Preprint typeset in JHEP style - HYPER VERSION

hep-th/0602254

Product Representation of Dyon Partition Function in CHL Models

Justin R. David, Dileep P. Jatkar and Ashoke Sen

Harish-Chandra Research Institute,
Chhatnag Road., Jhunsi,
Allahabad 211019, India.
justin,dileep,sen@mri.ernet.in

Abstract: A formula for the exact partition function of 1/4 BPS dyons in a class of CHL models has been proposed earlier. The formula involves inverse of Siegel modular forms of subgroups of $Sp(2,\mathbb{Z})$. In this paper we propose product formulae for these modular forms. This generalizes the result of Borcherds and Gritsenko and Nikulin for the weight 10 cusp form of the full $Sp(2,\mathbb{Z})$ group.

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1. Introduction and Summary

There exists a proposal for the exact degeneracy of dyons in toroidally compactified heterotic string theory[1, 2, 3, 4, 5] and also in toroidally compactified type II string theory[6]. These formulæ are invariant under the S-duality transformations of the theory, and also reproduce the entropy of a dyonic black hole in the limit of large charges[2]. In [7] this proposal was generalized to a class of CHL models[8, 9, 10, 11, 12, 13], obtained by modding out heterotic string theory on $T^2 \times T^4$ by a $\mathbb{Z}_N$ transformation that involves $1/N$ unit of translation along one of the circles of T^2 and a non-trivial action on the internal conformal field theory (CFT) describing heterotic string compactification on T^4. The values of N considered in [7] were N =2,3,5,7. Using string-string duality[14, 15, 16, 17, 18] one can relate these models to $\mathbb{Z}_N$ orbifolds of type IIA string theory on $T^2 \times K3$, with the $\mathbb{Z}_N$ transformation acting as $1/N$ unit of shift along a circle of T^2 together with an action on the internal CFT describing type IIA string compactification on $K3$.

The proposal of [7] may be summarized as follows. If we denote by Q_e and Q_m the electric and the magnetic charge vectors then the degeneracy $d(Q_e, Q_m)$ of dyons carrying charges (Q_e, Q_m) is of the form

$$d(Q_e, Q_m) = g\left(\frac{1}{2}Q_m^2, \frac{1}{2}Q_e^2, Q_e \cdot Q_m\right), \tag{1.1}$$

where $g(m, n, p)$ is defined through the Fourier expansion

$$\frac{1}{\widetilde{\Phi}_k(U, T, V)} = C_0 \sum_{\substack{m,n,p \\ m\geq -1, n\geq -1/N}} e^{2\pi i(mU+nT+pV)} g(m, n, p)\,. \tag{1.2}$$

Here C_0 is a numerical constant and $\widetilde{\Phi}_k(U, T, V)$ is a modular form of weight k under a subgroup $\widetilde{G}$ of $Sp(2, \mathbb{Z}) \equiv SO(2, 3; \mathbb{Z})$ where

$$k = \frac{24}{N+1} - 2\,. \tag{1.3}$$

An explicit algorithm for constructing the Fourier expansion of $\widetilde{\Phi}_k$ in the variables T, U and V was given in [7].

The degeneracy $d(Q_e, Q_m)$ defined through eqs.(1.1), (1.2) is invariant under the T- and S-duality symmetries of the theory. Furthermore it generates integer results for the degeneracies and its behaviour for large charges is consistent with the black hole entropy calculation[7, 19].

In this paper we use the method of [20, 21] to propose an alternative form of $\widetilde{\Phi}_k$ as an infinite product:

$$\begin{aligned}
\widetilde{\Phi}_k(U, T, V) = &-(i\sqrt{N})^{-k-2}\ \exp\left(2\pi i\left(\frac{1}{N}T + U + V\right)\right) \\
&\prod_{r=0}^{N-1} \prod_{\substack{l,b\in \mathbb{Z}, k'\in \mathbb{Z}+\frac{r}{N} \\ k',l,b>0}} \Big\{1 - \exp(2\pi i(k'T + lU + bV))\Big\}^{\frac{1}{2}\sum_{s=0}^{N-1} e^{-2\pi i l s/N}\, c^{(r,s)}(4lk'-b^2)} \\
&\prod_{r=0}^{N-1} \prod_{\substack{l,b\in \mathbb{Z}, k'\in \mathbb{Z}-\frac{r}{N} \\ k',l,b>0}} \Big\{1 - \exp(2\pi i(k'T + lU + bV))\Big\}^{\frac{1}{2}\sum_{s=0}^{N-1} e^{2\pi i l s/N}\, c^{(r,s)}(4lk'-b^2)}
\end{aligned} \tag{1.4}$$

where $(k', l, b) > 0$ means $k' > 0, l \geq 0, b \in \mathbb{Z}$ or $k' = 0, l > 0, b \in \mathbb{Z}$ or $k' = 0, l = 0, b < 0$ and $c^{(r,s)}(n)$ are some calculable coefficients related to the twisted elliptic genus of $K3$. If $\widetilde{g}$ denotes the generator of the $\mathbb{Z}_N$ action on $K3$ that is used in the construction of the CHL model, then we define the twisted elliptic genus of $K3$ as

$$F^{(r,s)}(\tau,z) = \frac{1}{N} Tr^{K3}_{RR;\widetilde{g}^r} \left((-1)^{F_{K3}} (-1)^{\bar{F}_{K3}} \widetilde{g}^s e^{2\pi i z F_{K3}} q^{L_0} \bar{q}^{\bar{L}_0} \right) , \quad 0 \le r, s \le (N-1) , \tag{1.5}$$

where $Tr^{K3}_{RR;\widetilde{g}^r}$ denotes trace in the superconformal field theory associated with target space $K3$ in the $\widetilde{g}^r$ twisted RR sector, $q = e^{2\pi i \tau}$, and F_{K3}, $\bar{F}_{K3}$ denote the left- and right-handed world-sheet fermion numbers in this theory. Here and throughout the rest of the paper L_0 and $\bar{L}_0$ include an additive factor of $-c/24$ so that the RR sector ground state has $L_0 = \bar{L}_0 = 0$. The coefficients $c^{(r,s)}(n)$ are then defined through the Fourier expansion of $F^{(r,s)}(\tau,z)$:

$$F^{(r,s)}(\tau,z) = \sum_{b\in\mathbb{Z},n} c^{(r,s)}(4n-b^2)\, q^n\, e^{2\pi i z b} . \tag{1.6}$$

Furthermore for the $N = 2$, $k = 6$ case we were able to explicitly compute the functions $F^{(r,s)}(\tau,z)$. They are given by

$$\begin{aligned} F^{(0,0)}(\tau,z) &= 4 \left[\frac{\vartheta_2(\tau,z)^2}{\vartheta_2(\tau,0)^2} + \frac{\vartheta_3(\tau,z)^2}{\vartheta_3(\tau,0)^2} + \frac{\vartheta_4(\tau,z)^2}{\vartheta_4(\tau,0)^2} \right] , \\ F^{(0,1)}(\tau,z) &= 4\, \frac{\vartheta_2(\tau,z)^2}{\vartheta_2(\tau,0)^2} , \quad F^{(1,0)}(\tau,z) = 4 \frac{\vartheta_4(\tau,z)^2}{\vartheta_4(\tau,0)^2} , \quad F^{(1,1)}(\tau,z) = 4\, \frac{\vartheta_3(\tau,z)^2}{\vartheta_3(\tau,0)^2} . \end{aligned} \tag{1.7}$$

For higher values of N we did not evaluate the functions $F^{(r,s)}(\tau,z)$ directly, but were able to guess their forms from general considerations. The results are:

$$\begin{aligned} F^{(0,0)}(\tau,z) &= \frac{8}{N} A(\tau,z) , \\ F^{(0,s)}(\tau,z) &= \frac{8}{N(N+1)} A(\tau,z) - \frac{2}{N+1} B(\tau,z)\, E_N(\tau) \qquad \text{for } 1 \le s \le (N-1) , \\ F^{(r,rk)}(\tau,z) &= \frac{8}{N(N+1)} A(\tau,z) + \frac{2}{N(N+1)} E_N\left(\frac{\tau+k}{N}\right) B(\tau,z) , \\ & \qquad\qquad \text{for } 1 \le r \le (N-1),\ 0 \le k \le (N-1) , \end{aligned} \tag{1.8}$$

where

$$A(\tau,z) = \left[\frac{\vartheta_2(\tau,z)^2}{\vartheta_2(\tau,0)^2} + \frac{\vartheta_3(\tau,z)^2}{\vartheta_3(\tau,0)^2} + \frac{\vartheta_4(\tau,z)^2}{\vartheta_4(\tau,0)^2} \right] , \tag{1.9}$$

$$B(\tau, z) = \eta(\tau)^{-6}\vartheta_1(\tau, z)^2\,, \tag{1.10}$$

and

$$E_N(\tau) = \frac{12i}{\pi(N-1)}\,\partial_\tau\left[\ln\eta(\tau) - \ln\eta(N\tau)\right] = 1 + \frac{24}{N-1}\sum_{\substack{n_1,n_2\geq 1\\ n_1\neq 0\, mod\, N}} n_1 e^{2\pi i n_1 n_2 \tau}\,. \tag{1.11}$$

Eq.(1.4) gives a generalization of Borcherds and Gritsenko and Nikulin's result[22, 23] of the product representation of $\widetilde{\Phi}_{10}$, – the unique cusp form of weight 10 of the group $Sp(2,\mathbb{Z})$. A systematic procedure for arriving at the product representation for $\widetilde{\Phi}_{10}$ was given in [20]. Our construction of $\widetilde{\Phi}_k$ is essentially based on a generalization of the techniques of [20].

Given the two different constructions of $\widetilde{\Phi}_k$, – one given in [7] and one in the present paper, it is natural to ask if they are the same. For the $N=2$, $k=6$ case we have compared 31 different Fourier expansion coefficients of the two proposals and found them to be the same.[1] For other values of N we have compared the expansions up to order $e^{4\pi i T}e^{4\pi i U}$ and all powers of $e^{2\pi i V}$. For general N we also verify that the behaviour of $\widetilde{\Phi}_k$ (and of Φ_k introduced in footnote 1) in the $V\to 0$ limit as well as in the $U\to i\infty$ limit agrees with the results found in [7].

The rest of the paper is organized as follows. In section 2 we outline the strategy that we shall be using for finding $\widetilde{\Phi}_k$. Sections 3 and 4 involve detailed calculations leading to the determination of $\widetilde{\Phi}_6$ associated with the $\mathbb{Z}_2$ orbifold theory. In section 5 we give the final form of $\widetilde{\Phi}_6$ and compare some of its properties with those found in [7]. Section 6 is devoted to the construction of the related quantity Φ_6 described in footnote 1 and its comparison with the corresponding quantity calculated in [7]. In section 7 we describe the construction of $\widetilde{\Phi}_k$ and Φ_k for a general k given in (1.3). The three appendices contain some technical details which were omitted from discussion in the main body of the paper.

