathcal{L} + \epsilon^{\mu_1\cdots\mu_D}\left(H_{\mu_1\ldots\mu_n} - \Omega_{\mu_1\ldots\mu_n}\right)\mathcal{H}_{\mu_{n+1}\mu_{n+2}\cdots\mu_D}\,, \tag{2.6.11}$$

and treat $H_{\mu_1\ldots\mu_n}$ and $\mathcal{B}_{\mu_1\ldots\mu_{D-n-1}}$ as independent fields. In this case the equation of motion of the $\mathcal{B}_{\mu_1\ldots\mu_{D-n-1}}$ field gives

$$\epsilon^{\mu_1\cdots\mu_D}\partial_{\mu_1}\left(H_{\mu_2\ldots\mu_{n+1}} - \Omega_{\mu_2\ldots\mu_{n+1}}\right) = 0\,, \tag{2.6.12}$$

whose general solution is of the form (2.6.5). On the other hand the equation of motion of $H_{\mu_1\ldots\mu_n}$ associated with the new action has the form

$$\frac{\delta\mathcal{S}}{\delta H_{\mu_1\ldots\mu_n}} + \epsilon^{\mu_1\cdots\mu_D}\,\partial_{\mu_{n+1}}\,\mathcal{B}_{\mu_{n+2}\ldots\mu_D} = 0\,. \tag{2.6.13}$$

This gives

$$\partial_{\mu_1}\frac{\delta\mathcal{S}}{\delta H_{\mu_1\ldots\mu_n}} = 0\,, \tag{2.6.14}$$

which is the equation of motion of the field $B_{\mu_1\ldots\mu_{n-1}}$ in the original theory. Furthermore, the equation of motion of any other field $\psi(x)$ computed from the new action $\int\sqrt{-\det g}\widetilde{\mathcal{L}}$ is the same as the one derived from the original action $\mathcal{S} = \int\sqrt{-\det g}\mathcal{L}$. To see this note that (2.6.11) gives an equation of motion of ψ of the form:

$$\left.\frac{\delta\mathcal{S}}{\delta\psi(x)}\right|_H - \int d^Dy\epsilon^{\mu_1\cdots\mu_D}\frac{\delta\Omega_{\mu_1\cdots\mu_n}(y)}{\delta\psi(x)}\mathcal{H}_{\mu_{n+1}\cdots\mu_D}(y) = 0\,, \tag{2.6.15}$$

where the subscript $_H$ denotes that we need to carry out the functional derivative treating $H_{\mu_1\cdots\mu_n}$ as an independent field. On the other hand the original equations of motion derived from the action $\mathcal{S}$, where we treat $B_{\mu_1\ldots\mu_{n-1}}$ as independent field, may be expressed as

$$\frac{\delta \mathcal{S}}{\delta\psi(x)}\bigg|_H + \int d^D y \frac{\delta\mathcal{S}}{\delta H_{\mu_1\cdots\mu_n}} \frac{\delta\Omega_{\mu_1\cdots\mu_n}(y)}{\delta\psi(x)} = 0\,, \tag{2.6.16}$$

where we have taken into account the fact that there may be a hidden dependence of $\mathcal{S}$ on ψ through the Chern-Simons form Ω in the definition of H. Using (2.6.13) one can verify that (2.6.15) and (2.6.16) are identical. Thus (2.6.11) is classically equivalent to the original Lagrangian density and we can use this new Lagrangian density to carry out the computation of black hole entropy in this theory.

Since $H_{\mu_1\ldots\mu_n}$ is now an independent field, and since the field strength $\mathcal{H}$ is defined as in (2.6.9), we see that in the new theory the definition of field strengths do not contain any Chern-Simons term. However the last term in the Lagrangian density,

$$-\,\epsilon^{\mu_1\ldots\mu_D}\,\Omega_{\mu_1\ldots\mu_n}\,\mathcal{H}_{\mu_{n+1}\mu_{n+2}\ldots\mu_D}\,, \tag{2.6.17}$$

is a Chern-Simons term. Thus the new Lagrangian density is of type 1 where the Lagrangian density has an explicit Chern-Simons term.

Let us now proceed to analyze Lagrangian densities of type 1. We shall find it useful to use the notation of differential forms rather than tensors. Chern-Simons terms which appear in string theory Lagrangian density have one of two forms

$$B^{(1)} \wedge dB^{(2)} \wedge \ldots dB^{(s)} \tag{2.6.18}$$

or

$$dB^{(1)} \wedge dB^{(2)} \wedge \ldots dB^{(s)} \wedge \Omega_{3L} \tag{2.6.19}$$

where $B^{(i)}$ are r_i-form fields with associated (r_i-1)-form gauge transformations as in (2.6.2), and Ω_{3L} is the Lorentz Chern-Simons 3-form.[9] The first term is manifestly invariant under the gauge transformations associated with the $B^{(2)}$, $B^{(3)}$, $\ldots$ $B^{(s)}$ fields, but fails to be invariant under the gauge transformation associated with the $B^{(1)}$ field. However by adding a total derivative term to the Lagrangian density we can transfer the exterior derivative from any of the other $B^{(i)}$ fields to $B^{(1)}$. In this case the term will be manifestly invariant under the gauge transformation associated with $B^{(1)}$,

[9]We are assuming that the relevant rank 1 gauge fields are abelian so that their Chern-Simons terms are of the form (2.6.18).

but will fail to be invariant under one of the other gauge transformations. The second term (2.6.19) is manifestly invariant under all the $B^{(i)}$ gauge transformations, but fails to be invariant under the local Lorentz transformation. Again by adding a total derivative term we can transfer the exterior deivative from one of the $B^{(i)}$'s to the Lorentz Chern-Simons term. The resulting Lagrangian density will be manifestly general coordinate and local Lorentz invariant since $d\Omega_{3L}$ transforms covariantly under these transformations, but will fail to be invariant under one of the $B^{(i)}$ gauge transformations.

Our proposal for dealing with the terms given in (2.6.18) and (2.6.19) is to dimensionally reduce the theory to two dimensions by regarding the sphere S^{D-2} as a compact direction, express the resulting action as the integral of a covariant Lagrangian density in two dimensions spanned by the r and t coordinates and then calculate its contribution to the entropy function in the usual manner. After dimensional reduction the magnetic flux p_a through S^{D-2} will appear as parameters labeling the two dimensional theory. Since we are interested in only $SO(D-1)$ invariant field configuration, the dimensional reduction is a straightforward process except in cases where the Lagrangian density, evaluated in the $SO(D-1)$ invariant background, has a term that is not manifestly $SO(D-1)$ invariant. This would happen if the Lagrangian density either contains a Lorentz Chern-Simons term which, evaluated for the sphere metric, is not manifestly $SO(D-1)$ invariant, or depends explicitly on a $B^{(i)}$ whose field strength $dB^{(i)}$ has a non-zero flux through S^{D-2} since in this case $B^{(i)}$ itself does not remain invariant under an $SO(D-1)$ rotation. Our strategy will be to avoid these terms to whatever extent possible by adding total derivative to the Lagrangian density *before dimensional reduction* to transfer the derivatives in appropriate places. Thus if in (2.6.18) and/or (2.6.19) there is even a single $B^{(i)}$ which does not have an associated flux through S^{D-2}, we can take the Lagrangian density in a form where that particular $B^{(i)}$ appears without a derivative, and every other factor has a manifestly covariant form. In this case the Lagrangian density evaluated for a generic $SO(D-1)$ invariant background will have a manifestly $SO(D-1)$ invariant form. Thus the only cases where the dimensional reduction is complicated is the one where all the $B^{(i)}$'s have flux through S^{D-2}. This requires all the $B^{(i)}$'s appearing in (2.6.18) and/or (2.6.19) to be $(D-3)$-form so that their field strengths are $(D-2)$-forms. Since the Lagrangian density must be a D-form, for (2.6.18) this gives

$$s(D-2)-1=D \qquad \text{i.e.} \qquad s=\frac{D+1}{D-2}\,. \tag{2.6.20}$$

On the other hand for (2.6.19) this gives

$$s(D-2)+3=D \qquad \text{i.e.} \qquad s=\frac{D-3}{D-2}\,. \tag{2.6.21}$$

First let us deal with the case (2.6.20). Since s must be an integer, the only possible cases are $D = 3$, $s = 4$ and $D = 5$, $s = 2$. For simplicity we shall explain how to deal with the second case; the analysis of the first case will proceed in an identical manner. The relevant term in the Lagrangian density is proportional to

$$B^{(1)} \wedge dB^{(2)} \,, \tag{2.6.22}$$

where $B^{(1)}$ and $B^{(2)}$ are 2-form fields. Note that $B^{(1)}$ and $B^{(2)}$ must be different fields since $B \wedge dB$ is a total derivative for any even form field B. Suppose $dB^{(1)}$ and $dB^{(2)}$ have flux p_1 and p_2 through S^3. We now define new 2-form fields

$$C^{(1)} = \frac{p_2 B^{(1)} - p_1 B^{(2)}}{\sqrt{p_1^2 + p_2^2}}, \qquad C^{(2)} = \frac{p_1 B^{(1)} + p_2 B^{(2)}}{\sqrt{p_1^2 + p_2^2}} \,. \tag{2.6.23}$$

In terms of these fields (2.6.22) can be wrtten as

$$C^{(1)} \wedge dC^{(2)} \tag{2.6.24}$$

plus total derivative terms. Furthermore the field $C^{(1)}$ has no flux through S^3 and $C^{(2)}$ has a flux proportional to $\sqrt{p_1^2 + p_2^2}$. Since the Lagrangian density does not involve $C^{(2)}$ explicitly, we can carry out the dimensional reduction of this term in a straightforward fashion, and get a term proportional to

$$\sqrt{p_1^2 + p_2^2}\; C^{(1)} \,, \tag{2.6.25}$$

in the two dimensional theory. Since the 2-form field $C^{(1)}$ can be regarded as a scalar field density in the two dimensional theory, (2.6.25) has a manifestly covariant form in two dimensions and we can use entropy function formalism to analyze extremal black hole solutions in this theory.

We note in passing that (2.6.25) is the only term in the two dimensional Lagrangian density which depends explicitly on $C^{(1)}$; the rest of the Lagrangian density depends on $C^{(1)}$ through $dC^{(1)}$ and hence vanishes in two dimensions. Requiring the action to be stationary with respect to $C^{(1)}$ then gives $p_1 = p_2 = 0$. This shows that in the presence of a term of the form (2.6.22) we cannot have an extremal black hole solution with magnetic charges associated with the $B^{(1)}$ and $B^{(2)}$ fields.

We now turn to a discussion of terms of the form (2.6.19) for which the only problematic case is (2.6.21). Requiring s to be integer gives $s = 0$, $D = 3$ as the only case. This corresponds to the presence of a gravitational Chern-Simons term in the three dimensional theory [46, 47]:

$$\sqrt{-\det g}\,\mathcal{L}^{(3)}_{CS} = K\,\epsilon^{\mu\nu\tau}\left[\frac{1}{2}\widehat{\Gamma}^{\rho}_{\mu\sigma}\partial_\nu\widehat{\Gamma}^{\sigma}_{\tau\rho} + \frac{1}{3}\widehat{\Gamma}^{\rho}_{\mu\sigma}\widehat{\Gamma}^{\sigma}_{\nu\eta}\widehat{\Gamma}^{\eta}_{\tau\rho}\right] \,, \tag{2.6.26}$$

where $\widehat{\Gamma}^{\mu}_{\nu\rho}$ denotes the Christoffel symbol and K is a constant. To deal with this term, we regard the horizon S^1 as a compact direction and carry out the dimensional reduction of this term by taking the ansatz:

$$g_{\mu\nu}dx^{\mu}dx^{\nu} = \phi\left[g^{(2)}_{\alpha\beta}dx^{\alpha}dx^{\beta} + (dy + a_{\alpha}dx^{\alpha})^2\right] . \tag{2.6.27}$$

Here $g^{(2)}_{\alpha\beta}$ $(0 \leq \alpha, \beta \leq 1)$ denotes a two dimensional metric, a_{α} denotes a two dimensional gauge field and ϕ denotes a two dimensional scalar field. The y coordinate labeling the horizon S^1 is taken to have period 2π. In terms of these two dimensional fields the lagrangian density (2.6.26), after dimensional reduction to two dimensions by integration over the y coordinate, takes the form [48,45]:

$$K\,\pi\,\left[\frac{1}{2}R^{(2)}\varepsilon^{\alpha\beta}f_{\alpha\beta} + \frac{1}{2}\varepsilon^{\alpha\beta}f_{\alpha\gamma}f^{\gamma\delta}f_{\delta\beta}\right] , \tag{2.6.28}$$

plus total derivative terms. Here $f_{\alpha\beta} = \partial_{\alpha}a_{\beta} - \partial_{\beta}a_{\alpha}$ is the field strength associated with the two dimensional gauge field a_{α}, $R^{(2)}$ denotes the Ricci scalar constructed out of the two dimensional metric $g^{(2)}_{\alpha\beta}$ and $\varepsilon^{\alpha\beta}$ is the totally anti-symmetric symbol with $\epsilon^{01} = 1$. Since the Lagrangian density (2.6.28) has a manifestly covariant form in two dimensions, we can apply the entropy function formalism on this lagrangian density. This will be illustrated in more detail in the context of BTZ black holes in §3.4.

This concludes our discussion on Chern-Simons terms in the context of spherically symmetric black holes in arbitrary dimensions. A similar analysis may be carried out for rotating black holes, but we shall not discuss this case here.

3 Explicit Computation of Black Hole Entropy

In this section we shall illustrate the entropy function formalism of §2 by using it to calculate the entropy of extremal black holes in a variety of theories. Many of these results can also be derived from other methods; in each of these cases the result obtained using entropy function method (naturally) agrees with the ones obtained by other methods. The analysis of this section will serve the twin purpose of illustrating the entropy function formalism and deriving specific results for black hole entropy which will later be compared with the statistical entropy in §5.

3.1 Black holes in $\mathcal{N} = 4$ supersymmetric theories in $D = 4$

There are various string compactifications which lead to $\mathcal{N} = 4$ supersymmetric theories in four dimensions. These theories have many scalar fields, known as moduli fields, whose potential vanishes identically by the requirement of supersymmetry. Thus they can take arbitrary vacuum expectation values (vev), and the space of vev of these scalar fields describe the moduli space. At a generic point in the moduli space the requirement of $\mathcal{N}=4$ supersymmetry completely determines the massless field content of the theory in terms of a single integer $r \geq 6$ which labels the number of $U(1)$ gauge fields in the theory. In particular the massless bosonic fields are the string metric $G_{\mu\nu}$, r abelian gauge fields $A_\mu^{(i)}$ $(i = 1, \ldots r)$, a complex scalar field $a + iS$ taking value in the upper half plane, and a set of $r \times r$ matrix valued scalar fields M subject to the constraint:

$$MLM^T = L, \quad M^T = M\,, \tag{3.1.1}$$

where L is a matrix with 6 eigenvalues $+1$ and $(r-6)$ eigenvalues -1. For $r \geq 12$, a convenient choice of L is

$$L = \begin{pmatrix} 0_6 & I_6 & \\ I_6 & 0_6 & \\ & & -I_{r-12} \end{pmatrix}\,, \tag{3.1.2}$$

where I_k denotes a $k \times k$ identity matrix and 0_k denote $k \times k$ zero matrix. The canonical metric $g_{\mu\nu}$ is related to the string metric $G_{\mu\nu}$ via the relation $G_{\mu\nu} = S g_{\mu\nu}$.

Our analysis in this section will mostly follow [4, 8].

3.1.1 Supergravity approximation

Requirement of $\mathcal{N}=4$ supersymmetry also fixes all the terms in the action containing at most two derivatives. The part of the action containing the massless bosonic fields is given by

$$\begin{aligned} \mathcal{S} &= \frac{1}{2\pi\alpha'} \int d^4x \sqrt{-\det G}\, S \,\Big[R_G + \frac{1}{S^2} G^{\mu\nu} (\partial_\mu S \partial_\nu S - \frac{1}{2} \partial_\mu a \partial_\nu a) + \frac{1}{8} G^{\mu\nu} Tr(\partial_\mu M L \partial_\nu M L) \\ & \quad - G^{\mu\mu'} G^{\nu\nu'} F^{(i)}_{\mu\nu} (LML)_{ij} F^{(j)}_{\mu'\nu'} - \frac{a}{S} G^{\mu\mu'} G^{\nu\nu'} F^{(i)}_{\mu\nu} L_{ij} \widetilde{F}^{(j)}_{\mu'\nu'} \Big]\,. \end{aligned} \tag{3.1.3}$$

Note that this action has an $SO(6, r-6)$ symmetry acting on M and $F^{(i)}_{\mu\nu}$:

$$M \to \Omega M \Omega^T, \quad F^{(i)}_{\mu\nu} \to \Omega_{ij} F^{(j)}_{\mu\nu}\,, \tag{3.1.4}$$

where Ω is an $r \times r$ matrix satisfying

$$\Omega^T L \Omega = L\,. \tag{3.1.5}$$

This corresponds to the continuous T-duality symmetry of the supergravity action. As will be discussed in §3.1.3, an appropriate discrete subgroup of this is an exact symmetry of the full string theory.

In this theory we look for a spherically symmetric extremal black hole solution carrying arbitrary electric charges q_i and magnetic charges p_i for $i = 1, \cdots r$. Following the analysis of §2.2 we look for a near horizon field configuration of the form:

$$\begin{aligned} ds^2 &= \frac{\alpha'}{16} v_1 \left(-r^2 dt^2 + \frac{dr^2}{r^2} \right) + \frac{\alpha'}{16} v_2 (d\theta^2 + \sin^2\theta\, d\phi^2)\,, \\ S &= u_S, \qquad a = u_a, \qquad M_{ij} = u_{Mij} \\ F^{(i)}_{rt} &= \frac{\sqrt{\alpha'}}{4}\, e_i, \quad F^{(i)}_{\theta\phi} = \frac{p_i\sqrt{\alpha'}}{16\pi}\, \sin\theta\,, \end{aligned} \tag{3.1.6}$$

where, for later convenience, we have included additional factors of $\alpha'/16$ multiplying v_1 and v_2 and additional factors of $\sqrt{\alpha'}/4$ multiplying e_i and p_i.[10] The effect of this will be to change the definition of the electric and magnetic charges. Eq.(3.1.6) agrees with the corresponding equations in [4] for $\alpha' = 16$. Substituting (3.1.6) into (3.1.3) and using (2.2.3) we get

$$\begin{aligned} & f(u_S, u_a, u_M, \vec v, \vec e, \vec p) \equiv \int d\theta d\phi \sqrt{-\det G}\, \mathcal{L} \\ = \ & \frac{1}{8} v_1\, v_2\, u_S \left[-\frac{2}{v_1} + \frac{2}{v_2} + \frac{2}{v_1^2} e_i (L u_M L)_{ij} e_j - \frac{1}{8\pi^2 v_2^2} p_i (L u_M L)_{ij} p_j + \frac{u_a}{\pi u_S v_1 v_2} e_i L_{ij} p_j \right]. \end{aligned} \tag{3.1.7}$$

Eq.(2.2.9) now gives

$$\begin{aligned} \mathcal{E}(u_S, u_a, u_M, \vec v, \vec e, \vec q, \vec p) &\equiv 2\pi \left(e_i q_i - f(u_S, u_a, u_M, \vec v, \vec e, \vec p) \right) \\ &= 2\pi \Bigg[e_i q_i - \frac{1}{8} v_1\, v_2\, u_S \Big\{ -\frac{2}{v_1} + \frac{2}{v_2} + \frac{2}{v_1^2} e_i (L u_M L)_{ij} e_j \\ &\qquad - \frac{1}{8\pi^2 v_2^2} p_i (L u_M L)_{ij} p_j + \frac{u_a}{\pi u_S v_1 v_2} e_i L_{ij} p_j \Big\} \Bigg]. \end{aligned} \tag{3.1.8}$$

Eliminating e_i from (3.1.8) using the equation $\partial\mathcal{E}/\partial e_i = 0$ we get:

$$\begin{aligned} \mathcal{E}(u_S, u_a, u_M, \vec v, \vec q, \vec p) &= 2\pi \Big[\frac{u_S}{4}(v_2 - v_1) + \frac{v_1}{v_2 u_S}\, q^T u_M q + \frac{v_1}{64\pi^2 v_2 u_S} (u_S^2 + u_a^2) p^T L u_M L p \\ &\qquad - \frac{v_1}{4\pi v_2 u_S}\, u_a\, q^T u_M L p \Big]. \end{aligned} \tag{3.1.9}$$

We can simplify the formulæ by defining new charge vectors:

$$Q_i = 2 q_i, \qquad P_i = \frac{1}{4\pi}\, L_{ij} p_j\,. \tag{3.1.10}$$

[10] We shall use the convention that the coordinates r, t, θ, ϕ and all the scalar fields are dimensionless, the gauge fields have dimension of length and the metric has dimension of length^2. With this convention the near horizon parameters v_1, v_2, u_a, u_S, u_{Mij}, e_i and p_j are dimensionless.

In terms of $\vec{Q}$ and $\vec{P}$ the entropy function $\mathcal{E}$ is given by:

$$\mathcal{E} = \frac{\pi}{2}\Big[u_S(v_2 - v_1) + \frac{v_1}{v_2 u_S}\left(Q^T u_M Q + (u_S^2 + u_a^2)\, P^T u_M P - 2\, u_a\, Q^T u_M P\right)\Big]\,. \tag{3.1.11}$$

We now need to find the extremum of $\mathcal{E}$ with respect to u_S, u_a, u_{Mij}, v_1 and v_2. In general this leads to a complicated set of equations. However we can simplify the analysis by noting that (3.1.4) induces the following transformation on the various parameters:

$$\begin{gathered} e_i \to \Omega_{ij} e_j, \qquad p_i \to \Omega_{ij} p_j, \qquad u_M \to \Omega u_M \Omega^T\,, \\ q_i \to (\Omega^T)^{-1}_{ij} q_j\,, \qquad Q_i \to (\Omega^T)^{-1}_{ij} Q_j, \qquad P_i \to (\Omega^T)^{-1}_{ij} P_j\,. \end{gathered} \tag{3.1.12}$$

The entropy function (3.1.11) is invariant under these transformations. Since at its extremum with respect to u_{Mij} the entropy function depends only on $\vec{P}$, $\vec{Q}$, v_1, v_2, u_S and u_a it must be a function of the $SO(6, r-6)$ invariant combinations:

$$Q^2 = Q_i L_{ij} Q_j, \quad P^2 = P_i L_{ij} P_j, \quad Q\cdot P = Q_i L_{ij} P_j\,, \tag{3.1.13}$$

besides v_1, v_2, u_S and u_a. Let us for definiteness take $Q^2 > 0$, $P^2 > 0$, and $(Q\cdot P)^2 < Q^2 P^2$. In that case with the help of an $SO(6, r-6)$ transformation we can make

$$(I_r - L)_{ij} Q_j = 0, \quad (I_r - L)_{ij} P_j = 0\,, \tag{3.1.14}$$

where I_r denotes the $r\times r$ identity matrix. This is most easily seen by diagonalizing L to the form $\begin{pmatrix} I_6 & \\ & -I_{r-6}\end{pmatrix}$. In this case $\vec{Q}$ and $\vec{P}$ satisfying (3.1.14) will have

$$Q_i = 0, \quad P_i = 0, \quad \text{for } 7 \le i \le r\,. \tag{3.1.15}$$

We shall now show that for $\vec{P}$ and $\vec{Q}$ satisfying this condition, every term in (3.1.11) is extremized with respect to u_M for

$$u_M = I_r\,. \tag{3.1.16}$$

Clearly a variation δu_{Mij} with either i or j in the range $[7, r]$ will give vanishing contribution to each term in $\delta\mathcal{E}$ computed from (3.1.11). On the other hand due to the constraint (3.1.1) on M, any variation δM_{ij} (and hence δu_{Mij}) with $1 \le i, j \le 6$ must vanish, since in this subspace (3.1.1) requires M to be both symmetric and orthogonal. Thus each term in $\delta\mathcal{E}$ vanishes under all the allowed variations of u_M.

(3.1.16) is not the only possible value of u_M that extremizes $\mathcal{E}$. Any u_M related to (3.1.16) by an $SO(6, r-6)$ transformation that preserves the vectors $\vec{Q}$ and $\vec{P}$ will extremize $\mathcal{E}$. Thus there is a family of extrema representing flat directions of $\mathcal{E}$. However as we have argued in §2.3, the value of the entropy is independent of the choice of u_M.

We note in passing that the entropy function (3.1.11) is also invariant under continuous S-duality transformation

$$\begin{pmatrix} Q' \\ P' \end{pmatrix} = \begin{pmatrix} m & n \\ r & s \end{pmatrix} \begin{pmatrix} Q \\ P \end{pmatrix}, \quad u'_a + iu'_S = \frac{m(u_a+iu_S)+n}{r(u_a+iu_S)+s}, \quad v'_i = \frac{u_S}{u'_S} v_i \,, \tag{3.1.17}$$

where m, n, r, s are real numbers satisfying $ms - nr = 1$. This is a reflection of the continuous S-duality covariance of the equations of motion derived from the action (3.1.3). We shall not make use of this symmetry in our analysis. However a discrete subgroup of this, to be introduced in §3.1.3, is an exact symmetry of string theory and will play an important role in our analysis.

Substituting (3.1.16) into (3.1.11) and using (3.1.13), (3.1.14), we get:

$$\mathcal{E} = \frac{\pi}{2}\left[u_S(v_2 - v_1) + \frac{v_1}{v_2}\left\{ \frac{Q^2}{u_S} + \frac{P^2}{u_S}(u_S^2 + u_a^2) - 2\,\frac{u_a}{u_S}\,Q\cdot P \right\}\right]. \tag{3.1.18}$$

Note that we have expressed the right hand side of this equation in an $SO(2,2)$ invariant form. Written in this manner, eq.(3.1.18) is valid for general $\vec{P}$, $\vec{Q}$ satisfying

$$P^2 > 0, \quad Q^2 > 0, \quad (Q\cdot P)^2 < Q^2 P^2 \,. \tag{3.1.19}$$

It remains to extremize $\mathcal{E}$ with respect to v_1, v_2, u_S and u_a. Extremization with respect to v_1 and v_2 give:

$$v_1 = v_2 = u_S^{-2}\,\left(Q^2 + P^2(u_S^2 + u_a^2) - 2u_a\,Q\cdot P\right). \tag{3.1.20}$$

Substituting this into (3.1.18) gives:

$$\mathcal{E} = \frac{\pi}{2}\frac{1}{u_S}\,\left\{Q^2 - 2\,u_a\,Q\cdot P + P^2(u_S^2 + u_a^2)\right\}. \tag{3.1.21}$$

Finally extremizing this with respect to u_a, u_S we get

$$u_S = \frac{\sqrt{Q^2P^2 - (Q\cdot P)^2}}{P^2}\,, \qquad u_a = \frac{Q\cdot P}{P^2}, \qquad v_1 = v_2 = 2P^2\,. \tag{3.1.22}$$

The black hole entropy, given by the value of $\mathcal{E}$ for this configuration, is

$$S_{BH} = \pi\,\sqrt{Q^2P^2 - (Q\cdot P)^2}\,. \tag{3.1.23}$$

Although this formula has been derived under the condition $P^2 > 0$, $Q^2 > 0$, $P^2Q^2 > (Q\cdot P)^2$, the final result for the entropy also holds for arbitrary P^2, Q^2 as long as $P^2Q^2 > (Q\cdot P)^2$. Extremal black holes with $P^2Q^2 > (Q\cdot P)^2$ are known to be supersymmetric although we cannot see this in our analysis. The entropy function formalism also allows us to calculate the entropy of extremal black holes with $P^2Q^2 < (Q\cdot P)^2$. We shall not go through the analysis here, but just quote the final result:

$$S_{BH} = \pi\sqrt{(Q\cdot P)^2 - Q^2P^2}\,. \tag{3.1.24}$$

3.1.2 A special class of $\mathcal{N}=4$ supersymmetric string theories

Our goal is to study the effect of higher derivative terms in the action on the entropy function. However, unlike in the case of two derivative terms, the higher derivative corrections do depend on the details of the string compactification that gives rise to the theory. Thus in order to study the effect of higher derivative corrections we need to consider specific theories. We shall restrict our analysis to a class of string theories where we begin with type IIB string theory on $\mathcal{M}\times\widetilde{S}^1\times S^1$ where $\mathcal{M}$ is either K3 or T^4 and $\widetilde{S}^1$ and S^1 are two circles, and take an orbifold of this theory by a $\mathbb{Z}_N$ symmetry group. The generator g of the $\mathbb{Z}_N$ group involves $1/N$ unit of shift along the circle S^1 together with an order N transformation $\widetilde{g}$ in $\mathcal{M}$. $\widetilde{g}$ is chosen so that it commutes with the $\mathcal{N}=4$ supersymmetry generators of the parent theory. Thus the final theory has $\mathcal{N}=4$ supersymmetry. Various properties of $\widetilde{g}$ coming from this requirement have been discussed in appendix B. In particular this requires $\mathcal{M}/\widetilde{g}$ to be an orbifold of $SU(2)$ holonomy.

The description of the theory given above will be referred to as the first description of the theory. Another useful description is obtained by making an S-duality transformation in the type IIB theory on $\mathcal{M}\times\widetilde{S}^1\times S^1$ that exchanges the NS 5-branes with D5-branes and fundamental strings with D-strings, followed by an $R\to 1/R$ duality on the circle $\widetilde{S}^1$ that takes type IIB theory on $\widetilde{S}^1$ to type IIA theory on the dual circle $\widehat{S}^1$, and then using six dimensional string-string duality to relate this to a heterotic string theory on $T^4\times\widehat{S}^1\times S^1$ for $\mathcal{M}=K3$ [49, 50, 51, 52, 53] and type IIA string theory on $T^4\times\widehat{S}^1\times S^1$ for $\mathcal{M}=T^4$ [54]. Under this duality the transformation $\widetilde{g}$ gets mapped to a transformation $\widehat{g}$ that acts only as a shift on the right-moving degrees of freedom on the world-sheet and as a shift plus rotation on the left-moving degrees of freedom. In the final theory, obtained by taking the orbifold of heterotic or type IIA string theory on $T^4\times\widehat{S}^1\times S^1$ by a $1/N$ unit of shift along S^1 together with the transformation $\widehat{g}$, all the space-time supersymmetries come from the right-moving sector of the world-sheet. We shall call this the second description of the theory.

The heterotic models were first discovered in the analysis of [55,56,57,58,59]. The type II versions were analyzed in [54].

At the level of two derivative terms, the effective action of each of these theories is given by (3.1.3) for appropriate choice of r determined from the details of compactification. The precise expression for r is given by

$$r = 2\,k + 8\,, \tag{3.1.25}$$

where k has been computed in appendix C (eq.(C.17)) and is equal to half the number of $\widetilde{g}$ invariant harmonic $(1,1)$ forms on $\mathcal{M}$. Explicit form of k for special cases can be found in (C.36), (C.38).

In order to facilitate later analysis where we compare the black hole entropy with the statistical entropy, it will be useful to know the correspondence between the various fields and charges appearing in (3.1.3) with the physical fields and charges in string theory. First of all the field $\tau = a + iS$ corresponds to the complex structure modulus of the torus $\widetilde{S}^1 \times S^1$ in the first description. By following the duality chain carefully one can see that it represents the usual axion-dilaton field in the second description, $-\,a$ being the scalar field obtained by dualizing the anti-symmetric tensor field in the NSNS sector, and S being $e^{-2\Phi}$ where Φ denotes the dilaton field. The matrix valued scalar field M encodes information about the shape and size of the compact space $T^4 \times \widehat{S}^1 \times S^1$ and the components of the NSNS sector 2-form field along $T^4 \times \widehat{S}^1 \times S^1$ in the second description. Finally in the second description the gauge fields appearing in the action (3.1.3) can be related directly to the ones coming from the dimensional reduction of the ten dimensional metric, NSNS sector anti-symmetric tensor field and gauge fields, without any further electric-magnetic duality transformation. As a result the elementary string states in this description carry electric charge vector $\vec{Q}$ and the various solitons carry magnetic charge vector $\vec{P}$. We shall carry on the rest of the discussion in this section in the second description, but following the chain of dualities relating the two descriptions one can easily work out the interpretation of various quantities in the first description.

Let x^4 and x^5 denote the coordinates along $\widehat{S}^1$ and S^1 respectively, both normalized to have period $2\pi\sqrt{\alpha'}$ after the $\mathbb{Z}_N$ orbifolding, and let x^μ $(0 \le \mu \le 3)$ denote the non-compact coordinates. For most of our analysis it will be useful to study in detail a subsector of the theory in which we include only those gauge fields which are associated with the 4μ and 5μ components of the metric and the anti-symmetric tensor field, only those components of M which encode information about the components of the metric and the anti-symmetric tensor field along $\widehat{S}^1 \times S^1$, the axion-dilaton field, and the four dimensional metric. In this subsector there are four gauge fields $A^{(i)}_\mu$ $(1 \le i \le 4)$ and a 4×4 matrix valued field M satisfying

$$M^T = M, \quad M L M^T = L, \quad L \equiv \begin{pmatrix} 0 & I_2 \\ I_2 & 0 \end{pmatrix}. \tag{3.1.26}$$

The fields $A^{(i)}_\mu$ and M are related to the ten dimensional string metric G_{MN} and 2-form field B_{MN}

via the relations [60, 61]:

$$
\begin{aligned}
&\widehat{G}_{mn} \equiv G^{(10)}_{mn}, \quad \widehat{B}_{mn} \equiv B^{(10)}_{mn}, \qquad m, n = 4, 5, \\
&M = \begin{pmatrix} \widehat{G}^{-1} & \widehat{G}^{-1} B \\ -\widehat{B}\widehat{G}^{-1} & \widehat{G} - \widehat{B}\widehat{G}^{-1}\widehat{B} \end{pmatrix} \\
&A^{(m-3)}_{\mu} = \frac{1}{2} (\widehat{G}^{-1})^{mn} G^{(10)}_{m\mu}, \quad A^{(m-1)}_{\mu} = \frac{1}{2} B^{(10)}_{m\mu} - \widehat{B}_{mn} A^{(m-3)}_{\mu}, \\
&\qquad 4 \le m, n \le 5, \quad 0 \le \mu, \nu \le 3 .
\end{aligned} \tag{3.1.27}
$$

The Lagrangian density involving the axion-dilaton field, the four gauge fields and the 4×4 matrix valued scalar field M has a form identical to the one given in (3.1.3) with L given as in (3.1.26). In fact this is a consistent truncation of the full $\mathcal{N} = 4$ supergravity theory.

With the normalization convention we have used for the charges $\vec{P}$ and $\vec{Q}$, and the sign conventions described in appendix A, a state with $\widehat{n}$ unit of momentum and $-\widehat{w}$ unit of winding along $\widehat{S}^1$, n' unit of momentum and $-w'$ unit of winding along S^1, $\widehat{N}$ unit of Kaluza-Klein monopole charge [62, 63] associated with $\widehat{S}^1$, $-\widehat{W}$ unit of NS 5-brane wrapped along $T^4 \times S^1$, N' unit of Kaluza-Klein monopole charge associated with S^1 and W' unit of NS 5-brane wrapped along $T^4 \times \widehat{S}^1$ describes a four dimensional charge vector of the form[11]

$$
Q = \begin{pmatrix} \widehat{n} \\ n' \\ \widehat{w} \\ w' \end{pmatrix}, \qquad P = \begin{pmatrix} \widehat{W} \\ W' \\ \widehat{N} \\ N' \end{pmatrix} . \tag{3.1.28}
$$

Thus we have

$$
Q^2 = 2(\widehat{n}\widehat{w} + n'w'), \qquad P^2 = 2(\widehat{N}\widehat{W} + N'W'), \qquad P \cdot Q = \widehat{N}\widehat{n} + \widehat{W}\widehat{w} + N'n' + W'w' . \tag{3.1.29}
$$

We shall denote by $\mathcal{V}$ the subspace spanned by charge vectors of the form (3.1.28).

The sign conventions for various charges have been described in detail in appendix A. Here we shall say a few words about the normalization of the various charges appearing in (3.1.28). First of all units of momentum along S^1 and $\widehat{S}^1$ will be taken to be $1/\sqrt{\alpha'}$. We take $\widehat{S}^1$ to have coordinate radius $2\pi\sqrt{\alpha'}$ and S^1 to have coordinate radius $2\pi\sqrt{\alpha'}N$ before orbifolding. Thus after orbifolding $S^1/\mathbb{Z}_N$ has coordinate radius $2\pi\sqrt{\alpha'}$, and various fields satisfy $\widehat{g}$ twisted boundary condition under

[11] A Kaluza-Klein monopole associated with a circle denotes a Taub-NUT space whose asymptotic geometry is the product of this circle and the three non-compact spatial directions. Also an NS 5-brane wrapped along $T^4 \times S^1$ will be said to carry one unit of H-monopole charge associated with $\widehat{S}^1$ and an NS 5-brane wrapped along $T^4 \times \widehat{S}^1$ will be said to carry -1 unit of H-monopole charge associated with S^1 [64].

a translation by $2\pi\sqrt{\alpha'}$ along S^1. As a result the momentum along S^1 is quantized in units of $1/(N\sqrt{\alpha'})$. Similar conventions are followed for all other quantum numbers. One unit of winding along S^1 will refer to a state such that as we go once around the string, its coordinate along S^1 shifts by $2\pi\sqrt{\alpha'}$. This represents a twisted sector state. An untwisted sector state whose coordinate along S^1 changes by multiples of $2\pi\sqrt{\alpha'}N$ will carry winding charge w' in multiples of N. A single H-monopole associated with S^1, with $W'=1$, will correspond to an array of NS 5-branes wrapped on $\widehat{S}^1 \times T^4$ and placed at intervals of $2\pi\sqrt{\alpha'}$ along S^1. Finally the original Kaluza-Klein monopole represented by a Taub-NUT space with an asymptotic circle of radius $N\sqrt{\alpha'}$, after the orbifolding, will develop a $\mathbb{Z}_N$ singularity at its centre and has to be regarded as carrying N units of Kaluza-Klein monopole charge associated with S^1. Thus the Kaluza-Klein monopole charge N' will be quantized in units of N. The charges $\widehat{n}$, $\widehat{w}$, $\widehat{N}$ and $\widehat{W}$ are all quantized in integer units since $\widehat{S}^1$ has period $2\pi\sqrt{\alpha'}$ and the orbifold group does not act on $\widehat{S}^1$. Similar convention must also be followed in the definition of various fields. For example in defining the matrix valued field M and the gauge fields $A^{(i)}_\mu$ via eq.(3.1.27), the coordinates x^4 and x^5 must be chosen so that x^4 has period $2\pi\sqrt{\alpha'}$ and x^5 has period $2\pi\sqrt{\alpha'}N$ before orbifolding.

We must also follow the same convention in identifying fields in the first description. For example if the physical radii of S^1 and $\widetilde{S}^1$ are R_0 and $\widetilde{R}$ before orbifolding, then the field $\tau = a + iS$ has to be regarded as the complex structure modulus of the torus $\widetilde{S}^1 \times S^1$ with S^1 direction regarded as having period $2\pi R_0/N$. Thus we shall have $\sqrt{a^2+S^2} = R_0/(N\widetilde{R})$, and $\tan^{-1}(a/S)$ will be given by the angle between the two circles.

3.1.3 Duality symmetries

The $\mathcal{N}=4$ supersymmetric string theories discussed here have T- and S-duality symmetries induced from the duality symmetries of the parent theory before orbifolding. Since classification of duality symmetries into T- and S-dualities depends on the description of the system we are using, we shall follow the convention that unless mentioned otherwise, whenever we refer to T- or S-duality symmetry of the theory we shall imply T- or S-duality symmetry in the second description. Similarly whenever we mention electric or magnetic charges we shall imply electric or magnetic charges in the second description. With this convention a general T-duality transformation acts non-trivially on the charges and the matrix valued scalar field M as:

$$M \to \Omega M \Omega^T, \quad Q \to (\Omega^T)^{-1} Q, \quad P \to (\Omega^T)^{-1} P\,, \tag{3.1.30}$$

where Ω is an $r \times r$ matrix that preserves the charge lattice and satisfies

$$\Omega L \Omega^T = L\,, \tag{3.1.31}$$

L being a matrix with 6 eigenvalues $+1$ and $(r-6)$ eigenvalues -1. Since $L^2 = 1$ it follows from (3.1.31) that $\Omega^T L \Omega = L$.

Since much of our analysis will involve states with electric and magnetic charges of the form given in (3.1.28), we shall explicitly determine the part of the T-duality group that acts on this subspace. This is the T-duality group associated with the torus $\widehat{S}^1 \times S^1$ in the second description. Before taking the $\mathbb{Z}_N$ orbifold this T-duality group was $SL(2, \mathbb{Z}) \times SL(2, \mathbb{Z})$. Taking the orbifold preserves a subgroup of this group which commutes with the orbifold action, i.e. commutes with translation by $2\pi\sqrt{\alpha'}$ along S^1 up to translations by $2\pi\sqrt{\alpha'}N$ and $2\pi\sqrt{\alpha'}$ along S^1 and $\widehat{S}^1$ respectively. This turns out to be isomorphic to the group $\Gamma_1(N) \times \Gamma_1(N)$ where $\Gamma_1(N)$ consists of 2×2 matrices of the form $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ satisfying

$$ad - bc = 1, \quad a, d \in N\,\mathbb{Z} + 1, \quad b \in \mathbb{Z}, \quad c \in N\,\mathbb{Z}\,. \tag{3.1.32}$$

The matrix Ω is expressed in terms of the pair of $\Gamma_1(N)$ matrices $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and $\begin{pmatrix} p & q \\ r & s \end{pmatrix}$ as

$$(\Omega^T)^{-1} = \begin{pmatrix} d & -c & 0 & 0 \\ -b & a & 0 & 0 \\ 0 & 0 & a & b \\ 0 & 0 & c & d \end{pmatrix} \begin{pmatrix} p & 0 & 0 & -q \\ 0 & p & q & 0 \\ 0 & r & s & 0 \\ -r & 0 & 0 & s \end{pmatrix}\,. \tag{3.1.33}$$

It is straightforward to verify that this matrix satisfies (3.1.31) and preserves the charge quantization laws described below (3.1.29).

If in the second description the theory is an asymmetric orbifold of heterotic string theory then there is an additional $\mathbb{Z}_2$ duality symmetry represented by the matrix

$$\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}\,. \tag{3.1.34}$$

Physically this represents the effect of $R \to 1/R$ duality on the circle $\widehat{S}^1$.

The S-duality symmetry of the theory is best described in the first description where it corresponds to the global diffeomorphism symmetry associated with the torus $\widetilde{S}^1 \times S^1$. Before orbifolding this symmetry is $SL(2, \mathbb{Z})$. However only those elements of $SL(2, \mathbb{Z})$ which commute with the orbifold group generator g are true symmetries of the theory. This again gives rise to the group $\Gamma_1(N)$ consisting of 2×2 matrices $\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$ satisfying the constraints:

$$\alpha\delta - \beta\gamma = 1, \quad \alpha, \delta \in N\,\mathbb{Z} + 1, \quad \beta \in \mathbb{Z}, \quad \gamma \in N\,\mathbb{Z}\,. \tag{3.1.35}$$

Its action on the axion-dilaton modulus $\tau = a + iS$ and the charges is given by

$$\tau \to \frac{\alpha\tau + \beta}{\gamma\tau + \delta}, \qquad \begin{pmatrix} Q \\ P \end{pmatrix} \to \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \begin{pmatrix} Q \\ P \end{pmatrix}. \tag{3.1.36}$$

The entropy function (3.1.18) obtained in the leading supergravity approximation is invariant under both these symmetries.

3.1.4 Corrections due to Gauss-Bonnet terms

In the first description of the theory the axion-dilaton field $\tau = a + iS$ has the interpretation of the complex structure modulus of the torus $\widetilde{S}^1 \times S^1$. As discussed in appendix H, one loop effective action in this theory contains a term of the form [65, 66]:

$$\Delta\mathcal{S} = \int d^4x\, \sqrt{-\det g}\, \Delta\mathcal{L}\,, \tag{3.1.37}$$

where

$$\Delta\mathcal{L} = \phi(a, S)\, \left\{ R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} - 4R_{\mu\nu}R^{\mu\nu} + R^2 \right\}\,. \tag{3.1.38}$$

Here $R_{\mu\nu\rho\sigma}$ is the Riemann tensor constructed from the canonical metric $g_{\mu\nu}$:

$$g_{\mu\nu} = S\, G_{\mu\nu}\,. \tag{3.1.39}$$

The function $\phi(a, S)$ appearing in (3.1.38) was originally computed in [8] using the formalism developed in [67]. This calculation has been reproduced in appendix H and the result is

$$\phi(a, S) = -\frac{1}{64\pi^2}\, \left((k+2)\ln S + \ln g(a+iS) + \ln g(-a+iS) \right) + \text{constant}\,, \tag{3.1.40}$$

where, as mentioned below (3.1.25), k is equal to half the number of $\widetilde{g}$ invariant harmonic (1,1) forms on $\mathcal{M}$, and $g(\tau)$, computed in (C.27), is given by:

$$g(\tau) = e^{2\pi i \widehat{\alpha}\tau} \prod_{n=1}^{\infty} \prod_{r=0}^{N-1} \left(1 - e^{2\pi i r/N} e^{2\pi i n\tau} \right)^{s_r}\,. \tag{3.1.41}$$

Here s_r counts the number of harmonic p-forms of $\mathcal{M}$ with $\widetilde{g}$ eigenvalue $e^{2\pi i r/N}$ weighted by $(-1)^p$ and $\widehat{\alpha}$, given in (C.20), is the Euler character of $\mathcal{M}$ divided by 24. Thus we have

$$\widehat{\alpha} = \begin{cases} 1 & \text{for} \quad \mathcal{M} = K3 \\ 0 & \text{for} \quad \mathcal{M} = T^4 \end{cases} . \tag{3.1.42}$$

Since in the second description of the theory $\mathcal{M} = K3$ corresponds to an orbifold of heterotic string theory on $T^4 \times \widehat{S}^1 \times S^1$ and $\mathcal{M} = T^4$ corresponds to an orbifold of type II string theory on $T^4 \times \widehat{S}^1 \times S^1$, we see that in this description

$$\widehat{\alpha} = \begin{cases} 1 & \text{for heterotic} \\ 0 & \text{for type II} \end{cases} . \tag{3.1.43}$$

For special cases explicit expressions for $g(\tau)$ in terms of Dedekind eta function can be found in in (C.35), (C.37).

As shown in (C.34), under a duality transformation $g(\tau)$ transforms as

$$g((a\tau + b)(c\tau + d)^{-1}) = (c\tau + d)^{k+2} g(\tau) . \tag{3.1.44}$$

Using this one can show that $\phi(a, S)$ is manifestly invariant under the S-duality transformation (3.1.36). Since a and S do not transform under a T-duality transformation of the form given in (3.1.4), this shows that (3.1.38) is invariant under both S- and T-duality transformations. Note that without the $\ln S$ term in its definition, $\phi(a, S)$ would not have been S-duality invariant. This is the only term in $\phi(a, S)$ that cannot be written as a sum of a holomorphic and an anti-holomorphic term, and has been called the holomorphic anomaly [68, 69, 70].

The effect of this additional term (3.1.38) in the Lagrangian density gives correction to the entropy of the black hole. This correction was first studied in [71] using an Euclidean action formalism; here we describe a systematic method for calculating this correction using the entropy function formalism. Using the definition of the entropy function it is easy to calculate the correction to the entropy function due to this additional term. It is given by

$$\Delta\mathcal{E} = -2\pi \int d\theta d\phi \sqrt{-\det g}\, \Delta\mathcal{L} = 64\pi^2 \phi(u_a, u_S) . \tag{3.1.45}$$

Together with (3.1.11) this gives

$$\begin{aligned} \mathcal{E} \quad = \quad & \frac{\pi}{2}\Big[u_S(v_2 - v_1) + \frac{v_1}{v_2 u_S} \left\{Q^T u_M Q + (u_S^2 + u_a^2)\, P^T u_M P \right. \\ & \left. -2\, u_a\, Q^T u_M P\right\} + 128\, \pi\, \phi(u_a, u_S)\Big] . \end{aligned} \tag{3.1.46}$$

Since the extra term is independent of u_M, v_1 and v_2, the extremization of $\mathcal{E}$ with respect to these variables can be carried out as before without any change. This gives, for $P^2 > 0$, $Q^2 > 0$, $P^2Q^2 > (Q\cdot P)^2$,

$$\mathcal{E} = \frac{\pi}{2}\Big[\frac{1}{u_S}\,\left\{Q^2 - 2\,u_a\,Q\cdot P + P^2(u_S^2+u_a^2)\right\} + 128\pi\phi(u_a,u_S)\Big]\,. \tag{3.1.47}$$

The values of u_a and u_S at the horizon are determined by extremizing $\mathcal{E}$ with respect to u_a and u_S. This gives:

$$\begin{aligned} P^2u_a - Q\cdot P + 64\,\pi\,u_S\,\frac{\partial\phi}{\partial u_a} &= 0\,, \\ -\frac{1}{u_S^2}\left(Q^2 - 2\,u_a\,Q\cdot P + P^2u_a^2\right) + P^2 + 128\,\pi\,\frac{\partial\phi}{\partial u_S} &= 0\,. \end{aligned} \tag{3.1.48}$$

Finally the value of $\mathcal{E}$ evaluated at the solution to eqs.(3.1.48) gives the entropy of the black hole. As mentioned earlier, these black holes are expected to be supersymmetric. Eqs.(3.1.47), (3.1.48) first appeared in [72] in the context of $\mathcal{N}=2$ supergravity theories.

Although it is difficult to solve the extremization equations (3.1.48) analytically, we can solve it iteratively. In particular, at the level of four derivative terms in the action, we are interested in corrections to the entropy which are suppressed compared to the leading contribution (3.1.23) by two powers of various charges, i.e. terms which remain invariant under a simultaneous rescaling of all the charges. For this we can simply evaluate the modified entropy function at the leading order solution (3.1.22) for u_S and u_a. This gives the following expression for the black hole entropy:

$$S_{BH} = \pi\,\sqrt{Q^2P^2-(Q\cdot P)^2} + 64\,\pi^2\,\phi\left(\frac{Q\cdot P}{P^2}, \frac{\sqrt{Q^2P^2-(Q\cdot P)^2}}{P^2}\right) + \cdots \tag{3.1.49}$$

where $\cdots$ denote correction terms which are suppressed by inverse powers of charges.

Although (3.1.49) give the correction to the black hole entropy due to the Gauss-Bonnet term, we should note that the effective action of string theory contains other four derivative terms besides the Gauss-Bonnet term. In principle their contribution to the entropy will be of the same order as that of the Gauss-Bonnet term. Thus one could question the significance of the result given in (3.1.49). At present there is no completely satisfactory answer to this question, but we shall try to summarize our current understanding of the situation. In order to set up the background for this analysis we shall first study (3.1.47), (3.1.48) in a particular limit. Let us consider a range of charges where the electric charge $\vec{Q}$ is much larger than the magnetic charge $\vec{P}$. In this case the leading order result (3.1.22) shows that u_S is large at the horizon and hence the string loop corrections in the second description, involving inverse powers of u_S, should be small. Thus we can expect that in the second description we only need to include corrections to the effective action at string tree level. In the particular context of the Gauss-Bonnet term, this corresponds to replacing $\phi(u_a, u_S)$ in (3.1.47), (3.1.48) by its expression for large u_S. Using (3.1.40), (3.1.41) we see that for large u_S

$$\phi(u_a, u_S) \simeq \frac{1}{16\pi}\,\widehat{\alpha}\,u_S\,. \tag{3.1.50}$$

Substituting this in (3.1.47) we get

$$\mathcal{E} = \frac{\pi}{2}\Big[\frac{1}{u_S}\,\left(Q^2 - 2\,u_a\,Q\cdot P + P^2(u_S^2+u_a^2)\right) + 8\,\widehat{\alpha}\,u_S\Big]\,. \tag{3.1.51}$$

Extremization of this function with respect to u_S and u_a can now be carried out analytically and, using (3.1.20), yields the answer

$$\begin{aligned} u_S &= \sqrt{\frac{Q^2P^2-(Q\cdot P)^2}{P^2(P^2+8\widehat{\alpha})}}\,, \qquad u_a = \frac{Q\cdot P}{P^2}\,, \\ v_1 &= v_2 = 2P^2 + 8\widehat{\alpha}\,, \end{aligned} \tag{3.1.52}$$

and

$$S_{BH} = \mathcal{E} = \pi\sqrt{\frac{(P^2+8\widehat{\alpha})(Q^2P^2-(Q\cdot P)^2)}{P^2}}\,. \tag{3.1.53}$$

For later use in §3.2.2 and §4.2 we shall write down the solution (3.1.52) for a special class of black holes for which

$$Q = \begin{pmatrix} \widehat{n} \\ 0 \\ \widehat{w} \\ 0 \end{pmatrix}, \qquad P = \begin{pmatrix} 0 \\ W' \\ 0 \\ N' \end{pmatrix}\,. \tag{3.1.54}$$

For definiteness we shall take

$$N', W' > 0, \quad \widehat{n}, \widehat{w} < 0\,, \tag{3.1.55}$$

so that (3.1.19) is satisfied. Let us further assume that $\widehat{n}\widehat{w} >> N'W'$ so that u_S at the horizon is large and hence $\phi(u_a, u_S)$ can be approximated as in (3.1.50). In this case (3.1.52), (3.1.53) take the form:

$$\begin{aligned} u_S &= \sqrt{\frac{\widehat{n}\widehat{w}}{N'W'+4\widehat{\alpha}}}\,, \qquad u_a = 0\,, \\ v_1 &= v_2 = 4(N'W'+2\widehat{\alpha})\,. \end{aligned} \tag{3.1.56}$$

The solution for M is also easy to calculate by extremizing (3.1.11) and we get

$$M = \begin{pmatrix} \widehat{w}/\widehat{n} & & & \\ & N'/W' & & \\ & & \widehat{n}/\widehat{w} & \\ & & & W'/N' \end{pmatrix}\,. \tag{3.1.57}$$

Comparing with eq.(3.1.27) we see that if $\widehat{R}$ and R denote the radii of $\widehat{S}^1$ and S^1 measured in the string metric after $\mathbb{Z}_N$ orbifolding then

$$\widehat{R} = \sqrt{\widehat{n}/\widehat{w}}, \qquad R = \sqrt{W'/N'}\,. \tag{3.1.58}$$

Finally (3.1.53) gives

$$S_{BH} = 2\pi\sqrt{\widehat{n}\widehat{w}(N'W' + 4\widehat{\alpha})}\,. \tag{3.1.59}$$

We shall see in §4 (see eq.(4.2.6) and the discussion below) that (3.1.59) is the exact answer for the entropy in the $\widehat{n}\widehat{w} >> N'W'$ limit. More generally one can show that (3.1.53) is exact in the limit $\sqrt{Q^2P^2 - (Q\cdot P)^2} >> P^2$.

Let us now turn to the question of validity of (3.1.49). The question is: how does the formula get corrected by other four derivative terms? To this end we make the following observations:

- A simple scaling argument shows that the contribution to the entropy from the four derivative terms must remain invariant when all charges are scaled by a common parameter. This is manifestly true for the contribution from the Gauss-Bonnet term. The contribution from the other four derivative terms must also satisfy this constraint.

- Since the answer for the entropy after inclusion of Gauss-Bonnet term is duality invariant, the additional contribution due to the other four derivative terms must be duality invariant by itself.

- For $\sqrt{Q^2P^2 - (Q\cdot P)^2} >> P^2$ the additional contribution must vanish since we know that in this limit the correction to the entropy due to the Gauss-Bonnet term captures the complete answer.

This gives strong constraints on the contribution from the additional four derivative terms. Having this contribution vanish is consistent with all these constraints. We would like to believe that these, together with some other constraints, can be used to argue that the additional term actually vanishes, but we do not have such a proof as of now. However we shall see in §5.6 that (3.1.49) agrees perfectly with the first non-leading correction to the statistical entropy.

3.1.5 Non-supersymmetric extremal black holes

In the supergravity approximation non-supersymmetric black holes arise for $P^2Q^2 < (Q\cdot P)^2$, and the entropy of the corresponding black hole has been given in (3.1.24). Let us consider a special class of such black holes with charge vector given in (3.1.54), for the range

$$N', W', \widehat{n} > 0, \qquad \widehat{w} < 0\,. \tag{3.1.60}$$

This differs from (3.1.19) by a flip $\widehat{n} \to -\widehat{n}$. Extremization of $\mathcal{E}$ can be done simply by flipping, in the solution corresponding to (3.1.54), (3.1.55), the sign of the electric field conjugate to $\widehat{n}$ keeping every other near horizon parameter unchanged. Thus the solution and the black hole entropy for $\widehat{n} > 0$ are given by eqs.(3.1.56)-(3.1.59) with $\widehat{n}$ replaced by $-\widehat{n}$. In particular the black hole entropy is given by

$$S_{BH}^{ns} = 2\pi\sqrt{|\widehat{n}\widehat{w}|(N'W' + 4\widehat{\alpha})}\,. \tag{3.1.61}$$

3.2 Black holes in $\mathcal{N} = 2$ supersymmetric theories in $D = 4$

In this section we shall apply the entropy function formalism to calculate the entropy of extremal black holes in a more general class of theories in four dimensions, namely $\mathcal{N} = 2$ supergravity theories. Our analysis will follow [73].

3.2.1 General considerations

The off-shell formulation of $\mathcal{N} = 2$ supergravity action in four dimensions was developed in [74, 75, 76, 77, 78, 79, 80, 81, 82]. Here we shall review this formulation following the notation of [83]. The basic bosonic fields in the theory are a set of $(N + 1)$ complex scalar fields X^I with $0 \le I \le N$ (of which one can be gauged away using a scaling symmetry), $(N + 1)$ gauge fields A^I_μ and the metric $g_{\mu\nu}$. Besides this the theory contains several non-dynamical fields. For the black hole solution we shall study, many of these fields can be set to zero using certain symmetries of the action and constraints. The relevant field which takes non-zero value near the horizon is a complex anti-self-dual antisymmetric tensor field $T^-_{\mu\nu}$. The lagrangian density $\mathcal{L}$ of the theory involving these fields is determined completely in terms of a single holomorphic function $F(\vec{X}, \widehat{A})$ of the scalars X^I and an auxiliary variable $\widehat{A}$, satisfying

$$F(\lambda\vec{X}, \lambda^2\widehat{A}) = \lambda^2\, F(\vec{X}, \widehat{A})\,. \tag{3.2.1}$$

The expression for $\mathcal{L}$ in terms of this function F has been reviewed in [83]; for brevity we shall not reproduce it here.

