HRI/ST/0916

Black Hole Hair Removal: Non-linear Analysis

Dileep P. Jatkar, Ashoke Sen and Yogesh K. Srivastava

Harish-Chandra Research Institute
Chhatnag Road, Jhusi, Allahabad 211019, INDIA

E-mail: dileep, sen, yogesh@mri.ernet.in

Abstract

BMPV black holes in flat transverse space and in Taub-NUT space have identical near horizon geometries but different microscopic degeneracies. It has been proposed that this difference can be accounted for by different contribution to the degeneracies of these black holes from hair modes, – degrees of freedom living outside the horizon. In this paper we explicitly construct the hair modes of these two black holes as finite bosonic and fermionic deformations of the black hole solution satisfying the full non-linear equations of motion of supergravity and preserving the supersymmetry of the original solutions. Special care is taken to ensure that these solutions do not have any curvature singularity at the future horizon when viewed as the full ten dimensional geometry. We show that after removing the contribution due to the hair degrees of freedom from the microscopic partition function, the partition functions of the two black holes agree.

Contents

1 Introduction

String theory has been successful in providing an explanation of the entropy of supersymmetric extremal black holes in terms of microscopic degrees of freedom. Initial studies focussed on black holes carrying large charges for which the classical two derivative action, and the associated formula for the entropy due to Bekenstein and Hawking, is sufficient to compute the entropy. This assumption can be relaxed to some extent using Wald's formula for black hole entropy[1, 2, 3, 4] that takes into account higher derivative corrections to the classical action. However a complete expression for the entropy of a black hole receives contribution from higher derivative corrections as well as quantum corrections. On general grounds one would expect that the generalization of Wald's formula to the full quantum theory will involve some computation in string theory on the near horizon geometry of the black hole and will not be sensitive to the nature of the solution away from the horizon[5]. Indeed Wald's classical formula for the entropy certainly satisfies this criterion.

This simple assumption has a non-trivial consequence: two different black holes with identical near horizon geometries have the same macroscopic entropy. The equality of the macroscopic and the microscopic entropy would then imply that they must have the same microscopic entropy. There is however a counterexample: a rotating black hole in type IIB string theory compactified on $K3 \times S^1$, known as the BMPV black hole[6], placed in a flat transverse space and in Taub-NUT space[7] have identical near horizon geometries[8] but different microscopic degeneracies[9, 10, 11, 12, 13, 14, 15]!

The following resolution to this puzzle was proposed in [16]. Whereas an appropriate computation in string theory in the near horizon geometry of the black hole would give the macroscopic entropy associated with the horizon, the full macroscopic entropy also involves contribution from the hair degrees of freedom – degrees of freedom living outside the horizon. For a supersymmetric black hole the latter can be computed by identifying classical supersymmetry preserving normalizable deformations[1] of the black hole solution with support outside the horizon, and then carrying out geometric quantization on the space of these solutions. Ref.[16] identified a class of such deformations both for the BMPV black hole in flat transverse space and BMPV black hole in Taub-NUT space and found that after removing the contribution from these hair degrees of freedom from the microscopic degeneracy formulæ, one obtains identical result for the two black holes. This can then be identified as the common contribution to the degeneracy coming from the horizon.

The purpose of this paper is to fill some of the gaps in the analysis of [16]. These are of three types:

1. Ref.[16] identified the bosonic deformations of the black hole solution by working with the linearized equations of motion. We extend them to the solutions to full non-linear equations of motion.

2. Ref.[16] gave a general argument for the existence of a certain set of fermionic deformations but did not construct them explicitly. We construct these fermionic modes by

[1]The deformations we shall consider will always be along a null vector or tensor and hence the norm will vanish identically. We shall call a deformation normalizable if it vanishes at infinity and produces a configuration with finite ADM mass / charge.

solving the equations of motion of the fermions around the BMPV black hole background.

3. Ref.[16] did not study supersymmetry properties of the deformations explicitly. We demonstrate that the deformations preserve the same number of supersymmetries as the original BMPV black hole background.

During this investigation we also found an unexpected result: one set of deformations for each black hole have mild curvature singularities in the future horizon when viewed as ten dimensional geometries[17, 18]. This forces us to remove these modes from the counting of the hair degrees of freedom. Fortunately however they give identical contribution to the partition function for both black holes and hence even after removing their contribution from the hair partition function, we continue to get agreement between the partition functions of the two black holes after hair removal.

In order to guide the reader through the rest of the paper we shall now briefly list the hair modes of both types of black holes which we shall construct. Since the solution is independent of the coordinate along S^1 it is often useful to regard this as a string like object extended along S^1. In this case a left-moving mode will represent a set of deformations labelled by an arbitrary function of the light-cone coordinate that describes propagation of a plane wave along the negative S^1 direction. We begin with BMPV black hole in flat transverse space. In this case the hair modes are expected to consist of *(i)* four left-moving bosonic modes describing the transverse oscillations of the black string and *(ii)* four left-moving fermionic modes describing propagation of the goldstino modes associated with some broken supersymmetries. On the other hand BMPV black hole in transverse Taub-NUT space is expected to carry *(i)* three left-moving bosonic modes describing the oscillation of the black string in three transverse directions,[2] *(ii)* 21 left-moving bosonic modes arising from certain oscillation modes of the 2-form fields, *(iii)* four left-moving fermionic modes describing propagation of the goldstino modes associated with some broken supersymmetries, and *(iv)* four more left-moving bosonic modes describing the transverse oscillation of the BMPV black string *relative to the* Taub-NUT space. We explicitly construct each of these modes in our analysis.[3] §2 is devoted to the construction of the hair modes of BMPV black hole in flat transverse space, §3 contains the

[2]Asymptotically Taub-NUT has the form of $\widetilde{S}^1 \times \mathbf{R}^3$; thus there are three transverse directions.

[3]In this context we would like to mention that since most of our argument towards the absence (or triviality) of higher order corrections to the solution is due to our inability to contract indices, and do not need to make explicit use of the form of the action, we expect these deformations to survive even after inclusion of higher derivative corrections.

construction of the hair modes of BMPV black hole in Taub-NUT space and §4 contains a proof that the modes constructed in §2 and §3 preserve all the supersymmetries of the undeformed background. However we show in appendix C following [17, 18, 19, 20] that the four bosonic modes describing the transverse oscillations of the black string in flat transverse space and the four bosonic modes describing the transverse oscillations of the black string relative to the Taub-NUT space have mild curvature singularity at the future even horizon, Thus they should not be counted as hair degrees of freedom. In §5 we compute the partition function associated with the horizons of the two black holes by dividing the microscopic partition function by the partition function associated with the hair and show that the results match.

Finally we note that besides the hair modes described above, both black holes carry twelve fermionic zero modes associated with the broken supersymmetry generators. The construction of these zero modes is straightforward[21]; we take a local supersymmetry transformation whose parameter approaches a constant spinor other than the Killing spinor at infinity and vanishes at the horizon, and apply it to the original black hole solution to generate a fermionic zero mode. Since there are 12 independent supersymmetry transformations whose parameters do not approach a Killing spinor at infinity, this generates 12 fermion zero modes. We shall not discuss the construction of these zero modes any further, but count them in computing the partition function of the hair modes in §5.

2 BMPV Black Hole Hair

In this section we shall analyze the deformations of the BMPV black hole representing its hair modes, i.e. deformations which live outside the horizon and do not change the near horizon geometry. The theory that we shall study is type IIB supergravity compactified on $K3$[22, 23, 24]. The effective six dimensional theory of massless fields that one gets has many fields but we shall list only those which will play a role in our analysis. We denote by Φ the ten dimensional dilaton, by G_{MN} $(0 \le M, N \le 5)$ the string metric in six dimensions, by $C^{(2)}$ the RR 2-form field and by $F^{(3)} = d\,C^{(2)}$ the associated field strength. The theory also has several other 2-form fields. One of them comes from the NSNS sector and has no constraint on its field strength, but there are 22 others obtained by dimensional reduction of the RR 4-form on 2-cycles of K3, of which 19 have anti-self-dual field strength and 3 have self-dual field strength. Including the RR 2-form field $C^{(2)}$ of the ten dimensional theory, we have altogether 21 2-form fields with anti-self-dual field strength and 5 2-form fields with self-dual field strength. We shall denote the self-dual and the anti-self-dual field strengths by $\bar{H}^k_{MNP}$ $(1 \leq k \leq 5)$ and H^s_{MNP} $(6 \leq s \leq 26)$ respectively, satisfying

$$\bar{H}^{kMNP} = \frac{1}{3!} \,|\det g|^{-1/2} \epsilon^{MNPQRS}\, \bar{H}^k_{QRS}, \qquad H^{sMNP} = -\frac{1}{3!} \,|\det g|^{-1/2} \epsilon^{MNPQRS}\, H^s_{QRS}\,, \quad (2.1)$$

where ϵ^{MNPQRS} is the totally anti-symmetric symbol. We shall describe our choice of the sign convention for ϵ shortly. The theory also contains a set of scalar fields besides the dilaton, coming from the moduli of $K3$, the RR scalar, as well as the components of the NSNS 2-form field and the RR 2- and 4-form fields along the two and four cycles of $K3$. Throughout this paper we shall set all the scalar fields including the dilaton to fixed values.[4] The fermion fields in this six dimensional theory consist of a set of four left-chiral gravitini Ψ^α_μ $(0 \leq \mu \leq 5$, $1 \leq \alpha \leq 4)$ and a set of 4×21 right-chiral spin 1/2 fermions $\chi^{\alpha r}$ $(1 \leq r \leq 21)$. The precise form of the chirality projection rules is given in (2.30), (2.31). Note that we have suppressed the Dirac indices.

The field strengths $\bar{H}^k_{MNP}$ and H^s_{MNP} will include the self-dual and anti-self-dual parts of $F^{(3)}$. We shall choose the convention where $\bar{H}^1$ and H^6 denote the self-dual and anti-self-dual components of $F^{(3)}$ up to a normalization. More precisely we choose

$$F^{(3)}_{MNP} = 2\, e^{-\Phi} \left(\bar{H}^1_{MNP} + H^6_{MNP}\right)\,, \quad (2.2)$$

where Φ is the constant value of the dilaton field. The self-dual-field strengths $\bar{H}^2, \cdots \bar{H}^5$ will be set to zero and will play no role throughout our analysis. In the sector where all the scalar fields are constants and fermions are set to zero, the bosonic equations of motion take the form:[5]

$$\begin{aligned} R_{MN} &= \bar{H}^k_{MPQ}\, \bar{H}^{kPQ}_N + H^s_{MPQ}\, H^{sPQ}_N \\ & \bar{H}^k_{MNP} H^{sMNP} = 0\,, \end{aligned} \quad (2.3)$$

where R_{MN} is the Ricci tensor defined in the sign convention in which on the sphere the Ricci scalar $G^{MN} R_{MN}$ is positive.

[4]When the background scalar fields are not constants, the self-dual and anti-self-dual field strengths are not closed but can be expressed as linear combinations of closed 3-forms with coefficients given by functions of the scalar fields[22, 23, 24]. This complication is absent when the scalars are constants in space-time.

[5]This requires appropriate normalization factors appearing in the definition of the $\bar{H}^k$'s and H^s's in terms of the fundamental fields of string theory. Typically these normalization factors will be functions of the various scalar fields in six dimensions but as long as the scalar fields are frozen to constant values we do not have to worry about these normalizations.

We now further compactify the theory on S^1 and consider a rotating black hole solution describing Q_5 D5-branes along $K3\times S^1$, Q_1 D1-branes along S^1, $-n$ units of momentum along S^1 and angular momentum J[6]. We denote by x^5 the coordinate of the circle S^1 with period $2\,\pi\,R_5$, by $(2\pi)^4\,V$ the volume of $K3$ measured in the string metric, and by λ the asymptotic value of the string coupling. As in [16] we shall set the asymptotic values of the scalar fields to their attractor values to keep the solution simple. We also denote by t the time coordinate and by w_i $(1\le i\le 4)$ the four non-compact spatial coordinates. Finally we denote by (r,θ,ϕ,x^4) the Gibbons-Hawking coordinates of the four dimensional space labelled by $\vec w$ so that we have

$$\begin{aligned} w^1 &= 2\sqrt{r}\cos\frac{\theta}{2}\cos\frac{x^4+\phi}{2}, &\qquad w^2 &= 2\sqrt{r}\cos\frac{\theta}{2}\sin\frac{x^4+\phi}{2},\\ w^3 &= 2\sqrt{r}\sin\frac{\theta}{2}\cos\frac{x^4-\phi}{2}, &\qquad w^4 &= 2\sqrt{r}\sin\frac{\theta}{2}\sin\frac{x^4-\phi}{2},\\ (\theta,\phi,x^4) &\equiv (2\pi-\theta,\phi+\pi,x^4+\pi) &\equiv\ & (\theta,\phi+2\pi,x^4+2\pi)\equiv(\theta,\phi,x^4+4\pi)\,. \end{aligned} \tag{2.4}$$