2. The Strategy

Our goal is to find a product representation for $\widetilde{\Phi}_k$. In attaining this goal we shall proceed as in the case of ordinary toroidal compactification of heterotic string theory

[1] Actually we compare not the Fourier expansion coefficients of $\widetilde{\Phi}_k$ but those of a closely related object $\Phi_k(U,T,V) = T^{-k}\,\widetilde{\Phi}_k(U - T^{-1}V^2, -T^{-1}, T^{-1}V)$.

or equivalently type II string theory on $T^2 \times K3$. This corresponds to the case $N = 1$, $k = 10$ and the associated modular form $\widetilde{\Phi}_{10}$ is the unique weight 10 cusp form of the Siegel modular group $Sp(2; \mathbb{Z})$. The steps leading to a systematic construction of the product representation of $\widetilde{\Phi}_{10}$ are as follows[20]:

1. We consider a superconformal σ-model with target space $T^2 \times K3$ with y^1, y^2 denoting the T^2 coordinates. We denote by F_{K3} and F_{T^2} the holomorphic parts of the world-sheet fermion number associated with the $K3$ and the T^2 parts and by $\bar{F}_{K3}$ and $\bar{F}_{T^2}$ the anti-holomorphic parts of the world-sheet fermion number associated with the $K3$ and the T^2 parts. We shall be considering an arbitrary T^2 parametrized by the Kähler modulus T and complex structure modulus U, and arbitrary Wilson lines A_1, A_2 corresponding to deforming the world-sheet theory by the marginal operator

$$\sum_{i=1}^{2} A_i \int d^2z \, \bar{\partial} Y^i J_{K3} \,, \tag{2.1}$$

where J_{K3} is the U(1) current corresponding to the charge F_{K3}. We shall denote by V the complex combination $A_2 - iA_1$. V is normalized so that $V \equiv V + 1$.

This theory has an $SO(2, 3; \mathbb{Z})$ T-duality group. If we denote by (m_1, m_2) the integers labeling momenta along y^1, y^2, by (n_1, n_2) the integers labeling winding along y^1, y^2, and by b the F_{K3} charge, then the $SO(2, 3; \mathbb{Z})$ transformation S acts on these charges and the parameters T, U, V as

$$\begin{pmatrix} m_1' \\ m_2' \\ n_1' \\ n_2' \\ b' \end{pmatrix} = S \begin{pmatrix} m_1 \\ m_2 \\ n_1 \\ n_2 \\ b \end{pmatrix} , \qquad \begin{pmatrix} T' \\ T'U' - V'^2 \\ -U' \\ 1 \\ 2V' \end{pmatrix} = \lambda \, S \begin{pmatrix} T \\ TU - V^2 \\ -U \\ 1 \\ 2V \end{pmatrix} \tag{2.2}$$

where S is a 5×5 matrix with integer entries, satisfying

$$S^T L S = L, \qquad L = \begin{pmatrix} 0 & I_2 & 0 \\ I_2 & 0 & 0 \\ 0 & 0 & 1/2 \end{pmatrix} , \tag{2.3}$$

and λ is a number to be adjusted so that the fourth element of the vector on the right hand side of (2.2) is 1. I_n denotes $n \times n$ identity matrix.

Using the equivalence between $SO(2,3)$ and $Sp(2)$ we can represent the T-duality group elements by $Sp(2,\mathbb{Z})$ matrices of the form $\begin{pmatrix} A & B \\ C & D \end{pmatrix}$ where A, B, C and D are each 2×2 matrix with integer entries satisfying

$$AB^T = BA^T, \qquad CD^T = DC^T, \qquad AD^T - BC^T = I_2\,. \tag{2.4}$$

If we define

$$\Omega = \begin{pmatrix} U & V \\ V & T \end{pmatrix}, \tag{2.5}$$

then the duality group acts on Ω as

$$\Omega \to (A\Omega + B)(C\Omega + D)^{-1}\,. \tag{2.6}$$

2. In this theory we define:

$$\mathcal{I}_0(U,T,V) = \int_{\mathcal{F}} \frac{d^2\tau}{\tau_2}\, Tr_{RR}\left((-1)^{(F_{K3}+F_{T^2})}(-1)^{(\bar F_{K3}+\bar F_{T^2})} F_{T^2}\bar F_{T^2} q^{L_0}\bar q^{\bar L_0}\right) \tag{2.7}$$

where $\mathcal{F}$ is the fundamental domain of $SL(2,\mathbb{Z})$ and $q = e^{2\pi i\tau}$. $\mathcal{I}(U,T,V)$ is expected to be invariant under $SO(2,3;\mathbb{Z})$ transformation.

3. Analysis of the integral given in (2.7) shows that it can be expressed in the form

$$\mathcal{I}_0 = -20\ln\det\mathrm{Im}\,\Omega - 2\ln\widetilde\Phi_{10}(\Omega) - 2\ln\widetilde\Phi_{10}(\bar\Omega)\ +\ \text{constant} \tag{2.8}$$

where $\widetilde\Phi_{10}(\Omega)$ is a holomorphic function of T, U and V with a product representation. Since under the duality transformation (2.6)

$$\det\mathrm{Im}\,\Omega \to (\det(C\Omega+D))^{-1}\,(\det(C\bar\Omega+D))^{-1}\,\det\mathrm{Im}\,\Omega\,, \tag{2.9}$$

and $\mathcal{I}_0$ is invariant, we must have[2]

$$\widetilde\Phi_{10}\left((A\Omega+B)(C\Omega+D)^{-1}\right) = (\det(C\Omega+D))^{10}\,\widetilde\Phi_{10}(\Omega)\,. \tag{2.10}$$

Thus $\widetilde\Phi_{10}(\Omega)$ must be a Siegel modular form of weight 10. This leads to the construction of the product representation of $\widetilde\Phi_{10}$.

[2]In principle there could be Ω independent phases on the right hand side of (2.10), but it is known that they are absent in this case.

Our goal is to construct a modular form $\widetilde{\Phi}_k$ of weight k of an appropriate subgroup $\widetilde{G}$ of $SO(2,3;\mathbb{Z})$ for k given in (1.3). The subgroup $\widetilde{G}$ is the T-duality group of the superconformal field theory with target space $(T^2 \times K3)/\mathbb{Z}_N$ where the $\mathbb{Z}_N$ acts as a $1/N$ unit of shift along a circle on T^2 and as a geometric transformation of order N on $K3$.[3] Thus only those $SO(2,3;\mathbb{Z})$ transformation which commute with the $1/N$ unit of shift along T^2 will be symmetries of the resulting theory.

We shall try to construct $\widetilde{\Phi}_k$ by first defining an analog of the integral $\mathcal{I}_0$ invariant under this subgroup and then splitting it into a sum of an holomorphic piece, an anti-holomorphic piece and a term proportional to $\ln\det \mathrm{Im}\,\Omega$ as in (2.8). A natural candidate integral is

$$\mathcal{I}(U,T,V) = \int_{\mathcal{F}} \frac{d^2\tau}{\tau_2}\, Tr_{RR}\left((-1)^{(F_{K3}+F_{T^2})}(-1)^{(\bar{F}_{K3}+\bar{F}_{T^2})} F_{T^2}\bar{F}_{T^2} q^{L_0}\bar{q}^{\bar{L}_0}\right) \qquad (2.11)$$

where the trace is taken over the states in this orbifold superconformal field theory.

For $V=0$ this integral has been calculated for the $\mathbb{Z}_2$ orbifold model in [24]. In the next few sections we shall describe computation of this integral for the $N=2$ case for non-zero V. This will enable us to determine the product form of $\widetilde{\Phi}_6$. Later we shall discuss generalization of this analysis to other values of N.

3. The Integrand for the $\mathbb{Z}_2$ Orbifold Theory

In this section we shall analyze the integrand in eq.(2.11) for the $\mathbb{Z}_2$ orbifold conformal field theory described earlier. We can decompose the contribution to the trace in (2.11) as a sum of the contribution from different sectors characterized by the five charges (m_1, n_1, m_2, n_2, b) introduced earlier.[4] In this case we can factor out the T, U and V dependence of the trace into an overall factor of $q^{p_L^2/2}\bar{q}^{p_R^2/2}$ where

$$\begin{aligned} \frac{1}{2}p_R^2 &= \frac{1}{4\det Im\Omega}|-m_1U+m_2+n_1T+n_2(TU-V^2)+bV|^2, \\ \frac{1}{2}p_L^2 &= \frac{1}{2}p_R^2 + m_1n_1 + m_2n_2 + \frac{1}{4}b^2\,. \end{aligned} \qquad (3.1)$$

[3]In order to preserve the $\mathcal{N}=4$ target space supersymmetry, the $\mathbb{Z}_N$ action on $K3$ must commute with the (4,4) superconformal symmetry possessed by a supersymmetric σ-model with target space $K3$.

[4]Note that now the twisted sector states carry half integer winding number n_1 along y^1.

Thus $\mathcal{I}(U,T,V)$ has the form

$$\mathcal{I}(U,T,V)=\int_{\mathcal{F}}\frac{d^2\tau}{\tau_2}\sum_{m_1,m_2,n_1,n_2,b}q^{p_L^2/2-b^2/4}\bar{q}^{p_R^2/2}F_{m_1,m_2,n_1,n_2;b}(\tau) \tag{3.2}$$

where $F_{m_1,m_2,n_1,n_2;b}(\tau)$ is independent of T, U and V and is given by

$$F_{m_1,m_2,n_1,n_2;b}(\tau)=Tr_{m_1,m_2,n_1,n_2;b;RR}\left((-1)^{(F_{K3}+F_{T^2})}(-1)^{(\bar{F}_{K3}+\bar{F}_{T^2})}F_{T^2}\bar{F}_{T^2}q^{L_0'}\bar{q}^{\bar{L}_0'}\right). \tag{3.3}$$

Here

$$L_0'=L_0-\frac{p_L^2}{2}+\frac{b^2}{4},\qquad \bar{L}_0'=\bar{L}_0-\frac{p_R^2}{2}, \tag{3.4}$$

are independent of T, U and V and $Tr_{m_1,m_2,n_1,n_2;b}$ denotes trace over a subspace of the Hilbert space carrying momentum (m_1,m_2) and winding (n_1,n_2) along T^2 and F_{K3} charge b. Note that we have included the $b^2/4$ term in L_0' so that for $V=0$ when the conformal field theories associated with $K3$ and T^2 parts decouple, L_0' and $\bar{L}_0'$ describe complete contribution from the CFT associated with $K3$ and oscillator contribution from the CFT associated with T^2. Since $F_{m_1,m_2,n_1,n_2;b}(\tau)$ is independent of T, U and V, we can set $V=0$ while evaluating (3.3).