In order to facilitate comparison with the results obtained by other approaches, *e.g.* in [83], we shall use a different normalization convention for the charges than the one used so far. We introduce charges $\widetilde{q}_I$, $\widetilde{p}^I$ related to the charges q_I, p^I of the earlier convention via the relations:

$$\widetilde{q}_I = -2\, q_I, \qquad \widetilde{p}^I = \frac{p^I}{4\pi}\,. \tag{3.2.2}$$

With this normalization convention a general extremal black hole in this theory has a near horizon geometry of the form:

$$\begin{aligned} & ds^2 = v_1(-r^2 dt^2 + dr^2/r^2) + v_2(d\theta^2 + \sin^2\theta d\phi^2) \\ & F^I_{rt} = e_I, \quad F^I_{\theta\phi} = \widetilde{p}^I \sin\theta, \quad X^I = x^I, \quad T^-_{rt} = v_1\, w\,. \end{aligned} \tag{3.2.3}$$

Using the known Lagrangian density $\mathcal{L}$, we can calculate the entropy function associated with this black hole:

$$\mathcal{E}(v_1, v_2, w, \vec{x}, \vec{e}, \vec{\widetilde{q}}, \vec{\widetilde{p}}) = 2\pi \left(-\frac{1}{2}\, \widetilde{q}_I e^I - \int d\theta d\phi \sqrt{-\det g}\, \mathcal{L} \right) . \tag{3.2.4}$$

The result is [73]:

$$\begin{aligned} \mathcal{E} \;&=\; -\pi \widetilde{q}_I e^I - \pi\, g(v_1, v_2, w, \vec{x}, \vec{e}, \vec{\widetilde{p}}) \\ g(v_1, v_2, w, \vec{x}, \vec{e}, \vec{\widetilde{p}}) \;&=\; v_1 v_2 \Bigg[i(v_1^{-1} - v_2^{-1})(x^I \bar{F}_I - \bar{x}^I F_I) \\ & -\Big\{ \frac{i}{4} v_1^{-2} F_{IJ} (e^I - i v_1 v_2^{-1} \widetilde{p}^I - \frac{1}{2} \bar{x}^I v_1 w)(e^J - i v_1 v_2^{-1} \widetilde{p}^J - \frac{1}{2} \bar{x}^J v_1 w) + h.c. \Big\} \\ & -\Big\{ \frac{i}{4} v_1^{-1} w \bar{F}_I (e^I - i v_1 v_2^{-1} \widetilde{p}^I - \frac{1}{2} \bar{x}^I v_1 w) + h.c. \Big\} \\ & +\Big\{ \frac{i}{8} \bar{w}^2 F + h.c. \Big\} + 8\, i\, \bar{w} w \Big(-v_1^{-1} - v_2^{-1} + \frac{1}{8} \bar{w} w \Big) \Big(F_{\widehat{A}} - \bar{F}_{\widehat{A}} \Big) \\ & +64\, i\, (v_1^{-1} - v_2^{-1})^2 \Big(F_{\widehat{A}} - \bar{F}_{\widehat{A}} \Big) \Bigg]_{\widehat{A} = -4w^2} , \end{aligned} \tag{3.2.5}$$

where

$$F_I = \frac{\partial F}{\partial x^I}, \quad F_{\widehat{A}} = \frac{\partial F}{\partial \widehat{A}}, \quad F_{IJ} = \frac{\partial^2 F}{\partial x^I \partial x^J}, \quad F_{\widehat{A}I} = \frac{\partial^2 F}{\partial x^I \partial \widehat{A}}, \quad F_{\widehat{A}\widehat{A}} = \frac{\partial^2 F}{\partial \widehat{A}^2}\,, \tag{3.2.6}$$

and bar denotes complex conjugation. This entropy function has a scale invariance

$$x^I \to \lambda x^I, \quad v_i \to \lambda^{-1} \bar{\lambda}^{-1} v_i, \quad e^I \to e^I. \quad w \to \lambda w, \quad \widetilde{q}_I \to \widetilde{q}_I, \quad \widetilde{p}^I \to \widetilde{p}^I\,. \tag{3.2.7}$$

This descends from the invariance of the lagrangian density of $\mathcal{N} = 2$ supergravity theories under local scale transformation, and is usually eliminated by using some gauge fixing condition. We shall however find it convenient to work with the gauge invariant equations of motion obtained by extremizing (3.2.5) with respect to v_1, v_2, w, $\vec{x}$ and $\vec{e}$.

It can be easily seen that the entropy function is extremized if we set

$$v_1 = v_2 = \frac{16}{\bar{w}w}\,, \tag{3.2.8}$$

$$e^I - i v_1 v_2^{-1} \widetilde{p}^I - \frac{1}{2}\bar{x}^I v_1 w = 0 \tag{3.2.9}$$

$$(\bar{w}^{-1}\bar{F}_I - w^{-1}F_I) = -\frac{i}{4}\,\widetilde{q}_I\,. \tag{3.2.10}$$

Taking the real and imaginary parts of eq.(2.2.20) gives

$$e^I = 4(\bar{w}^{-1}\bar{x}^I + w^{-1}x^I)\,, \tag{3.2.11}$$

and

$$(\bar{w}^{-1}\bar{x}^I - w^{-1}x^I) = -\frac{1}{4}\,i\,\widetilde{p}^I\,. \tag{3.2.12}$$

The black hole entropy computed by evaluating the entropy function in this background is given by

$$S_{BH} = 2\pi\left[-\frac{1}{2}\,\vec{\widetilde{q}}\cdot\vec{e} - 16\,i\,(w^{-2}F - \bar{w}^{-2}\bar{F})\right]\,. \tag{3.2.13}$$

If we choose w=constant gauge (which corresponds to $\widehat{A}$=$-4w^2$=constant), then eqs.(3.2.8)-(3.2.12) describe the usual supersymmetric attractor equations for the near horizon geometry of extremal black holes, and (3.2.13) gives the expression for the entropy of these black holes as written down in [84, 85, 86, 72, 87, 83, 88, 89, 2]. For example (3.2.13) shows that in the gauge w =real constant, the Legendre transform of the black hole entropy with respect to the electric charges $\widetilde{q}_I$ is proportional to the imaginary part of the prepotential F. Furthermore eqs.(3.2.8), (3.2.9) shows that the argument x^I of the prepotential is proportional to $e^I + i\widetilde{p}^I$, i.e. its real part is the variable conjugate to the electric charge $\widetilde{q}_I$ and its imaginary part is the magnetic charge $\widetilde{p}^I$. These observations were originally made in [2].

The attractor equations (3.2.8)-(3.2.12) provide sufficient but not necessary conditions for extremizing the entropy function. We can have other near horizon configurations which extremize the entropy function but do not satisfy eqs.(3.2.8)-(3.2.12). These will typically describe non-supersymmetric extremal black holes.

3.2.2 S-T-U model

A special $\mathcal{N}=2$ supergravity theory which will be of interest to us is a model with four complex scalars $X^0, \ldots X^3$, described by the prepotential

$$F(X^0, X^1, X^2, X^3, \widehat{A}) = -\frac{X^1 X^2 X^3}{X^0} - \frac{\widehat{\alpha}}{64}\widehat{A}\,\frac{X^1}{X^0}\,, \tag{3.2.14}$$

where $\widehat{\alpha}$ is a constant. The scalar fields take value in the range

$$Im\left(\frac{X^a}{X^0}\right) > 0 \quad \text{for} \quad a = 1, 2, 3\,. \tag{3.2.15}$$

For $\widehat{\alpha} = 0$ this theory describes a subsector of the $\mathcal{N}=4$ supergravity theory with two derivative terms as described in eqs.(3.1.26)-(3.1.29). In this case the dilaton-axion pair (S, a) appearing in (3.1.3) correspond respectively to the imaginary and real parts of X^1/X^0:

$$\frac{X^1}{X^0} = a + iS\,. \tag{3.2.16}$$

Various components of the 4×4 matrix valued field M can be identified with various combinations of the four real fields associated with X^2/X^0 and X^3/X^0. In particular in terms of the components $\widehat{G}_{mn}$ and $\widehat{B}_{mn}$ ($4 \le m, n \le 5$) appearing in (3.1.27) we have

$$\frac{X^2}{X^0} = \widehat{B}_{45} + i\sqrt{\det \widehat{G}}, \qquad \frac{X^3}{X^0} = (\widehat{G}_{45} + i\widehat{G}_{44})/\sqrt{\det \widehat{G}}\,. \tag{3.2.17}$$

These relations take simple form if the fields X^2/X^0 and X^3/X^0 are purely imaginary. In this case the corresponding matrix M is diagonal. If we parametrize X^2/X^0 and X^3/X^0 as

$$\frac{X^2}{X^0} = iR\widehat{R}, \qquad \frac{X^3}{X^0} = i\frac{\widehat{R}}{R} \tag{3.2.18}$$

then

$$M = \begin{pmatrix} \widehat{R}^{-2} & 0 & 0 & 0 \\ 0 & R^{-2} & 0 & 0 \\ 0 & 0 & \widehat{R}^2 & 0 \\ 0 & 0 & 0 & R^2 \end{pmatrix}\,. \tag{3.2.19}$$

As can be seen from (3.2.17) and (3.2.18), in this case $\widehat{R}$ and R have the interpretation of the radii of $\widehat{S}^1$ and $S^1/\mathbb{Z}_N$ measured in the string metric. Finally the four sets of gauge field strengths $F^{(i)}_{\mu\nu}$ are related to the four sets of gauge field strengths $\mathcal{F}^I_{\mu\nu}$ of the $\mathcal{N}=2$ supergravity theory under study via a complicated set of duality transformations. The relations between the gauge fields are best summarized by relating the electric and magnetic charges $\widetilde{q}_I$, $\widetilde{p}^I$ of the $\mathcal{N}=2$ supergravity theory with the charges Q_i and P_i appearing in §3.1:[12]

$$\begin{aligned} &Q_4 = \widetilde{q}_2, \quad Q_3 = -\widetilde{p}^1, \quad Q_2 = \widetilde{q}_3, \quad Q_1 = \widetilde{q}_0\,, \\ &P_4 = \widetilde{p}^3, \quad P_3 = \widetilde{p}^0, \quad P_2 = \widetilde{p}^2, \quad P_1 = \widetilde{q}_1\,. \end{aligned} \tag{3.2.20}$$

After eliminating the electric field variables e^I from (3.2.5) using the $\partial\mathcal{E}/\partial e^I = 0$ equations and fixing an appropriate 'gauge' for the symmetry (3.2.7), the entropy function (3.2.5) of the S-T-U model reduces to the one given in (3.1.11).

We have seen in §3.1 that the T-duality invariant combination of the charges are

$$Q^2 = 2(Q_4Q_2 + Q_3Q_1), \quad P^2 = 2(P_4P_2 + P_3P_1), \quad Q\cdot P = (Q_4P_2 + Q_3P_1 + Q_2P_4 + Q_1P_3)\,. \tag{3.2.21}$$

One can explicitly verify that for

$$P^2Q^2 > (Q\cdot P)^2, \quad P_2, P_4 > 0, \quad Q_1, Q_3 < 0, \quad P_3 = 0 \tag{3.2.22}$$

the supersymmetric attractor equations (3.2.8)-(3.2.12) can be solved by setting [85, 73][13]

$$\begin{aligned} x^0 &= -\frac{1}{8}\frac{Q_3P^2}{\sqrt{P^2Q^2-(P\cdot Q)^2}} \\ \frac{x^1}{x^0} &= -\frac{P\cdot Q}{P^2} + i\frac{\sqrt{P^2Q^2-(P\cdot Q)^2}}{P^2} \\ \frac{x^2}{x^0} &= -\frac{1}{2Q_3P_4}(Q_3P_1 + Q_4P_2 - P_4Q_2) - i\frac{P_2}{Q_3}\frac{\sqrt{P^2Q^2-(P\cdot Q)^2}}{P^2} \\ \frac{x^3}{x^0} &= -\frac{1}{2Q_3P_2}(Q_3P_1 - Q_4P_2 + P_4Q_2) - i\frac{P_4}{Q_3}\frac{\sqrt{P^2Q^2-(P\cdot Q)^2}}{P^2} \\ v_1 &= v_2 = 16, \qquad e^I = 8\,Re(x^I) \quad \text{for} \quad 0\le I\le 3, \qquad w = 1\,, \end{aligned} \tag{3.2.23}$$

Evaluating the entropy function at the solution and rewriting the result in terms of the duality invariant combinations P^2, Q^2 and $Q\cdot P$, we get back the result

$$\mathcal{E} = \pi\sqrt{Q^2P^2 - (Q\cdot P)^2}\,, \tag{3.2.24}$$

in agreement with (3.1.23). The parameters v_1, v_2 given in (3.2.23) do not agree with the ones given in (3.1.20) even though the two theories are supposed to be equivalent. This can be traced to the

[12]This differs from the identification made in [73] by a transformation $(Q_1, Q_2, Q_3, Q_4) \to (Q_4, Q_3, Q_2, Q_1)$, $(P_1, P_2, P_3, P_4) \to (P_4, P_3, P_2, P_1)$.

[13]Here $w = 1$ is gauge condition.

fact that in the $w = 1$ gauge being used here, the metric in the two descriptions are related by a field dependent scale transformation.

Finally note that although the expression (3.2.24) for the entropy was derived for a special choice of the charges ($P_3 = 0$), T-duality invariance guarantees that (3.2.24) continues to hold even when we deform P_3 away from 0.

We shall now consider the effect of switching on the constant $\widehat{\alpha}$. In the $\mathcal{N} = 2$ supergravity theory this gives rise to various higher derivative terms in the action. Of them is a term proportional to S times the square of the Riemann tensor, and the coefficient is identical to the one appearing in (3.1.38) for $\phi(a, S)$ given in (3.1.50). However the two actions are not physically equivalent i.e. they cannot be related by a field redefinition even at the level of four derivative terms. If we solve the attractor equations (3.2.8)-(3.2.12) and calculate the entropy by evaluating the entropy function at the solution, we get

$$\begin{aligned}
x^0 &= -\frac{1}{8}Q_3\sqrt{\frac{P^2(P^2+8\widehat{\alpha})}{P^2Q^2-(P\cdot Q)^2}} \\
\frac{x^1}{x^0} &= -\frac{P\cdot Q}{P^2} + i\sqrt{\frac{P^2Q^2-(P\cdot Q)^2}{P^2(P^2+8\widehat{\alpha})}} \\
\frac{x^2}{x^0} &= -\frac{1}{2Q_3P_4}(Q_3P_1+Q_4P_2-P_4Q_2) - i\frac{P_2}{Q_3}\sqrt{\frac{P^2Q^2-(P\cdot Q)^2}{P^2(P^2+8\widehat{\alpha})}} \\
\frac{x^3}{x^0} &= -\frac{1}{2Q_3P_2}(Q_3P_1-Q_4P_2+P_4Q_2) - i\frac{P_4}{Q_3}\sqrt{\frac{P^2Q^2-(P\cdot Q)^2}{P^2(P^2+8\widehat{\alpha})}} \\
v_1 &= v_2 = 16, \qquad e^I = 8\,Re(x^I) \quad \text{for} \quad 0 \le I \le 3, \qquad w = 1\,,
\end{aligned} \tag{3.2.25}$$

$$S_{BH} = \pi\sqrt{P^2Q^2-(P\cdot Q)^2}\sqrt{1+\frac{8\widehat{\alpha}}{P^2}}\,. \tag{3.2.26}$$

Surprisingly, the result for the entropy agrees with the one given in (3.1.53).

For a special class of black holes for which

$$Q = \begin{pmatrix} \widehat{n} \\ 0 \\ \widehat{w} \\ 0 \end{pmatrix}, \qquad P = \begin{pmatrix} 0 \\ W' \\ 0 \\ N' \end{pmatrix}, \tag{3.2.27}$$

$$N', W' > 0, \qquad \widehat{n}, \widehat{w} < 0\,, \tag{3.2.28}$$

the solution (3.2.25) gives

$$\frac{x^1}{x^0} = i\sqrt{\frac{\widehat{n}\widehat{w}}{N'W'+4\widehat{\alpha}}}, \qquad \frac{x^2}{x^0} = -i\frac{W'}{\widehat{w}}\sqrt{\frac{\widehat{n}\widehat{w}}{N'W'+4\widehat{\alpha}}}, \qquad \frac{x^3}{x^0} = -i\frac{N'}{\widehat{w}}\sqrt{\frac{\widehat{n}\widehat{w}}{N'W'+4\widehat{\alpha}}}\,. \tag{3.2.29}$$

Using (3.2.16) and (3.2.18) we get

$$\widehat{R} = \sqrt{\frac{\widehat{n}}{\widehat{w}}}\sqrt{\frac{N'W'}{N'W'+4\widehat{\alpha}}}, \qquad R = \sqrt{\frac{W'}{N'}}\,, \qquad u_S = \sqrt{\frac{\widehat{n}\widehat{w}}{N'W'+4\widehat{\alpha}}}\,. \tag{3.2.30}$$

Finally (3.2.26) gives

$$S_{BH} = 2\pi\sqrt{\widehat{n}\widehat{w}(N'W'+4\widehat{\alpha})}\,. \tag{3.2.31}$$

For nonvanishing $\widehat{\alpha}$ the solution (3.2.30) differs from the solution given in (3.1.56), (3.1.58) for $\mathcal{N}=4$ supergravity theory with Gauss-Bonnet term, but the formulæ (3.1.59) and (3.2.31) for the entropy continue to agree [90, 4].

We would like to note that neither the Gauss-Bonnet correction given in (3.1.38), nor the correction to the prepotential proportional to $\widehat{\alpha}$ given in (3.2.14), describes the complete set of four derivative corrections in tree level string theory. Nevertheless we shall argue in §4 that the result (3.1.59) or (3.2.31) does not change after inclusion of additional terms in the effective action at string tree level, i.e. for large S. Translated to a condition on the charges, this means that the result (3.1.59) or (3.2.31) become a good approximation in the limit when the electric charges are much larger than the magnetic charges.

Given this surprising agreement between the $\mathcal{N}=4$ and $\mathcal{N}=2$ results one might ask whether it is possible to also reproduce the more general results (3.1.47), (3.1.48) from the $\mathcal{N}=2$ viewpoint. This cannot be done completely rigorously since due to the presence of holomorphic anomaly term proportional to $\ln S$ in the coefficient (3.1.40) of the curvature squared term there is no known generalization of this term into an $\mathcal{N}=2$ supersymmentric lagrangian. Nevertheless the form of these corrections were guessed in [72] by first examining the contribution from the rest of the terms in $\phi(a,S)$ and then requiring the result to be duality invariant.

3.2.3 Non-supersymmetric extremal black holes

We can also construct non-supersymmetric extremal black holes in this theory by directly solving the equations corresponding to extremization of $\mathcal{E}$ rather than solving the attractor equations (3.2.8)-(3.2.12). Since for $\widehat{\alpha}=0$ the theory is equivalent to the $\mathcal{N}=4$ supergravity theory described in §3.1.1, a convenient starting point for constructing non-supersymmetric solution is to start with the non-supersymmetric solution described in §3.1.5 for $\widehat{\alpha}=0$. Thus we consider the charge vectors of the form given in (3.2.27) with $\widehat{n}$, $\widehat{w}$, N', W' in the range

$$N', W', \widehat{n} > 0, \quad \widehat{w} < 0\,. \tag{3.2.32}$$

The solution and the entropy for $\widehat{\alpha}=0$ are obtained simply by replacing $\widehat{n}\to-\widehat{n}$ and setting $\widehat{\alpha}=0$ in eqs.(3.1.56)-(3.1.59). Using the relations (3.2.16), (3.2.18) and (3.2.19) between the scalar fields in the $\mathcal{N}=4$ description and the $\mathcal{N}=2$ description, we get

$$\frac{x^1}{x^0}=u_a+iu_S=i\sqrt{\frac{|\widehat{n}\widehat{w}|}{N'W'}},\quad \frac{x^2}{x^0}=iR\widehat{R}=i\sqrt{\left|\frac{\widehat{n}W'}{\widehat{w}N'}\right|},\quad \frac{x^3}{x^0}=i\frac{\widehat{R}}{R}=i\sqrt{\left|\frac{\widehat{n}N'}{\widehat{w}W'}\right|}, \tag{3.2.33}$$

and

$$S_{BH}=2\pi\sqrt{|\widehat{n}\widehat{w}|N'W'}\,. \tag{3.2.34}$$

Note that we cannot use (3.1.56) to determine the values of v_1 and v_2 in the $\mathcal{N}=2$ description since they are gauge dependent in this description. Instead v_1, v_2 and w are found by directly solving the extremization equations of the $\mathcal{N}=2$ theory after fixing an appropriate gauge. Beginning with this leading order solution, we can solve the extremization equations for $\mathcal{E}$ given in (3.2.5) in a power series expansion in $\widehat{\alpha}$. We shall not describe the details of the calculation but only give the final result for the entropy calculated by this method [73]:

$$S^{ns}_{BH}=2\pi\sqrt{|\widehat{n}\widehat{w}|N'W'}(1+80u-3712u^2-243712u^3-18325504u^4-9538502656u^5+\mathcal{O}(u^6))\,, \tag{3.2.35}$$

where

$$u=\frac{\widehat{\alpha}}{128N'W'}\,. \tag{3.2.36}$$

3.3 Small black holes

In this section we shall focus on a special subset of the black holes described in §3.1 and §3.2.2, – those with vanishing magnetic charge $\vec{P}$. These black holes, being purely electrically charged, have the same quantum numbers as the elementary string excitations in the second description of these theories. In particular the extremal supersymmetric black holes should correspond to BPS states [91, 92] in the spectrum of elementary string. A simple class of such states, corresponding to $\vec{P}=0$ and $\vec{Q}$ of the form given in (3.2.27) with $\widehat{n},\widehat{w}<0$, represent BPS elementary string states carrying $\widehat{n}$ units of momentum and $-\widehat{w}$ units of winding along $\widehat{S}^1$. In the light-cone gauge Green-Schwarz formulation of the fundamental string world-sheet theory, we denote by L_0' and $\bar{L}_0'$ the vacuum and oscillator contribution to the zero modes L_0 and $\bar{L}_0$ of the left- and the right-handed Virasoro generators. On the other hand the contributions to L_0 and $\bar{L}_0$ from the momentum and winding along $\widehat{S}^1$ are given by $1/4(\widehat{n}\sqrt{\alpha'}/\widehat{R}\mp\widehat{w}\widehat{R}/\sqrt{\alpha'})^2$, where $\widehat{R}$ is the radius of $\widehat{S}^1$. Thus the contribution to $L_0-\bar{L}_0$ from these charges is given by $-\widehat{n}\widehat{w}$, and the left-right level matching condition tells us that these states must have $L_0' - \bar{L}_0' = \widehat{n}\widehat{w}$. On the other hand since space-time supersymmetry originates in the right-moving sector of the world-sheet theory, the BPS condition tells us that $\bar{L}_0'$ must vanish. Thus we have $L_0' = \widehat{n}\widehat{w}$. For a more general electric charge vector $\vec{Q}$, by matching the quantum numbers of the black hole with that of the elementary string and imposing the left-right level matching condition one can easily verify that a supersymmetric black hole with charge $\vec{Q}$ would correspond to elementary string excitations where the right-moving oscillators are in the $\bar{L}_0' = 0$ state and the left-moving oscillators are excited to a level $L_0' = Q^2/2$ [93, 94, 95]. The degeneracy $d(Q)$ of these states for large Q^2 can be computed using the Cardy formula (1.4). Since for the heterotic string theory $c_L = 24$ and for the type II string theory $c_L = 12$, we have

$$S_{stat}(Q) \equiv \ln d(Q) \simeq 4\pi\sqrt{Q^2/2} \tag{3.3.1}$$

for heterotic string theory, and

$$S_{stat}(Q) \equiv \ln d(Q) \simeq 2\sqrt{2}\,\pi\sqrt{Q^2/2} \tag{3.3.2}$$

for type II string theory. We would like to test if the entropy of the corresponding black hole solution, computed by extremizing the entropy funcion, agrees with this statistical entropy. This is the problem we shall now address. The analysis will have two parts. First we shall use symmetry principles to argue that the black hole entropy, if non-zero, will have the same dependence on the charges as in (3.3.1), (3.3.2) [93, 94]. Then we shall describe computation of the overall coefficient [96].

Since the black hole carries zero magnetic charges, the near horizon background of the black hole is of the form

$$\begin{aligned} ds^2 &= v_1\left(-r^2 dt^2 + \frac{dr^2}{r^2}\right) + v_2(d\theta^2 + \sin^2\theta\, d\phi^2)\,, \\ S &= u_S, \qquad a = u_a, \qquad M_{ij} = u_{Mij} \\ F^{(i)}_{rt} &= e_i, \quad F^{(i)}_{\theta\phi} = 0\,, \end{aligned} \tag{3.3.3}$$

and in the leading supergravity approximation the function f given in (3.1.7) becomes:

$$\begin{aligned} f(u_S, u_a, u_M, \vec{v}, \vec{e}) &\equiv \int d\theta d\phi\,\sqrt{-\det G}\,\mathcal{L} \\ &= \frac{1}{8}\,v_1\,v_2\,u_S\left[-\frac{2}{v_1} + \frac{2}{v_2} + \frac{2}{v_1^2}e_i(Lu_M L)_{ij}e_j\right]\,. \end{aligned} \tag{3.3.4}$$

It is easy to see that the entropy function computed from this function f has no non-trivial extremum. Indeed, if we set $\vec{P} = 0$ in (3.1.22) we get singular solution, and the entropy given in (3.1.23) vanishes.

It is in principle possible that once higher derivative and/or string loop corrections to the supergravity action are included, we might get a non-singular solution. However since the leading solution is singular, these corrections are no longer small, and we may have to include all possible corrections to the effective action. The goal of this section will be to analyze the effect of these corrections in detail and see if we can extract some useful information about the entropy of such black holes.

In order to get some insight into the problem we note from (3.1.22) that if we naively take $\vec{P} \to 0$ in these equations then $u_S \to \infty$ whereas $v_i \to 0$. Since u_S denotes the inverse string coupling square this implies that the string coupling constant becomes small at the horizon. Thus we could expect that the string loop corrections are not important. On the other hand since the curvatures of AdS_2 and S^2 measured in the string metric are inversely proportional to v_1 and v_2 respectively, they become large in this limit and we expect that higher derivative corrections to the action become important near the horizon of the small black hole. Thus we should in principle include all tree level higher derivative corrections. For the time being we shall proceed with this ansatz and study the effect of tree level higher derivative corrections to the black hole solution. Later we shall verify that the solution obtained with this assumption is self-consistent, namely that the effect of string loop corrections on these solutions are indeed small.

The effective action of the tree level string theory, with all the Ramond-Ramond (RR) fields set to zero, has two important properties which will be important for our analysis:

1. The full tree level effective action of string theory is invariant under a continuous $SO(6, r-6)$ T-duality symmetry. As a result the complete entropy function

$$\mathcal{E} = 2\pi \left[\frac{1}{2} e_i Q_i - f(u_S, u_a, u_M, \vec{v}, \vec{e}) \right] \tag{3.3.5}$$

computed with the tree level effective action of string theory will be invariant under the transformation

$$e_i \to \Omega_{ij} e_j, \qquad u_M \to \Omega u_M \Omega^T, \qquad Q_i \to (\Omega^T)^{-1}_{ij} Q_j \, . \tag{3.3.6}$$

2. The tree level effective action picks up a constant multiplicative factor λ under $S \to \lambda S$, $a \to \lambda a$. As a result

$$f(\lambda u_S, \lambda u_a, u_M, \vec{v}, \vec{e}) = \lambda\, f(u_S, u_a, u_M, \vec{v}, \vec{e}) \, , \tag{3.3.7}$$

and

$$\mathcal{E} \to \lambda \mathcal{E} \quad \text{under} \quad Q_i \to \lambda Q_i, \quad u_S \to \lambda u_S, \quad u_a \to \lambda u_a \, . \tag{3.3.8}$$

These two properties imply that the black hole entropy $S_{BH}(Q)$, obtained by extremizing the entropy function with respect to the variables e_i, v_1, v_2, u_{Mij}, u_S and u_a, has the following two properties:

1. S_{BH} depends on $\vec{Q}$ only through the duality invariant combination $Q^2 \equiv Q_i L_{ij} Q_j$.

2. S_{BH} has the scaling property

$$S_{BH} \to \lambda\, S_{BH} \quad \text{under} \quad Q_i \to \lambda Q_i\,. \tag{3.3.9}$$

This gives

$$S_{BH} = C\,\sqrt{Q^2}\,, \tag{3.3.10}$$

for some constant C. This agrees with the expression for the statistical entropy given in (3.3.1), (3.3.2) up to an overall normalization constant C which cannot be determined from this simple scaling argument.

Eq.(3.3.10) can in fact be derived even without assuming an $AdS_2 \times S^{D-2}$ near horizon geometry [93, 94]. The main additional complication in this analysis is that in absence of a near horizon AdS_2 geometry and the associated attractor mechanism we can no longer assume that the entropy is independent of the asymptotic moduli; we have to prove this by explicitly examining the supergravity solution describing the near horizon geometry of small black holes. This was carried out in [93,94,97].

The same scaling argument also tells us that at the extremum

$$v_1 = c_1, \quad v_2 = c_2, \quad e_i = c_3\, L_{ij} Q_j/\sqrt{Q^2}, \quad u_S = c_4\sqrt{Q^2}\,, \tag{3.3.11}$$

where c_i's are numerical constants independent of $\vec{Q}$. This shows that u_S is large for large Q^2 and hence string loop corrections are indeed small near the horizon. On the other hand since v_1, v_2 and e_i are of order unity, higher derivative corrections are important, and we must include all tree level higher derivative corrections to the action to get a reliable estimate of the constant C appearing in (3.3.10). Nevertheless it is instructive to see what value of C we get just by including the Gauss-Bonnet term in the action. For this we set $Q \cdot P = 0$ and then take the $P^2 \to 0$ limit of (3.1.53) (or equivalently (3.2.26) if we want to do this computation in $\mathcal{N} = 2$ supersymmetric S-T-U model with the prepotential given in (3.2.14)) which was derived for the case when electric charges are large compared to magnetic charges. This gives [96, 98, 99, 100, 101]

$$S_{BH} = 4\pi\sqrt{\widehat{\alpha}\, Q^2/2}\,. \tag{3.3.12}$$

Using (3.1.43) we see that for heterotic string compactification, the constant $\widehat{\alpha} = 1$. Thus the result for the entropy calculated with the Gauss-Bonnet term agrees exactly with the statistical entropy given in (3.3.1). On the other hand for type II string theories the constant $\widehat{\alpha}$ vanishes, showing that the entropy vanishes to this order. This is in disagreement with the statistical entropy (3.3.2). We should keep in mind however that at this point there is no reason to expect that including just the Gauss-Bonnet correction will give the correct value of the constant C. Hence at this stage neither the agreement for the heterotic string theory, nor the disagreement for the type II string theory should be taken seriously. We shall return to a more detailed discussion on this point in §4.4.

The scaling analysis carried out here can be easily generalized to higher dimensional small black holes [94, 97]. It can also be generalized to include elementary string states carrying angular momentum [102] assuming that the near horizon geometry of such an object has the structure of a black ring with $AdS_2 \times S^1 \times S^{d-3}$ near horizon geometry [103, 104].

There is one subtle point about the small black hole that requires special mention. According to the analysis described in this section, the near horizon geometry of a small black hole is given by $AdS_2 \times S^2$. As seen from (3.3.3), there is an electric flux through AdS_2, but there is no flux through S^2. This creates a puzzle. If the background given in (3.3.3) is what couples to the sigma model describing string propagation in this background, then the sigma model will be a direct sum of two conformal field theories, – one associated with AdS_2 together with the electric flux through it, and the other associated with S^2 without any flux. However it is well known that the sigma model associated with S^2 does not give rise to a conformal field theory; instead under the renormalization group flow the radius of S^2 goes to zero in the infrared. This would seem to contradict the result that the near horizon geometry of the black hole is given by (3.3.3). The only consistent resolution to this puzzle seems to be that the metric and other field variables in terms of which the solution takes the form (3.3.3) are not the ones which couple directly to the world-sheet sigma model.[14] Instead the fields which couple to the sigma model must be related to the ones appearing in (3.3.3) by an appropriate field redefinition which becomes singular when evaluated on the solution of the entropy function extremization equations. For example the sigma model metric could be related to the one in (3.3.3) by multiplication by a T-duality invariant function of the gauge field strengths and the matrix M, such that this factor vanishes at the solution. In that case although the metric appearing in (3.3.3) is finite at the solution, the metric that couples to the sigma model would correspond to a sphere of zero radius in accordance with known results.

An interesting problem is to construct the conformal field theory describing the near horizon geometry of a small black hole. As should be clear from the above discussion, we do not expect this to be a sum of two decoupled conformal field theories; instead it must be described as a single

[14] See refs. [105, 106, 107, 108] for further insight into this issue.

conformal field theory involving the four non-compact space time coordinates as well as the internal coordinates responsible for the electric charge of the black hole. Indeed, in order to argue that the entropy of the black hole has the form given in (3.3.10), it is not necessary to assume a near horizon geometry of the form (3.3.3); it is enough to assume the existence of a non-singular conformal field theory associated with the near horizon geometry of the black hole [93, 94, 95]. Furthermore this argument can be generalized to small black holes in higher dimensions as well. Various proposals for this conformal field theory have recently been made in [105, 106, 107, 108].

Finally, note that the above analysis holds only for the Neveu-Schwarz formulation of the conformal field theory. If instead we use the light-cone gauge Green-Schwarz formulation of the theory then the world-sheet fermions transform in a spinor representation of the tangent space group of the target manifold, and as a result the conformal field theories associated with the S^2 part does not decouple from the rest of the conformal field theory. Thus in this case there is no argument showing that the σ-model target space manifold cannot be $AdS_2 \times S^2$ (or, as we shall discuss in §4.4, $AdS_3 \times S^2$).

3.4 Extremal BTZ black holes with gauge and gravitational Chern-Simons terms

BTZ solution describes a rotating black hole in three dimensional theory of gravity with negative cosmological constant [109] and often appears as a factor in the near horizon geometry of higher dimensional black holes in string theory [110, 111, 112]. For this reason it has provided us with a useful tool for relating black hole entropy to the degeneracy of microstates of the black hole, both in three dimensional theories of gravity and also in string theory [111, 113]. In this section we shall apply the entropy function method to compute the entropy of a BTZ black hole in a theory of three dimensional gravity coupled to a set of abelian gauge fields and neutral scalar fields, with an arbitrary general coordinate invariant and gauge invariant action. In particular we shall include Lorentz Chern-Simons terms of the form described in (2.6.26) and gauge Chern-Simons terms. In the spirit of the discussion in §2.5 this will be done by treating the angular coordinate along which the brane rotates as a compact direction. The analysis presented here will be a generalization of the one given in [45] where we considered the case of a purely gravitational theory, and will borrow insights from the analysis of [114, 115]. Computation of the entropy of BTZ black holes in a general higher derivative theory of gravity without Chern-Simons terms has been carried out previously in [116]. Entropy of BTZ black holes has also been analyzed using the Euclidean action formalism in [117, 19, 20, 118].

Let us consider a three dimensional theory of gravity with metric G_{MN} ($0 \leq M, N \leq 2$), $U(1)$

gauge fields $A_M^{(i)}$ $(1 \le i \le n_1)$ and neutral scalar fields $\{\phi_s\}$ $(1 \le s \le n_2)$ and a general action of the form:[15]

$$S = \int d^3x \sqrt{-\det G}\left[\mathcal{L}_0^{(3)} + \mathcal{L}_1^{(3)} + \mathcal{L}_2^{(3)}\right] . \tag{3.4.1}$$

Here $\mathcal{L}_0^{(3)}$ denotes an arbitrary scalar function of the metric, scalar fields, gauge field strengths $F_{MN}^{(i)} = \partial_M A_N^{(i)} - \partial_N A_M^{(i)}$, the Riemann tensor and covariant derivatives of these quantities. $\sqrt{-\det G}\,\mathcal{L}_1^{(3)}$ denotes the gravitational Chern-Simons term:

$$\sqrt{-\det G}\,\mathcal{L}_1^{(3)} = K\,\Omega_3(\widehat{\Gamma})\,, \tag{3.4.2}$$

where K is a constant, $\widehat{\Gamma}$ is the Christoffel connection constructed out of the metric G_{MN} and

$$\Omega_3(\widehat{\Gamma}) = \epsilon^{MNP}\left[\frac{1}{2}\widehat{\Gamma}^R_{MS}\partial_N\widehat{\Gamma}^S_{PR} + \frac{1}{3}\widehat{\Gamma}^R_{MS}\widehat{\Gamma}^S_{NT}\widehat{\Gamma}^T_{PR}\right] . \tag{3.4.3}$$

ϵ is the totally anti-symmetric symbol with $\epsilon^{012} = 1$. Finally $\sqrt{-\det G}\,\mathcal{L}_2^{(3)}$ denotes a gauge Chern-Simons term of the form:

$$\sqrt{-\det G}\,\mathcal{L}_2^{(3)} = \frac{1}{2}\,C_{ij}\,\epsilon^{MNP}\,A_M^{(i)}\,F_{NP}^{(j)}\,, \tag{3.4.4}$$

for some constants $C_{ij} = C_{ji}$.

We shall consider field configurations where one of the coordinates (say $y \equiv x^2$) is compact with period 2π and the metric is independent of this compact direction. In this case we can define two dimensional fields through the relation:[16]

$$\begin{aligned} G_{MN}dx^M dx^N &= \phi\left[g_{\mu\nu}dx^\mu dx^\nu + (dy + a_\mu dx^\mu)^2\right]\,, \qquad 0 \le \mu,\nu \le 1 \\ A_M^{(i)}dx^M &= \chi^{(i)}(dy + a_\mu dx^\mu) + a_\mu^{(i)}dx^\mu\,. \end{aligned} \tag{3.4.5}$$

Here $g_{\mu\nu}$ denotes a two dimensional metric, a_μ and $a_\mu^{(i)}$ denote two dimensional gauge fields and ϕ and $\chi^{(i)}$ denote two dimensional scalar fields. In terms of these two dimensional fields

$$\frac{1}{2}F_{MN}^{(i)}dx^M \wedge dx^N = d\chi^{(i)} \wedge (dy + a_\mu dx^\mu) + \frac{1}{2}\,\left(\chi^{(i)}f_{\mu\nu} + f_{\mu\nu}^{(i)}\right)dx^\mu \wedge dx^\nu\,, \tag{3.4.6}$$

where

$$f_{\mu\nu}^{(i)} = \partial_\mu a_\nu^{(i)} - \partial_\nu a_\mu^{(i)}\,, \tag{3.4.7}$$

[15]We could add any number of scalar fields without changing the final result since they must be frozen to constant values in order to comply with the homogeneity of the BTZ configuration.

[16]The two dimensional view of the three dimensional black holes was first discussed in [119].

and

$$f_{\mu\nu} = \partial_\mu a_\nu - \partial_\nu a_\mu \,. \tag{3.4.8}$$

The action takes the form:

$$S = \int d^2x \sqrt{-\det g} \left[\mathcal{L}_0^{(2)} + \mathcal{L}_1^{(2)} + \mathcal{L}_2^{(2)} \right] \tag{3.4.9}$$

where

$$\sqrt{-\det g}\, \mathcal{L}_0^{(2)} = \int dy \sqrt{-\det G}\, \mathcal{L}_0^{(3)} = 2\pi \sqrt{-\det G}\, \mathcal{L}_0^{(3)} \,, \tag{3.4.10}$$

$$\sqrt{-\det g}\, \mathcal{L}_1^{(2)} = K\, \pi \, \left[\frac{1}{2} R \varepsilon^{\mu\nu} f_{\mu\nu} + \frac{1}{2} \varepsilon^{\mu\nu} f_{\mu\tau} f^{\tau\sigma} f_{\sigma\nu} \right] \,. \tag{3.4.11}$$

and

$$\sqrt{-\det g}\, \mathcal{L}_2^{(2)} = C_{ij}\, \varepsilon^{\mu\nu} \, \left(\chi^{(i)} f^{(j)}_{\mu\nu} + \frac{1}{2} \chi^{(i)} \chi^{(j)} f_{\mu\nu} \right) \,. \tag{3.4.12}$$

Here R is the scalar curvature of the two dimensional metric $g_{\mu\nu}$ and $\varepsilon^{\mu\nu}$ is the totally antisymmetric symbol with $\varepsilon^{01} = 1$. (3.4.10) is a straightforward dimensional reduction of the $\mathcal{L}_0^{(3)}$ term. (3.4.11) comes from dimensional reduction of the gravitational Chern-Simons term after throwing away total derivative terms and was worked out in [48]. (3.4.12) comes from dimensional reduction of (3.4.4) after throwing away total derivative terms. (3.4.11), (3.4.12) show that although the Chern-Simons terms cannot be expressed in a manifestly covariant form in three dimensions, they do reduce to manifestly covariant expressions in two dimensions.

We shall define a general extremal black hole in the two dimensional theory to be the one whose near horizon geometry is AdS_2 and for which the scalar fields ϕ, $\{\phi_s\}$ and $\{\chi^{(i)}\}$ and the gauge field strengths $f_{\mu\nu}$ and $f^{(i)}_{\mu\nu}$ are invariant under the $SO(2,1)$ isometry of the AdS_2 background. The most general near horizon background consistent with this requirement is

$$\begin{aligned} & g_{\mu\nu} dx^\mu dx^\nu = v \left(-r^2 dt^2 + \frac{dr^2}{r^2} \right), \qquad f_{rt} = e, \qquad \phi = u \,, \\ & \phi_s = u_s, \quad f^{(i)}_{rt} = e_i, \quad \chi^{(i)} = w_i \,, \end{aligned} \tag{3.4.13}$$

where v, e, u, $\{u_s\}$, $\{e_i\}$ and $\{w_i\}$ are constants. This corresponds to a three dimensional configuration of the form

$$\begin{aligned} & G_{MN} dx^M dx^N = v\, u\, (-r^2 dt^2 + \frac{dr^2}{r^2}) + u\, (dy + e r dt)^2 \\ & A^{(i)}_M dx^M = w_i\, (dy + e r dt) + e_i r dt, \qquad \phi_s = u_s \\ & \frac{1}{2} F^{(i)}_{MN} dx^M \wedge dx^N = (e_i + w_i e)\, dr \wedge dt \,. \end{aligned} \tag{3.4.14}$$

Following the procedure described in §2 we define

$$f(u,v,e,\vec{u},\vec{e},\vec{w}) = \sqrt{-\det g}\,(\mathcal{L}_0^{(2)} + \mathcal{L}_1^{(2)} + \mathcal{L}_2^{(2)})\,, \tag{3.4.15}$$

evaluated in the background (3.4.13), and

$$\mathcal{E}(u,v,e,\vec{u},\vec{e},\vec{w},q,\vec{q}) = 2\pi(eq + e_i q_i - f(u,v,e,\vec{u},\vec{e},\vec{w}))\,. \tag{3.4.16}$$

The near horizon values of u, v and e, $\{u_s\}$, $\{e_i\}$ and $\{w_i\}$ for an extremal black hole with electric charges q, $\{q_i\}$ are obtained by extremizing the entropy function $\mathcal{E}$ with respect to these variables. Furthermore, Wald's entropy for this black hole is given by the value of the function $\mathcal{E}$ at this extremum.

Using eqs.(3.4.10), (3.4.11), (3.4.12) and (3.4.13), (3.4.15) we see that for the theory considered here,

$$f(u,v,e,\vec{u},\vec{e},\vec{w}) = f_0(u,v,e,\vec{u},\vec{e}+e\vec{w}) + \pi\,K\,(2\,e\,v^{-1} - e^3\,v^{-2}) - 2\,C_{ij}\,\left(w_i e_j + \frac{1}{2} e w_i w_j\right)\,, \tag{3.4.17}$$

where

$$f_0(u,v,e,\vec{u},\vec{e}+e\vec{w}) = 2\pi\,\sqrt{-\det G}\,\mathcal{L}_0^{(3)} \tag{3.4.18}$$

evaluated in the background (3.4.14). Note that f_0 depends on $\vec{e}$ and $\vec{w}$ only through the combination $\vec{e}+e\vec{w}$ since this is the combination that enters the expression for $F^{(i)}_{MN}$ given in (3.4.14). Substituting (3.4.17) into (3.4.16) we get

$$\begin{aligned}\mathcal{E} \;\; &= \;\; 2\pi\Big[eq + e_i q_i - f_0(u,v,e,\vec{u},\vec{e}+e\vec{w}) - \pi\,K\,(2\,e\,v^{-1} - e^3\,v^{-2}) \\ &\qquad +2\,C_{ij}\,\left(w_i e_j + \frac{1}{2} e w_i w_j\right)\Big]\,.\end{aligned} \tag{3.4.19}$$

We shall carry out the extremizaton of $\mathcal{E}$ in stages. First we shall eliminate w_i and e_i using their equations of motion. Extremization of $\mathcal{E}$ with respect to w_i gives

$$-\,e\frac{\partial f_0}{\partial e_i} + 2\,C_{ij}\,(e_j + e w_j) = 0\,. \tag{3.4.20}$$

Now the terms in f_0 involving $\vec{e} + e\,\vec{w}$ must involve quadratic and higher powers of $e_i + e w_i$ since these come from terms in $\mathcal{L}_0^{(3)}$ involving the gauge fields $F^{(i)}_{MN}$. Thus $\partial f_0/\partial e_i$ vanishes for $\vec{e}+e\,\vec{w} = 0$, This shows that (3.4.20) can be solved by choosing

$$e_i + e\,w_i = 0\,. \tag{3.4.21}$$

Extremization of $\mathcal{E}$ with respect to e_i now gives

$$w_i = -\frac{1}{2} C^{-1}_{ij}\, q_j\,, \tag{3.4.22}$$

assuming that C is invertible as a matrix. Substituting (3.4.21) and (3.4.22) into the expression (3.4.19) for $\mathcal{E}$ we now get

$$\mathcal{E} = 2\pi\, \left[e \left(q - \frac{1}{4}\, C^{-1}_{ij}\, q_i q_j \right) - f_0(u, v, e, \vec{u}, \vec{0}) - \pi\, K\, (2\, e\, v^{-1} - e^3\, v^{-2}) \right]. \tag{3.4.23}$$

Eq.(3.4.21) together with (3.4.14) tells us that the three dimensional gauge field strengths $F^{(i)}_{MN}$ vanish, although the gauge field components $A^{(i)}_y$ have non-zero constant values proportional to w_i.

We now turn to the extremization of $\mathcal{E}$ with respect to u, v, e and $\vec{u}$. It turns out that if v and e satisfy the relation

$$v = e^2 \tag{3.4.24}$$

then the background (3.4.14) actually describes a locally AdS_3 space-time. Since a local AdS_3 space has a higher degree of isometry than a local AdS_2 space, setting $v = e^2$ is a consistent truncation of the theory and hence we can extremize the entropy function within this class of configurations. This corresponds to a three dimensional field configuration of the form:

$$\begin{aligned} &G_{MN} dx^M dx^N = u\, e^2 \left(-r^2 dt^2 + \frac{dr^2}{r^2} \right) + u\, (dy + e\, r\, dt)^2 \\ &\phi_s = u_s, \qquad \frac{1}{2} F^{(i)}_{MN} dx^M \wedge dx^N = 0\,. \end{aligned} \tag{3.4.25}$$

By making a coordinate change $y = e\, z$ we can see that the metric depends only on the combination ue^2. Thus all the scalars constructed out of the metric and the Riemann tensor must be a function of this combination only. The entropy function will still depend on e and ue^2 since the new coordinate z will have e dependent period. We shall proceed by choosing e and

$$l = 2\sqrt{ue^2}\,, \tag{3.4.26}$$

as independent variables. Since $\mathcal{L}^{(3)}_0$ is a scalar, and hence is a function of the combination ue^2 only, we can define a function $h(l, \vec{u})$ via the relation

$$h(l, \vec{u}) = \mathcal{L}^{(3)}_0 \tag{3.4.27}$$

evaluated in the background (3.4.25). Eq.(3.4.25) now gives

$$f_0 \equiv 2\pi\, \sqrt{-\det G}\, \mathcal{L}^{(3)}_0 = \frac{1}{|e|} g(l, \vec{u})\,, \tag{3.4.28}$$

where

$$g(l,\vec{u}) = \frac{\pi\, l^3\, h(l,\vec{u})}{4}\,. \tag{3.4.29}$$

Plugging all these results back into eq.(3.4.16) we now get

$$\mathcal{E} = 2\pi\left(\widehat{q}\, e - \frac{1}{|e|} g(l,\vec{u}) - \frac{\pi\, K}{e}\right)\,, \tag{3.4.30}$$

where

$$\widehat{q} = q - \frac{1}{4}\, C^{-1}_{ij} q_i q_j\,. \tag{3.4.31}$$

We need to extremize $\mathcal{E}$ with respect to l, u_s and e. The extremization with respect to l and u_s clearly requires extremization of $g(l,\vec{u})$ with respect to l and u_s. Defining

$$C = -\frac{1}{\pi} g(l,\vec{u}) \tag{3.4.32}$$

at the extremum of g we get

$$\mathcal{E} = 2\pi\left(\widehat{q}\, e + \frac{\pi\, C}{|e|} - \frac{\pi\, K}{e}\right)\,. \tag{3.4.33}$$

We shall assume that $C \geq |K|$. Extremizing (3.4.33) with respect to e we now get:

$$\begin{aligned} e &= \sqrt{\frac{\pi(C-K)}{\widehat{q}}} \quad \text{for}\, \widehat{q} > 0\,, \\ &= \sqrt{\frac{\pi(C+K)}{|\widehat{q}|}} \quad \text{for}\, \widehat{q} < 0\,. \end{aligned} \tag{3.4.34}$$

Furthermore, at the extremum,

$$\begin{aligned} \mathcal{E} &= 2\pi\sqrt{\frac{c_R\,\widehat{q}}{6}} \quad \text{for}\, \widehat{q} > 0\,, \\ &= 2\pi\sqrt{\frac{c_L\,|\widehat{q}|}{6}} \quad \text{for}\, \widehat{q} < 0\,, \end{aligned} \tag{3.4.35}$$

where we have defined

$$c_L = 24\,\pi\,(C+K)\,, \qquad c_R = 24\,\pi\,(C-K)\,. \tag{3.4.36}$$

(3.4.35) gives the entropy of extremal BTZ black hole as a function of the charges q and $\{q_i\}$. Physically q_i are the charges conjugate to the gauge fields $A^{(i)}_M$ while q labels the angular momentum of the black hole.

Note that the Chern-Simons term plays no role in the determination of the parameters l, $\vec{u}$ and

$$c_L + c_R = 48\,\pi\,C = -48\,g(l,\vec{u})\,. \tag{3.4.37}$$

This is a reflection of the fact that in three dimensions the effect of the Chern-Simons term on the equations of motion involves covariant derivative of the Ricci tensor [48] which vanishes for BTZ solution. On the other hand

$$c_L - c_R = 48\,\pi\,K \tag{3.4.38}$$

is insensitive to the detailed structure of the higher derivative terms and is determined completely by the coefficient of the Chern-Simons term. This is a consequence of the fact that $c_L - c_R$ is determined by the parity odd part of the action evaluated on the near horizon geometry of the BTZ black hole, and this contribution comes solely from the Chern-Simons term.

4 Black Holes with an AdS_3 Factor in the Near Horizon Geometry

For some extremal black holes in string theory the AdS_2 component of the near horizon geometry, together with an internal circle, describes a locally AdS_3 space. More accurately the near horizon geometry of these extremal black holes correspond to that of extremal BTZ black holes of the type discussed in §3.4 with the momentum along the internal circle representing the angular momentum of the black hole. In such situations the enhanced isometry group of the AdS_3 space allows us to get a more detailed information about the entropy of the system and prove certain non-renormalization theorems [20, 19, 21, 22] for the entropy of supersymmetric as well as non-supersymmetric black holes. In this section we will outline these arguments and carry out a comparison between the two approaches when both methods are available. Our discussion will follow closely the one given in [32].

We shall divide the discussion into three parts. In §4.1 we shall describe the origin of the AdS_3 factor and the information it provides for the black hole entropy. In §4.2 we shall discuss consequences of this result for a specific class of black holes described in §3.1-3.3. In §4.3 we shall discuss possible limitations of this approach.

4.1 Origin and consequences of AdS_3 factor

We begin by reviewing the origin of the AdS_3 geometry. For this we focus on the AdS_2 part of the near horizon geometry together with the electric flux through it. By choosing the basis of gauge fields appropriately we can arrange that only one gauge field has non-vanishing electric field along the AdS_2; let us denote this gauge field strength by $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$. Then the relevant part of the near horizon background takes the form:

$$ds^2 \equiv g_{\alpha\beta}dx^\alpha dx^\beta = v(-r^2dt^2 + r^{-2}dr^2), \quad F_{rt} = e\,. \tag{4.1.1}$$

We shall assume that there is an appropriate duality frame in which we can regard the gauge field component A_μ as coming from the component of a three dimensional metric along certain internal circle labelled by a coordinate y. Let us use the convention in which the two dimensional metric $g_{\alpha\beta}$ is related to the three dimensional metric G_{MN} in the r, t, y space via the dimensional reduction formula given in (3.4.5). In this case the solution (4.1.1) has the same structure as the one described in §3.4. In particular if we choose

$$v = e^2\,, \tag{4.1.2}$$

then the three dimensional metric in the r-t-y plane describes a locally AdS_3 space. More precisely, due to the compact nature of the coordinate y it becomes the quotient of the AdS_3 space by a translation by 2π along y. The effect of taking this quotient is to break the $SO(2,2)$ isometry group of AdS_3 to $SO(2,1)\times U(1)$ [120], – the symmetries of an $AdS_2 \times S^1$ manifold. Since the physical radius of the y circle is given by $\sqrt{G_{yy}} = \sqrt{u}$, we expect that the effect of this symmetry breaking will be small for large u.

Let us for the time being ignore the effect of this symmetry breaking and suppose that the background has full symmetries of the AdS_3 space. In this case we expect that the dynamics of the theory in this background will be governed by an effective three dimensional action, obtained by treating all the other directions, including the angular coordinates labeling the non-compact part of space, as compact. As described in (3.4.1) the effective Lagrangian density will have a piece $\mathcal{L}_0^{(3)}$ which is a scalar function of the metric, Riemann tensor and covariant derivatives of the Riemann tensor and a gravitational Chern-Simons term with coefficient K. The resulting entropy will also have the same form as (3.4.35). The quantum number q now has the interpretation of electric charge associated with the gauge field A_μ rather than angular momentum. We shall, for simplicity, ignore the presence of additional gauge fields (with vanishing electric field but nonvanishing Wilson lines) so that $\widehat{q}$ appearing in (3.4.35) is equal to q.

Although the above analysis gives a general form of the entropy of a black hole with an AdS_3 factor in the near horizon geometry, this analysis by itself does not determine the constants c_L and c_R appearing in (3.4.36). While $(c_L - c_R)$ can be determined in terms of the coefficient of the gravitational Chern-Simons term via eq.(3.4.38), the computation of $c_L + c_R$ via eq.(3.4.37) will, in general, require detailed knowledge of higher derivative terms in the action. However the situation simplifies if the underlying three dimensional theory has at least (0,4) supersymmetry, – in this case using AdS/CFT correspondence [121, 122, 123] one can also determine $(c_L + c_R)$ in terms of the coefficient of a Chern-Simons term in the action [19, 20, 118]. The argument proceeds as follows. The constants c_L and c_R given in (3.4.36) can be interpreted as the left- and right-moving central charges of the two dimensional CFT living on the boundary of the AdS_3 [124, 19, 20, 118]. If the boundary theory happens to have $(0, 4)$ supersymmetry, then the central charge c_R is related to the central charge of an $SU(2)_R$ current algebra which is also a part of the $(0, 4)$ supersymmetry algebra. Associated with the $SU(2)_R$ currents there will be $SU(2)$ gauge fields in the bulk which typically arise from the dimensional reduction of the full string theory on the transverse sphere and the central charge of the $SU(2)_R$ current algebra will be determined in terms of the coefficient of the gauge Chern-Simons term in the bulk theory. This determines c_R in terms of the coefficient of the gauge Chern-Simons term in the bulk theory [19, 20]. On the other hand we have already seen in (3.4.38) that $c_L - c_R$ is determined in terms of the coefficient K of the gravitational Chern-Simons term. Since both c_L and c_R are determined in terms of the coefficients of the Chern-Simons term in the bulk theory, they do not receive any higher derivative corrections. This completely determines the entropy from (3.4.35).

This result is somewhat surprising from the point of view of the bulk theory, since for a given three dimensional theory of gravity the entropy does have non-trivial dependence on all the higher derivative terms. Thus one could wonder how the dependence of the entropy on these higher derivative terms disappears by imposing the requirement of (0,4) supersymmetry. There is however a simple explanation of this fact even in the bulk theory: (0,4) supersymmetry prevents the addition of any higher derivative terms in the supergravity action (except those which can be removed by field redefinition[17]) and hence the entropy computed using the three dimensional supergravity theory with the Chern-Simons terms is the exact answer.

This non-renormalization theorem may be proved as follows [126]. In AdS/CFT correspondence the boundary operators dual to the fields in the supergravity multiplet are just the superconformal currents associated with the (0,4) supersymmetry algebra. The correlation functions of these opera-

[17]As has been discussed in [125], in a general theory of gravity in three dimensions with negative cosmological constant we can find explicit field redefinition that removes all the higher derivative corrections. However this does not rule out the possibility that the cosmological constant is renormalized during this field redefinition, – we need additional input from supersymmetry to establish this. Furthermore as far as we can see this argument holds only if the original action was local, containing powers of Riemann tensor and their covariant derivatives. In contrast our argument based of AdS_3/CFT correspondence applies to the full quantum corrected effective action including non-local terms as long as we have a global AdS_3 space.

tors in the boundary theory are determined completely in terms of the central charges c_L, c_R of the left-moving Virasoro algebra and the right-moving super-Virasoro algebra. Of these c_R is related to the central charge k_R of the right-moving SU(2) currents which form the R-symmetry currents of the super-Virasoro algebra and hence to the coefficient of the Chern-Simons term of the associated SU(2) gauge fields in the bulk theory. On the other hand $c_L - c_R$ is determined in terms of the coefficient of the gravitational Chern-Simons term in the bulk theory. Thus the knowledge of the gauge and gravitational Chern-Simons terms in the bulk theory determines all the correlation functions of (0,4) superconformal currents in the boundary theory. Since by AdS/CFT correspondence [121] these correlation functions in the boundary theory determine completely the boundary S-matrix of the supergravity fields [122, 123], we conclude that the coefficients of the gauge and gravitational Chern-Simons terms in the bulk theory determine completely the boundary S-matrix elements in this theory.

Now the boundary S-matrix elements are the only perturbative observables of the bulk theory. Thus we expect that two different theories with the same boundary S-matrix must be related by a field redefinition. Combining this with the observation made in the last paragraph we see that two different gravity theories, both with (0,4) supersymmetry and the same coefficients of the gauge and gravitational Chern-Simons terms, must be related by field redefinition. Put another way, once we have constructed a classical supergravity theory with (0,4) supersymmetry and given coefficients of the Chern-Simons terms, there cannot be any higher derivative corrections to the action involving fields in the gravity supermultiplet except for those which can be removed by field redefinition. The non-renormalization of the entropy of the BTZ black hole then follows trivially from this fact. The complete theory in the bulk of course will have other matter multiplets whose action will receive higher derivative corrections. However since restriction to the fields in the gravity supermultiplet provides a consistent truncation of the theory, and since the BTZ black hole is embedded in this subsector, its entropy will not be affected by these additional higher derivative terms.

We shall now describe this unique (0,4) supergravity action and compute the constants c_L and c_R from this action. The action was constructed in [127, 128] (generalizing earlier work of [129, 130, 131, 132] for supergravity actions based on $Osp(p|2;R) \times Osp(q|2;R)$ group) by regarding the supergravity as a gauge theory based on $SU(1,1) \times SU(1,1|2)$ algebra. If Γ_L and Γ_R denote the (super-)connections in the $SU(1,1)$ and $SU(1,1|2)$ algebras respectively, then the action is taken to be a Chern-Simons action of the form:

$$\mathcal{S} \;=\; -a_L \int d^3x \left[Tr(\Gamma_L \wedge d\Gamma_L + \frac{2}{3}\Gamma_L \wedge \Gamma_L \wedge \Gamma_L \right]$$

$$+a_R \int d^3x \left[Str(\Gamma_R \wedge d\Gamma_R + \frac{2}{3}\Gamma_R \wedge \Gamma_R \wedge \Gamma_R \right] \,, \tag{4.1.3}$$

where a_L and a_R are constants. Note that the usual metric degrees of freedom are encoded in the connections Γ_L and Γ_R. Thus there is no obvious way to add $SU(1,1)\times SU(1,1|2)$ invariant higher derivative terms in the action involving the field strengths associated with the connections Γ_L and Γ_R. From this viewpoint also it is natural that the supergravity action does not receive any higher derivative corrections.