In this case we have

$$r=\frac{1}{4}\,w^iw^i\,,\quad dw^idw^i = r\,(dx^4+\cos\theta d\phi)^2+\frac{1}{r}\,(dr^2+r^2d\theta^2+r^2\,\sin^2\theta\,d\phi^2)\,, \tag{2.5}$$

and the solution takes the form

$$\begin{aligned} ds^2 &\equiv G_{MN}dx^Mdx^N\\ &= \psi^{-1}(r)\left[du\,dv+(\psi(r)-1)\,dv^2+\chi_i(r)\,dv\,dw^i\right]+\psi(r)dw_idw_i\\ e^{-2\Phi} &= \lambda^{-2}\,,\\ F^{(3)} &\equiv \frac{1}{6}\,F^{(3)}_{MNP}dx^M\wedge dx^N\wedge dx^P=\frac{r_0}{\lambda}\left(\epsilon_3+*_6\epsilon_3+\frac{1}{r_0}\psi^{-1}(r)dv\wedge d\zeta\right)\,, \end{aligned} \tag{2.6}$$

where G_{MN} is the six dimensional string metric and

$$\begin{aligned} u &\equiv x^5-t,\quad v\equiv x^5+t,\\ \psi(r) &\equiv \left(1+\frac{r_0}{r}\right)\,,\\ \chi_i\,dw^i &\equiv -2\,\zeta\,,\quad \zeta\equiv-\frac{\widetilde J}{8r}\,(dx^4+\cos\theta d\phi)\,,\\ \epsilon_3 &\equiv \sin\theta\,dx^4\wedge d\theta\wedge d\phi\,. \end{aligned} \tag{2.7}$$

Here $*_6$ denotes Hodge dual in the six dimensions spanned by t, x^5, x^4, r, θ and ϕ with the convention $\epsilon^{t54r\theta\phi}=1$. The constants r_0 and $\widetilde J$ are given in terms of the charges and the asymptotic values of the moduli fields as follows:

$$r_0 = \frac{\lambda(Q_1 - Q_5)}{4V} = \frac{\lambda Q_5}{4} = \frac{\lambda^2 |n|}{4R_5^2 V}\,, \qquad \widetilde{J} = \frac{J\,\lambda^2}{2\,R_5\,V}\,. \tag{2.8}$$

Eq.(2.8) gives specific relations between V, λ and R_5 reflecting the fact we have chosen them to coincide with the attractor values instead of keeping them general. For later use we note that the background metric and the three form field strengths can be expressed as

$$\begin{aligned} ds^2 &= -(e^0)^2 + (e^1)^2 + (e^2)^2 + (e^3)^2 + (e^4)^2 + (e^5)^2\,, \\ F^{(3)} &= \frac{r_0}{\lambda\, r^2}\,\Big[\psi^{-3/2}(r)\, r^{1/2}\,(e^2\wedge e^4\wedge e^5 + e^0\wedge e^1\wedge e^3) \\ &\quad + \frac{\widetilde{J}}{8\,r_0}\,\psi^{-2}(r)\,(-e^0\wedge e^2\wedge e^3 + e^0\wedge e^4\wedge e^5 - e^1\wedge e^2\wedge e^3 + e^1\wedge e^4\wedge e^5)\Big]\,, \end{aligned} \tag{2.9}$$

where

$$\begin{aligned} e^0 &= \psi^{-1}(r)\,(dt+\zeta), \\ e^1 &= \big(dx^5 + dt - \psi^{-1}(r)(dt+\zeta)\big)\,, \\ e^2 &= \psi^{1/2}(r)\,r^{1/2}(dx^4 + \cos\theta d\phi), \\ e^3 &= \psi^{1/2}(r)\,r^{-1/2}\,dr\,, \\ e^4 &= \psi^{1/2}(r)\,r^{1/2}\,d\theta\,, \\ e^5 &= \psi^{1/2}(r)\,r^{1/2}\,\sin\theta\,d\phi\,. \end{aligned} \tag{2.10}$$

The one forms e^A are related to the vielbeins e_M^A via the relations

$$e^A = e_M^A dx^M\,. \tag{2.11}$$

Here A labels a tangent space index. From (2.9) it follows that the fields strength $F^{(3)}$ appearing in (2.6) is self-dual. Thus in the black hole background all the anti-self-dual field strengths H^s_{MNP}'s vanish.

The near horizon geometry of (2.6) is obtained by introducing new coordinates ρ, τ via:

$$r = r_0\,\beta\rho, \quad t = \tau/\beta, \tag{2.12}$$

and taking the limit $\beta \to 0$ keeping τ, v, ρ, x^4, θ and ϕ finite. In this limit (2.6) takes the form

$$ds^2 = r_0\frac{d\rho^2}{\rho^2} + dv^2 + r_0(dx^4+\cos\theta d\phi)^2 + \frac{\widetilde{J}}{4r_0}dv(dx^4+\cos\theta d\phi) - 2\rho\, dv d\tau$$

$$
\begin{aligned}
& +r_0\left(d\theta^2+\sin^2\theta d\phi^2\right)\,, \\
e^{\Phi} &= \lambda\,, \\
F^{(3)} &= \frac{r_0}{\lambda}\left[\epsilon_3 + *\epsilon_3 + \frac{\widetilde{J}}{8\,r_0^2}\,dv\wedge\left(\frac{1}{\rho}\,d\rho\wedge(d\,x^4+\cos\theta\,d\phi)+\sin\theta\,d\theta\wedge d\phi\right)\right]\,. \quad (2.13)
\end{aligned}
$$

We shall now analyze various bosonic and fermionic deformations of this solution which live outside the horizon. This in particular will require that when expressed in terms of the new coordinate system (2.12) the deformations should vanish as $\beta\to 0$. Geometric quantization of these deformations are supposed to generate the degeneracies associated with the hair modes. The bosonic deformations representing transverse oscillation of the black hole were constructed in [16] at the linearized level. Here however we shall go beyond the linearized approximation and construct the fully backreacted solution.

2.1 Bosonic deformations representing transverse oscillation of the BMPV black hole

In this section we shall follow [25, 26, 27, 28, 29] to construct deformations describing left-moving transverse oscillations of the black hole. Even though these deformations will turn out to be singular at the future horizon[17, 18] and hence will not be counted among the hair degrees of freedom, we shall go through it carefully as similar deformations of the four dimensional solution will turn out to be non-singular and hence will correspond to hair degrees of freedom.

Given a space-time with metric G_{MN} satisfying the supergravity equations and a null, killing and hypersurface orthogonal vector field k_M, *i.e.*, satisfying the following properties

$$
k^M k_M = 0,\; k_{M;N}+k_{N;M}=0,\; k_{M;N}=\frac{1}{2}(k_M A_{,N}-k_N A_{,M}) \quad (2.14)
$$

for some scalar function A, one can construct a new exact solution of the equations of motion by defining[25]

$$
G'_{MN} = G_{MN} + e^{-A} T k_M k_N \quad (2.15)
$$

where the function T satisfies

$$
\nabla^2 T = 0\;,\; k^M\partial_M T = 0\,. \quad (2.16)
$$

The new metric G'_{MN} describes a gravitational wave on the background of the original metric provided the matter fields, if any, satisfy some conditions. We take $(\frac{\partial}{\partial u})$ as our null killing vector. Since $G_{uu}=0$, it is obviously null and since the metric coefficients do not depend on u, it is also killing. For our case, only non-zero component of killing one-form is $k_v = G_{uv} = \psi^{-1}$ and the hypersurface-orthogonality condition (the last equation in (2.14)) is satisfied by choosing $e^{-A}=\psi$. Applying the transform we get[26, 27, 28]

$$ds^2 = \psi^{-1}(r)\left\{dudv + (\psi - 1 + T(v,\vec{w}))dv^2 + \chi_i(r)dvdw_i\right\} + \psi(r)dw_i dw_i \tag{2.17}$$

where $T(v,\vec{w})$ satisfies the flat four dimensional Laplace equation:

$$\partial_{w^i}\partial_{w^i}T(v,\vec{w}) = 0\,. \tag{2.18}$$

It also follows from the analysis of [25] that we do not need to modify the dilaton and the 2-form fields. A simple way to see this is as follows. For any component of a covariant tensor, we define the weight of that component as the number of v indices minus the number of u indices carried by the tensor. For a component of the contravariant tensor we define the weight to be the number of u indices minus the number of v indices. Then any tensor can be decomposed as a sum of tensors of fixed weights and in the contraction of covariant and contravariant indices the weight is preserved. Now by examining the background (2.6) we see that each term in the solution has weight zero or positive. On the other hand the term proportional to $T(v,\vec{w})$ in (2.17) has weight 2. Furthermore the original background as well as the deformation generated by $T(v,\vec{w})$ are u independent; hence we cannot reduce the weight by taking u derivative of the background. Thus the term proportional to $T(v,\vec{w})$ can only produce terms in the equation of motion of weight two or more. In other words it can only generate terms for which the number of covariant v indices is larger then the number of covariant u indices by at least 2. This is impossible for the dilaton equation of motion which carries no index. The equation of motion for the 2-form field has two indices, but it is anti-symmetric in these two indices. Thus it is impossible to have more than one covariant v index. The only equation of motion that can be affected by the $T(v,\vec{w})\,dv^2$ term is the vv component of the metric equation, leading to (2.18).

We can write down the general solution to (2.18) as an expansion in spherical harmonics on S^3, but after requiring regularity at the origin and at infinity and dropping terms which can be removed by coordinate transformation, we can choose

$$T(v,\vec{w}) = \vec{f}(v)\cdot\vec{w}\,, \qquad \int_0^{2\pi R_5} f_i(v)dv = 0\,, \tag{2.19}$$

for some arbitrary set of four functions $(f_1(v), \cdots f_4(v))$ subject to the restriction given above. The corresponding metric

$$ds^2 = \psi^{-1}(r)\left[dudv + \left\{\psi - 1 + \vec{f}(v)\cdot\vec{w}\right\}dv^2 + \chi_i(r)dvdw_i\right] + \psi(r)dw_i dw_i \tag{2.20}$$

is apparently not asymptotically flat but can be made so by the following coordinate transformations[6]

$$\begin{aligned} v &= v' \\ \vec{w} &= \vec{w'} + \vec{F} \\ u &= u' - 2\dot{F}_i w'_i - 2\dot{F}_i F_i + \int^{v'} \dot{F}^2(v'')dv''. \end{aligned} \tag{2.21}$$

Here $\vec{f}(v) = 2\ddot{\vec{F}}$ and dot refers to derivative with respect to v. Making this change of coordinates, the terms in metric change as follows

$$\begin{aligned} dudv &= du'dv' - 2\dot{F}_i dw'_i dv' - \dot{F}_i \dot{F}_i \, dv'^2 - 2\ddot{F}_i \left(w'_i + F_i\right) dv'^2 \\ dv\, dw_i &= dv' \left(dw'_i + \dot{F}_i \, dv'\right) \\ dw_j dw_j &= dw'_j dw'_j + \dot{F}_i \dot{F}_i \, dv'^2 + 2\dot{F}_i \, dw'_i \, dv' \,. \end{aligned} \tag{2.22}$$

Removing the primes, we write the above metric as

$$ds^2 = H^{-1} dudv + H dw_j^2 + A_j dw_j dv + K dv^2 \tag{2.23}$$

where

$$\begin{aligned} &H = 1 + \frac{4\, r_0}{|\vec{w} + \vec{F}(v)|^2}, \quad K = 1 - H^{-1} + (H - H^{-1})\dot{F}^2(v) + H^{-1}\, \chi_j \dot{F}_j(v), \\ &A_j = H^{-1}\chi_j + 2\dot{F}_j (H - H^{-1})\,. \end{aligned} \tag{2.24}$$

Since $A_j \to 0$, $K \to 0$ and $H \to 1$ as $|\vec{w}| \to \infty$, the metric is asymptotically flat. Note however that this change of coordinates changes the location of the horizon, and hence it is not apparent that the deformation lives outside the horizon. To overcome this we shall make the coordinate transformation that takes the form given in (2.21) for large r but which becomes identity near the horizon. In this case the coordinates near the horizon are the original coordinates $(v, u, \vec{w})$ and the metric takes the form given in (2.20). Since in the new coordinate system (2.12) $\psi^{-1}(r)\vec{f}(v) \cdot \vec{w} \sim \beta^{3/2}$, the deformation vanishes as $\beta \to 0$. Thus the deformations generated by T does not affect the near horizon geometry of the black hole, and represent candidates for hair degrees of freedom.

[6] Note that in order that the deformations preserve the asymptotic geometry, the shifted coordinates (u', v') should be identified with $(x^5 \mp t)$. Thus for example the identification under $x^5 \to x^5 + 2\pi R_5$ will act as $(u', v') \to (u' + 2\pi R_5, v' + 2\pi R_5)$.

To linear order in $\vec{f}(v)$ the solution given in (2.20) can be shown to be related by a coordinate transformation to the deformation described in [16] representing transverse motion of the BMPV black hole. Thus the solution (2.20) represents, physically, finite amplitude oscillations of the BMPV black hole in the transverse direction after taking into account the backreaction of the gravitational and other fields.

Since the deformation parameters $\vec{f}(v)$ transform as a vector under the SO(4) rotation in the transverse space, we expect the deformations to carry angular momentum. This is visible explicitly in the additional term proportional to $\dot{F}_j$ appearing in the expression for A_j. Since this modifies the coefficient of the $dw^j dv$ term in the asymptotic metric given in (2.23), the deformed configuration carries additional angular momentum besides the one associated with the undeformed solution.

We shall however see in appendix C that even though these modes apparently vanish at the horizon, they in fact have curvature singularities at the future horizon. Thus they should be excluded from the counting of the hair modes.

2.2 Fermionic deformations associated with the broken supersymmetry generators of the BMPV black hole

Since the black hole solution breaks twelve of the sixteen space-time supersymmetries, we expect to have twelve fermionic zero modes living on the black hole, forming part of the black hole hair. It was argued in [16] that four of these lift to full left-moving fields on the two dimensional world volume of the black hole spanned by t and x^5. In that case we should be able to construct solutions to the equations of motion of the fermion fields labelled by four independent functions of v. We shall now explicitly construct these solutions in the undeformed background (2.6) and then argue that the solutions remain unaffected by the deformation described in (2.17). We shall follow the notation of [24].