Let us define

$$F_{m_1,m_2,n_1,n_2}(\tau,z)=\sum_b F_{m_1,m_2,n_1,n_2;b}(\tau)\,e^{2\pi ibz}. \tag{3.5}$$

It then follows from (3.3) that

$$F_{m_1,m_2,n_1,n_2}(\tau,z)=Tr_{m_1,m_2,n_1,n_2;RR}\left((-1)^{(F_{K3}+F_{T^2})}(-1)^{(\bar{F}_{K3}+\bar{F}_{T^2})}F_{T^2}\bar{F}_{T^2}e^{2\pi izF_{K3}}\,q^{L_0'}\bar{q}^{\bar{L}_0'}\right). \tag{3.6}$$

We shall first compute $F_{m_1,m_2,n_1,n_2}(\tau,z)$ and then extract $F_{m_1,m_2,n_1,n_2;b}(\tau)$ using eq.(3.5). Since the contribution to (3.6) from the T^2 part is somewhat trivial, it is useful to separate out this contribution. For this we denote by g' the generator of the $\mathbb{Z}_2$ group with which we take the orbifold of $K3\times T^2$. Then

$$\begin{aligned}&F_{m_1,m_2,n_1,n_2}(\tau,z)\\ &=\frac{1}{2}\sum_{r,s=0}^{1}Tr^{K3\times T^2}_{m_1,m_2,n_1,n_2;RR;(g')^r}\left((-1)^{(F_{K3}+F_{T^2})}(-1)^{(\bar{F}_{K3}+\bar{F}_{T^2})}F_{T^2}\bar{F}_{T^2}e^{2\pi izF_{K3}}\,q^{L_0'}\bar{q}^{\bar{L}_0'}(g')^s\right),\end{aligned} \tag{3.7}$$

where the superscript $K3\times T^2$ in Tr indicates that the trace is taken in the superconformal field theory with target space $K3\times T^2$, and the subscript $(g')^r$ in Tr indicates that the trace is over the sector twisted by $(g')^r$. We now split g' as

$$g' = \hat{g}\,\widetilde{g}\,, \tag{3.8}$$

where $\hat{g}$ and $\widetilde{g}$ represent the action of g' on the T^2 and $K3$ parts respectively. Twisting by $\hat{g}^r$ makes the winding number $n_1 \in \mathbb{Z}+\frac{r}{2}$, and hence the right hand side of (3.7) vanishes unless $n_1-\frac{r}{2}\in\mathbb{Z}$. The $(\hat{g})^s$ factor inside the trace produces a factor of $(-1)^{m_1 s}$. The non-zero mode bosonic and fermionic oscillator contributions from the T^2 factor always cancel since they are neutral under $\hat{g}$. The fermion zero modes associated with T^2 give a factor of 4 due to 2-fold degeneracy each from the holomorphic and anti-holomorphic sectors, but this cancels with the factor of 1/4 coming from the $F_{T^2}\bar{F}_{T^2}$ factor inside the trace. Thus we can write

$$F_{m_1,m_2,n_1,n_2}(\tau,z) = \sum_{s=0}^{1} (-1)^{m_1\, s}\, F^{(r,s)}(\tau,z) \qquad \text{for } n_1\in\mathbb{Z}+\tfrac{r}{2}, \quad r=0,1 \tag{3.9}$$

where

$$F^{(r,s)}(\tau,z) = \frac{1}{2}\, Tr^{K3}_{RR;\widetilde{g}^r}\left((-1)^{F_{K3}}(-1)^{\bar{F}_{K3}}\,\widetilde{g}^s\, e^{2\pi i z F_{K3}}\, q^{L_0}\bar{q}^{\bar{L}_0}\right)\,. \tag{3.10}$$

Here $Tr^{K3}_{RR;\widetilde{g}^r}$ denotes trace in the superconformal field theory associated with target space $K3$ in the $\widetilde{g}^r$ twisted RR sector, and L_0, $\bar{L}_0$ inside the trace now includes contribution from $K3$ only. This is twisted elliptic genus of $K3$. These quantities were introduced in [25] in order to calculate the elliptic genus of $\widetilde{g}$ orbifold of $K3$. This would be given by $\sum_{r,s=0}^{1} F^{(r,s)}(\tau,z)$. Here however we need the individual $F^{(r,s)}(\tau,z)$ since we shall be using them for a different purpose.

From the definitions given in (3.10) it follows that[25]

$$F^{(r,s)}\left(\frac{a\tau+b}{c\tau+d},\frac{z}{c\tau+d}\right) = \exp\left(2\pi i\frac{c\,z^2}{c\tau+d}\right) F^{(cs+ar,ds+br)}(\tau,z)\,, \tag{3.11}$$

for

$$a,b,c,d\in\mathbb{Z}, \qquad ad-bc=1\,. \tag{3.12}$$

In (3.11) the indices $cs+ar$ and $ds+br$ are to be taken mod 2.

$F_{m_1,m_2,n_1,n_2}(\tau, z)$ has been calculated in appendix A using an orbifold description of $K3$ and the result is given in eq.(A.15). Comparing this with eq.(3.9) we get

$$\begin{aligned}
F^{(0,0)}(\tau, z) &= 4\left[\frac{\vartheta_2(\tau,z)^2}{\vartheta_2(\tau,0)^2} + \frac{\vartheta_3(\tau,z)^2}{\vartheta_3(\tau,0)^2} + \frac{\vartheta_4(\tau,z)^2}{\vartheta_4(\tau,0)^2}\right], \\
F^{(0,1)}(\tau, z) &= 4\,\frac{\vartheta_2(\tau,z)^2}{\vartheta_2(\tau,0)^2}, \quad F^{(1,0)}(\tau, z) = 4\frac{\vartheta_4(\tau,z)^2}{\vartheta_4(\tau,0)^2} \quad F^{(1,1)}(\tau, z) = 4\,\frac{\vartheta_3(\tau,z)^2}{\vartheta_3(\tau,0)^2}.
\end{aligned} \tag{3.13}$$

Using the known modular transformation laws of $\vartheta_i(\tau, z)$ we can verify that $F^{(r,s)}(\tau, z)$ given in (3.13) satisfy (3.11).

We now use the relations:

$$\begin{aligned}
\vartheta_1^2(\tau, z) &= \vartheta_2(2\tau, 0)\vartheta_3(2\tau, 2z) - \vartheta_3(2\tau, 0)\vartheta_2(2\tau, 2z) \\
\vartheta_2^2(\tau, z) &= \vartheta_2(2\tau, 0)\vartheta_3(2\tau, 2z) + \vartheta_3(2\tau, 0)\vartheta_2(2\tau, 2z) \\
\vartheta_3^2(\tau, z) &= \vartheta_3(2\tau, 0)\vartheta_3(2\tau, 2z) + \vartheta_2(2\tau, 0)\vartheta_2(2\tau, 2z) \\
\vartheta_4^2(\tau, z) &= \vartheta_3(2\tau, 0)\vartheta_3(2\tau, 2z) - \vartheta_2(2\tau, 0)\vartheta_2(2\tau, 2z)
\end{aligned} \tag{3.14}$$

to rewrite (3.13) as

$$F^{(r,s)}(\tau, z) = h_0^{(r,s)}(\tau)\,\vartheta_3(2\tau, 2z) + h_1^{(r,s)}(\tau)\,\vartheta_2(2\tau, 2z) \tag{3.15}$$

where

$$\begin{aligned}
h_0^{(0,0)}(\tau) &= 8\,\frac{\vartheta_3(2\tau,0)^3}{\vartheta_3(\tau,0)^2\vartheta_4(\tau,0)^2} + 2\,\frac{1}{\vartheta_3(2\tau,0)} \\
h_1^{(0,0)}(\tau) &= -8\,\frac{\vartheta_2(2\tau,0)^3}{\vartheta_3(\tau,0)^2\vartheta_4(\tau,0)^2} + 2\,\frac{1}{\vartheta_2(2\tau,0)} \\
h_0^{(0,1)}(\tau) &= 2\,\frac{1}{\vartheta_3(2\tau,0)}, \qquad h_1^{(0,1)}(\tau) = 2\,\frac{1}{\vartheta_2(2\tau,0)}, \\
h_0^{(1,0)}(\tau) &= 4\frac{\vartheta_3(2\tau,0)}{\vartheta_4(\tau,0)^2}, \qquad h_1^{(1,0)}(\tau) = -4\frac{\vartheta_2(2\tau,0)}{\vartheta_4(\tau,0)^2}, \\
h_0^{(1,1)}(\tau) &= 4\frac{\vartheta_3(2\tau,0)}{\vartheta_3(\tau,0)^2}, \qquad h_1^{(1,1)}(\tau) = 4\frac{\vartheta_2(2\tau,0)}{\vartheta_3(\tau,0)^2},
\end{aligned} \tag{3.16}$$

Since

$$\vartheta_3(2\tau, 2z) = \sum_{b\in 2\mathbb{Z}} e^{2\pi i b z} q^{b^2/4}, \qquad \vartheta_2(2\tau, 2z) = \sum_{b\in 2\mathbb{Z}+1} e^{2\pi i b z} q^{b^2/4}, \tag{3.17}$$

we see, by comparing (3.5) and (3.9), (3.15) that

$$F_{m_1,m_2,n_1,n_2;b}(\tau) = q^{b^2/4} \sum_{s=0}^{1} (-1)^{m_1 s}\, h_l^{(r,s)}(\tau) \quad \text{for } n_1 \in \mathbb{Z} + \tfrac{r}{2},\ b \in 2\mathbb{Z} + l$$
$$r, l = 0, 1\,. \tag{3.18}$$

Using (3.18) the original integral $\mathcal{I}(U,T,V)$ given in eq.(3.2) may be written as

$$\mathcal{I}(U,T,V) = \sum_{l,r,s=0}^{1} \mathcal{I}_{r,s,l} \tag{3.19}$$

where

$$\mathcal{I}_{r,s,l} = \int_{\mathcal{F}} \frac{d^2\tau}{\tau_2} \sum_{\substack{m_1,m_2,n_2 \in \mathbb{Z} \\ n_1 \in \mathbb{Z}+\frac{r}{2}, b \in 2\mathbb{Z}+l}} q^{p_L^2/2} \bar{q}^{p_R^2/2}\, (-1)^{m_1 s}\, h_l^{(r,s)}(\tau)\,. \tag{3.20}$$

From this we see that those $SO(2,3;\mathbb{Z})$ transformations which, acting on a vector (m_1, m_2, n_1, n_2, b) with m_1, m_2, n_2, b integers and n_1 half-integer, preserves m_1 modulo 2, n_1, m_2, n_2 modulo 1 and b modulo 2, will be symmetries of $\mathcal{I}$. This defines the subgroup $\widetilde{G}$.

For later use we define the coefficients $c^{(r,s)}(4n)$ through the expansion

$$h_0^{(r,s)}(\tau) = \sum_n c^{(r,s)}(4n) q^n\,, \qquad h_1^{(r,s)}(\tau) = \sum_n c^{(r,s)}(4n) q^n\,. \tag{3.21}$$

By examining (3.16) we see that in the expansion of $h_l^{(r,s)}$, $n \in \mathbb{Z} - \frac{l}{4}$ for $r = 0$ and $n \in \frac{1}{2}\mathbb{Z} - \frac{l}{4}$ for $r = 1$. Note that we have used the same symbol $c^{(r,s)}(4n)$ for describing the expansion of $h_0^{(r,s)}(\tau)$ and $h_1^{(r,s)}(\tau)$. This is possible since $c^{(r,s)}(4n)$ has different support for $l = 0$ and $l = 1$.

Using eq.(3.15) and the Fourier expansion (3.17) of ϑ_3 and ϑ_2 we can write the double Fourier expansion of $F^{(r,s)}(\tau, z)$

$$F^{(r,s)}(\tau, z) = \sum_{b \in \mathbb{Z}, n} c^{(r,s)}(4n - b^2)\, q^n\, e^{2\pi i z b}\,, \tag{3.22}$$

where $n \in \mathbb{Z}$ for $r = 0$ and $\frac{1}{2}\mathbb{Z}$ for $r = 1$.