The bosonic fields of this theory include the metric G_{MN} and an SU(2) gauge field $\mathbf{A}_M$ ($0 \le M \le 2$), represented as a linear combination of 2×2 anti-hermitian matrices. After expressing the action in the component notation and eliminating auxiliary fields using their equations of motion as in [127, 128, 129, 130, 131] we arrive at the action

$$\mathcal{S} = \int d^3x \left[\sqrt{-\det G}\,\left[R + 2m^2\right] + K\,\Omega_3(\widehat{\Gamma}) - \frac{k_R}{4\pi}\,\epsilon^{MNP} Tr\left(\mathbf{A}_M \partial_N \mathbf{A}_P + \frac{2}{3}\mathbf{A}_M \mathbf{A}_N \mathbf{A}_P \right) \right] \tag{4.1.4}$$

where

$$\frac{1}{m} = \frac{1}{2}(a_R + a_L), \qquad K = \frac{1}{2}(a_L - a_R)\,, \tag{4.1.5}$$

$$k_R = 4\pi a_R = 4\pi\left(\frac{1}{m} - K\right)\,, \tag{4.1.6}$$

and $\Omega_3(\widehat{\Gamma})$ is the gravitational Chern-Simons term defined in (3.4.3).

We shall now compute the constants c_L and c_R in this theory by using the general results of §3.4. We have in this theory

$$h(l) = (-6l^{-2} + 2m^2)\,, \tag{4.1.7}$$

$$g(l) = \frac{\pi}{4} l^3\,(-6l^{-2} + 2m^2)\,, \tag{4.1.8}$$

$$l_0 = \frac{1}{m}\,, \tag{4.1.9}$$

$$C = \frac{1}{m} \tag{4.1.10}$$

and

$$c_L = 24\,\pi\left(\frac{1}{m} + K\right) = 24\,\pi\,a_L, \qquad c_R = 24\,\pi\left(\frac{1}{m} - K\right) = 24\,\pi\,a_R\,, \tag{4.1.11}$$

where in (4.1.11) we have used (4.1.5). Using (4.1.6) we get

$$c_R = 6\,k_R, \qquad c_L = 48\,\pi\,K + 6\,k_R\,. \tag{4.1.12}$$

This gives the expressions for c_L and c_R in terms of the coefficients k_R and K of the gauge and gravitational Chern-Simons terms. By the argument outlined earlier, this result will not be modified by higher derivative corrections in the theory.

The argument given above can be easily generalized to the cases where the theory contains additional U(1) gauge fields with non-degenerate Chern-Simons terms and the black hole is charged under these gauge fields. The analysis of §3.4 shows that the expressions for c_L and c_R are independent of the action involving these gauge fields and hence will continue to be given by eqs.(4.1.12). The quantity $\widehat{q}$ appearing in the expression (3.4.35) the black hole entropy will however depend on the additional gauge charges via eq.(3.4.31).

The results for black hole entropy computed using the general arguments outlined above have been verified by explicit computation in heterotic string theory after including all tree level four derivative corrections [133, 44] and also in five dimensional supergravity theories [134, 135, 136] with curvature squared corrections [137].

4.2 Applications to black holes in string theory

We shall now apply the observations made in §4.1 to the study of black holes discussed in §3.1.2-§3.1.5. We shall use the second description of the theory, – as a $\mathbb{Z}_N$ orbifold of heterotic or type II string theory on $T^4 \times \widehat{S}^1 \times S^1$, – and consider a black hole with $\widehat{n}$ units of momentum and $-\widehat{w}$ units of fundamental string winding charge along $\widehat{S}^1$ and N' units of Kaluza-Klein monopole charge and $-W'$ units of H-monopole charge associated with the circle S^1. From (3.1.28) we see that such a state has

$$Q = \begin{pmatrix} \widehat{n} \\ 0 \\ \widehat{w} \\ 0 \end{pmatrix}, \qquad P = \begin{pmatrix} 0 \\ W' \\ 0 \\ N' \end{pmatrix} \tag{4.2.1}$$

giving

$$Q^2 = 2\widehat{n}\widehat{w}, \qquad P^2 = 2\,W'\,N', \qquad Q \cdot P = 0\,. \tag{4.2.2}$$

In order to apply the formalism described in §4.1 we need to ensure that the only electric field carried by the solution comes from the dimensional reduction of the metric along a compact circle. In this case however there are two sets of electric fields corresponding to the dimensional reduction of the metric along $\widehat{S}^1$ and the dimensionl reduction of the NS sector 2-form field along $\widehat{S}^1$. To avoid this problem we take the heterotic or type II string theory on $T^4 \times S^1/\,\mathbb{Z}_N$ and dualize the 2-form field B_{MN} in the five remaining dimensions into a gauge field $\mathcal{B}_M$. We then compactify the resulting theory on $\widehat{S}^1$. In this description the fundamental string winding charge $\widehat{w}$ along $\widehat{S}^1$ appears as a magnetic charge of the four dimensional gauge field $\mathcal{B}_\mu$. As a result the quantum numbers N', W' and $\widehat{w}$ appear as magnetic charges whereas the quantum number $\widehat{n}$ appears as electric charge.

As in §4.1 we proceed by making the ansatz that the AdS_2 factor in the near horizon geometry of the black hole combines with $\widehat{S}^1$ to produce an AdS_3 factor. If this ansatz leads to a finite size AdS_3 then our ansatz is self-consistent. In this case we can regard the near horizon geometry of the black hole as that of an extremal BTZ black hole in a three dimensional theory of gravity, obtained by compactifying the full string theory on $(T^4 \times S^1)/\,\mathbb{Z}_N \times S^2$. The three dimensional theory obtained this way turns out to have a (0,4) supersymmetry and associated SU(2) gauge fields which come from the isometries of S^2. Thus due to the arguments outlined in §4.1, the central charges c_L and c_R appearing in the formula for the entropy can be calculated from the knowledge of the Lorentz and SU(2) Chern-Simons terms. The coefficients of the Chern-Simons terms in turn can be calculated by carefully keeping track of various non-covariant terms appearing in the original Lagrangian density as well those appearing in the process of dimensional reduction. The final result for $N', W' > 0$, $\widehat{w} < 0$ is [19, 20, 22]

$$\begin{aligned} c_R &= -6(N'W'\widehat{w} + 2\widehat{\alpha}\widehat{w}), \\ c_L &= -6(N'W'\widehat{w} + 4\widehat{\alpha}\widehat{w})\,, \end{aligned} \tag{4.2.3}$$

where $\widehat{\alpha} = 1$ for heterotic string theory and 0 for type II string theory. On the other hand with the convention we have chosen the quantity q appearing in (3.4.35) is given by

$$q = \widehat{n}\,. \tag{4.2.4}$$

Thus if we take the supersymmetric configuration

$$N', W' > 0, \qquad \widehat{n}, \widehat{w} < 0\,, \tag{4.2.5}$$

then from (3.4.35) we get

$$S_{BH} = 2\pi\sqrt{(N'W' + 4\widehat{\alpha})\,\widehat{w}\,\widehat{n}}\,. \tag{4.2.6}$$

This agrees with (3.1.59) and (3.2.31). By switching on the electric charges associated with various gauge fields in the way described in §3.4 and using T-duality invariance one can in fact argue that the more general result (3.1.53) (and the corresponding result (3.2.26) in $\mathcal{N} = 2$ supergravity theory) is exact in the limit $\sqrt{Q^2P^2 - (Q\cdot P)^2} >> P^2$. This exact agreement for supersymmetric black holes is somewhat mysterious since neither the Gauss-Bonnet term nor the $\mathcal{N} = 2$ supergravity analysis described in §3.2 captures the complete set of terms even at the four derivative level. Neither do they satisfy the condition which led to (4.2.6), namely neither of these theories come from dimensional reduction on $\widehat{S}^1$ of a supersymmetric action in one higher dimension.

For the non-supersymmetric configuration

$$N', W', \widehat{n} > 0, \qquad \widehat{w} < 0\,, \tag{4.2.7}$$

we get

$$S_{BH}^{ns} = 2\pi\sqrt{(N'W' + 2\widehat{\alpha})|\widehat{n}\widehat{w}|}\,. \tag{4.2.8}$$

This does not agree with either (3.1.61) or (3.2.35).

As long as the near horizon geometry of the black hole is given by a locally AdS_3 space, the results (4.2.6), (4.2.8) are exact. Indeed, this has now been verified by explicit computation of the black hole entropy [44, 133] keeping all the four derivative terms in the effective action [138, 139] and comparing the result for extremal black hole entropy with the expansion of (4.2.6), (4.2.8) up to first non-leading order in $1/|N'W'|$. In §4.3 we shall examine under what condition this approximation breaks down.

4.3 Limitations of AdS_3 based approach

Clearly the existence of an AdS_3 factor in the near horizon geometry gives us results which are much stronger than the ones which can be derived based on the existence of only an AdS_2 factor. In this section we shall discuss the approximation under which (3.4.35) holds and possible corrections to this formula.

The main underlying assumption behind (3.4.35) is that the black hole solution is described by an effective three dimensional theory of gravity with a generally covariant action in three dimensions of the form (3.4.1). In this case we can look for solutions in this three dimensional theory with SO(2,2) isometry which corresponds to an AdS_3 space. However this SO(2,2) isometry of the near horizon background is only an approximate symmetry since due to the compactness of the angular coordinate of the BTZ black hole the actual space is a quotient of the AdS_3 space by the group of 2π translation along this coordinate. The true symmetry of the background is a subgroup of SO(2,2) that commutes with the translation and this is simply SO(2,1)×U(1), – the product of the isometry group of AdS_2 and the group of translations along the compact direction. Let us denote the coordinate along this compact direction by y. As long as the physical radius of the y coordinate is large we expect that the effect of the breaking of SO(2,2) isometry will be small and (3.4.35) will be valid. However if this radius is of order unity, then the SO(2,2) symmetry of AdS_3 to be broken strongly and it will be more appropriate to regard the background as a two dimensional background by dimensionally reducing the theory along y. The effective two dimensional action governing the dynamics in AdS_2 space, besides having a 'local' piece of the form (3.4.1) with three dimensional general coordinate invariance, contains additional terms which cannot be written as dimensional reduction of a generally covariant three dimensional action.[18] There are various sources of these additional terms, *e.g.* due to the quantization of the momenta along the y direction, contribution to the effective action from various euclidean branes wrapping the y circle, etc. In the presence of such terms there will be additional contribution to the entropy which are not of the form (3.4.35). These additional corrections can be interpreted as due to the corrections to the full string theory partition function on thermal AdS_3 [21, 22] or equivalently as corrections to the Cardy formula in the CFT living on the boundary of AdS_3, but there is no simple way to calculate these corrections without knowing the details of this CFT.

We will illustrate this in the context of the black holes discussed in §4.2. As can be seen from eqs.(3.1.56), (3.1.58) for $\widehat{\alpha} = 0$, in the leading supergravity approximation the near horizon values of the radii $\widehat{R}$ and R of $\widehat{S}^1$ and S^1 and field S representing square of the inverse string coupling are given by

$$\widehat{R} = \sqrt{\left|\frac{\widehat{n}}{\widehat{w}}\right|}, \qquad R = \sqrt{\left|\frac{W'}{N'}\right|}, \qquad u_S = \sqrt{\left|\frac{\widehat{n}\widehat{w}}{N'W'}\right|}. \tag{4.3.1}$$

From this we see that if we take $|\widehat{n}|$ large keeping the other charges fixed, the radius $\widehat{R}$ of the circle $\widehat{S}^1$ becomes large. Thus we expect that in this limit the $SO(2,2)$ isometry of the near horizon geometry will be a good approximation and the entropy will have the form given in (4.2.6) even after inclusion of higher derivative corrections. However when all charges are of the same order then the radius of $\widehat{S}^1$ becomes of order unity and the higher derivative corrections to the action will contain terms which cannot be regarded as the dimensional reduction of a three dimensional general coordinate invariant action of the form given in (3.4.2). Consequently the higher derivative corrections to the entropy will cease to be of the form given in (4.2.6).

This can be seen explicitly by taking into account the effect of the four derivative Gauss-Bonnet term in the four dimensional effective action describing heterotic string compactification on $T^4 \times$

[18]This can be seen even in ordinary Kaluza-Klein compactification of flat space-time. If the space-time contains a compact circle of radius R, then the quantum effective action will typically involve terms with complicated dependence on R. Since the dimensional reduction of a higher dimensional generally covariant action on a circle of radius R produces a Lagrangian density proportional to R, not all the terms in the quantum effective action can be viewed as coming from the dimensional reduction of a higher dimensional generally covariant action.

$\widehat{S}^1 \times S^1$. An expression for the entropy of a black hole in the presence of a Gauss-Bonnet term has been given in (3.1.49). For large $|\widehat{n}|$, u_S given in (4.3.1) is large. In this case we can approximate $\phi(u_a, u_S)$ by its large u_S limit given in (3.1.50) and the corresponding entropy (3.1.59) agrees with the result (4.2.6). However if all the charges are of the same order, then u_S given in (4.3.1) is of order unity and we cannot approximate $\phi(u_a, u_S)$ by its large u_S limit. Instead we need to use the complete expression given in (3.1.49) with Q^2, P^2 and $Q \cdot P$ given in (4.2.2):

$$S_{BH} \simeq 2\pi\sqrt{\widehat{n}\widehat{w}N'W'} + 64\pi^2\phi\left(0, \sqrt{\left|\frac{\widehat{n}\widehat{w}}{N'W'}\right|}\right) . \tag{4.3.2}$$

For the specific case of heterotic string theory on $T^4 \times \widehat{S}^1 \times S^1$ we get from (3.1.40), (3.1.41) for the case $\mathcal{M} = K3$, $N = 1$,

$$\phi(a, S) = -\frac{3}{16\pi^2} \ln\left(2S|\eta(a + iS)|^4\right) , \tag{4.3.3}$$

up to an additive constant. This gives

$$\Delta S_{BH} \simeq 64\,\pi^2\,\phi(0, u_S)\big|_{u_S=\sqrt{|\widehat{n}\widehat{w}/N'W'|}} = -12\,\ln\left[2\sqrt{\left|\frac{\widehat{n}\widehat{w}}{N'W'}\right|}\eta\left(i\sqrt{\left|\frac{\widehat{n}\widehat{w}}{N'W'}\right|}\right)^4\right] . \tag{4.3.4}$$

This clearly has a complicated $\widehat{n}$-dependence and is not in agreement with the simple form (4.2.6).

It is instructive to study the origin of the terms which break the SO(2,2) symmetry of AdS_3 in this specific example. First of all (4.3.4) contains a correction term proportional to $\ln S \sim \ln\left|\frac{\widehat{n}\widehat{w}}{N'W'}\right|$. This can be traced to the effect of replacing the continuous integral over the momentum along S^1 by a discrete sum. There are also additional corrections involving powers of $e^{-2\pi u_S}$. These can be traced to the effect of Euclidean 5-branes wrapped on $K3 \times \widehat{S}^1 \times S^1$ [66]. Since the 5-brane has one of its legs along $\widehat{S}^1$, it breaks the SO(3,1) isometry of Euclidean AdS_3.

The above example also illustrates the basic difference between the approximation schemes used by the AdS_3 and AdS_2 based approaches. The AdS_3 based approach is useful when we take the momentum along the AdS_3 circle S^1 to be large keeping the other charges fixed. In this limit the size of S^1 becomes large (see eq.(4.3.1)) and hence the $SO(2,2)$ symmetry of AdS_3 is broken weakly. As a result the entropy has the form (3.4.35). In the CFT living on the boundary of AdS_3, this corresponds to a state with large L_0 (or $\bar{L}_0$) eigenvalue, keeping the central charge fixed. This is precisely the limit in which the Cardy formula for the degeneracy of states is valid. On the other hand the AdS_2 based approach is useful if all the charges are large since in this limit the AdS_2 has small curvature, and we can use the derivative expansion of the effective action to find a systematic expansion of the entropy and the entropy function in inverse powers of charges.

4.4 Small black hole revisited

Given this understanding of the range of validity of the AdS_3 based approach, let us now return to the case of small black holes. In this case we have $N' = 0$, $W' = 0$. Eq.(4.2.6) shows that for $\widehat{n}\widehat{w} > 0$, i.e. for supersymmetric small black holes, the entropy is given by

$$S_{BH} = 4\pi\sqrt{\widehat{\alpha}\widehat{w}\widehat{n}}\,, \tag{4.4.1}$$

if we take the large $\widehat{n}$ limit at fixed $\widehat{w}$. In fact since we have already argued earlier that for large charges the result for the entropy depends only on the combination $Q^2 = 2\widehat{n}\widehat{w}$, (4.4.1) must be valid for large $\widehat{n}\widehat{w}$. For heterotic string theory $\widehat{\alpha} = 1$ and the result is in perfect agreement with the statistical entropy (3.3.1). However for type II string theory $\widehat{\alpha} = 0$ and the result is in disagreement with the statistical entropy formula (3.3.2).

The origin of this discrepancy for type II string theories is not completely clear at this stage. Since (4.4.1) gives $S_{BH} = 0$, the most conservative point of view would be that one cannot find a solution to the equations of motion with the ansatz that there is an underlying locally AdS_3 factor. This still leaves open the possibility that type II string theory admits a small black hole solution whose near horizon geometry has an AdS_2 factor, and finite entropy which can be computed using the entropy function method after taking into account all the higher derivative corrections to the tree level effective action. A different approach to this problem has been described in [105, 106, 107, 108].

One can also carry out a similar analysis for non-supersymmetric small black holes. If we set $N' = W' = 0$ in eq.(4.2.8) we get

$$S^{ns}_{BH} = 2\sqrt{2}\pi\sqrt{\widehat{\alpha}|\widehat{n}\widehat{w}|}\,. \tag{4.4.2}$$

For $\widehat{\alpha} = 1$ this agrees with the statistical entropy of these black holes computed from the spectrum of elementary string states carrying right-moving excitations with $\bar{L}_0 = |\widehat{n}\widehat{w}|$ and no left-moving excitations. However for $\widehat{\alpha} = 0$ the result again fails to agree with the statistical entropy of small non-supersymmetric black holes in type II string theory which is given by $2\sqrt{2}\pi\sqrt{|\widehat{n}\widehat{w}|}$.

5 Precision Counting of Dyon States

In this section we shall compute the statistical entropy of a class of dyonic black holes in the theories described in §3.1.2. We shall carry out our analysis by first counting states of a configuration carrying specific charge vectors $(\vec{Q}, \vec{P})$ in a specific corner of the moduli space and then extend the results to more general charges. In §5.1 we shall describe this specific microscopic configuration. Also, in order to guide the reader through the rest of the section, we shall summarize in this section the results of §5.2-§5.7. In §5.2 we carry out the computation of BPS states for the microscopic configuration described in §5.1. In §5.3 we describe how our results can be extended to more general charge vectors. In §5.4 we discuss how the spectrum of the theory could change discontinuously as we move across walls of marginal stability in the moduli space, and use these results to determine the region of moduli space in which our formula for the degeneracy of dyons remains valid. In §5.5 we study T- and S-duality transformation properties of the degeneracy formula, and use the requirement of duality invariance to determine how the degeneracy changes as we move across a wall of marginal stability. In §5.6 we calculate the statistical entropy of the system, – given by the logarithm of the degeneracy of states, – by expanding the result of §5.2 in a series expansion in inverse powers of charges and compare the results with the results for black hole entropy given in §3.1.4. Finally in §5.7 we demonstrate how the change in the degeneracy across walls of marginal stability can be related to (dis)appearance of 2-centered black holes as we cross the marginal stability walls [140, 141, 142, 143, 144, 145, 146, 147].

Throughout this section we shall set $\alpha' = 1$ unless mentioned otherwise. Our counting of states will follow [6, 7, 8]. The original formula for the degeneracy was first proposed in [9] for the special case of heterotic string theory compactified on T^6, and extended to more general models in [10,11,12]. Various alternative approaches to proving these formulæ have been explored in [13, 14, 15, 16, 17].

5.1 Summary of the results

As in §3.1.2 we consider type IIB string theory on $\mathcal{M} \times \widetilde{S}^1 \times S^1$ where $\mathcal{M}$ is either K3 or T^4, and mod out this theory by a $\mathbb{Z}_N$ symmetry group generated by a transformation g that involves $1/N$ unit of shift along the circle S^1 together with an order N transformation $\widetilde{g}$ in $\mathcal{M}$. $\widetilde{g}$ is chosen in such a way that the final theory has $\mathcal{N} = 4$ supersymmetry. In keeping with the convention described below (3.1.29) we shall take the coordinate radii of $S^1/\mathbb{Z}_N$ and $\widetilde{S}^1$ to be 1. In this convention the original S^1 before orbifolding has coordinate radius N and the $\mathbb{Z}_N$ action involves 2π translation along S^1. Since under this translation various modes get transformed by $\widetilde{g}$ twist instead of remaining invariant, the momentum along S^1 is quantized in multiples of $1/N$ instead of being integers. Following [13] we consider in this theory a configuration with a single D5-brane wrapped on $\mathcal{M} \times S^1$, Q_1 D1-branes wrapped on S^1, a single Kaluza-Klein monopole associated with the circle $\widetilde{S}^1$ with negative magnetic charge, momentum $-n/N$ along S^1 and momentum J along $\widetilde{S}^1$.[19] Since a D5-brane wrapped on $\mathcal{M}$ carries, besides the D5-brane charge, $-\beta$ units of D1-brane charge with β given by the Euler

[19] In short, we have a BMPV black hole [148] at the center of Taub-NUT space.

character of $\mathcal{M}$ divided by 24 [149], the net D1-brane charge carried by the system is $Q_1 - \beta$.

As has already been discussed in §3.1.2, there is a second description of the theory obtained by an S-duality transformation of type IIB string theory, followed by a T-duality transformation along $\widetilde{S}^1$ that takes us to type IIA string theory on $(\mathcal{M} \times \widehat{S}^1 \times S^1)/\mathbb{Z}_N$, and finally a string-string duality transformation that takes us to heterotic (type IIA) string theory on $(T^4 \times \widehat{S}^1 \times S^1)/\mathbb{Z}_N$ for $\mathcal{M} = K3$ ($\mathcal{M} = T^4$). By following the duality transformation rules and the sign conventions given in appendix A we can find the physical interpretation of various charges carried by the system in the second description. We find that it corresponds to a state with momentum $-n/N$ along S^1, a single Kaluza-Klein monopole associated with $\widehat{S}^1$, $(-Q_1 + \beta)$ units of NS 5-brane charge along $T^4 \times S^1$, $-J$ units of NS 5-brane charge along $T^4 \times \widehat{S}^1$ and a single fundamental string wound along S^1 [6]. In particular the Kaluza-Klein monopole charge associated with $\widetilde{S}^1$ in the first description gets mapped to the fundamental string winding number along S^1 in the second description and the D5-brane wrapped on $\mathcal{M} \times S^1$ in the first description gets mapped to Kaluza-Klein monopole charge associated with $\widehat{S}^1$ in the second description. Using the convention of (3.1.28) we see that this corresponds to the charge vectors[20]

$$Q = \begin{pmatrix} 0 \\ -n/N \\ 0 \\ -1 \end{pmatrix}, \quad P = \begin{pmatrix} Q_1 - \beta \\ -J \\ 1 \\ 0 \end{pmatrix} . \tag{5.1.1}$$

This gives

$$Q^2 = 2n/N, \qquad P^2 = 2(Q_1 - \beta), \qquad Q \cdot P = J\,. \tag{5.1.2}$$

In the rest of this section we shall summarize the results of §5.2 - §5.7.

We denote by $d(\vec{Q}, \vec{P})$ the number of bosonic minus fermionic quarter BPS supermultiplets carrying a given set of charges $(\vec{Q}, \vec{P})$, a supermultiplet being considered bosonic (fermionic) if it is obtained by tensoring the basic 64 dimensional quarter BPS supermultiplet, with helicity ranging from $-\frac{3}{2}$ to $\frac{3}{2}$, with a supersymmetry singlet bosonic (fermionic) state. For the charge vector given in (5.1.1) and in the region of the moduli space where the type IIB string coupling in the first description

[20] Ref. [6] actually considered a more general charge vector where Q_5, representing the number of D5-branes wrapped along $K3 \times S^1$, was arbitrary and found that $d(\vec{Q}, \vec{P})$, expressed as a function of Q^2, P^2 and $Q \cdot P$, continues to be given by the same function (5.1.3). However the analysis of dyon spectrum becomes simpler for $Q_5 = 1$. For this reason we have set $Q_5 = 1$. We shall comment on the more general case at the end of §5.3 (see paragraph containing eq.(5.3.32).).

of the theory is small, our result for $d(\vec{Q},\vec{P})$ is[21]

$$d(\vec{Q},\vec{P}) = (-1)^{Q\cdot P+1}\,\frac{1}{N}\int_{\mathcal{C}} d\widetilde{\rho}\, d\widetilde{\sigma}\, d\widetilde{v}\, e^{-\pi i(N\widetilde{\rho}Q^2+\widetilde{\sigma}P^2/N+2\widetilde{v}Q\cdot P)}\,\frac{1}{\widetilde{\Phi}(\widetilde{\rho},\widetilde{\sigma},\widetilde{v})}\,, \tag{5.1.3}$$

where $\mathcal{C}$ is a three real dimensional subspace of the three complex dimensional space labelled by $(\widetilde{\rho},\widetilde{\sigma},\widetilde{v}) \equiv (\widetilde{\rho}_1+i\widetilde{\rho}_2, \widetilde{\sigma}_1+i\widetilde{\sigma}_2, \widetilde{v}_1+i\widetilde{v}_2)$, given by

$$\begin{aligned} &\widetilde{\rho}_2 = M_1, \quad \widetilde{\sigma}_2 = M_2, \quad \widetilde{v}_2 = -M_3, \\ 0 \le \widetilde{\rho}_1 \le 1, \quad &0 \le \widetilde{\sigma}_1 \le N, \quad 0 \le \widetilde{v}_1 \le 1\,, \end{aligned} \tag{5.1.4}$$

M_1, M_2 and M_3 being large but fixed positive numbers with $M_3 << M_1, M_2$, and

$$\begin{aligned} \widetilde{\Phi}(\widetilde{\rho},\widetilde{\sigma},\widetilde{v}) &= e^{2\pi i(\widetilde{\alpha}\widetilde{\rho}+\widetilde{\gamma}\widetilde{\sigma}+\widetilde{v})} \\ &\times \prod_{b=0}^{1}\prod_{r=0}^{N-1}\prod_{\substack{k'\in\mathbb{Z}+\frac{r}{N}, l\in\mathbb{Z}, j\in 2\mathbb{Z}+b \\ k',l\ge 0, j<0 \text{ for } k'=l=0}} [1-\exp\{2\pi i(k'\widetilde{\sigma}+l\widetilde{\rho}+j\widetilde{v})\}]^{\sum_{s=0}^{N-1} e^{-2\pi i s l/N} c_b^{(r,s)}(4k'l-j^2)}\,. \end{aligned} \tag{5.1.5}$$

The coefficients $c_b^{(r,s)}(u)$ have been defined through eqs. (B.2), (B.6):

$$\begin{aligned} F^{(r,s)}(\tau,z) &\equiv \sum_{b=0}^{1}\sum_{j\in 2\mathbb{Z}+b, n\in\mathbb{Z}/N} c_b^{(r,s)}(4n-j^2)e^{2\pi i n\tau+2\pi i j z} \\ &= \frac{1}{N}Tr_{RR;\widetilde{g}^r}\left(\widetilde{g}^s(-1)^{F_L+F_R}e^{2\pi i\tau L_0}e^{-2\pi i\bar{\tau}\bar{L}_0}e^{2\pi i F_L z}\right), \qquad 0\le r,s\le N-1\,, \end{aligned} \tag{5.1.6}$$

where $Tr_{RR;\widetilde{g}^r}$ denotes trace over $\widetilde{g}^r$ twisted RR-sector states in the two dimensional superconformal field theory with target space $\mathcal{M}$, F_L, F_R denote the left- and right-handed fermion numbers in this world-sheet theory, and L_n and $\bar{L}_n$ denote the left- and right-handed Virasoro generators. The additive constants in L_0 and $\bar{L}_0$ are adjusted so that supersymmetric ground states in the RR sector have $L_0 = \bar{L}_0 = 0$, – this is a convention we shall follow throughout the rest of the article. The constants $\widetilde{\alpha}$ and $\widetilde{\gamma}$ are given in terms of the coefficients $c_b^{(r,s)}(u)$ via eqs.(B.20), (C.20):

$$Q_{r,s} = N\,\left(c_0^{(r,s)}(0) + 2\,c_1^{(r,s)}(-1)\right)\,, \tag{5.1.7}$$

[21] The overall factor of $(-1)^{Q\cdot P+1}$ was left out in the analysis of [6,7,8]. The $(-1)^{Q\cdot P}$ factor appeared previously in [15,150] and reflects the difference in statistics between the four and five dimensional viewpoint for modes carrying odd units of $Q\cdot P$ quantum number. As we shall see in (5.2.21), the -1 factor appears from the spectrum of bound state of a D1-D5 system to the Kaluza-Klein monopole.

$$\widetilde{\alpha} = \frac{1}{24N}\,Q_{0,0} - \frac{1}{2N}\sum_{s=1}^{N-1} Q_{0,s}\,\frac{e^{-2\pi i s/N}}{(1-e^{-2\pi i s/N})^2}\,, \qquad \widetilde{\gamma} = \frac{1}{24N}\,Q_{0,0} = \frac{1}{24N}\,\chi(\mathcal{M})\,. \tag{5.1.8}$$

As has been shown in (C.24), $\widetilde{\Phi}$ satisfies the periodicity conditions

$$\widetilde{\Phi}(\widetilde{\rho}+1,\widetilde{\sigma},\widetilde{v}) = \widetilde{\Phi}(\widetilde{\rho},\widetilde{\sigma}+N,\widetilde{v}) = \widetilde{\Phi}(\widetilde{\rho},\widetilde{\sigma},\widetilde{v}+1) = \widetilde{\Phi}(\widetilde{\rho},\widetilde{\sigma},\widetilde{v})\,. \tag{5.1.9}$$

Using this we can express $d(\vec{Q},\vec{P})$ as

$$d(\vec{Q},\vec{P}) = (-1)^{Q\cdot P+1}\, g\left(\frac{N}{2}Q^2, \frac{1}{2\,N}\,P^2, Q\cdot P\right)\,, \tag{5.1.10}$$

where $g(m,n,p)$ are the coefficients of Fourier expansion of the function $1/\widetilde{\Phi}(\widetilde{\rho},\widetilde{\sigma},\widetilde{v})$:

$$\frac{1}{\widetilde{\Phi}(\widetilde{\rho},\widetilde{\sigma},\widetilde{v})} = \sum_{m,n,p} g(m,n,p)\, e^{2\pi i(m\,\widetilde{\rho}+n\,\widetilde{\sigma}+p\,\widetilde{v})}\,. \tag{5.1.11}$$

Although the formulæ (5.1.3), (5.1.10) were originally derived for charge vectors of the form (5.1.1), we show in §5.3 that the same formula holds for a more general class of charge vectors for weak type IIB string coupling in the first description [151]. In the subspace $\mathcal{V}$ introduced in (3.1.28) this general charge vector takes the form

$$Q = \begin{pmatrix} k_3 \\ k_4 \\ k_5 \\ -1 \end{pmatrix}, \qquad P = \begin{pmatrix} l_3 \\ l_4 \\ 1 \\ 0 \end{pmatrix}, \qquad k_4 \in \mathbb{Z}/N, \quad k_i, l_i \in \mathbb{Z} \quad \text{otherwise}\,. \tag{5.1.12}$$

In writing down the formula (5.1.3) we have made an implicit assumption. Eqs.(5.1.3) and (5.1.10) are equivalent only if the sums over m, n, p in (5.1.11) are convergent for large imaginary $\widetilde{\rho}$, $\widetilde{\sigma}$ and $-\widetilde{v}$, – the region in which the contour $\mathcal{C}$ lies. This in particular requires that the sum over m and n are bounded from below, and that for fixed m and n the sum over p is bounded from above. By examining the formula (5.1.5) for $\widetilde{\Phi}$ we can see that the sum over m and n are indeed bounded from below. Furthermore, using the fact that the coefficients $c_b^{(r,s)}(u)$ are non-zero only for $4u \geq -b^2$, we can verify that with the exception of the contribution from the $k' = l = 0$ term in this product, the other terms, when expanded in a power series expansion in $e^{2\pi i\widetilde{\rho}}$, $e^{2\pi i\widetilde{\sigma}}$ and $e^{2\pi i\widetilde{v}}$, does have the form of (5.1.11) with p bounded from above (and below) for fixed m, n. However for the $k' = l = 0$ term, which gives a contribution $e^{-2\pi i\widetilde{v}}/(1-e^{-2\pi i\widetilde{v}})^2$, there is an ambiguity in carrying out the series expansion. We could either use the form given above and expand the denominator in a series expansion in $e^{-2\pi i\widetilde{v}}$, or express it as $e^{2\pi i\widetilde{v}}/(1-e^{2\pi i\widetilde{v}})^2$ and expand it in a series expansion in $e^{2\pi i\widetilde{v}}$. As will be discussed below (5.2.21), depending on the angle between S^1 and $\widetilde{S}^1$ in the first description, only one of these expansions produce the degeneracy formula correctly via (5.1.11) [6]. The physical spectrum actually changes as this angle passes through 90° since at this point the system is only marginally stable. On the other hand our degeneracy formula (5.1.3), (5.1.4) implicitly requires that we expand this factor in powers of $e^{-2\pi i\widetilde{v}}$ since only in this case the sum over p in (5.1.11) is bounded from above for fixed m, n. Thus as it stands the formula is valid for a specific range of values of the angle between S^1 and $\widetilde{S}^1$. In the second description of the system this corresponds to a region in the moduli space where the axion field a, obtained by dualizing the NS sector 2-form field, has a positive sign. For the negative sign of the axion the correct formula for $d(\vec{Q},\vec{P})$ is obtained by expanding $e^{-2\pi i\widetilde{v}}/(1-e^{-2\pi i\widetilde{v}})^2$ in positive powers of $e^{2\pi i\widetilde{v}}$. In this case the sum over p in (5.1.11) is bounded from below, and in (5.1.3) we need to take a different contour $\widehat{C}$ to get the correct formula for the degeneracy:

$$\begin{gathered} \widetilde{\rho}_2 = M_1, \quad \widetilde{\sigma}_2 = M_2, \quad \widetilde{v}_2 = M_3, \\ 0 \le \widetilde{\rho}_1 \le 1, \quad 0 \le \widetilde{\sigma}_1 \le N, \quad 0 \le \widetilde{v}_1 \le 1\,, \end{gathered} \tag{5.1.13}$$

where M_1, M_2 and M_3 are large positive numbers with $M_3 << M_1, M_2$.

It turns out that walls of marginal stability, – codimension one subspaces of the asymptotic moduli space on which the BPS mass of the system becomes equal to the sum of masses of two or more other states carrying the same total charge, – are quite generic for quarter BPS states in $\mathcal{N}=4$ supersymmetric string theories [152]. The shapes of these walls have been analyzed in detail in §5.4. For fixed values of the other moduli, the marginal stability walls in the axion-dilaton moduli space are either circles or straight lines, with the property that they never intersect in the interior of the upper half plane, but can intersect either at $i\infty$ or at rational points on the real axis. Thus a given domain bounded by the walls of marginal stability has vertices either at rational points on the real axis or at $i\infty$. As we vary the other moduli, the shapes of the walls in the axion-dilaton moduli space changes, but the vertices do not change. Thus every domain may be given an invariant characterization by specifying the vertices of the domain. While comparing these domains for different states carrying different charges and / or different asymptotic values of the other moduli, we shall call them the same if their vertices in the axion-dilaton moduli space coincide.

We expect the spectrum of quarter BPS states to change discontinuously as the asymptotic moduli fields pass through any of these walls of marginal stability.[22] Thus the expression for the degeneracy

[22]What we refer to as a wall is actually a codimension one subspace of the full moduli space. If a state becomes marginally stable on a surface of codimension ≥ 2, then we can always move around this subspace in going from one point to another and hence the spectrum cannot change discontinuously.

given above holds only in a finite domain of the moduli space, bounded by the walls of marginal stability. More precisely, we have the formula for the degeneracy in two different domains separated by the domain wall on which the axion field in the second description vanishes. We denote by $\mathcal{R}$ the domain in which the original formula (5.1.3), (5.1.4) is valid, and by $\mathcal{L}$ the domain in which the same formula with the modified integration contour given in (5.1.13) is valid. An important question is: how does the degeneracy formula look inside other domains? It turns out that invariance of the theory under S- and T-duality symmetries gives non-trivial information about the degeneracy formula inside other domains and for other charge vectors. However before describing the logic behind this analysis, we need to say a few words about duality invariance.

First note that that although (5.1.3) has been expressed as a function of the T-duality invariant combinations P^2, Q^2 and $Q \cdot P$, it was derived initially for special charge vectors $\vec{Q}$, $\vec{P}$ described in (5.1.1), and extended to more general charge vectors of the form (5.1.12) in §5.3. Even then this is not the most general charge vector of the theory. One can try to extend this to more general charge vectors using T-duality symmetry of the theory. However here we encounter two problems. First of all two charge vectors carrying the same values of P^2, Q^2 and $Q \cdot P$ may not necessarily be related by a T-duality transformation.[23] In that case the degeneracy of states for these two charge vectors could be different. An example of this is that a state carrying fundamental heterotic string winding charge w' along S^1 with $w' \neq 0 \bmod N$ can never be related to a state carrying $w' = 0 \bmod N$ even if they have the same values of Q^2, P^2 and $Q \cdot P$, since the former carries twisted sector electric charge and the latter carries untwisted sector electric charge. Thus the degeneracy formula we have derived holds at best for charges which are in the same T-duality orbit as the general charge vector (5.1.12). Second, even though we expect T-duality to be a symmetry of the theory, we should remember that it acts not only on the charges but also on the asymptotic moduli. Had the spectrum been independent of the asymptotic moduli, we could have demanded that the spectrum remains invariant under T-duality transformation of the charges. However if a T-duality transformation takes the asymptotic moduli fields across a wall of marginal stability, then all we can say is that the spectrum remains unchanged under a simultaneous T-duality transformation of the moduli fields and the charges. We show in §5.5 that a T-duality transformation on the moduli space preserves the domains bounded by marginal stability walls in the sense that it preserves the vertices of the domain while changing the shapes of the walls. Hence we do expect that the degeneracy formula within a given domain characterized by a fixed set of vertices will be invariant under a T-duality transformation acting on

[23] Generically they are related by a continuous T-duality transformation but only a discrete subgroup of this is a genuine symmetry of the theory.

the charges only. In the subpace $\mathcal{V}$ of electric and magnetic charges introduced in (3.1.28), – spanned by the momenta, fundamental string winding charge, H- and Kaluza-Klein monopole charges along the circles $\widehat{S}^1$ and S^1 in the second description, – the T-duality orbit of (5.1.12) has been analyzed at the end of §5.3. We find that the orbits contain four dimensional electric and magnetic charge vectors $\vec{Q}$ and $\vec{P}$ in this subspace satisfying charge quantization laws and the following additional restrictions:

1. The electric charge vector must correspond to the charge carried by a g twisted state, i.e. in the second description the fundamental string winding charge $-k_6$ along S^1 must be 1 mod N.

2. The Kaluza-Klein monopole charge l_5 associated with the circle $\widehat{S}^1$ in the second description must be 1 mod N.

3. The electric and the magnetic charge vectors $Q = \begin{pmatrix} k_3 \\ k_4 \\ k_5 \\ k_6 \end{pmatrix}$ and $P = \begin{pmatrix} l_3 \\ l_4 \\ l_5 \\ l_6 \end{pmatrix}$ must satisfy the primitivity conditions:

$$\text{g.c.d.}(Nk_3l_4 - Nk_4l_3, k_5l_6 - k_6l_5, k_3l_5 - k_5l_3 + k_4l_6 - k_6l_4) = 1\,. \tag{5.1.14}$$

These are necessary conditions for the charge vectors to be in the orbit of (5.1.12), but we have not proven that these conditions are sufficient. Thus our formula (5.1.3), (5.1.4) holds in the domain $\mathcal{R}$ for charge vectors satisfying these criteria, and possibly some additional criteria.[24]. For the same charge vectors, the formula (5.1.3) with the contour $\mathcal{C}$ replaced by a new contour $\widehat{\mathcal{C}}$ given in (5.1.13), hold in the domain $\mathcal{L}$.

For the choice $\mathcal{M} = K3$, i.e. theories for which the second description is based on orbifolds of heterotic string theory, we can relax the conditions somewhat by taking an initial configuration with multiple D5-branes [6], with the number of D1 and D5-branes being relatively prime. This analysis has been described in appendix E and, after being combined with the analysis of §5.3, shows that our degeneracy formula (5.1.3), (5.1.4) holds in the domain $\mathcal{R}$ for a general charge vector of the form

$$Q = \begin{pmatrix} k_3 \\ k_4 \\ k_5 \\ -1 \end{pmatrix}, \qquad P = \begin{pmatrix} l_3 \\ l_4 \\ l_5 \\ 0 \end{pmatrix}, \qquad k_4 \in \mathbb{Z}/N, \quad k_i, l_i \in \mathbb{Z} \quad \text{otherwise}, \quad \text{g.c.d.}(l_3, l_5) = 1\,. \tag{5.1.15}$$

[24]Possible dependence of the degeneracy formula on invariants other than the continuous T-duality invariants have been anticipated in [153]

Thus the degeneracy formula will continue to hold for any charge vector related to (5.1.15) by a T-duality transformation. This allows us to relax the condition 2 on l_5 given above. This also relaxes the condition (5.1.14) to

$$\text{g.c.d.}\{k_i l_j - k_j l_i,\, N k_4 l_s - N k_s l_4,\, k_4 l_6 - k_6 l_4; \quad i, j = 3, 5, 6,\, s = 3, 5\} = 1\,. \tag{5.1.16}$$

It may also be possible to relax these conditions in a similar manner for orbifolds of type IIB string theory on $T^4 \times \widetilde{S}^1 \times S^1$ by taking multiple D5-branes as in the case of $K3 \times \widetilde{S}^1 \times S^1$. However a careful analysis, taking into account the dynamics of Wilson lines on multiple D5-branes, has not been carried out so far.

Given this, we can now study the consequences of S-duality invariance. It turns out that unlike T-duality, typically an S-duality transformation takes us from one domain to another. Thus invariance under S-duality can be used to derive the formula for the degeneracy in domains other than the original domain where it was computed. We find that the degeneracies inside the other domains formally look the same as (5.1.3) but the contour $\mathcal{C}$ over which we need to carry out the integration is different in different domains. On the other hand there are a few S-duality transformations which preserve a given domain. These must be symmetries of the degeneracy formula, – namely two charge vectors related by such an S-duality transformation must have the same degeneracy. This is indeed borne out by explicit computation. These issues have been discussed in detail in §5.5.

Finally let us turn to the comparison between statistical entropy – defined as the logarithm of the degeneracy of states – with the black hole entropy. For this we need to extract the behaviour of the degeneracy $d(\vec{Q}, \vec{P})$ for large charges. By performing one of the integrals in (5.1.3) by picking up residues at the poles of the integrand, and other two integrals by saddle point approximation, we can extract this behaviour. The result is that up to first non-leading order, the entropy is given by extremizing a statistical entropy function

$$-\widetilde{\Gamma}_B(\vec{\tau}) = \frac{\pi}{2\tau_2}\,|Q - \tau P|^2 - \ln g(\tau) - \ln g(-\bar{\tau}) - (k+2)\ln(2\tau_2) + \text{constant} + \mathcal{O}(Q^{-2}) \tag{5.1.17}$$

with respect to real and imaginary parts of the complex variable $\tau = \tau_1 + i\tau_2$. The functon $g(\tau)$ and the constant k are the same as the ones which appear in the expression (3.1.40) for the function $\phi(a, S)$ multiplying the coefficient of the Gauss-Bonnet term in the low energy effective action.

With the identification $\tau = u_a + i u_S$ the statistical entropy function (5.1.17) matches the black hole entropy function of the same theory given in (3.1.47). Thus the statistical entropy and the black hole entropy, given by the values of the corresponding entropy functions at their extrema, also agree to this order.

Since the expression for $d(\vec{Q},\vec{P})$ changes across the walls of marginal stability, one might wonder how this affects the large $\vec{Q},\vec{P}$ behaviour of $d(\vec{Q},\vec{P})$. One finds that these changes are exponentially small compared to the leading contribution. Nevertheless one could ask if the changes in $d(\vec{Q},\vec{P})$ across walls of marginal stability can be seen on the black hole side. It turns out that this is indeed possible. First of all, due to the attractor mechanism the entropy of a single centered extremal black hole of the kind analyzed in §3.1 does not change as the asymptotic values of the moduli vary across a wall of marginal stability. However on the black hole side the contribution to the total entropy comes not only from single centered black holes but also from multi-centered black holes carrying the same total charge. It turns out that as we cross a wall of marginal stability, typically a two centered black hole solution (dis)appears, i.e. these solutions exist only on one side of the marginal stability wall [140, 141, 142, 144, 145, 146]. As a result there is a change in the total entropy on the black hole side as well. This change precisely agrees with the result predicted from the exact degeneracy formula [154, 150]. We shall illustrate this in §5.7.

5.2 The counting

We shall now describe the counting of BPS states of the configuration described in (5.1.1), – this will eventually lead to the expressions (5.1.3), (5.1.4) for $d(\vec{Q},\vec{P})$. For this configuration the charges $(\vec{Q},\vec{P})$ are labelled by the set of integers Q_1, n and J. The other two charges, namely the number of D5-branes along $\mathcal{M}\times S^1$ and the number of Kaluza-Klein monopoles associated with the circle $\widetilde{S}^1$ in the first description have been taken to be 1. We shall denote by $h(Q_1,n,J)$ the number of bosonic supermultiplets minus the number of fermionic supermultiplets carrying quantum numbers (Q_1,n,J). Computation of $h(Q_1,n,J)$ is best done in the weak coupling limit of the first description of the system where the quantum numbers n and J can arise from three different sources [6]: the excitations of the Kaluza-Klein monopole carrying certain amount of momentum $-l_0'/N$ along S^1,[25] the overall motion of the D1-D5 system in the background of the Kaluza-Klein monopole carrying certain amount of momentum $-l_0/N$ along S^1 and j_0 along $\widetilde{S}^1$ and the motion of the Q_1 D1-branes in the plane of the D5-brane carrying total momentum $-L/N$ along S^1 and J' along $\widetilde{S}^1$. Thus we have

$$l_0' + l_0 + L = n, \qquad j_0 + J' = J\,. \tag{5.2.1}$$

[25] A Kaluza-Klein monopole associated with the compactification circle $\widetilde{S}^1$ cannot carry any momentum along $\widetilde{S}^1$ since the solution is invariant under translation along $\widetilde{S}^1$ [155].

Let

$$f(\widetilde{\rho},\widetilde{\sigma},\widetilde{v}) = \sum_{Q_1,n,J} (-1)^{J+1}\, h(Q_1,n,J) e^{2\pi i(\widetilde{\rho}n+\widetilde{\sigma}Q_1/N+\widetilde{v}J)}\,, \tag{5.2.2}$$

denote the partition function of the system. Then in the weak coupling limit we can ignore the interaction between the three different sets of degrees of freedom described above, and $f(\widetilde{\rho},\widetilde{\sigma},\widetilde{v})$ is obtained as a product of three separate partition functions:[26]

$$\begin{aligned} f(\widetilde{\rho},\widetilde{\sigma},\widetilde{v}) &= -\frac{1}{64}\left(\sum_{Q_1,L,J'} (-1)^{J'}\, d_{D1}(Q_1,L,J') e^{2\pi i(\widetilde{\sigma}Q_1/N+\widetilde{\rho}L+\widetilde{v}J')}\right) \\ &\qquad \left(\sum_{l_0,j_0} (-1)^{j_0} d_{CM}(l_0,j_0) e^{2\pi i l_0\widetilde{\rho}+2\pi i j_0\widetilde{v}}\right)\left(\sum_{l_0'} d_{KK}(l_0') e^{2\pi i l_0'\widetilde{\rho}}\right)\,, \end{aligned} \tag{5.2.3}$$

where $d_{D1}(Q_1,L,J')$ is the degeneracy of Q_1 D1-branes moving in the plane of the D5-brane carrying momenta $(-L/N, J')$ along $(S^1,\widetilde{S}^1)$, $d_{CM}(l_0,j_0)$ is the degeneracy associated with the overall motion of the D1-D5 system in the background of the Kaluza-Klein monopole carrying momenta $(-l_0/N, j_0)$ along $(S^1,\widetilde{S}^1)$ and $d_{KK}(l_0')$ denotes the degeneracy associated with the excitations of a Kaluza-Klein monopole carrying momentum $-l_0'/N$ along S^1. The factor of $1/64$ in (5.2.3) accounts for the fact that a single quarter BPS supermultiplet has 64 states. In each of these sectors we count the degeneracy weighted by $(-1)^F$ with F denoting space-time fermion number of the state, except for the parts obtained by quantizing the fermion zero-modes associated with the broken supersymmetry generators. Since a Kaluza-Klein monopole in type IIB string theory on $\mathcal{M}\times\widetilde{S}^1\times S^1/\mathbb{Z}_N$ breaks 8 of the 16 supersymmetries, quantization of the fermion zero modes associated with the broken supersymmetry generators give rise to a $2^{8/2}=16$-fold degeneracy with equal number of bosonic and fermionic states. This appears as a factor in $d_{KK}(l_0')$. Furthermore since a D1-D5 system in the background of such a Kaluza-Klein monopole breaks 4 of the 8 remaining supersymmetry generators, we get, from the associated fermion zero modes, a $2^{4/2}=4$-fold degeneracy appearing as a factor in $d_{CM}(l_0,j_0)$, with equal number of fermionic and bosonic states. This factor of 16×4 cancel the $1/64$ factor in (5.2.3). After separating out this factor, we count the contribution to the degeneracy from the rest of the degrees of freedom weighted by a factor of $(-1)^F$.

We shall now compute each of the three pieces, $d_{KK}(l_0')$, $d_{CM}(l_0,j_0)$ and $d_{D1}(Q_1,L,J')$ separately. Before we go on however, we shall fix some conventions. The world-volume of the Kaluza-Klein

[26]Even though the mutual interaction between these three systems vanish, the individual systems may be interacting. In particular we shall see that the dynamics of the D1-D5 system in the Kaluza-Klein monopole background has a strongly interacting component that is responsible for binding the D1-D5 system to the Kaluza-Klein monopole.

monopole as well as that of the D5-brane is 5+1 dimensional with the five spatial directions lying along $\mathcal{M}\times S^1$. By taking the size of $\mathcal{M}$ to be much smaller than that of S^1 we can regard these as 1+1 dimensional world-sheet theories, obtained by dimensional reduction of the original 5+1 dimensional theory on $\mathcal{M}$. We shall follow this viewpoint throughout the rest of this section although we shall often refer to this as the world-volume theory. In particular left- and right-moving modes on the world-volume theory will refer respectively to the modes which move to the left- and right along S^1. The D1-brane world volume theory is of course naturally 1+1 dimensional. We shall choose a convention in which the four unbroken supersymmetry generators of the full configuration act on the right-moving modes in the world-volume theory. This in turn means that the supersymmetry transformation parameters themselves are represented by left-chiral spinors since the transformation laws of the scalars, being proportional to $\bar{\epsilon}\psi$, is non-zero only if the supersymmetry transformation parameter ϵ and the fermion field ψ have opposite chirality. In contrast if a supersymmetry is spontaneously broken then the associated goldstino fermion field ψ on the world-volume has the same chirality as the transformation parameter ϵ due to the transformation law $\delta\psi \propto \epsilon$.

5.2.1 Counting states of the Kaluza-Klein monopole

We consider type IIB string theory in the background $\mathcal{M}\times TN\times S^1$ where TN denotes Taub-NUT space described by the metric [156]

$$ds^2 = \left(1+\frac{R_0}{r}\right)\left(dr^2 + r^2(d\theta^2+\sin^2\theta d\phi^2)\right) + R_0^2\left(1+\frac{R_0}{r}\right)^{-1}(2\,d\psi + \cos\theta d\phi)^2 \tag{5.2.4}$$

with the identifications:

$$(\theta,\phi,\psi) \equiv (2\pi-\theta,\phi+\pi,\psi+\frac{\pi}{2}) \equiv (\theta,\phi+2\pi,\psi+\pi) \equiv (\theta,\phi,\psi+2\pi)\,. \tag{5.2.5}$$

Here R_0 is a constant determining the size of the Taub-NUT space. This describes type IIB string theory compactified on $\mathcal{M}\times\widetilde{S}^1\times S^1$ in the presence of a Kaluza-Klein monopole, with $\widetilde{S}^1$ identified with the asymptotic circle of the Taub-NUT space labeled by the coordinate ψ in (5.2.4). The metric (5.2.4) admits a normalizable self-dual harmonic form ω, given by [157, 158]

$$\omega \propto \frac{r}{r+R_0}d\sigma_3 + \frac{R_0}{(r+R_0)^2}dr\wedge\sigma_3\,, \qquad \sigma_3 \equiv \left(d\psi + \frac{1}{2}\cos\theta d\phi\right)\,. \tag{5.2.6}$$

We now take an orbifold of the theory by a $\mathbb{Z}_N$ group generated by the transformation g. Our goal is to compute the degeneracy of the half-BPS states of the Kaluza-Klein monopole carrying momentum $-l_0'/N$ along S^1. For $\mathcal{M}=K3$, the world-volume supersymmetry on the Kaluza-Klein monopole will be chiral since the supersymmetry generators of type IIB string theory on K3 are chiral. According to our convention there will be eight left-chiral supersymmetry transformation parameters, acting only on the right-moving degrees of freedom. Thus the BPS states of the Kaluza-Klein monopole will correspond to states in this field theory where the right-moving oscillators are in their ground state. For $\mathcal{M} = T^4$ the world-volume theory of the Kaluza-Klein monopole will have eight left-chiral and eight right-chiral supersymmetry transformation parameters. However right-chiral supersymmetries will be broken once we take the $\mathbb{Z}_N$ orbifold, and the unbroken supersymmetries of the theory will again come from the left-chiral spinors. Since the latter act on the right-moving modes, the BPS condition will again require that the right-moving oscillators are in their ground state.

In order to count these states we proceed as follows:

1. First we determine the spectrum of massless fields in the world-volume theory of the Kaluza-Klein monopole solution described above. In particular we show that the world-volume theory always contains eight right-moving massless scalar fields and eight right-moving massless fermion fields. In addition for $\mathcal{M} = K3$ there are twenty four left-moving massless scalar fields whereas for $\mathcal{M} = T^4$ there are eight left-moving massless scalar fields and eight left-moving massless fermion fields.

2. Next we identify the transformation laws of various fields under the orbifold group generator $\widetilde{g}$. We show that all the right-moving fields are $\widetilde{g}$ invariant, whereas the action of $\widetilde{g}$ on the left-moving massless bosonic (fermionic) fields is identical to the action of $\widetilde{g}$ on the even (odd) degree harmonic forms of $\mathcal{M}$.

3. We now use this information to determine all the g-invariant modes on the Kaluza-Klein monopole. This will essentially require that a field that picks up a $\widetilde{g}$ phase $e^{2\pi ik/N}$ must carry momentum $n - k/N$ ($n, k \in \mathbb{Z}$) along S^1 so that the phase obtained due to translation along S^1 cancels the $\widetilde{g}$ phase.

4. Finally we count the number of ways a total momentum $-l_0'/N$ along S^1 can be partitioned into these various g-invariant modes, taking into account that part of this momentum comes from the momenum of the Kaluza-Klein monopole vacuum without any excitations. This vacuum momentum is calculated by mapping the Kaluza-Klein monopole to a fundamental string state in a dual description of the theory.

We begin by analyzing the spectrum of massless fields in the world-volume theory. First of all there are three non-chiral massless scalar fields associated with oscillations in the three transverse directions of the Kaluza-Klein monopole solution. There are two additional non-chiral scalar fields obtained by reducing the two 2-form fields of type IIB string theory along the harmonic 2-form (5.2.6). Finally, the self-dual four form field of type IIB theory, reduced along the tensor product of the harmonic 2-form (5.2.6) and a harmonic 2-form on $\mathcal{M}$, can give rise to a chiral scalar field on the world-volume. The chirality of the scalar field is correlated with whether the corresponding harmonic 2-form on $\mathcal{M}$ is self-dual or anti-self-dual. Since T^4 has three self-dual and three anti-selfdual harmonic 2-forms and $K3$ has three self-dual and nineteen anti-selfdual harmonic 2-forms, we get 3 right-moving and P left-moving scalars where $P = 3$ for $\mathcal{M} = T^4$ and 19 for $\mathcal{M} = K3$. Thus we have altogether 8 right-moving massless scalar fields and $P + 5$ left-moving massless scalar fields on the world-volume of the Kaluza-Klein monopole.

Next we turn to the spectrum of massless fermions in this world-volume theory. These typically arise from the Goldstino fermions associated with broken supersymmetry generators. Since type IIB string theory on K3 has 16 unbroken supersymmetries[27] of which 8 are broken in the presence of the Taub-NUT space, we have 8 massless goldstino fermion fields on the world-volume of the Kaluza-Klein monopole. Since according to our convention the eight unbroken supersymmetry transformation parameters are left-chiral on S^1, the broken supersymmetry transformation parameters must be right-chiral. As a result the goldstino fermion fields associated with broken supersymmetries are also right-moving on the world-volume. On the other hand if we take type IIB on T^4 we have altogether 32 unbroken supercharges of which 16 are broken in the presence of the Taub-NUT space. This produces 16 goldstino fermion fields on the world-volume of the Kaluza-Klein monopole. Since type IIB on T^4 is a non-chiral theory, eight of these fermion fields are right-moving and eight are left-moving.

To summarize, the world-volume theory describing the dynamics of the Kaluza-Klein monopole always contains 8 bosonic and 8 fermionic right-moving massless fields. For $\mathcal{M} = K3$ the world-volume theory has 24 left-moving massles bosonic fields and no left-moving massless fermionic fields whereas for $\mathcal{M} = T^4$ the world-volume theory has 8 left-moving bosonic and 8 left-moving fermionic fields. This is consistent with the fact the under the duality transformation that takes us from the first to the second description of the theory, the Kaluza-Klein monopole associated with $\widetilde{S}^1$ is mapped

[27] In this section we shall refer to unbroken supersymmetries in various context. Some time it may refer to the symmetry of the given compactification, and some time it will refer to the symmetry of a given brane configuration. Also some time it may refer to the number of unbroken supersymmetries before taking the $\mathbb{Z}_N$ orbifold and at other times it may refer to the number of unbroken supersymmetries in the orbifold theory. The reader must carefully examine the context in which the symmetry is being discussed, since the number of unbroken generators and their action on various fields depend crucially on this information.

to a fundamental heterotic (type IIA) string wrapped on S^1 for $\mathcal{M} = K3$ ($\mathcal{M} = T^4$).