The linearized equation of motion of Ψ_M^α and $\chi^{\alpha r}$ in the background where all the scalars are constants and $\chi^{\alpha r}$ are set to zero are

$$\begin{aligned}
&\Gamma^{MNP} D_N \Psi_P^\alpha - \bar{H}^{kMNP} \Gamma_N \,\widehat{\Gamma}^k_{\alpha\beta} \Psi_P^\beta = 0, \\
&H^{sMNP}\, \Gamma_{MN} \Psi_P^\alpha = 0\,,
\end{aligned} \tag{2.25}$$

where

$$D_M \Psi_P^\alpha = \partial_M \Psi_P^\alpha - \Gamma^N_{MP} \Psi_N^\alpha + \frac{1}{4}\,\omega_M^{AB}\,\widetilde{\Gamma}^{AB}\,\Psi_P^\alpha\,, \tag{2.26}$$

$$\Gamma^M_{NP} \equiv \frac{1}{2} G^{MR} (\partial_N G_{PR} + \partial_P G_{NR} - \partial_R G_{NP}), \qquad \omega_M^{AB} \equiv -G^{NP} e_N{}^B \partial_M e_P{}^A + e_N{}^A e_P{}^B G^{PQ} \Gamma^N_{QM} \,. \tag{2.27}$$

Since in our background $F^{(3)}$ is self-dual, we have $H^{sMNP} = 0$ and hence the second set of equations in (2.25) is automatically satisfied. The first set of equations involves only the self-dual part of the 3-form denoted by $\bar{H}^k_{MNP}$ for $1 \leq k \leq 5$. In (2.25) Γ^M's $(0 \leq M \leq 5)$ denote 8×8 $SO(5,1)$ gamma matrices written in the coordinate basis and $\widehat{\Gamma}^i$ denote the 4×4 $SO(5)$ gamma matrices, satisfying

$$\{\Gamma^M, \Gamma^N\} = 2\, G^{MN}, \qquad \{\widehat{\Gamma}^k, \widehat{\Gamma}^l\} = 2\, \delta_{kl} \,, \tag{2.28}$$

and $\Gamma^{M_1 \cdots M_k}$ is the totally anti-symmetric product of $\Gamma^{M_1}, \cdots \Gamma^{M_k}$. It will also be useful to introduce the gamma matrices $\widetilde{\Gamma}^A$ carrying SO(5,1) tangent space indices:

$$\widetilde{\Gamma}^A = e_M^A \Gamma^M, \qquad \{\widetilde{\Gamma}^A, \widetilde{\Gamma}^B\} = 2\, \eta^{AB} \,. \tag{2.29}$$

In this convention the fields Ψ^α_M and $\chi^{\alpha r}$ satisfiy chirality projection conditions

$$\left(\frac{1}{6!} \, |\det g|^{-1/2} \, \epsilon^{MNPQRS} \Gamma_{MNPQRS} + 1 \right) \Psi^\alpha_M = 0 \quad \rightarrow \quad (\widetilde{\Gamma}_{012345} + 1)\, \Psi^\alpha_M = 0 \,, \tag{2.30}$$

$$\left(\frac{1}{6!} \, |\det g|^{-1/2} \, \epsilon^{MNPQRS} \Gamma_{MNPQRS} - 1 \right) \chi^{\alpha r} = 0 \,. \tag{2.31}$$

Note that the tangent space indices are raised and lowered by the flat metric η_{AB}. It follows from (2.10) that

$$\widetilde{\Gamma}^0 + \widetilde{\Gamma}^1 = \Gamma^v \,. \tag{2.32}$$

To solve (2.25), we make the following ansatz for the gravitino fields:

$$\Psi^\alpha_M = 0 \quad \text{for} \quad M \neq v \,, \tag{2.33}$$

and furthermore that Ψ^α_v is u-independent. We also impose a gauge condition on Ψ^α_M

$$\Gamma^M \Psi^\alpha_M = 0 \quad \rightarrow \quad \Gamma^v \Psi^\alpha_v = 0. \tag{2.34}$$

Using (2.32) this may be expressed as

$$(\widetilde{\Gamma}^0 + \widetilde{\Gamma}^1) \Psi^\alpha_v = 0 \quad \rightarrow \quad \widetilde{\Gamma}^0 \, \widetilde{\Gamma}^1 \, \Psi^\alpha_v = \Psi^\alpha_v \,. \tag{2.35}$$

Since the only non-vanishing component of the gravitino is Ψ^α_v, we see that in the convention described below (2.18) the fermionic deformation has weight 1. Note that we do not assign any weight to the SO(5,1) or SO(5) spinor indices. Consider now a term in the equation of motion that is linear in the gravitino field. Since the fields in the original background are all of weight ≥ 0, multiplying the gravitino by these fields cannot reduce the weight. Furthermore since Ψ_v^α as well as all other background fields is u independent, we cannot reduce the weight by acting with a u derivative on the gravitino. Finally we also cannot reduce the weight by acting with a Γ^v on the gravitino due to eq.(2.34).[7] Thus we conclude that any term in the equation of motion that involves at least one power of the gravitino must be of weight ≥ 1. This in turn shows that the only non-trivial component of the equation of motion (2.25) is the one associated with the choice $M = u$. For this choice (2.25) takes the form:

$$\Gamma^{uiv}\left(\partial_i + \frac{1}{4}\,\omega_i^{AB}\,\widetilde{\Gamma}^{AB}\right)\Psi_v^\alpha - \bar{H}^{kuiv}\Gamma_i\,\widehat{\Gamma}^k_{\alpha\beta}\Psi_v^\beta = 0\,. \tag{2.36}$$

The above analysis also tells us that in computing the right hand side of (2.36) we only need to keep terms in the background fields of weight zero. Thus we can ignore the terms proportional to dv^2 and $dvdw^i$ in the metric and the term proportional to $\widetilde{J}$ in $F^{(3)}$. This allows us to choose the vielbeins to be of the form:

$$\begin{aligned}
&e^0 = \frac{1}{2}(dv - \psi^{-1}du), \qquad e^1 = \frac{1}{2}(dv + \psi^{-1}du),\\
&e^2 = \psi^{1/2}r^{1/2}(dx^4 + \cos\theta d\phi), \qquad e^3 = \psi^{1/2}r^{-1/2}dr,\\
&e^4 = \psi^{1/2}r^{1/2}d\theta, \qquad e^5 = \psi^{1/2}r^{1/2}\sin\theta\, d\phi\,.
\end{aligned} \tag{2.37}$$

The associated non-vanishing components of the spin connection are given by

$$\begin{aligned}
&\omega_r^{01} = -\frac{1}{2}\,\frac{\psi'}{\psi}\,, \quad \omega_{x^4}^{23} = \frac{1}{2}\,\frac{(r\psi)'}{\psi}\,, \quad \omega_{x^4}^{45} = \frac{1}{2}, \quad \omega_\phi^{23} = \frac{1}{2}\,\frac{(r\psi)'}{\psi}\,\cos\theta, \quad \omega_\phi^{24} = -\frac{1}{2}\,\sin\theta,\\
&\omega_\phi^{35} = -\frac{1}{2}\,\frac{(r\psi)'}{\psi}\,\sin\theta, \quad \omega_\phi^{45} = -\frac{1}{2}\,\cos\theta\,, \quad \omega_\theta^{25} = \frac{1}{2}, \quad \omega_\theta^{34} = -\frac{1}{2}\,\frac{(r\psi)'}{\psi}\,.
\end{aligned} \tag{2.38}$$

The same argument implies that terms quadratic and higher powers in the gravitino fields, being of weight two or more, cannot affect the gravitino field equations. The only equation it could possibly affect is the vv component of the metric equation, but the projection condition (2.35) rules this out since it makes it impossible to construct gravitino bilinears without any spinor index unless one uses insertion of a Γ^u that increases the weight further. Thus a solution to (2.36) will give an exact solution to the equations of motion.

[7] If there are a set of other gamma matrices between the Γ^v and the gravitino, we can still bring Γ^v next to the gravitino using eq.(2.28), and none of the extra terms generated in this process can reduce the weight.

Eq.(2.36) can be manipulated as follows. First of all the Γ^{uiv} factor may be expressed as a sum of six terms with each term containing a different arrangement of Γ^u, Γ^i and Γ^v. The terms where Γ^v is to the extreme right vanish due to (2.34). In the other terms we can bring Γ^v to the extreme right using (2.28) and then use (2.34) again. This allows us to reduce the Γ^{uiv} factor to a single gamma matrix and leads to the equation:

$$\Gamma^i\, G^{uv} \left(\partial_i + \frac{1}{4}\,\omega_i^{AB}\,\widetilde{\Gamma}^{AB}\right)\Psi_v^\alpha - \Gamma^i\, G^{uv}\, G^{vu}\,\bar{H}^k_{viu}\,\widehat{\Gamma}^k_{\alpha\beta}\Psi_v^\beta = 0\,. \tag{2.39}$$

Now dropping an overall G^{uv} factor, using (2.2) and the fact that $F^{(3)}$ is self-dual, we get

$$\Gamma^i \left(\partial_i + \frac{1}{4}\,\omega_i^{AB}\,\widetilde{\Gamma}^{AB}\right)\Psi_v^\alpha + \frac{\lambda}{2}\, G^{vu}\, F^{(3)}_{ivu}\,\Gamma^i\widehat{\Gamma}^1_{\alpha\beta}\Psi_v^\beta = 0\,. \tag{2.40}$$

These equations are written in a covariant form in the transverse coordinates. Thus the sum over i can be taken either over the coordinates $(w^1, \cdots w^4)$ or over the coordinates (r, x^4, θ, ϕ). We shall use the (r, x^4, θ, ϕ) coordinates. Using eqs.(2.30), (2.35) we arrive at the following equation:

$$\begin{aligned}&\psi^{-1/2} r^{1/2}\widetilde{\Gamma}^3\left(\partial_r + \frac{\psi'}{\psi} + \frac{1}{r} - \frac{1}{2}\frac{\psi'}{\psi}\,\widehat{\Gamma}^1\right)\Psi_v + (r\psi)^{-1/2}(\sin\theta)^{-1}\widetilde{\Gamma}^5\partial_\phi\Psi_v \\ &+(r\psi)^{-1/2}(\widetilde{\Gamma}^2 - \cot\theta\,\widetilde{\Gamma}^5)\,\partial_{x^4}\Psi_v + (r\psi)^{-1/2}\widetilde{\Gamma}^4\left(\partial_\theta + \frac{1}{2}\cot\theta\right)\Psi_v = 0\,.\end{aligned} \tag{2.41}$$

In looking for solutions to these equations we use the fact that the gravitino deformation we are looking for carries no x^4 momentum and carries $\pm 1/2$ units of ϕ momentum[16]. Thus we can require

$$\partial_{x^4}\Psi_v = 0, \qquad \partial_\phi\Psi_v = i\,m\,\Psi_v, \quad m = \pm\frac{1}{2}\,. \tag{2.42}$$

Substituting this into (2.41) we get

$$\begin{aligned}&\psi^{-1/2} r^{1/2}\widetilde{\Gamma}^3\left(\partial_r + \frac{\psi'}{\psi} + \frac{1}{r} - \frac{1}{2}\frac{\psi'}{\psi}\,\widehat{\Gamma}^1\right)\Psi_v \\ &+i\,m\,(r\psi)^{-1/2}(\sin\theta)^{-1}\widetilde{\Gamma}^5\,\Psi_v + (r\psi)^{-1/2}\widetilde{\Gamma}^4\left(\partial_\theta + \frac{1}{2}\cot\theta\right)\Psi_v = 0\,.\end{aligned} \tag{2.43}$$

We shall now rewrite this equation as

$$\begin{aligned}&\psi^{-1/2} r^{1/2}\widetilde{\Gamma}^3\left(\partial_r + \frac{\psi'}{\psi} - \frac{1}{2}\frac{\psi'}{\psi}\,\widehat{\Gamma}^1\right)\Psi_v \\ &+(r\psi)^{-1/2}\widetilde{\Gamma}^4\left[i\,m\,(\sin\theta)^{-1}\widetilde{\Gamma}^4\widetilde{\Gamma}^5 + \widetilde{\Gamma}^4\widetilde{\Gamma}^3 + \left(\partial_\theta + \frac{1}{2}\cot\theta\right)\right]\Psi_v = 0\,.\end{aligned} \tag{2.44}$$

We shall now find solutions to this equation by separately setting to zero the terms in the two lines. For this we use the following representation[8] of $\widetilde{\Gamma}^3$, $\widetilde{\Gamma}^4$ and $\widetilde{\Gamma}^5$:

$$\widetilde{\Gamma}^4 = \sigma^1, \quad \widetilde{\Gamma}^5 = \sigma^2, \quad \widetilde{\Gamma}^3 = \sigma^3 . \tag{2.45}$$

Setting the second line of (2.44) to zero then gives

$$\left[\partial_\theta + \frac{1}{2}\cot\theta - m\,(\sin\theta)^{-1}\,\sigma^3 - i\sigma^2\right]\Psi_v = 0 . \tag{2.46}$$

This has the following non-singular solutions:

$$\begin{aligned}
\Psi_v &\propto e^{i\phi/2}\begin{pmatrix}\cos(\theta/2)\\ -\sin(\theta/2)\end{pmatrix} \quad \text{for} \quad m = \frac{1}{2}\,, \\
\Psi_v &\propto e^{-i\phi/2}\begin{pmatrix}\sin(\theta/2)\\ \cos(\theta/2)\end{pmatrix} \quad \text{for} \quad m = -\frac{1}{2}\,,
\end{aligned} \tag{2.47}$$

where the 'constants' of proportionality could involve arbitrary functions of r and v. Note that we have included in (2.47) the ϕ dependence of Ψ_v. On the other hand the equation obtained by setting to zero the first line of (2.44) gives

$$\left[\partial_r + \frac{\psi'}{\psi} - \frac{1}{2}\frac{\psi'}{\psi}\widehat{\Gamma}^1\right]\Psi_v = 0 . \tag{2.48}$$