4. The Integral

We shall now proceed to evaluate the integral (3.20). We define

$$Y = \det \mathrm{Im}\Omega = T_2\, U_2 - (V_2)^2, \qquad T_2 > 0, \quad U_2 > 0, \quad Y > 0\,. \tag{4.1}$$

where for any complex number u, we denote by u_1 and u_2 its real and imaginary parts respectively. Substituting the values of p_L^2 and p_R^2 from (3.1) into (3.20) we obtain

$$\mathcal{I}_{r,s,l} = \int_{\mathcal{F}} \frac{d^2\tau}{\tau_2} \sum_{m_1,m_2,n_2\in\mathbb{Z}, n_1\in\mathbb{Z}+\frac{r}{2}, b\in 2\mathbb{Z}+l} \exp\left[2\pi i\tau(m_1n_1+m_2n_2+\frac{b^2}{4})\right]\times$$
$$\exp\left[\frac{-\pi\tau_2}{Y}\left|n_2(TU-V^2)+bV+n_1T-Um_1+m_2\right|^2\right](-1)^{m_1 s}\, h_l^{(r,s)}(\tau)\,. \tag{4.2}$$

To evaluate the integral we first perform the Poisson resummation over the momenta m_1, m_2. The basic formula for Poisson resummation we will use is

$$\sum_{m\in\mathbb{Z}} f(m)e^{2\pi i s m/N} = \sum_{k\in\mathbb{Z}+\frac{s}{N}} \int_{-\infty}^{\infty} du f(u)\exp(2\pi i k u) \tag{4.3}$$

for any integer N. Now performing the Poisson resummation over m_1, m_2 and performing the Gaussian integration over the corresponding variables u_1, u_2, we obtain the following

$$\mathcal{I}_{r,s,l} = \int_{\mathcal{F}} \frac{d^2\tau}{\tau_2^2}\frac{Y}{U_2} \sum_{n_2,k_2\in\mathbb{Z}, n_1\in\mathbb{Z}+\frac{r}{2}, k_1\in\mathbb{Z}+\frac{s}{2}, b\in 2\mathbb{Z}+l} h_l^{(r,s)}(\tau)\exp\left[\mathcal{G}(\vec{n},\vec{k},b)\right] \tag{4.4}$$

where

$$\begin{aligned}\mathcal{G}(\vec{n},\vec{k},b) &= -\frac{\pi Y}{U_2^2\tau_2}|\mathcal{A}|^2 - 2\pi i T\det A\\ &+ \frac{\pi b}{U_2}(V\tilde{\mathcal{A}}-\bar{V}\mathcal{A}) - \frac{\pi n_2}{U_2}(V^2\tilde{\mathcal{A}}-\bar{V}^2\mathcal{A})\\ &+ \frac{2\pi i V_2^2}{U_2^2}(n_1+n_2\bar{U})\mathcal{A} + 2\pi i\tau\frac{b^2}{4}\,,\end{aligned} \tag{4.5}$$

$$A = \begin{pmatrix} n_1 & k_1 \\ n_2 & k_2 \end{pmatrix}, \tag{4.6}$$

$$\mathcal{A} = (1,U)A\begin{pmatrix}\tau\\1\end{pmatrix}, \qquad \tilde{\mathcal{A}} = (1,\bar{U})A\begin{pmatrix}\tau\\1\end{pmatrix}. \tag{4.7}$$

Using (4.5) we can represent the sum over b in (4.4) as

$$\sum_{b\in 2\mathbb{Z}+l} e^{2\pi i\tau\frac{b^2}{4}+\frac{\pi b}{U_2}(V\tilde{\mathcal{A}}-\bar{V}\mathcal{A})} = \begin{cases}\vartheta_3(2\tau, -i\frac{V\tilde{\mathcal{A}}-\bar{V}\mathcal{A}}{U_2}) & \text{for } l=0\\ \vartheta_2(2\tau, -i\frac{V\tilde{\mathcal{A}}-\bar{V}\mathcal{A}}{U_2}) & \text{for } l=1\end{cases} \tag{4.8}$$

Substituting this into (4.4) and using (3.15) we get

$$\mathcal{I} \equiv \sum_{l,r,s=0}^{1} = \int_{\mathcal{F}} \frac{d^2\tau}{\tau_2^2} \sum_{r,s=0}^{1} \sum_{n_2,k_2\in\mathbb{Z},n_1\in\mathbb{Z}+\frac{r}{2},k_1\in\mathbb{Z}+\frac{s}{2}} \mathcal{J}(A,\tau)\,, \tag{4.9}$$

where

$$\begin{aligned}\mathcal{J}(A,\tau) = \frac{Y}{U_2}\exp\Bigg(&-\frac{\pi Y}{U_2^2\tau_2}|\mathcal{A}|^2 - 2\pi i T \det A \\ &-\frac{\pi n_2}{U_2}(V^2\tilde{\mathcal{A}} - \bar{V}^2\mathcal{A}) + \frac{2\pi i V_2^2}{U_2^2}(n_1+n_2\bar{U})\mathcal{A}\Bigg) F^{(r,s)}\left(\tau, -i\frac{V\tilde{\mathcal{A}}-\bar{V}\mathcal{A}}{2\,U_2}\right) \\ &r = 2\,n_1 \bmod 2, \quad s = 2\,k_1 \bmod 2\,.\end{aligned} \tag{4.10}$$

In order to interpret the right hand side as a function of the matrix A we need to use eqs.(4.6), (4.7). We may now interpret the sum over r,s and $\vec{n},\vec{k}$ in the right hand side of eq.(4.9) as a sum over all matrices A of the form (4.6) with n_2, k_2 integer, and n_1, k_1 integer or half-integer. (4.9) may then be rewritten as

$$\mathcal{I} = \int_{\mathcal{F}} \frac{d^2\tau}{\tau_2^2} \sum_A \mathcal{J}(A,\tau)\,. \tag{4.11}$$

Now it follows from the modular transformation laws (3.11) and the definition of $\mathcal{J}(A,\tau)$ given in (4.10) that

$$\mathcal{J}\left(A, \frac{a\tau+b}{c\tau+d}\right) = \mathcal{J}\left(A\begin{pmatrix} a & b \\ c & d\end{pmatrix}, \tau\right)\,. \tag{4.12}$$

Using this symmetry, we can extend the integration over the fundamental domain to its images under $SL(2,\mathbb{Z})$ and at the same time restrict the summation over A to summation over inequivalent $SL(2,\mathbb{Z})$ orbits. If we denote by $\sum_A'$ the sum over inequivalent $SL(2,\mathbb{Z})$ orbits then we can express $\mathcal{I}$ as

$$\begin{aligned}\mathcal{I} = \sum_A{}' \int_{\mathcal{F}_A} \frac{d^2\tau}{\tau_2^2}\frac{Y}{U_2}\exp\Bigg(&-\frac{\pi Y}{U_2^2\tau_2}|\mathcal{A}|^2 - 2\pi i T\det A \\ &-\frac{\pi n_2}{U_2}(V^2\tilde{\mathcal{A}} - \bar{V}^2\mathcal{A}) + \frac{2\pi i V_2^2}{U_2^2}(n_1+n_2\bar{U})\mathcal{A}\Bigg) F^{(r,s)}\left(\tau, -i\frac{V\tilde{\mathcal{A}}-\bar{V}\mathcal{A}}{2\,U_2}\right)\end{aligned} \tag{4.13}$$

where now r, s in the label of $F^{(r,s)}$ are to be interpreted as $2\,n_1$ mod 2 and $2\,k_1$ mod 2 respectively. The region of integration $\mathcal{F}_A$ depends on the orbit represented by A.

Following the same procedure as in [28] we now split the integration into the three orbits. These are the zero orbit

$$A = 0\,, \tag{4.14}$$

the non-degenerate orbit

$$A = \begin{pmatrix} k & j \\ 0 & p \end{pmatrix}, \quad 2k-1 \geq 2j \geq 0, \quad p \neq 0, \quad k, j \in \frac{1}{2}\mathbb{Z}, \quad p \in \mathbb{Z}\,, \tag{4.15}$$

and the degenerate orbit

$$A = \begin{pmatrix} 0 & j \\ 0 & p \end{pmatrix}, \quad (j,p) \neq (0,0), \quad j \in \frac{1}{2}\mathbb{Z}, \quad p \in \mathbb{Z}\,. \tag{4.16}$$

The contribution from these orbits has been evaluated in appendix B. The final result, as given in (B.39), takes the form

$$\begin{aligned}
\mathcal{I} &= -2\ln\Bigg[\kappa(\det \mathrm{Im}\Omega)^6 \Bigg|\exp\left(2\pi i\left(\frac{1}{2}T + U + V\right)\right) \\
&\qquad \prod_{r,s=0}^{1} \prod_{\substack{(l,b)\in\mathbb{Z}, k\in\mathbb{Z}+\frac{r}{2} \\ (k,l,b)>0}} \left\{(1-\exp(2\pi i(kT + lU + bV))^{(-1)^{ls}\, c^{(r,s)}(4kl-b^2)}\right\}\Bigg|^2\Bigg] \\
\kappa &= \left(\frac{8\pi}{3\sqrt{3}}e^{1-\gamma_E}\right)^6
\end{aligned} \tag{4.17}$$

and $(k,l,b) > 0$ means $k > 0, l \geq 0, b \in \mathbb{Z}$ or $k = 0, l > 0, b \in \mathbb{Z}$ or $k = 0, l = 0, b < 0$.

5. $\widetilde{\Phi}_6$ and its $V \to 0$ Limit

Eq.(4.17) can be written as

$$\mathcal{I} = -2\left[6\ln\det \mathrm{Im}\Omega + \ln\widetilde{\Phi}_6 + \ln\bar{\widetilde{\Phi}}_6 + \ln\kappa + 8\ln 2\right]\,, \tag{5.1}$$

where

$$\begin{aligned}
\widetilde{\Phi}_6(\Omega) &= \frac{1}{16}\,\exp\left(2\pi i\left(\frac{1}{2}\,T + U + V\right)\right) \\
&\qquad \prod_{r,s=0}^{1} \prod_{\substack{l,b\in\mathbb{Z}, k\in\mathbb{Z}+\frac{r}{2} \\ k,l,b>0}} \Big[1 - \exp\left\{2\pi i(kT + lU + bV)\right\}\Big]^{(-1)^{ls}c^{(r,s)}(4lk-b^2)}\,.
\end{aligned} \tag{5.2}$$

Note that we have normalized $\widetilde{\Phi}_6$ so that the coefficient of $\exp(2\pi i(\frac{1}{2}\,T + U + V))$ is $1/16$. This agrees with the normalization convention of [7].

Since under a duality transformation by an element $\begin{pmatrix} A & B \\ C & D \end{pmatrix}$ of $\widetilde{G} \subset Sp(2, \mathbb{Z})$

$$\det \mathrm{Im}\Omega \to |\det(C\Omega + D)|^{-2} \det \mathrm{Im}\Omega \,, \tag{5.3}$$

we must have

$$\widetilde{\Phi}_6((A\Omega + B)(C\Omega + D)^{-1}) = \det(C\Omega + D)^6 \widetilde{\Phi}_6(\Omega) \,, \tag{5.4}$$

in order that $\mathcal{I}$ given in (5.1) is invariant under this transformation. Thus $\widetilde{\Phi}_6$ transforms as a modular form of weight 6 under $\widetilde{G}$.

We shall now analyze the $V \to 0$ limit of (5.2) and compare this with the corresponding result in [7]. This analysis is facilitated by examining the relation (3.22) at $z = 0$:

$$\sum_n \sum_b c^{(r,s)}(4n - b^2) q^n = F^{(r,s)}(\tau, 0) = \begin{cases} 12 & \text{for} \quad (r,s) = (0,0) \\ 4 & \text{for} \quad (r,s) \neq (0,0) \end{cases} . \tag{5.5}$$

This gives

$$\sum_b c^{(r,s)}(4n - b^2) = \begin{cases} 12\, \delta_{n,0} & \text{for} \quad (r,s) = (0,0) \\ 4\, \delta_{n,0} & \text{for} \quad (r,s) \neq (0,0) \end{cases} . \tag{5.6}$$

Taking $V \to 0$ limit in (5.2) we now get

$$\widetilde{\Phi}_6(U,T,V) \simeq -\frac{4\pi^2 V^2}{16} \, e^{2\pi i \left(\frac{1}{2}T + U\right)} \prod_{\substack{k=1 \\ k \in \mathbb{Z}}}^{\infty} \left\{ \left(1 - e^{2\pi i k T}\right)^8 \left(1 - e^{\pi i k T}\right)^8 \right\}$$
$$\prod_{\substack{l=1 \\ l \in \mathbb{Z}}}^{\infty} \left\{ \left(1 - e^{2\pi i l U}\right)^8 \left(1 - e^{4\pi i l U}\right)^8 \right\} \tag{5.7}$$

where the $-4\pi^2 V^2$ term comes from the $k = l = 0$, $b = -1$ term. This can be rewritten as

$$\widetilde{\Phi}_6(U,T,V) \simeq -\frac{1}{4} \pi^2 \, V^2 \eta(T/2)^8 \eta(T)^8 \eta(U)^8 \eta(2U)^8 \,. \tag{5.8}$$

This factorization property, including the overall normalization of $-\frac{1}{4}\pi^2$, agrees with that found in [7].