We shall now determine the $\widetilde{g}$ transformation properties of these modes. For this we note that irrespective of whether $\mathcal{M}$ is T^4 or $K3$, after taking the $\mathbb{Z}_N$ orbifold the unbroken supersymmetries of the theory are in one to one correspondence with those of type IIB string theory on $K3 \times \widetilde{S}^1 \times S^1$. Thus $\widetilde{g}$ commutes with the supersymmetries of type IIB on $K3$. Half of these $\widetilde{g}$ invariant supersymmetries are broken in the presence of Kaluza-Klein monopole and give rise to right-moving goldstino fermions on the world-volume of the Kaluza-Klein monopole. Thus these fermions must be neutral under $\widetilde{g}$. The other half of the $\widetilde{g}$ invariant supersymmetry generators, which remain unbroken in the presence of the Kaluza-Klein monopole, transform the right-moving fermions into right-moving world-volume scalars. Thus the eight right-moving scalars must also be invariant under $\widetilde{g}$. Five of the left-moving scalars, associated with the 3 transverse degree of freedom and the modes of the 2-form fields along the harmonic 2-form of TN are also invariant under $\widetilde{g}$ since $\widetilde{g}$ acts trivially on the Taub-NUT space. The action of $\widetilde{g}$ on the other P left-moving scalars, associated with the modes of the 4-form field along the tensor product of the harmonic 2-form of TN and the P left-handed harmonic 2-forms of $\mathcal{M}$, is represented by the action of $\widetilde{g}$ on the P left-handed 2-forms on $\mathcal{M}$. This completely determines the action of $\widetilde{g}$ on all the $P + 5$ left-moving scalars. Now it has been shown in appendix B that $\widetilde{g}$ leaves invariant the harmonic 0-form, 4-form and all the three right-handed 2-forms on $\mathcal{M}$. Associating these five $\widetilde{g}$-invariant harmonic forms with the five $\widetilde{g}$ invariant left-moving scalars found above, we can represent the net action of $\widetilde{g}$ on the $(P + 5)$ left-handed scalar fields by the action of $\widetilde{g}$ on the $(P + 5)$ even degree harmonic forms of $\mathcal{M}$.

What remains is to determine the action of $\widetilde{g}$ on the left-moving fermions. We shall now show that this can be represented by the action of $\widetilde{g}$ on the harmonic 1- and 3-forms of $\mathcal{M}$. For $\mathcal{M} = K3$ there are no 1- or 3-forms and no left-moving fermions on the world-volume of the Kaluza-Klein monopole. Hence the result holds trivially. For $\mathcal{M} = T^4$ there are eight left-moving fermions and eight right-moving fermions associated with the sixteen supersymmetry generators which are broken in the presence of a Kaluza-Klein monopole in type IIB string theory on $T^4 \times \widetilde{S}^1 \times S^1$. Although we have already argued that the right-moving fermions are $\widetilde{g}$ neutral, let us forget this result for a while and analyze the $\widetilde{g}$ transformation properties of the full set of 16 fermions. Clearly these transform in the spinor representation of the tangent space $SO(4)_\parallel$ group associated with the T^4 direction. Now $\widetilde{g}$ is an element of this group describing $2\pi/N$ rotation in one plane and $-2\pi/N$ rotation in an orthogonal plane. Translating this into the spinor representation we see that $\widetilde{g}$ must leave half of the sixteen fermions invariant, rotate two pairs of fermions by $2\pi/N$ and rotate the other two pairs of fermions by $-2\pi/N$. Now we use the information that the right-moving fermions are neutral under

$\widetilde{g}$. Thus the action of $\widetilde{g}$ on the left-moving fermions is to rotate two pairs of fermions by $2\pi/N$ and another two pairs of fermions by $-2\pi/N$. This is identical to the action of $\widetilde{g}$ on the harmonic 1- and 3-forms of T^4 given in (B.12) and (B.14):

$$\begin{aligned}
&dz^1 \to e^{2\pi i/N} dz^1, \quad dz^2 \to e^{-2\pi i/N} dz^2\,, \\
&dz^1 \wedge dz^2 \wedge d\bar{z}^1 \to e^{-2\pi i/N}\, dz^1 \wedge dz^2 \wedge d\bar{z}^1, \quad dz^1 \wedge dz^2 \wedge d\bar{z}^2 \to e^{2\pi i/N}\, dz^1 \wedge dz^2 \wedge d\bar{z}^2,
\end{aligned} \tag{5.2.7}$$

where (z^1, z^2) denote complex coordinates on T^4.

Thus the problem of studying the $\widetilde{g}$ transformation properties of the left-moving bosonic and fermionic degrees of freedom on the world-volume reduces to the problem of finding the action of $\widetilde{g}$ on the even and odd degree harmonic forms of $\mathcal{M}$. Let q_s be the difference between the number of even degree harmonic forms and odd degree harmonic forms, weighted by $\widetilde{g}^s$. It has been shown in appendix B (see the discussion below (B.20)) that q_s is equal to $Q_{0,s}$ defined in (5.1.7). Combining this with the results of our previous analysis we now get

$$\begin{aligned}
Q_{0,s} \;&=\; \text{number of left handed bosons weighted by } \widetilde{g}^s \\
&\quad\; -\text{number of left handed fermions weighted by } \widetilde{g}^s
\end{aligned} \tag{5.2.8}$$

on the world-volume of the Kaluza-Klein monopole. Let n_l be the number of left-handed bosons minus fermions carrying $\widetilde{g}$ quantum number $e^{2\pi i l/N}$. Then we have from (5.2.8), (5.1.7)

$$n_l = \frac{1}{N} \sum_{s=0}^{N-1} e^{-2\pi i l s/N}\, Q_{0,s} \;= \sum_{s=0}^{N-1} e^{-2\pi i l s/N} \,\left(c_0^{(0,s)}(0) + 2 c_1^{(0,s)}(-1) \right). \tag{5.2.9}$$

Clearly n_l is invariant under $l \to l + N$.

This finishes our analysis of the spectrum of massless fields in the world-volume theory of the Kaluza-Klein monopole. We now turn to the problem of counting the spectrum of BPS excitations of the Kaluza-Klein monopole. First of all note that since there are eight right-moving fermions neutral under $\widetilde{g}$, the zero modes of these fermions are $\mathbb{Z}_N$ invariant. These eight fermionic zero modes may be associated with the eight supersymmetry generators of type IIB on $(\mathcal{M} \times S^1)/\,\mathbb{Z}_N$ which are broken in the presence of the Kaluza-Klein monopole. Upon quantization this produces a 16-fold degeneracy of states with equal number of bosonic and fermionic states. This is the correct degeneracy of a single irreducible short multiplet representing half BPS states in type IIB string theory compactified on $\mathcal{M} \times S^1/\,\mathbb{Z}_N$, and will eventually become part of the 64-fold degeneracy of a 1/4 BPS supermultiplet once we tensor this state with the state of the D1-D5 system. Thus in computing the index we are interested in, we must associate weight one with each of the sixteen states irrespective of whether the state is bosonic or fermionic. Since supersymmetry acts on the right-moving sector of the world-volume theory, BPS condition requires that all the non-zero mode right-moving oscillators are in their ground state. Thus the spectrum of BPS states is obtained by taking the tensor product of the irreducible 16 dimensional supermultiplet with either fermionic or bosonic excitations involving the left-moving degrees of freedom on the world-volume of the Kaluza-Klein monopole. We shall denote by $d_{KK}(l'_0)/16$ the degeneracy of states associated with left-moving oscillator excitations carrying total momentum $-l'_0/N$, weighted by $(-1)^{F_L}$. Thus $d_{KK}(l'_0)$ measures the total degeneracy of half-BPS states weighted by $(-1)^{F'}$ where for a given half-BPS supermultiplet F' denotes the fermion number of the middle helicity state of the supermultiplet.

In order to calculate $d_{KK}(l'_0)$ we need to count the number of ways the total momentum $-l'_0/N$ can be distributed among the different $\mathbb{Z}_N$ invariant left-moving oscillator excitations. Since a mode carrying momentum $-l/N$ along S^1 picks up a phase of $e^{-2\pi i l/N}$ under 2π translation along S^1, it must pick up a phase of $e^{2\pi i l/N}$ under $\widetilde{g}$. Thus the number of left-handed bosonic minus fermionic modes carrying momentum $-l/N$ along S^1 is equal to the number n_l given in eq.(5.2.9). The number $d_{KK}(l'_0)/16$ can now be identified as the number of different ways the total momentum $-l'_0/N$ can be distributed among different oscillators, there being n_l oscillators carrying momentum $-l/N$. This gives

$$\sum_{l'_0} d_{KK}(l'_0) e^{2\pi i \widetilde{\rho} l'_0} = 16\, e^{2\pi i N C \widetilde{\rho}} \prod_{l=1}^{\infty} (1 - e^{2\pi i l \widetilde{\rho}})^{-n_l} \,. \tag{5.2.10}$$

The constant C represents the l'_0/N quantum number of the vacuum of the Kaluza-Klein monopole when all oscillators are in their ground state. In order to determine C let us consider the second description of the system where the Kaluza-Klein monopole gets mapped to an elementary heterotic or type IIA string wound along S^1, and C represents the contribution to the ground state L_0 eigenvalue from all the left-moving oscillators. If $\widehat{g}$ denotes the image of $\widetilde{g}$ in this description, then the elementary string wound once along S^1 is in the sector twisted by $\widehat{g}$. Since the modes of the Kaluza-Klein monopole get mapped to the degrees of freedom of the fundamental heterotic or type IIA string, there are n_l left moving bosonic minus fermionic modes which pick up a phase of $e^{2\pi i l/N}$ under the action of $\widehat{g}$. Since a bosonic and a fermionic mode twisted by a phase of $e^{2\pi i \varphi}$ for $0 \le \varphi \le 1$ gives a contribution of $-\frac{1}{24} + \frac{1}{4}\,\varphi\,(1-\varphi)$ and $\frac{1}{24} - \frac{1}{4}\,\varphi\,(1-\varphi)$ respectively to the ground state L_0 eigenvalue,[28]

[28] We are counting the contribution from a mode and its complex conjugate separately.

we have

$$
\begin{aligned}
C &= -\frac{1}{24}\sum_{l=0}^{N-1} n_l + \frac{1}{4}\sum_{l=0}^{N-1} n_l\,\frac{l}{N}\left(1-\frac{l}{N}\right) \\
&= -\frac{1}{24N}\sum_{s=0}^{N-1} Q_{0,s}\sum_{l=0}^{N-1} e^{-2\pi i l s/N} + \frac{1}{4N}\sum_{s=0}^{N-1} Q_{0,s}\sum_{l=0}^{N-1}\frac{l}{N}\left(1-\frac{l}{N}\right)e^{-2\pi i l s/N}
\end{aligned}
\tag{5.2.11}
$$

where in the last step we have used the expression for n_l given in (5.2.9). The sum over l can be performed separately for $s=0$ and $s\neq 0$, and yields the answer

$$
C = -\widetilde{\alpha}/N\,, \tag{5.2.12}
$$

with $\widetilde{\alpha}$ defined as in (5.1.8). The left-right level matching condition in the second description guarantees that $N\,C$ and hence $\widetilde{\alpha}$ must be an integer. Using (5.2.9), (5.2.12) we can rewrite (5.2.10) as

$$
\sum_{l_0'} d_{KK}(l_0')e^{2\pi i\widetilde{\rho}l_0'} = 16\,e^{-2\pi i\widetilde{\alpha}\widetilde{\rho}}\prod_{l=1}^{\infty}(1-e^{2\pi i l\widetilde{\rho}})^{-\sum_{s=0}^{N-1}e^{-2\pi i l s/N}\left(c_0^{(0,s)}(0)+2c_1^{(0,s)}(-1)\right)}\,. \tag{5.2.13}
$$

5.2.2 Counting states associated with the overall motion of the D1-D5 system

We shall now analyze the contribution to the partition function from the overall motion of the D1-D5 system. This has two components, – the center of mass motion of the D1-D5 system along the Taub-NUT space transverse to the plane of the D5-brane, and the dynamics of the Wilson lines on the D5-brane along $\mathcal{M}$. The first component is present irrespective of the choice of $\mathcal{M}$ but the second component exits only if $\mathcal{M}$ has non-contractible one cycles, i.e. for $\mathcal{M} = T^4$.

Dynamics of D1-D5 motion in Taub-NUT space: The contribution from this component is independent of the choice of $\mathcal{M}$. We shall take $\mathcal{M} = K3$ and analyze the contribution following [6]. The D1-D5-system wrapped on $K3$ in flat transverse space has four massless scalar fields associated with the four transverse coordinates and eight massless goldstino fermionic fields associated with the breaking of eight out of sixteen supersymmetries of type IIB string theory on $K3$ by the D1-D5 system. Thus when the transverse space is Taub-NUT, we expect the low energy dynamics of this system to be described by a (1+1) dimensional supersymmetric field theory with four scalar and eight fermion fields, with the scalar fields taking value in the Taub-NUT target space. Since $\widetilde{g}$ does not act on the Taub-NUT space, the scalar fields are $\widetilde{g}$ invariant. Furthermore, since $\widetilde{g}$ commutes with the supersymmetry generators of type IIB string theory on K3, all the massless fermions, being associated with the broken supersymmetry generators, are also $\widetilde{g}$ invariant.[29]

Our goal will be to calculate the spectrum of BPS states in this theory carrying momentum $-l_0/N$ along S^1 and j_0 along the asymptotic circle $\widetilde{S}^1$ of the Taub-NUT space after taking the $\mathbb{Z}_N$ orbifold. This will be done as follows:

1. We first find the interpretation of the quantum numbers l_0 and j_0 in this (1+1) dimensional supersymmetric field theory and then determine the values of l_0 and j_0 quantum numbers carried by various world-volume fields.

2. Since a σ-model with Taub-NUT target space is an interacting theory, we cannot carry out the counting of states by regarding the world-volume theory as free. We show that the world-volume theory actually contains two mutually non-interacting pieces, – a theory of free left-moving fermions and an interacting theory of scalars and right-moving fermions.

3. The contribution to the partition function from the free left-moving fermions is easily computed. In computing the contribution from the scalars and the right-moving fermions, we split the system into two parts: the zero mode part and the non-zero mode part. By taking the size of the Taub-NUT space to be large we argue that the non-zero mode part can be treated essentially as a free field theory and we can evaluate the contribution to the partition function by simple counting.

4. The problem of studying the effect of the zero mode part can be mapped to counting of bound states in a supersymmetric quantum mechanics describing the motion of a superparticle in Taub-NUT space. This problem had been studied earlier in [158, 159]. Using the results of these papers we compute the contribution to the partition function from the zero modes.

5. Finally multiplying the contribution to the partition function from different sources we get the net contribution to the partition function from the overall motion of the D1-D5 system in the Taub-NUT space.

6. Note that since all the world-volume fields involved in this analysis are neutral under $\widetilde{g}$, the orbifold projection will force the momentum of various modes along S^1 to be integers. Other than that it plays no role.

[29]This is consistent with the fact that the unbroken supersymmetry generators – also $\widetilde{g}$ invariant – transform the scalar fields to fermions and vice versa.

We begin by identifying the quantum numbers l_0 and j_0 in the world-volume theory. $-l_0/N$ is, by definition, the momentum along S^1. According to the point 6 above, all the modes on the world-volume carry integer values of l_0/N. Conversely, for every world-volume field all non-negative integer values of l_0/N are allowed, – the positivity constraint being a consequence of the BPS condition which allows only left-moving modes carrying negative momentum along S^1 to be excited.

To identify the quantum number j_0 we need to examine closely the metric of the Taub-NUT space given in (5.2.4), (5.2.5). Close to the origin $r = 0$ the metric reduces to that of flat space $\mathbb{R}^4$ written in terms of Euler angles θ, ϕ, ψ and the radial coordinate $\rho \equiv \sqrt{r}$, while for large r it is that of $\mathbb{R}^3 \times \widetilde{S}^1$, with $\widetilde{S}^1$ parametrized by the angular coordinate ψ and $\mathbb{R}^3$ parametrized by the spherical polar coordinates (r, θ, ϕ). In terms of the coordinates

$$
\begin{aligned}
x^1 &= 2\sqrt{r}\cos\frac{\theta}{2}\cos\left(\psi + \frac{\phi}{2}\right), \qquad & x^2 &= 2\sqrt{r}\cos\frac{\theta}{2}\sin\left(\psi + \frac{\phi}{2}\right), \\
x^3 &= 2\sqrt{r}\sin\frac{\theta}{2}\cos\left(\psi - \frac{\phi}{2}\right), \qquad & x^4 &= 2\sqrt{r}\sin\frac{\theta}{2}\sin\left(\psi - \frac{\phi}{2}\right)
\end{aligned}
\tag{5.2.14}
$$

the metric at the origin $r = 0$ takes the form of the flat Euclidean metric written in Cartesian coordinates. As a result it has the usual $SO(4) \equiv SU(2)_L \times SU(2)_R$ rotation symmetry acting on the x^i's as:

$$
\begin{pmatrix} x^1 + ix^2 & x^3 + ix^4 \\ x^3 - ix^4 & x^1 - ix^2 \end{pmatrix} \to U_L \begin{pmatrix} x^1 + ix^2 & x^3 + ix^4 \\ x^3 - ix^4 & x^1 - ix^2 \end{pmatrix} U_R^T, \qquad U_L, U_R \in SU(2)\,. \tag{5.2.15}
$$

It is easy to see that only the $U(1)_L \times SU(2)_R$ subgroup of this is a symmetry of the the full metric (5.2.4). The $SU(2)_R$ symmetry generated by the matrix U_R acts as the usual rotation group on the three dimensional space labelled by (r, θ, ϕ). The $U(1)_L$ symmetry generated by $\mathrm{diag}(e^{i\epsilon/2}, e^{-i\epsilon/2})$ acts as

$$
\psi \to \psi + \frac{1}{2}\epsilon\,, \tag{5.2.16}
$$

with no action on any of the other coordinates. From the point of view of an asymptotic observer this is just a translation along the compact circle $\widetilde{S}^1$ parametrized by ψ, and the corresponding conserved charge is the quantum number $j_0/2$. On the other hand using (5.2.14) we see that near the origin the ψ translation acts as simultaneous rotation in the 1-2 and 3-4 planes. Thus near the origin the contribution to the j_0 charge can be identified as the sum of the angular momentum in the 1-2 and 3-4 planes [160].

Since the metric at the origin is the usual four dimensional Euclidean metric, we can describe it in terms of a set of vierbeins proportional to the identity matrix. The $SU(2)_L \times SU(2)_R$ transformation described in (5.2.15) will leave the vierbeins invariant only if they are accompanied by a compensating $SU(2)_L^T \times SU(2)_R^T$ rotation in the tangent space. In particular the global $U(1)_L$ symmetry (5.2.16) will induce a tangent space rotation belonging to the $U(1)_L^T \subset SU(2)_L^T$ group, and the quantum number j_0 can be interpreted as a $U(1)_L^T$ charge $j_0/2$. This has a subtle effect on the statistics of various degrees of freedom carrying j_0 charge. From the point of view of an asymptotic observer, the angular momentum generators are those of $SU(2)_R$ and hence the statistics $(-1)^F$ of an excitation is equal to $(-1)^{2J_{3R}}$. On the other hand the same excitation, viewed from the center of the Kaluza-Klein monopole will have statistics $(-1)^{2J_{3L}+2J_{3R}} = (-1)^F(-1)^{j_0}$. Since we shall be interested in identifying the statistics of the states from the point of view of the asymptotic observer, but a large part of the actual counting of states will be done by analyzing the modes near the center of Taub-NUT, we must take into account this difference in our analysis.

Since the massless scalar fields describing transverse oscillations of the D1-D5 system take values in the Taub-NUT target space, (5.2.16) automatically determines the transformation laws of these scalar fields under the $U(1)_L$ transformation and hence the j_0 charges carried by these fields. In particular at the origin of the Taub-NUT space the scalar fields are in one to one correspondence with the coordinates x^i, and belong to the $(2,2)$ representation of the $SU(2)_L \times SU(2)_R$ group. Identifying j_0 with $2U(1)_L$ we see that two of these fields carry j_0 charge 1 and other two fields carry j_0 charge -1. On the other hand, since the fermions transform in the $(1,2)+(2,1)$ representation of the tangent space $SU(2)_L^T \times SU(2)_R^T$ group, half of the fermions are neutral under $SU(2)_L^T$ and hence also under the global $U(1)_L$. As a result these do not carry any j_0 quantum number. The other half of the fermions are neutral under $SU(2)_R$ but transform in the spinor representation of the $SU(2)_L$ group. Thus they carry $j_0 = \pm 1$.

The world-volume field theory involving these bosons and fermions will in general be an interacting field theory since the target space metric (5.2.4) is non-trivial. The bosonic part of the theory is just that of a σ-model with Taub-NUT as the target space. The coupling of the fermions may be determined as follows. As discussed in §5.2.1, type IIB string theory on $K3 \times TN$ has 8 unbroken left-chiral supersymmetries on the 1+1 dimensional world-volume. These are singlets of the holonomy group of $K3 \times TN$ and must also commute with the generator of translation along $\widetilde{S}^1$. Thus they carry zero j_0 charge. The presence of D1-D5 system breaks 4 of these supersymmetries. The associated goldstino fermions must be left-moving and carry zero j_0 charge. Furthermore, being goldstino fermions they must be non-interacting in the low energy limit. The four unbroken supersymmetry generators, which are also j_0 neutral and left-moving, would mix the four bosons with four right-moving fermions carrying the same j_0 charge as the bosons. Furthermore since the bosons are interacting, their superpartner right-moving fermions must also be interacting. Thus we have four interacting right-moving fermions carrying j_0 charges ± 1. This correctly accounts for all the eight fermions and their j_0 charges on the D1-D5 world-volume.

To summarize, the (1+1) dimensional world-volume theory associated with the overall motion of the D1-D5 system in the Taub-NUT target space is described by a set of four free left-moving $U(1)_L$ invariant fermion fields, together with an interacting theory of four bosons and four right-moving $U(1)_L$ non-invariant fermions. For a D1-D5 system placed at the origin of the Taub-NUT space, two of the bosons and two of the right-moving fermions carry $j_0 = 1$, and the other two bosons and right-moving fermions carry $j_0 = -1$. The unbroken supersymmetry transformations leave the free left-moving fermions untouched but acts on the scalars and the right-moving fermions. All the fields carry integer momenta along S^1. The above classification of various fields into fermions and bosons is from the point of view of a five dimensional observer sitting at the center of Taub-NUT space, – this is related to the statistics measured from the point of view of the asymptotic four dimensional observer by a multiplicative factor of $(-1)^{j_0}$.

We now turn to the computation of the partition function. Let us first calculate the contribution to the partition function due to the free left-moving fermions. Since these fermons do not carry any j_0 charge, and carry l_0 quantum numbers in units of N, their contribution is given by:

$$Z_{\text{free}}(\widetilde{\rho}) \equiv \text{Tr}_{\text{free left-moving fermions}}((-1)^F(-1)^{j_0} e^{2\pi i\widetilde{\rho} l_0} e^{2\pi i\widetilde{v} j_0}) = 4\prod_{n=1}^{\infty}(1-e^{2\pi i n N\widetilde{\rho}})^4\,, \tag{5.2.17}$$

where F denotes the total contribution to the space-time fermion number, – except from the fermion zero-modes associated with the broken supersymmetry generators, – from the point of view of an asymptotic four dimensional observer. The multiplicative factor of 4 comes from the quantization of the free fermion zero modes. The latter in turn can be interpreted as due to the four broken supersymmetries of the D1-D5-system on $K3 \times TN \times S^1$.

Now we turn to the interacting part of the theory. Since we are computing an index we can assume that it does not change under continuous variation of the moduli parameters. Let us take the size R_0 of the Taub-NUT space to be large so that the metric is almost flat and in a local region of the Taub-NUT space the world-volume theory of the D1-D5 system is almost free. In this case we should be able to compute the contribution due to the non-zero mode bosonic and fermonic oscillators by placing the D1-D5 system at the origin of the Taub-NUT space and treating them as oscillators of free fields. The contribution from the zero modes however is sensitive to the global geometry of the Taub-NUT space and should be computed separately.

Since supersymmetry acts on the right-moving bosons and fermions, in order to get a BPS state the right-moving bosonic and fermionic oscillators must be in their ground state. Thus as far as the contribution due to the non-zero mode oscillators are concerned, we only need to examine the effect of left-moving bosonic oscillators carrying momentum $-l_0/N$ along S^1 and angular momentum j_0. We have already argued that two of the four bosons carry $j_0 = 1$ and the other two bosons carry $j_0 = -1$, and that each of these bosons carry arbitrary positive integer values of l_0/N. The contribution to the partition function from these oscillators can be easily computed [161] and yields the answer

$$\begin{aligned} Z_{\rm osc}(\widetilde{\rho}, \widetilde{v}) &\equiv \mathrm{Tr}_{\rm oscillators}((-1)^F (-1)^{j_0} e^{2\pi i \widetilde{\rho} l_0 + 2\pi i \widetilde{v} j_0}) \\ &= \prod_{n=1}^{\infty} \frac{1}{(1 - e^{2\pi i n N \widetilde{\rho} + 2\pi i \widetilde{v}})^2 (1 - e^{2\pi i n N \widetilde{\rho} - 2\pi i \widetilde{v}})^2} \, . \end{aligned} \tag{5.2.18}$$

In arriving at this equation we have used the fact that since these oscillators are bosonic from the five dimensional viewpoint, they have statistics $(-1)^F = (-1)^{j_0}$ from the four dimensional viewpoint.

Finally we turn to the contribution $Z_{\rm zero}$ to the partition function from the bosonic and fermionic zero modes of the interacting part of the theory. Since the intercting theory has four bosonic and four fermionic fields, the dynamics of zero modes is that of a superparticle with four bosonic and four fermionic coordinates moving in Taub-NUT space. Under the holonomy group $SU(2)_L^T$ of the Taub-NUT space the fermions and hence also their superpartner bosons transform in a pair of spinor representations. This system is described by an $\mathcal{N} = 4$ supersymmetric quantum mechanics. Thus in order to look for BPS states of the D1-D5 system we need to look for supersymmetric ground states of this quantum mechanics.

So far we have been working at a special point in the moduli space of the CHL string theory where the circles S^1 and $\widetilde{S}^1$ are orthonormal in the asymptotic geometry. In this case the BPS mass of the D1-D5-Kaluza-Klein monopole system is equal to the sum of the BPS masses of the D1-D5 system and the Kaluza-Klein monopole. As a result there is no potential term in the D1-D5 world-volume action and analysis of bound states is difficult. But this is not a generic situation. As we shall see, once we switch on a component of the metric that mixes S^1 and $\widetilde{S}^1$ we get a potential that binds the D1-D5 system to the Kaluza-Klein monopole and the problem is easier to analyze. On the other hand by taking the potential to be sufficiently mild we can ensure that the analysis of the dynamics of non-zero modes will not be affected by this modification. Hence $Z_{\rm free}$ and $Z_{\rm osc}$ should remain unchanged.

The mixing between S^1 and $\widetilde{S}^1$ can be achieved by replacing the $d\psi$ term in the expression for the metric given in (5.2.4) by $d\psi + \lambda dy$ where y is the coordinate along S^1 and λ is a small deformation parameter. This clearly remains a solution of the equations of motion but gives an r dependent contribution to the yy component of the metric:

$$\Delta g_{yy} = 4\,\lambda^2\,R_0^2\left(1+\frac{R_0}{r}\right)^{-1}\,. \tag{5.2.19}$$

As a result the tension of the D1-D5 system, being proportional to $\sqrt{g_{yy}}$, acquires an r-dependent contribution proportional to

$$\lambda^2 R_0^2\left(1+\frac{R_0}{r}\right)^{-1} \tag{5.2.20}$$

to first order in λ^2. Supersymmetrization of this term gives rise to other fermionic terms.

Thus we have to analyze the dynamics of a superparticle with $\mathcal{N}=4$ supersymmetry moving in Taub-NUT space under a potential proportional to (5.2.20). This is precisely the problem solved in [158, 159]. The result of this analysis can be summarized as follows. Depending on the sign of the deformation parameter λ we have supersymmetric bound states for $j_0>0$ or $j_0<0$, where j_0 is the momentum conjugate to the coordinate ψ.[30] In the weak coupling limit the number of bound states for a given value of j_0 is given by $|j_0|$ and they carry angular momentum $(|j_0|-1)/2$.[31] Thus for these states $(-1)^F=(-1)^{j_0-1}$. If for definiteness we choose the sign of λ such that we get bound states for positive j_0, then this gives the zero mode partition function

$$Z_{\rm zero}(\widetilde v) \equiv Tr_{\rm zero\,modes}\left((-1)^F(-1)^{j_0}e^{2\pi i\widetilde v j_0}\right) = -\sum_{j_0=1}^\infty j_0\, e^{2\pi i\widetilde v j_0} = -\frac{e^{2\pi i\widetilde v}}{(1-e^{2\pi i\widetilde v})^2}\,. \tag{5.2.21}$$

Since this is invariant under $\widetilde v\to-\widetilde v$ we shall get the same answer if we had chosen to work with the opposite sign of λ that produces bound states with negative j_0. However in order to extract the degeneracy of the states from the partition function we have to make a decision as to whether we should expand the right hand side of (5.2.21) in powers of $e^{2\pi i\widetilde v}$ or $e^{-2\pi i\widetilde v}$, and this depends on the sign of λ. Since in the heterotic description the complex structure modulus of the torus $\widetilde S^1\times S^1/\,\mathbb{Z}_N$ corresponds to the axion-dilaton field, the sign of λ would correspond to the sign of the asymptotic value of the axion field. A careful analysis shows that for positive sign of the axion

[30] Since the potential given in (5.2.20) does not depend on the sign of λ, the reader may wonder why the spectrum depends on the sign of λ. It turns out that other terms in the world-volume action related to (5.2.20) by supersymmetry do depend on the sign of λ.

[31] Strictly speaking there is an upper bound on the possible value of $|j_0|$ which goes to ∞ as the type IIB coupling goes to zero. Put another way, the degeneracy formula given by the partition function (5.2.21) is valid only for type IIB coupling below a certain value determined by the magnitude of j_0. This bound is related to the existence of the walls of marginal stability to be discussed in §5.4.

field the degeneracy is given by expanding (5.2.21) in powers of $e^{-2\pi i\widetilde{v}}$, whereas for negative sign of the axion field we need to expand (5.2.21) in powers of $e^{2\pi i\widetilde{v}}$.

Finally putting all the ingredients together the partition function of states associated with the centre of mass motion of the D1-D5 system in Taub-NUT space is given by

$$\begin{aligned}\sum_{l_0,j_0} d_{transverse}(l_0,j_0)(-1)^{j_0}e^{2\pi i l_0\widetilde{\rho}+2\pi i j_0\widetilde{v}} &= Z_{\rm free}(\widetilde{\rho})Z_{\rm osc}(\widetilde{\rho},\widetilde{v})Z_{\rm zero}(\widetilde{v}) \\ &= -4e^{-2\pi i\widetilde{v}}(1-e^{-2\pi i\widetilde{v}})^{-2} \\ &\times\prod_{n=1}^{\infty}\{(1-e^{2\pi i nN\widetilde{\rho}})^4(1-e^{2\pi i nN\widetilde{\rho}+2\pi i\widetilde{v}})^{-2}(1-e^{2\pi i nN\widetilde{\rho}-2\pi i\widetilde{v}})^{-2}\}.\end{aligned} \tag{5.2.22}$$

The dynamics of Wilson lines along $\mathcal{M}$: Let us now compute the contribution to the partition function from the dynamics of the Wilson lines along $\mathcal{M}=T^4$. For this we can ignore the presence of the Kaluza-Klein monopole and the D1-branes, and consider the dynamics of a D5-brane wrapped on $T^4\times S^1$, – the sole effect of the Kaluza-Klein monopole will be in the identification between the angular momentum carried by the system from the point of view of a five dimensional observer sitting at the center of Taub-NUT and the momentum along $\widetilde{S}^1$ from the point of view of an asymptotic four dimensional observer. Taking the T^4 to have small size we can regard the world-volume theory of the D5-brane as (1+1) dimensional. This contains eight scalars associated with four Wilson lines and four transverse coordinates. It also has a total of 16 massless fermions of which eight are left-moving and eight are right-moving, – these can be regarded as the goldstino fermions associated with 16 broken supersymmetry generators of type IIB string theory on T^4 in the presence of the D5-brane. The eight bosons and the sixteen fermions mix under the action of the sixteen unbroken supersymmetry generators on the D5-brane world-volume. However only eight of these generators commute with the orbifold group generator $\widetilde{g}$, – these coincide with the unbroken supersymmetries of a D5-brane wrapped on K3 and consist of four left-chiral and four right-chiral generators. Under these $\widetilde{g}$ invariant supersymmetry transformations the scalars associated with the coordinates transverse to the D5-brane mix with eight of the sixteen fermions on the D5-brane world-volume. The scalars associated with the Wilson lines mix with the other eight fermions. Since the contribution to the partition function from the first set of fermions and scalars, – associated with the transverse coordinates and their superpartners, – have already been taken into account in (5.2.22), we shall focus on the second set of world-volume fields consisting of the Wilson lines and their superpartners. Since the $\widetilde{g}$ invariant supersymmetry generators are non-chiral, the superpartners of the Wilson line must also be non-chiral. Thus this set consists of four left-moving and four right-moving fermion fields.

Our goal is to compute the contribution to the partition function of BPS states from this sector. We proceed in the following steps:

1. First we shall determine the $\widetilde{g}$ transformation properties as well as the l_0 and j_0 quantum numbers of various world-volume fields.

2. Since the bosonic world-volume fields represent Wilson lines along T^4, the world-volume theory is free. Thus once we have determined the quantum numbers carried by various fields, computation of the partition function may be done by simple counting.

We begin by determining the various quantum numbers carried by the world-volume fields. Since $\widetilde{g}$ represents a $2\pi/N$ rotation in one plane of T^4 and $-2\pi/N$ rotation in an orthogonal plane, $\widetilde{g}$ acts as a rotation by $2\pi/N$ on one pair of Wilson lines and as a rotation by $-2\pi/N$ on the other pair. Thus it must act in the same way on the left- and right-moving fermionic fields related to these Wilson lines by $\widetilde{g}$ invariant supersymmetry transformations. In order to be $\mathbb{Z}_N$ invariant, the modes which pick a phase of $e^{2\pi i/N}$ under $\widetilde{g}$ must carry momentum along S^1 of the form $k - \frac{1}{N}$ for integer k, whereas modes which pick a phase of $e^{-2\pi i/N}$ under $\widetilde{g}$ must carry momentum along S^1 of the form $k + \frac{1}{N}$ for integer k. As a result, both in the left and the right-moving sector, we have a pair of bosons and a pair of fermions carrying S^1 momentum of the form $k + \frac{1}{N}$, and a pair of bosons and a pair of fermions carrying S^1 momentum of the form $k - \frac{1}{N}$.

Eventually when we place this in the background of the Kaluza-Klein monopole, only the left-moving supersymmetry acting on the right-moving modes remain unbroken. Thus in order to get a BPS state of the final supersymmetry algebra we must put all the right-moving oscillators in their ground state and consider only left moving excitations.

In order to calculate the partition function associated with these modes we also need information about their j_0 quantum numbers. Since the system is eventually placed at the centre of Taub-NUT space which converts angular momentum at the centre into momentum along $\widetilde{S}^1$ at infinity, j_0 is still given by the sum of angular momenta in the 1-2 and 3-4 planes transverse to the D5-brane world-volume. Since the scalars represented by the Wilson lines along T^4 are neutral under rotation in the transverse plane, they do not carry any j_0 quantum number. The story however is different for the fermions. First of all, since type IIB string theory on T^4 is a non-chiral theory, there should be no correlation between the $SU(2)_L$ quantum number j_0 and the world-volume chirality of the massless fermions living on a D5-brane on T^4. Now from our previous analysis of the D1-D5 system in the background of $K3 \times TN$ we know that the world-volume fermions in this system have the property that the left-movers have $j_0 = 0$ and the right-movers carry j_0 quantum numbers ± 1.

Such fermions also exist on a D5-brane on T^4 as partners of the transverse coordinates under $\widetilde{g}$ invariant supersymmetry transformation, but the contribution to the partition function from these fermions have already been taken into account in (5.2.22). The rest of the fermions must have opposite correlation between $SU(2)_L^T$ quantum numbers and world-volume chirality, i.e. the right-movers must have $j_0 = 0$ and the left-movers must carry j_0 quantum numbers ± 1. Furthermore, since $\widetilde{g}$ has no action on the transverse coordinates, it commutes with $SU(2)_L^T$ and there should be no correlation between the $\widetilde{g}$ quantum numbers and the sign of the $U(1)_L \subset SU(2)_L^T$ quantum numbers. Thus the two left-moving fermions carrying $\widetilde{g}$ quantum number $e^{2\pi i/N}$ must have $j_0 = \pm 1$ and the two left-moving fermions carrying $\widetilde{g}$ quantum number $e^{-2\pi i/N}$ must have $j_0 = \pm 1$.

To summarize, when we choose $\mathcal{M} = T^4$ instead of $K3$, the additional left-moving excitations on the D5-brane world-volume consist of four bosonic and four fermionic modes. Invariance under the orbifold group generator g requires that two of the four bosonic modes carry momentum along S^1 of the form $k + \frac{1}{N}$ and the other two carry momentum along S^1 of the form $k - \frac{1}{N}$, but neither of them carry any momentum along $\widetilde{S}^1$. On the other hand two of the left-moving fermionic modes carry momentum along S^1 of the form $k + \frac{1}{N}$ and ± 1 unit of momentum along $\widetilde{S}^1$, and the other two left-moving fermionic modes carry momentum along S^1 of the form $k - \frac{1}{N}$ and ± 1 unit of momentum along $\widetilde{S}^1$. As before the statistics of these oscillators are altered by a factor of $(-1)^{j_0}$ as we come down from five to four dimensions. Thus if $d_{wilson}(l_0, j_0)$ denotes the number of bosonic minus fermionic states associated with these modes carrying total momentum $-l_0/N$ along S^1 and total momentum j_0 along $\widetilde{S}^1$, then

$$
\begin{aligned}
&\sum_{l_0, j_0} d_{wilson}(l_0, j_0)(-1)^{j_0} e^{2\pi i l_0 \widetilde{\rho} + 2\pi i j_0 \widetilde{v}} \\
= \; & \prod_{\substack{l \in N\mathbb{Z}+1 \\ l>0}} (1 - e^{2\pi i l \widetilde{\rho}})^{-2} \prod_{\substack{l \in N\mathbb{Z}-1 \\ l>0}} (1 - e^{2\pi i l \widetilde{\rho}})^{-2} \prod_{\substack{l \in N\mathbb{Z}+1 \\ l>0}} (1 - e^{2\pi i l \widetilde{\rho} + 2\pi i \widetilde{v}}) \\
& \prod_{\substack{l \in N\mathbb{Z}+1 \\ l>0}} (1 - e^{2\pi i l \widetilde{\rho} - 2\pi i \widetilde{v}}) \prod_{\substack{l \in N\mathbb{Z}-1 \\ l>0}} (1 - e^{2\pi i l \widetilde{\rho} + 2\pi i \widetilde{v}}) \prod_{\substack{l \in N\mathbb{Z}-1 \\ l>0}} (1 - e^{2\pi i l \widetilde{\rho} - 2\pi i \widetilde{v}}) \, . \qquad (5.2.23)
\end{aligned}
$$

The partition function associated with the overall dynamics of the D1-D5 system is given by the product of the contribution (5.2.22) from the dynamics of the transverse modes and (in case $\mathcal{M} = T^4$) the contribution (5.2.23) from the dynamics of the Wilson lines along T^4. The final result can be written in a compact form using the coefficients $c_b^{(r,s)}(u)$ introduced in (5.1.7). In particular if we use the relations (B.11), (B.16):

$$
c_1^{(0,s)}(-1) = \begin{cases} \frac{2}{N} & \text{for } \mathcal{M} = K3 \\ \frac{1}{N}\left(2 - e^{2\pi i s/N} - e^{-2\pi i s/N}\right) & \text{for } \mathcal{M} = T^4 \end{cases} \qquad (5.2.24)
$$

then the product of (5.2.22) and (for $\mathcal{M} = T^4$) (5.2.23) can be written as

$$\sum_{l_0,j_0} d_{CM}(l_0,j_0)(-1)^{j_0}\, e^{2\pi i l_0\widetilde{\rho}+2\pi i j_0\widetilde{v}} = -4\, e^{-2\pi i\widetilde{v}} \prod_{l=1}^{\infty}(1-e^{2\pi i l\widetilde{\rho}})^{2\sum_{s=0}^{N-1} e^{-2\pi i l s/N}\, c_1^{(0,s)}(-1)}$$
$$\prod_{l=1}^{\infty}(1-e^{2\pi i l\widetilde{\rho}+2\pi i\widetilde{v}})^{-\sum_{s=0}^{N-1} e^{-2\pi i l s/N}\, c_1^{(0,s)}(-1)} \prod_{l=0}^{\infty}(1-e^{2\pi i l\widetilde{\rho}-2\pi i\widetilde{v}})^{-\sum_{s=0}^{N-1} e^{-2\pi i l s/N}\, c_1^{(0,s)}(-1)} \tag{5.2.25}$$

both for $\mathcal{M} = K3$ and $\mathcal{M} = T^4$.

5.2.3 Counting states associated with the relative motion of the D1-D5 system

Finally we turn to the problem of counting states associated with the motion of the D1-brane in the plane of the D5-brane. This is a well known system that appears in the original analysis of the five dimensional black holes in [1] (see [162] for a review of this system). Our analysis will follow the approach described in [6], which in turn is a generalization of the analysis given in [163, 13] for the case of type IIB string theory on $K3 \times S^1$.

At the special point in the moduli space at which we have been working so far, the D1-D5 system in the absence of the Kaluza-Klein monopole is marginally bound and hence the D1-brane can move freely in directions transverse to D5. This makes it difficult to analyze this system. We shall avoid this problem by switching on a small amount of NS-NS sector 2-form field along the D5-brane world-volume. This binds the D1-brane on to the D5-brane, – the D1-brane being identified as non-commutative U(1) instanton of the gauge theory on the D5-brane world-volume [164, 165]. Thus the only possible motion of the D1-brane inside the D5-brane is along the directions tangential to the D5-brane.

We can now proceed in the following steps:

1. We first analyze the world-volume theory of a single D1-brane inside a D5-brane. This is given by a superconformal field theory with target space $\mathcal{M}$. $\widetilde{g}$ represents a $\mathbb{Z}_N$ symmetry of this superconformal field theory. We also identify the momenta along S^1 and $\widetilde{S}^1$ as specific quantum numbers in this superconformal field theory.

2. We then compute of the degeneracy $n(w,l,j)$ of a single D1-brane moving inside the D5-brane, wound w times along $S^1/\mathbb{Z}_N$, and carrying momenta $-l/N$ along S^1 and j along $\widetilde{S}^1$. The result is expressed in terms of the set of coefficients $c_b^{(r,s)}(u)$ introduced in (5.1.6).

3. Finally we consider the contribution to the partition function from multiple D1-branes moving inside the D5-brane and, using straightforward combinatoric analysis, express the result in terms of the degeneracies $n(w,l,j)$ of a single D1-brane.

We begin our analysis with a single D1-brane moving inside a D5-brane. Let σ denote the coordinate along the length of the D1-brane, σ being normalized so that it coincides with the target space coordinate in which $S^1/\,\mathbb{Z}_N$ has period 2π. If the D1-brane winds w times along $S^1/\,\mathbb{Z}_N$, then σ changes by $2\pi w$ as we traverse the whole length of the string, regarded as a configuration in the orbifold. Under $\sigma \to \sigma + 2\pi w$, the physical coordinate of the D1-brane shifts by $2\pi r$ along S^1 where

$$r = w \mod N\,. \tag{5.2.26}$$

Identification in $\mathcal{M}\times S^1$ under $\mathbb{Z}_N$ then requires that under $\sigma \to \sigma + 2\pi w$ the location of the D1-brane along $\mathcal{M}$ gets transformed by $\widetilde{g}^r = \widetilde{g}^w$.

Since in the weak coupling limit the dynamics of the D1-brane inside a D5-brane is insensitive to the presence of the Kaluza-Klein monopole, the 2-dimensional theory describing this system has (4,4) supersymmetry. Thus we expect the low energy dynamics of this D-brane system to be described by a superconformal field theory (SCFT) with target space $\mathcal{M}$ subject to the above boundary condition. In particular the state must be twisted by $\widetilde{g}^r$. Furthermore, since the supersymmetry generators commute with $\widetilde{g}$, the supercurrents will satisfy periodic boundary condition under $\sigma \to \sigma + 2\pi w$. Thus the state belongs to the RR sector. Since the D1-brane has coordinate length $2\pi w$, the momentum along S^1 can be identified as the $(\bar{L}_0 - L_0)/w$ eigenvalue of this state. Thus a total momentum $-l/N$ corresponds to $\bar{L}_0 - L_0$ eigenvalue $-lw/N$. On the other hand the BPS condition on the D1-brane requires $\bar{L}_0$ to vanish.[32] Thus we are looking for a state in the $\widetilde{g}^r$ twisted RR sector of this SCFT with

$$L_0 = lw/N, \qquad \bar{L}_0 = 0\,. \tag{5.2.27}$$

Eqs.(5.2.26), (5.2.27) give interpretation of the quantum numbers w and l in the D1-brane world-volume theory. What about the quantum number j? This superconformal field theory has an R-symmetry group $SO(4)^T = SU(2)^T_L \times SU(2)^T_R$ associated with tangent space rotation along directions transverse to the D1-brane and the D5-brane. All the bosonic fields in the world-volume theory are neutral under this group but the fermions transform non-trivially. As discussed in the paragraphs below (5.2.16), in the presence of the Kaluza-Klein monopole background a translation ϵ along $\widetilde{S}^1$

[32] Even though the D1-D5 system has supersymmetry acting on both the right- and the left-moving fields, only the supersymmetries acting on the right-moving fields survive when we place the system in a Kaluza-Klein monopole background. Thus we only require invariance under the supersymmetries acting on the right-moving fields.

must be accompanied by a rotation 2ϵ in $U(1)_L \subset SU(2)_L$. Thus if F_L denotes twice the $U(1)_L$ generator, then the quantum number j can be identified as the F_L eigenvalue of the state [33]. F_L is also referred to as the world-sheet fermion number associated with the left-moving sector of the (4,4) superconformal field theory.

Let F_R denote twice the $U(1)_R \subset SU(2)_R$ generator, or equivalently, the world-sheet fermion number associated with the right-moving modes of the (4,4) superconformal field theory. Since in our system the world-sheet supersymmetry originates from space-time supersymmetry, the total world-sheet fermion number $F_L + F_R$ can be interpreted as the space-time fermion number from the point of view of a five dimensional observer at the center of Taub-NUT space. Taking into account the fact that the four and five dimensional statistics differ by a factor of $(-1)^j$ we see that in four dimensions, in counting the total number of bosonic minus fermionic states weighted by $(-1)^j$ with a given set of charges, we must calculate the number of states weighted by $(-1)^{F_L+F_R}$.

Finally we must remember that not all the states of the superconformal field theory are allowed states of the D-brane, – we must pick $\mathbb{Z}_N$ invariant states. Since the total momentum along S^1 is $-l/N$, under translation by 2π along S^1 this state picks up a phase $e^{-2\pi i l/N}$. Thus the projection operator onto $\mathbb{Z}_N$ invariant states is given by

$$\frac{1}{N}\sum_{s=0}^{N-1} e^{-2\pi i s l/N} \widetilde{g}^s \, . \tag{5.2.28}$$

Putting these results together we see that the total number of $\mathbb{Z}_N$ invariant bosonic minus fermionic states weighted by $(-1)^j$ of the single D1-brane carrying quantum numbers w, l, j is given by

$$n(w,l,j) \equiv \frac{1}{N}\sum_{s=0}^{N-1} e^{-2\pi i s l/N} Tr_{RR;\widetilde{g}^r}\left(\widetilde{g}^s (-1)^{F_L+F_R} \delta_{NL_0, lw} \delta_{F_L, j}\right), \quad r = w \text{ mod } N \, . \tag{5.2.29}$$

Here $Tr_{RR;\widetilde{g}^r}$ denotes trace, in the superconformal σ-model with target space $\mathcal{M}$, over RR sector states twisted by $\widetilde{g}^r$. Insertion of $(-1)^{F_R}$ in the trace automatically projects onto $\bar{L}_0 = 0$ states, – so we do not need to insert a $\delta_{\bar{L}_0,0}$ factor.

Let us define [11]

$$F^{(r,s)}(\tau, z) \equiv \frac{1}{N} Tr_{RR;\widetilde{g}^r}\left(\widetilde{g}^s (-1)^{F_L+F_R} e^{2\pi i \tau L_0} e^{2\pi i F_L z}\right) \, . \tag{5.2.30}$$

As shown in (B.6), $F^{(r,s)}(\tau, z)$ has power series expansion of the form

$$F^{(r,s)}(\tau, z) = \sum_{b=0}^{1} \sum_{j \in 2\mathbb{Z}+b, n \in \mathbb{Z}/N} c_b^{(r,s)}(4n - j^2) e^{2\pi i n \tau} e^{2\pi i j z} \, , \tag{5.2.31}$$

for appropriate coefficients $c_b^{(r,s)}(4n-j^2)$. From (5.2.30), (5.2.31) it now follows that

$$\frac{1}{N}\, tr_{RR,\widetilde{g}^r}\left(\widetilde{g}^s(-1)^{F_L+F_R}\delta_{NL_0,lw}\delta_{F_L,j}\right) = c_b^{(r,s)}(4lw/N - j^2)\,, \qquad b = j \bmod 2\,. \tag{5.2.32}$$

Hence (5.2.29) gives

$$n(w,l,j) = \sum_{s=0}^{N-1} e^{-2\pi i s l/N} c_b^{(r,s)}(4lw/N - j^2)\,, \qquad r = w \bmod N, \quad b = j \bmod 2\,. \tag{5.2.33}$$

Using this result for single D1-brane spectrum, we can now evaluate the degeneracy of multiple D1-branes moving inside the D5-brane. Let $d_{D1}(W,L,J')$ denote the total number of bosonic minus fermionic states of this system, weighted by $(-1)^{J'}$ and carrying total D1-brane charge W, total momentum $-L/N$ along S^1 and total momentum J' along $\widetilde{S}^1$. This represents the number of ways we can distribute the quantum numbers W, L and J' into individual D1-branes carrying quantum number (w_i, l_i, j_i) subject to the constraint

$$W = \sum_i w_i, \qquad L = \sum_i l_i, \qquad J' = \sum_i j_i, \qquad w_i, l_i, j_i \in \mathbb{Z}, \qquad w_i \geq 1, \qquad l_i \geq 0\,. \tag{5.2.34}$$

A straightforward combinatoric analysis shows that

$$\sum_{W,L,J'} d_{D1}(W,L,J')(-1)^{J'}\, e^{2\pi i(\widetilde{\sigma}W/N + \widetilde{\rho}L + \widetilde{v}J')} = \prod_{\substack{w,l,j\in\mathbb{Z}\\ w>0,l\geq 0}} \left(1 - e^{2\pi i(\widetilde{\sigma}w/N + \widetilde{\rho}l + \widetilde{v}j)}\right)^{-n(w,l,j)}\,. \tag{5.2.35}$$

Using (5.2.33) and defining $k' = w/N$, we can express (5.2.35) as

$$\begin{aligned}&\sum_{W,L,J'} d_{D1}(W,L,J')(-1)^{J'}\, e^{2\pi i(\widetilde{\sigma}W/N + \widetilde{\rho}L + \widetilde{v}J')} \\ &= \prod_{r=0}^{N-1}\prod_{b=0}^{1}\prod_{\substack{k'\in\mathbb{Z}+\frac{r}{N}, l\in\mathbb{Z}, j\in 2\mathbb{Z}+b\\ k'>0, l\geq 0}} \left(1 - e^{2\pi i(\widetilde{\sigma}k' + \widetilde{\rho}l + \widetilde{v}j)}\right)^{-\sum_{s=0}^{N-1} e^{-2\pi i s l/N} c_b^{(r,s)}(4lk'-j^2)}\,.\end{aligned} \tag{5.2.36}$$

5.2.4 The full partition function

Using (5.2.3), (5.2.13), (5.2.25) and (5.2.36) we now get the full partition function:

$$f(\widetilde{\rho},\widetilde{\sigma},\widetilde{v}) = e^{-2\pi i(\widetilde{\alpha}\widetilde{\rho}+\widetilde{v})} \prod_{b=0}^{1}\prod_{r=0}^{N-1}\prod_{\substack{k'\in\mathbb{Z}+\frac{r}{N}, l\in\mathbb{Z}, j\in 2\mathbb{Z}+b\\ k', l\geq 0, j<0 \text{ for } k'=l=0}} \left(1 - e^{2\pi i(\widetilde{\sigma}k' + \widetilde{\rho}l + \widetilde{v}j)}\right)^{-\sum_{s=0}^{N-1} e^{-2\pi i s l/N} c_b^{(r,s)}(4lk'-j^2)}\,. \tag{5.2.37}$$

The multiplicative factor $e^{-2\pi i(\widetilde{\alpha}\widetilde{\rho}+\widetilde{v})}$ as well as the $k'=0$ term in this expression comes from the terms involving $d_{CM}(l_0, j_0)$ and $d_{KK}(l'_0)$. Comparing the right hand side of this equation with the expression for the function $\widetilde{\Phi}(\widetilde{\rho}, \widetilde{\sigma}, \widetilde{v})$ given in (5.1.5) we can rewrite (5.2.37) as

$$f(\widetilde{\rho}, \widetilde{\sigma}, \widetilde{v}) = \frac{e^{2\pi i \widetilde{\gamma}\widetilde{\sigma}}}{\widetilde{\Phi}(\widetilde{\rho}, \widetilde{\sigma}, \widetilde{v})} \,. \tag{5.2.38}$$

According to (5.2.2) we can identify $(-1)^{J+1}h(Q_1, n, J)$, – the number of bosonic minus fermionic quarter BPS supermultiplets weighted by $(-1)^{J+1}$, carrying Q_1 units of D1-brane winding charge along S^1, $-n/N$ units of momentum along S^1 and J units of momentum along $\widetilde{S}^1$, – as the coefficients of the expansion of $f(\widetilde{\rho}, \widetilde{\sigma}, \widetilde{v})$ in powers of $e^{2\pi i\widetilde{\rho}}$, $e^{2\pi i\widetilde{\sigma}}$ and $e^{2\pi i\widetilde{v}}$. Except for the overall multiplicative factor of $e^{-2\pi i\widetilde{\alpha}\widetilde{\rho}}$ in (5.2.37), this expansion involves positive powers of $e^{2\pi i\widetilde{\rho}}$ and $e^{2\pi i\widetilde{\sigma}}$. Furthermore, except for the $k'=l=0$ term, the power of $e^{2\pi i\widetilde{v}}$ for any given power of $e^{2\pi i\widetilde{\rho}}$ and $e^{2\pi i\widetilde{\sigma}}$ is bounded both from above and below. For the $k'=l=0$ term we need to carry out the expansion in positive or negative powers of $e^{2\pi i\widetilde{v}}$ depending on the sign of the angle between S^1 and $\widetilde{S}^1$. If the expansion is in positive powers of $e^{-2\pi i\widetilde{v}}$, – as in the case of positive value of the axion field, – we can extract the Fourier coefficient $h(Q_1, n, J)$ from the equation:

$$h(Q_1, n, J) = (-1)^{J+1}\,\frac{1}{N}\int_{\mathcal{C}} d\widetilde{\rho}\, d\widetilde{\sigma}\, d\widetilde{v}\, e^{-2\pi i(\widetilde{\rho} n + \widetilde{\sigma}(Q_1 - \widetilde{\gamma} N)/N + \widetilde{v} J)}\,\frac{1}{\widetilde{\Phi}(\widetilde{\rho}, \widetilde{\sigma}, \widetilde{v})}\,, \tag{5.2.39}$$

where $\mathcal{C}$ is a three real dimensional subspace of the three complex dimensional space labelled by $(\widetilde{\rho}, \widetilde{\sigma}, \widetilde{v})$, given by

$$\begin{aligned} &\widetilde{\rho}_2 = M_1, \quad \widetilde{\sigma}_2 = M_2, \quad \widetilde{v}_2 = -M_3, \\ &0 \le \widetilde{\rho}_1 \le 1, \quad 0 \le \widetilde{\sigma}_1 \le N, \quad 0 \le \widetilde{v}_1 \le 1\,. \end{aligned} \tag{5.2.40}$$

M_1, M_2 and M_3 are large but fixed positive numbers with $M_3 << M_1, M_2$. The choice of the M_i's is determined from the requirement that the Fourier expansion is convergent in the region of integration. On the other hand if the $k'=l=0$ term in $f(\widetilde{\rho}, \widetilde{\sigma}, \widetilde{v})$ is to be expanded in positive powers of $e^{2\pi i\widetilde{v}}$, – as in the case of negative value of the axion field, – then $h(Q_1, n, J)$ is given by an expression similar to (5.2.39), except that $\widetilde{v}_2$ is now set equal to a positive number M_3 instead of a negative number $-M_3$.

Identifying $h(Q_1, n, J)$ with the degeneracy $d(\vec{Q}, \vec{P})$, using (5.1.2), and noting that β, being equal to $\chi(\mathcal{M})/24$, is equal to $\widetilde{\gamma}N$ given in (5.1.8), we can rewrite (5.2.39) as

$$d(\vec{Q}, \vec{P}) = (-1)^{Q\cdot P+1}\,\frac{1}{N}\int_{\mathcal{C}} d\widetilde{\rho}\, d\widetilde{\sigma}\, d\widetilde{v}\, e^{-\pi i(N\widetilde{\rho} Q^2 + \widetilde{\sigma} P^2/N + 2\widetilde{v} Q\cdot P)}\,\frac{1}{\widetilde{\Phi}(\widetilde{\rho}, \widetilde{\sigma}, \widetilde{v})}\,. \tag{5.2.41}$$

Various useful properties of the function $\widetilde{\Phi}(\widetilde{\rho}, \widetilde{\sigma}, \widetilde{v})$ and a related function $\widehat{\Phi}(\rho, \sigma, v)$ have been discussed in appendices C and D.

5.3 Additional charges from collective modes

The analysis so far has been carried out for a restricted class of charge vectors. We shall now extend our result to a more general class of charge vectors by considering collective excitations of the system analyzed above. Our analysis will follow [151]. For simplicity we shall restrict our analysis to type II string theory compactified on $K3 \times \widetilde{S}^1 \times S^1$ or equivalently heterotic string theory on $T^4 \times \widehat{S}^1 \times S^1$. Generalizing this to the case of $\mathcal{N} = 4$ supersymmetric orbifolds of this theory will require setting some of the charges, which are not invariant under the orbifold group, to zero. The analysis for $\mathcal{N} = 4$ supersymmetric $\mathbb{Z}_N$ orbifolds of type II string theory compactified on T^6 can also be done in an identical manner.