Now since $(\widehat{\Gamma}^1)^2 = 1$, $\widehat{\Gamma}^1$ has eigenvalues ± 1. Thus we can try to solve this equation separately in the sector with $\widehat{\Gamma}^1$ eigenvalue 1 and $\widehat{\Gamma}^1$ eigenvalue -1. The solutions are

$$\begin{aligned}
\Psi_v &= \psi^{-3/2}\,\eta(v,\theta,\phi) \quad \text{for} \quad \widehat{\Gamma}^1\,\eta = -\eta\,, \\
\Psi_v &= \psi^{-1/2}\,\eta(v,\theta,\phi) \quad \text{for} \quad \widehat{\Gamma}^1\eta = \eta\,,
\end{aligned} \tag{2.49}$$

where $\eta(v,\theta,\phi)$ is an SO(5,1) spinor and also an $SO(5)$ spinor. The (θ,ϕ) dependence of $\eta(v,\theta,\phi)$ was computed in (2.47) and the v dependence is arbitrary except for the periodicity requirement imposed by the period of the coordinate x^5. Both these solutions vanish as we approach the horizon although the first solution vanishes more rapidly. Thus at this stage both would appear to be acceptable solutions. However we shall see in §4 that only the first solution preserves supersymmetry and hence only these deformations will contribute to the

[8] If we want to append $\widetilde{\Gamma}^2$ to this list we can take the direct product of the matrices given in (2.45) with σ_3 and represent $\widetilde{\Gamma}^2$ as $I_2 \times \sigma_1$. This construction can be easily extended to include $\widetilde{\Gamma}^0$ and $\widetilde{\Gamma}^1$ as well but will not affect the analysis following (2.45).

index. Furthermore we shall see in appendix C that the second solution is singular at the future horizon and hence should not be counted as a true hair degree of freedom. Thus Ψ_v satisfies

$$\widehat{\Gamma}^1 \Psi_v = -\Psi_v \,, \tag{2.50}$$

and the deformations associated with the gravitino field take the form:

$$\Psi_v = \psi^{-3/2}(r)\, \eta(v,\theta,\phi), \qquad (\widetilde{\Gamma}^0 + \widetilde{\Gamma}^1)\, \eta(v,\theta,\phi) = 0\,, \qquad \widehat{\Gamma}^1 \eta(v,\theta,\phi) = -\eta(v,\theta,\phi)\,. \tag{2.51}$$

Note that since $\psi \to 1$ as $r \to \infty$, the solution is r independent at infinity and hence is not normalizable. This however can be rectified by making a local supersymmetry transformation with a parameter that approaches $-\int^v dv'\eta(v',\theta,\phi)$ as $r \to \infty$ and which vanishes sufficiently fast as we approach the horizon. This sets the r independent part of the gravitino to zero at infinity but does not affect the mode near the horizon.

Let us now count the number of independent functions characterizing this deformation. To begin with Ψ_v is an $8 \times 4 = 32$ dimensional complex spinor. But since Ψ_v is a chiral spinor of $SO(5,1)$ only 16 of the 32 components are independent. The two additional conditions listed in (2.35), (2.50) cut down the number of independent complex components to 4. Finally we need to recall that the gravitino field satisfies a symplectic Majorana condition[22, 23, 24]. This gives altogether four independent real functions of v labelling the deformation, as expected. It follows from (2.47) that half of these deformations carry 1/2 unit of ϕ momentum and the other half carries $-1/2$ unit of ϕ momentum.

Finally note that if we switch on the deformation (2.17) then the extra terms containing at least one power of $T(v,\vec{w})$ and one power of Ψ_v will be of weight 3 and higher since the deformation associated with $T(v,\vec{w})$ is of weight 2. However the gravitino equation (2.25) does not have any weight 3 component. Thus we conclude that the extra terms proportional to (2.17) cannot affect the solution for the gravitino.

3 Four Dimensional Black Hole Hair

We shall now consider the case of four dimensional black hole obtained by placing the five dimensional black hole at the center of the Taub-NUT space. For this we introduce the Taub-NUT metric

$$ds^2_{TN} = \left(\frac{4}{R_4^2} + \frac{1}{r}\right)^{-1} (dx^4 + \cos\theta d\phi)^2 + \left(\frac{4}{R_4^2} + \frac{1}{r}\right) (dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta\, d\phi^2)\,, \tag{3.1}$$

and replace the metric $dw^i dw^i$ in (2.6) by this Taub-NUT metric. The full solution is given by[9]

$$\begin{aligned}
ds^2 &= \psi^{-1}(r)\left[du\,dv + (\psi(r)-1)\,dv^2 - 2\widetilde{\zeta}\,dv\right] + \psi(r)\,ds^2_{TN}\,, \\
e^{-2\Phi} &= \lambda^{-2}\,, \\
F^{(3)} &\equiv \frac{1}{6}\,F^{(3)}_{MNP} dx^M \wedge dx^N \wedge dx^P \\
&= \frac{r_0}{\lambda}\left[(\epsilon_3 + *_6\epsilon_3) + \frac{1}{r_0}\left(1+\frac{r_0}{r}\right)^{-1}(dx^5+dt)\wedge d\widetilde{\zeta}\right]\,, \\
\psi(r) &\equiv \left(1+\frac{r_0}{r}\right)\,, \\
\widetilde{\zeta} &\equiv -\frac{\widetilde{J}}{8}\left(\frac{1}{r}+\frac{4}{R_4^2}\right)(dx^4 + \cos\theta d\phi)\,, \\
\epsilon_3 &\equiv \sin\theta\,dx^4\wedge d\theta\wedge d\phi\,, \quad u\equiv x^5 - t, \quad v\equiv x^5+t\,.
\end{aligned} \tag{3.2}$$

It will be convenient to also introduce the coordinates

$$y^1 = r\,\sin\theta\,\cos\phi, \qquad y^2 = r\,\sin\theta\,\sin\phi, \qquad y^3 = r\,\cos\theta\,. \tag{3.3}$$

For $r >> R_4$ the asymptotic space-time locally has the form of $K3\times S^1\times\widetilde{S}^1\times\ \mathbb{R}^{3,1}$, with x^4 labelling the coordinate along the circle $\widetilde{S}^1$ and (y^1,y^2,y^3) labelling the space-like directions of $\mathbb{R}^{3,1}$. $\widetilde{S}^1$ is non-trivially fibered over the boundary S^2 of $\mathbb{R}^3$ reflecting that the space is actually Taub-NUT. Even in this modified background the field strength $F^{(3)}$ is self-dual, and hence all the anti-self-dual field strengths H^s_{MNP}'s continue to vanish.[10]

We can take the near horizon limit of (3.2) using the same coordinates introduced in (2.12) and taking the $\beta\to 0$ limit. It is easy to see that in this limit the solution (3.2) reduces to (2.13). Thus (2.6) and (3.2) have the same near horizon geometry[8, 16]. We also show in appendix C that these solutions are non-singular at the future horizon.

[9]For $R_4^6 < \widetilde{J}^2$ the projection of this metric in the $x^4 - x^5$ plane develops a negative eigenvalue, giving rise to closed time-like curves. We shall take $R_4^6 > \widetilde{J}^2$ to avoid this situation.

[10]Note that the asymptotic metric has a $dx^4 dt$ component. This can be removed by a shift of the x^4 coordinate proportional to t followed by a rescaling of t.

3.1 Bosonic deformations representing transverse oscillation of the black hole

We can generate deformations describing the oscillation of the black hole in the three transverse non-compact direction as in §2.1. In particular we deform the metric to

$$ds^2 \;=\; \psi^{-1}(r)\left[du\,dv + \left(\psi(r) - 1 + \widetilde{T}(v,\vec{y},x^4)\right)dv^2 - 2\widetilde{\zeta}\,dv\right] + \psi(r)\,ds_{TN}^2\,. \tag{3.4}$$

Again the argument below (2.18) tells us that without any modification of the scalar and the 3-form fields, (3.4) is guaranteed to be a solution to the equations of motion if $\widetilde{T}(v,\vec{y},x^4)$ is harmonic in the Taub-NUT space. Now one can verify that acting on an x^4 independent configuration the Laplacian in the Taub-NUT space is proportional to $\vec{\nabla}_y^2$ i.e. the laplacian in flat three dimensional space labelled by the Cartesian coordinates (y^1, y^2, y^3).

$$\widetilde{T}(v,\vec{y},x^4) \equiv \widetilde{T}(v,\vec{y}) = \vec{g}(v)\cdot\vec{y}\,, \qquad \int_0^{2\pi R_5} g_i(v)dv = 0\,, \tag{3.5}$$

where $(g_1(v), g_2(v), g_3(v))$ are three arbitrary functions subject to the restriction described above. These generate deformations representing transverse oscillation of the black hole with finite amplitude. Again one can show that even though the corresponding metric is not asymptotically flat, one can make a coordinate transformation

$$\begin{aligned}
&v = v', \qquad \vec{y} = \vec{y}\,' + \vec{F}, \qquad u = u' - \frac{8}{R_4^2}\dot{F}_i\, y'^i - \frac{8}{R_4^2}\dot{F}_i\, F_i + \frac{4}{R_4^2}\int^{v'} \dot{F}_i(v'')\,\dot{F}_i(v'')dv'', \\
&\frac{8}{R_4^2}\ddot{F}_i(v) \equiv g_i(v)\,,
\end{aligned} \tag{3.6}$$

to bring it to the asymptotically flat form. Furthermore to linear order in $\vec{g}(v)$ these deformations reduce to those given in [16] after a coordinate transformation. Finally to check that the deformations represent hair modes we note that in the coordinate system (2.12) they scale as β^2 and hence vanish as $\beta \to 0$. We also show in appendix C that unlike the deformations described in §2.1, these solutions are non-singular at the future horizon. The main difference between the deformations described in §2.1 and those described here is due to the fact that the former were proportional to $\vec{f}\cdot\vec{w}$ which vanish as $\sqrt{r}$ as $r \to 0$, whereas the latter, being proportional to $\vec{g}\cdot\vec{y}$, vanish as r as $r \to 0$.

3.2 Bosonic deformation representing the oscillation of the 2-form fields

Taub-NUT space has a self-dual harmonic form ω_{TN} given by

$$\omega_{TN} = -\frac{r}{4r+R_4^2}\sin\theta d\theta\wedge d\phi + \frac{R_4^2}{(4r+R_4^2)^2}\, dr\wedge(dx^4+\cos\theta d\phi)\,. \tag{3.7}$$

Now as was discussed at the beginning of §2, in type IIB string theory on $K3$ we have 21 2-form fields with anti-self-dual field strength, collectively denoted as H^s_{MNP} $(1\le s\le 21)$. We now switch on a deformation of the form[16]

$$\delta(ds^2) = \psi^{-1}(r)\left(\widetilde{T}(v,\vec{y}) + \widetilde{S}(v,\vec{y},x^4)\right)dv^2, \qquad \delta H^s = h^s(v)\; dv\wedge\omega_{TN}\,, \tag{3.8}$$

where $h^s(v)$ are arbitrary functions, $\widetilde{T}(v,\vec{y})$ is given in (3.5), and $\widetilde{S}$ is quadratic in h^s and will be determined below. We can verify, first of all, that the deformations δH^s given in (3.8) are closed and satisfy the requirement of anti-self-duality even in the presence of the metric deformation parametrized by $\widetilde{T}+\widetilde{S}$. Now it was shown in [16] that to linearized order the deformation (3.8) satisfy the equations of motion without any need to modify the background metric or the scalars due to the relation

$$F^{(3)}_{MPQ}\delta H_N^{sPQ} = 0\,, \tag{3.9}$$

so that the cross terms between the background 3-form field and the deformations do not produce a source for the metric and scalars. Thus we only need to analyze the contribution to the equations of motion from higher order terms. It follows from the arguments below (2.18) and that fact that under (2.18) the deformations δH^s and $\delta(ds^2)$ have weights one and two respectively that the only non-trivial equation that we need to check at quadratic and higher order in the deformation is the vv component of the metric equation. Furthermore the equation should involve at most linear terms in $\widetilde{T}$ and $\widetilde{S}$ without any power of h^s or two powers of h^s without any factor of $\widetilde{T}$ or $\widetilde{S}$. Expressing the metric equation as $R_{MN}\propto T_{MN}$ and the fact that the contribution to T_{MN} from the H^s fields is proportional to $H^s_{MPQ}H_N^{sPQ}$ at the quadratic order, we get[11]

$$\nabla_\perp^2\widetilde{S}(v,\vec{y},x^4) = C(v)\,R_4^2\,(4r+R_4^2)^{-4}\,, \qquad C(v)\equiv 8\,h^s(v)\,h^s(v)\,, \tag{3.10}$$

[11]It is a little easier to use the metric equation of the form $R^u_v\propto T^u_v$. In this case $\delta R^u_v = \psi^{-1}\nabla_\perp^2(\widetilde{T}+\widetilde{S}) = \psi^{-1}\nabla^2_\perp\widetilde{S}$ and $\delta T^u_v\propto\psi^{-1}(4r+R_4^2)^{-4}$, leading to (3.10).

where $\nabla_\perp^2$ denotes the Laplacian in the Taub-NUT space. In arriving at (3.10) we have used the fact that $\nabla_\perp^2 \widetilde{T}(v,\vec{y}) = 0$. Since any solution to the source free equation can be absorbed into $\widetilde{T}$ we only need to look for a particular solution. The following solution describes a normalizable deformation of the metric living outside the horizon:

$$\widetilde{S}(v,\vec{y},x^4) = \frac{C(v)r}{2R_4^2(4r+R_4^2)} \tag{3.11}$$

The $\widetilde{S}(v)$ given in (3.11) does not vanish at infinity but this can be easily repaired by an appropriate reparametrization which takes the form $u \to u - \frac{1}{8R_4^2}\int^v C(v')dv'$ as $r \to \infty$ and $u \to u$ as $r \to 0$. Finally to check that these deformations represent hair modes we note that in the coordinate system (2.12), δH^s given in (3.8) scales as β and hence vanishes as $\beta \to 0$.