6. Construction of Φ_6

In the analysis of [7] we introduced another function Φ_6 related to $\widetilde{\Phi}_6$ by:

$$\widetilde{\Phi}_6(U,T,V) = T^{-6} \Phi_6 \left(U - \frac{V^2}{T}, -\frac{1}{T}, \frac{V}{T} \right) , \tag{6.1}$$

or equivalently

$$\Phi_6(U,T,V) = T^{-6}\widetilde{\Phi}_6\left(U - \frac{V^2}{T}, -\frac{1}{T}, \frac{V}{T}\right) . \tag{6.2}$$

From the expressions for $\mathcal{I}_{r,s,l}$ given in (4.2) we see that this transformation may be implemented by

$$m_2 \to n_1, \quad n_1 \to -m_2, \quad m_1 \to -n_2, \quad n_2 \to m_1 . \tag{6.3}$$

Thus in order to find an expression for Φ_6 we can replace $\mathcal{I}_{r,s,l}$ given in (4.2) by $\mathcal{I}'_{r,s,l}$ in which we sum over $m_2 \in \mathbb{Z} + \frac{r}{2}$ instead of $n_1 \in \mathbb{Z} + \frac{r}{2}$, and replace the $(-1)^{m_1 s}$ factor in the summand by $(-1)^{n_2 s}$:

$$\begin{aligned} \mathcal{I}'_{r,s,l} = \int_{\mathcal{F}} \frac{d^2\tau}{\tau_2} \sum_{m_1,n_1,n_2\in\mathbb{Z}, m_2\in\mathbb{Z}+\frac{r}{2}, b\in 2\mathbb{Z}+l} \exp\left[2\pi i\tau(m_1n_1+m_2n_2+\frac{b^2}{4})\right] \times \\ \exp\left[\frac{-\pi\tau_2}{Y}\left|n_2(TU-V^2)+bV+n_1T-Um_1+m_2\right|^2\right] (-1)^{n_2 s}\, h_l^{(r,s)}(\tau) . \end{aligned} \tag{6.4}$$

After Poisson resummation this amounts to summing over only integer values of n_1, n_2, k_1, k_2 and including a factor of

$$(-1)^{k_2 r}(-1)^{n_2 s} , \tag{6.5}$$

in the summand. The integral can now be evaluated following exactly the same procedure as in appendix B, the only difference being that the sum over p in eqs.(B.11), (B.21), (B.26) will contain an additional factor of $(-1)^{pr}$.[5] The net contribution to the full integral comes out to be

$$\begin{aligned} \mathcal{I}' = -2\ln\Bigg[2^8\,\kappa(\det \mathrm{Im}\Omega)^6\Bigg| \exp(2\pi i(T+U+V)) \\ \prod_{r,s=0}^{1} \prod_{\substack{(k,l.b)\in\mathbb{Z} \\ (k,l,b)>0}} \left\{1-(-1)^r \exp(2\pi i(kT+lU+bV)\right\}^{c^{(r,s)}(4kl-b^2)}\Bigg|^2\Bigg] . \end{aligned} \tag{6.6}$$

[5]An apparent additional complication arises due to the fact that the Fourier expansions of $F^{(1,0)}$ and $F^{(1,1)}$ as given in (3.22) have half integer powers of q. Thus the sum over j in eq.(B.8) will not vanish for non-integer n/k. However since $F^{(1,0)}+F^{(1,1)}$ is invariant under $\tau\to\tau+1$ due to the modular properties described in (3.11), it has Fourier expansion in integer powers of q. Thus if in analyzing the sum over j in (B.8) we consider the contribution from $F^{(1,0)}$ and $F^{(1,1)}$ together, the sum over j will force n to be a multiple of k.

We can rewrite this as

$$\mathcal{I}' = -2\left[6\ln\det\mathrm{Im}\Omega + \ln\Phi_6 + \ln\bar{\Phi}_6 + \ln\kappa + 8\ln 2\right] , \tag{6.7}$$

where

$$\Phi_6(\Omega) = -\exp(2\pi i(T+U+V)) \\ \prod_{r,s=0}^{1} \prod_{\substack{(k,l.b)\in \mathbb{Z} \\ (k,l,b)>0}} \left\{1-(-1)^r \exp(2\pi i(kT+lU+bV)\right\}^{c^{(r,s)}(4kl-b^2)} . \tag{6.8}$$

The normalization of Φ_6 is not arbitrary; it has been chosen so that we have the same additive constant $8\ln 2$ in (6.7) as in (5.1). The phase of Φ_6 can be adjusted. With the choice of phase given in (6.8) the coefficient of the $e^{2\pi i(T+U+V)}$ term matches with that of the corresponding expression in [7]. Following the same argument as in the case of $\widetilde{\Phi}_6$ we can argue that Φ_6 transforms as a modular form of weight 6 under a subgroup G of $Sp(2,\mathbb{Z})$ which is related to the earlier subgroup $\widetilde{G}$ by the conjugation described in (6.3).

Study of the $V\to 0$ limit of this expression is also straightforward. Using the relations (5.6) and the explicit expressions for the coefficients $c^{(r,s)}(0)$ and $c^{(r,s)}(-1)$ given in (B.35), we get

$$\Phi_6(U,T,V) \simeq 4\pi^2\, V^2\, \eta(T)^8\eta(2T)^8\eta(U)^8\eta(2U)^8 . \tag{6.9}$$

This is the same behaviour as found in [7].

We can also carry out a more detailed comparison between the Φ_6 defined here and those in [7]. The algorithm given in [7] goes as follows:

- We first define a set of coefficients f_n $(n\geq 1)$ through the relation:

  $$\sum_{n\geq 1} f_n e^{2\pi i\tau(n-\frac{1}{4})} = \eta(\tau)^2\,\eta(2\tau)^8 , \tag{6.10}$$

  where $\eta(\tau)$ is the Dedekind function.

- Next we define the coefficients $C(m)$ through

  $$C(m) = (-1)^m \sum_{\substack{s,n\in \mathbb{Z} \\ n\geq 1}} f_n \delta_{4n+s^2-1,m} . \tag{6.11}$$

- Φ_6 is now given by

$$\Phi_6(U,T,V) = \sum_{\substack{n,m,r\in\mathbb{Z}\\ n,m\geq 1,\, r^2<4mn}} a(n,m,r)\, e^{2\pi i(nU+mT+rV)}\,, \tag{6.12}$$

where

$$a(n,m,r) = \sum_{\substack{\alpha\in 2\mathbb{Z}+1\\ \alpha|(n,m,r),\alpha>0}} \alpha^{k-1}\, C\left(\frac{4mn-r^2}{\alpha^2}\right)\,, \tag{6.13}$$

We have compared 31 different coefficients $a(n,m,r)$ defined in (6.13) with the ones obtained from (6.8) and found them to be the same. These results for $a(n,m,r)$ are given in appendix C.

7. Construction of Φ_k and $\widetilde{\Phi}_k$

Generalization of the modular form $\widetilde{\Phi}_6$ to describe the degeneracy of dyons in a $\mathbb{Z}_N$ orbifold of $T^2\times K3$ for $N=2,3,5,7$ was also introduced in [7]. The generator g' of the $\mathbb{Z}_N$ is given by

$$g' = \hat{g}\,\widetilde{g}\,, \tag{7.1}$$

where $\hat{g}$ represents $1/N$ unit of shift along T^2 (which we shall take to be in the y^1 direction) and $\widetilde{g}$ denotes an appropriate $\mathbb{Z}_N$ action on $K3$. $\widetilde{g}$ preserves the harmonic (0,0)-form, (2,2)-form, (0,2)-form and (2,0)-form. Furthermore for each $r\neq 0$, there are $24/(N+1)$ (1,1)-forms with $\widetilde{g}$ eigenvalue $e^{2\pi ir/N}$. The rest of the $20-24(N-1)/(N+1)$ of the (1,1)-forms are invariant under $\widetilde{g}$.

The generating function for the degeneracy is given by $(\widetilde{\Phi}_k)^{-1}$ where

$$k = \frac{24}{N+1} - 2\,, \tag{7.2}$$

and $\widetilde{\Phi}_k$ is a weight k modular form of a subgroup $\widetilde{G}$ of $Sp(2,\mathbb{Z}) = SO(2,3;\mathbb{Z})$ that commutes with $1/N$ unit of shift along a circle of T^2. Associated with $\widetilde{\Phi}_k$ there is a modular form Φ_k of a different subgroup G of $Sp(2,\mathbb{Z})$, related to G by conjugation described in (6.2):

$$\Phi_k(U,T,V) = T^{-k}\widetilde{\Phi}_k\left(U-\frac{V^2}{T}, -\frac{1}{T}, \frac{V}{T}\right)\,. \tag{7.3}$$

Our goal is to find a product representation of Φ_k and $\widetilde{\Phi}_k$. For this we shall start with an analog of eq.(2.11) for the superconformal field theory associated with the $\mathbb{Z}_N$ orbifold of $K3 \times T^2$ and express it as a sum of a holomorphic and an anti-holomorphic term and a term proportional to $\ln \det \mathrm{Im}\Omega$. The holomorphic part can then be identified with Φ_k. Proceeding as in section 2 we arrive at the analog of eq.(3.9), (3.10)

$$F_{m_1,m_2,n_1,n_2}(\tau,z) = \sum_{s=0}^{N-1} e^{2\pi i m_1 s/N}\; F^{(r,s)}(\tau,z) \qquad \text{for } n_1 \in \mathbb{Z} + \tfrac{r}{N}, \quad r = 0, 1, \ldots (N-1)\,, \tag{7.4}$$

where

$$F^{(r,s)}(\tau,z) = \frac{1}{N}\, Tr^{K3}_{RR;\widetilde{g}^r}\left((-1)^{F_{K3}}(-1)^{\bar F_{K3}}\, \widetilde{g}^s\, e^{2\pi i z F_{K3}}\, q^{L_0}\bar q^{\bar L_0}\right)\,. \tag{7.5}$$

From these definitions it follows that

$$F^{(r,s)}\left(\frac{a\tau+b}{c\tau+d}, \frac{z}{c\tau+d}\right) = \exp\left(2\pi i \frac{cz^2}{c\tau+d}\right) F^{(cs+ar,ds+br)}(\tau,z)\,, \tag{7.6}$$

for

$$a,b,c,d \in \mathbb{Z}, \qquad ad-bc = 1\,. \tag{7.7}$$

In (7.6) the indices $cs+ar$ and $ds+br$ are to be taken mod N. Thus for each (r,s), $F^{(r,s)}(\tau,z)$ transforms as a weak Jacobi form[26] of weight zero and index 1 under the group $\Gamma(N)$.