The compactified theory has 28 U(1) gauge fields and hence a given state is characterized by 28 dimensional electric and magnetic charge vectors $\vec{Q}$ and $\vec{P}$ as defined in §3.1.1. We shall choose a basis in which the matrix L has the form

$$L = \begin{pmatrix} \widehat{L} & & & & \\ & 0 & 1 & & \\ & 1 & 0 & & \\ & & & 0_2 & I_2 \\ & & & I_2 & 0_2 \end{pmatrix}, \tag{5.3.1}$$

where $\widehat{L}$ is a matrix with 3 eigenvalues $+1$ and 19 eigenvalues -1. The charge vectors will be labelled as

$$Q = \begin{pmatrix} \widehat{Q} \\ k_1 \\ k_2 \\ k_3 \\ k_4 \\ k_5 \\ k_6 \end{pmatrix}, \qquad P = \begin{pmatrix} \widehat{P} \\ l_1 \\ l_2 \\ l_3 \\ l_4 \\ l_5 \\ l_6 \end{pmatrix}. \tag{5.3.2}$$

The last four elements of $\vec{Q}$ and $\vec{P}$ are to be identified with the four dimensional electric and magnetic charge vectors introduced in (3.1.28):

$$\begin{pmatrix} k_3 \\ k_4 \\ k_5 \\ k_6 \end{pmatrix} = \begin{pmatrix} \widehat{n} \\ n' \\ \widehat{w} \\ w' \end{pmatrix}, \qquad \begin{pmatrix} l_3 \\ l_4 \\ l_5 \\ l_6 \end{pmatrix} = \begin{pmatrix} \widehat{W} \\ W' \\ \widehat{N} \\ N' \end{pmatrix}. \tag{5.3.3}$$

Thus in the second description of the theory k_3, k_4, $-k_5$ and $-k_6$ label respectively the momenta along $\widehat{S}^1$, S^1 and fundamental string winding along $\widehat{S}^1$ and S^1. On the other hand $-l_3$, l_4, l_5 and l_6 label respectively the number of NS 5-branes wrapped along $S^1 \times T^4$ and $\widehat{S}^1 \times T^4$, and Kaluza-Klein monopole charges associated with $\widehat{S}^1$ and S^1. The rest of the charges label the momentum/winding or monopole charges associated with the other internal directions. By following the duality chain that relates the first and second description of the theory and the sign conventions described in appendix A, the different components of $\vec{P}$ and $\vec{Q}$ can be given the following interpretation in the first description of the theory. k_3 represents the D-string winding charge along $\widetilde{S}^1$, k_4 is the momentum along S^1, k_5 is the D5-brane charge along $K3 \times \widetilde{S}^1$, k_6 is the number of Kaluza-Klein monopoles associated with the compact circle $\widetilde{S}^1$, l_3 is the D-string winding charge along S^1, $-l_4$ is the momentum along $\widetilde{S}^1$, l_5 is the D5-brane charge along $K3 \times S^1$ and l_6 is the number of Kaluza-Klein monopoles associated with the compact circle S^1. Other components of $\vec{Q}$ ($\vec{P}$) represent various other branes of type IIB string theory wrapped on $\widetilde{S}^1$ (S^1) times various cycles of $K3$. We shall choose a convention in which the 22-dimensional charge vector $\widehat{Q}$ represents 3-branes wrapped along the 22 2-cycles of K3 times $\tilde{S}^1$, k_1 represents fundamental type IIB string winding charge along $\widetilde{S}^1$, k_2 represents the number of NS 5-branes of type IIB wrapped along $K3 \times \widetilde{S}^1$, the 22-dimensional charge vector $\widehat{P}$ represents 3-branes wrapped along the 22 2-cycles of K3 times S^1, l_1 represents fundamental type IIB string winding charge along S^1 and l_2 represents the number of NS 5-branes of type IIB wrapped along $K3 \times S^1$. In this convention $\widehat{L}$ represents the intersection matrix of 2-cycles of $K3$.

In our new notation the original configuration described in (5.1.1) has charge vectors of the form:

$$Q = \begin{pmatrix} \widehat{0} \\ 0 \\ 0 \\ 0 \\ -n \\ 0 \\ -1 \end{pmatrix}, \qquad P = \begin{pmatrix} \widehat{0} \\ 0 \\ 0 \\ Q_1 - Q_5 = Q_1 - 1 \\ -J \\ Q_5 = 1 \\ 0 \end{pmatrix}, \tag{5.3.4}$$

with

$$Q^2 = 2n, \qquad P^2 = 2(Q_1 - 1), \qquad Q \cdot P = J\,. \tag{5.3.5}$$

Note that we have set $N = 1$. The degeneracy of this system calculated in §5.2.4 may be written as:

$$d(\vec{Q}, \vec{P}) = h(Q_1, n, J) = h\left(\frac{1}{2}P^2 + 1, \frac{1}{2}Q^2, Q \cdot P\right), \tag{5.3.6}$$

where the function h is given in (5.2.39) with the choice $\mathcal{M} = K3$ and $N = 1$. Our goal will be to consider charge vectors more general than the ones given in (5.3.4) and check if the degeneracy is still given by (5.3.6). We shall do this by adding charges to the existing system by exciting appropriate collective modes of the system. These collective modes come from three sources:

1. The original configuraion in the type IIB theory contains a Kaluza-Klein monopole associated with the circle $\widetilde{S}^1$. This solution has been given in (5.2.4), with the coordinate ψ labelling the coordinate of $\widetilde{S}^1$, and (r, θ, ϕ) representing spherical polar coordinates of the non-compact space. These coordinates label the geometry of the space 'transverse' to the Kaluza-Klein monopole. The world-volume of the Kaluza-Klein monopole spans the $K3$ surface, the circle S^1 which we shall label by y, and time t. As in §5.2 we shall take the size of $K3$ to be small compared to that of S^1 and use dimensional reduction on K3 to regard the world-volume as two dimensional, spanned by y and t.

   Type IIB string theory compactified on K3 has various 2-form fields, – the original NSNS and RR 2-form fields B and $C^{(2)}$ of the ten dimensional type IIB string theory as well as the components of the 4-form field $C^{(4)}$ along various 2-cycles of $K3$. Given any such 2-form field C_{MN}, we can introduce a scalar mode ϕ by considering deformations of the form [155]:

$$C = \phi(y, t)\, \omega\,, \tag{5.3.7}$$

   where ω is the harmonic 2-form of Taub-NUT space introduced in (5.2.6):

$$\omega \propto \frac{r}{r + R_0} d\sigma_3 + \frac{R_0}{(r + R_0)^2} dr \wedge \sigma_3\,, \qquad \sigma_3 \equiv \left(d\psi + \frac{1}{2}\cos\theta d\phi\right)\,. \tag{5.3.8}$$

   If the field strength dC associated with C is self-dual or anti-selfdual in six dimensions transverse to $K3$ then the corresponding scalar field ϕ is chiral in the $y - t$ space; otherwise it represents a non-chiral scalar field. The non-zero mode oscillators associated with the left-moving components of these scalar fields were used in §5.2.1 for counting the number of BPS states of the Kaluza-Klein monopole. Our focus in this section will be on the zero modes of these scalar fields. In particular we shall consider configurations which carry momentum conjugate to ϕ or winding number along y of ϕ, represented by a solution where ϕ is linear in t or y. In the six dimensional language this corresponds to $dC \propto dt \wedge \omega$ or $dy \wedge \omega$. From (5.3.8) we see that $dC \propto dt \wedge \omega$ will have a component proportional to $r^{-2}\, dt \wedge dr \wedge d\psi$ for large r, and hence the coefficient of this term represents the charge associated with a string, electrically charged under C, wrapped along $\widetilde{S}^1$. On the other hand $dC \propto dy \wedge \omega$ will have a component proportional to $\sin\theta\, dy \wedge d\theta \wedge d\phi$ and the coefficient of this term will represent the charge associated with a string, magnetically charged under C, wrapped along $\widetilde{S}^1$. If the

2-form field C represents the original RR or NSNS 2-form field of type IIB string theory in ten dimensions, then the electrically charged string would correspond to a D-string or a fundamental type IIB string and the magnetically charged string would correspond to a D5-brane or NS 5-brane wrapped on $K3$. On the other hand if the 2-form C represents the component of the 4-form field along a 2-cycle of K3, then the corresponding string represents a D3-brane wrapped on a 2-cycle times $\widetilde{S}^1$. Recalling the interpretation of the charges $\widehat{Q}$ and k_i appearing in (5.3.2) we now see that the momentum and winding modes of ϕ correspond to the charges $\widehat{Q}$, k_1, k_2, k_3 and k_5. More specifically, after taking into account the sign conventions described in appendix A, these charges correspond to switching on deformations of the form:

$$
\begin{aligned}
&dB \propto -k_1 dt \wedge \omega, \quad dB \propto k_2 dy \wedge \omega, \quad dC^{(2)} \propto -k_3 dt \wedge \omega, \quad dC^{(2)} \propto k_5 dy \wedge \omega, \\
&dC^{(4)} \propto \sum_\alpha \widehat{Q}_\alpha (1 + *)\, \Omega_\alpha \wedge dy \wedge \omega \,,
\end{aligned} \tag{5.3.9}
$$

where $\{\Omega_\alpha\}$ denote a basis of harmonic 2-forms on $K3$ $(1 \le \alpha \le 22)$ satisfying $\int_{K3} \Omega_\alpha \wedge \Omega_\beta = \widehat{L}_{\alpha\beta}$. Thus in the presence of these deformations we have a more general electric charge vector of the form

$$
Q_0 = \begin{pmatrix} \widehat{Q} \\ k_1 \\ k_2 \\ k_3 \\ -n \\ k_5 \\ -1 \end{pmatrix} . \tag{5.3.10}
$$

As can be easily seen from (5.3.9), k_2 represents NS 5-brane charge wrapped along $K3 \times \widetilde{S}^1$. However for weakly coupled type IIB string theory, the presence of this charge could have large backreaction on the geometry. We can avoid this by choosing

$$
k_2 = 0 \,. \tag{5.3.11}
$$

Alternatively by taking the radius of S^1 to be large we could make the Kaluza-Klein monopole much heavier than the NS 5-brane wrapped on $K3 \times \widetilde{S}^1$. This will keep the backreaction of the NS 5-brane under control. We shall continue to take $k_2 = 0$ for simplicity.

2. The original configuration considered in §5.1 also contains a D5-brane wrapped around $K3 \times S^1$. We can switch on flux of gauge field strengths $\mathcal{F}$ on the D5-brane world-volume along the

various 2-cycles of K3 that it wraps. The net coupling of the RR gauge fields to the D-brane world-volume in the presence of the gauge fields may be expressed as [166]

$$\int \left[C^{(6)} + C^{(4)} \wedge \mathcal{F} + \frac{1}{2} C^{(2)} \wedge \mathcal{F} \wedge \mathcal{F} + \cdots \right] , \tag{5.3.12}$$

up to a constant of proportionality. The integral is over the D5-brane world-volume spanned by y, t and the coordinates of $K3$. Via the coupling

$$\int C^{(4)} \wedge \mathcal{F} , \tag{5.3.13}$$

the gauge field configuration will produce the charges of a D3-brane wrapped on a 2-cycle of $K3$ times S^1, – i.e. the 22 dimensional magneic charge vector $\widehat{P}$ appearing in (5.3.2).

In order to be compatible with our convention of appendix A that the D5-brane wrapped on $K3 \times S^1$ carries negative $(dC^{(6)})_{(K3)yrt}$ asymptotically, we need to take the integration measure in the yt plane in (5.3.12) as $dy \wedge dt$, i.e. $\epsilon^{yt} > 0$. Using this information one can show that the gauge field flux required to produce a specific magnetic charge vector $\widehat{P}$ is

$$\mathcal{F} \propto -\sum_\alpha \widehat{P}_\alpha \, \Omega_\alpha \, . \tag{5.3.14}$$

3. The D1-D5 system can also carry electric flux along S^1. This will induce the charge of a fundamental type IIB string wrapped along S^1. According to the physical interpretation of various charges given earlier, this gives the component l_1 of the magnetic charge vector P.

   The net result of switching on both the electric and magnetic flux along the D5-brane world-volume is to generate a magnetic charge vector of the form:

$$P_0 = \begin{pmatrix} \widehat{P} \\ l_1 \\ 0 \\ Q_1 - 1 \\ -J \\ 1 \\ 0 \end{pmatrix} . \tag{5.3.15}$$

This however is not the end of the story. So far we have discussed the effect of the various collective mode excitations on the charge vector to first order in the charges, without taking into account the effect of the interaction of the deformations produced by the collective modes with the background fields already present in the system, or the background fields produced by other collective modes. Taking into account these effects produces further shifts in the charge vector as described below.

1. As seen from (5.3.12), the D5-brane world-volume theory has a coupling proportional to $\int C^{(2)} \wedge \mathcal{F} \wedge \mathcal{F}$. Thus in the presence of magnetic flux $\mathcal{F}$ the D5-brane wrapped on $K3 \times S^1$ acts as a source of the D1-brane charge wrapped on S^1. The effect is a shift in the magnetic charge quantum number l_3 that is quadratic in $\mathcal{F}$ and hence quadratic in $\widehat{P}$ due to (5.3.14). A careful calculation, taking into various signs and normalization factors, shows that the net effect of this term is to give an additional contribution to l_3 of the form:
$$\Delta_1 l_3 = -\widehat{P}^2/2 \,. \tag{5.3.16}$$

2. Let C be a 2-form in the six dimensional theory obtained by compactifying type IIB string theory on $K3$ and $F = dC$ be its field strength. As summarized in (5.3.9), switching on various components of the electric charge vector $\vec{Q}$ requires us to switch on F proportional to $dt \wedge \omega$ or $dy \wedge \omega$. The presence of such background induces a coupling proportional to
$$-\int \sqrt{-\det g}\, g^{yt} F_{ymn} F_t{}^{mn} \tag{5.3.17}$$
with the indices m, n running over the coordinates of the Taub-NUT space. This produces a source for g^{yt}, i.e. momentum along S^1. The effect of such terms is to shift the component k_4 of the charge vector $\vec{Q}$. A careful calculation shows that the net change in k_4 induced due to this coupling is given by
$$\Delta_2 k_4 = k_3 k_5 + \widehat{Q}^2/2 \,, \tag{5.3.18}$$
where we have used the fact that k_2 has been set to zero. The $k_3 k_5$ term comes from taking F in (5.3.17) to be the field strength of the RR 2-form field, and $\widehat{Q}^2/2$ term comes from taking F to be the field strength of the components of the RR 4-form field along various 2-cycles of $K3$. For non-zero k_2 there will also be an additive contribution of $k_1 k_2$ to the right hand side of eq.(5.3.18).

3. The D5-brane wrapped on $K3 \times S^1$ or the magnetic flux on this brane along any of the 2-cycles of $K3$ produces a magnetic type 2-form field configuration of the form:
$$F \equiv dC \propto \sin\theta \, d\psi \wedge d\theta \wedge d\phi \,, \tag{5.3.19}$$

where C is any of the RR 2-form fields in six dimensional theory obtained by compactifying type IIB string theory on $K3$. One can verify that the 3-form appearing on the right hand side of (5.3.19) is both closed and co-closed in the Taub-NUT background and hence F given in (5.3.19) satisfies both Bianchi identity and the linearized equations of motion. The coefficients of the term given in (5.3.19) for various 2-form fields C are determined in terms of $\widehat{P}$ and the D5-brane charge along $K3 \times S^1$ which has been set equal to 1. This together with the term in F proportional to $dt \wedge \omega$ coming from the collective coordinate excitation of the Kaluza-Klein monopole generates a source of the component $g^{\psi t}$ of the metric via the coupling proportional to

$$-\int \sqrt{-\det g}\, g^{\psi t} F_{\psi mn} F_t{}^{mn} \tag{5.3.20}$$

This induces a net momentum along $\widetilde{S}^1$ and gives a contribution to the component l_4 of the magnetic charge vector P. A careful calculation shows that the net additional contribution to l_4 due to this coupling is given by

$$\Delta_3 l_4 = k_3 + \widehat{Q} \cdot \widehat{P} \,. \tag{5.3.21}$$

In this expression the contribution proportional to k_3 comes from taking F in (5.3.20) to be the field strength associated with the RR 2-form field of IIB, whereas the term proportional to $\widehat{Q} \cdot \widehat{P}$ arises from taking F to be the field strength associated with the components of the RR 4-form field along various 2-cycles of $K3$.

4. Eqs.(5.3.15) and (5.3.16) show that we have a net D1-brane charge along S^1 equal to

$$l_3 = Q_1 - 1 - \widehat{P}^2/2 \,. \tag{5.3.22}$$

If we denote by $C^{(2)}$ the 2-form field of the original ten dimensional type IIB string theory, then the effect of this charge is to produce a background of the form:

$$dC^{(2)} \propto (Q_1 - 1 - \widehat{P}^2/2)\, r^{-2}\, dr \wedge dt \wedge dy \,. \tag{5.3.23}$$

Again one can verify explicitly that the right hand side of (5.3.23) is both closed and co-closed in the Taub-NUT background. If we superimpose this on the background

$$dC^{(2)} \propto k_5\, dy \wedge \omega \,, \tag{5.3.24}$$

coming from the excitation of the collective coordinate of the Kaluza-Klein monopole, then we get a source term for $g^{\psi t}$ via the coupling proportional to

$$-\int \sqrt{-\det g}\, g^{\psi t} F_{\psi r y} F_t{}^{ry} \tag{5.3.25}$$

This gives an additional contribution to the charge l_4 of the form

$$\Delta_4 l_4 = k_5(Q_1 - 1 - \widehat{P}^2/2)\,. \tag{5.3.26}$$

For non-zero k_2 there will also be an additional contribution of $k_2 l_1$ to l_4 from the $-\int g^{\psi t}(dB)_{\psi r y}(dB)_t{}^{ry}$ term in the action.

This finishes our analysis of the possible additional sources produced by the quadratic terms in the fields. What about higher order terms? It is straightforward to show that the possible effect of the higher order terms on the shift in the charges will involve one or more powers of the type IIB string coupling. Since the shift in the charges must be quantized, they cannot depend on continuous moduli. Thus at least in the weakly coupled type IIB string theory there are no additional corrections to the charges.

Combining all the results we see that we have a net electric charge vector $\vec{Q}$ and a magnetic charge vector $\vec{P}$ of the form:

$$Q = \begin{pmatrix} \widehat{Q} \\ k_1 \\ 0 \\ k_3 \\ -n + k_3 k_5 + \widehat{Q}^2/2 \\ k_5 \\ -1 \end{pmatrix}, \qquad P = \begin{pmatrix} \widehat{P} \\ l_1 \\ 0 \\ Q_1 - 1 - \widehat{P}^2/2 \\ -J + k_3 + \widehat{Q}\cdot\widehat{P} + k_5(Q_1 - 1 - \widehat{P}^2/2) \\ 1 \\ 0 \end{pmatrix}. \tag{5.3.27}$$

This has

$$Q^2 = 2n, \qquad P^2 = 2(Q_1 - 1), \qquad Q\cdot P = J\,. \tag{5.3.28}$$

Thus the additional charges do not affect the relationship between the invariants Q^2, P^2, $Q\cdot P$ and the original quantum numbers n, Q_1 and J.

Let us now turn to the analysis of the dyon spectrum in the presence of these charges. For this we recall that in §5.2 the dyon spectrum was computed from three mutually non-interacting parts, – the dynamics of the Kaluza-Klein monopole, the overall motion of the D1-D5 system in the background of the Kaluza-Klein monopole and the motion of the D1-branes relative to the D5-brane. The precise dynamics of the D1-branes relative to the D5-brane is affected by the presence of the gauge field flux on the D5-brane since it changes the non-commutativity parameter describing the dynamics of the gauge field on the D5-brane world-volume [165]. As a result the moduli space of D1-branes, described as non-commutative instantons in this gauge theory [164], gets deformed. However we do not expect this to change the elliptic genus of the corresponding conformal field theory [163] that enters the degeneracy formula. With the exception of the zero mode associated with the D1-D5 center of mass motion in the Kaluza-Klein monopole background, the rest of the contribution to the degeneracy came from the excitations involving non-zero mode oscillators and this is not affected either by switching on gauge field fluxes on the D5-brane world-volume or the momenta or winding number of the collective coordinates of the Kaluza-Klein monopole. On the other hand the dynamics of the D1-D5-brane center of mass motion in the background geometry is also not expected to be modified in the weakly coupled type IIB string theory since in this limit the additional background fields due to the additional charges are small compared to the one due to the Kaluza-Klein monopole. (For this it is important that the additional charges do not involve any other Kaluza-Klein monopole or NS 5-brane charge.) Thus we expect the degeneracy to be given the same function $h(Q_1, n, J)$ that appeared in the absence of the additional charges. Using (5.3.28) we can now write

$$d(\vec{Q}, \vec{P}) = h\left(\frac{1}{2}P^2 + 1, \frac{1}{2}Q^2, Q \cdot P\right) . \tag{5.3.29}$$

This is a generalization of (5.3.6) and shows that for the charge vectors given in (5.3.27), the degeneracy $d(\vec{Q}, \vec{P})$ depend on the charges only through the combination Q^2, P^2 and $Q \cdot P$.

This analysis easily generalizes to $\mathbb{Z}_N$ orbifolds of type IIB string theory on $K3 \times \widetilde{S}^1 \times S^1$, with the only change that the quantum number n, instead of being an integer, will be an integer multiple of $1/N$ and the charge vectors $\vec{Q}$, $\vec{P}$ are restricted to the $\mathbb{Z}_N$ invariant subspace. For $\mathbb{Z}_N$ orbifolds of type IIB on $T^4 \times \widetilde{S}^1 \times S^1$ there is an additional change, – the $(Q_1 - 1)$ factors in (5.3.27), (5.3.28) are replaced by Q_1.

It is instructive to focus on the four dimensional subspace of the charge lattice spanned by the last four elements of $\vec{Q}$ and $\vec{P}$. In the second description of the theory these correspond to momenta and fundamental string winding charges and H- and Kaluza-Klein monopole charges along the circles $\widehat{S}^1$ and S^1. It follows from (5.3.27), generalized to the $\mathbb{Z}_N$ orbifold cases so as to allow n to be multiples of $1/N$, that our formula for the degeneracy is valid for a general charge vector of the form

$$Q = \begin{pmatrix} k_3 \\ k_4 \\ k_5 \\ -1 \end{pmatrix}, \qquad P = \begin{pmatrix} l_3 \\ l_4 \\ 1 \\ 0 \end{pmatrix}, \qquad k_4 \in \mathbb{Z}/N, \quad k_i, l_i \in \mathbb{Z} \quad \text{otherwise}\,, \tag{5.3.30}$$

in this subspace.

For use in §5.5 we shall now analyze the T-duality orbit of these charge vectors. This is equivalent to the question: What is the most general charge vector that can be reached from (5.3.30) by a T-duality transformation acting within this four dimensional subspace? For this recall that T-duality transformation is the $\Gamma_1(N)\times\Gamma_1(N)$ subgroup of $SO(2,2;\ \mathbb{Z})$ matrices described in (3.1.33). Such matrices, acting on the set (5.3.30), produce charge vectors of the form:[33]

$$
\begin{aligned}
& Q=\begin{pmatrix} k_3\\ k_4\\ k_5\\ k_6 \end{pmatrix}, \quad P=\begin{pmatrix} l_3\\ l_4\\ l_5\\ l_6 \end{pmatrix}, \\
& k_4\in\ \mathbb{Z}/N, \quad k_6\in N\ \mathbb{Z}-1, \quad l_5\in N\ \mathbb{Z}+1, \quad l_6\in N\ \mathbb{Z}, \quad k_i,l_i\in\ \mathbb{Z}\ \text{otherwise}\,, \\
& \text{g.c.d.}(Nk_3l_4-Nk_4l_3, k_5l_6-k_6l_5, k_3l_5-k_5l_3+k_4l_6-k_6l_4)=1\,. \qquad (5.3.31)
\end{aligned}
$$

The condition involving the g.c.d. comes from the observation that the left hand side is preserved under the T-duality transformation[34] generated by the matrices (3.1.33) and for the initial charge vector (5.3.30) the left hand side is 1 since $k_5l_6-k_6l_5=1$. Note that quantization law allows k_6 to be an arbitrary integer but a T-duality transformation on the original charge vector can only produce those k_6 which are -1 modulo N. Since in the second description $-k_6$ measures the fundamental string winding charge w' along S^1 and the orbifold group generator involves translation by 2π along S^1, requiring k_6 to be -1 mod N corresponds to restriction to states whose electric charge vector lies in the sector twisted by a single power of the orbifold group. Similarly quantization law allows l_5 to be an arbitrary integer but a T-duality transformation of (5.3.30) can only produce charge vectors for which $l_5=1$ modulo N. In the second description this corresponds to requiring the total Kaluza-Klein monopole charge associated with the $\widehat{S}^1$ direction to be 1 modulo N. We shall use the T-duality invariance of the theory to argue in §5.5 that our results for degeneracy are valid for the class of charge vectors given in (5.3.31) in the same domain of the moduli space where the original calculation was performed.

For orbifolds of type IIB string theory on $K3\times\widetilde{S}^1\times S^1$ the condition $l_5\in N\ \mathbb{Z}+1$ can be relaxed by considering a more general configuration with arbitrary number Q_5 of D5-branes instead of a single D5-brane, subject to the condition g.c.d.(Q_5,Q_1)=1. The counting of quarter BPS states for this more general configuration has been carried out in [6] and reproduced in appendix E. Also in this case the g.c.d. appearing in (5.3.31) is no longer 1 since for the initial charge vector $l_5=Q_5$ and hence $k_5l_6-k_6l_5=Q_5$. Nevertheless the condition g.c.d.$(Q_5,Q_1-Q_5)=1$, – which translates to

[33]While we do not know of any further constraint on the charges, we have not proven that (5.3.31) is the complete set of conditions on the T-duality orbit of (5.3.30). Thus it is possible that the actual T-duality orbit has additional conditions on k_i, l_i.

[34]This in turn follows from the fact that under the transformation (3.1.33) the arguments of g.c.d. transform into linear combinations of each other with integer coefficients. Thus the final g.c.d. must be an integer multiple of the initial g.c.d.. Applying the inverse of the transformation (3.1.33) on the final variables we can prove that the initial g.c.d. is an integer multiple of the final g.c.d. Thus the initial and final g.c.d.'s must be equal.

g.c.d.$(l_3, l_5) = 1$, – can be used to argue that the charge vectors on the T-duality orbit of the initial charge vector still satisfy the condition [167]

$$\text{g.c.d.}\{k_i l_j - k_j l_i,\, N k_4 l_s - N k_s l_4,\, k_4 l_6 - k_6 l_4;\quad i,j = 3,5,6,\, s = 3,5\} = 1\,. \qquad (5.3.32)$$

This follows from the fact that the left hand side is invariant under T-duality and that it takes the value 1 for the initial charge vector. These relaxed constraints are also consistent with the existence of the extra duality transformation (3.1.34) in these theories. This duality transformation does not preserve the $l_5 = 1$ modulo N condition, nor does it preserve the g.c.d. appering in (5.3.31).

It may also be possible to relax the $l_5 \in N\,\mathbb{Z} + 1$ condition and the last condition in (5.3.31) for orbifolds of type IIB string theory on $T^4 \times \widetilde{S}^1 \times S^1$ by taking multiple D5-branes as in the case of $K3 \times \widetilde{S}^1 \times S^1$. This would require a careful analysis taking into account the dynamics of Wilson lines on multiple D5-branes.

Let us now consider the effect of S-duality transformation on these charge vectors. The action of this transformation on the charges has been described in (3.1.35), (3.1.36). It follows from this that an S-duality transformation acting on a charge vector of the form given in (5.3.31) gives us back another charge vector of the same form. Thus the subset of the charge lattice consisting of elements of the form (5.3.31) is invariant under both the $\Gamma_1(N)$ S-duality and $\Gamma_1(N) \times \Gamma_1(N)$ T-duality transformations. This is also the case if we relax the $l_5 \in N\,\mathbb{Z} + 1$ condition and replace the last condition on (5.3.31) by the condition (5.3.32), – in this case the conditions are also invariant under the additional $\mathbb{Z}_2$ T-duality transformation given in (3.1.34).

Since we have not shown that T-duality orbits of (5.3.30) generate all charge vectors of the form (5.3.31) (see footnote 33), it will be useful to prove a slightly different result, – namely that the set of charge vectors in the T-duality orbits of the charges of the form (5.3.30) is closed under S-duality transformation. For this we need to prove that an arbitrary S-duality transform of any charge vector in the orbit can be brought to the form (5.3.30) by a T-duality transformation. To see that let us consider a charge vector obtained from (5.3.30) by left multiplication by a T-duality transformation matrix Ω_0, followed by an arbitrary S-duality transformation $\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$. This produces a charge vector:

$$Q = \Omega_0 \begin{pmatrix} \alpha k_3 + \beta l_3 \\ \alpha k_4 + \beta l_4 \\ \alpha k_5 + \beta \\ -\alpha \end{pmatrix}, \qquad P = \Omega_0 \begin{pmatrix} \gamma k_3 + \delta l_3 \\ \gamma k_4 + \delta l_4 \\ \gamma k_5 + \delta \\ -\gamma \end{pmatrix}. \qquad (5.3.33)$$

It is easy to see that left multiplication by the T-duality transformation matrix

$$\begin{pmatrix} \gamma k_5 + \delta & -\gamma & 0 & 0 \\ -\alpha k_5 - \beta & \alpha & 0 & 0 \\ 0 & 0 & \alpha & \alpha k_5 + \beta \\ 0 & 0 & \gamma & \gamma k_5 + \delta \end{pmatrix} \Omega_0^{-1} \, , \tag{5.3.34}$$

brings (5.3.33) to the form (5.3.30) with $k_5 = 0$. Thus the set of charge vectors which can be obtained from charge vectors of the form (5.3.30) by T-duality transformations is closed under an S-duality transformation.

5.4 Walls of marginal stability

As has been briefly mentioned in §5.1, the degeneracy formula derived here is expected to be valid within a certain region of the moduli space bounded by codimension one subspaces on which the BPS state under consideration becomes marginally stable. As we cross this subspace of the moduli space, the spectrum can change discontinuously. In this section we shall study in some detail the locations of these walls of marginal stability so that we can identify the region within which our degeneracy formula will remain valid. Our analysis will follow the one given in [168]. Some related work can be found in [167, 150, 169, 170].

Let us consider a state carrying electric charge $\vec{Q}$ and magnetic charge $\vec{P}$ and examine under what condition it can decay into a pair of half-BPS states.[35] This happens when its mass is equal to the sum of the masses of a pair of half BPS states whose electric and magnetic charges add up to $\vec{Q}$ and $\vec{P}$ respectively. Since for half BPS states the electric and magnetic charges must be parallel, these pair of states must have charge vectors of the form $(a\vec{M}, c\vec{M})$ and $(b\vec{N}, d\vec{N})$ for some constants a, b, c, d and a pair of r-dimensional vectors $\vec{M}$, $\vec{N}$. We shall normalize $\vec{M}$, $\vec{N}$ such that

$$ad - bc = 1 \, . \tag{5.4.1}$$

Then the requirement that the charges add up to $(\vec{Q}, \vec{P})$ gives

$$\vec{M} = d\vec{Q} - b\vec{P}, \qquad \vec{N} = -c\vec{Q} + a\vec{P} \, . \tag{5.4.2}$$

[35]There are also subspaces of the asymptotic moduli space where the mass of a quarter BPS state becomes equal to the sum of the masses of a pair of *quarter* BPS states, or a quarter BPS state and a half-BPS state or more than two half or quarter BPS states. However it has been shown in [169, 170] that such subspaces are of codimension larger than one. Hence in going from one generic point in the moduli space to another one can avoid them by going around them. As a result we do not expect them to affect the dyon spectrum.

Thus the charges of the decay products are given by

$$(ad\vec{Q} - ab\vec{P}, cd\vec{Q} - cb\vec{P}) \quad \text{and} \quad (-bc\vec{Q} + ab\vec{P}, -cd\vec{Q} + ad\vec{P})\,. \tag{5.4.3}$$

Under the scale transformation

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \to \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix} \tag{5.4.4}$$

eqs.(5.4.1) and (5.4.3) remain unchanged. There is also a discrete transformation

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \to \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \tag{5.4.5}$$

which leaves (5.4.1) unchanged and exchanges the two decay products in (5.4.3). Thus a pair of matrices $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ related by (5.4.4) or (5.4.5) describe identical decay channels.

In order that the charge vectors of the decay products given in (5.4.3) satisfy the charge quantization rules we must ensure that $a\vec{M} = ad\vec{Q} - ab\vec{P}$ and $b\vec{N} = -bc\vec{Q} + ab\vec{P}$ belong to the lattice of electric charges and that $c\vec{M} = cd\vec{Q} - cb\vec{P}$ and $d\vec{N} = -cd\vec{Q} + ad\vec{P}$ belong to the lattice of magnetic charges. For the charge vectors $\vec{Q}$, $\vec{P}$ given in (5.1.1), or more generally (5.3.27) or (5.3.31), this would require

$$ad, ab, bc \in \mathbb{Z}, \qquad cd \in N\,\mathbb{Z}\,. \tag{5.4.6}$$

The condition $cd \in N\,\mathbb{Z}$ comes from the requirement that $cd\vec{Q} - cb\vec{P}$ is an allowed magnetic charge. In particular for a $\vec{Q}$ of the form (5.1.1), a magnetic charge $cd\vec{Q}$ represents, in either description, a state with Kaluza-Klein monopole charge $-cd$ associated with S^1. Since this charge is quantized in units of N, cd must be a multiple of N. We shall denote by $\mathcal{A}$ the set of matrices $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ subject to the equivalence relations (5.4.4), (5.4.5) and satisfying (5.4.1), (5.4.6). One can show that using the scaling freedom (5.4.4) one can always choose a, b, c, d to be integers satisfying (5.4.1) [168]. Eq.(5.4.6) then gives further restriction on the integers c and d.

We shall now determine the wall of marginal stability corresponding to the decay channel given in (5.4.3). Our starting point will be the formula for the mass $m(\vec{Q}, \vec{P})$ of a BPS state carrying electric charge $\vec{Q}$ and magnetic charge $\vec{P}$ [171, 172]

$$\begin{aligned} m(\vec{Q}, \vec{P})^2 &= \frac{1}{S_\infty}(Q - \bar{\tau}_\infty P)^T (M_\infty + L)(Q - \tau_\infty P) \\ &\quad + 2\left[(Q^T(M_\infty + L)Q)(P^T(M_\infty + L)P) - (P^T(M_\infty + L)Q)^2\right]^{1/2}, \end{aligned} \tag{5.4.7}$$

where $\tau = a + iS$ and the subscript ∞ denotes asymptotic values of various fields. This expression is manifestly invariant under the T- and S-duality transformations described in eqs.(3.1.30)-(3.1.36) in §3.1.3. In order that the state $(\vec{Q}, \vec{P})$ is marginally stable against decay into $(ad\vec{Q} - ab\vec{P}, cd\vec{Q} - cb\vec{P})$ and $(-bc\vec{Q} + ab\vec{P}, -cd\vec{Q} + ad\vec{P})$, we need

$$m(\vec{Q}, \vec{P}) = m(ad\vec{Q} - ab\vec{P}, cd\vec{Q} - cb\vec{P}) + m(-bc\vec{Q} + ab\vec{P}, -cd\vec{Q} + ad\vec{P}) \,. \tag{5.4.8}$$

Using (5.4.7), (5.4.8) and some tedious algebra, we arrive at the condition

$$\left(a_\infty - \frac{ad+bc}{2cd}\right)^2 + \left(S_\infty + \frac{E}{2cd}\right)^2 = \frac{1}{4c^2d^2}(1+E^2)\,, \tag{5.4.9}$$

where

$$E \equiv \frac{cd(Q^T(M_\infty + L)Q) + ab(P^T(M_\infty + L)P) - (ad+bc)(P^T(M_\infty + L)Q)}{[(Q^T(M_\infty + L)Q)(P^T(M_\infty + L)P) - (P^T(M_\infty + L)Q)^2]^{1/2}}\,. \tag{5.4.10}$$

Note that E depends on M_∞, the constants a, b, c, d and the charges $\vec{Q}$, $\vec{P}$, but is independent of τ_∞. Thus for fixed $\vec{P}$, $\vec{Q}$ and M_∞, the wall of marginal stability describes a circle in the (a_∞, S_∞) plane with radius

$$R = \sqrt{1+E^2}/2|cd|\,, \tag{5.4.11}$$

and center at

$$C = \left(\frac{ad+bc}{2cd}, -\frac{E}{2cd}\right)\,. \tag{5.4.12}$$

This circle intersects the real τ_∞ axis at

$$a/c \quad \text{and} \quad b/d\,. \tag{5.4.13}$$

The cases where either c or d vanish require special attention. First consider the case $c = 0$. In this case the condition $ad - bc = 1$ implies that $ad = 1$, i.e. either $a = d = 1$ or $a = d = -1$. Using the scaling freedom (5.4.4) we can choose $a = d = 1$. By taking the $c \to 0$ limit of (5.4.9), (5.4.10) we see that the wall of marginal stability becomes a straight line in the (a_∞, S_∞) plane for a fixed M_∞:

$$a_\infty - \frac{b(P^T(M_\infty + L)P) - (P^T(M_\infty + L)Q)}{[(Q^T(M_\infty + L)Q)(P^T(M_\infty + L)P) - (P^T(M_\infty + L)Q)^2]^{1/2}}\, S_\infty - b = 0\,. \tag{5.4.14}$$

The $d = 0$ case is related to the $c = 0$ case by the equivalence relation (5.4.5), and hence do not give rise to new walls of marginal stability.

In order to get some insight into the geometric structure of the domain bounded by these marginal stability walls it will be useful to study the possible intersection points of these walls in the upper half τ_∞ plane [168]. By a careful analysis one finds that these walls never intersect in the interior of the upper half plane. The only possible intersections are on the real axis, at $i\infty$ or in the lower half plane. Furthermore due to (5.4.13) the intersection points on the real axis are independent of the charges or the moduli M_∞. Thus a domain bounded by the marginal stability walls has universal vertices although the precise shape of the walls do depend on the moduli M_∞ as well as the charges $(\vec{Q}, \vec{P})$. This allows us to give a universal classification of domains in terms of their vertices. In fact since the integers a, b, c, d are related to the charges of the decay products on the corresponding wall via (5.4.3), this universal characterization of a domain corresponds to specifying how the charges of a decay product are related to the charges of the original system on the different walls bordering a particular domain. Various other geometric properties of these walls have been discussed in [170].

This finishes our general analysis of marginal stability walls and domains bounded by them. An important question that we need to address now is: in which of the many domains in the τ_∞ plane is our degeneracy formula given in (5.1.3), (5.1.4) valid? This question can be answered by recalling that in the first description we work in the weak coupling limit of type IIB string theory at finite values of the other moduli. Following the chain of dualities one can translate this to information about τ_∞ and the matrix M_∞ in the second description, and work out how this region is situated with respect to various marginal stability lines in the τ_∞ plane. It turns out [168] in this limit a_∞, S_∞ are finite and $P^T(M_\infty + L)P \sim |Q^T(M_\infty + L)P| << Q^T(M_\infty + L)Q$. As a result for $cd \neq 0$, E defined in (5.4.10) is large and in the upper half plane the circles (5.4.10) lie close to the real axis. On the other hand the straight lines (5.4.14) become almost vertical lines passing through the integers b. Thus for $-1 < a_\infty < 1$ the weak coupling region in the first description gets mapped to one of two domains, – the right domain $\mathcal{R}$ bounded by the lines corresponding to $b = 0$ and $b = 1$ in (5.4.14) together with a set of circle segments at the bottom, and the left domain $\mathcal{L}$ bounded by the lines corresponding to $b = -1$ and $b = 0$ in (5.4.14) together with a set of circle segments at the bottom. The marginal stability wall corresponding to $b = 0$ corresponds to the wall of marginal stability encountered in the analysis of the supersymmetric quantum mechanics describing the D1-D5 center of mass motion in the Kaluza-Klein monopole background. For the right domain $\mathcal{R}$ our original formula given in (5.1.3), (5.1.4) is valid. For the left domain $\mathcal{L}$ we need to change the integration contour to the one given in (5.1.13). We shall denote by $\mathcal{B}_R$ the set of matrices $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ corresponding to the marginal stability walls which form the boundary of the region $\mathcal{R}$ and by $\mathcal{B}_L$ the set of matrices describing the marginal stability walls which form the boundary of the region $\mathcal{L}$. These sets have been determined explicitly in [168] for various values of N. For example for $N = 1$ the set $\mathcal{B}_R$ is given by

$$\mathcal{B}_R : \quad \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \right\}, \tag{5.4.15}$$

representing respectively a straight line passing through 0, a straight line passing through 1 and a circle passing through 0 and 1. Thus the vertices of $\mathcal{R}$ are at 0, 1 and ∞. One can show that for $N = 2$ the vertices of $\mathcal{R}$ are at 0, 1/2, 1 and ∞ and for $N = 3$ they are at 0, 1/3, 1/2, 2/3 and ∞ [168]. For $N \geq 4$ each domain bounded by walls of marginal stability has infinite number of vertices.

An interesting question is: how does the degeneracy formula change as we cross other marginal stability walls? We shall argue in §5.5 that the changes are such that the expression for the degeneracy in the other domains can also be expressed as an integral of the type (5.1.3), but with an integration contour that is different from (5.1.4).

5.5 Duality transformation of the degeneracy formula

As noted in §5.1, the degeneracy formula (5.1.3), (5.1.4) has been written in terms of T-duality invariant combinations Q^2, P^2 and $Q \cdot P$ although we have derived the formula only for a special class of charge vectors (5.1.1) or more generally (5.3.27). In this section we shall discuss what information about the degeneracy formula can be extracted using the T- and S-duality symmetries of the theory.

We begin by studying the consequences of the T-duality symmetries of the theory. It follows from (3.1.30), (3.1.31) that if a T-duality transformation takes a charge vector $(\vec{Q}, \vec{P})$ to $(\vec{Q}', \vec{P}')$ then

$$Q'^2 = Q^2, \quad P'^2 = P^2, \quad Q' \cdot P' = Q \cdot P \,. \tag{5.5.1}$$

However there may be pairs of charge vectors with the same Q^2, P^2 and $Q \cdot P$ which are not related by a T-duality transformation. Clearly T-duality invariance of the theory cannot give us any relation between the degeneracies associated with such a pair of charge vectors. In what follows we shall focus on charge vectors $(\vec{Q}', \vec{P}')$ which are in the same T-duality orbit of a charge vector $(\vec{Q}, \vec{P})$ for which we have derived (5.1.3).

In §5.4 we have denoted by $\mathcal{R}$ the domain of the region of the moduli space in which the original formula (5.1.3), (5.1.4) for $d(\vec{Q}, \vec{P})$ is valid. It is bounded by a set of marginal stability walls labelled by $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathcal{B}_R$. Let $\mathcal{R}'$ denote the image of $\mathcal{R}$ under some particular T-duality map. In this case we expect $d(\vec{Q}', \vec{P}')$ in the region $\mathcal{R}'$ to be equal to $d(\vec{Q}, \vec{P})$ given in (5.1.3):

$$\begin{aligned} d(\vec{Q}', \vec{P}') &= (-1)^{Q\cdot P+1}\, \frac{1}{N} \int_{\mathcal{C}} d\widetilde{\rho}\, d\widetilde{\sigma}\, d\widetilde{v}\, e^{-\pi i (N\widetilde{\rho} Q^2 + \widetilde{\sigma} P^2/N + 2\widetilde{v} Q\cdot P)}\, \frac{1}{\widetilde{\Phi}(\widetilde{\rho}, \widetilde{\sigma}, \widetilde{v})}\,, \\ &= (-1)^{Q'\cdot P'+1}\, \frac{1}{N} \int_{\mathcal{C}} d\widetilde{\rho}\, d\widetilde{\sigma}\, d\widetilde{v}\, e^{-\pi i (N\widetilde{\rho} Q'^2 + \widetilde{\sigma} P'^2 2/N + 2\widetilde{v} Q'\cdot P')}\, \frac{1}{\widetilde{\Phi}(\widetilde{\rho}, \widetilde{\sigma}, \widetilde{v})}\,, \end{aligned} \tag{5.5.2}$$

where $\mathcal{C}$ has been defined in (5.1.4). In going from the first to the second line of (5.5.2) we have used (5.5.1).

Let us now determine the region $\mathcal{R}'$. Since under a T-duality transformation $M \to \Omega M \Omega^T$, and since $\mathcal{R}'$ is the image of $\mathcal{R}$ under this map, $\mathcal{R}'$ is bounded by walls of marginal stability described in (5.4.9), (5.4.10) with $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathcal{B}_R$ and M_∞ in (5.4.10) replaced by $\Omega^{-1} M_\infty (\Omega^T)^{-1}$. Using (3.1.30) we see that this effectively replaces $(\vec{Q}, \vec{P})$ by $(\vec{Q}', \vec{P}')$ in (5.4.10). Thus $\mathcal{R}'$ is the region of the upper half plane bounded by the circles:

$$\left(a_\infty - \frac{ad+bc}{2cd}\right)^2 + \left(S_\infty + \frac{E'}{2cd}\right)^2 = \frac{1}{4c^2d^2}(1+E'^2)\,, \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathcal{B}_R\,, \tag{5.5.3}$$

where

$$E' \equiv \frac{cd(Q'^T(M_\infty+L)Q') + ab(P'^T(M_\infty+L)P') - (ad+bc)(P'^T(M_\infty+L)Q')}{[(Q'^T(M_\infty+L)Q')(P'^T(M_\infty+L)P') - (P'^T(M_\infty+L)Q')^2]^{1/2}}\,. \tag{5.5.4}$$

In particular $\mathcal{R}'$ has the same vertices a/c and b/d as $\mathcal{R}$. Thus in the universal classification scheme described in §5.4, $\mathcal{R}'$ and $\mathcal{R}$ correspond to the same domains although the precise shape of the domain walls differ for $\mathcal{R}$ and $\mathcal{R}'$ due to the replacement of $(\vec{Q}, \vec{P})$ by the new charge vector $(\vec{Q}', \vec{P}')$. Since eqs.(5.5.2)-(5.5.4) are valid for any charge vector $(\vec{Q}', \vec{P}')$ which can be related to the charge vectors given in (5.3.27) via a T-duality transformation, we conclude that any $(\vec{Q}, \vec{P})$ which is in the T-duality orbit of the charge vectors (5.3.27), the degeneracy $d(\vec{Q}, \vec{P})$ is given by (5.1.3), (5.1.4) in the domain $\mathcal{R}$. In the four dimensional subspace of charge vectors, consisting of momenta, fundamental string winding charge, H-monopole charge and Kaluza-Klein monopole charge along the circles $\widehat{S}^1$ and S^1 in the second description, the T-duality orbit consists of charge vectors of the type given in (5.3.31). Thus for these charge vectors, eqs.(5.1.3), (5.1.4) give us the correct expression for $d(\vec{Q}, \vec{P})$ in the domain $\mathcal{R}$.

Next we shall analyze the consequences of S-duality symmetry. We begin with a charge vector $(\vec{Q}, \vec{P})$ for which (5.1.3), (5.1.4) hold in the domain $\mathcal{R}$, *e.g.* a charge vector in the T-duality orbit of (5.3.30). An S-duality transformation changes the vector $(\vec{Q}, \vec{P})$ to another vector $(\vec{Q}'', \vec{P}'')$ and τ to

τ'' via the formulæ (3.1.35), (3.1.36). Thus if $\mathcal{R}''$ denotes the image of the region $\mathcal{R}$ under the map (3.1.36), then S-duality invariance implies that inside $\mathcal{R}''$ the degeneracy $d(\vec{Q}'', \vec{P}'')$ is given by the same expression (5.1.3) for $d(\vec{Q}, \vec{P})$:

$$d(\vec{Q}'', \vec{P}'') = (-1)^{Q\cdot P+1}\,\frac{1}{N}\int_{\mathcal{C}} d\widetilde{\rho}\, d\widetilde{\sigma}\, d\widetilde{v}\, e^{-\pi i(N\widetilde{\rho}Q^2+\widetilde{\sigma}P^2/N+2\widetilde{v}Q\cdot P)}\,\frac{1}{\widetilde{\Phi}(\widetilde{\rho},\widetilde{\sigma},\widetilde{v})}\,. \tag{5.5.5}$$

We would like to express the right hand side of (5.5.5) in terms of the vectors $\vec{Q}''$ and $\vec{P}''$. For this we define

$$\begin{pmatrix} \tilde\alpha & \tilde\beta \\ \tilde\gamma & \tilde\delta \end{pmatrix} = \begin{pmatrix} \delta & \gamma/N \\ \beta N & \alpha \end{pmatrix} \in \Gamma_1(N)\,, \tag{5.5.6}$$

and

$$\begin{pmatrix} \widetilde{\rho}'' \\ \widetilde{\sigma}'' \\ \widetilde{v}'' \end{pmatrix} \equiv \begin{pmatrix} \widetilde{\rho}_1'' + i\widetilde{\rho}_2'' \\ \widetilde{\sigma}_1'' + i\widetilde{\sigma}_2'' \\ \widetilde{v}_1'' + i\widetilde{v}_2'' \end{pmatrix} = \begin{pmatrix} \tilde\alpha^2 & \tilde\beta^2 & -2\tilde\alpha\tilde\beta \\ \tilde\gamma^2 & \tilde\delta^2 & -2\tilde\gamma\tilde\delta \\ -\tilde\alpha\tilde\gamma & -\tilde\beta\tilde\delta & (\tilde\alpha\tilde\delta+\tilde\beta\tilde\gamma) \end{pmatrix} \begin{pmatrix} \widetilde{\rho} \\ \widetilde{\sigma} \\ \widetilde{v} \end{pmatrix}. \tag{5.5.7}$$

Using (3.1.35), (3.1.36), (5.5.6), (5.5.7) one can easily verify that

$$e^{-\pi i(N\widetilde{\rho}Q^2+\widetilde{\sigma}P^2/N+2\widetilde{v}Q\cdot P)} = e^{-\pi i(N\widetilde{\rho}''Q''^2+\widetilde{\sigma}''P''^2/N+2\widetilde{v}''Q''\cdot P'')}\,, \tag{5.5.8}$$

and

$$d\widetilde{\rho}\, d\widetilde{\sigma}\, d\widetilde{v} = d\widetilde{\rho}''\, d\widetilde{\sigma}''\, d\widetilde{v}''\,. \tag{5.5.9}$$

Furthermore, with the help of eq.(C.22) one can show that [8]

$$\widetilde{\Phi}(\widetilde{\rho}'',\widetilde{\sigma}'',\widetilde{v}'') = \widetilde{\Phi}(\widetilde{\rho},\widetilde{\sigma},\widetilde{v})\,. \tag{5.5.10}$$

If $\mathcal{C}''$ denotes the image of $\mathcal{C}$ under the map (5.5.7) then eqs.(5.5.8)-(5.5.10) allow us to express (5.5.5) as[36]

$$d(\vec{Q}'', \vec{P}'') = (-1)^{Q''\cdot P''+1}\,\frac{1}{N}\int_{\mathcal{C}''} d\widetilde{\rho}''\, d\widetilde{\sigma}''\, d\widetilde{v}''\, e^{-\pi i(N\widetilde{\rho}''Q''^2+\widetilde{\sigma}''P''^2/N+2\widetilde{v}Q''\cdot P'')}\,\frac{1}{\widetilde{\Phi}(\widetilde{\rho}'',\widetilde{\sigma}'',\widetilde{v}'')}\,. \tag{5.5.11}$$

To find the location of $\mathcal{C}''$ we note that under the map (5.5.7) the real parts of $\widetilde{\rho}$, $\widetilde{\sigma}$ and $\widetilde{v}$ mix among themselves and the imaginary parts of $\widetilde{\rho}$, $\widetilde{\sigma}$ and $\widetilde{v}$ mix among themselves. The initial contour $\mathcal{C}$ corresponded to a unit cell of the cubic lattice in the ($\widetilde{\rho}_1$,$\widetilde{\sigma}_1$,$\widetilde{v}_1$) space spanned by the basis vectors $(1,0,0)$, $(0,N,0)$ and $(0,0,1)$. The unimodular map (5.5.7) transforms this into a different unit cell of the same lattice. We can now use the shift symmetries

$$\widetilde{\Phi}(\widetilde{\rho},\widetilde{\sigma},\widetilde{v}) = \widetilde{\Phi}(\widetilde{\rho}+1,\widetilde{\sigma},\widetilde{v}) = \widetilde{\Phi}(\widetilde{\rho},\widetilde{\sigma}+N,\widetilde{v}) = \widetilde{\Phi}(\widetilde{\rho},\widetilde{\sigma},\widetilde{v}+1)\,, \tag{5.5.12}$$

[36]In arriving at (5.5.11) we have used that $(-1)^{Q\cdot P} = (-1)^{Q''.P''}$. This follows from the S-duality transformation laws of the charges and the observation that NQ^2 and P^2 are even integers.

which are manifest from (5.1.5), to bring the integration region back to the original unit cell. Thus $\mathcal{C}''$ and $\mathcal{C}$ differ only in the values of the imaginary parts of $\widetilde{\rho}$, $\widetilde{\sigma}$ and $\widetilde{v}$. Using (5.1.4), (5.5.7) we see that for the contour $\mathcal{C}''$,

$$\begin{aligned}
\widetilde{\rho}_2'' &= \tilde{\alpha}^2\, M_1 + \tilde{\beta}^2\, M_2 + 2\tilde{\alpha}\tilde{\beta}\, M_3\,, \\
\widetilde{\sigma}_2'' &= \tilde{\gamma}^2\, M_1 + \tilde{\delta}^2\, M_2 + 2\tilde{\gamma}\tilde{\delta}\, M_3\,, \\
v_2'' &= -\tilde{\alpha}\tilde{\gamma}\, M_1 - \tilde{\beta}\tilde{\delta}\, M_2 - (\tilde{\alpha}\tilde{\delta} + \tilde{\beta}\tilde{\gamma})\, M_3\,.
\end{aligned} \tag{5.5.13}$$

Thus $\mathcal{C}''$ is not identical to $\mathcal{C}$. We could try to deform $\mathcal{C}''$ back to $\mathcal{C}$, but in that process we might pick up contribution from the residues at the poles of $1/\widetilde{\Phi}(\widetilde{\rho}'', \widetilde{\sigma}'', \widetilde{v}'')$. Thus we see that the degeneracy formula (5.5.11) for $d(\vec{Q}'', \vec{P}'')$ is not obtained by simply replacing $(\vec{Q}, \vec{P})$ by $(\vec{Q}'', \vec{P}'')$ in the expression for $d(\vec{Q}, \vec{P})$. The integration contour $\mathcal{C}$ also gets deformed to a new contour $\mathcal{C}''$.

Let us now analyze the region $\mathcal{R}''$ of the asymptotic moduli space in which (5.5.11) is valid. This is obtained by taking the image of the region $\mathcal{R}$ under the transformation (3.1.36). To determine this region we need to first study the images of the curves described in (5.4.9) in the $a_\infty - S_\infty$ plane. A straightforward analysis shows that the image of (5.4.9) is described by the curve

$$\left(a_\infty - \frac{a''d'' + b''c''}{2c''d''}\right)^2 + \left(S_\infty + \frac{E''}{2c''d''}\right)^2 = \frac{1}{4c''^2 d''^2}(1 + E''^2)\,, \tag{5.5.14}$$

where

$$\begin{pmatrix} a'' & b'' \\ c'' & d'' \end{pmatrix} = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix}\,, \tag{5.5.15}$$

and

$$E'' \equiv \frac{c''d''(Q''^T(M_\infty + L)Q'') + a''b''(P''^T(M_\infty + L)P'') - (a''d'' + b''c'')(P''^T(M_\infty + L)Q'')}{[(Q''^T(M_\infty + L)Q'')(P''^T(M_\infty + L)P'') - (P''^T(M_\infty + L)Q'')^2]^{1/2}}\,. \tag{5.5.16}$$

This is identical in form to (5.4.9) with $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ replaced by $\begin{pmatrix} a'' & b'' \\ c'' & d'' \end{pmatrix}$ and $(\vec{Q}, \vec{P})$ replaced by $(\vec{Q}'', \vec{P}'')$. Since the original domain $\mathcal{R}$ was bounded by a set of marginal stability walls $\left\{\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right\} \in \mathcal{B}_R$, the domain $\mathcal{R}''$ is bounded by the collection of walls described by (5.5.14), (5.5.16) with $\begin{pmatrix} a'' & b'' \\ c'' & d'' \end{pmatrix} \in \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \mathcal{B}_R$. Since these walls end on vertices $a''/c'' \neq a/c$ and $b''/d'' \neq b/d$, under the universal classification scheme $\mathcal{R}''$ and $\mathcal{R}$ describe different domains. Thus the result of this analysis may be summarized in the statement that $d(\vec{Q}'', \vec{P}'')$ inside the domain $\mathcal{R}''$ is given by the same integral formula (5.1.3) with $(\vec{Q}, \vec{P})$ replaced by $(\vec{Q}'', \vec{P}'')$ and the contour $\mathcal{C}$ replaced by the new contour $\mathcal{C}''$ given in (5.5.13).

Now, as has been argued at the end of §5.3, an S-duality transformation acting on a charge vector in the T-duality orbit of (5.3.30) gives us back another charge vector in the T-duality orbit of (5.3.30). Thus $(\vec{Q}'', \vec{P}'')$ is in the T-duality orbit of (5.3.30), and $d(\vec{Q}'', \vec{P}'')$ inside the domain $\mathcal{R}$ would have been given by eqs.(5.1.3), (5.1.4) with $(\vec{Q}, \vec{P})$ replaced by $(\vec{Q}'', \vec{P}'')$. Thus we now have expressions for $d(\vec{Q}'', \vec{P}'')$ in two different domains, – $\mathcal{R}$ and $\mathcal{R}''$. In both domains the degeneracy is given by an integral. The integrand in both cases are same, but in one case the integration contour is $\mathcal{C}$ while in the other case it is $\mathcal{C}''$. This shows that as we cross the walls of marginal stability to move from the domain $\mathcal{R}$ to $\mathcal{R}''$ the expression for $d(\vec{Q}'', \vec{P}'')$ changes by a modification in the location of the contour of integration. The precise modification of the contour for a given wall crossing will be given by (5.5.13). Following this line of argument one finds that for charge vectors in the T-duality orbit of (5.3.30) the degeneracy formula in different domains bounded by walls of marginal stability are given by the three dimensional contour integral of the same integrand as in (5.1.3) but with different choices of the integration contour.

The jump in the degeneracy across a marginal stability wall can be calculated in terms of residues of the integrand at the poles we encounter as we deform the relevant contour on one side of the wall to the relevant contour on the other side of the wall. One finds that if the wall corresponds to the decay of the original dyon into a pair of half BPS dyons with charges $(\vec{Q}_1, \vec{P}_1)$ and $(\vec{Q}_2, \vec{P}_2)$, and if it is related to the wall corresponding to the decay into $(\vec{Q}, 0) + (0, \vec{P})$ by an S-duality transformation, then up to a sign that depends on the direction in which we cross the wall, the jump is given by [168]

$$(-1)^{\vec{Q}_1\cdot\vec{P}_2 - \vec{Q}_2\cdot\vec{P}_1+1} (\vec{Q}_1 \cdot \vec{P}_2 - \vec{Q}_2 \cdot \vec{P}_1)\, d_{half}(\vec{Q}_1, \vec{P}_1)\, d_{half}(\vec{Q}_2, \vec{P}_2)\,, \tag{5.5.17}$$

where $d_{half}(\vec{Q}_i, \vec{P}_i)$ denotes the number of bosonic minus fermionic half BPS supermultiplets carrying charges $(\vec{Q}_i, \vec{P}_i)$. For $(\vec{Q}_1, \vec{P}_1) = (\vec{Q}, 0)$ and $(\vec{Q}_2, \vec{P}_2) = (0, \vec{P})$ this result will be proved in eq.(5.7.5). For the other cases the formula can be obtained by duality transformation of this result as long as the corresponding marginal stability wall is related to the wall associated with the decay into $(\vec{Q}, 0)$ and $(0, \vec{P})$ by an S-duality transformation. For $N = 1, 2, 3$, i.e. for heterotic string theory on T^6 and asymmetric $\mathbb{Z}_2$ and $\mathbb{Z}_3$ orbifolds of heterotic or type II string theory on T^6, this includes all the walls [168]. The result (5.5.17) is identical to the wall crossing formula proposed in [145] for $\mathcal{N} = 2$ supersymmetric string theories and will be relevant for the analysis in §5.7.

So far we have used S-duality invariance to determine the locations of the integration contour in the degeneracy formula in different domains in the moduli space, but have not carried out any test of S-duality. We shall now describe some tests of S-duality that one could perform.

1. If there is an S-duality transformation that leaves the set $\mathcal{B}_R$ invariant, then under such a

transformation the contour $\mathcal{C}$ either should not transform, or should transform to another contour that is continuously deformable to $\mathcal{C}$ without passing through any poles.