This shows that we have a family of finite deformations, labelled by the 24 functions $(\vec{g}(v), \{h^s(v)\})$, of the original four dimensional black hole solution. Furthermore these deformations are supported outside the horizon and do not affect the horizon geometry. We also show in appendix C that these solutions are non-singular at the future horizon.

3.3 Fermionic deformations

Construction of the left-moving fermionic deformations proceeds in the same way as in §2.2. The analysis up to (2.40) is more or less identical except that we now have different expressions for the simplified form of vielbeins and the spin connections after dropping terms of weight > 0. Thus equations (2.37), (2.38) get replaced by:

$$\begin{aligned}
&e^0 = \frac{1}{2}(dv - \psi^{-1}du), \qquad e^1 = \frac{1}{2}(dv + \psi^{-1}du),\\
&e^2 = \psi^{1/2}r^{1/2}\chi^{-1/2}(dx^4 + \cos\theta d\phi), \qquad e^3 = \psi^{1/2}r^{-1/2}\chi^{1/2}dr,\\
&e^4 = \psi^{1/2}r^{1/2}\chi^{1/2}d\theta, \qquad e^5 = \psi^{1/2}r^{1/2}\chi^{1/2}\sin\theta\, d\phi\,,
\end{aligned} \tag{3.12}$$

$$\chi \equiv \left(1 + \frac{4r}{R_4^2}\right)\,, \tag{3.13}$$

$$\begin{aligned}
&\omega_r^{01} = -\frac{1}{2}\frac{\psi'}{\psi}\,, \quad \omega_{x^4}^{23} = \frac{1}{2}\frac{(r\psi\chi^{-1})'}{\psi}\,, \quad \omega_{x^4}^{45} = \frac{1}{2}\chi^{-2}, \quad \omega_\phi^{23} = \frac{1}{2}\frac{(r\psi\chi^{-1})'}{\psi}\cos\theta,\\
&\omega_\phi^{24} = -\frac{1}{2}\chi^{-1}\sin\theta, \quad \omega_\phi^{35} = -\frac{1}{2}\frac{(r\psi\chi)'}{\psi\chi}\sin\theta, \quad \omega_\phi^{45} = -\frac{1}{2}\left(2-\chi^{-2}\right)\cos\theta\,,\\
&\omega_\theta^{25} = \frac{1}{2}\chi^{-1}, \quad \omega_\theta^{34} = -\frac{1}{2}\frac{(r\psi\chi)'}{\psi\chi}\,.
\end{aligned} \tag{3.14}$$

Substituting these into the gravitino equation and assuming that these modes have no x^4 dependence we get the following form.

$$\sqrt{\frac{r}{\psi\chi}}\tilde{\Gamma}^3\left[\partial_r + \frac{\psi'}{\psi} - \frac{\psi'}{2\psi}\widehat{\Gamma}^1 + \frac{3}{4r} + \frac{\chi'}{4\chi} + \frac{1}{4r\chi}\right]\Psi_v$$
$$+\frac{1}{\sqrt{r\psi\chi}}\tilde{\Gamma}^4\left[\partial_\theta + \frac{\cos\theta}{2\sin\theta}\right]\Psi_v + \frac{1}{\sqrt{r\psi\chi}\sin\theta}\tilde{\Gamma}^5\partial_\phi\Psi_v = 0\,. \tag{3.15}$$

Using the fact that

$$\frac{\chi'}{4\,\chi} + \frac{1}{4\,r\,\chi} = \frac{1}{4\,r}\,, \tag{3.16}$$

we can see that the R_4 dependence drops out and hence eq.(3.15) is identical to the corresponding equations for the BMPV black hole in flat transverse space. The gravitino modes are therefore unaffected by the Taub-NUT space! Finally, as in the case of BMPV black holes in flat space, the gravitino zero modes in this case are also non-singular at the future horizon.

When we switch on the deformations described in §3.2, H^{sMNP} no longer vanish and we need to examine the second equation of (2.25). However since the deformation given in (3.8) has weight ≥ 1, it can only contribute to equations with weight ≥ 1. The right hand side of the second equations in (2.25) however has weight 0. Thus we conclude that this equation is not affected by the deformations given in (3.8).

3.4 Bosonic deformation representing relative oscillation between the BMPV black hole and KK monopole

It was argued in [16] that the BMPV black hole in Taub-NUT space contains another set of hair degrees of freedom which represent the left-moving oscillations of the BMPV black hole relative to the Taub-NUT background. In the limit when the Taub-NUT radius goes to infinity these modes coincide with the transverse left-moving oscillation modes of the BMPV black hole in flat space-time, constructed in §2.1. Also since near the origin the Taub-NUT metric looks like flat metric, we expect that near the horizon these modes will have the same behaviour as the transverse oscillation modes of the BMPV black hole in flat space-time.

We have not tried to construct these modes explicitly for finite R_4 since, as these modes have identical near horizon behaviour as those of §2.1, they will have a curvature singularity at the future horizon. Thus we shall not count these modes among the hair degrees of freedom of the BMPV black hole in Taub-NUT space.

4 Supersymmetry of the Deformed Configuration

In order to study the supersymmetry of the deformed background we need to examine the equations which set to zero the supersymmetry variation of all the fields. Since we shall always be working in a background where scalar fields are constants and the spin 1/2 fields $\chi^{\alpha r}$ are zero, we shall write down the equations in this background. The equations obtained by setting to zero the supersymmetry variation of the metric, the gravitino, the 3-form field strength $\bar{H}^k_{MNP}$ and the spin 1/2 field $\chi^{\alpha r}$ take the form[22, 23, 24]:

$$
\begin{aligned}
&\eta_{AB}\,\bar\epsilon\,\widetilde\Gamma^A\,\Psi_M\,e_N^{\ B} + \eta_{AB}\,e_M^{\ A}\,\bar\epsilon\,\widetilde\Gamma^B\,\Psi_N = 0\,,\\
&D_M\epsilon - \frac{1}{4}\bar H^i_{MNP}\,\Gamma^{NP}\,\widehat\Gamma^i\epsilon = 0\,,\\
&\partial_{[P}\left(\bar\epsilon\,\Gamma_M\,\widehat\Gamma^i\Psi_{N]}\right) = 0\,,\\
&\Gamma^{MNP}H^s_{MNP}\epsilon = 0\,,
\end{aligned}
\tag{4.1}
$$

where ϵ is the supersymmetry transformation parameter satisfying

$$
(\widetilde\Gamma_{012345}+1)\epsilon = 0, \qquad \bar\epsilon = \epsilon^T\,C\,\Omega\,, \tag{4.2}
$$

C and Ω being the $SO(5,1)$ and $SO(5)$ charge conjugation matrices satisfying

$$
(C\widetilde\Gamma^A)^T = -C\widetilde\Gamma^A, \qquad (\Omega\widehat\Gamma^i)^T = -\Omega\widehat\Gamma^i, \qquad \Omega^T = -\Omega\,. \tag{4.3}
$$

The equations obtained by setting to zero the variations of the fields strengths H^s_{MNP} and the scalar fields are automatically satisfied in this restricted class of backgrounds. Note that we have suppressed the SO(5,1) and SO(5) spinor indices in eqs.(4.1).

Now suppose $\epsilon_{(0)}$ is a Killing spinor of the original background which could be either the BMPV black hole in flat transverse space or BMPV black hole in Taub-NUT space. The explicit form of $\epsilon_{(0)}$ has been given in appendix A, but we only need to use the fact that it satisfies the projection conditions

$$
(\widetilde\Gamma^0+\widetilde\Gamma^1)\,\epsilon_{(0)} = 0 \quad \to \quad \Gamma^v\,\epsilon_{(0)} = 0\,, \tag{4.4}
$$

and

$$
\widehat\Gamma^1\epsilon_{(0)} = \epsilon_{(0)}\,. \tag{4.5}
$$

Since $\epsilon_{(0)}$ does not have any space-time index we can assign to it weight zero in the convention described below (2.18). Furthermore due to (4.4) we cannot reduce the weight of any expression containing $\epsilon_{(0)}$ by acting on it by Γ^v. We shall now see that this guarantees that $\epsilon_{(0)}$ automatically satisfies (4.1) even in the presence of the deformations.

We begin with the metric deformations. Since these deformation are proportional to dv^2, they carry weight 2. Thus any term involving these deformations will generate terms of weight ≥ 2. Examining (4.1) we see that the only term that carries weight ≥ 2 is the weight 2 term obtained by choosing $M = N = v$ in the first equation. But this is linear in gravitino deformation Ψ_v which already carries weight 1. Since a term linear in Ψ_v and also in δG_{vv} has weight 3, we see that δG_{vv} cannot affect the first equation of (4.1). Thus we conclude that the metric deformations considered here are invariant under $\epsilon_{(0)}$.

Next we turn to the deformations involving three form field strengths H^r_{MNP} as described in §3.2. The only equation in (4.1) which involves H^r_{MNP} is the last equation. However from (3.8) we see that $H^r_{MNP}\Gamma^{MNP}$ is proportional to Γ^v and hence the last term in (4.1) vanishes identically due to (4.4). On the other hand the arguement given in the previous paragraph shows that the induced metric (3.11) does not affect the Killing spinor equation. Thus these deformations also do not destroy the supersymmetry of the background.

Finally we turn to the fermionic deformations involving Ψ_v. Since this has weight 1, it can only affect terms in the equation with weight ≥ 1. The relevant equations are the first and third equation of (4.1). In the first equation we can choose MN to be either vv or vw^i, and in the third equation we need to choose PMN to be $w^i w^j v$. Now the first equation involves terms of the form $\bar{\epsilon}_{(0)}\widetilde{\Gamma}^A\Psi_v$. Since $\epsilon_{(0)}$ and Ψ_v satisfy opposite $\widehat{\Gamma}^1$ projection (see eqs.(2.50) and (4.5)) these terms vanish. Thus we only need to examine the left hand side of the third equation. It follows from (2.30), (2.35) and (4.2), (4.4) that $\epsilon_{(0)}$ and Ψ_v satisfiy the same $SO(4)$ projection rules:

$$\widetilde{\Gamma}^{2345}\Psi_v = \Psi_v, \qquad \widetilde{\Gamma}^{2345}\epsilon_{(0)} = \epsilon_{(0)} \,. \tag{4.6}$$

As a result $\bar{\epsilon}_{(0)}\widetilde{\Gamma}^i\widehat{\Gamma}^k\Psi_v$ vanishes for $i = 2, 3, 4, 5$. This in turn shows that the left hand side of the third term also vanishes.

5 Partition Function After Hair Removal

In this section we shall briefly analyze the partition functions of the five and the four dimensional black hole entropy after hair removal. The analysis will be similar to that in [16] except that we shall now take into account the fact that the plane waves describing the transverse oscillation of the BMPV black hole have curvature singularities at the future horizon and hence should not be counted as part of the hair degrees of freedom. Also for simplicity we shall ignore the contribution to the degeneracies from small black hole core dressed by hair, – a detailed discussion on this can be found in [16]. The net effect of this is to remove from the final partition functions (5.3), (5.6) the contribution from the half-BPS states. Even at the intermediate stages of the analysis these contributions are exponentially suppressed compared to the leading contribution.

We consider the case where there is a single D5-brane, and introduce the variables (ρ, σ, v) as conjugates to the D1-brane charge along S^1, momentum along S^1 and the momentum along x^4. The index is related to the partition function Z by Fourier transform.[12] Then the microscopic partition function of the five dimensional system is computed by multiplying the partition function associated with the oscillations of the D1-branes relative to the D5-brane[9] and the center of mass oscillation of the combined system. The result is[16][13]

$$\begin{aligned} Z_{5D}(\rho, \sigma, v) &= e^{-2\pi i\rho - 2\pi i\sigma} \prod_{\substack{k,l,j\in \mathbb{Z} \\ k\geq 1, l\geq 0}} \left(1 - e^{2\pi i(\sigma k + \rho l + vj)}\right)^{-c(4lk - j^2)} \\ &\quad \times \left\{ \prod_{l\geq 1} (1 - e^{2\pi i(l\rho + v)})^{-2} \, (1 - e^{2\pi i(l\rho - v)})^{-2} \, (1 - e^{2\pi i l\rho})^4 \right\} (-1) \, (e^{\pi i v} - e^{-\pi i v})^2 \, . \end{aligned} \tag{5.1}$$

The hair of the five dimensional black hole contains a set of 12 gravitino zero modes. Their quantum numbers can be easily read out from the quantum numbers associated with the broken supersymmetries. Four of these zero modes do not carry any x^4 momentum (which in five dimensions is a particular component of the angular momentum) – they are used in soaking up the fermion zero modes in the computation of the helicity trace[16]. The rest carry x^4 momentum $\pm 1/2$ and gives a contribution to the partition function of the form $(e^{\pi i v} - e^{-\pi i v})^4$[16]. Finally there are 4 left-moving gravitino modes carrying no x^4 momentum as described in §2.2. They give a contribution of $\prod_{l\geq 1} (1 - e^{2\pi i l\rho})^4$. Combining these two contributions we get

$$Z_{5D}^{hair}(\rho, \sigma, v) = (e^{\pi i v} - e^{-\pi i v})^4 \prod_{l\geq 1} (1 - e^{2\pi i l\rho})^4 \, . \tag{5.2}$$

[12] For a precise definition of what index and partition function we are computing see ref.[16].