We can now define the coefficients $c^{(r,s)}(n)$ in a manner analogous to (3.22)[6]

$$F^{(r,s)}(\tau,z) = \sum_{b\in\mathbb{Z},\, n\in\mathbb{Z}/N} c^{(r,s)}(4n-b^2)\, q^n\, e^{2\pi i z b}\,. \tag{7.8}$$

Contribution to $c^{(0,s)}(l)$ for $l = 0,\, -1$ comes from geometric data of $K3$ and can be computed easily. In particular untwisted sector states with $n=0$, $b=0$ are associated with (1,1)-forms, those with $n=0$, $b=1$ are associated with the (2,2) and the (2,0)-forms, and those with $n=0$, $b=-1$ are associated with the (0,0) and the (0,2)-forms. Thus $Nc^{(0,s)}(0)$ measures the trace of $\widetilde{g}^s$ on the (1,1)-forms of $K3$ and $Nc^{(0,s)}(-1)$ measures the trace of $\widetilde{g}^s$ on the (0,0), (0,2) or (2,0), (2,2)-forms of

[6] In order that $F^{(r,s)}(\tau,z)$ has an expansion of the form given in (7.8) we need to ensure that this can be expressed as a linear combination of $\vartheta_3(2\tau,2z)$ and $\vartheta_2(2\tau,2z)$ with z-independent coefficients as in (3.15). This follows from the fact that the z-dependence of $F^{(r,s)}(\tau,z)$ comes from the SU(2) current algebra associated with the superconformal field theory, and this commutes with the $\mathbb{Z}_N$ generator $\widetilde{g}$. $\vartheta_3(2\tau,2z)$ and $\vartheta_2(2\tau,2z)$ simply represent the contributions from the even and odd F_{K3} charge sector of this SU(2) sector of the theory.

$K3$. These can be easily computed from the $\widetilde{g}$ action of the cycles described earlier, and we get

$$c^{(0,0)}(0) = \frac{20}{N}\,, \qquad c^{(0,0)}(-1) = \frac{2}{N}\,,$$
$$c^{(0,s)}(0) = \frac{1}{N}\left(20 - \frac{24N}{N+1}\right)\,, \qquad c^{(0,s)}(-1) = \frac{2}{N}\,, \quad \text{for } s = 1, 2, \ldots (N-1)\,. \tag{7.9}$$

Several other useful properties of $c^{(r,s)}$ may be derived without explicitly computing $F^{(r,s)}(\tau, z)$. First note that $F^{(0,0}(\tau, z)$ is $1/N$ times the elliptic genus of $K3$. Hence it is given by

$$F^{(0,0)}(\tau, z) = \frac{8}{N}\left[\frac{\vartheta_2(\tau, z)^2}{\vartheta_2(\tau, 0)^2} + \frac{\vartheta_3(\tau, z)^2}{\vartheta_3(\tau, 0)^2} + \frac{\vartheta_4(\tau, z)^2}{\vartheta_4(\tau, 0)^2}\right]\,. \tag{7.10}$$

Next it follows from the definition (7.5) that $F^{(0,s)}(\tau, 0)$ is τ independent since it receives contribution only from the $L_0 = \bar{L}_0 = 0$ states. The modular transformation laws (7.6) together with (7.9) then imply that

$$F^{(r,s)}(\tau, 0) = F^{(0,t)}(\tau, 0)|_{t=g.c.d.(r,s)} = c^{(0,t)}(0) + 2\,c^{(0,t)}(-1) = \frac{24}{N(N+1)} \quad \text{for} \quad (r,s) \neq (0,0)\,. \tag{7.11}$$

Substituting (7.10), (7.11) into the expansion (7.8) we get the analog of eq.(5.6)

$$\sum_b c^{(r,s)}(4n - b^2) = \begin{cases} \frac{24}{N}\,\delta_{n,0} & \text{for} \quad (r,s) = (0,0) \\ \frac{24}{N(N+1)}\,\delta_{n,0} & \text{for} \quad (r,s) \neq (0,0) \end{cases}\,. \tag{7.12}$$

Further information about these coefficients comes from the fact that $\sum_{r,s=0}^{N-1} F^{(r,s)}(\tau, z)$ represent the elliptic genus of the super-conformal σ-model with target space $K3/\mathbb{Z}_N$ with the $\mathbb{Z}_N$ generated by $\widetilde{g}$. However for any N this gives us back the superconformal field theory with target space $K3$, and hence $\sum_{r,s=0}^{N-1} F^{(r,s)}(\tau, z)$ must give us the elliptic genus of $K3$. This in turn is just $NF^{(0,0)}(\tau, z)$. Thus we have

$$\sum_{r,s=0}^{N-1} F^{(r,s)}(\tau, z) = N\,F^{(0,0)}(\tau, z)\,. \tag{7.13}$$

Furthermore the contribution $\sum_{s=0}^{N-1} F^{(r,s)}(\tau, z)$ for a fixed r may be interpreted as the contribution to the elliptic genus from the sector twisted by $\widetilde{g}^r$. For prime values of N, $\widetilde{g}^r$ is an order N transformation for all $r \neq 0$ mod N. Hence we expect the sectors twisted by $\widetilde{g}^r$ to give the same contribution to the elliptic genus for all $r \neq 0$ mod N. This, together with (7.13), gives

$$\sum_{s=0}^{N-1} F^{(r,s)}(\tau, z) = \frac{1}{N-1}\left[N\, F^{(0,0)}(\tau, z) - \sum_{s=0}^{N-1} F^{(0,s)}(\tau, z)\right] \qquad r \neq 0 \bmod N\,. \tag{7.14}$$

Translated to a condition on the coefficients $c^{(r,s)}(m)$, this gives

$$\sum_{s=0}^{N-1} c^{(r,s)}(m) = \frac{1}{N-1}\left[N\, c^{(0,0)}(m) - \sum_{s=0}^{N-1} c^{(0,s)}(m)\right] \qquad \text{for any } m, \quad r \neq 0 \bmod N\,. \tag{7.15}$$

For $m = 0, -1$ we can explicitly evaluate the right hand side of this equation using (7.9). In particular setting $m = -1$ we get

$$\sum_{s=0}^{N-1} c^{(r,s)}(-1) = 0\,, \qquad \text{for } r \neq 0 \bmod N\,. \tag{7.16}$$

Although for $N = 3, 5, 7$ we have not been able to compute $F^{(r,s)}(\tau, z)$ directly, a set of $F^{(r,s)}(\tau, z)$ satisfying the requirements given above are as follows. Let us define

$$A(\tau, z) = \left[\frac{\vartheta_2(\tau,z)^2}{\vartheta_2(\tau,0)^2} + \frac{\vartheta_3(\tau,z)^2}{\vartheta_3(\tau,0)^2} + \frac{\vartheta_4(\tau,z)^2}{\vartheta_4(\tau,0)^2}\right]\,, \tag{7.17}$$

$$B(\tau, z) = \eta(\tau)^{-6}\vartheta_1(\tau, z)^2\,, \tag{7.18}$$

and

$$E_N(\tau) = \frac{12i}{\pi(N-1)}\,\partial_\tau\left[\ln\eta(\tau) - \ln\eta(N\tau)\right] = 1 + \frac{24}{N-1}\sum_{\substack{n_1,n_2\geq 1\\ n_1\neq 0 \bmod N}} n_1 e^{2\pi i n_1 n_2 \tau}\,. \tag{7.19}$$

Then under an $SL(2,\mathbb{Z})$ transformation $A(\tau, z)$ transforms as a weak Jacobi form of weight 0 and index 1 and $B(\tau, z)$ transforms as a weak Jacobi form of weight -2 and index 1. Furthermore

$$E_N(\tau+1) = E_N(\tau), \qquad E_N(-1/\tau) = -\tau^2\frac{1}{N}E_N(\tau/N)\,. \tag{7.20}$$

From this it follows that $E_N(\tau)$ is a modular form of weight 2 of the group $\Gamma_0(N)$ and hence also of $\Gamma(N)$[27]. Using these properties one can show that the following choice of $F^{r,s}(\tau,z)$ satisfy all the requirements described above:

$$\begin{aligned}
F^{(0,0)}(\tau,z) &= \frac{8}{N} A(\tau,z)\,, \\
F^{(0,s)}(\tau,z) &= \frac{8}{N(N+1)}\, A(\tau,z) - \frac{2}{N+1}\, B(\tau,z)\, E_N(\tau) \qquad \text{for } 1\le s\le (N-1)\,, \\
F^{(r,rk)}(\tau,z) &= \frac{8}{N(N+1)}\, A(\tau,z) + \frac{2}{N(N+1)}\, E_N\left(\frac{\tau+k}{N}\right)\, B(\tau,z)\,, \\
&\qquad\qquad\qquad \text{for } 1\le r\le (N-1),\ 0\le k\le (N-1)\,.
\end{aligned} \tag{7.21}$$

The rest of the analysis now proceeds exactly as in the $N=2$ case. We arrive at an analog of eq.(4.2) for $\mathcal{I}_{r,s,l}$:

$$\begin{aligned}
\mathcal{I}_{r,s,l} = \int_{\mathcal{F}} \frac{d^2\tau}{\tau_2} & \sum_{m_1,m_2,n_2\in\mathbb{Z},n_1\in\mathbb{Z}+\frac{r}{N},b\in 2\mathbb{Z}+l} \exp\left[2\pi i\tau(m_1n_1+m_2n_2+\frac{b^2}{4})\right]\times \\
& \exp\left(\frac{-\pi\tau_2}{Y}\left|n_2(TU-V^2)+bV+n_1T-Um_1+m_2\right|^2\right)\, e^{2\pi i m_1 s/N}\, h_l^{(r,s)}(\tau)\,, \\
& \qquad 0\le r,s\le (N-1)\,.
\end{aligned} \tag{7.22}$$

This can then be Poisson resummed and analyzed using the techniques described in appendix B and be split into holomorphic and anti-holomorphic parts to extract the expression for $\widetilde{\Phi}_k$. On the other hand if we want information about Φ_k we need to use the operation eq.(6.3) to consider a new integral

$$\begin{aligned}
\mathcal{I}'_{r,s,l} = \int_{\mathcal{F}} \frac{d^2\tau}{\tau_2} & \sum_{m_1,n_1,n_2\in\mathbb{Z},m_2\in\mathbb{Z}-\frac{r}{N},b\in 2\mathbb{Z}+l} \exp\left[2\pi i\tau(m_1n_1+m_2n_2+\frac{b^2}{4})\right]\times \\
& \exp\left(\frac{-\pi\tau_2}{Y}\left|n_2(TU-V^2)+bV+n_1T-Um_1+m_2\right|^2\right)\, e^{-2\pi i n_2 s/N}\, h_l^{(r,s)}(\tau)\,, \\
& \qquad 0\le r,s\le (N-1)\,.
\end{aligned} \tag{7.23}$$

In this case Poisson resummation over m_1, m_2 will give rise to an additional factor of $\exp(2\pi i k_2 r/N)$ and the final sum will be over integer values of n_1, n_2, k_1, k_2. This can again be analyzed using the techniques described in appendix B.