2. Analysis of §5.2 has shown that inside the left domain $\mathcal{L}$ corresponding to the set of matrices $\mathcal{B}_L$, the degeneracy is obtained by performing integration over the contour $\widehat{\mathcal{C}}$ described in (5.1.13). Thus if there is an S-duality transformation that maps the set $\mathcal{B}_R$ to the set $\mathcal{B}_L$ then such a transformation must map the contour $\mathcal{C}$ to the contour $\widehat{\mathcal{C}}$ or another contour deformable to $\widehat{\mathcal{C}}$ without passing through any pole.

In fact for all values of N one can identify a pair of S-duality transformations which map $\mathcal{B}_R$ to $\mathcal{B}_L$ [168]. They are given by

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} \equiv g_1, \qquad \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -N & 1 \end{pmatrix} \equiv g_2 \,. \tag{5.5.18}$$

This in turn implies that $g_1 g_2^{-1}$ maps the set $\mathcal{B}_R$ to itself. One can show that under both transformations given in (5.5.18) the contour $\mathcal{C}$ gets mapped to another contour that is deformable to $\widehat{\mathcal{C}}$ [168]. This in turn implies that $g_1 g_2^{-1}$ takes the contour $\mathcal{C}$ to another contour deformable to $\mathcal{C}$. This provides non-trivial test of S-duality symmetry of the degeneracy formula. For $N \leq 6$ all transformations which map $\mathcal{B}_R$ to $\mathcal{B}_R$ may be obtained by taking positive and negative powers of $g_1 g_2^{-1}$, and those which takes $\mathcal{B}_R$ to $\mathcal{B}_L$ are obtained by taking positive and negative powers of $g_1 g_2^{-1}$ followed by a single power of g_1 (or g_2) [168]. Thus for these cases our test of S-duality is complete.

5.6 The statistical entropy function

Although (5.1.3) gives an exact formula for the degeneracy of dyons, it is hard to compare this directly with the results for the black hole entropy derived earlier in §3.1. In this section we shall describe a systematic procedure for extracting the behaviour of $d(\vec{Q}, \vec{P})$ for large charges:

$$Q^2 >> 0, \qquad P^2 >> 0, \qquad Q^2 P^2 - (Q \cdot P)^2 >> 0, \tag{5.6.1}$$

and also explicitly compute the first order corrections to the leading asymptotic formula. Our analysis will follow the one given in [6]; this in turn is based on the earlier analysis of [9, 18, 10, 173].

Our starting point will be the general expression for $d(\vec{Q}, \vec{P})$

$$d(\vec{Q}, \vec{P}) = (-1)^{Q \cdot P + 1} \frac{1}{N} \int d\widetilde{\rho}\, d\widetilde{\sigma}\, d\widetilde{v}\, e^{-\pi i (N \widetilde{\rho} Q^2 + \widetilde{\sigma} P^2 / N + 2 \widetilde{v} Q \cdot P)} \, \frac{1}{\widetilde{\widetilde{\Phi}}(\widetilde{\rho}, \widetilde{\sigma}, \widetilde{v})} \,, \tag{5.6.2}$$

with the integration contour chosen according to the domain in which we want to compute the degeneracy. As we shall discuss at the end of this section, the change in the degeneracy across a wall of marginal stability is exponentially suppressed compared to the leading term in the limit of large charges. Thus the asymptotic expansion of the statistical entropy in inverse powers of charges will be independent of the domain in which we calculate the entropy. For definiteness we shall choose our asymptotic moduli to lie inside the domain $\mathcal{R}$ introduced at the end of §5.4. In this case the contour of integration in (5.6.2) will be taken along fixed but large positive values M_1 and M_2 of $\widetilde{\rho}_2$ and $\widetilde{\sigma}_2$, and fixed but large negative value $-M_3$ of $\widetilde{v}_2$, with $|M_3| << M_1, M_2$. At a typical point on the contour the exponent in the integrand is quadratic in Q and P with large coefficients governed by the M_i's. However the phase of the integrand oscillates rapidly and so we cannot estimate the integral from the magnitude of the integrand. We remedy this problem by deforming the integration contour so as to make the factor in the exponent as small as possible. In particular if we bring the integration contour to a new position $\widetilde{\mathcal{C}}$, defined by

$$\widetilde{\rho}_2 = \eta_1, \quad \widetilde{\sigma}_2 = \eta_2, \quad \widetilde{v}_2 = \eta_3 \, , \tag{5.6.3}$$

where η_1, η_2 are small but fixed positive numbers and η_3 is a small positive or negative number, then the exponential factor in the new integrand is quadratic in the charges with small coefficients. Since the expected entropy should grow quadratically with charges with finite coefficients, we conclude that the integral over the contour $\widetilde{\mathcal{C}}$ is exponentially suppressed compared to the leading contribution. Thus the dominant contribution must come from the residue at the poles through which the contour passes as we deform it from $\mathcal{C}$ to $\widetilde{\mathcal{C}}$. For this reason we shall tentatively neglect the contribution from $\widetilde{\mathcal{C}}$ and focus on the contribution from the poles, – as long as this contribution to the statistical entropy grows quadratically with the charges with finite coefficients, our ansatz is self-consistent.

Let $\mathcal{D}$ denote the four dimensional region bounded by $\mathcal{C}$ and $\widetilde{\mathcal{C}}$ along which we deform the contour. For evaluating the contribution to the integral from the residues at the poles we need to locate the poles of the integrand in (5.6.2) inside the region $\mathcal{D}$. These poles in turn must come from the zeroes of $\widetilde{\Phi}(\widetilde{\rho}, \widetilde{\sigma}, \widetilde{v})$. According to the analysis given in appendix D (eq.(D.19)), $\widetilde{\Phi}(\widetilde{\rho}, \widetilde{\sigma}, \widetilde{v})$ has second order zeroes at

$$\begin{aligned} & n_2(\widetilde{\sigma}\widetilde{\rho} - \widetilde{v}^2) + j\widetilde{v} + n_1\widetilde{\sigma} - m_1\widetilde{\rho} + m_2 = 0, \\ & \text{for} \quad m_1 \in N\,\mathbb{Z},\, n_1 \in \mathbb{Z},\, j \in 2\,\mathbb{Z}+1,\, m_2, n_2 \in \mathbb{Z}, \quad m_1 n_1 + m_2 n_2 + \frac{j^2}{4} = \frac{1}{4}\,. \end{aligned} \tag{5.6.4}$$

For fixed integers m_1, m_2, n_1, n_2 and j this describes a surface of real codimension two, i.e. of real dimension four. Typically this will intersect $\mathcal{D}$ in a two dimensional subspace $\mathcal{B}$ bounded by the intersection of $\widetilde{\mathcal{C}} - \mathcal{C}$ with (5.6.4), and the contribution to the integral from the pole at (5.6.4) is obtained by integrating the residue over $\mathcal{B}$. In fact since we expect the integrand to be non-singular over the original contour $\mathcal{C}$, $\mathcal{C}$ should not have any intersection with (5.6.4) representing locations of the poles of the integrand. Thus the boundary of the two dimensional subspace $\mathcal{B}$ is a one dimensional curve $\partial\mathcal{B}$ given by the intersection of $\widetilde{\mathcal{C}}$ with (5.6.4).

Since $\mathcal{B}$ is an open two dimensional space we cannot again use the residue theorem to carry out the integral over the region $\mathcal{B}$. Neither can we estimate the integral by examining the maximum value of the integrand inside $\mathcal{B}$ since typically the integrand will still have rapid oscillations inside $\mathcal{B}$ and there are large cancellations. We remedy this by using the saddle point method. Keeping the boundary $\partial\mathcal{B}$ fixed we deform the integration region $\mathcal{B}$ to a new region $\mathcal{B}'$ inside the subspace (5.6.4) such that the maximum value of the integrand inside $\mathcal{B}'$ takes the minimum possible value.[37] The location of this maximum is a saddle point, – the integrand decreases as we move away from the saddle point along $\mathcal{B}'$ and increases as we move away from the saddle point in directions transverse to $\mathcal{B}'$. We can now evaluate the leading contribution to the integral as well as power law corrections to it by systematically expanding the integrand around the saddle point and carrying out the integration.

In order to carry out this procedure we shall proceed in three steps.

1. First we shall determine which of the zeroes of $\widetilde{\Phi}$ described in (5.6.4) would give dominant contribution to the integrand at the saddle point.

2. Then we shall verify that this particular zero has a non-trivial intersection $\mathcal{B}$ with the region $\mathcal{D}$, and that this two dimensional surface $\mathcal{B}$ can be deformed to another surface $\mathcal{B}'$ passing through the saddle point so that the integrand on $\mathcal{B}'$ has maximum magnitude at the saddle point.

3. Finally we shall evaluate the contribution from the integral using the saddle point method.

We begin with the first step. Let us define

$$A = n_2, \quad B = (n_1, -m_1, \frac{1}{2}j), \quad y = (\widetilde{\rho}, \widetilde{\sigma}, -\widetilde{v}), \qquad C = m_2\,, \quad q = (P^2/N, NQ^2, Q\cdot P)\,, \tag{5.6.5}$$

and denote by $\cdot$ the $SO(2,1)$ invariant inner product

$$(x^1, x^2, x^3)\cdot(y^1, y^2, y^3) = x^1 y^2 + x^2 y^1 - 2x^3 y^3\,. \tag{5.6.6}$$

Then we have

$$y^2 \equiv y\cdot y = 2(\widetilde{\rho}\widetilde{\sigma} - \widetilde{v}^2), \qquad B\cdot y = j\widetilde{v} + n_1\widetilde{\sigma} - m_1\widetilde{\rho}\,, \tag{5.6.7}$$

[37] In terms of the original construction such deformations of $\mathcal{B}$ amounts to deforming the region $\mathcal{D}$ bounded by $\mathcal{C}$ and $\widetilde{\mathcal{C}}$ so that it contains the saddle point of the integrand.

and the first equation of (5.6.4) may be rewritten as

$$\frac{1}{2}Ay^2 + B \cdot y + C = 0 \,. \tag{5.6.8}$$

Picking up residue at the pole forces us to evaluate the exponent in (5.6.2)

$$-\, i\pi \left(\widetilde{\rho} N Q^2 + \widetilde{\sigma} P^2/N + 2\widetilde{v} Q \cdot P\right) = -i\,\pi\, q \cdot y \,, \tag{5.6.9}$$

at (5.6.8). For a given zero of $\widetilde{\Phi}$ labelled by $(\vec{m}, \vec{n}, j)$ the location of the saddle point to leading approximation is now determined by extremizing (5.6.9) with respect to y subject to the condition (5.6.8). This gives, for $n_2 \neq 0$,

$$q + \lambda(Ay + B) = 0 \,, \tag{5.6.10}$$

where λ is a lagrange multiplier. (5.6.8) and (5.6.10) now give:

$$\lambda = \pm\sqrt{\frac{q^2}{B^2 - 2AC}}, \qquad y = -\frac{1}{A}\left(\frac{q}{\lambda} + B\right) \,. \tag{5.6.11}$$

Since

$$B^2 - 2AC = -2(m_1 n_1 + m_2 n_2 + \frac{j^2}{4}) = -\frac{1}{2} \tag{5.6.12}$$

due to the last equation in (5.6.4), we get

$$\lambda = \pm\, i\, \sqrt{2q^2} \tag{5.6.13}$$

for $q^2 \equiv 2(Q^2 P^2 - (Q \cdot P)^2) > 0$. The correct sign in (5.6.13) is determined as follows. First of all, note that (5.6.4) describes the same surface if we change the signs of m_i, n_i and j. Using this freedom we can choose n_2, i.e. A to be positive. Since Q^2 and P^2 are positive, (5.6.5), (5.6.11) shows that in order for $\widetilde{\rho}_2$ and $\widetilde{\sigma}_2$ to be positive we must have $Im\ \lambda > 0$. Thus we have

$$\lambda = i\,\sqrt{2q^2}\,, \tag{5.6.14}$$

and at the saddle point the exponential $e^{-i\pi q \cdot y}$ takes the form:

$$E \equiv e^{-i\pi\, q \cdot y} = e^{i\pi(q^2/\lambda + q \cdot B)/A} = e^{\left(\pi\sqrt{q^2/2} + i\pi q \cdot B\right)/A} \,. \tag{5.6.15}$$

Since $q \cdot B/A$ is a rational number, the second term only gives a phase. Hence

$$|E| = e^{\frac{\pi}{A}\sqrt{q^2/2}} = e^{\pi\sqrt{Q^2 P^2 - (Q \cdot P)^2}/n_2} \,. \tag{5.6.16}$$

Eq.(5.6.16) shows that the leading contribution to the integral comes from the saddle point corresponding to $n_2 = 1$. In this case a $\widetilde{\rho} \to \widetilde{\rho} + 1$ transformation in (5.6.4), which is a symmetry of $\widetilde{\Phi}(\widetilde{\rho}, \widetilde{\sigma}, \widetilde{v})$ due to (5.1.9), induces $n_1 \to n_1 + 1$, $m_2 \to m_2 - m_1$. Since $n_1 \in \mathbb{Z}$, we can use this symmetry to bring the saddle point to $n_1 = 0$. On the other hand a $\widetilde{\sigma} \to \widetilde{\sigma} + N$ transformation in eq.(5.6.4) induces $m_1 \to m_1 - N$, $m_2 \to m_2 + n_1 N$. Since $m_1 \in N\,\mathbb{Z}$, we can use this transformation to bring m_1 to 0. Finally the $\widetilde{v} \to \widetilde{v} + 1$ transformation in (5.6.4) induces $j \to j - 2$, $m_2 \to m_2 + j - 1$. Since $j \in 2\,\mathbb{Z} + 1$, we can use this transformation to set $j = 1$. m_2 is now determined to be zero from the last equation in (5.6.4). Thus we have

$$m_1 = m_2 = n_1 = 0, \quad n_2 = 1, \quad j = 1\,. \tag{5.6.17}$$

The corresponding zero of $\widetilde{\Phi}$ is at

$$\widetilde{\rho}\widetilde{\sigma} - \widetilde{v}^2 + \widetilde{v} = 0\,. \tag{5.6.18}$$

Also eqs.(5.6.5), (5.6.11) give the location of the saddle point to be at

$$\widetilde{\rho} = i\frac{P^2}{2N\sqrt{Q^2P^2 - (Q\cdot P)^2}}\,, \qquad \widetilde{\sigma} = i\frac{NQ^2}{2\sqrt{Q^2P^2 - (Q\cdot P)^2}}, \qquad \widetilde{v} = \frac{1}{2} - i\frac{Q\cdot P}{2\sqrt{Q^2P^2 - (Q\cdot P)^2}}\,. \tag{5.6.19}$$

Since we have used the freedom of $\widetilde{\rho}$, $\widetilde{\sigma}$ and $\widetilde{v}$ translations to pick this particular pole, we can no longer choose the range of $\widetilde{\rho}_1$, $\widetilde{\sigma}_1$ and $\widetilde{v}_1$ to be (0,1), $(0, N)$ and $(0, 1)$ respectively. Instead we should allow them to run all the way from $-\infty$ to ∞ and intersection with (5.6.18) will pick up the appropriate subspace over which the integration needs to be performed.

Let us now verify that the region $\mathcal{D}$ has a non-trivial intersection with (5.6.18). This will be done by showing that $\widetilde{\mathcal{C}}$ and hence $\partial\mathcal{D}$ has a non-trivial intersection with (5.6.18). The equation determining the intersection of (5.6.3) with (5.6.18) is given by[38]

$$\begin{aligned}
&\widetilde{\rho}_2 = \eta_1, \quad \widetilde{\sigma}_2 = \eta_2, \quad \widetilde{v}_2 = \eta_3, \\
&\widetilde{\rho}_1 = -\frac{\eta_1\widetilde{\sigma}_1 - (2\widetilde{v}_1 - 1)\eta_3}{\eta_2}\,, \quad \frac{\eta_1}{\eta_2}(\widetilde{\sigma}_1)^2 + \left(\widetilde{v}_1 - \frac{1}{2}\right)^2 - 2\,\frac{\eta_3}{\eta_2}\,\widetilde{\sigma}_1\left(\widetilde{v}_1 - \frac{1}{2}\right) = \frac{1}{4} - (\eta_1\eta_2 - \eta_3^2)\,.
\end{aligned} \tag{5.6.20}$$

As long as $0 < 4(\eta_1\eta_2 - \eta_3^2) < 1$, this describes an ellipse in the $\widetilde{\sigma}_1$-$\widetilde{v}_1$ plane. Thus the two dimensional surface $\mathcal{B}$ will be a surface inside the 4-dimensional subspace (5.6.18), bounded by the curve (5.6.20).

Finally we need to show that there exists a surface $\mathcal{B}'$ inside (5.6.18) bounded by the same curve (5.6.20) such that it passes through the saddle point (5.6.19) and the integrand has a global maximum

[38] I wish to thank Justin David for pointing out an error in eq.(5.6.20) in an earlier version of the manuscript.

at the saddle point. In that case we can deform the surface $\mathcal{B}$ to $\mathcal{B}'$ and compute the integral over $\mathcal{B}'$ by expanding the integrand around the saddle point. We can explicitly construct such a surface $\mathcal{B}'$ by considering the family of curves $\mathcal{C}(\lambda)$ defined as

$$\begin{aligned}
&\widetilde{\rho}_2 = \eta_1(\lambda), \quad \widetilde{\sigma}_2 = \eta_2(\lambda), \quad \widetilde{v}_2 = \eta_3(\lambda), \qquad \widetilde{\rho}_1 = -\frac{\eta_1(\lambda)\widetilde{\sigma}_1 - (2\widetilde{v}_1 - 1)\eta_3(\lambda)}{\eta_2(\lambda)}\,, \\
&\frac{\eta_1(\lambda)}{\eta_2(\lambda)}(\widetilde{\sigma}_1)^2 + \left(\widetilde{v}_1 - \frac{1}{2}\right)^2 - 2\,\frac{\eta_3(\lambda)}{\eta_2(\lambda)}\,\widetilde{\sigma}_1\,\left(\widetilde{v}_1 - \frac{1}{2}\right) = \frac{1}{4} - \left(\eta_1(\lambda)\eta_2(\lambda) - \eta_3(\lambda)^2\right),
\end{aligned} \tag{5.6.21}$$

where

$$\begin{aligned}
\eta_1(\lambda) &= \lambda\frac{P^2}{2N\sqrt{Q^2P^2 - (Q\cdot P)^2}}\,, \qquad \eta_2(\lambda) = \lambda\frac{NQ^2}{2\sqrt{Q^2P^2 - (Q\cdot P)^2}}, \\
\eta_3(\lambda) &= -\lambda\frac{Q\cdot P}{2\sqrt{Q^2P^2 - (Q\cdot P)^2}}\,.
\end{aligned} \tag{5.6.22}$$

Since (5.6.21) is the same as eq.(5.6.20) with η_i replaced by $\eta_i(\lambda)$, the curve (5.6.21) lies on the surface (5.6.18). Identifying $\eta_i(\epsilon)$ with η_i for some small positive number ϵ, we see that at $\lambda = \epsilon$ the curve coincides with $\widetilde{C}$. On the other hand at $\lambda = 1$ the curve (5.6.21) shrinks to the saddle point (5.6.19). Finally, the magnitude of the exponential factor in the integrand, – which depends only on the imaginary parts of $\widetilde{\rho}$, $\widetilde{\sigma}$ and $\widetilde{v}$, – takes value $\exp(\pi\lambda\sqrt{Q^2P^2 - (Q\cdot P)^2})$ on the curve (5.6.21). Thus in the range $\epsilon \le \lambda \le 1$ it reaches a maximum at $\lambda = 1$, i.e. at the saddle point (5.6.19). From this we see that the surface foliated by the family of curves $\mathcal{C}(\lambda)$ for $\epsilon \le \lambda \le 1$ has all the right properties to be identified as the surface $\mathcal{B}'$.

We now turn to the evaluation of the contribution to the integral from the residue at (5.6.18). For this we introduce a new set of variables (ρ, σ, v) related to $(\widetilde{\rho}, \widetilde{\sigma}, \widetilde{v})$ via the relations

$$\widetilde{\rho} = \frac{1}{N}\,\frac{1}{2v - \rho - \sigma}, \qquad \widetilde{\sigma} = N\,\frac{v^2 - \rho\sigma}{2v - \rho - \sigma}, \qquad \widetilde{v} = \frac{v - \rho}{2v - \rho - \sigma}\,, \tag{5.6.23}$$

or equivalently,

$$\rho = \frac{\widetilde{\rho}\widetilde{\sigma} - \widetilde{v}^2}{N\widetilde{\rho}}, \qquad \sigma = \frac{\widetilde{\rho}\widetilde{\sigma} - (\widetilde{v} - 1)^2}{N\widetilde{\rho}}, \qquad v = \frac{\widetilde{\rho}\widetilde{\sigma} - \widetilde{v}^2 + \widetilde{v}}{N\widetilde{\rho}}\,. \tag{5.6.24}$$

Under this map (5.6.18) takes the form

$$v = 0\,. \tag{5.6.25}$$

Now it has been shown in eq.(C.21) that

$$\widetilde{\Phi}(\widetilde{\rho}, \widetilde{\sigma}, \widetilde{v}) = -(i)^k\, C_1\, (2v - \rho - \sigma)^k\, \widehat{\Phi}(\rho, \sigma, v) \tag{5.6.26}$$

where C_1 is a real positive constant and $\widehat\Phi(\rho,\sigma,v)$ is a new function defined in (C.19):

$$\begin{aligned}
\widehat\Phi(\rho,\sigma,v) &= e^{2\pi i(\widehat\alpha\rho+\widehat\gamma\sigma+v)} \\
&\quad \prod_{b=0}^{1}\prod_{r=0}^{N-1}\prod_{\substack{(k',l)\in\mathbb{Z},j\in 2\mathbb{Z}+b\\ k',l\ge 0,j<0 \text{ for } k'=l=0}} \left\{1-e^{2\pi ir/N}\, e^{2\pi i(k'\sigma+l\rho+jv)}\right\}^{\sum_{s=0}^{N-1} e^{-2\pi isr/N} c_b^{(0,s)}(4k'l-j^2)} \\
&\quad \widehat\alpha = \widehat\gamma = \frac{1}{24}\,\chi(\mathcal{M})\,.
\end{aligned} \tag{5.6.27}$$

Using (5.6.26) and the identity

$$d\widetilde\rho\wedge d\widetilde\sigma\wedge d\widetilde v = -(2v-\rho-\sigma)^{-3} d\rho\wedge d\sigma\wedge dv\,, \tag{5.6.28}$$

we can express the contribution to (5.6.2) from the residue at the pole at $v=0$ as

$$\begin{aligned}
d(\vec Q,\vec P) &\simeq (-1)^{Q\cdot P+1}\frac{(i)^{-k}}{N\,C_1}\int_{\mathcal{C}'} d\rho\wedge d\sigma\wedge dv\,(2v-\rho-\sigma)^{-k-3}\,\frac{1}{\widehat\Phi(\rho,\sigma,v)} \\
&\quad \exp\left[-i\pi\left\{\frac{v^2-\rho\sigma}{2v-\rho-\sigma}P^2+\frac{1}{2v-\rho-\sigma}Q^2+\frac{2(v-\rho)}{2v-\rho-\sigma}Q\cdot P\right\}\right]
\end{aligned} \tag{5.6.29}$$

where $\mathcal{C}'$ denotes a contour around $v=0$. Note that we have used the wedge product notation to keep track of the orientation of the integration region and have implicitly used the convention that the original integral over $\widetilde\rho_1$, $\widetilde\sigma_1$ and $\widetilde v_1$ corresponds to the measure $d\widetilde\rho\wedge d\widetilde\sigma\wedge d\widetilde v$.

We shall evaluate this integral by first performing the v integral using Cauchy's formula and then carrying out the ρ and σ integrals by saddle point approximation. According to (C.25) near $v=0$

$$\widehat\Phi(\rho,\sigma,v) = -4\pi^2\, v^2\, g(\rho)\, g(\sigma) + \mathcal{O}(v^4)\,, \tag{5.6.30}$$

where the function $g(\rho)$ has been introduced in (3.1.41). Thus the contribution to (5.6.29) from the pole at $v=0$ is given by

$$\begin{aligned}
d(\vec Q,\vec P) &\simeq -(-1)^{Q\cdot P+1}\frac{(i)^{-k}}{4\pi^2 NC_1}\int_{\mathcal{C}'} d\rho\wedge d\sigma\wedge dv\, v^{-2}\,(2v-\rho-\sigma)^{-k-3}\,(g(\rho)g(\sigma))^{-1} \\
&\quad \exp\left[-i\pi\left\{\frac{v^2-\rho\sigma}{2v-\rho-\sigma}P^2+\frac{1}{2v-\rho-\sigma}Q^2+\frac{2(v-\rho)}{2v-\rho-\sigma}Q\cdot P\right\}\right]\,.
\end{aligned} \tag{5.6.31}$$

Evaluating the v integral in (5.6.31) by Cauchy's formula, we get

$$d(\vec Q,\vec P) \simeq \frac{(i)^{-k+1}\gamma}{2\pi NC_1}\,(-1)^k\int\frac{d\rho\wedge d\sigma}{(\rho+\sigma)^2}$$

$$
\begin{aligned}
&\left[-2(k+3) + 2\pi i \left\{ \frac{\rho\sigma}{\rho+\sigma} P^2 - \frac{1}{\rho+\sigma} Q^2 + \frac{\rho-\sigma}{\rho+\sigma} Q\cdot P \right\}\right] \\
&\exp\left[- i\pi \left\{ \frac{\rho\sigma}{\rho+\sigma} P^2 - \frac{1}{\rho+\sigma} Q^2 + \frac{\rho-\sigma}{\rho+\sigma} Q\cdot P \right\} \right. \\
&\left. - \ln g(\rho) - \ln g(\sigma) - (k+2)\ln(\rho+\sigma) \right], \qquad (5.6.32)
\end{aligned}
$$

where $\gamma = 1$ or -1 depending on whether the $\widetilde{v}$ (or v) contour encloses the pole anti-clockwise or clockwise, or equivalently whether the pole from which the dominant contribution comes was above or below the $\widetilde{v}$ integration contour for the original contour $\mathcal{C}$. It may be determined as follows. First note that the zeroes of $\widetilde{\Phi}$ at $\widetilde{\rho}\widetilde{\sigma} - \widetilde{v}^2 + \widetilde{v} = 0$ correspond to $\widetilde{v} = \frac{1}{2} \pm \sqrt{\frac{1}{4} + \widetilde{\rho}\widetilde{\sigma}}$. For the original contour $\mathcal{C}$, $\widetilde{\rho}_2$ and $\widetilde{\sigma}_2$ are large and hence the poles are located at large positive or negative imaginary values of $\widetilde{v}$. As we reduce $\widetilde{\rho}_2$ and $\widetilde{\sigma}_2$ in the process of deforming the $\widetilde{\rho}$ and $\widetilde{\sigma}$ contours, these poles approach the $\widetilde{v}$ integration contour. We can however avoid them by deforming the $\widetilde{v}$ integration contour with the ultimate goal that we try to minimize the maximum value of the integrand over the integration contour. For $Q\cdot P > 0$ this can be done by deforming the $\widetilde{v}$ contour into the lower half plane since the exponential factor decreases as we reduce $\widetilde{v}_2$. Thus we can always avoid the pole coming down from above; but at some point we encounter the pole coming up from below and the major contribution to the integral would come from the residue at this pole. Since this pole was below the original $\widetilde{v}$ integration contour, the residue from this is calculated by enclosing it in the clockwise direction. This gives $\gamma = -1$. An exactly similar argument shows that for $Q\cdot P < 0$, the main obstruction to our ability to reduce the integrand by deforming the $\widetilde{v}$ contour comes from the pole above the original $\widetilde{v}$ contour. This gives $\gamma = 1$. Thus we have

$$
\gamma = -Sign(Q\cdot P)\,. \qquad (5.6.33)
$$

The correction to (5.6.32) involves contribution from other poles for which $n_2 \neq 1$, and are suppressed by fractional powers of $e^{-\pi\sqrt{Q^2P^2-(Q\cdot P)^2}}$. Thus (5.6.32) contains information not only about the leading contribution to the statistical entropy but also about all the subleading corrections involving inverse powers of charges.

Let us now introduce new complex variables τ_1 and τ_2 through the relations:

$$
\rho = \tau_1 + i\tau_2, \qquad \sigma = -\tau_1 + i\tau_2\,. \qquad (5.6.34)
$$

Then (5.6.32) may be rewritten as

$$
d(\vec{Q},\vec{P}) \simeq \frac{\gamma}{4\pi N C_1} \int \frac{d\tau_1\wedge d\tau_2}{\tau_2^2}
$$

$$
\left[2(k+3) + \frac{\pi}{\tau_2}\left\{(\tau_1^2+\tau_2^2)P^2 + Q^2 - 2\tau_1 Q\cdot P\right\}\right]
$$
$$
\exp\left[\frac{\pi}{2\tau_2}\left\{(\tau_1^2+\tau_2^2)P^2 + Q^2 - 2\tau_1 Q\cdot P\right\}\right.
$$
$$
\left. - \ln g(\tau_1+i\tau_2) - \ln g(-\tau_1+i\tau_2) - (k+2)\ln(2\tau_2)\right]. \tag{5.6.35}
$$

The integral in (5.6.35) is carried out over the image of the surface (5.6.21) in the complex τ_1-τ_2 coordinate system. In the leading approximation the saddle point, obtained by extremizing

$$
\frac{\pi}{2\tau_2}\left\{(\tau_1^2+\tau_2^2)P^2 + Q^2 - 2\tau_1 Q\cdot P\right\} \tag{5.6.36}
$$

occurs at

$$
\tau_1 = Q\cdot P/P^2, \qquad \tau_2 = \sqrt{Q^2P^2 - (Q\cdot P)^2}/P^2\,. \tag{5.6.37}
$$

On the other hand the steepest descent of the integrand occurs as we move away from the saddle point along the imaginary directions in the τ_1 and τ_2 plane. Hence the integration contour is such that it passes through the saddle point (5.6.37) and lies along imaginary τ_1 and τ_2 direction at the saddle point. If we now regard τ_1 and τ_2 as real variables and define

$$
\tau = \tau_1 + i\tau_2\,, \tag{5.6.38}
$$

then (5.6.35) may be formally reexpressed as

$$
\begin{aligned}
d(\vec{Q},\vec{P}) \;\simeq\; & \frac{\gamma}{4\pi N C_1}\int \frac{d\tau_1\wedge d\tau_2}{\tau_2^2}\left[2(k+3) + \frac{\pi}{\tau_2}|Q-\tau P|^2\right] \\
& \exp\left[\frac{\pi}{2\tau_2}|Q-\tau P|^2 - \ln g(\tau) - \ln g(-\bar\tau) - (k+2)\ln(2\tau_2)\right].
\end{aligned} \tag{5.6.39}
$$

In evaluating this integral we must first express the integrand in terms of τ_1 and τ_2 treating them as real variables, and then carry out the integral by analytically rotating the integration contours along the imaginary axis both in the complex τ_1 and complex τ_2 plane.

In order to determine the overall sign, we need to determine whether $d\tau_1\wedge d\tau_2$ defines a positive or negative integration measure along the deformed contour at the saddle point. This can be determined by expressing the original $(\widetilde\rho,\widetilde\sigma,\widetilde v)$ coordinates in terms of the new coordinates (τ_1,τ_2). Setting $v=0$ in (5.6.23) and using (5.6.34) we get

$$
\widetilde\rho = \frac{i}{2N\tau_2}, \qquad \widetilde\sigma = iN\frac{\tau_1^2+\tau_2^2}{2\tau_2}, \qquad \widetilde v = \frac{1}{2} - i\frac{\tau_1}{2\tau_2}\,. \tag{5.6.40}
$$

From this we see that

$$d\widetilde{\rho}\wedge d\widetilde{\sigma} = -\frac{\tau_1}{2\tau_2^3} d\tau_1 \wedge d\tau_2 \,. \tag{5.6.41}$$

Since $d\widetilde{\rho}\wedge d\widetilde{\sigma}$ represents positive integration measure $d\widetilde{\rho}_1 d\widetilde{\sigma}_1$ according to the convention introduced below (5.6.29), we see that around the saddle point $d\tau_1\wedge d\tau_2$ describes positive (negative) integration measure if τ_1 is negative (positive), ı.e, $Q\cdot P$ is negative (positive). Combining this with (5.6.33) we see that (5.6.39) may be expressed as

$$\begin{aligned} d(\vec{Q},\vec{P}) &\simeq \frac{1}{4\pi N C_1}\int \frac{d\tau_1 d\tau_2}{\tau_2^2}\left[2(k+3)+\frac{\pi}{\tau_2}|Q-\tau P|^2\right] \\ &\quad \exp\left[\frac{\pi}{2\tau_2}\,|Q-\tau P|^2 - \ln g(\tau) - \ln g(-\bar\tau) - (k+2)\ln(2\tau_2)\right], \end{aligned} \tag{5.6.42}$$

where $d\tau_1 d\tau_2$ is defined such that it describes a positive integration measure when the integration contour lies along imaginary τ_1 and τ_2 direction at the saddle point.

Identifying $d(\vec{Q},\vec{P})$ with $e^{S_{stat}(\vec{Q},\vec{P})}$ where S_{stat} denotes the statistical entropy, we can rewrite (5.6.42) as

$$e^{S_{stat}(\vec{Q},\vec{P})} \simeq \int \frac{d^2\tau}{\tau_2^2}\, e^{-F(\vec{\tau})}\,, \tag{5.6.43}$$

where $\vec{\tau}=(\tau_1,\tau_2)$ or $(\tau,\bar\tau)$ depending on the basis we choose to use, and

$$\begin{aligned} F(\vec{\tau}) &= -\left[\frac{\pi}{2\tau_2}\,|Q-\tau P|^2 - \ln g(\tau) - \ln g(-\bar\tau) - (k+2)\ln(2\tau_2)\right. \\ &\quad \left. + \ln\left\{K_0\,\left(2(k+3)+\frac{\pi}{\tau_2}|Q-\tau P|^2\right)\right\}\right], \\ K_0 &\equiv \frac{1}{4\pi N C_1}\,. \end{aligned} \tag{5.6.44}$$

Note that $F(\vec{\tau})$ also depends on the charge vectors $\vec{Q}$, $\vec{P}$, but we have not explicitly displayed these in its argument.

As described in eq.(3.1.44) (and proved in appendix C, eq.(C.34)), $g(\tau)$ transforms as a modular form of weight $(k+2)$ under the $\Gamma_1(N)$ group. Using this fact, and (5.6.44), one can show that in (5.6.43) the integration measure $d^2\tau/(\tau_2)^2$ as well as the integrand $e^{-F(\vec{\tau})}$ are manifestly invariant under the $\Gamma_1(N)$ transformation:

$$\begin{aligned} &\vec{Q}\to\alpha\vec{Q}+\beta\vec{P}, \quad \vec{P}\to\gamma\vec{Q}+\delta\vec{P}, \quad \tau\to\frac{\alpha\tau+\beta}{\gamma\tau+\delta}\,, \\ &\alpha,\beta,\gamma,\delta\in\mathbb{Z}, \quad \alpha\delta-\beta\gamma=1, \quad \alpha,\delta=1 \bmod N, \quad \gamma=0 \bmod N\,. \end{aligned} \tag{5.6.45}$$

Thus $S_{stat}(\vec{Q}, \vec{P})$ computed from (5.6.43) is invariant under $\Gamma_1(N)$. From (3.1.36) one can see that this corresponds to the S-duality symmetry of the theory in the second description.

We shall now describe a systematic procedure for evaluating S_{stat} given in (5.6.43) as an expansion in inverse powers of the charges. For this we introduce the generating function:

$$e^{W(\vec{J})} = \int \frac{d^2\tau}{\tau_2^2} \, e^{-F(\vec{\tau}) + \vec{J}\cdot\vec{\tau}} \, , \tag{5.6.46}$$

for a two dimensional vector $\vec{J}$, and define $\Gamma(\vec{u})$ as the Legendre transform of $W(\vec{J})$:

$$\Gamma(\vec{u}) = \vec{J}\cdot\vec{u} - W(\vec{J}) \, , \qquad u_i = \frac{\partial W(\vec{J})}{\partial J_i} \, . \tag{5.6.47}$$

It follows from (5.6.47) that

$$J_i = \frac{\partial \Gamma(\vec{u})}{\partial u_i} \, . \tag{5.6.48}$$

As a result if

$$\frac{\partial \Gamma(\vec{u})}{\partial \vec{u}_i} = 0 \quad \text{at } \vec{u} = \vec{u}_0 \, , \tag{5.6.49}$$

then it follows from (5.6.46)-(5.6.48), (5.6.43) that

$$\Gamma(\vec{u}_0) = -W(\vec{J} = 0) = -S_{stat} \, . \tag{5.6.50}$$

Thus the computation of S_{stat} can be done by first calculating $\Gamma(\vec{u})$ and then evaluating it at its extremum. $\Gamma(\vec{u})$ in turn can be calculated by regarding this as a sum of one particle irreducible (1PI) Feynman diagrams in the zero dimensional field theory with action $F(\vec{\tau}) + 2\ln\tau_2$. Since S_{stat} is given by the value of the function $-\Gamma(\vec{u})$ at its extremum, we can identify $-\Gamma(\vec{u})$ as the entropy function for the statistical entropy in analogy with the corresponding result for black hole entropy [3, 4].

A convenient method of calculating $\Gamma(\vec{u})$ is the so called background field method. For this we choose some arbitrary base point $\vec{\tau}_B$ and define

$$e^{W_B(\vec{\tau}_B, \vec{J})} = \int \frac{d^2\eta}{(\tau_{B2} + \eta_2)^2} \, e^{-F(\vec{\tau}_B + \vec{\eta}) + \vec{J}\cdot\vec{\eta}} \, , \tag{5.6.51}$$

$$\Gamma_B(\vec{\tau}_B, \vec{\chi}) = \vec{J}\cdot\vec{\chi} - W_B(\vec{\tau}_B, \vec{J}) \, , \qquad \chi_i = \frac{\partial W_B(\vec{\tau}_B, \vec{J})}{\partial J_i} \, . \tag{5.6.52}$$

By shifting the integration variable in (5.6.51) to $\vec{\tau} = \vec{\tau}_B + \vec{\eta}$ it follows easily that

$$W_B(\vec{\tau}_B, \vec{J}) = W(\vec{J}) - \vec{\tau}_B \cdot \vec{J} \, , \tag{5.6.53}$$

and hence

$$\Gamma_B(\vec{\tau}_B, \vec{\chi}) = \Gamma(\vec{\tau}_B + \vec{\chi})\,. \tag{5.6.54}$$

Thus the computation of $\Gamma(\vec{u})$ reduces to the computation of $\Gamma_B(\vec{u}, \vec{\chi} = 0)$. The latter in turn can be computed as the sum of 1PI vacuum diagrams in the 0-dimensional field theory with action $F(\vec{u} + \vec{\eta}) + 2\ln(u_2 + \eta_2)$, with $\vec{\eta}$ regarded as fundamental fields, and $\vec{u}$ regarded as some fixed background.

While this gives a definition of the statistical entropy function whose extremization leads to the statistical entropy, the entropy function constructed this way is not manifestly invariant under the S-duality transformation given in (3.1.36). This is due to the fact that since the S-duality transformation has a non-linear action on (τ_1, τ_2), the generating function $W(\vec{J})$ defined in (5.6.46) and hence also the effective action $\Gamma(\vec{u})$ defined in (5.6.47) does not have manifest S-duality symmetries. Of course the statistical entropy obtained by extremizing $\Gamma(\vec{u})$ will be duality invariant since this is given in terms of the manifestly duality invariant integral (5.6.44). In appendix F we have described a slightly different construction based on Riemann normal coordinates which yields a manifestly duality invariant statistical entropy function. The result of this analysis is that instead of using the function $-\Gamma_B(\vec{\tau})$ as the statistical entropy function we can use a different manifestly duality invariant function $-\widetilde{\Gamma}_B(\vec{\tau})$ as the statistical entropy function. $\widetilde{\Gamma}_B(\vec{\tau}_B)$ is defined as the sum of 1PI vacuum diagrams computed from the action

$$-\ln\left(\frac{1}{|\vec{\xi}|}\sinh|\vec{\xi}|\right) - \sum_{n=0}^{\infty}\frac{1}{n!}(\tau_{B2})^n \xi_{i_1}\ldots\xi_{i_n}\, D_{i_1}\cdots D_{i_n}F(\vec{\tau})\Big|_{\vec{\tau}=\vec{\tau}_B}\,, \tag{5.6.55}$$

where $\xi = \xi_1 + i\xi_2$ is to be regarded as a zero dimensional complex quantum field, τ_B is a fixed zero dimensional complex background field and D_τ, $D_{\bar{\tau}}$ are duality invariant covariant derivatives defined recursively through the relation:

$$\begin{aligned} D_\tau(D_\tau^m D_{\bar\tau}^n F(\vec{\tau})) &= (\partial_\tau - im/\tau_2)(D_\tau^m D_{\bar\tau}^n F(\vec{\tau})), \\ D_{\bar\tau}(D_\tau^m D_{\bar\tau}^n F(\vec{\tau})) &= (\partial_{\bar\tau} + in/\tau_2)(D_\tau^m D_{\bar\tau}^n F(\vec{\tau}))\,, \end{aligned} \tag{5.6.56}$$

for any arbitrary ordering of D_τ and $D_{\bar\tau}$ in $D_\tau^m D_{\bar\tau}^n F(\vec{\tau})$. The result of this computation expresses $\widetilde{\Gamma}_B(\vec{\tau}_B)$ in terms of manifestly duality invariant quantity $F(\vec{\tau})$ and its duality covariant derivatives. It has been shown in appendix F that explicit evaluation of $\widetilde{\Gamma}_B(\vec{\tau})$ gives

$$-\widetilde{\Gamma}_B(\vec{\tau}) = \frac{\pi}{2\tau_2}|Q - \tau P|^2 - \ln g(\tau) - \ln g(-\bar\tau) - (k+2)\ln(2\tau_2) + \ln(4\pi K_0) + \mathcal{O}(Q^{-2})\,. \tag{5.6.57}$$

In order to see how good this approximation is, we give below the exact results for $d(\vec{Q}, \vec{P})$ calculated from (5.1.3) and $S_{stat} \equiv \ln d(\vec{Q}, \vec{P})$ for certain values of Q^2, P^2 and $Q \cdot P$ in the case of heterotic string theory on T^6, i.e. the $N = 1$, $\mathcal{M} = K3$ model.[39] We also give the approximate statistical entropies $S^{(0)}_{stat}$ calculated with the 'tree level' statistical entropy function, and $S^{(1)}_{stat}$ calculated with the 'tree level' plus 'one loop' statistical entropy function.[40]

Q^2	P^2	$Q \cdot P$	$d(Q, P)$	S_{stat}	$S^{(0)}_{stat}$	$S^{(1)}_{stat}$
2	2	0	50064	10.82	6.28	10.62
4	4	0	32861184	17.31	12.57	16.90
6	6	0	16193130552	23.51	18.85	23.19
6	6	1	11232685725	23.14	18.59	22.88
6	6	2	4173501828	22.15	17.77	21.94
6	6	3	920577636	20.64	16.32	20.41
6	6	4	110910300	18.52	14.05	18.40

Up to an additive constant (5.6.57) agrees with the black hole entropy function for these models as given in (3.1.47) if we identify τ as $u_a + iu_S$. This in turn implies agreement between black hole entropy and statistical entropy to this order. We should however keep in mind that the computation of the black hole entropy function has been done by including in the effective action only a certain subset of four derivative terms proportional to the Gauss-Bonnet term. As discussed at the end of §3.1.4, for large u_S or equivalently $\sqrt{Q^2P^2 - (Q \cdot P)^2} >> P^2$, the argument of [19,20,22] tell us that the result computed using the Gauss-Bonnet term is the correct answer to this order. But there is no such concrete result for finite u_S. Nevertheless the agreement between the black hole and the statistical entropy indicates that there might exist such a non-renormalization theorem even for finite u_S.

Finally we would like to note that even though the complete spectrum changes discontinuously as we cross a wall of marginal stability, the large charge expansion is not affected by this change. We

[39] In this case $K_0 = 1/4\pi$ and the additive constant in (5.6.57) vanishes.

[40] In this table the sign of $Q \cdot P$ has been chosen in such a way that in deforming the original contour lying at $\widetilde{v}_2 < 0$ to the final contour corresponding to $\widetilde{\rho}_2$, $\widetilde{v}_2$, $\widetilde{\sigma}_2$ given by $\eta_1(\epsilon)$, $\eta_2(\epsilon)$, $\eta_3(\epsilon)$ given in (5.6.22) we do not pass through the pole at $\widetilde{v} = 0$. For the other choice of the sign of $Q \cdot P$ we need to pass through this pole. This will give an additional contribution to S_{stat}. Although this contribution is exponentially suppressed for large charges, it may not be negligible for the charges used in this table.

can illustrate this by considering the example of the marginal stability wall separating the domains $\mathcal{R}$ and $\mathcal{L}$ defined at the end of §5.4. As we cross this wall, the prescription for the integration contour changes from $\widetilde{v}_2 = -M_3$ to $\widetilde{v}_2 = M_3$, keeping $\widetilde{\rho}_2$ and $\widetilde{\sigma}_2$ at fixed values $M_1, M_2 >> M_3$. It is easy to see that the only pole through which the contour passes during this deformation is the one at $\widetilde{v} = 0$. Thus the difference between the two cases is the residue of the integrand at $\widetilde{v} = 0$. Since this pole is not in the class given in (5.6.4) with $n_2 = 1$, the contribution from this pole is exponentially suppressed compared to the leading contribution. A careful analysis shows that the same story is repeated as we cross other marginal stability walls [168].

This result is consistent with the fact that on the black hole side the attractor point described in §3.1.1 is a stable supersymmetric attractor for $P^2 > 0$, $Q^2 > 0$, $P^2Q^2 > (Q \cdot P)^2$. Thus the near horizon geometry of these black holes is always given by this attractor point and is independent of the asymptotic moduli even if the asymptotic moduli cross one or more walls of marginal stability. We shall see in §5.7 that the exponentially suppressed changes in the degeneracy across the walls of marginal stability can be related to the (dis)appearance of two centered black hole solutions as we cross these walls.

5.7 Jump in degeneracy and two centered black holes

Given the success of the analysis of the previous section relating the black hole entropy to the statistical entropy, we can now ask: can we understand the jump in the degeneracy across walls of marginal stability on the black hole side? The issue is somewhat tricky since these jumps in the degeneracy are exponentially small compared to the leading contribution to the entropy. Nevertheless since the change is discontinuous, one might hope that there is a clear mechanism on the black hole side which produces these discontinuous changes across the walls of marginal stability and if we can identify this mechanism then we may be able to reproduce these jumps on the black hole side. In this section we shall show that there is indeed a clear mechanism on the black hole side that describes these jumps, – this is the phenomenon of (dis)appearance of multicentered black hole solutions for a given total charge as we cross various walls of marginal stability in the space of asymptotic values of the moduli fields [140, 142, 143, 144, 145, 146]. In particular the exponential of the entropy associated with these multi-centered black holes will reproduce the jump in the degeneracy computed from the exact dyon spectrum. Our analysis will follow [154]. Related observations have been made in [167, 150].

In order to keep the analysis simple we shall restrict our analysis to the four dimensional charge vectors of the type given in (5.1.1)

$$\vec{Q} = \begin{pmatrix} 0 \\ -n/N \\ 0 \\ -1 \end{pmatrix}, \qquad \vec{P} = \begin{pmatrix} Q_1 - 1 \\ -J \\ Q_5 \\ 0 \end{pmatrix}, \qquad n, J, Q_1, Q_5 \in \mathbb{Z}, \quad n, Q_1 \geq 0, \quad Q_5 > 0\,, \tag{5.7.1}$$

and consider M_∞ of the form:

$$M_\infty = \begin{pmatrix} \widehat{R}^{-2} & & & \\ & R^{-2} & & \\ & & \widehat{R}^2 & \\ & & & R^2 \end{pmatrix}. \tag{5.7.2}$$

In this case eq.(5.4.14) describing the marginal stability wall takes the form

$$a_\infty = a_c, \quad a_c \equiv -\frac{J\,\widehat{R}}{R\{Q_1 - 1 + \widehat{R}^2 Q_5\}}\, S_\infty \tag{5.7.3}$$

for $b = 0$. By following carefully the duality chain relating the first and the second description one finds that the weak coupling region of the first description corresponds to large values of R parametrizing the matrix M_∞ in (5.7.2). In this region the degeneracy formulæ for $a_\infty > a_c$ and $a_\infty < a_c$ are given respectively by (5.1.3), (5.1.4) and a similar formula with the contour $\mathcal{C}$ replaced by the contour $\widehat{\mathcal{C}}$ given in (5.1.13). If we denote them by $d_>(\vec{Q}, \vec{P})$ and $d_<(\vec{Q}, \vec{P})$ respectively, then the difference between them can be computed by evaluating the contribution from the pole of the integrand at $\widetilde{v} = 0$ since this is the pole we encounter while deforming $\mathcal{C}$ to $\widehat{\mathcal{C}}$ [168]. Now from (C.26) we know that for $\widetilde{v} \simeq 0$ $\widetilde{\Phi}$ takes the form:

$$\widetilde{\Phi}(\widetilde{\rho}, \widetilde{\sigma}, \widetilde{v}) = -4\pi^2\, \widetilde{v}^2\, f_1(N\widetilde{\rho}) f_2(\widetilde{\sigma}/N) + \mathcal{O}(\widetilde{v}^4)\,, \tag{5.7.4}$$

where $(f_1(\tau))^{-1}$ and $(f_2(\tau))^{-1}$ have the interpretation of the generating function for the degeneracies of purely electric half-BPS states and purely magnetic half-BPS states respectively. Using (5.7.4) and evaluating the residue at the pole of the integrand in (5.1.3) at $\widetilde{v} = 0$ we get

$$d_>(Q, P) - d_<(Q, P) = -(-1)^{Q\cdot P+1}\, Q \cdot P\, d_{el}(Q)\, d_{mag}(P)\,, \tag{5.7.5}$$

where

$$d_{el}(Q) = \int_0^1 d\widetilde{\rho}\, e^{-i\pi N\widetilde{\rho} Q^2}\, (f_1(N\widetilde{\rho}))^{-1}\,, \qquad d_{mag}(P) = \frac{1}{N}\int_0^N d\widetilde{\sigma}\, e^{-i\pi\widetilde{\sigma} P^2/N}\, (f_2(\widetilde{\sigma}/N))^{-1} \tag{5.7.6}$$

are the degeneracies of purely electric and purely magnetic half-BPS states carrying charges Q and P respectively. Thus $\ln d_{el}(Q)$ and $\ln d_{mag}(P)$ can be identified as the entropies of small black holes of electric charge Q and magnetic charge P. Since $\ln|Q\cdot P|$ is subleading compared to these entropies for large Q^2 and P^2 i.e. for

$$n, Q_1, Q_5 >> 1\,, \tag{5.7.7}$$

we see that $\ln|d_>(Q,P) - d_<(Q,P)|$ can be identified as the sum of the entropies of a small electric black hole of charge Q and a small magnetic black hole of charge P. In carrying out the analysis on the black hole side we shall choose charge vectors satisfying (5.7.7).

It is known from the general analysis of [143, 145, 147] that a two centered black hole carrying charges $(Q,0)$ and $(0,P)$ contributes to the index with a sign $(-1)^{Q\cdot P+1}$. Indeed, according to these papers the contribution to the index due to a two centered black hole is given precisely by $(-1)^{Q\cdot P+1}\,|Q\cdot P|\,d_{el}(Q)\,d_{mag}(P)$.[41] Taking into account the sign of the right hand side of (5.7.5), and assuming that this phenomenon has a description in terms of (dis)appearance of a two centered black hole carrying charges $(Q,0)$ and $(0,P)$, we can draw the following conclusion:[42]

For $J(=Q\cdot P)>0$, as we cross the wall of marginal stability (5.7.3) from $a_\infty > a_c$ to $a_\infty < a_c$, a new two centered small black hole solution should appear with an electric center of charge Q and a magnetic center of charge P. On the other hand for $J(=Q\cdot P)<0$, as we cross the wall of marginal stability (5.7.3) from $a_\infty < a_c$ to $a_\infty > a_c$, a new two centered small black hole solution should appear with an electric center of charge Q and a magnetic center of charge P.

We shall now verify this explicitly. For describing the two centered black hole we shall use the $\mathcal{N}=2$ supersymmetric description of the same system given in §3.2.2 in terms of the metric $g_{\mu\nu}$, four scalar fields X^I, four gauge fields $\mathcal{A}^I_\mu$ and some additional auxiliary fields. We shall work in the supergravity approximation in which case the prepotential (3.2.14) takes the form:

$$F = -\frac{X^1X^2X^3}{X^0}\,. \tag{5.7.8}$$

Using the relations (3.2.20) between the charges $\widetilde{q}_I$, $\widetilde{p}^I$ in the STU model and the charge vector Q, P in the $\mathcal{N}=4$ description we see that for the configuration (5.7.1) we have

$$(\widetilde{q}_0,\widetilde{q}_1,\widetilde{q}_2,\widetilde{q}_3) = (0, Q_1-1, -1, -n/N), \qquad (\widetilde{p}^0,\widetilde{p}^1,\widetilde{p}^2,\widetilde{p}^3) = (Q_5, 0, -J, 0)\,. \tag{5.7.9}$$

[41] This formula was derived for black holes in $\mathcal{N}=2$ supersymmetric string theories, – we shall assume that it holds also in the $\mathcal{N}=4$ theories. This is not unreasonable since these black holes can be embedded in the $\mathcal{N}=2$ supersymmetric S-T-U models.

[42] Note that when a new configuration with same charge appears in the black hole system, its degeneracy (or more precisely the index), i.e. exponential of the entropy, will add to the degeneracy of the other configurations of the same charge.

As discussed in §3.2.2, the theory has an underlying 'gauge invariance' that allows for a scaling of all the X^I's by a complex function. We shall fix this gauge using the gauge condition:

$$i(\bar{X}^I F_I - X^I \bar{F}_I) = 1\,, \qquad F_I \equiv \partial F/\partial X^I\,. \tag{5.7.10}$$

This fixes the normalization but not the overall phase of the X^I's. While studying a black hole solution carrying a given set of charges, it will be convenient to fix the overall phase of the X^I's such that

$$Arg(\widetilde{q}_I X^I - \widetilde{p}^I F_I) = \pi \quad \text{at } \vec{r} = \infty\,. \tag{5.7.11}$$

In this gauge one can construct a general multi-centered black hole solution with charges $(\widetilde{q}^{(s)}, \widetilde{p}^{(s)})$ located at $\vec{r}_s$ following the procedure described in [140,141,142,144]. The locations $\vec{r}_s$ are constrained by the equations [140, 141, 142, 144]

$$h_I \widetilde{p}^{(s)I} - h^I \widetilde{q}_I^{(s)} + \sum_{t\neq s} \frac{\widetilde{p}^{(s)I}\widetilde{q}_I^{(t)} - \widetilde{q}_I^{(s)}\widetilde{p}^{(t)I}}{|\vec{r}_s - \vec{r}_t|} = 0 \tag{5.7.12}$$

where h^I and h_I are constants defined through the equations

$$X^I_\infty - \bar{X}^I_\infty = i h^I\,, \qquad F_{I\infty} - \bar{F}_{I\infty} = i h_I\,. \tag{5.7.13}$$

We shall not give the complete solution but will give the solution for the scalars for illustration. They are obtained by solving the equations:

$$X^I - \bar{X}^I = i\left(h^I + \sum_s \frac{\widetilde{p}^{(s)I}}{|\vec{r} - \vec{r}_s|}\right), \qquad F_I - \bar{F}_I = i\left(h_I + \sum_s \frac{\widetilde{q}_I^{(s)}}{|\vec{r} - \vec{r}_s|}\right). \tag{5.7.14}$$

If we define α and β via the relations

$$X^0_\infty = \alpha + i\beta\,, \tag{5.7.15}$$

then using (3.2.16), (3.2.18), (5.7.8) and (5.7.13) we get

$$\begin{aligned}
&h^0 = 2\beta, \quad h^1 = 2(\beta a_\infty + \alpha S_\infty), \quad h^2 = 2\widehat{R}R\alpha, \quad h^3 = 2\widehat{R}\alpha/R, \\
&h_0 = -2\widehat{R}^2(\alpha S_\infty + \beta a_\infty), \quad h_1 = 2\beta\widehat{R}^2, \quad h_2 = 2\widehat{R}(\beta S_\infty - \alpha a_\infty)/R, \\
&h_3 = 2\widehat{R}R(\beta S_\infty - \alpha a_\infty)\,.
\end{aligned} \tag{5.7.16}$$

The gauge condition (5.7.10) gives

$$\alpha^2 + \beta^2 = (8\widehat{R}^2 S_\infty)^{-1}\,. \tag{5.7.17}$$

To proceed further we need to focus on a specific two centered solution with electric charge Q at one center and a magnetic charge P at the other center.[43] Using (3.2.20), (5.7.1) we see that the charges at the two centers are given by:

$$\widetilde{q}^{(1)} = (0,0,-1,-n/N), \quad \widetilde{p}^{(1)} = (0,0,0,0), \quad \widetilde{q}^{(2)} = (0,Q_1-1,0,0), \quad \widetilde{p}^{(2)} = (Q_5,0,-J,0)\,. \quad (5.7.18)$$

Eqs.(5.7.12) for $s=1$ and 2 now give:

$$h^2 + \frac{n}{N}h^3 = \frac{J}{L}, \quad (5.7.19)$$

$$h_0 Q_5 - h_2 J - h^1(Q_1-1) + \frac{J}{L} = 0\,, \quad (5.7.20)$$

where $L = |\vec{r}_1 - \vec{r}_2|$ is the separation between the two centers.[44] Using (5.7.16) and (5.7.19) we get

$$\alpha = \frac{J}{2L}\frac{1}{R\widehat{R} + \frac{n}{N}\frac{\widehat{R}}{R}}\,. \quad (5.7.21)$$

Using (5.7.16) and (5.7.21) we may now express (5.7.20) as

$$\beta\left(a_\infty(Q_1 - 1 + \widehat{R}^2 Q_5) + \frac{\widehat{R}JS_\infty}{R}\right) + \alpha\left((Q_1 - 1 + \widehat{R}^2 Q_5)S_\infty - \widehat{R}R - \frac{n}{N}\frac{\widehat{R}}{R} - \frac{\widehat{R}Ja_\infty}{R}\right) = 0\,. \quad (5.7.22)$$

Substituting α and β computed from (5.7.21), (5.7.22) into (5.7.17) we can determine L. The ambiguity in determining the sign of L can be fixed using (5.7.11).

We are interested in determining under what conditions the two centered black hole solution described above exists. For this we note that a sensible solution should have positive value of L.

[43] In the supergravity approximation the solution is singular at each center, but once higher derivative corrections are taken into account each center is transformed into the near horizon geometry of a non-singular extremal black hole with finite entropy equal to the statistical entropy of the corresponding microstates as described in §3.3, §4.4. As we have seen, this phenomenon can be demonstrated explicitly for purely electrically charged small black holes representing fundamental heterotic string [96, 98, 99, 100, 101, 174, 175, 176, 19] and hence also their S-dual purely magnetic states. In this case the modifications of the two centered solution due to higher derivative corrections can be found using the method developed in [88]. This approach fails for small black holes describing fundamental type II string compactification and hence also their S-dual purely magnetic states. However it is expected that once the effect of full set of higher derivative terms are taken into account the entropy of a small black hole in type II string theory will also reproduce the statistical entropy of the corresponding microstates [93, 95], – see [106] for some recent progress on this issue.