[13] The terms in the expansion whose ρ dependence is of the form $e^{-2\pi i\rho}$ represent contribution from half BPS states. We can remove this contribution by an appropriate subtraction as was done in [16], but for simplicity we shall ignore this complication.

Hence the partition function associated with the horizon is given by

$$\begin{aligned} Z_{5D}^{hor}(\rho,\sigma,v) &= Z_{5D}/Z_{5D}^{hair} \\ &= -e^{-2\pi i\rho-2\pi i\sigma}\left(e^{\pi iv}-e^{-\pi iv}\right)^{-2}\prod_{\substack{k,l,j\in\mathbb{Z}\\ k\geq 1,l\geq 0}}\left(1-e^{2\pi i(\sigma k+\rho l+vj)}\right)^{-c(4lk-j^2)} \\ &\quad \left\{\prod_{l\geq 1}(1-e^{2\pi i(l\rho+v)})^{-2}\,(1-e^{2\pi i(l\rho-v)})^{-2}\right\}. \end{aligned} \tag{5.3}$$

Let us now repeat the analysis for the BMPV black hole in transverse Taub-NUT space. The microscopic partition function is given by[10, 11, 12, 13, 14]

$$Z_{4D}(\rho,\sigma,v) = -e^{-2\pi i\rho-2\pi i\sigma-2\pi iv}\prod_{\substack{k,l,j\in\mathbb{Z}\\ k,l\geq 0, j<0\, for\, k=l=0}}\left(1-e^{2\pi i(\sigma k+\rho l+vj)}\right)^{-c(4lk-j^2)}. \tag{5.4}$$

In this case the hair modes include 12 fermion zero modes all of which are used in saturating the helicity factors inserted into the helicity trace. Besides these there are 21 left-moving bosonic modes associated with the 2-form deformations and 3 left-moving bosonic modes associated with the transverse oscillation of the black hole, all of which are neutral under the x^4 translation. Finally there are four left-moving gravitini modes, also neutral under x^4. These four fermionic modes cancel the contribution from four of the bosonic modes and we are left with the contribution:

$$Z_{4D}^{hair}(\rho,\sigma,v) = \prod_{l=1}^{\infty}\left(1-e^{2\pi il\rho}\right)^{-20}. \tag{5.5}$$

Thus we get

$$\begin{aligned} Z_{4D}^{hor}(\rho,\sigma,v) &= Z_{4D}/Z_{4D}^{hair} \\ &= -e^{-2\pi i\rho-2\pi i\sigma}\left(e^{\pi iv}-e^{-\pi iv}\right)^{-2}\prod_{\substack{k,l,j\in\mathbb{Z}\\ k\geq 1,l\geq 0}}\left(1-e^{2\pi i(\sigma k+\rho l+vj)}\right)^{-c(4lk-j^2)} \\ &\quad \left\{\prod_{l\geq 1}(1-e^{2\pi i(l\rho+v)})^{-2}\,(1-e^{2\pi i(l\rho-v)})^{-2}\right\}, \end{aligned} \tag{5.6}$$

where is the last step we have used $c(-1)=2$, $c(0)=20$. Comparing (5.6) with (5.3) we see that

$$Z_{5D}^{hor}(\rho,\sigma,v) = Z_{4D}^{hor}(\rho,\sigma,v). \tag{5.7}$$

Acknowledgement: We would like to thank Nabamita Banerjee, Borundev Chowdhury, Justin David, Suresh Govindarajan, Ipsita Mandal, Samir Mathur, Ashish Saxena and Nemani

Suryanarayana for useful discussions. DPJ and YS would like to thank CHEP, Bangalore for warm hospitality during the course of this work. DPJ would also like to thank IIT, Madras and IMSc, Chennai for warm hospitality. A.S. would like to thank GGI, Florence and University of Roma, Tor Vergata for warm hospitality during the course of this work. This work was supported by the project 11-R& D-HRI-5.02-0304 and the J.C.Bose fellowship of the Department of Science and Technology.

A Killing Spinors

The Killing spinor equation in the BMPV black hole and BMPV black hole in the Taub-NUT space, obtained by setting $\delta\Psi^{\alpha}_{M}=0$, is

$$D_M\epsilon-\frac{1}{4}\bar{H}^{i}_{MNP}\Gamma^{NP}\widehat{\Gamma}^{i}\epsilon=0\,, \tag{A.1}$$

where $\bar{H}^{i}_{MNP}$ for $1\leq i\leq 5$ are self-dual field strengths of 2-form fields in six dimensions. The equations obtained by setting $\delta\chi^{\alpha r}=0$ involve anti-self-dual components of the 3-form field strength and are automatically satisfied in this background. The three form field strength $F^{(3)}$ appearing in the BMPV black hole in the flat transverse space and BMPV in the Taub-NUT space is expressed in terms of $\bar{H}^{i}$ as

$$F^{(3)}_{MNP}=2\,\lambda^{-1}\,\bar{H}^{1}_{MNP}\,. \tag{A.2}$$

As described below (2.18), we can decompose the background field configuration as sum of different components carrying different weights. The weight of any term appearing in the left hand side of the Killing spinor equation is greater than or equal to the sum of the weights of the various field components which enter that term. This is due to the fact that the only way to reduce the weight of a given combination of fields is to contract one of the covariant v index with a Γ^{v} but we prevent this from happening by demanding that

$$\Gamma^{v}\epsilon=0\quad\rightarrow\quad\widetilde{\Gamma}^{0}\widetilde{\Gamma}^{1}\epsilon=\epsilon\,. \tag{A.3}$$

Now since the v component of the Killing spinor equation has weight 1, the u component has weight -1 and the other components have weight zero we conclude that for $M=v$ the left hand side of eq.(A.1) can receive contribution only from terms of weight zero or one in the field configuration, for $M=u$ the equation must be identically satisfied and for $M\neq u,v$ only terms of weight 0 can contribute. In particular since terms in $F^{(3)}$ which are independent of $\tilde{J}$

are of weight zero and the $\tilde{J}$ dependent terms have weight one, we see that $\tilde{J}$ dependent pieces do not contribute to the Killing spinor equation for $M \neq v$. However $\tilde{J}$ dependent terms could potentially contribute to the v component of the Killing spinor equation.

Let us first look at the v component of the Killing spinor equation. Requiring the Killing spinor to be v independent we find that for the BMPV black hole in flat transverse space this equation takes the form

$$\left[\frac{\psi'}{8\psi^2}\Gamma^{ru} - \frac{1}{8}\partial_{w^i}(\psi^{-1}\chi_j)\Gamma^{ij} - \frac{\psi'}{8\psi^2}\Gamma^{ru}\widehat{\Gamma}^1 + \frac{1}{8}\partial_{w^i}(\psi^{-1}\chi_j)\Gamma^{ij}\widehat{\Gamma}^1\right]\epsilon = 0\,. \tag{A.4}$$

This can be satisfied by choosing

$$\widehat{\Gamma}^1\epsilon = \epsilon\,. \tag{A.5}$$

For BMPV black hole in Taub-NUT space χ_i in eq.(A.4) is replaced by $-2\widetilde{\zeta}_i$, but (A.5) still provides a solution to this equation.

Next we examine the r, θ, ϕ and x^4 components of the Killing spinor equation. Using (A.5) and the spin connection components given in (2.38) and (3.14) for the BMPV black hole in the flat space and the BMPV black hole in TN space respectively, we get the following form of these equations for both black holes:

$$\begin{aligned}
\left(\partial_r + \frac{\psi'}{2\psi}\right)\epsilon &= 0\,, &(A.6)\\
\left(\partial_\theta - \frac{1}{2}\widetilde{\Gamma}^{34}\right)\epsilon &= 0\,, &(A.7)\\
\partial_{x^4}\epsilon &= 0\,, &(A.8)\\
\left(\partial_\phi - \frac{1}{2}\sin\theta\,\widetilde{\Gamma}^{35} - \frac{1}{2}\cos\theta\,\widetilde{\Gamma}^{45}\right)\epsilon &= 0\,. &(A.9)
\end{aligned}$$

Using the gamma matrix representations given in (2.45) we find the following solutions:

$$\epsilon = \psi(r)^{-1/2}\,e^{i\phi/2}\begin{pmatrix}\cos(\theta/2)\\ -\sin(\theta/2)\end{pmatrix}, \qquad \epsilon = \psi(r)^{-1/2}\,e^{-i\phi/2}\begin{pmatrix}\sin(\theta/2)\\ \cos(\theta/2)\end{pmatrix}. \tag{A.10}$$

To count the number of independent Killing spinors we note that to begin with ϵ is an $8\times4 = 32$ dimensional complex spinor. The chirality projection condition (4.2) and the two dynamical constraints given in (A.3), (A.5) reduce this number to 4 complex parameters. Finally a reality condition (symplectic Majorana) reduces the number to 4 real parameters.

Note that the Killing spinor (A.10) is independent of whether we consider BMPV black hole on flat transverse space or Taub-NUT space. This behaviour can be explained using the following reasons. The Taub-NUT space has $SU(2)$ holonomy, which by convention is identified with $SU(2)_L$ subgroup of its $SO(4)$ tangent space symmetry. Fermions in the Taub-NUT space transform as $(2,1)+(1,2)$ under $SO(4) = SU(2)_L \times SU(2)_R$. Thus half of the fermions are neutral under $SU(2)_L$ and hence behave as free fermions as far as the Taub-NUT space is concerned. In our six dimensional space, the $SO(1,1)$ chirality is correlated with the $SO(4)$ chirality in the following manner: $SU(2)_L$ singlets are left moving with respect to $SO(1,1)$ and $SU(2)_R$ singlets are right moving. Since Killing spinors corresponds to unbroken supersymmetry, which in our convention are left moving spinors of $SO(1,1)$, they are singlet of the Taub-NUT holonomy group $SU(2)_L$. As a result the Killing spinors are unaffected when we replace flat space by the Taub-NUT space.

B Black Hole Metric in Non-singular Coordinate System

In this appendix, following [19, 20] we write the black hole metric in coordinates in which it is regular and analytic at the future horizon. This coordinate system will then be used in appendix C to analyse regularity of the modes at the location of the horizon. For simplicity we will be working with $\tilde{J} = 0$ solution. Later we shall briefly discuss the extension to the $\widetilde{J} \neq 0$ case.

The original metric given in (2.6) may be expressed as

$$ds^2 = \psi^{-1}(dudv+Kdv^2)+\psi\left(r^{-1}dr^2 + 4\,r\,d\Omega_3^2\right)\,, \quad d\Omega_3^2 \equiv \frac{1}{4}\left((dx^4+\cos\theta d\phi)^2 + d\theta^2 + \sin^2\theta d\phi^2\right)\,, \tag{B.1}$$

where,

$$\psi = 1 + \frac{r_0}{r}, \qquad K = \psi - 1\,. \tag{B.2}$$

Following [19] we will now do the following coordinate transformation:

$$V = -\sqrt{r_0}\exp(-\frac{v}{\sqrt{r_0}}), \qquad W = \frac{1}{R}\exp(\frac{v}{2\sqrt{r_0}}), \qquad U = u + \frac{R^2}{2\sqrt{r_0}} + 2v\,, \tag{B.3}$$

$$R \equiv 2\sqrt{r_0\left(1+\frac{r_0}{r}\right)}\,. \tag{B.4}$$

Note that the region outside the horizon has $V < 0$. In these new coordinates the metric becomes

$$ds^2 = 4\,r_0\,\left[W^2 dU dV + dV^2\left\{\frac{3\sqrt{r_0}W^2}{V} - \frac{(1-Z^{-3})}{4V^2}\right\} - dV dW\left(\frac{1-Z^{-3}}{VW}\right) + \frac{dW^2}{W^2Z^3} + \frac{d\Omega_3^2}{Z}\right], \tag{B.5}$$

where

$$Z \equiv 1 + 4\sqrt{r_0}VW^2. \tag{B.6}$$

To see that metric is regular at $V = 0$, we expand Z in (B.5) to get

$$\begin{aligned} ds^2 \;&=\; 4\,r_0\,\Big[W^2 dU dV + dV^2 r_0 W^4 Z^{-3}(24 + 128\sqrt{r_0}VW^2 + 192 r_0 V^2 W^4) \\ &\quad - dV dW\, 4\sqrt{r_0} W Z^{-3}(3 + 12\sqrt{r_0}VW^2 + 16 r_0 V^2 W^4) + W^{-2}Z^{-3}dW^2 + Z^{-1}d\Omega_3^2\Big]. \end{aligned} \tag{B.7}$$

It is now easy to see that the metric is regular at the future horizon $V = 0$. In fact the metric components are polynomials in V and therefore they are analytic functions of V. Thus all derivatives of the metric components, and hence the Riemann tensor, remain finite at the horizon for finite W.[14]

We can also write down the three form field strength in terms of new coordinates. For $\tilde{J} = 0$ we get

$$F^{(3)} = \frac{r_0}{\lambda}\,\left[\sin\theta\, dx^4 \wedge d\theta \wedge d\phi + 4W dW \wedge dV \wedge dU\right]. \tag{B.8}$$

In the near horizon limit, $F^{(3)}$ is well behaved and independent of V.

Next we now look at the behaviour of the $\widetilde{J} = 0$ black hole in the Taub-NUT space. The G_{uv} component is the same as in the case of the $\widetilde{J} = 0$ black hole in flat transverse space. The difference comes in the components involving x^4, θ, ϕ and r coordinates. Most of these differences vanish near the horizon both in the original coordinate system and in the new coordinate system and hence do not spoil the regularity property of the metric in the new coordinates. The only additional term that apparently diverges at the horizon is the contribution

$$\delta(ds^2) = \frac{4\psi}{R_4^2}\,dr^2\,, \tag{B.9}$$

which in the new coordinate system becomes

$$\delta(ds^2) = \frac{(4r_0)^3}{R_4^2 Z^4}\left(-\frac{W^2\,dV^2}{4\sqrt{r_0}V} - \frac{V\,dW^2}{\sqrt{r_0}} - \frac{W dW dV}{\sqrt{r_0}}\right). \tag{B.10}$$

[14]For $V \to 0$ the metric reduces to that of $AdS_3 \times S^3$ locally[19].