We shall not give the details of the analysis but write down the final expression. The expressions for Φ_k and $\widetilde{\Phi}_k$ obtained this way are:

$$\Phi_k(U,T,V) = -\exp\left\{2\pi i\,(T+U+V)\right\}$$

$$\prod_{r,s=0}^{N-1} \prod_{\substack{(k',l,b)\in \mathbb{Z}\\ (k',l,b)>0}} \left\{1 - e^{2\pi ir/N}\ \exp(2\pi i(k'T + lU + bV)\right\}^{\frac{1}{2}c^{(r,s)}(4k'l-b^2)}$$

$$\prod_{r,s=0}^{N-1} \prod_{\substack{(k',l,b)\in \mathbb{Z}\\ (k',l,b)>0}} \left\{1 - e^{-2\pi ir/N}\ \exp(2\pi i(k'T + lU + bV)\right\}^{\frac{1}{2}c^{(r,s)}(4k'l-b^2)} \tag{7.24}$$

$$\begin{aligned}\widetilde{\Phi}_k(U,T,V) = &-(i\sqrt{N})^{-k-2}\ \exp\left(2\pi i\left(\frac{1}{N}T + U + V\right)\right)\\ &\prod_{r=0}^{N-1} \prod_{\substack{l,b\in \mathbb{Z}, k'\in \mathbb{Z}+\frac{r}{N}\\ k',l,b>0}} \left\{1 - \exp(2\pi i(k'T + lU + bV))\right\}^{\frac{1}{2}\sum_{s=0}^{N-1} e^{-2\pi ils/N}\ c^{(r,s)}(4lk'-b^2)}\\ &\prod_{r=0}^{N-1} \prod_{\substack{l,b\in \mathbb{Z}, k'\in \mathbb{Z}-\frac{r}{N}\\ k',l,b>0}} \left\{1 - \exp(2\pi i(k'T + lU + bV))\right\}^{\frac{1}{2}\sum_{s=0}^{N-1} e^{2\pi ils/N}\ c^{(r,s)}(4lk'-b^2)}\end{aligned} \tag{7.25}$$

Φ_k has been normalized so that the coefficient of the $\exp(2\pi i(T+U+V))$ is -1. $\widetilde{\Phi}_k$ is normalized so that the coefficient of the $\exp(2\pi i(\frac{1}{N}T+U+V))$ term is $-(i\sqrt{N})^{-k-2}$. These conventions agree with the one used in [7].

The weight k of the modular form, determined by examining the term proportional to $\ln\det \mathrm{Im}\Omega$ in the final expression for the integral, is given by

$$k = \frac{1}{2}\sum_{s=0}^{N-1} c^{(0,s)}(0) = \frac{24}{N+1} - 2\,, \tag{7.26}$$

where we have used eq.(7.9). This agrees with (7.2). Furthermore, using eqs.(7.9), (7.12) and (7.16) we can study the $V \to 0$ limits of Φ_k and $\widetilde{\Phi}_k$. We get

$$\Phi_k(U,T,V) \simeq 4\pi^2 V^2 (\eta(T)\eta(NT))^{k+2}(\eta(U)\eta(NU))^{k+2}\,, \tag{7.27}$$

and

$$\widetilde{\Phi}_k(U,T,V) \simeq (i\sqrt{N})^{-k-2}\, 4\pi^2 V^2\, (\eta(T)\eta(T/N))^{k+2}(\eta(U)\eta(NU))^{k+2}\,, \tag{7.28}$$

in agreement with [7].

Another important consistency check for eqs.(7.24), (7.25) comes from looking at the coefficient of the terms involving a single power of $e^{2\pi iU}$ and all powers of T

and V. For Φ_k this is given by

$$e^{2\pi iU}\,\eta(T)^{k-4}\,\eta(NT)^{k+2}\,\vartheta_1(T,V)^2\,, \tag{7.29}$$

and for $\widetilde{\Phi}_k$ this is given by

$$(i\sqrt{N})^{-k-2}\,e^{2\pi iU}\,\eta(T)^{k-4}\,\eta(T/N)^{k+2}\,\vartheta_1(T,V)^2\,. \tag{7.30}$$

These agree with the corresponding expressions found in [7].

We have also compared a few terms in the expansion of Φ_k given in (7.24) with the one given in [7]. The results are given in appendix C.

Acknowledgments

We would like to thank A. Dabholkar, D. Gaiotto, E. Gava, R. Gopakumar, S. Gun, K. S. Narain, B. Ramakrishnan and D. Suryaramana for useful discussions. D. J. would like to acknowledge the hospitality of TIFR and APCTP where part of this work was done.

A. Calculation of the Elliptic Genus

In this appendix we shall calculate

$$\begin{aligned}&F_{m_1,m_2,n_1,n_2}(\tau,z)\\ &= Tr_{RR;m_1,m_2,n_1,n_2}\left((-1)^{(F_{K3}+F_{T^2})}(-1)^{(\bar{F}_{K3}+\bar{F}_{T^2})}F_{T^2}\bar{F}_{T^2}e^{2\pi izF_{K3}}\,q^{L_0'}\bar{q}^{\bar{L}_0'}\right)\,,\end{aligned} \tag{A.1}$$

in the superconformal field theory with target space $(K3\times T^2)/\mathbb{Z}_2$. For this we shall use an orbifold description of $K3$. We consider a superconformal σ-model with target space $T^2\times T^4$ with y^1, y^2 denoting the T^2 coordinates and y^3, y^4, y^5, y^6 denoting the T^4 coordinates, and mod out the theory by a $\mathbb{Z}_2\times\mathbb{Z}_2$ symmetry generated by elements g and g'. The action of g and g' are given by:

$$\begin{aligned}g:&\quad (y^1,y^2,y^3,y^4,y^5,y^6)\to(y^1,y^2,-y^3,-y^4,-y^5,-y^6)\\ g':&\quad (y^1,y^2,y^3,y^4,y^5,y^6)\to(y^1+\pi,y^2,y^3+\pi,y^4,y^5,y^6)\,.\end{aligned} \tag{A.2}$$

Orbifolding by g produces a $K3\times T^2$ manifold. Further orbifolding by g' produces $(K3\times T^2)/\mathbb{Z}_2$ where the $\mathbb{Z}_2$ generator involves a shift along T^2 and a $\mathbb{Z}_2$ involution in $K3$ that preserves the (4,4) superconformal symmetry of the corresponding world-sheet theory. We denote by F_{T^4} and F_{T^2} holomorphic parts of the world-sheet fermion number associated with the T^4 and the T^2 parts and by $\bar F_{T^4}$ and $\bar F_{T^2}$ the anti-holomorphic parts of the world-sheet fermion number associated with the T^4 and the T^2 parts. We shall be considering an arbitrary T^2 parametrized by the Kähler modulus T and complex structure modulus U, and arbitrary Wilson lines A_1, A_2 corresponding to deforming the world-sheet theory by the marginal operator

$$\sum_{i=1}^{2} A_i \int d^2z\, \bar\partial Y^i J_{T^4}\,, \tag{A.3}$$

where J_{T^4} is the U(1) current corresponding to the charge F_{T^4}. We shall denote by V the complex combination $A_2 - iA_1$.

We now define

$$\begin{aligned}
&F_{m_1,m_2,n_1,n_2}(a,b;c,d;\tau,z)\\
&= Tr^{T^4\times T^2}_{m_1,m_2,n_1,n_2;RR;g^a,g'^b}\left((-1)^{(F_{T^4}+F_{T^2})}(-1)^{(\bar F_{T^4}+\bar F_{T^2})}F_{T^2}\bar F_{T^2}e^{2\pi i z F_{T^4}}q^{L_0}\bar q^{\bar L_0}g^c g'^d\right)\,,
\end{aligned} \tag{A.4}$$

where L_0', $\bar L_0'$ have been defined in eqs.(3.1), (3.4). Here a,b,c,d take values 0 or 1. $Tr^{T^4\times T^2}_{m_1,m_2,n_1,n_2;RR;g^a,g'^b}$ denotes trace in the original CFT associated with the $T^2\times T^4$ target space over RR sector states twisted by $g^a g'^b$ and carrying (m_1,m_2) units of momentum and (n^1,n^2) units of winding along (y^1,y^2). The quantity $F_{m_1,m_2,n_1,n_2}(\tau,z)$ is then given by

$$F_{m_1,m_2,n_1,n_2}(\tau,z) = \frac{1}{4}\sum_{a,b,c,d=0}^{1} F_{m_1,m_2,n_1,n_2}(a,b;c,d;\tau,z)\,. \tag{A.5}$$

We shall now calculate $F_{m_1,m_2,n_1,n_2}(a,b;c,d;\tau,z)$. First we note that

$$F_{m_1,m_2,n_1,n_2}(0,0;0,d;\tau,z) = 0 \quad \text{for } d=0,1 \tag{A.6}$$

due to the fermion zero modes associated with the $3,4,5,6$ directions.

Next we have

$$\begin{aligned}
F_{m_1,m_2,n_1,n_2}(0,0;1,d;\tau,z) &= (-1)^{m_1 d}\, 4\,(1+e^{2\pi i z})(1+e^{-2\pi i z})\\
&\qquad \frac{\prod_{n=1}^{\infty}(1+q^n e^{2\pi i z})^2(1+q^n e^{-2\pi i z})^2}{\prod_{n=1}^{\infty}(1+q^n)^4}\\
&= (-1)^{m_1 d}\, 16\, \frac{\vartheta_2(\tau,z)^2}{\vartheta_2(\tau,0)^2}\,.
\end{aligned} \tag{A.7}$$

In the first line the factor of 4 comes from the anti-holomorphic fermion zero modes associated with the 3,4,5,6 directions and the factor of $(1+e^{2\pi iz})(1+e^{-2\pi iz})$ comes from the holomorphic fermion zero-modes. In the second line the numerator comes from the holomorphic non-zero mode fermionic oscillators associated with the 3,4,5,6 directions and the denominator comes from the holomorphic non-zero mode bosonic oscillators associated with the same directions. The contribution from the bosonic and fermionic oscillators associated with the 1 and 2 directions cancel. Also the contributions from all the non-zero mode fermion and bosonic oscillators in the anti-holomorphic sector always cancel. In arriving at (A.7) we have used that the action of g' on the state carrying m_1 units of momentum along y^1 gives a factor of $(-1)^{m_1}$ and the action of g changes the signs of the fermionic and the bosonic oscillators associated with T^4. Also since the action of g reverses the direction of momentum along the 3,4,5,6 directions, only states carrying zero momentum along T^4 contributes to the trace and hence the result is independent of the moduli of T^4. This will be a generic feature of all the terms; either they will vanish due to fermion zero modes or only the zero momentum mode will contribute due to either a g insertion or a twist under g.

Let us now turn to the twisted sector states. First note that there are 16 twisted sector states under g, located as $y^m = 0,\pi$ for $m = 3,4,5,6$. g' (and also gg') exchanges these states pairwise. Thus the action of g' and gg' on these states is off-diagonal and hence the trace of g' and gg' over these states vanish. This gives

$$F_{m_1,m_2,n_1,n_2}(1,0;c,1;\tau,z) = 0 \quad \text{for } c = 0,1\,. \tag{A.8}$$

On the other hand we have

$$\begin{aligned} F_{m_1,m_2,n_1,n_2}(1,0;c,0;\tau,z) &= 16\frac{\prod_{n=0}^{\infty}(1-q^{n+\frac{1}{2}}e^{2\pi iz+i\pi c})^2(1-q^{n+\frac{1}{2}}e^{-2\pi iz+i\pi c})^2}{\prod_{n=0}^{\infty}(1-e^{i\pi c}\,q^{n+\frac{1}{2}})^4} \\ &= \begin{cases} 16\,\vartheta_4(\tau,z)^2/\vartheta_4(\tau,0)^2 & \text{for } c=0 \\ 16\,\vartheta_3(\tau,z)^2/\vartheta_3(\tau,0)^2 & \text{for } c=1 \end{cases}. \end{aligned} \tag{A.9}$$

The factor of 16 is due to the existence of 16 twisted sector states.