[44] Note that this is the coordinate separation. In order to express this in physical units *e.g.* string length, we need to examine the asymptotic metric associated with this solution and also the relation between the metric appearing in the $\mathcal{N}=2$ supersymmetric S-T-U model and the string metric $G_{\mu\nu}$ that appears naturally in the $\mathcal{N}=4$ supergravity action (3.1.3).

Typically as we change the values of the asymptotic moduli keeping the charges fixed, the value of L changes. On some subspace of codimension 1 the value of L becomes infinite and beyond that the solution gives negative values of L which means that a physical solution does not exist. To determine this codimension 1 subspace we simply need to determine the conditions on the asymptotic moduli for which $L = \infty$. From (5.7.21) we see that in this case $\alpha = 0$. Since eq.(5.7.17) now requires β to be non-zero, we see from (5.7.22) that

$$a_\infty(Q_1 - 1 + \widehat{R}^2 Q_5) + \widehat{R} J S_\infty / R = 0\,. \tag{5.7.23}$$

This is identical to the condition (5.7.3) for marginal stability [140]. Thus we conclude that as a_∞ passes through a_c, the two centered black hole solution carrying an entropy equal to the sum of the entropies of a small electric black hole of charge Q and a small magnetic black hole of charge P, (dis)appears from the spectrum. This is precisely what was predicted earlier by analyzing the exact formula for the degeneracy of dyons.

In order to complete the verification of the predictions made earlier we need to determine on which side of the $a_\infty = a_c$ line the two centered solution exists. For this we use eq.(5.7.11). For the solution under consideration this gives, using (5.7.22),

$$\alpha \left(a_\infty(Q_1 - 1 + \widehat{R}^2 Q_5) + \frac{\widehat{R} J S_\infty}{R} \right) \left\{ 1 + \frac{\left((Q_1 - 1 + \widehat{R}^2 Q_5) S_\infty - \widehat{R} R - \frac{n}{N} \frac{\widehat{R}}{R} - \frac{\widehat{R} J a_\infty}{R} \right)^2}{\left(a_\infty(Q_1 - 1 + \widehat{R}^2 Q_5) + \frac{\widehat{R} J S_\infty}{R} \right)^2} \right\} < 0\,. \tag{5.7.24}$$

First consider the case $J > 0$. Since L must be positive for the two centered solution to exist, we see from (5.7.21) that $\alpha > 0$. In this case the term on the left hand side of (5.7.24) is negative for $a_\infty < a_c$ and positive for $a_\infty > a_c$. Thus the inequality is satisfied only for $a_\infty < a_c$, leading to the conclusion that the two centered black hole exists only for $a_\infty < a_c$. A similar analysis shows that for $J < 0$, the two centered black hole exists only for $a_\infty > a_c$. This is exactly what has been predicted earlier from the analysis of the exact dyon spectrum of the theory.

6 Open Questions and Speculation on $\mathcal{N} = 2$

We end by reviewing some of the questions left open in our analysis. Some of these issues are technical in nature while some others are conceptual. We also speculate on the degeneracy of dyons in $\mathcal{N} = 2$ supersymmetric string theories.

- **Non-locality of the 1PI action:** The entropy function formalism described in this article gives us a way to calculate the entropy of an extremal black hole with a given set of charges in a given theory assuming that such a black hole solution exists. The main ingredient in this computation is the assumption that the underlying theory is described by a local generally covariant and gauge invariant action and also that the near horizon geometry of an extremal black hole has $SO(2,1)$ isometry associated with an AdS_2 factor.

  It is natural to expect that the formalism also works for studying quantum corrected entropy provided we replace the classical Lagrangian density in the expression for the entropy function by the quantum Lagrangian density associated with one particle irreducible effective action. Indeed this was an underlying assumption in studying the effect of Gauss-Bonnet term on the entropy of $\mathcal{N}=4$ supersymmetric black holes, since some part of the Gauss-Bonnet term comes from quantum corrections. This is also natural from the point of view of duality symmetries since classical higher derivative corrections in one description of the theory may appear as quantum effects in a different description and the entropy of the black hole must be given by a duality invariant expression. However this assumption suffers from an intrinsic problem that sooner or later we shall encounter non-local terms in the quantum effective action, and the current formulation of the entropy function does not allow us to deal with non-local terms in the action. It seems natural that the way to deal with such terms is to generalize the notion of entropy being an extremum of entropy function to the entropy being the result of an appropriate functional integral in the background of an AdS_2 geometry. It will be interesting to make this into a more precise conjecture.

- $\mathcal{N}=2$ **dyon spectrum:** Given the success in finding the dyon spectrum in $\mathcal{N}=4$ supersymmetric string compactifications, one could hope that the dyon degeneracy in $\mathcal{N}=2$ supersymmetric string theories will also be given by a similar formula:

$$d(\vec{Q},\vec{P}) = \int_{\mathcal{C}} dM\, f(\vec{Q},\vec{P},M)\,, \tag{6.1}$$

  where M denotes a set of complex variables, $\mathcal{C}$ is a contour in the complex manifold labelled by the variables M, and $f(\vec{Q},\vec{P},M)$ is an appropriate function of the charges and the complex variables M. One could further speculate that $f(\vec{Q},\vec{P},M)$ has the structure

$$f(\vec{Q},\vec{P},M) = \exp\left(f^{(1)}_{ij}(M)Q_iQ_j + f^{(2)}_{ij}(M)P_iP_j + f^{(3)}_{ij}(M)Q_iP_j\right)\, g(M)\,, \tag{6.2}$$

  where $f^{(s)}_{ij}(M)$ are simple functions of M and $g(M)$ encodes all the non-trivial properties of the theory. The degeneracies in different domains in the moduli space bounded by walls of marginal stability will correspond to different choices of contour $\mathcal{C}$, and as we cross the walls of marginal stability the choice of the contour will change. The jump in the degeneracy can then be computed by evaluating the residues of the integrand at the poles that we encounter as we deform the contour. On the other hand from the analysis of [143, 145, 147] we know that for a decay $(\vec{Q}, \vec{P}) \to (\vec{Q}_1, \vec{P}_1) + (\vec{Q} - \vec{Q}_1, \vec{P} - \vec{P}_1)$ this jump is given by

$$\Delta d(\vec{Q}, \vec{P}) = (-1)^{Q_1 \cdot P - Q \cdot P_1 + 1} \, |Q_1 \cdot P - Q \cdot P_1| \, d(\vec{Q}_1, \vec{P}_1) d(\vec{Q} - \vec{Q}_1, \vec{P} - \vec{P}_1) \,. \tag{6.3}$$

Since for $\mathcal{N} = 2$ supersymmetric string theories all BPS states are half-BPS, the above equation would relate the residues of the function $f(\vec{Q}, \vec{P}, M)$ at various poles to contour integrals of the functions $f(\vec{Q}_1, \vec{P}_1, M)$ and $f(\vec{Q} - \vec{Q}_1, \vec{P} - \vec{P}_1, M)$. This in effect would give a set of bootstrap relations involving the integrands $f(\vec{Q}, \vec{P}, M)$ for different $\vec{Q}$, $\vec{P}$, M. Under favourble conditions we may even be able to solve these bootstrap equations to extract the form of the functions $f(\vec{Q}, \vec{P}, M)$.

- **Non-renormalization of the Gauss-Bonnet contribution:** One of the small miracles in our analysis has been that in a class of string theories, by taking into account either the tree level Gauss-Bonnet term or the $\mathcal{N} = 2$ supersymmetric generalization of the curvature squared terms, we recover the exact results for the entropy of a dyonic black hole in the large electric charge approximation. This is surprising because there are additional corrections to the tree level effective action which could affect the calculation of the entropy. Indeed for a similar non-supersymmetric black hole, related to the supersymmetric black hole by the reversal of sign of one of the charges, neither the Gauss-Bonnet term nor the supersymmetric generalization of the curvature squared term gives the correct answer for the entropy. It will be important to understand the origin of the underlying non-renormalization theorem in more detail. It is especially important in view of the fact that for the case where the electric and the magnetic charges are comparable the complete answer for the black hole entropy is not known. Thus if there is an underlying non-renormalization theorem that tells us that including just the effect of Gauss-Bonnet term is sufficient even in this case, then it would give us a way to calculate the entropy of the dyonic black holes for comparable electric and magnetic charges. The agreement between the statistical entropy and the answer for the entropy computed from the Gauss-Bonnet action seems to point towards the existence of such a non-renormalization theorem.

- **Small black holes:** The statistical entropy of a small black hole in heterotic string theory, whose microscopic description involves excitations of a fundamental heterotic string, agree with the black hole entropy after taking into account the effect of higher derivative corrections in the effective action. Although initial analysis took into account only a small part of the higher derivative corrections and the agreement between the two answers was puzzling, we now have a good understanding of this agreement under the assumption that the near horizon geometry of these black holes have an underlying locally AdS_3 space. Unfortunately the same analysis does not produce a similar agreement for small black holes describing excited states of the fundamental type II string. The most naive explanation seems to be that these black holes do not have an underlying AdS_3 factor in their near horizon geometry; however a different explanation has been suggested in [105, 106, 107, 108]. It will be useful to resolve this issue one way or the other and also to extend the analysis to small black holes in higher dimensions.

- **OSV conjecture:** One question that naturally comes to mind is: is there a possible connection between the results on black hole entropy discussed here with the OSV conjecture [2]? There are two aspects of the OSV conjecture. The first one gives a prescription for relating the microscopic degeneracy of states to the topological string partition function. Most of the precision tests of the OSV conjecture has been carried out in this context. Since we have an exact expression for the degeneracy of states in a class of $\mathcal{N} = 4$ supersymmetric string theories, one can in principle try to see if this can be reproduced using the OSV conjecture. This would be a test of the first aspect of the conjecture. Some attempts in this direction have been made in [15].

  The second aspect of OSV conjecture – which is much less studied[45] – relates Wald's entropy to the statistical entropy. However the statistical entropy, instead of being computed in the microcanonical ensemble, is computed in a mixed ensemble where magnetic charges and the chemical potentials dual to the electric charges are fixed. If both aspects of the OSV conjecture are correct then this would amount to the statement that Wald's entropy of a black hole in $\mathcal{N} \geq 2$ supersymmetric string theory receives contribution only from the F-terms modulo some corrections discussed in [15, 145]. This seems strange since the effective action is known to receive other corrections. It is conceivable (although by no means proven) that if one computes Wald's entropy using the Wilsonian effective action then only the F-terms contribute to the entropy. We have not followed this approach. Instead we calculate Wald's entropy using the one particle irreducible (1PI) effective action[46] and relate it directly to the statistical entropy

[45]Even the precision tests of [175, 176] involving small black holes really tests the relation between the topological string partition function and the microscopic degeneracy since the 'Wald entropy' that was used in this test was computed using only the F-term corrections, and in this approximation it is directly related to the topological string partition function.

[46]This is reflected for example in the $\ln S$ term in the coefficient of the Gauss-Bonnet term as given in eqs.(3.1.38),

computed in the microcanonical ensemble. This approach has the advantage that both the Wald's entropy computed using the one particle irreducible (1PI) effective action and the microcanonical entropy are manifestly duality invariant in the sense described at the end of §2.3. Thus we compare two duality invariant quantities. The disadvantage of this approach is the intrinsic non-local nature of the 1PI effective action discussed earlier.

Before concluding I would like to emphasize again that this article reviews only some special aspects of extremal black hole entropy. Various other recent references on entropy function and attractor mechanism can be found in [177, 178, 179, 180, 181, 182, 183, 184, 185, 186, 187, 188, 189, 190, 191, 192, 193, 194, 195, 196, 197, 198, 199, 200, 201, 202, 203, 204, 205, 206, 207, 208, 209, 210, 211, 212, 213, 214, 215, 216, 217, 218, 219, 220, 221, 222, 223, 224, 225, 226, 227, 228, 229, 230, 231, 232, 233, 234, 235, 236, 237, 238, 239, 240, 241, 242, 243, 244, 245, 246].

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A The Sign Conventions

In this appendix we shall fix the sign conventions for the charges carried by various branes, as well as the sign convention for the duality transformations relating the two descriptions of the $\mathcal{N}=4$ supersymmetric string theories introduced in §3.1.2: as a $\mathbb{Z}_N$ orbifold of type IIB string theory on $\mathcal{M}\times\widetilde{S}^1\times S^1$ for $\mathcal{M}=K3$ or T^4 and as an asymmetric $\mathbb{Z}_N$ orbifold of heterotic or type IIA string theory on $T^4\times\widehat{S}^1\times S^1$. The series of duality transformations taking us from the first to the second description are: an S-duality transformation on type IIB string theory, a T-duality on $\widetilde{S}^1$ mapping the theory to a $\mathbb{Z}_N$ orbifold of type IIA string theory on $\mathcal{M}\times\widehat{S}^1\times S^1$, and finally a

(3.1.40).

string-string duality transformation taking this to an asymmetric $\mathbb{Z}_N$ orbifold of heterotic / type IIA string theory on $T^4 \times \widehat{S}^1 \times S^1$ for $\mathcal{M} = K3$ / $\mathcal{M} = T^4$.

We denote by y the coordinate along S^1, by ψ the coordinate along $\widetilde{S}^1$, by χ the coordinate along the circle $\widehat{S}^1$ dual to $\widetilde{S}^1$, by t the time coordinate and by (r, θ, ϕ) the spherical polar coordinates of the non-compact space. Let B_{MN} denote the NSNS 2-form fields, g_{MN} denote the metric, and $C^{(k)}_{MN}$ denote the RR k-form field subject to the relations

$$* \, dC^{(k)} = (-1)^{k(k-1)/2} \, dC^{(8-k)} + \cdots \,, \tag{A.1}$$

where $*$ denotes Hodge dual and $\cdots$ represent non-linear terms. In defining the Hodge dual in the first description we choose the ϵ tensor such that after dimensional reduction on K3, $\epsilon^{ty\psi r\theta\phi} > 0$. The chain of duality transformations taking us from the first to the second description are chosen so that at the linearized level the first S-duality transformation of IIB acts as $C^{(2)} \to B$, $B \to -C^{(2)}$, the next $R \to 1/R$ duality transformations of $\widetilde{S}^1$ acts as $g_{\psi\mu} \to -B_{\chi\mu}$, $B_{\psi\mu} \to -g_{\chi\mu}$ together with appropriate transformations on the various RR gauge fields, and the final string string duality transformation acts via a Hodge duality transformation in six dimensions on the NS sector 3-form field strength with $\epsilon^{t\chi yr\theta\phi} > 0$, and maps various four dimensional gauge fields arising from various components of the RR sector fields to the 24 (8) NS sector gauge fields in heterotic (type IIA) string theory on T^4. Finally, we use the following convention for the signs of the charges carried by various branes. If $F^{(3)} \equiv dC^{(2)}$ denotes the RR 3-form field strength, then asymptotically a D1-brane along S^1 carries positive $F^{(3)}_{yrt}$, a D5-brane along $\widetilde{S}^1 \times K3$ carries positive $F^{(3)}_{\theta y\phi}$, a D1-brane along $\widetilde{S}^1$ carries positive $F^{(3)}_{\psi rt}$ and a D5-brane along $S^1 \times K3$ carries negative $F^{(3)}_{\theta\psi\phi}$. The same convention is followed for fundamental string and NS 5-brane with $F^{(3)}$ replaced by the NSNS 3-form field strength $H = dB$. A state carrying positive momentum along S^1 or $\widetilde{S}^1$ is defined to be the one which produces positive $\partial_r g_{yt}$ or $\partial_r g_{\psi t}$, and a positively charged Kaluza-Klein monopole associated with the circle S^1 or $\widetilde{S}^1$ is defined to be the one that carries positive $\partial_\theta g_{y\phi}$ or $\partial_\theta g_{\psi\phi}$. Using the relation between $C^{(2)}$ and $C^{(6)}$ given in (A.1) one can verify that a D5-brane wrapped on $K3 \times S^1$ carries negative $(dC^{(6)})_{K3,yrt}$ asymptotically.

B A Class of (4,4) Superconformal Field Theories

In this appendix we shall introduce a class of (4,4) superconformal field theories and study their properties. These have been used in §5 for precision counting of dyon states in $\mathcal{N} = 4$ supersymmetric string theories based on $\mathbb{Z}_N$ orbifolds. These will also be used in appendix H for computing the coefficient of the Gauss-Bonnet term in the effective action of these string theories.

Let $\mathcal{M}$ be either a $K3$ or a T^4 manifold, and let $\widetilde{g}$ be an order N discrete symmetry transformation acting on $\mathcal{M}$. We shall choose $\widetilde{g}$ in such a way that it satisfies the following properties (not all of which are independent):

1. We require that in an appropriate complex coordinate system of $\mathcal{M}$, $\widetilde{g}$ preserves the (0,2) and (2,0) harmonic forms of $\mathcal{M}$.

2. Let $\widetilde{\mathbb{Z}}_N$ denote the group generated by $\widetilde{g}$. We shall require that the orbifold $\widehat{\mathcal{M}} = \mathcal{M}/\widetilde{\mathbb{Z}}_N$ has SU(2) holonomy.

3. Let ω_i denote the harmonic 2-forms of $\mathcal{M}$ and

$$I_{ij} = \int_{\mathcal{M}} \omega_i \wedge \omega_j \tag{B.1}$$

denote the intersection matrix of these 2-forms in $\mathcal{M}$. When we diagonalize I we get 3 eigenvalues -1 and a certain number (say P) of the eigenvalues $+1$ ($P = 19$ for $K3$ and 3 for T^4). We call the 2-forms carrying eigenvalue -1 right-handed or self-dual 2-forms and the 2-forms carrying eigenvalues $+1$ left-handed or anti-selfdual 2-forms. We shall choose $\widetilde{g}$ such that it leaves invariant all the right-handed 2-forms.[47]

4. The $(4,4)$ superconformal field theory with target space $\mathcal{M}$ has $SU(2)_L \times SU(2)_R$ R-symmetry group. We shall require that the transformation $\widetilde{g}$ commutes with the (4,4) superconformal symmetry and the $SU(2)_L \times SU(2)_R$ R-symmetry group of the theory. (For $\mathcal{M} = T^4$ the supersymmetry and the R-symmetry groups are bigger, but $\widetilde{g}$ must be such that only the (4,4) superconformal symmetry and the $SU(2)_L \times SU(2)_R$ part of the R-symmetry group commute with $\widetilde{g}$.)

Let us now take an orbifold of this (4,4) superconformal field theory by the group $\widetilde{\mathbb{Z}}_N$ generated by the transformation $\widetilde{g}$, and define [247]

$$F^{(r,s)}(\tau, z) \equiv \frac{1}{N} Tr_{RR;\widetilde{g}^r} \left(\widetilde{g}^s (-1)^{F_L+F_R} e^{2\pi i \tau L_0} e^{-2\pi i \bar{\tau} \bar{L}_0} e^{2\pi i F_L z} \right), \qquad 0 \le r, s \le N-1\,, \tag{B.2}$$

where $Tr_{RR;\widetilde{g}^r}$ denotes trace over all the RR sector states twisted by $\widetilde{g}^r$ in the SCFT described above *before we project on to $\widetilde{g}$ invariant states*, L_n, $\bar{L}_n$ denote the left- and right-moving Virasoro

[47]We should note that there is no correlation between left- and right-handed 2-forms and left- and right-moving degrees of freedom on the world-sheet of the sigma model with target space $\mathcal{M}$.

generators and F_L and F_R denote the world-sheet fermion numbers associated with left and right-moving sectors in this SCFT.[48] As mentioned earlier we include in the definition of L_0, $\bar{L}_0$ additive factors of $-c_L/24$ and $-c_R/24$ respectively, so that RR sector ground state has $L_0 = \bar{L}_0 = 0$. Due to the insertion of $(-1)^{F_R}$ factor in the trace the contribution to $F^{(r,s)}$ comes only from the $\bar{L}_0 = 0$ states. As a result $F^{(r,s)}$ does not depend on $\bar{\tau}$.

The quantities F_L, F_R can be identified as twice the third generators of the $SU(2)_L$ and $SU(2)_R$ R-symmetry algebras respectively. As a result the z-dependence of $F^{(r,s)}(\tau, z)$ is determined by the characters of the $SU(2)_L$ current algebra. Since for the SCFT under consideration the $SU(2)_L$, $SU(2)_R$ current algebras have level 1, the only $SU(2)_L$ primaries which can appear in the spectrum are those corresponding to isospins 0 and $\frac{1}{2}$. The associated characters are given by $\vartheta_3(2\tau, 2z)$ and $\vartheta_2(2\tau, 2z)$ respectively. Thus the functions $F^{(r,s)}(\tau, z)$ have the form

$$F^{(r,s)}(\tau, z) = h_0^{(r,s)}(\tau)\, \vartheta_3(2\tau, 2z) + h_1^{(r,s)}(\tau)\, \vartheta_2(2\tau, 2z) \tag{B.3}$$

for some functions $h_0^{(r,s)}(\tau)$ and $h_1^{(r,s)}(\tau)$. The functions $h_b^{(r,s)}(\tau)$ in turn have expansions of the form

$$h_b^{(r,s)}(\tau) = \sum_{k \in \frac{1}{N}\mathbb{Z} - \frac{b^2}{4}} c_b^{(r,s)}(4k) e^{2\pi i k \tau} \,. \tag{B.4}$$

This defines the coefficients $c_b^{(r,s)}(u)$. We shall justify the restriction on the allowed values of k shortly. Using the known expansion of ϑ_3 and ϑ_2:

$$\vartheta_3(2\tau, 2z) = \sum_{j \in 2\mathbb{Z}} e^{2\pi i j z} e^{\pi i \tau j^2/2}, \qquad \vartheta_2(2\tau, 2z) = \sum_{j \in 2\mathbb{Z}+1} e^{2\pi i j z} e^{\pi i \tau j^2/2}, \tag{B.5}$$

we get

$$F^{(r,s)}(\tau, z) = \sum_{b=0}^{1} \sum_{j \in 2\mathbb{Z}+b, n \in \mathbb{Z}/N} c_b^{(r,s)}(4n - j^2) e^{2\pi i n \tau + 2\pi i j z} \,. \tag{B.6}$$

$F^{(r,s)}(\tau, 0)$ defined from (B.2) can be regarded as the partition function of the superconformal field theory on a torus with appropriate twisted boundary condition along the a and the b cycles. From this it follows that

$$F^{(r,s)}\left(\frac{a\tau + b}{c\tau + d}, 0\right) = F^{(cs+ar, ds+br)}(\tau, 0) \,. \tag{B.7}$$

[48] At this stage we are describing an abstract conformal field theory without connecting it to string theory. In all cases where we use this conformal field theory to describe a fundamental string world-sheet theory or world-volume theory of some soliton, we shall use the Green-Schwarz formulation. Thus the world-sheet fermion number of this SCFT will represent the space-time fermion number in string theory.

Using (B.3) and the known modular transformation properties of Jacobi ϑ-functions it then follows that

$$F^{(r,s)}\left(\frac{a\tau+b}{c\tau+d},\frac{z}{c\tau+d}\right)=\exp\left(2\pi i\frac{cz^2}{c\tau+d}\right)F^{(cs+ar,ds+br)}(\tau,z)\,. \tag{B.8}$$

The functions $F^{(r,s)}(\tau,z)$ or equivalently the coefficients $c_b^{(r,s)}(u)$ contain information about the spectrum of $\widetilde{g}^r$ twisted $\bar{L}_0=0$ states of the superconformal field theory, carrying definite $\widetilde{g}$, L_0 and F_L quantum numbers. In the rest of this appendix we shall study various properties of these coefficients. First of all, since in the RR sector the L_0 eigenvalue is ≥ 0 for any state, it follows from (B.3)-(B.5) that

$$c_0^{(r,s)}(u)=0 \quad \text{for } u<0, \qquad c_1^{(r,s)}(u)=0 \quad \text{for } u<-1\,. \tag{B.9}$$

Using $\begin{pmatrix} a & b \\ c & d\end{pmatrix}=\begin{pmatrix} -1 & 0 \\ 0 & -1\end{pmatrix}$ in (B.8) we get $F^{(r,s)}(\tau,z)=F^{(-r,-s)}(\tau,-z)$. It then follows from (B.3), (B.6), that

$$h_b^{(r,s)}(\tau)=h_b^{(-r,-s)}(\tau)\,, \qquad c_b^{(r,s)}(u)=c_b^{(-r,-s)}(u)\,, \qquad \text{for } b=0,1\,. \tag{B.10}$$

Furthermore, taking $\begin{pmatrix} a & b \\ c & d\end{pmatrix}=\begin{pmatrix} 1 & 1 \\ 0 & 1\end{pmatrix}$ in (B.8) we get $F^{(r,s)}(\tau+1,z)=F^{(r,s+r)}(\tau,z)$. Since (r,s) are defined modulo N we get $F^{(r,s)}(\tau,z)=F^{(r,s)}(\tau+N,z)$. This is the physical origin of the restriction $n\in\mathbb{Z}/N$ in (B.6) and $k\in\mathbb{Z}/N-b^2/4$ in (B.4).

The $n=0$ terms in the expansion (B.6) is given by the contribution to (B.2) from the RR sector states with $L_0=\bar{L}_0=0$. For $r=0$, i.e. in the untwisted sector, these states are in one to one correspondence with harmonic (p,q) forms on $\mathcal{M}$, with $(p-1)$ and $(q-1)$ measuring the quantum numbers F_L and F_R [248, 249]. Comparing (B.2) with (B.6) we now see that $N\,c_0^{(0,s)}(0)$, being $N\times$ the coefficient of the $n=0$, $j=0$ term in (B.6), measures the number of harmonic $(1,q)$ forms weighted by $(-1)^{q-1}\widetilde{g}^s$ and summed over q, and $N\,c_1^{(0,s)}(-1)$, being $N\times$ the coefficient of the $n=0$, $j=-1$ (or $j=1$) term in (B.6), measures the number of harmonic $(0,q)$ (or $(2,q)$) forms weighted by $(-1)^q\widetilde{g}^s$ and summed over q. If $\mathcal{M}=K3$ then the only $(0,q)$ forms are $(0,0)$ and $(0,2)$ forms both of which are invariant under $\widetilde{g}$. Thus we have

$$c_1^{(0,s)}(-1)=\frac{2}{N} \quad \text{for } \mathcal{M}=K3\,. \tag{B.11}$$

On the other hand for $\mathcal{M}=T^4$ one can represent the explicit action of $\widetilde{g}$ in an appropriate complex coordinate system (z^1,z^2) as

$$dz^1\to e^{2\pi i/N}dz^1, \quad dz^2\to e^{-2\pi i/N}dz^2\,, \quad d\bar{z}^1\to e^{-2\pi i/N}d\bar{z}^1, \quad d\bar{z}^2\to e^{2\pi i/N}d\bar{z}^2\,. \tag{B.12}$$

Using this one can work out its action on all the 2-, 3- and 4-forms:

$$\begin{aligned} &dz^1 \wedge dz^2 \to dz^1 \wedge dz^2, \quad dz^1 \wedge d\bar{z}^1 \to dz^1 \wedge d\bar{z}^1, \quad dz^1 \wedge d\bar{z}^2 \to e^{4\pi i/N}\, dz^1 \wedge d\bar{z}^2, \\ &d\bar{z}^1 \wedge d\bar{z}^2 \to d\bar{z}^1 \wedge d\bar{z}^2, \quad dz^2 \wedge d\bar{z}^2 \to dz^2 \wedge d\bar{z}^2, \quad d\bar{z}^1 \wedge dz^2 \to e^{-4\pi i/N}\, d\bar{z}^1 \wedge dz^2 \,, \end{aligned} \tag{B.13}$$

$$\begin{aligned} &dz^1 \wedge dz^2 \wedge d\bar{z}^1 \to e^{-2\pi i/N}\, dz^1 \wedge dz^2 \wedge d\bar{z}^1, \quad dz^1 \wedge dz^2 \wedge d\bar{z}^2 \to e^{2\pi i/N}\, dz^1 \wedge dz^2 \wedge d\bar{z}^2, \\ &d\bar{z}^1 \wedge d\bar{z}^2 \wedge dz^1 \to e^{2\pi i/N}\, d\bar{z}^1 \wedge d\bar{z}^2 \wedge dz^1, \quad d\bar{z}^1 \wedge d\bar{z}^2 \wedge dz^2 \to e^{-2\pi i/N}\, d\bar{z}^1 \wedge d\bar{z}^2 \wedge dz^2, \end{aligned} \tag{B.14}$$

$$dz^1 \wedge dz^2 \wedge d\bar{z}^1 \wedge d\bar{z}^2 \to dz^1 \wedge dz^2 \wedge d\bar{z}^1 \wedge d\bar{z}^2 \,. \tag{B.15}$$

This shows that the $(0,0)$ and $(0,2)$ forms are invariant under $\widetilde{g}$ but the two $(0,1)$ forms carry $\widetilde{g}$ eigenvalues $e^{\pm 2\pi i/N}$. Thus we have

$$c_1^{(0,s)}(-1) = \frac{1}{N}\left(2 - e^{2\pi i s/N} - e^{-2\pi i s/N}\right) \quad \text{for } \mathcal{M} = T^4 \,. \tag{B.16}$$

(B.13) also shows that $\widetilde{g}$ acts trivially on four of the 2-forms, and acts as a rotation by $4\pi/N$ in the two dimensional subspace spanned by the other two 2-forms. By writing the 2-forms in the real basis one can easily verify that the 2-forms which transform non-trivially under $\widetilde{g}$ correspond to left-handed 2-forms. This is one of the requirements on $\widetilde{g}$ listed at the beginning of this appendix.

Another useful set of results emerges by taking the $z \to 0$ limit of eqs.(B.2) and (B.6). This gives

$$\sum_{b=0}^{1} \sum_{j\in 2\mathbf{Z}+b, n\in \mathbf{Z}/N} c_b^{(r,s)}(4n - j^2) e^{2\pi i n\tau} = \frac{1}{N}\, Q_{r,s} \,, \tag{B.17}$$

where

$$Q_{r,s} = Tr_{RR;\widetilde{g}^r}\left(\widetilde{g}^s (-1)^{F_L+F_R} e^{2\pi i \tau L_0} e^{-2\pi i \bar{\tau} \bar{L}_0}\right), \qquad 0 \le r, s \le N-1 \,. \tag{B.18}$$

$Q_{r,s}$ is independent of τ and $\bar{\tau}$ since the $(-1)^{F_L+F_R}$ insertion in the trace makes the contribution from the $(L_0, \bar{L}_0) \neq (0,0)$ states cancel. Thus (B.17) gives

$$\sum_{b=0}^{1} \sum_{j\in 2\mathbf{Z}+b} c_b^{(r,s)}(4n - j^2) = \frac{1}{N}\, Q_{r,s}\, \delta_{n,0} \,. \tag{B.19}$$

Setting $n = 0$ in the above equation and using eq.(B.9) we get

$$Q_{r,s} = N\, \left(c_0^{(r,s)}(0) + 2\, c_1^{(r,s)}(-1)\right) \,. \tag{B.20}$$

For $r = 0$, i.e. in the untwisted sector, the trace in (B.18) reduces to a sum over the harmonic forms of $\mathcal{M}$. Since $F_L + F_R + 2$ is mapped to the degree of the harmonic form, $Q_{0,s}$ has the interpretation of the trace of $\widetilde{g}^s$ over the even degree harmonic forms of $\mathcal{M}$ minus the trace of $\widetilde{g}^s$ over the odd degree harmonic forms of $\mathcal{M}$. In particular we have

$$Q_{0,0} = \chi(\mathcal{M})\,, \tag{B.21}$$

where $\chi(\mathcal{M})$ denotes the Euler number of $\mathcal{M}$.

So far our definitions of various quantities have been somewhat abstract. For $\mathcal{M} = K3$ and prime values of N ($N = 1$, 2, 3, 5, 7) the functions $F^{(r,s)}(\tau)$ are known explicitly and are given by [11]

$$\begin{aligned}
F^{(0,0)}(\tau,z) &= \frac{8}{N} A(\tau,z)\,, \\
F^{(0,s)}(\tau,z) &= \frac{8}{N(N+1)}\, A(\tau,z) - \frac{2}{N+1}\, B(\tau,z)\, E_N(\tau) \qquad \text{for } 1 \le s \le (N-1)\,, \\
F^{(r,rk)}(\tau,z) &= \frac{8}{N(N+1)}\, A(\tau,z) + \frac{2}{N(N+1)}\, E_N\left(\frac{\tau+k}{N}\right)\, B(\tau,z)\,, \\
&\qquad\qquad \text{for } 1 \le r \le (N-1), \quad 0 \le k \le (N-1)\,,
\end{aligned} \tag{B.22}$$

where

$$A(\tau,z) = \left[\frac{\vartheta_2(\tau,z)^2}{\vartheta_2(\tau,0)^2} + \frac{\vartheta_3(\tau,z)^2}{\vartheta_3(\tau,0)^2} + \frac{\vartheta_4(\tau,z)^2}{\vartheta_4(\tau,0)^2}\right]\,, \tag{B.23}$$

$$B(\tau,z) = \eta(\tau)^{-6} \vartheta_1(\tau,z)^2\,, \tag{B.24}$$

and

$$E_N(\tau) = \frac{12i}{\pi(N-1)}\, \partial_\tau \left[\ln \eta(\tau) - \ln \eta(N\tau)\right] = 1 + \frac{24}{N-1} \sum_{\substack{n_1,n_2 \ge 1 \\ n_1 \ne 0 \, mod \, N}} n_1 e^{2\pi i n_1 n_2 \tau}\,. \tag{B.25}$$

On the other hand for $\mathcal{M} = T^4$ the function $F^{(r,s)}(\tau,z)$ are known for $N = 2, 3$ and are given by [7]

$$\begin{aligned}
F^{(0,s)}(\tau,z) &= \frac{16}{N} \sin^4\left(\frac{\pi s}{N}\right) \frac{\vartheta_1\left(\tau, z + \frac{s}{N}\right) \vartheta_1\left(\tau, -z + \frac{s}{N}\right)}{\vartheta_1\left(\tau, \frac{s}{N}\right)^2} \\
F^{(r,s)}(\tau,z) &= \frac{4\,N}{(N-1)^2} \frac{\vartheta_1\left(\tau, z + \frac{s}{N} + \frac{r}{N}\tau\right) \vartheta_1\left(\tau, -z + \frac{s}{N} + \frac{r}{N}\tau\right)}{\vartheta_1\left(\tau, \frac{s}{N} + \frac{r}{N}\tau\right)^2}\,, \\
&\qquad \text{for } 1 \le r \le N-1,\ 0 \le s \le N-1\,.
\end{aligned} \tag{B.26}$$

Using these results we can compute the coefficients $c_b^{(r,s)}(u)$ and $Q_{r,s}$ for these theories.

C Siegel Modular Forms from Threshold Integrals

In this section we shall prove various properties of the function $\widetilde{\Phi}(\widetilde{\rho}, \widetilde{\sigma}, \widetilde{v})$ defined in (5.1.5) by relating it to a 'threshold integral'. These techniques were first developed in [250,251,252,253] and generalized to the cases under study in [11].

We begin by defining:

$$\Omega = \begin{pmatrix} \rho & v \\ v & \sigma \end{pmatrix} , \tag{C.1}$$

and

$$\begin{aligned} \frac{1}{2}p_R^2 &= \frac{1}{4 \det Im\Omega} | - m_1\rho + m_2 + n_1\sigma + n_2(\sigma\rho - v^2) + jv|^2 , \\ \frac{1}{2}p_L^2 &= \frac{1}{2}p_R^2 + m_1 n_1 + m_2 n_2 + \frac{1}{4}j^2 , \end{aligned} \tag{C.2}$$

where ρ, σ and v are three complex variables. We now consider the 'threshold integrals'

$$\widetilde{\mathcal{I}}(\rho, \sigma, v) = \sum_{r,s=0}^{N-1} \sum_{b=0}^{1} \widetilde{\mathcal{I}}_{r,s,b} , \qquad \widehat{\mathcal{I}}(\rho, \sigma, v) = \sum_{r,s=0}^{N-1} \sum_{b=0}^{1} \widehat{\mathcal{I}}_{r,s,b} , \tag{C.3}$$

where

$$\widetilde{\mathcal{I}}_{r,s,b} = \int_{\mathcal{F}} \frac{d^2\tau}{\tau_2} \left[\sum_{\substack{m_1, m_2, n_2 \in \mathbb{Z} \\ n_1 \in \mathbb{Z} + \frac{r}{N}, j \in 2\mathbb{Z}+b}} q^{p_L^2/2} \bar{q}^{p_R^2/2} e^{2\pi i m_1 s/N} h_b^{(r,s)}(\tau) - \delta_{b,0}\delta_{r,0} c_0^{(0,s)}(0) \right] , \tag{C.4}$$

and

$$\widehat{\mathcal{I}}_{r,s,b} = \int_{\mathcal{F}} \frac{d^2\tau}{\tau_2} \left[\sum_{\substack{m_1, n_1 \in \mathbb{Z}, m_2 \in \mathbb{Z}/N \\ n_2 \in N\mathbb{Z}+r, j \in 2\mathbb{Z}+b}} q^{p_L^2/2} \bar{q}^{p_R^2/2} e^{2\pi i m_2 s} h_b^{(r,s)}(\tau) - \delta_{b,0}\delta_{r,0} c_0^{(0,s)}(0) \right] , \tag{C.5}$$

with

$$q \equiv e^{2\pi i \tau} . \tag{C.6}$$

$\mathcal{F}$ denotes the fundamental region of $SL(2, \mathbb{Z})$ in the upper half plane. The subtraction terms proportional to $c_0^{(0,s)}(0)$ have been chosen so that the integrand vanishes faster than $1/\tau_2$ in the $\tau \to i\infty$ limit, rendering the integral finite. Let us now introduce another set of variables $(\widetilde{\rho}, \widetilde{\sigma}, \widetilde{v})$ related to (ρ, σ, v) via the relations

$$\widetilde{\rho} = \frac{1}{N} \frac{1}{2v - \rho - \sigma}, \qquad \widetilde{\sigma} = N \frac{v^2 - \rho\sigma}{2v - \rho - \sigma}, \qquad \widetilde{v} = \frac{v - \rho}{2v - \rho - \sigma} , \tag{C.7}$$

or equivalently,

$$\rho = \frac{\widetilde{\rho}\widetilde{\sigma} - \widetilde{v}^2}{N\widetilde{\rho}}, \qquad \sigma = \frac{\widetilde{\rho}\widetilde{\sigma} - (\widetilde{v}-1)^2}{N\widetilde{\rho}}, \qquad v = \frac{\widetilde{\rho}\widetilde{\sigma} - \widetilde{v}^2 + \widetilde{v}}{N\widetilde{\rho}} . \tag{C.8}$$

We also define

$$\widetilde{\Omega} = \begin{pmatrix} \widetilde{\rho} & \widetilde{v} \\ \widetilde{v} & \widetilde{\sigma} \end{pmatrix} . \tag{C.9}$$

Then we have the relations:

$$\begin{aligned} & (\det Im\Omega)^{-1} \, | - m_1\rho + m_2 + n_1\sigma + n_2(\sigma\rho - v^2) + jv|^2 \\ = \; & (\det Im\widetilde{\Omega})^{-1} \, | - m_1'\widetilde{\rho} + m_2' + n_1'\widetilde{\sigma} + n_2'(\widetilde{\sigma}\widetilde{\rho} - \widetilde{v}^2) + j'\widetilde{v}|^2 \\ & m_1 n_1 + m_2 n_2 + \frac{1}{4} j^2 = m_1' n_1' + m_2' n_2' + \frac{1}{4} j'^2 \end{aligned} \tag{C.10}$$

where

$$m_1' = m_2 N, \quad m_2' = n_1, \quad n_1' = n_2/N, \quad n_2' = m_1 - n_1 - j, \quad j' = -j - 2n_1 . \tag{C.11}$$

Using these relations and relabeling the indices m_1, m_2, n_1, n_2 in eqs.(C.4)-(C.5) one can easily prove the relations

$$\widehat{\mathcal{I}}(\rho, \sigma, v) = \widetilde{\mathcal{I}}(\widetilde{\rho}, \widetilde{\sigma}, \widetilde{v}) . \tag{C.12}$$

In the same way one can show that under a transformation of the form

$$\Omega \to (A\Omega + B)(C\Omega + D)^{-1} , \tag{C.13}$$

$\widehat{\mathcal{I}}(\rho, \sigma, v)$ remains invariant for the following choices of the matrices A, B, C, D:

$$\begin{aligned} \begin{pmatrix} A & B \\ C & D \end{pmatrix} &= \begin{pmatrix} a & 0 & b & 0 \\ 0 & 1 & 0 & 0 \\ c & 0 & d & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} , \qquad ad - bc = 1, \quad c = 0 \text{ mod } N, \quad a, d = 1 \text{ mod } N \\ \begin{pmatrix} A & B \\ C & D \end{pmatrix} &= \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix} , \\ \begin{pmatrix} A & B \\ C & D \end{pmatrix} &= \begin{pmatrix} 1 & 0 & 0 & \mu \\ \lambda & 1 & \mu & 0 \\ 0 & 0 & 1 & -\lambda \\ 0 & 0 & 0 & 1 \end{pmatrix} , \qquad \lambda, \mu \in \mathbb{Z}. \end{aligned} \tag{C.14}$$

The group of transformations generated by these matrices is a subgroup of the Siegel modular group $Sp(2,\ \mathbb{Z})$; we shall denote this subgroup by $\widehat{G}$. Via eq.(C.12) this also induces a group of symmetry transformations of $\widetilde{\mathcal{I}}(\widetilde{\rho},\widetilde{\sigma},\widetilde{v})$; we shall denote this group by $\widetilde{G}$.

We can now use the modular property of $F^{(r,s)}(\tau,z)$ given in (B.8) to evaluate the integrals $\widetilde{\mathcal{I}}$ and $\widehat{\mathcal{I}}$. We refer the reader to [11] for detailed calculations in a specific example and quote here the final results:

$$\widetilde{\mathcal{I}}(\rho,\sigma,v) = -2\ln\left[(\det\,\mathrm{Im}\Omega)^k\right] - 2\ln\widetilde{\Phi}(\rho,\sigma,v) - 2\ln\bar{\widetilde{\Phi}}(\rho,\sigma,v) + \mathrm{constant} \tag{C.15}$$

and

$$\widehat{\mathcal{I}}(\rho,\sigma,v) = -2\ln\left[(\det\,\mathrm{Im}\Omega)^k\right] - 2\ln\widehat{\Phi}(\rho,\sigma,v) - 2\ln\bar{\widehat{\Phi}}(\rho,\sigma,v) + \mathrm{constant} \tag{C.16}$$

where

$$k = \frac{1}{2}\sum_{s=0}^{N-1} c_0^{(0,s)}(0)\,, \tag{C.17}$$

$$\begin{aligned}\widetilde{\Phi}(\rho,\sigma,v) &= e^{2\pi i(\widetilde{\alpha}\rho+\widetilde{\gamma}\sigma+v)}\\ &\times\prod_{b=0}^{1}\prod_{r=0}^{N-1}\prod_{\substack{k'\in\mathbb{Z}+\frac{r}{N},l\in\mathbb{Z},j\in2\mathbb{Z}+b\\ k',l\geq0,j<0\ \mathrm{for}\ k'=l=0}}\left(1-e^{2\pi i(k'\sigma+l\rho+jv)}\right)^{\sum_{s=0}^{N-1}e^{-2\pi isl/N}c_b^{(r,s)}(4k'l-j^2)}\end{aligned} \tag{C.18}$$

and

$$\begin{aligned}\widehat{\Phi}(\rho,\sigma,v) &= e^{2\pi i\left(\widehat{\alpha}\rho+\widehat{\gamma}\sigma+\widehat{\beta}v\right)}\\ &\prod_{b=0}^{1}\prod_{r=0}^{N-1}\prod_{\substack{(k',l)\in\mathbb{Z},j\in2\mathbb{Z}+b\\ k',l\geq0,j<0\ \mathrm{for}\ k'=l=0}}\left\{1-e^{2\pi ir/N}\,e^{2\pi i(k'\sigma+l\rho+jv)}\right\}^{\sum_{s=0}^{N-1}e^{-2\pi irs/N}c_b^{(0,s)}(4k'l-j^2)}\end{aligned} \tag{C.19}$$

$$\begin{aligned}&\widetilde{\alpha} = \frac{1}{24N}\,Q_{0,0} - \frac{1}{2N}\sum_{s=1}^{N-1}Q_{0,s}\,\frac{e^{-2\pi is/N}}{(1-e^{-2\pi is/N})^2}\,, \qquad \widetilde{\gamma} = \frac{1}{24N}\,Q_{0,0} = \frac{1}{24N}\,\chi(\mathcal{M}),\\ &\widehat{\alpha} = \widehat{\beta} = \widehat{\gamma} = \frac{1}{24}\,Q_{0,0} = \frac{1}{24}\,\chi(\mathcal{M})\,.\end{aligned} \tag{C.20}$$

The quantities $Q_{r,s}$ have been given in terms of the coefficients $c^{(r,s)}(u)$ in (B.20). In arriving at (C.18),(C.19) one needs to use the relations (B.10), (B.19), (B.21) and also (B.11), (B.16). Since

$Nc_0^{(0,s)}(0)$ measures the number of harmonic $(1,q)$-forms weighted by $(-1)^q \widetilde{g}^s$ and summed over q, the constant k defined in (C.17) has the interpretation of being half the number of $\widetilde{g}$ invariant $(1,q)$ forms weighted by $(-1)^{q+1}$ and summer over q. Since both for $K3$ and T^4 the only $\widetilde{g}$ invariant harmonic $(1,q)$ forms are (1,1) forms (see eqs.(B.12)-(B.15)), k can be regarded as half the number of $\widetilde{g}$ invariant harmonic (1,1) forms on $\mathcal{M}$.

It follows from (C.12), (C.15) and (C.16) that

$$\widetilde{\Phi}(\widetilde{\rho},\widetilde{\sigma},\widetilde{v}) = -(i)^k\, C_1\,(2v-\rho-\sigma)^k\,\widehat{\Phi}(\rho,\sigma,v) \tag{C.21}$$

where C_1 is a constant. The factor of $-(i)^k$ has been included to ensure that C_1 is real and positive. To see this we can consider the case where ρ, σ and v are all purely imaginary, with $|v_2| << \rho_2, \sigma_2$. In this case $\widehat{\Phi}(\rho,\sigma,v)$ defined in (C.19) is real and positive. On the other hand from (C.7) we get $\widetilde{\rho}$ and $\widetilde{\sigma}$ purely imaginary and $\widetilde{v}$ real. Eq.(C.18), together with the relation $\sum_{s=0}^{N-1} c_1^{(0,s)}(-1) = 2$ then tells us that $\widetilde{\Phi}(\widetilde{\rho},\widetilde{\sigma},\widetilde{v})$ is real and negative. Hence in order to satisfy (C.21) we must have C_1 real and positive. The magnitude of C_1 can be calculated by carefully evaluating the constants in eqs.(C.15), (C.16) but we shall not do it here.

Furthermore given the invariance of $\widetilde{\mathcal{I}}$ and $\widehat{\mathcal{I}}$ under the groups $\widetilde{G}$ and $\widehat{G}$, it follows from eqs.(C.15), (C.16) that

$$\widetilde{\Phi}((A\Omega+B)(C\Omega+D)^{-1}) = \det(C\Omega+D)^k\,\widetilde{\Phi}(\Omega) \quad \text{for} \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \widetilde{G}\,, \tag{C.22}$$

and

$$\widehat{\Phi}((A\Omega+B)(C\Omega+D)^{-1}) = \det(C\Omega+D)^k\,\widehat{\Phi}(\Omega) \quad \text{for} \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \widehat{G}\,. \tag{C.23}$$

In principle these transformation laws could have arbitrary phases but one can show that the phases are trivial. Thus $\widetilde{\Phi}$ and $\widehat{\Phi}$ transform as modular forms of weight k under the groups $\widetilde{G}$ and $\widehat{G}$ respectively. As special cases of (C.22), we have

$$\widetilde{\Phi}(\widetilde{\rho}+1,\widetilde{\sigma},\widetilde{v}) = \widetilde{\Phi}(\widetilde{\rho},\widetilde{\sigma}+N,\widetilde{v}) = \widetilde{\Phi}(\widetilde{\rho},\widetilde{\sigma},\widetilde{v}+1) = \widetilde{\Phi}(\widetilde{\rho},\widetilde{\sigma},\widetilde{v})\,. \tag{C.24}$$

This also follows from (C.18) and the integrality of $\widetilde{\alpha}$ and $N\widetilde{\gamma}$. Integrality of $N\widetilde{\gamma}$ is manifest from (C.20) and that of $\widetilde{\alpha}$ has been argued below (5.2.12).

From (C.18), (C.19), (B.19), (B.20), (B.11) and (B.16) it is easy to see that for small v

$$\widehat{\Phi}(\rho,\sigma,v) = -4\pi^2\,v^2\,g(\rho)\,g(\sigma) + \mathcal{O}(v^4) \tag{C.25}$$

$$\widetilde{\Phi}(\rho,\sigma,v) = -4\pi^2\,v^2\,f_1(N\rho)f_2(\sigma/N) + \mathcal{O}(v^4)\,, \tag{C.26}$$

where

$$g(\rho) = e^{2\pi i\widehat{\alpha}\rho} \prod_{n=1}^{\infty} \prod_{r=0}^{N-1} \left(1 - e^{2\pi i r/N} e^{2\pi i n\rho}\right)^{s_r} , \tag{C.27}$$

$$f_1(N\rho) = e^{2\pi i\widetilde{\alpha}\rho} \prod_{l=1}^{\infty} (1 - e^{2\pi i l\rho})^{s_l} , \tag{C.28}$$

$$f_2(\sigma/N) = e^{2\pi i\widetilde{\gamma}\sigma} \prod_{r=0}^{N-1} \prod_{\substack{k' \in \mathbb{Z}+r/N \\ k'>0}} (1 - e^{2\pi i k'\sigma})^{t_r} , \tag{C.29}$$

$$s_r = \frac{1}{N} \sum_{s'=0}^{N-1} e^{-2\pi i r s'/N} \, Q_{0,s'} = \sum_{s'=0}^{N-1} e^{-2\pi i r s'/N} \, \left(c_0^{(0,s')}(0) + 2c_1^{(0,s')}(-1)\right) , \tag{C.30}$$

$$t_r = \frac{1}{N} \sum_{s=0}^{N-1} Q_{r,s} = \sum_{s=0}^{N-1} \left(c_0^{(r,s)}(0) + 2c_1^{(r,s)}(-1)\right) . \tag{C.31}$$

Eq.(C.21) then gives, for small v, i.e. small $\widetilde{\rho}\widetilde{\sigma} - \widetilde{v}^2 + \widetilde{v}$,

$$\widetilde{\Phi}(\widetilde{\rho}, \widetilde{\sigma}, \widetilde{v}) = -4\pi^2 \, C_1 \, (2v - \rho - \sigma)^k \, v^2 g(\rho) \, g(\sigma) + \mathcal{O}(v^4) \, . \tag{C.32}$$

Since $Q_{0,s}$ is the trace of $(-1)^p \widetilde{g}^s$ over the harmonic p-forms of $\mathcal{M}$, s_r has the interpretation of being the number of harmonic p-forms in $\mathcal{M}$ with $\widetilde{g}$ eigenvalue $e^{2\pi i r/N}$ weighted by $(-1)^p$. Thus it is an integer. On the other hand it follows from the definition (B.18) of $Q_{r,s}$ that b_r is the number of $\widetilde{g}$ invariant, $\widetilde{g}^r$ twisted state. Thus it is also an integer.

$(f_1(N\widetilde{\rho}))^{-1}$ computed from (C.28) coincides with the partition function (5.2.13) of a single Kaluza-Klein monopole in the first description. Since this corresponds to an elementary twisted sector string in the second description, we see that $(f_1(\tau))^{-1}$ can be interpreted as the partition function of purely electrically charged twisted sector states in the second description. On the other hand $(f_2(\widetilde{\sigma}/N))^{-1}$ coincides with the $l = 0$ term in (5.2.36) with $\widetilde{v} = 0$. This leads to the conclusion that $(f_2(\widetilde{\sigma}/N))^{-1}$ can be interpreted as the partition function of the D1-D5 system in the absence of Kaluza-Klein monopole, with arbitrary angular momentum, zero momentum along $\widetilde{S}^1$ and zero momentum along S^1.[49] Since in the second description this gets mapped to a purely magnetically charged half-BPS state, we conclude that $(f_2(\tau))^{-1}$ describes the partition function of purely magnetically charged half-BPS states in the second description.

[49]Note that in the absence of the Kaluza-Klein monopole the momentum along $\widetilde{S}^1$ can no longer be identified with angular momentum.

Using

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} a & 0 & b & 0 \\ 0 & 1 & 0 & 0 \\ c & 0 & d & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \subset \widehat{G},$$

$$ad - bc = 1, \quad a, b, c, d \in \mathbb{Z}, \quad a, d = 1 \bmod N, \quad c = 0 \bmod N, \tag{C.33}$$

in (C.23), and eq.(C.25), one can show that

$$g((a\rho + b)(c\rho + d)^{-1}) = (c\rho + d)^{k+2} g(\rho)\,. \tag{C.34}$$

Thus $g(\rho)$ transforms as a modular form of weight $(k + 2)$ under $\Gamma_1(N)$. The behaviour of $g(\rho)$ for large ρ_2 is governed by the constant $\widehat{\alpha}$ defined in (C.20).

For $\mathcal{M} = K3$ and prime values of N ($N = 1,\ 2,\ 3,\ 5,\ 7$) we can use (B.22) to find the explicit expressions for $g(\tau)$ and k [11]:

$$g(\tau) = \eta(\tau)^{k+2}\, \eta(N\tau)^{k+2}\,, \tag{C.35}$$

and

$$k = \frac{24}{N + 1} - 2\,. \tag{C.36}$$

On the other hand for $\mathcal{M} = T^4$ and $N = 2, 3$ we get, using (B.26) [7],

$$g(\tau) = \eta(\tau)^{2N(k+2)/(N-1)} \eta(N\tau)^{-2(k+2)/(N-1)}\,, \tag{C.37}$$

and

$$k = \frac{12}{N + 1} - 2\,. \tag{C.38}$$

For $\mathcal{M} = K3$ and $N = 1$, the function $\widetilde{\Phi}$ constructed in this appendix is the well known weight 10 cusp form of the genus two Siegel modular group [254, 255, 256, 257]. For $\mathcal{M} = K3$ and $N = 2, 3$ the function $\widetilde{\Phi}$ was found in [258, 259, 260]. A general discussion on construction of Siegel modular forms can be found in [261]. Different ways of constructing the same functions $\widetilde{\Phi}$ can be found in [262, 263, 10, 12].

D Zeroes and Poles of $\widetilde{\Phi}$

In this appendix we shalll determine the zeroes and poles of the function $\widetilde{\Phi}(\widetilde{\rho}, \widetilde{\sigma}, \widetilde{v})$ introduced in appendix C. Via eq.(C.21) this also determines the zeroes and poles of the function $\widehat{\Phi}(\rho, \sigma, v)$. The zeroes of $\widetilde{\Phi}$ found in (C.26) and (C.32) will be special cases of the general set of zeroes we shall find.

Using (C.3), (C.4), (C.15) we see that the Siegel modular form $\widetilde{\Phi}(\widetilde{\rho}, \widetilde{\sigma}, \widetilde{v})$ satisfies the relation:

$$
\begin{aligned}
& -2 \ln \widetilde{\Phi}(\widetilde{\rho}, \widetilde{\sigma}, \widetilde{v}) - 2 \ln \bar{\widetilde{\Phi}}(\widetilde{\rho}, \widetilde{\sigma}, \widetilde{v}) - 2\, k\, \ln \det Im\widetilde{\Omega} + \text{constant} \\
= \ & \int_{\mathcal{F}} \frac{d^2\tau}{\tau_2} \Bigg[\sum_{r,s=0}^{N-1} \sum_{b=0}^{1} \sum_{\substack{m_1, m_2, n_2 \in \mathbb{Z}, \\ n_1 \in \mathbb{Z} + \frac{r}{N}, j \in 2\mathbb{Z}+b}} \exp\left[2\pi i \tau (m_1 n_1 + m_2 n_2 + \frac{j^2}{4}) \right] \times \\
& \exp\left(\frac{-\pi\tau_2}{\widetilde{Y}} \left| n_2 (\widetilde{\sigma}\widetilde{\rho} - \widetilde{v}^2) + j\widetilde{v} + n_1 \widetilde{\sigma} - m_1 \widetilde{\rho} + m_2 \right|^2 \right) e^{2\pi i m_1 s/N}\, h_b^{(r,s)}(\tau) \\
& - \sum_{s=0}^{N-1} c_0^{(0,s)}(0) \Bigg] ,
\end{aligned} \tag{D.1}
$$

where

$$
\widetilde{\Omega} = \begin{pmatrix} \widetilde{\sigma} & \widetilde{v} \\ \widetilde{v} & \widetilde{\rho} \end{pmatrix} , \qquad \widetilde{Y} = \det Im\widetilde{\Omega} . \tag{D.2}
$$

Eq.(D.1) shows that the zeroes and poles of $\widetilde{\Phi}$ appear only when the τ integral on the right hand side of this equation diverges from the region near $\tau = i\infty$. Now, if we consider a term proportional to $e^{2\pi i n \tau}$ in the expansion of $h_b^{(r,s)}$, then the τ_1 dependent term in the integrand is of the form $\exp\left(2\pi i \tau_1 (n + m_1 n_1 + m_2 n_2 + \frac{1}{4} j^2)\right)$. Since for large τ_2 the τ_1 integral runs from $-\frac{1}{2}$ to $\frac{1}{2}$, it gives a non-vanishing answer only if

$$
n + m_1 n_1 + m_2 n_2 + \frac{j^2}{4} = 0 . \tag{D.3}
$$

Thus after performing the τ_1 integral, the only τ_2 dependence of the integrand in the large τ_2 region comes from the

$$
\exp\left[-\frac{\pi\tau_2}{\widetilde{Y}} \left| n_2 (\widetilde{\sigma}\widetilde{\rho} - \widetilde{v}^2) + j\widetilde{v} + n_1 \widetilde{\sigma} - \widetilde{\rho} m_1 + m_2 \right|^2 \right] \tag{D.4}
$$

factor. As long as the coefficient of τ_2 in the exponent is non-zero the integrand is exponentially suppressed for large τ_2 and as a result the integral is convergent. Thus the only way the integral can diverge from the large τ_2 region is if this vanishes:

$$
n_2 (\widetilde{\sigma}\widetilde{\rho} - \widetilde{v}^2) + j\widetilde{v} + n_1 \widetilde{\sigma} - \widetilde{\rho} m_1 + m_2 = 0 , \tag{D.5}
$$

for some m_1, m_2, n_1, n_2, j appearing in the sum in (D.1).