Among these, only the first term is singular at the horizon. This singular contribution, however, can be removed by a shift in the U coordinate of the form $U \to U + 4r_0^{3/2} R_4^{-2} \ln(-V/\sqrt{r_0})$. We can combine this shift with the one given in (B.3) to write

$$U = u + \frac{R^2}{2\sqrt{r_0}} + 2v + \frac{(4r_0)}{R_4^2} v\,. \tag{B.11}$$

It is also easy to check that $F^{(3)}$ has the same form as (B.8) and hence is non-singular at the horizon.

For completeness we shall now briefly discuss the effect of switching on the parameter $\widetilde{J}$ in the original solution, labelling the angular momentum carried by the black hole. In this case the metric has the form:

$$\begin{aligned} ds^2 &= \psi^{-1}\left(dudv + Kdv^2 + \frac{\widetilde{J}}{4r}\left(dx^4 + \cos\theta d\phi\right)dv\right) \\ &\quad +\psi\left[\frac{dr^2}{r} + r\left\{(dx^4+\cos\theta d\phi)^2 + d\theta^2 + \sin^2\theta d\phi^2\right\}\right]. \end{aligned} \tag{B.12}$$

In order to introduce coordinates in which the metric at the future horizon is non-singular, we first shift $x^4 \to x^4 - \frac{\widetilde{J}}{8r_0^2} v$ so that the cross term between dv and $(dx^4 + \cos\theta d\phi)$ has a zero at $r = 0$.[15] This brings the metric to the form:

$$\begin{aligned} ds^2 &= \psi^{-1}\left[dudv + \left(K + \frac{\widetilde{J}^2}{64r_0^4} r\psi^2 - \frac{\widetilde{J}^2}{32r_0^2\, r}\right)dv^2 + \left(\frac{\widetilde{J}}{4r} - \frac{\widetilde{J}r}{4r_0^2}\psi^2\right)\left(dx^4 + \cos\theta d\phi\right)dv\right] \\ &\quad +\psi\left[\frac{dr^2}{r} + r\left\{(dx^4+\cos\theta d\phi)^2 + d\theta^2 + \sin^2\theta d\phi^2\right\}\right]. \end{aligned} \tag{B.13}$$

In the next step we carry out a rescaling $u \to (1 - \frac{\widetilde{J}^2}{64r_0^3})^{1/2} u$, $v \to v/(1 - \frac{\widetilde{J}^2}{64r_0^3})^{1/2}$ so that the coefficient of the dv^2 term in the metric coincides with that in the $\widetilde{J} = 0$ case as $r \to 0$. This gives:[16]

$$ds^2 = \psi^{-1}\left[dudv + \left(K + \frac{\widetilde{J}^2}{64r_0^4} r\psi^2 - \frac{\widetilde{J}^2}{32r_0^2 r}\right)\left(1 - \frac{\widetilde{J}^2}{64r_0^3}\right)^{-1} dv^2\right.$$

[15] Note that the periodic identification $x^5 \equiv x^5 + 2\pi R_5$ takes the form $(x^4, x^5) \equiv (x^4 + \frac{\widetilde{J}}{8r_0^2} 2\pi R_5, x^5 + 2\pi R_5)$ in the new coordinate system. This however does not affect our analysis.

[16] In order to be able to carry out this rescaling we need $\widetilde{J}^2 < 64r_0^3$. From (2.8) it follows that this is the condition $(Q_1 - Q_5)Q_5 n > J^2$ that guarantees that the original black hole solution is supersymmetric. This is also the condition needed for the absence of closed time-like curves[6].

$$+\left(\frac{\widetilde{J}}{4r}-\frac{\widetilde{J}r}{4r_0^2}\psi^2\right)\left(1-\frac{\widetilde{J}^2}{64r_0^3}\right)^{-1/2}(dx^4+\cos\theta d\phi)\,dv\Bigg]$$

$$+\psi\left[\frac{dr^2}{r}+r\left\{(dx^4+\cos\theta d\phi)^2+d\theta^2+\sin^2\theta d\phi^2\right\}\right]. \tag{B.14}$$

Let us denote the difference between this metric and the non-rotating black hole metric (B.1) by $\Delta(ds^2)$. We have

$$\begin{aligned}\Delta(ds^2) &= \psi^{-1}\left[\left(K+\frac{\widetilde{J}^2}{64r_0^4}r\psi^2-\frac{\widetilde{J}^2}{32r_0^2r}\right)\left(1-\frac{\widetilde{J}^2}{64r_0^3}\right)^{-1}-K\right]dv^2\\ &\quad+\psi^{-1}\left(\frac{\widetilde{J}}{4r}-\frac{\widetilde{J}r}{4r_0^2}\psi^2\right)\left(1-\frac{\widetilde{J}^2}{64r_0^3}\right)^{-1/2}(dx^4+\cos\theta d\phi)\,dv\,.\end{aligned} \tag{B.15}$$

Expressing this in the new coordinate system (B.3) we get

$$\Delta(ds^2)=-\frac{\widetilde{J}^2}{8r_0^{3/2}}\left(1-\frac{\widetilde{J}^2}{64r_0^3}\right)^{-1}W^2V^{-1}dV^2+\text{n.s.}\,, \tag{B.16}$$

where n.s. denotes terms which are non-singular as $V\to 0$. From the form of the $\widetilde{J}=0$ metric given in (B.7) we see that the singular term in (B.16) can be removed by a shift in the U coordinate of the form

$$U\to U+\frac{\widetilde{J}^2}{32r_0^{5/2}}\left(1-\frac{\widetilde{J}^2}{64r_0^3}\right)^{-1}\ln(-V)\,. \tag{B.17}$$

This makes the metric non-singular.

The three form field strength in this new coordinate system takes the form

$$\begin{aligned}F^{(3)} &= \frac{r_0}{\lambda}\Bigg[\sin\theta\,dx^4\wedge d\theta\wedge d\phi+4W\,dW\wedge dV\wedge dU\\ &\quad-\frac{\tilde{J}}{r_0}\left(1-\frac{\tilde{J}^2}{64r_0^3}\right)^{-1/2}W\,dW\wedge dV\wedge(dx^4+\cos\theta d\phi)\\ &\quad-\frac{\tilde{J}}{2r_0}\left(1-\frac{\tilde{J}^2}{64r_0^3}\right)^{-1/2}W^2\sin\theta\,dV\wedge d\theta\wedge d\phi\Bigg].\end{aligned} \tag{B.18}$$

The analysis for $\widetilde{J}\neq 0$ black hole in Taub-NUT proceeds along similar lines. The only difference is in the final step – the shift in U given in (B.17) is now replaced by

$$U \;\to\; U+\frac{\widetilde{J}^2}{32r_0^{5/2}}\left(1-\frac{\widetilde{J}^2}{64r_0^3}\right)^{-1}\ln(-V)+4\,r_0^{3/2}\,R_4^{-2}\,\ln(-V)$$

$$-\frac{3\widetilde{J}^2}{16 r_0^{3/2} R_4^2}\left(1-\frac{\widetilde{J}^2}{64 r_0^3}\right)^{-1}\ln(-V)\,. \qquad \text{(B.19)}$$

C Regularity of the Deformed Solution

In this appendix we shall check whether the deformations we have obtained by turning on various modes in §2 and §3 produce regular field configuration at the future horizon. Since for deformation of the type we are considering the possible singularities are null singularities, they will not show up in the invariant scalars constructed out of the field strengths and the Riemann tensor. Instead we need to work in a coordinate system in which the metric and the other background fields are continuous at the horizon, and then check whether the components of the Riemann tensor and other field strengths are finite in this coordinate system[18]. We will systematically carry out this analysis for all the modes, but not in the same order in which they were analyzed in the text. As in the previous appendix, we will be working with the $\tilde{J}=0$ solution for the sake of simplicity, but the generalization to $\widetilde{J}\neq 0$ case is straightforward. We shall however continue to refer to the $\widetilde{J}=0$ black holes as BMPV black holes.

We will start with the deformations described in §3.2. These are generated by the 2-form fields in BMPV black hole in Taub-NUT space. The deformation in the 2-form field given in (3.8) near the origin behaves as

$$\delta H^s \simeq \frac{1}{R_4^2} h^s dv \wedge \left[-r\sin\theta d\theta d\phi + dr\wedge(dx^4+\cos\theta d\phi)\right]\,. \qquad \text{(C.1)}$$

In the new coordinate system it takes the form

$$\delta\, H^s \simeq \frac{4\,r_0^2}{R_4^2} h^s\, dV\wedge\left[-W^2\,\sin\theta\, d\theta\wedge d\phi + 2\,W\,dW\wedge(dx^4+\cos\theta\, d\phi)\right]\,. \qquad \text{(C.2)}$$

This is clearly non-singular near the horizon $V=0$. Let us now examine the metric deformation generated by these modes as given in eq.(3.11):

$$\psi^{-1}\,\widetilde{S}(v,\vec{y})\,dv^2 = \psi^{-1}\,\frac{C(v)\,r}{2R_4^2(4r+R_4^2)}\,dv^2. \qquad \text{(C.3)}$$

This term in the new coordinates becomes

$$\psi^{-1}\,\widetilde{S}(v,\vec{y})\,dv^2 = C(v)\,dV^2\,\frac{8r_0^3W^4}{R_4^2\left\{R_4^2(1+4r_0^{1/2}VW^2) - 16r_0^{3/2}VW^2\right\}}\,. \qquad \text{(C.4)}$$

This is also regular at the horizon $V=0$.

Note however that since $v \propto \ln(-V)$ for small $|V|$, v is a rapidly varying function of V near the horizon. Since $C(v)$ is an oscillatory function of v with finite period (set by the period of the x^5 coordinate) $C(v)$ and hence G_{VV} is a rapidly varying oscillatory function of V for small V. This can be remedied by a shift in the U coordinate of the form:

$$U \to U - F(V,W), \qquad F(V,W) \equiv \frac{2r_0^2 W^2}{R_4^4} \int_0^V C(v')dV' \,. \tag{C.5}$$

In this coordinate system G_{VV} vanishes at $V=0$, but we get an additional term in the $dWdV$ component proportional to $4\,r_0 W^2 \partial_W F(V,W)$. Since $F(V,W)$ vanishes at $V=0$ for all W, this additional term vanishes at the horizon. Thus it does not alter the behaviour of the metric at the horizon. We note however that $\partial_V F \propto W^2 C(v')$ is rapidly oscillating and as a result $\partial_V^2 F$ diverges at the horizon. This could give a potential divergence in the Riemann tensor which involves two derivatives of the metric. However it can be seen using the argument below (2.18) that the Riemann tensor never involves $\partial_V^2 G_{WV}$. The latter term is of weight 3 (we now replace (u,v) by (U,V) in counting weight) whereas the Riemann tensor, written in a covariant form can have at most two indices set equal to V and hence can at most be of weight 2. Thus $\partial_V^2 G_{WV}$ cannot appear in the expression for the Riemann tensor and the latter is finite at the horizon.

Next we look at the gravitino modes. For definiteness we shall consider the case of BMPV black holes in flat transverse space, but an identical analysis can be carried out for Taub-NUT space. The gravitino modes in the original metric are non-vanishing only for the v components and these components take the form

$$\Psi_v = \psi^{-3/2}(r)\eta(v,\theta,\phi), \qquad (\widetilde{\Gamma}^0 + \widetilde{\Gamma}^1)\eta(v,\theta,\phi) = 0\,, \qquad \widehat{\Gamma}^1\eta(v,\theta,\phi) = -\eta(v,\theta,\phi)\,, \tag{C.6}$$

where $\eta(v,\theta,\phi)$ is an SO(5,1) spinor and also an $SO(5)$ spinor. The (θ,ϕ) dependence of $\eta(v,\theta,\phi)$ was computed in §2.2 and the v dependence is arbitrary except for the periodicity requirement imposed by the period of the coordinate x^5. In the new coordinate system the gravitino field takes the form

$$\Psi_V = (4\,r_0)^{5/4}\, W^3(-2V)^{1/2}\, \eta(v,\theta,\phi)\,. \tag{C.7}$$

This however is not the end of the story. The gravitino field configuration (C.6) was computed using the set of vielbeins (2.10) which become singular near the horizon. This can be seen by expressing them in the new coordinate system:

$$\begin{aligned}
e^{+} &\equiv e^{0}+e^{1}=-r_0^{1/2}\,\frac{dV}{V}\,,\\
e^{-} &\equiv e^{1}-e^{0}=-4\,r_0^{1/2}\,V\,W^{2}\,dU+4\,r_0^{1/2}\,\frac{dW}{W}+\left(\frac{r_0^{1/2}}{V}-12\,r_0W^{2}\right)dV\\
e^{3} &= 2\,r_0^{1/2}\,(1+4\,r_0^{1/2}\,V\,W^{2})^{-3/2}\left(\frac{dW}{W}+\frac{dV}{2V}\right)\,.
\end{aligned}\tag{C.8}$$

They are clearly singular at $V=0$. Thus we must make a local Lorentz transformation to make them non-singular. From the metric (B.7) we see that a non-singular choice of vielbeins will correspond to

$$\begin{aligned}
\tilde{e}^{+} &= 2\,r_0^{1/2}\,dV,\\
\tilde{e}^{-} &= 2r_0^{1/2}\Big[W^{2}dU+8r_0W^{4}Z^{-3}(3+16r_0^{1/2}\,V\,W^{2}+24r_0V^{2}W^{4})dV\\
&\quad -4r_0^{1/2}WZ^{-3}(3+12r_0^{1/2}VW^{2}+16r_0V^{2}W^{4})dW\Big]\,,\\
\tilde{e}^{3} &= 2r_0^{1/2}W^{-1}Z^{-3/2}dW\,.
\end{aligned}\tag{C.9}$$