Now we have the identity

$$
m_1 n_1 + m_2 n_2 + \frac{j^2}{4} = \frac{1}{2} (p_L^2 - p_R^2) , \tag{D.6}
$$

where

$$\begin{aligned}
p_R^2 &= \frac{1}{2\widetilde{Y}} \left| n_2(\widetilde{\sigma}\widetilde{\rho} - \widetilde{v}^2) + j\widetilde{v} + n_1\widetilde{\sigma} - \widetilde{\rho}m_1 + m_2 \right|^2 , \\
p_L^2 &= \frac{1}{2\widetilde{Y}} \left\{ m_2 + n_2(\widetilde{\sigma}_2\widetilde{\rho}_2 + \widetilde{\sigma}_1\widetilde{\rho}_1 - \widetilde{v}_1^2 - \widetilde{v}_2^2) - m_1\widetilde{\rho}_1 + n_1\widetilde{\sigma}_1 + j\widetilde{v}_1 \right\}^2 \\
&\quad + \frac{1}{2\widetilde{Y}} \left\{ n_2 \left(\widetilde{\sigma}_1\widetilde{\rho}_1 - \widetilde{\sigma}_1\widetilde{\rho}_2 + 2\widetilde{v}_1\widetilde{v}_2 - \frac{2\widetilde{v}_2^2\widetilde{\rho}_1}{\widetilde{\rho}_2} \right) + m_1\widetilde{\rho}_2 + n_1 \left(\widetilde{\sigma}_2 - \frac{2\widetilde{v}_2^2}{\widetilde{\rho}_2} \right) - j\widetilde{v}_2 \right\}^2 \\
&\quad + 2 \left\{ \frac{j}{2} + n_1 \frac{\widetilde{v}_2}{\widetilde{\rho}_2} - n_2\widetilde{v}_1 + n_2 \frac{\widetilde{v}_2\widetilde{\rho}_1}{\widetilde{\rho}_2} \right\}^2 .
\end{aligned} \tag{D.7}$$

Since p_L^2 is positive semi-definite, and since p_R^2 vanishes when (D.5) holds, (D.6) shows that we must have

$$m_1 n_1 + m_2 n_2 + \frac{j^2}{4} \geq 0 . \tag{D.8}$$

Furthermore the equality sign holds only when p_L^2 also vanishes. This requires $m_1 = m_2 = n_1 = n_2 = j = 0$. The corresponding divergence is present for all $\widetilde{\sigma}$, $\widetilde{\rho}$, $\widetilde{v}$ and is removed by the subtraction term proportional to $\sum_s c_0^{(0,s)}(0)$ in (D.1). Thus the divergences which depend on $\widetilde{\sigma}$, $\widetilde{\rho}$, $\widetilde{v}$ come from those values of m_i, n_i, j which satisfy (D.5) and for which

$$m_1 n_1 + m_2 n_2 + \frac{j^2}{4} > 0 . \tag{D.9}$$

This, together with eq.(D.3), now show that we must have

$$n < 0 . \tag{D.10}$$

In other words the only terms in the expansion of $h_b^{(r,s)}$ responsible for a divergent contribution to the integral (D.1) are the ones involving negative powers of $e^{2\pi i \tau}$. Eqs.(B.4), (B.9) now imply that the divergent contribution comes from terms proportional to $c_1^{(r,s)}(-1)$ for all N, and $c_1^{(r,s)}(-1 + \frac{4p}{N})$ ($p \in \mathbb{Z}$, $\frac{N}{4} > p \geq 1$) for $N \geq 5$. The corresponding values of n are $-\frac{1}{4}$ and $-\frac{1}{4} + \frac{p}{N}$ respectively.

First consider the contribution from the $c_1^{(r,s)}(-1)$ term. Putting $n = -1/4$ in (D.3) we get

$$m_1 n_1 + m_2 n_2 + \frac{j^2}{4} = \frac{1}{4} . \tag{D.11}$$

By estimating the τ_2 integral in the right hand side of (D.1) for $n_2(\widetilde{\sigma}\widetilde{\rho} - \widetilde{v}^2) + j\widetilde{v} + n_1\widetilde{\sigma} - \widetilde{\rho}m_1 + m_2 \simeq 0$, one easily finds that the divergent contribution is given by

$$\begin{aligned}
&- 2 \sum_{s=0}^{N-1} e^{2\pi i m_1 s/N} \, c_1^{(r,s)}(-1) \, \ln \left| n_2(\widetilde{\sigma}\widetilde{\rho} - \widetilde{v}^2) + j\widetilde{v} + n_1\widetilde{\sigma} - \widetilde{\rho}m_1 + m_2 \right|^2 , \\
&\qquad\qquad r = n_1 N \bmod N, \; j = 1 \bmod 2 ,
\end{aligned} \tag{D.12}$$

where we have included a factor of 2 due to the fact that the lattice vectors $(\vec{m}, \vec{n}, j)$ and $(-\vec{m}, -\vec{n}, -j)$ give identical divergent contribution. Comparing this with the left-hand side of (D.1) we see that near this region $\widetilde{\Phi}$ behaves as

$$\begin{aligned}
&\widetilde{\Phi} \sim \left(n_2(\widetilde{\sigma}\widetilde{\rho} - \widetilde{v}^2) + j\widetilde{v} + n_1\widetilde{\sigma} - \widetilde{\rho}m_1 + m_2\right)^{\sum_{s=0}^{N-1} e^{2\pi i m_1 s/N} c_1^{(r,s)}(-1)}, \\
&m_1, m_2, n_2 \in \mathbb{Z}, \quad j \in 2\,\mathbb{Z}+1, \quad n_1 \in \mathbb{Z} + \frac{r}{N}, \quad m_1 n_1 + m_2 n_2 + \frac{j^2}{4} = \frac{1}{4}\,.
\end{aligned} \tag{D.13}$$

For $N \geq 5$ we also have divergent contribution to (D.1) from the $c_1^{(r,s)}(-1 + \frac{4p}{N})$ term. In this case (D.3) gives

$$m_1 n_1 + m_2 n_2 + \frac{j^2}{4} = \frac{1}{4} - \frac{p}{N}\,. \tag{D.14}$$

The divergent contribution takes the form

$$\begin{aligned}
&-2\sum_{s=0}^{N-1} e^{2\pi i m_1 s/N}\, c_1^{(r,s)}(-1 + \frac{4p}{N})\, \ln\left|n_2(\widetilde{\sigma}\widetilde{\rho} - \widetilde{v}^2) + j\widetilde{v} + n_1\widetilde{\sigma} - \widetilde{\rho}m_1 + m_2\right|^2, \\
&\qquad\qquad\qquad\qquad r = n_1 N \bmod N,\ j = 1 \bmod 2\,.
\end{aligned} \tag{D.15}$$

Thus $\widetilde{\Phi}$ behaves as

$$\begin{aligned}
&\widetilde{\Phi} \sim \left(n_2(\widetilde{\sigma}\widetilde{\rho} - \widetilde{v}^2) + j\widetilde{v} + n_1\widetilde{\sigma} - \widetilde{\rho}m_1 + m_2\right)^{\sum_{s=0}^{N-1} e^{2\pi i m_1 s/N} c_1^{(r,s)}(-1+\frac{4p}{N})}, \\
&m_1, m_2, n_2 \in \mathbb{Z}, \quad j \in 2\,\mathbb{Z}+1, \quad n_1 \in \mathbb{Z} + \frac{r}{N}, \quad m_1 n_1 + m_2 n_2 + \frac{j^2}{4} = \frac{1}{4} - \frac{p}{N}\,.
\end{aligned} \tag{D.16}$$

From (B.2), (B.6) it follows that the exponent in (D.13) has the interpretation as the number of $\widetilde{g}^r$ twisted states with $\widetilde{g}$ eigenvalue $e^{-2\pi i m_1/N}$, $F_L = 1$ (or $F_L = -1$) and $L_0 = \bar{L}_0 = 0$, weighted by $(-1)^{F_L+F_R}$. On the other hand the exponent in (D.16) has the interpretation as the number of $\widetilde{g}^r$ twisted states with $\widetilde{g}$ eigenvalue $e^{-2\pi i m_1/N}$, $F_L = 1$ (or $F_L = -1$), $L_0 = p/N$ and $\bar{L}_0 = 0$, weighted by $(-1)^{F_L+F_R}$. Thus both numbers are integers.

For the analysis in §5.6 we need to know which exponents are positive, corresponding to the zeroes of $\widetilde{\Phi}$, and which exponents are negative, corresponding to the poles of $\widetilde{\Phi}$. First consider the case $r = 0$, i.e. $n_1 \in \mathbb{Z}$. In this case using eqs.(B.11) we see that the exponent in (D.13) for $\mathcal{M} = K3$ is given by

$$\sum_{s=0}^{N-1} e^{2\pi i m_1 s/N}\, c_1^{(0,s)}(-1) = \begin{cases} 2 \text{ for } m_1 \in N\,\mathbb{Z} \\ 0 \text{ otherwise} \end{cases}. \tag{D.17}$$

On the other hand using eq.(B.16) we see that for $\mathcal{M} = T^4$ the exponent in (D.13) for $\mathcal{M} = T^4$ is given by

$$\sum_{s=0}^{N-1} e^{2\pi i m_1 s/N} \, c_1^{(0,s)}(-1) = \begin{cases} 2 \text{ for } m_1 \in N\,\mathbb{Z} \\ -1 \text{ for } m_1 \in N\,\mathbb{Z} \pm 1 \\ 0 \text{ otherwise} \end{cases} . \tag{D.18}$$

In the special case of $N = 2$, the sets $N\,\mathbb{Z} \pm 1$ coincide, and the exponent becomes equal to -2 instead of -1 for $m_1 \in N\,\mathbb{Z} \pm 1$.

Since for $r = 0$, $m_1, n_1, m_2, n_2 \in \mathbb{Z}$, and $j \in 2\,\mathbb{Z} + 1$, the only way to satisfy (D.14) is to take $p = 0$. Thus there are no zeroes or poles of the type given in (D.16) with $p \neq 0$.

The zeroes and poles originating in the $r \neq 0$ mod N sector are more difficult to evaluate since these require counting twisted sector states with specific $\widetilde{g}$ quantum numbers. However one can extract some general information by noting that since the coefficients $c_b^{(r,s)}(4n)$ do not depend on the shape and size of $\mathcal{M}$, we can compute them by taking the size of $\mathcal{M}$ to be large so that near any fixed point of $\widetilde{g}$ the orbifold can be regarded as that of $\mathbf{R}^4$. Thus the contribution from a given twisted sector associated with a given fixed point can be computed in a free super-conformal field theory. Locally the action of $\widetilde{g}$ may be represented as rotation by $2\pi/N$ in one two dimensional plane and rotation by $-2\pi/N$ in an orthogonal two dimensional plane. Thus in a twisted RR sector, all the bosons and fermions will be twisted and there are no zero modes. As a result the ground state with $L_0 = \bar{L}_0 = 0$ is unique, carrying $F_L = F_R = 0$. Even after we apply left-moving oscillators to create excited BPS states, these states will continue to have $F_R = 0$. Now since the computation of the exponents in (D.13), (D.16) involves counting BPS states with $F_L = 1$ (or -1), the weight factor $(-1)^{F_L + F_R}$ is given by $-(-1)^{F_R} = -1$ for each of these states. Thus the exponents in (D.13) or (D.16) are always negative for $r \neq 0$ mod N. These correspond to poles of $\widetilde{\Phi}$ rather than zeroes.

The net conclusion of this analysis is that both for $\mathcal{M} = K3$ and $\mathcal{M} = T^4$, the only zeroes of $\widetilde{\Phi}(\widetilde{\rho}, \widetilde{\sigma}, \widetilde{v})$ are of the form:

$$\begin{aligned} &\widetilde{\Phi} \sim \left(n_2(\widetilde{\sigma}\widetilde{\rho} - \widetilde{v}^2) + j\widetilde{v} + n_1\widetilde{\sigma} - \widetilde{\rho} m_1 + m_2\right)^2 , \\ &m_1 \in N\,\mathbb{Z}, \quad n_1, m_2, n_2 \in \mathbb{Z}, \quad j \in 2\,\mathbb{Z} + 1, \quad m_1 n_1 + m_2 n_2 + \frac{j^2}{4} = \frac{1}{4} . \end{aligned} \tag{D.19}$$

The rest are poles.

For $\mathcal{M} = K3$ and prime values of N ($N = 1, 2, 3, 5, 7$) we can use eq.(B.22) to explicitly compute the exponents appearing in (D.13) and (D.16). We get [6]

$$\sum_{s=0}^{N-1} e^{2\pi i m_1 s/N} \, c_1^{(r,s)}(-1) = \begin{cases} 2 & \text{for } r = 0 \text{ mod } N, \, m_1 = 0 \text{ mod } N \\ 0 & \text{otherwise} \end{cases} , \tag{D.20}$$

$$\sum_{s=0}^{N-1} e^{2\pi i m_1 s/N}\, c_1^{(r,s)}(-1+\frac{4}{N}) = \begin{cases} -48/(N^2-1) & \text{for } m_1 r = -1 \bmod N \\ 0 & \text{otherwise} \end{cases} . \tag{D.21}$$

For $\mathcal{M} = T^4$ and $N = 2, 3$ we can use eq.(B.26) to explicitly compute the exponents appearing in (D.13) and (D.16). The result is [7]

$$\sum_{s=0}^{N-1} e^{2\pi i m_1 s/N}\, c_1^{(r,s)}(-1) = \begin{cases} 2 & \text{for } r = 0 \bmod N,\ m_1 = 0 \bmod N \\ -\frac{2}{N-1} & \text{for } r = 0 \bmod N,\ m_1 = \pm 1 \bmod N \\ 0 & \text{otherwise} \end{cases} . \tag{D.22}$$

E The Case of Multiple D5 branes

In this appendix we extend the counting of states associated with the relative motion of the D1-D5 system to the case when the number of D5-branes is $Q_5 \geq 1$. We shall restrict our analysis to the case when $\mathcal{M} = K3$ and follow the analysis given in [6].

We shall choose the world-volume space coordinate σ along each of the D1-branes to coincide with the coordinate along S^1. Due to the $\mathbb{Z}_N$ orbifolding each D1-brane satisfies a twisted boundary condition, – under $\sigma \to \sigma + 2\pi$ its location along $K3$ must get transformed by $\widetilde{g}$. We shall first ignore the effect of this twist and pretend that the D1-brane satisfies periodic boundary condition under $\sigma \to \sigma + 2\pi$, and later take into account the effect of the twist.

The dynamics of the relative motion of Q_5 D5-branes wrapped on $K3 \times S^1$ and Q_1 D1-branes wrapped on S^1 is captured by the $\mathcal{N} = (4, 4)$ superconformal σ-model with the symmetric product of $W = Q_5(Q_1 - Q_5) + 1$ copies of $K3$ as the target space as long as Q_5 and Q_1 do not have a common factor [264]. We shall denote this target space by $S^W K3 \equiv (K3)^W / S_W$, where S_W refers to the permutation group of W elements. The world-sheet coordinate σ of this conformal field theory is identified with the coordinate along S^1. We shall first review various aspects of the superconformal field theory with target space $(K3)^W / S_W$ [163], and then discuss the effect of the $\mathbb{Z}_N$ twist that is required to describe a D1-D5-brane configuration on the CHL orbifold.

Let g be an element of S_W and $[g]$ denote the conjugacy class of g. Then the Hilbert space of the SCFT with target space $(K3)^W / S_W$ decomposes into a direct sum of twisted sectors labelled by the conjugacy classes of S_W:

$$\mathcal{H} = \oplus_{[g]} \mathcal{H}_g^{(\mathcal{C}_g)} \tag{E.1}$$

where $\mathcal{C}_g$ denotes the centralizer of $[g]$ and $\mathcal{H}_g^{(\mathcal{C}_g)}$ refers to the Hilbert space in the g twisted sector projected by $\mathcal{C}_g$. The conjugacy classes of S_W may be labelled as

$$[g] = (1)^{P_1} (2)^{P_2} \cdots (s)^{P_s} \tag{E.2}$$

where (w) denotes cyclic permutation of w elements and P_w is the number of copies of (w) in g. Thus these conjugacy classes are characterized by partitions P_w of W such that

$$\sum_w wP_w = W\,. \tag{E.3}$$

The centralizer $\mathcal{C}_g$ of the conjugacy class $[g]$ given in (E.2) is given by

$$\mathcal{C}_g = S_{P_1} \times (S_{P_2} \times \mathbb{Z}_2^{P_2}) \times \cdots \times (S_{P_s} \times \mathbb{Z}_s^{P_s})\,. \tag{E.4}$$

Let us denote by $\mathcal{H}_w$ the Hilbert space of states twisted by the generator ω of the $\mathbb{Z}_w$ group of cyclic permutation of w elements, and projected by the same $\mathbb{Z}_w$ group. Then (E.4) shows that for the conjugacy class $[g]$ given in (E.2)

$$\mathcal{H}_g^{(\mathcal{C}_g)} = \otimes_{w>0} S^{P_w}\mathcal{H}_w \tag{E.5}$$

Consider first the Hilbert space $\mathcal{H}_w$. This twisted sector is represented by the Hilbert space of the sigma model of w coordinate fields $X_i(\sigma) \in K3$ with the cyclic boundary condition

$$X_i(\sigma + 2\pi) = \omega X_i(\sigma) = X_{i+1}(\sigma), \quad i \in (1, \ldots, w)\,, \tag{E.6}$$

where ω acts by $\omega : X_i \to X_{i+1}$. Therefore the w coordinate fields can be glued together as a single field but in the interval $0 \le \sigma \le 2\pi w$, moving in the target space K3. Thus we now have a string of length $2\pi w$, – commonly known as the long string, – moving in K3. Whereas for $Q_5 = 1$ the quantum number w can be identified with the winding charge of the D-string, this is not so for $Q_5 > 1$. Thus we should not regard the long string as a D-string, – rather it provides some effective description of the dynamics.

Once we know the spectrum of $\mathcal{H}_w$, – which can be found from the spectrum of an SCFT with target space $K3$ after a rescaling of the L_0 and $\bar{L}_0$ eigenvalues by $1/w$ to take into account the effect of the length of the string, – the full spectrum of the CFT of the D1-D5 system is obtained by taking the direct product of the spectrum of $\mathcal{H}_w$'s and then carrying out appropriate symmetrization described in (E.5).

We now turn to the effect of the $\mathbb{Z}_N$ twisted boundary condition that is required in order to get a state of the D1-D5 system in the $\mathbb{Z}_N$ CHL model. For this we need to change (E.6) to $X_i(\sigma + 2\pi) = \widetilde{g}X_{i+1}(\sigma)$. Thus effectively we modify the generator of $\mathbb{Z}_w$ by an additional $\widetilde{g}$ transformation leaving unchanged the rest of the analysis. Since the long string has length $2\pi w$, as we go once around the long string the boundary condition is twisted by $\widetilde{g}^w = \widetilde{g}^r$ where $r = w$ mod N. Let us denote by $\mathcal{H}'_w$ the Hilbert space of states of the long string with $\widetilde{g}^r$ twisted boundary condition, and projected by the new $\mathbb{Z}_w$ group. Then the full Hilbert space of the D1-D5 system will be obtained simply by replacing $\mathcal{H}_w$ by $\mathcal{H}'_w$ in (E.5):

$$\otimes_{w>0} S^{P_w}\mathcal{H}'_w \,. \tag{E.7}$$

Clearly $\mathcal{H}'_w$ can be identified with the Hilbert space of $\widetilde{g}^r$ twisted states in the SCFT described in appendix B. Since the string has length $2\pi w$, a physical momentum $-l/N$ along S^1 would correspond to $L_0 - \bar{L}_0$ eigenvalue of lw/N in this SCFT. Since supersymmetry requires $\bar{L}_0$ to vanish, we have $L_0 = lw/N$. Let F_L denote the left-moving world-sheet fermion number of this SCFT. By the standard argument, the presence of Kaluza-Klein monopole background converts F_L eigenvalues into momenta along $\widetilde{S}^1$. Since the projection operator for $\mathbb{Z}_N$ invariant states with physical momentum $-l/N$ along S^1 is

$$\frac{1}{N}\sum_s e^{-2\pi i l s/N}\widetilde{g}^s \,, \tag{E.8}$$

the total number of bosonic minus fermionic states in the single long string Hilbert space, carrying momentum $-l/N$ along S^1 and momentum j along $\widetilde{S}^1$ is given by:

$$n(w,l,j) = \frac{1}{N}\sum_s e^{-2\pi i l s/N} Tr_{RR;\widetilde{g}^r}\big(\widetilde{g}^s(-1)^{F_L+F_R}\delta_{NL_0,lw}\delta_{F_L,j}\big)\,. \tag{E.9}$$

Using (5.2.32) this may be written as

$$n(w,l,j) = \sum_s e^{-2\pi i l s/N} c_b^{(r,s)}(4lw/N - j^2), \quad b = j \text{ mod } 2\,. \tag{E.10}$$

According to (E.7) the next step is the evaluation of the partition function for the symmetrized tensor products of the Hilbert spaces $\mathcal{H}'_w$. For this we use the following formula from [163]. If $d_{sym}(P_w, w, L, J')$ denotes the number of bosonic minus fermionic states in $S^{P_w}\mathcal{H}'_w$ carrying total momentum $-L/N$ along S^1 and total momentum J' along $\widetilde{S}^1$, then

$$\sum_{P_w=0}^{\infty}\sum_{L,J'} d_{sym}(P_w,w,L,J')e^{2\pi i L\widetilde{\rho}+2\pi i J'\widetilde{v}+2\pi i\widetilde{\sigma}P_w/N} = \prod_{\substack{l,j\in\mathbb{Z}\\ l\geq 0}}\left(1-e^{2\pi i\widetilde{\sigma}/N+2\pi i l\widetilde{\rho}+2\pi i j\widetilde{v}}\right)^{-n(w,l,j)}\,. \tag{E.11}$$

Using the identity in (E.11) we can evaluate the generating function for the bosonic minus fermionic states for the relative dynamics of the D1-D5 system. Eq.(E.7) shows that all we need to do is to take the product over w of the right hand side of (E.11). More specifically, if $d_{D1D5}(W, L, J')$ denotes the total number of bosonic minus fermionic states carrying total string length $2\pi W =$

$2\pi\sum_{w>0} wP_w$ (counting a single long string with quantum number w to have length $2\pi w$), total momentum $-L/N$ along S^1 and total momentum J' along $\widetilde{S}^1$, then we have

$$\begin{aligned}
&\sum_{W,L,J'} d_{D1D5}(W,L,J') e^{2\pi i(\widetilde{\rho}L+\widetilde{\sigma}W/N+\widetilde{v}J')} \\
&= \prod_{\substack{w\in\mathbb{Z}\\ w>0}} \prod_{\substack{l,j\in\mathbb{Z}\\ l\geq 0}} \left(1-e^{2\pi i\widetilde{\sigma}w/N+2\pi il\widetilde{\rho}+2\pi ij\widetilde{v}}\right)^{-n(w,l,j)} \\
&= \prod_{r=0}^{N-1}\prod_{b=0}^{1} \prod_{\substack{k'\in\mathbb{Z}+\frac{r}{N},l\in\mathbb{Z},j\in 2\mathbb{Z}+b\\ k'>0,l\geq 0}} (1-e^{2\pi i(k'\widetilde{\sigma}+l\widetilde{\rho}+j\widetilde{v})})^{-\sum_{s=0}^{N-1} e^{-2\pi isl/N} c_b^{(r,s)}(4k'l-j^2)}\,.
\end{aligned} \tag{E.12}$$

In arriving at the last expression in (E.12) we have defined $k' = w/N$. Physically $d_{D1D5}(W,L,J')$ counts the number of states with fixed number Q_1 and Q_5 of D1 and D5-branes, and fixed momenta $-L/N$ and J' along S^1 and $\widetilde{S}^1$, with W identified with the number $Q_5(Q_1-Q_5)+1$. Eq.(E.12) replaces (5.2.36) for general Q_5, and reduces to (5.2.36) for $Q_5=1$. Subsequent analysis leading to the full partition function of the system proceeds in a manner identical to the one described in §5.2.4 and the final result for $d(\vec{Q},\vec{P})$ has the form of (5.2.41) with P^2 given by $2(W-1)=2Q_5(Q_1-Q_5)$.

F Riemann Normal Coordinates and Duality Invariant Statistical Entropy Function

In section 5.6 we considered $\vec{\eta} = \vec{\tau} - \vec{\tau}_B$ for some fixed base point $\vec{\tau}_B$ as the fundamental field in defining $W_B(\vec{\tau}_B,\vec{J})$ and $\Gamma_B(\vec{\tau}_B,\vec{\chi})$. In this appendix we shall try to generalize this by treating

$$\vec{\xi} = \vec{g}(\vec{\eta}) \tag{F.1}$$

as a fundamental field. Here $\vec{g}(\vec{\eta})$ is an arbitrary function of $\vec{\eta}$ with a Taylor series expansion starting with the linear terms (i.e. $\vec{g}(\vec{0})=\vec{0}$). In this case the generating function of $\vec{\xi}$ correlation functions will be given by

$$e^{\widetilde{W}_B(\vec{\tau}_B,\vec{J})} = \int \frac{d^2\eta}{(\tau_{B2}+\eta_2)^2}\, e^{-F(\vec{\tau}_B+\vec{\eta})+\vec{J}\cdot\vec{g}(\vec{\eta})}\,. \tag{F.2}$$

As before $\widetilde{W}_B(\vec{\tau}_B,\vec{0}) = S_{stat}$. The corresponding effective action is defined via the equations

$$\psi_i = \frac{\partial\widetilde{W}_B(\vec{\tau}_B,\vec{J})}{\partial J_i}\,, \qquad \widetilde{\Gamma}_B(\vec{\tau}_B,\vec{\psi}) = \vec{J}\cdot\vec{\psi} - \widetilde{W}_B(\vec{\tau}_B,\vec{J})\,. \tag{F.3}$$

From this it follows that

$$J_i = \frac{\partial \widetilde{\Gamma}_B(\vec{\tau}_B, \vec{\psi})}{\partial \psi_i}\,. \tag{F.4}$$

Now suppose $\vec{\tau}^{(0)}$ is a specific value of $\vec{\tau}_B$ for which

$$\frac{\partial \widetilde{\Gamma}_B(\vec{\tau}^{(0)}, \vec{\psi})}{\partial \psi_i}\bigg|_{\vec{\psi}=0} = 0 \quad \text{i.e.} \quad \frac{\partial \widetilde{W}_B(\vec{\tau}^{(0)}, \vec{J})}{\partial J_i}\bigg|_{\vec{J}=0} = 0\,. \tag{F.5}$$

In this case we have $\vec{J} = 0$ for $\vec{\psi} = 0$, and hence

$$\widetilde{\Gamma}_B(\vec{\tau}^{(0)}, \vec{0}) = -\widetilde{W}_B(\vec{\tau}^{(0)}, \vec{0}) = -S_{stat}\,. \tag{F.6}$$

We shall now show that $\widetilde{\Gamma}_B(\vec{\tau}_B, \vec{0})$, regarded as a function of $\vec{\tau}_B$, has an extremum at $\vec{\tau}_B = \vec{\tau}^{(0)}$. From (F.3), (F.4) we see that

$$\widetilde{\Gamma}_B(\vec{\tau}^{(0)} + \vec{\epsilon}, \vec{0}) = -\widetilde{W}_B\left(\vec{\tau}^{(0)} + \vec{\epsilon}, \vec{J} = \partial\widetilde{\Gamma}_B(\vec{\tau}^{(0)} + \vec{\epsilon}, \vec{\psi})/\partial\vec{\psi}\right)\bigg|_{\vec{\psi}=\vec{0}}\,. \tag{F.7}$$

Now

$$e^{\widetilde{W}_B(\vec{\tau}_B + \vec{\epsilon}, \vec{J})} = \int \frac{d^2\eta}{(\tau_{B2} + \epsilon_2 + \eta_2)^2}\, e^{-F(\vec{\tau}_B + \vec{\epsilon} + \vec{\eta}) + \vec{J}\cdot\vec{g}(\vec{\eta})} = \int \frac{d^2\eta}{(\tau_{B2} + \eta_2)^2}\, e^{-F(\vec{\tau}_B + \vec{\eta}) + \vec{J}\cdot\vec{g}(\vec{\eta} - \vec{\epsilon})}\,, \tag{F.8}$$

where in the second step we have made a change of variables $\vec{\eta} \to \vec{\eta} - \vec{\epsilon}$. Since $g(\vec{\eta} - \vec{\epsilon}) = g(\vec{\eta}) + O(\vec{\epsilon})$, this shows that

$$\widetilde{W}_B(\vec{\tau}_B + \vec{\epsilon}, \vec{J}) = \widetilde{W}_B(\vec{\tau}_B, \vec{J}) + O(\epsilon_i J_k)\,. \tag{F.9}$$

Using this information in (F.7) we get

$$\widetilde{\Gamma}_B(\vec{\tau}^{(0)} + \vec{\epsilon}, \vec{0}) = -\widetilde{W}_B(\vec{\tau}^{(0)}, \vec{J} = \partial\widetilde{\Gamma}_B(\vec{\tau}^{(0)} + \vec{\epsilon}, \vec{\psi})/\partial\vec{\psi})|_{\vec{\psi}=\vec{0}} + O\left(\epsilon_i \frac{\partial \widetilde{\Gamma}_B(\vec{\tau}^{(0)} + \vec{\epsilon}, \vec{\psi})}{\partial \psi_j}\bigg|_{\vec{\psi}=0}\right)\,. \tag{F.10}$$

Eq.(F.5) now shows that the second term on the right hand side of this equation is of order ϵ^2, and $\vec{J}$ appearing in the argument of the first term is of order ϵ. Expanding the first term in a Taylor series expansion in $\vec{J}$, and noting that the linear term vanishes due to (F.5), we get

$$\widetilde{\Gamma}_B(\vec{\tau}^{(0)} + \vec{\epsilon}, \vec{0}) = -\widetilde{W}_B(\vec{\tau}^{(0)}, \vec{J} = \vec{0}) + O(\epsilon^2) = \widetilde{\Gamma}_B(\vec{\tau}^{(0)}, \vec{0}) + O(\epsilon^2)\,. \tag{F.11}$$

Thus

$$\frac{\partial \widetilde{\Gamma}_B(\vec{\tau}, \vec{0})}{\partial \tau_i} = 0 \quad \text{at} \quad \vec{\tau} = \vec{\tau}^{(0)}\,. \tag{F.12}$$

Using (F.6) and (F.12) we see that the statistical entropy is given by the value of $-\widetilde{\Gamma}_B(\vec{\tau},\vec{0})$ at its extremum $\vec{\tau} = \vec{\tau}^{(0)}$. Thus we can identify $-\widetilde{\Gamma}_B(\vec{\tau},\vec{0})$ as the statistical entropy function. This is computed as the sum of 1PI vacuum amplitudes in the theory with ξ_i regarded as the fundamental fields.

We shall now show that for a suitable choice of the coordinates $\vec{\xi}$, the statistical entropy function $-\widetilde{\Gamma}_B(\vec{\tau},\vec{0})$ defined this way can be made manifestly duality invariant. This is done by choosing $\vec{\xi}$ as Riemann normal coordinates. For a given base point $\vec{\tau}_B$ the coordinate $\vec{\xi}$ for a given point $\vec{\tau}$ in the upper half plane is defined as follows. We introduce the duality invariant metric on the upper half plane

$$ds^2 = (\tau_2)^{-2}(d\tau_1^2 + d\tau_2^2)\,, \tag{F.13}$$

and draw a geodesic connecting $\vec{\tau}_B$ and $\vec{\tau}$. The coordinate $\vec{\xi}$ corresponding to the point $\vec{\tau}$ is then defined by identifying $|\vec{\xi}|$ as the distance between $\vec{\tau}_B$ and $\vec{\tau}$ along the geodesic and the direction of $\vec{\xi}$ is taken to be the direction of the tangent vector to the geodesic at $\vec{\tau}_B$.[50] Since the metric (F.13) is invariant under a duality transformation, it is clear that if a duality transformation maps $\vec{\tau}_B$ to $\vec{\tau}_B'$ and $\vec{\tau}$ to $\vec{\tau}'$, then the Riemann normal coordinate $\vec{\xi}'$ of $\vec{\tau}'$ with respect to $\vec{\tau}_B'$ will have the property that $|\vec{\xi}'| = |\vec{\xi}|$. Thus $\vec{\xi}$ and $\vec{\xi}'$ are related by a rotation. In order to determine the angle of rotation, we note that under a duality transformation (5.6.45),

$$d\tau' = (\gamma\tau + \delta)^{-2} d\tau\,. \tag{F.14}$$

Thus

$$\frac{d\tau'}{|d\tau'|} = \frac{|\gamma\tau+\delta|^2}{(\gamma\tau+\delta)^2}\,\frac{d\tau}{|d\tau|}\,. \tag{F.15}$$

This shows that a geodesic passing through τ_B gets rotated by a phase $|\gamma\tau_B+\delta|^2/(\gamma\tau_B+\delta)^2$ under a duality transformation. Hence

$$\xi' = \frac{|\gamma\tau_B+\delta|^2}{(\gamma\tau_B+\delta)^2}\,\xi = \frac{\gamma\bar{\tau}_B+\delta}{\gamma\tau_B+\delta}\,\xi\,, \tag{F.16}$$

where

$$\xi = \xi_1 + i\xi_2, \qquad \xi' = \xi_1' + i\xi_2'\,. \tag{F.17}$$

Since for given τ_B the duality transformation acts linearly on $\vec{\xi}$, the corresponding generating function $\widetilde{W}_B(\vec{\tau}_B,\vec{J})$ and the effective action $\widetilde{\Gamma}_B(\vec{\tau}_B,\vec{\psi})$ will be manifestly duality invariant under

[50] Often one uses the convention that the distance along the geodesic is $\sqrt{g_{ij}(\vec{\tau}_B)\xi^i\xi^j}$. This definition differs from the one used here by a multiplicative factor of τ_{2B}. Since this transforms covariantly under a duality transformation, both choices of $\vec{\xi}$ would give manifestly covariant Feynman rules.

simultaneous transformation of $\vec{\tau}_B$, $\vec{J}$ or $\vec{\psi}$ and of course the charges $\vec{Q}$ and $\vec{P}$. In particular the 1PI vacuum amplitude $\widetilde{\Gamma}_B(\vec{\tau},\vec{0})$ will be invariant under the duality transformation (5.6.45).

We shall now give an algorithm for explicitly generating duality covariant vertices in this 0-dimensional field theory. For this we need to expand the duality invariant function $F(\vec{\tau})$ in a Taylor series expansion in $\vec{\xi}$. This is given by:

$$F(\vec{\tau}) = \sum_{n=0}^{\infty} \frac{1}{n!} (\tau_{B2})^n \xi_{i_1} \ldots \xi_{i_n} \, D_{i_1} \cdots D_{i_n} F(\vec{\tau}) \Big|_{\vec{\tau}=\vec{\tau}_B} , \tag{F.18}$$

where D_i denotes covariant derivative with respect to τ_i, computed using the affine connection Γ^i_{jk} constructed from the metric (F.13). We arrive at (F.18) by using the result that in the $\vec{\xi}$ coordinate system symmetrized covariant derivatives are equal to ordinary derivatives. Using this we can replace ordinary derivatives in the Taylor series expansion by covariant derivatives with respect to ξ_i. In the second step we use the tensorial transformation properties of covariant derivatives to convert covariant derivative with respect to ξ_i to covariant derivative with respect to τ_i. The $(\tau_{B2})^n$ factor in (F.18) arises due to the fact that near $\vec{\tau} = \vec{\tau}_B$,

$$d\tau_i = \tau_{B2} d\xi_i \, . \tag{F.19}$$

In the $(\tau, \bar{\tau})$ coordinate system the non-zero components of the connection are

$$\Gamma^{\tau}_{\tau\tau} = \frac{i}{\tau_2}, \qquad \Gamma^{\bar{\tau}}_{\bar{\tau}\bar{\tau}} = -\frac{i}{\tau_2} \, . \tag{F.20}$$

The curvature tensor computed from this connection has the form

$$R^i{}_{jkl} = -(\delta^i_k g_{jl} - \delta^i_l g_{jk}) \, , \tag{F.21}$$

which shows that the metric (F.13) describes a constant negative curvature metric. From (F.20) it follows that

$$\begin{aligned} D_\tau (D^m_\tau D^n_{\bar{\tau}} F(\vec{\tau})) &= (\partial_\tau - im/\tau_2)(D^m_\tau D^n_{\bar{\tau}} F(\vec{\tau})), \\ D_{\bar{\tau}} (D^m_\tau D^n_{\bar{\tau}} F(\vec{\tau})) &= (\partial_{\bar{\tau}} + in/\tau_2)(D^m_\tau D^n_{\bar{\tau}} F(\vec{\tau})) \, , \end{aligned} \tag{F.22}$$

for any arbitrary ordering of D_τ and $D_{\bar{\tau}}$ in $D^m_\tau D^n_{\bar{\tau}} F(\vec{\tau})$. (F.22) provides us with explicit expressions for the covariant derivatives of F appearing in (F.18). Also using (F.22) one can prove iteratively that under a duality transformation

$$(\tau_2)^{m+n} \, D^m_\tau D^n_{\bar{\tau}} F(\vec{\tau}) \to \left(\frac{\gamma\tau + \delta}{\gamma\bar{\tau} + \delta} \right)^{m-n} (\tau_2)^{m+n} \, D^m_\tau D^n_{\bar{\tau}} F(\vec{\tau}) \, . \tag{F.23}$$

This establishes manifest duality covariance of the vertices constructed from (F.18).

We also need to worry about the contribution from the integration measure. The original measure $d^2\tau/(\tau_2)^2$ was duality invariant. Since duality transformation acts on $\vec{\xi}$ as a rotation, $d^2\xi$ is also a duality invariant measure. Thus we must have

$$\frac{d^2\tau}{(\tau_2)^2} = \mathcal{J}(\vec{\tau}_B, \vec{\xi})\, d^2\xi\,, \tag{F.24}$$

for some duality invariant function $\mathcal{J}(\vec{\tau}_B, \vec{\xi})$. It has been shown in appendix G that

$$\mathcal{J}(\vec{\tau}_B, \vec{\xi}) = \frac{1}{|\vec{\xi}|} \sinh |\vec{\xi}|\,. \tag{F.25}$$

We can now regard $-\ln \mathcal{J}(\vec{\tau}_B, \vec{\xi})$ as an additional contribution to the action and expand this in a power series expansion in $\vec{\xi}$ to generate additional vertices. Using the expression for $F(\vec{\tau})$ given in (5.6.44) we now see that the full 'action' is given by

$$\begin{aligned} F(\vec{\tau}) - \ln \mathcal{J}(\vec{\tau}_B, \vec{\xi}) \;\; = \;\; & -\bigg[\frac{\pi}{2\tau_2}\, |Q - \tau P|^2 - \ln g(\tau) - \ln g(-\bar{\tau}) - (k+2)\ln(2\tau_2) \\ & + \ln\left\{K_0 \frac{\pi}{\tau_2} |Q - \tau P|^2\right\} + \ln \mathcal{J}(\vec{\tau}_B, \vec{\xi}) \\ & + \ln\left(1 + \frac{2(k+3)\tau_2}{\pi |Q - \tau P|^2}\right)\bigg]\,. \end{aligned} \tag{F.26}$$

In this expression the first term inside the square bracket is quadratic in the charges, the last term contains terms of order Q^{-2n} for $n \geq 1$, and the other terms are of order Q^0. Thus in order to carry out a systematic expansion in powers of inverse charges we need to regard the first term as the tree level contribution, the last term as two and higher loop contributions and the other terms as the 1-loop contribution.

We can now evaluate the effective action $\widetilde{\Gamma}_B(\vec{\tau}_B)$ in a systematic loop expansion. The leading term in the effective action is then just the first term in (F.26) evaluated at $\vec{\tau} = \vec{\tau}_B$:

$$\widetilde{\Gamma}_0(\vec{\tau}_B) = -\frac{\pi}{2\tau_{2B}}\, |Q - \tau_B P|^2\,. \tag{F.27}$$

At the next order we need to include the tree level contribution from the rest of the terms in the action (except the last term which is higher order) and one loop contribution from the first term.

The former corresponds to these terms being evaluated at $\vec{\tau} = \vec{\tau}_B$, i.e. $\vec{\xi} = 0$. Since $\mathcal{J}(\vec{\tau}_B, \vec{\xi} = 0) = 1$, we get this contribution to be

$$\ln g(\tau_B) + \ln g(-\bar{\tau}_B) + (k+2)\ln(2\tau_{2B}) - \ln\left\{K_0 \frac{\pi}{\tau_{2B}}|Q - \tau_B P|^2\right\}. \tag{F.28}$$

For the one loop contribution from the first term in the action we need to expand this term to quadratic order in $\vec{\xi}$ using eqs.(F.18), (F.22). The order $\vec{\xi}$ and ξ^2 terms are given by

$$-\frac{i\pi}{4\tau_{2B}}\left\{\bar{\xi}(Q - \tau_B P)^2 + \xi(Q - \bar{\tau}_B P)^2\right\} - \frac{\pi}{4\tau_{2B}}|Q - \tau_B P|^2\,\bar{\xi}\xi\,. \tag{F.29}$$

The linear term in $\vec{\xi}$ do not give any contribution to the 1PI amplitudes. The contribution from the quadratic term gives

$$\ln\left(\frac{1}{4\tau_{2B}}|Q - \tau_B P|^2\right). \tag{F.30}$$

Thus the complete one loop contribution to the effective action is given by

$$\widetilde{\Gamma}_1(\vec{\tau}_B) = \ln g(\tau_B) + \ln g(-\bar{\tau}_B) + (k+2)\ln(2\tau_{2B}) - \ln(4\pi K_0)\,. \tag{F.31}$$

Up to an additive constant $-\widetilde{\Gamma}_0(\vec{\tau}_B) - \widetilde{\Gamma}_1(\vec{\tau}_B)$ agrees with the black hole entropy function given in (3.1.40), (3.1.47) if we identify τ_B as $u_a + iu_S$.

G The Integration Measure $\mathcal{J}(\vec{\tau}_B, \vec{\xi})$

In this appendix we shall compute the integration measure $\mathcal{J}(\vec{\tau}_B, \vec{\xi})$ which arises from a change of variables from τ_1, τ_2 to the Riemann normal coordinates:

$$\frac{d^2\tau}{(\tau_2)^2} = \mathcal{J}(\vec{\tau}_B, \vec{\xi})\, d^2\xi\,. \tag{G.1}$$

We first note that the duality invariant metric

$$\frac{1}{\tau_2^2}(d\tau_1^2 + d\tau_2^2) \tag{G.2}$$

describes a metric of constant negative curvature -1. Since this is a homogeneous space, $\mathcal{J}(\vec{\tau}_B, \vec{\xi})$ cannot depend on $\vec{\tau}_B$. Now, by defining new coordinate θ, ϕ via the relations

$$\tanh\frac{\theta}{2}\, e^{i\phi} = \frac{1 + i\tau}{1 - i\tau}\,, \tag{G.3}$$

we can express the metric (G.2) and the measure (G.1) as

$$ds^2 = d\theta^2 + \sinh^2\theta d\phi^2\,, \qquad \frac{d^2\tau}{\tau_2^2} = \sinh\theta\, d\theta\, d\phi\,. \tag{G.4}$$

Since $\mathcal{J}$ is independent of the base point $\vec{\tau}_B$, we can calculate it by taking the base point to be at $\theta = 0$. The geodesics passing through this point are constant ϕ lines, and the length measured along such a geodesic is given by θ. Thus we have

$$\vec{\xi} = (\theta\cos\phi, \theta\sin\phi)\,. \tag{G.5}$$

This gives

$$d^2\xi = \theta d\theta d\phi\,. \tag{G.6}$$

Comparing this with (G.4) we get

$$\frac{d^2\tau}{\tau_2^2} = \frac{\sinh\theta}{\theta} d^2\xi = \frac{1}{|\vec{\xi}|}\sinh|\vec{\xi}|\, d^2\xi\,. \tag{G.7}$$

Thus

$$\mathcal{J}(\vec{\tau}_B, \vec{\xi}) = \frac{1}{|\vec{\xi}|}\sinh|\vec{\xi}|\,. \tag{G.8}$$

H The Coefficient of the Gauss-Bonnet Term in Type IIB on $(\mathcal{M} \times \widetilde{S}^1 \times S^1)/\,\mathbb{Z}_N$

An important four derivative correction to the effective action in the $\mathcal{N} = 4$ supersymmetric string theories analyzed in this review is the Gauss-Bonnet term. In this appendix we shall describe the computation of this term.

The calculation is best carried out in the original description of the theory as type IIB string theory compactified on $(\mathcal{M} \times \widetilde{S}^1 \times S^1)/\,\mathbb{Z}_N$. We shall denote by $t = t_1 + it_2$ and $u = u_1 + iu_2$ the Kahler and complex structure moduli of the torus $\widetilde{S}^1 \times S^1$, and use the normalization convention that is appropriate for the orbifold theory as described below (3.1.29). Thus for example if $\widetilde{R}$ and R_0 denote the radii of $\widetilde{S}^1$ and S^1 measured in the string metric, and if the off-diagonal components of the metric and the anti-symmetric tensor field are zero, then we shall take $t_2 = \widetilde{R}R_0/N$ and $u_2 = R_0/(\widetilde{R}N)$, taking into account the fact that in the orbifold theory the various fields have $\widetilde{g}$-twisted boundary condition under a $2\pi R_0/N$ translation along S^1 and $2\pi\widetilde{R}$ translation along $\widetilde{S}^1$. In the same spirit we shall choose the units of momentum along S^1 and $\widetilde{S}^1$ to be N/R_0 and $1/\widetilde{R}$

respectively, and unit of winding charge along S^1 and $\widetilde{S}^1$ to be $2\pi R_0/N$ and $2\pi\widetilde{R}$ respectively. As a result one unit of winding charge along S^1 actually represents a twisted sector state, with twist g.

It is known that one loop quantum corrections in this theory give rise to a Gauss-Bonnet contribution to the effective Lagrangian density of the form [67]:

$$\Delta\mathcal{L} = \phi(u_1, u_2)\,\left\{R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} - 4R_{\mu\nu}R^{\mu\nu} + R^2\right\}\,, \tag{H.1}$$

where $\phi(u_1, u_2)$ is a function to be determined, and $R_{\mu\nu\rho\sigma}$, $R_{\mu\nu}$ and R denote respectively the Riemann tensor, Ricci tensor and scalar curvature computed from the canonical Einstein metric. Note in particular that ϕ is independent of the Kahler modulus t of $\widetilde{S}^1\times S^1$. The analysis of [67] shows that $\phi(u_1, u_2)$ is given by the relation:

$$\partial_u\phi(u_1, u_2) = \int_{\mathcal{F}} \frac{d^2\tau}{\tau_2}\,\partial_u B_4\,, \tag{H.2}$$

where B_4 is defined as follows. Let us consider type IIB string theory compactified on $(\mathcal{M}\times\widetilde{S}^1\times S^1)/\,\mathbb{Z}_N$ in the light-cone gauge Green-Schwarz formulation, denote by Tr^f the trace over all states in this world-sheet theory carrying some fixed momentum along the non-compact directions and denote by L_0^f, $\bar{L}_0^f$ the Virasoro generators associated with the left and the right-moving modes, excluding the contribution from the momenta along the non-compact directions. We also define F_L^f, F_R^f to be the contribution to the space-time fermion numbers from the left and the right-moving modes on the world-sheet. In this case

$$B_4 = K\,Tr^f\left(q^{L_0^f}\bar{q}^{\bar{L}_0^f}(-1)^{F_L^f+F_R^f}h^4\right), \qquad q\equiv e^{2\pi i\tau}\,, \tag{H.3}$$

where K is a constant to be determined later and h denotes the total helicity of the state. In defining L_0^f, $\bar{L}_0^f$ we subtract the constants $c_L/24$ and $c_R/24$, so that the RR vacuum has $L_0^f = \bar{L}_0^f = 0$.

The evaluation of the right hand side of (H.3) proceeds as follows. We first note that since $(\mathcal{M}\times\widetilde{S}^1\times S^1)/\,\mathbb{Z}_N$ has $SU(2)$ holonomy, and since a spinor representation of $SO(8)$ transforms as a pair of doublets and four singlets under this $SU(2)$ group, we have four free left-moving fermions and four free right-moving fermions associated with the singlets of $SU(2)$. These give rise to altogether eight fermion zero modes. Since quantization of a conjugate pair of fermion zero modes $(\psi_0, \psi_0^\dagger)$ gives rise to a pair of states with opposite $(-1)^{F_L^f+F_R^f}$, without the h^4 term the trace in (H.3) will vanish. This can be avoided if we insert a factor of h in the trace and pick the contribution to h from this particular conjugate pair of fermions, – in this case the two states have the same $(-1)^{F_L^f+F_R^f}\,h$ quantum numbers. This can be repeated for every pair of conjugate fermions. Altogether we need four factors of h to soak up all the eight fermion zero modes. Thus in effect we can simplify (H.3) by expressing it as

$$B_4 = K' Tr^{f\prime} \left(q^{L_0^f} \bar{q}^{\bar{L}_0^f} (-1)^{F_L^f + F_R^f} \right) \tag{H.4}$$

where K' is a different normalization constant and the prime in the trace denotes that we should ignore the effect of fermion zero modes in evaluating the trace.

Since we are using the Green-Schwarz formulation, the 4 left-moving and 4 right-moving fermions which are neutral under the holonomy group satisfy periodic boundary condition. Thus the effect of the non-zero mode oscillators associated with these fermions cancel against the contribution from the non-zero mode bosonic oscillators associated with the circles S^1 and $\widetilde{S}^1$ and the two non-compact directions. This leads to a further simplification in which the trace can be taken over only the degrees of freedom associated with the compact space $\mathcal{M}$ and the bosonic zero modes associated with the circles S^1 and $\widetilde{S}^1$. The latter includes the quantum numbers m_1 and m_2 denoting the number of units of momentum along $\widetilde{S}^1$ and S^1, and the quantum numbers n_1 and n_2 denoting the number of units of winding along $\widetilde{S}^1$ and S^1. The units of momentum and winding along the two circles are chosen according to the convention described earlier. Thus for example m_2 unit of momentum along S^1 will correspond to a physical momentum of Nm_2/R_0 in string units. This shows that m_2 can be fractional, being quantized in units of $1/N$. On the other hand a sector with n_2 unit of winding along S^1 describes a fundamental string of length $2\pi n_2 R_0/N$, and hence this state belongs to a sector twisted by g^{n_2}.

In this convention the contributions to $\bar{L}_0^f$ and L_0^f from the bosonic zero modes associated with $\widetilde{S}^1 \times S^1$ are given by, respectively,

$$\frac{1}{2}k_R^2 = \frac{1}{4t_2u_2}|-m_1u + m_2 + n_1t + n_2tu|^2 \,, \qquad \frac{1}{2}k_L^2 = \frac{1}{2}k_R^2 + m_1n_1 + m_2n_2 \,. \tag{H.5}$$

Furthermore, since under the $SU(2)$ holonomy group a vector in the tangent space of $\mathcal{M}$ also transform as a pair of doublets, the fermions in our system which transform as doublets of $SU(2)$ may be regarded as tangent space vectors. As a result, these fermions, together with the scalars associated with the coordinates of $\mathcal{M}$, describe a superconformal field theory with target space $\mathcal{M}$. Thus (H.4) may now be rewritten as

$$B_4 = \frac{K'}{N} \sum_{r=0}^{N-1} \sum_{s=0}^{N-1} \sum_{\substack{m_1,n_1\in\mathbb{Z}, m_2\in\mathbb{Z}/N \\ n_2\in N\mathbb{Z}+r}} q^{k_L^2/2} \bar{q}^{k_R^2/2} e^{2\pi i m_2 s} Tr_{RR,\widetilde{g}^r} \left((-1)^{F_L+F_R} \widetilde{g}^s q^{L_0} \bar{q}^{\bar{L}_0} \right) \,, \tag{H.6}$$

where $Tr_{RR;\widetilde{g}^r}$ denotes trace over the $\widetilde{g}^r$-twisted sector RR states of the (4,4) superconformal field theory with target space $\mathcal{M}$ and L_0 and $\bar{L}_0$ denote contribution to the Virasoro generators in this superconformal field theory with $c_L/24$ and $c_R/24$ subtracted. The sum over s in (H.6) arises from the insertion of the projection operator $\frac{1}{N}\sum_{s=0}^{N-1} g^s$ in the trace, while the sum over r represents the sum over various twisted sector states. As required, the quantum number n_2 that determines the part of g-twist along S^1 is correlated with the integer r that determines the amount of g-twist along $\mathcal{M}$. The $e^{2\pi i m_2 s}$ factor represents part of g^s that acts as translation along S^1 while the action of g^s on $\mathcal{M}$ is represented by the operator $\widetilde{g}^s$ inserted into the trace.

We now note that the trace part in (H.6) is precisely the quantity $N\, F^{(r,s)}(\tau, z=0)$ defined in (B.2) for $q = e^{2\pi i\tau}$. Thus we can rewrite (H.6) as

$$B_4 = K' \sum_{r=0}^{N-1}\sum_{s=0}^{N-1} \sum_{\substack{m_1,n_1\in\mathbb{Z}, m_2\in\mathbb{Z}/N \\ n_2\in N\mathbb{Z}+r}} q^{k_L^2/2}\bar{q}^{k_R^2/2} e^{2\pi i m_2 s} F^{(r,s)}(\tau,0)\,. \tag{H.7}$$

We shall now compare (H.7) with the expression for $\widehat{\mathcal{I}}(\widetilde{\rho},\widetilde{\sigma},\widetilde{v})$ given in (C.3), (C.5) at $\widetilde{\rho}=u$, $\widetilde{\sigma}=t$ and $\widetilde{v}=0$. In this case p_R^2, p_L^2 defined in (C.2) reduces to k_R^2 and $k_L^2+\frac{1}{2}j^2$ respectively, with k_R^2, k_L^2 given in (H.5). As a result we have

$$\begin{aligned}
\widehat{\mathcal{I}}(u,t,0) = \int_{\mathcal{F}} \frac{d^2\tau}{\tau_2}\Big[\sum_{r,s=0}^{N-1}\sum_{b=0}^{1} \sum_{\substack{m_1,n_1\in\mathbb{Z}, m_2\in\mathbb{Z}/N \\ n_2\in N\mathbb{Z}+r, j\in 2\mathbb{Z}+b}} q^{k_L^2/2}\bar{q}^{k_R^2/2} q^{j^2/4} e^{2\pi i m_2 s} h_b^{(r,s)}(\tau) - \sum_{s=0}^{N-1} c_0^{(0,s)}(0)\Big] \\
= \int_{\mathcal{F}} \frac{d^2\tau}{\tau_2}\Big[\sum_{r,s=0}^{N-1} \sum_{\substack{m_1,n_1\in\mathbb{Z}, m_2\in\mathbb{Z}/N \\ n_2\in N\mathbb{Z}+r}} q^{k_L^2/2}\bar{q}^{k_R^2/2} e^{2\pi i m_2 s}(\vartheta_3(2\tau,0) h_0^{(r,s)}(\tau) \\
+\vartheta_2(2\tau,0) h_1^{(r,s)}(\tau)) - \sum_{s=0}^{N-1} c_0^{(0,s)}(0)\Big] \\
= \int_{\mathcal{F}} \frac{d^2\tau}{\tau_2}\Big[\sum_{r,s=0}^{N-1} \sum_{\substack{m_1,n_1\in\mathbb{Z}, m_2\in\mathbb{Z}/N \\ n_2\in N\mathbb{Z}+r}} q^{k_L^2/2}\bar{q}^{k_R^2/2} e^{2\pi i m_2 s} F^{(r,s)}(\tau,0) - \sum_{s=0}^{N-1} c_0^{(0,s)}(0)\Big]\,,
\end{aligned} \tag{H.8}$$

where in the second step we have expressed the result of summing over j in terms of Jacobi ϑ-functions, and in the last step we have used eq.(B.3). Comparing (H.7) with (H.8) we see that

$$\int_{\mathcal{F}} \frac{d^2\tau}{\tau_2}\left(B_4 - K'\sum_{s=0}^{N-1} c_0^{(0,s)}(0)\right) = K'\,\widehat{\mathcal{I}}(u,t,0)\,. \tag{H.9}$$

Using (C.16) and (C.25) we get

$$\int_{\mathcal{F}} \frac{d^2\tau}{\tau_2}\left(B_4 - K'\sum_{s=0}^{N-1} c_0^{(0,s)}(0)\right) \quad = \quad -2\,K' \lim_{v\to 0}\Big[(k\ln t_2 + k\ln u_2 + 2\ln v + 2\ln\bar{v}$$

$$+\ln g(t)+\ln\overline{g(t)}+\ln g(u)+\ln\overline{g(u)}\Big)\Big]+\text{constant}\,. \tag{H.10}$$

Naively the right hand side diverges in the $v\to 0$ limit. The origin of this infinity lies in the fact that $\int d^2\tau B_4/\tau_2$ has divergences from integration over the large τ_2 region for $\vec m=\vec n=0$, and this divergence is not completely removed by the subtraction term proportional to $K'\sum_s c_0^{(0,s)}(0)$ in the integrand. The correct subtraction term in the integrand must be equal to $K'\lim_{\tau\to i\infty}F^{(0,s)}(\tau,0)$, – from (B.6) we see that this is given by $K'\sum_s\left(c_0^{(0,s)}(0)+2c_1^{(0,s)}(-1)\right)$. The extra counteterm proportional to $c_1^{(0,s)}(-1)$ is not needed for regulating $\widehat{\mathcal{I}}$ since the correponding potential divergence from the term $m_1=n_1=m_2=n_2=0$, $j=\pm1$ in (C.5). takes the form:

$$\int\frac{d^2\tau}{\tau_2^2}\exp\left(-\frac{2\pi}{t_2u_2-v_2^2}|v|^2\sum_{s=0}^{N-1}c_1^{(0,s)}(-1)\right)\simeq -2\sum_{s=0}^{N-1}c_1^{(0,s)}(-1)\ln\frac{|v|^2}{t_2u_2}+\text{constant}\simeq -4\ln\frac{|v|^2}{t_2u_2} \tag{H.11}$$

for small v. This is divergent in the $v\to 0$ limit but finite for small but finite v. Thus in order to recover

$$\int\frac{d^2\tau}{\tau_2}\left[B_4-K'\sum_s\left(c_0^{(0,s)}(0)+2c_1^{(0,s)}(-1)\right)\right] \tag{H.12}$$

from $\widehat{\mathcal{I}}$ we need to first remove the contribution $-4\ln\frac{|v|^2}{t_2u_2}$ from $\widehat{\mathcal{I}}$ and then take the $v\to 0$ limit. This gives

$$\begin{aligned}&\int_{\mathcal{F}}\frac{d^2\tau}{\tau_2}\left(B_4-K'\sum_{s=0}^{N-1}\left(c_0^{(0,s)}(0)+2c_1^{(0,s)}(-1)\right)\right)\\ =\;&-2\,K'\left((k+2)\ln t_2+(k+2)\ln u_2+\ln g(t)+\ln g(-\bar t)+\ln g(u)+\ln g(-\bar u)\right)+\text{constant}\,.\end{aligned} \tag{H.13}$$

In writing down (H.13) we have used $\overline{g(t)}=g(-\bar t)$, – this follows from the definition (C.27) of $g(\rho)$ and the identity $s_r=s_{-r}$. Comparing (H.2) with (H.13) we now get

$$\phi(u_1,u_2)=-2\,K'\ \left((k+2)\ln u_2+\ln g(u)+\ln g(-\bar u)\right)+\text{constant}\,. \tag{H.14}$$

We now turn to the determination of K'. This constant is universal independent of the specific theory we are analysing. Thus we can find it by working with the type IIB string theory compactified on $K3\times\widetilde S^1\times S^1$. In this case $k=10$ and $g(\tau)=\eta(\tau)^{24}$. This matches with the known answer [72,83] for $\phi(u_1,u_2)$ if we choose $K'=1/(128\pi^2)$. Thus we have

$$\phi(u_1,u_2)=-\frac{1}{64\pi^2}\ \left((k+2)\ln u_2+\ln g(u)+\ln g(-\bar u)\right)+\text{constant}\,. \tag{H.15}$$

Under the duality map that relates type IIB string theory on the $\mathbb{Z}_N$ orbifold of $\mathcal{M} \times \widetilde{S}^1 \times S^1$ to an asymmetric $\mathbb{Z}_N$ orbifold of heterotic or type IIA string theory on T^6, the modulus u of the original type IIB string theory gets related to the axion-dilaton modulus $a+iS$ of the final asymmetric orbifold theory. Thus in this description the Gauss-Bonnet term in the effective Lagrangian density takes the form

$$\Delta\mathcal{L} = \phi(a, S) \left\{ R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma} - 4 R_{\mu\nu} R^{\mu\nu} + R^2 \right\} . \tag{H.16}$$

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