The metric can be expressed as

$$ds^{2}=e^{+}e^{-}+(e^{3})^{2}+(e^{2})^{2}+(e^{4})^{2}+(e^{5})^{2}=\tilde{e}^{+}\tilde{e}^{-}+(\tilde{e}^{3})^{2}+(e^{2})^{2}+(e^{4})^{2}+(e^{5})^{2}\,.\tag{C.10}$$

We must now find the Lorentz transformation relating the two sets of vielbeins. This is done in two steps. First we apply a boost on (e^{+},e^{-}) that produces a new set of vielbeins:

$$\hat{e}^{+}=-2\,V\,e^{+},\quad \hat{e}^{-}=-\frac{1}{2\,V}e^{-},\qquad \hat{e}^{3}=e^{3}\,.\tag{C.11}$$

The vielbeins $(\hat{e}^{\pm},\hat{e}^{3})$ can now be shown to be related to $(\tilde{e}^{\pm},\tilde{e}^{3})$ by a Galilean transformation:

$$\tilde{e}^{+}=\hat{e}^{+},\qquad \tilde{e}^{-}=\hat{e}^{-}-2\,\beta\,\hat{e}^{3}-\beta^{2}\,\hat{e}^{+},\quad \tilde{e}^{3}=\hat{e}^{3}+\beta\,\hat{e}^{+}\,,\tag{C.12}$$

where

$$\beta=-\frac{1}{2\,Z^{3/2}\,V}\,.\tag{C.13}$$

These local Lorentz transformation will also act on the gravitino fields. First of all the boost transformation (C.11) transforms the gravitino to

$$\widehat{\Psi}_V=(-2V)^{-1/2}\,\Psi_V=(4\,r_0)^{5/4}\,W^{3}\,\eta(v,\theta,\phi)\,.\tag{C.14}$$

The simplest way to see this is to note that the under this boost we must have

$$\overline{\Psi}_V \widetilde{\Gamma}^- \Psi_V \, e^+ = \overline{\widehat{\Psi}}_V \widetilde{\Gamma}^- \widehat{\Psi}_V \, \hat{e}^+ \, , \qquad \widetilde{\Gamma}^\pm \equiv (\widetilde{\Gamma}^1 \pm \widetilde{\Gamma}^0) \, , \tag{C.15}$$

where we have used the $\widetilde{\Gamma}^+ \Psi_V = 0$ condition to infer that the $\overline{\Psi}_V \widetilde{\Gamma}^m \Psi_V$ is non-vanishing only for $m = -$. Since $\hat{e}^+ = -2\,V\,e^+$ this implies that $\widehat{\Psi}_V = (-2V)^{-1/2}\,\Psi_V$. On the other hand the Galilean transformation (C.12) does not act on Ψ_V since it is generated by $\widetilde{\Gamma}^{3+}$ and $\widetilde{\Gamma}^+$ annihilates Ψ_V. Thus the gravitino in the frame given in (C.9) takes the form:

$$\widetilde{\Psi}_V = \widehat{\Psi}_V = (4\,r_0)^{5/4}\,W^3\,\eta(v,\theta,\phi) \, . \tag{C.16}$$

Although this does not vanish at the horizon, we can make it vanish using a local supersymmetry transformation by a parameter proportional to $W^3 \int_0^V \eta(v',\theta,\phi)\,dV'$.

We note in passing that in this coordinate system the Killing spinor behaves as $W\,V^{1/2}$ near $V = 0$. After the local Lorentz transformation described above it goes as W and hence is well defined at the horizon. We also note that the second solution given in (2.49), for which $\Psi_v \sim r^{1/2}$ near $r = 0$, diverges as $V \to 0$ in the new coordinate system. Hence it is not an allowed deformation.

Let us now look at the modes generated by the Garfinkle-Vachaspati transformation of the original black hole metric. We begin with the centre of mass motion modes of the four dimensional black hole as described in §3.1. These modes are described by the metric perturbation

$$\psi^{-1}\, g_i(v) y^i dv^2 = \psi^{-1}\, r\, n^i g_i(v)\, dv^2, \qquad \vec{n} \equiv (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta) \, . \tag{C.17}$$

In the new coordinate system this deformation is given by

$$16 r_0^3 n^i g_i(v) (1 + 4\sqrt{r_0}\, V W^2)^{-1}\, W^4 dV^2 \, . \tag{C.18}$$

As in the case of the deformation (C.4), (C.18) takes finite value on the horizon but oscillates rapidly as $V \to 0$, We can make δG_{VV} vanish by the coordinate transformation

$$U \to U - H(V,W,\vec{n}), \qquad H(V,W,\vec{n}) = 4\,r_0^2\,W^2 \int_0^V dV'\, n^i g_i(v') (1 + 4\sqrt{r_0} V' W^2)^{-1} \, . \tag{C.19}$$

This generates a term $-4 r_0 W^2 (\partial_W H dV dW + \partial_\theta H dV d\theta + \partial_\phi H dV d\phi)$ in the metric but this vanishes at $V = 0$ since $H(V,W,\vec{n})$ vanishes at $V = 0$ for all W, θ, ϕ.[17] $\partial_V^2 H$ diverges at $V = 0$,

[17] H depends on θ, ϕ through $\vec{n}$.

but as argued before $\partial_V^2 G_{WV}$, $\partial_V^2 G_{\theta V}$ and $\partial_V^2 G_{\phi V}$ do not appear in the Riemann tensor for the type of metric we are considering. Thus the metric and the Riemann tensor are non-singular at $V = 0$ for this deformation.

Finally we shall carry out this analysis for the transverse oscillation of the BMPV metric in the flat space, described in §2.1. The same analysis also holds for the modes describing the transverse oscillation of the BMPV black hole relative to the Taub-NUT space as described in §3.4 since they are expected to have the same form near the horizon. In this case the deformation is given by:

$$\delta(ds^2) = \psi^{-1}\, \vec{f}(v)\cdot \vec{w}\, dv^2 = 2\, r^{1/2}\, \psi^{-1}\, \vec{f}(v)\cdot \vec{m}\, dv^2, \qquad \vec{m} = \vec{w}/|\vec{w}|\,. \tag{C.20}$$

In the new coordinate system this deformation takes the form

$$\delta(ds^2) = 16\, \frac{r_0^{9/4} W^3}{\sqrt{1+4\sqrt{r_0}W^2 V}}\, \frac{dV^2}{\sqrt{-V}}\, \vec{m}\cdot \vec{f}(v)\,. \tag{C.21}$$

The metric is singular at $V = 0$. However this term can be removed by the following shift of U:

$$U \to U - G(V, W, \vec{m}), \quad G(V, W, \vec{m}) = 4\, r_0^{5/4}\, W \int_0^V dV'\, (1+4\sqrt{r_0}W^2 V')^{-1/2}\, (-V')^{-1/2}\, \vec{m}\cdot \vec{f}(v')\,. \tag{C.22}$$

This shift however generates a term in the metric of the form

$$-\, 4\, r_0\, W^2\, \partial_W\, G(V, W, \vec{m})\, dW\, dV - 4\, r_0\, W^2\, \partial_i\, G(V, W, \vec{m})\, d\theta^i\, dV\,, \tag{C.23}$$

where θ^i denotes any of the angular coordinates. These vanish at the horizon, but their first V derivatives diverge at the horizon as $\vec{f}(v)/(-V)^{1/2}$. This could give rise to divergences in the Riemann tensor. Explicit computation shows that some of the components of the Riemann tensor do indeed diverge at $V = 0$. For example we find

$$R_{VWVW} = -2r_0\; W^{-1}\partial_W \partial_V (W^3 \partial_W G(W, V)) \;+\; \text{n.s.} = -24\, r_0^{9/4}\, W\, (-V)^{-1/2}\, \vec{m}\cdot \vec{f} \;+\; \text{n.s.}\,, \tag{C.24}$$

where n.s. denotes non-singular terms. This diverges as $V \to 0$. Thus we conclude that these modes should not be counted among the hair degrees of freedom of the black hole.

References

[1] R. M. Wald, "Black hole entropy in the Noether charge," Phys. Rev. D **48**, 3427 (1993) `arXiv:gr-qc/9307038`.

[2] T. Jacobson, G. Kang and R. C. Myers, "On Black Hole Entropy," Phys. Rev. D **49**, 6587 (1994) `arXiv:gr-qc/9312023`.

[3] V. Iyer and R. M. Wald, "Some properties of Noether charge and a proposal for dynamical black hole entropy," Phys. Rev. D **50**, 846 (1994) `arXiv:gr-qc/9403028`.

[4] T. Jacobson, G. Kang and R. C. Myers, "Black hole entropy in higher curvature gravity," `arXiv:gr-qc/9502009`.

[5] A. Sen, "Quantum Entropy Function from AdS(2)/CFT(1) Correspondence," `arXiv:0809.3304 [hep-th]`.

[6] J. C. Breckenridge, R. C. Myers, A. W. Peet and C. Vafa, "D-branes and spinning black holes," Phys. Lett. B **391**, 93 (1997) `arXiv:hep-th/9602065`.

[7] J. P. Gauntlett, J. B. Gutowski, C. M. Hull, S. Pakis and H. S. Reall, "All supersymmetric solutions of minimal supergravity in five dimensions," Class. Quant. Grav. **20**, 4587 (2003) `arXiv:hep-th/0209114`.

[8] D. Gaiotto, A. Strominger and X. Yin, "New connections between 4D and 5D black holes," JHEP **0602**, 024 (2006) `arXiv:hep-th/0503217`.

[9] R. Dijkgraaf, G. W. Moore, E. P. Verlinde and H. L. Verlinde, "Elliptic genera of symmetric products and second quantized strings," Commun. Math. Phys. **185**, 197 (1997) `arXiv:hep-th/9608096`.

[10] R. Dijkgraaf, E. P. Verlinde and H. L. Verlinde, "Counting dyons in N = 4 string theory," Nucl. Phys. B **484**, 543 (1997) `arXiv:hep-th/9607026`.

[11] G. L. Cardoso, B. de Wit, J. Kappeli and T. Mohaupt, "Asymptotic degeneracy of dyonic N = 4 string states and black hole entropy," JHEP **0412**, 075 (2004) `arXiv:hep-th/0412287`.

[12] D. Shih, A. Strominger and X. Yin, "Recounting dyons in N = 4 string theory," JHEP **0610**, 087 (2006) `arXiv:hep-th/0505094`.

[13] D. Gaiotto, "Re-recounting dyons in N = 4 string theory," `arXiv:hep-th/0506249`.

[14] J. R. David and A. Sen, "CHL dyons and statistical entropy function from D1-D5 system," arXiv:hep-th/0605210.

[15] A. Sen, "Black Hole Entropy Function, Attractors and Precision Counting of Microstates," arXiv:0708.1270 [hep-th].

[16] N. Banerjee, I. Mandal and A. Sen, "Black Hole Hair Removal," arXiv:0901.0359 [hep-th].

[17] N. Kaloper, R. C. Myers and H. Roussel, "Wavy strings: Black or bright?," Phys. Rev. D **55**, 7625 (1997) arXiv:hep-th/9612248.

[18] G. T. Horowitz and H. s. Yang, "Black strings and classical hair," Phys. Rev. D **55**, 7618 (1997) arXiv:hep-th/9701077.

[19] G. T. Horowitz and D. Marolf, "Counting states of black strings with traveling waves," Phys. Rev. D **55**, 835 (1997) arXiv:hep-th/9605224.

[20] G. T. Horowitz and D. Marolf, "Counting states of black strings with traveling waves. II," Phys. Rev. D **55**, 846 (1997) arXiv:hep-th/9606113.

[21] R. Brooks, R. Kallosh and T. Ortin, "Fermion zero modes and black hole hypermultiplet with rigid supersymmetry," Phys. Rev. D **52**, 5797 (1995) arXiv:hep-th/9505116.

[22] L. J. Romans, "Selfduality For Interacting Fields: Covariant Field Equations For Six-Dimensional Chiral Supergravities," Nucl. Phys. B **276**, 71 (1986).

[23] F. Riccioni, "Tensor multiplets in six-dimensional (2,0) supergravity," Phys. Lett. B **422**, 126 (1998) arXiv:hep-th/9712176.

[24] S. Deger, A. Kaya, E. Sezgin and P. Sundell, "Spectrum of D = 6, N = 4b supergravity on AdS(3) x S(3)," Nucl. Phys. B **536**, 110 (1998) arXiv:hep-th/9804166.

[25] D. Garfinkle and T. Vachaspati, "Cosmic string traveling waves," Phys. Rev. D **42**, 1960 (1990).

[26] F. Larsen and F. Wilczek, "Internal Structure of Black Holes," Phys. Lett. B **375**, 37 (1996) arXiv:hep-th/9511064.

[27] F. Larsen and F. Wilczek, "Classical Hair in String Theory I: General Formulation," Nucl. Phys. B **475**, 627 (1996) arXiv:hep-th/9604134.

[28] F. Larsen and F. Wilczek, "Classical hair in string theory. II: Explicit calculations," Nucl. Phys. B **488**, 261 (1997) arXiv:hep-th/9609084.

[29] M. Cvetic and A. A. Tseytlin, "Solitonic strings and BPS saturated dyonic black holes," Phys. Rev. D **53**, 5619 (1996) [Erratum-ibid. D **55**, 3907 (1997)] arXiv:hep-th/9512031.