Next we consider sectors twisted by g'. In this case the winding number n_1 along y^1 must be half integer and similarly the winding number along y^3 must also be half integer. Since the g' twist just involves a shift and does not affect the world-sheet fermions, the fermion zero modes associated with the 3-6 directions make the contribution vanish unless the g projection is inserted into the trace. This gives:

$$F_{m_1,m_2,n_1,n_2}(0,1;0,d;\tau,z) = 0 \quad \text{for } d = 0, 1\,. \tag{A.10}$$

On the other hand the action of g as well as of gg' reverses the sign of the winding number along y^3 and hence these elements are off-diagonal in the sector twisted by g'. This gives

$$F_{m_1,m_2,n_1,n_2}(0,1;1,d;\tau,z) = 0 \quad \text{for } d = 0, 1\,. \tag{A.11}$$

Finally let us turn to the sector twisted under gg'. Action of gg' on y^3, y^4, y^5, y^6 gives fixed points at $y^3 = \pi/2, 3\pi/2$, $y^m = 0, \pi$ for $m = 4, 5, 6$. Although this are not real fixed points due to the shift action $y^2 \to y^2 + \pi$, we can label the 16 twisted sectors by these would be fixed points. Both g and g' exchange these fixed points pairwise and hence are represented by off-diagonal matrices. This gives

$$\begin{aligned} F_{m_1,m_2,n_1,n_2}(1,1;1,0;\tau,z) &= 0\,, \\ F_{m_1,m_2,n_1,n_2}(1,1;0,1;\tau,z) &= 0\,. \end{aligned} \tag{A.12}$$

On the other hand both the identity element and gg' leave the fixed points invariant and give non-zero answers. We have

$$\begin{aligned} F_{m_1,m_2,n_1,n_2}(1,1;0,0;\tau,z) &= 16\,\frac{\prod_{n=0}^{\infty}(1-q^{n+\frac{1}{2}}e^{2\pi i z})^2(1-q^{n+\frac{1}{2}}e^{-2\pi i z})^2}{\prod_{n=0}^{\infty}(1-q^{n+\frac{1}{2}})^4} \\ &= 16\,\frac{\vartheta_4(\tau,z)^2}{\vartheta_4(\tau,0)^2}\,, \end{aligned} \tag{A.13}$$

and

$$\begin{aligned} F_{m_1,m_2,n_1,n_2}(1,1;1,1;\tau,z) &= 16\,(-1)^{m_1}\,\frac{\prod_{n=0}^{\infty}(1-q^{n+\frac{1}{2}}e^{2\pi i z+i\pi})^2(1-q^{n+\frac{1}{2}}e^{-2\pi i z+i\pi})^2}{\prod_{n=0}^{\infty}(1-e^{i\pi}q^{n+\frac{1}{2}})^4} \\ &= 16\,(-1)^{m_1}\,\frac{\vartheta_3(\tau,z)^2}{\vartheta_3(\tau,0)^2}\,. \end{aligned} \tag{A.14}$$

Using eqs.(A.4)-(A.14) we now get

$$\begin{aligned} F_{m_1,m_2,n_1,n_2}(\tau,z) &= 4\left[\frac{\vartheta_2(\tau,z)^2}{\vartheta_2(\tau,0)^2} + \frac{\vartheta_3(\tau,z)^2}{\vartheta_3(\tau,0)^2} + \frac{\vartheta_4(\tau,z)^2}{\vartheta_4(\tau,0)^2}\right] \\ &\quad +4(-1)^{m_1}\frac{\vartheta_2(\tau,z)^2}{\vartheta_2(\tau,0)^2} \qquad\qquad \text{for } n_1 \in \mathbb{Z} \\ &= 4\frac{\vartheta_4(\tau,z)^2}{\vartheta_4(\tau,0)^2} + 4\,(-1)^{m_1}\,\frac{\vartheta_3(\tau,z)^2}{\vartheta_3(\tau,0)^2} \qquad \text{for } n_1 \in \mathbb{Z}+\tfrac{1}{2} \end{aligned} \tag{A.15}$$

B. Evaluation of the Integral

In this appendix we shall evaluate the integral (4.13)

$$\mathcal{I} = \sum_A{}' \int_{\mathcal{F}_A} \frac{d^2\tau}{\tau_2^2}\,\frac{Y}{U_2}\,\exp\Bigg(-\frac{\pi Y}{U_2^2\tau_2}|\mathcal{A}|^2 - 2\pi i T \det A \\ -\frac{\pi n_2}{U_2}(V^2\tilde{\mathcal{A}} - \bar{V}^2\mathcal{A}) + \frac{2\pi i V_2^2}{U_2^2}(n_1 + n_2\bar{U})\mathcal{A}\Bigg)\, F^{(r,s)}\left(\tau, -i\frac{V\tilde{\mathcal{A}} - \bar{V}\mathcal{A}}{2\,U_2}\right)\,. \tag{B.1}$$

The sum over A runs over all integer valued 2×2 matrices of the form (4.6) which are not related to each other by an $SL(2,\mathbb{Z})$ transformation acting from the right. $\mathcal{F}_A$ is the union of images of the fundamental region $\mathcal{F}$ under $SL(2,\mathbb{Z})$ transformations which act non-trivially on A. $\mathcal{A}$, $\tilde{\mathcal{A}}$ are defined in (4.7) and $(r,s) = (2n_1, 2k_1)$ mod 2.

In carrying out the integral we need to introduce some regularization and subtraction scheme. Following [28] we regularize possible divergences in the integral by including a factor of $(1-\exp(-\Lambda/\tau_2))$ in the integrand. For $\tau_2 << \Lambda$ this factor is close to unity, but for $\tau_2 >> \Lambda$ it is close to zero. We also add to the integral a term

$$-\left(c^{(0,0)}(0) + c^{(0,1)}(0)\right)\int_{\mathcal{F}} \frac{d^2\tau}{\tau_2^2}\,(1-\exp(-\Lambda/\tau_2))\,. \tag{B.2}$$

As we shall see, this is necessary for getting a finite $\Lambda \to \infty$ limit.

Following the same procedure as in [28] we split the integration into the three orbits.

1. Contribution $\mathcal{I}_1$ from the zero orbit

For $A = 0$ we have $(r,s) = (0,0)$ and $\mathcal{F}_A = \mathcal{F}$, – the fundamental region of $SL(2,\mathbb{Z})$. The integral (4.13) reduces to

$$\mathcal{I}_1 = \frac{Y}{U_2}\int_{\mathcal{F}} \frac{d^2\tau}{\tau_2^2} F^{(0,0)}(\tau,0) = \frac{Y}{U_2}\,\frac{\pi}{3}\,12\,, \tag{B.3}$$

using the expression for $F^{(0,0)}(\tau,z)$ given in (3.13).

2. Contribution $\mathcal{I}_2$ from the non-degenerate orbit

Here we consider A to be

$$A = \begin{pmatrix} k & j \\ 0 & p \end{pmatrix}, \quad 2k-1 \geq 2j \geq 0, \quad p \neq 0, \quad k,j \in \frac{1}{2}\mathbb{Z}, \quad p \in \mathbb{Z}\,. \tag{B.4}$$

In this case the region $\mathcal{F}_A$ corresponds to two copies of the upper-half plane (coming from A and $-A$) and the indices (r,s) in (B.1) are given by

$$(r,s) = (2k \bmod 2, 2j \bmod 2)\,. \tag{B.5}$$

Note that for the above form of A,

$$\det A = kp, \quad \mathcal{A} = k\tau + j + pU, \quad \tilde{\mathcal{A}} = k\tau + j + p\bar{U}\,. \tag{B.6}$$

Let us first consider the case $k \in \mathbb{Z}$, $j \in \mathbb{Z}$. In this case j runs from 0 to $k-1$ in steps of 1. The relevant $F^{(r,s)}$ is $F^{(0,0)}$. In order to carry out the integral we replace $F^{(0,0)}(\tau,z)$ in (B.1) by its Fourier expansion (3.22). If we now change the integration variable from τ_1 to

$$\tau_1' = k\tau_1 + j + pU_1\,, \tag{B.7}$$

then $\mathcal{A}$, $\tilde{\mathcal{A}}$ and hence also the exponential factor in (B.1), expressed as a function of τ_1' and τ_2, will have no j dependence. The only j dependence comes from the term

$$\exp(2\pi i n\tau_1) = \exp\left(2\pi i n\frac{1}{k}(\tau_1' - j - pU_1)\right) \tag{B.8}$$

which arises from the factor $c^{(0,0)}(4n-b^2)\exp(2\pi i\tau n)$ in the expansion (3.22) of $F^{(0,0)}(\tau,z)$). Since in this case n is an integer, the summation over j from 0 to $k-1$ in steps of 1 imposes the condition $n = n'k$ where n' is an integer. Furthermore since $n \ge 0$ and $k > 0$, we have $n' \ge 0$. The summation over j also produces a factor of k which cancels the $1/k$ factor arising due to the change of variables from τ_1 to τ_1' in the measure.

Using (B.6)-(B.8) we see that the integration over τ_1' in (B.1) is just a Gaussian integration. The result of carrying out this integral is

$$\begin{aligned}
\mathcal{I}_{2;k,j\in\mathbb{Z}} &= \sum_{\substack{n',k\in\mathbb{Z},\, b,p\in\mathbb{Z}\\ n'\ge 0, k>0, p\ne 0}} \sqrt{Y}\int_0^\infty \frac{d\tau_2}{\tau_2^{3/2}}\,\exp(\mathcal{F})\,c^{(0,0)}(4n'k-b^2) \\
\mathcal{F} &\equiv -2\pi\tau_2 n'k - \frac{\pi Y}{U_2^2\tau_2}(k\tau_2 + pU_2)^2 - 2\pi i T kp - 2\pi i p n' U_1 \\
&\quad + \frac{\pi b}{U_2}(-2V_2 k\tau_2 - 2ipU_2V_1) \\
&\quad - \frac{2\pi V_2^2}{U_2^2}(k^2\tau_2 + kpU_2) - \frac{\pi B^2 U_2^2\tau_2}{Y} \\
B &\equiv n' + \frac{bV_2}{U_2} + \frac{V_2^2}{U_2^2}k
\end{aligned} \tag{B.9}$$

The τ_2 integral is of the Bessel form and can be performed using

$$\int_0^\infty \frac{du}{u^{3/2}}\, e^{-au-bu^{-1}} = e^{-2\sqrt{ab}}\sqrt{\frac{\pi}{b}}\,. \tag{B.10}$$

This gives

$$\begin{aligned}\mathcal{I}_{2;k,j\in\mathbb{Z}} = & \sum_{\substack{n',k,b\in\mathbb{Z},\, p\in\mathbb{Z}\\ n'\geq 0,k>0,p\neq 0}} \frac{1}{|p|} c^{(0,0)}(4n'k-b^2)\exp\{-2\pi iTkp - 2\pi k|p|T_2 - 2\pi kpT_2 \\ & -2\pi ipn'U_1 - 2\pi|p|U_2n' - 2\pi ibpV_1 - 2\pi|p|bV_2\} \\ = & -\ln \prod_{\substack{n',k,b\in\mathbb{Z}\\ n'\geq 0,k>0}} \left\{ \left|1-\exp(2\pi i(kT+n'U+bV))\right|^{2c^{(0,0)}(4n'k-b^2)} \right\}\end{aligned} \tag{B.11}$$

Next we consider the contribution from the $k\in\mathbb{Z}$, $j\in\mathbb{Z}+\frac{1}{2}$ terms. In this case j takes values from $\frac{1}{2}$ to $k-\frac{1}{2}$ in steps of 1 and $(r,s)=(0,1)$. The analysis proceeds as in the previous case, the only difference being that the sum over j of (B.8) gives an additional factor of $(-1)^{n'}$ besides forcing the condition $n=n'k$ with $n'\in\mathbb{Z}$. The analog of eq.(B.11) is then

$$\mathcal{I}_{2;k\in\mathbb{Z},j\in\mathbb{Z}+\frac{1}{2}} = -\ln \prod_{\substack{n',k,b\in\mathbb{Z}\\ n'\geq 0,k>0}} \left\{ \left|1-\exp(2\pi i(kT+n'U+bV))\right|^{2(-1)^{n'}c^{(0,1)}(4n'k-b^2)} \right\} \tag{B.12}$$

Finally let us consider the case $k\in\mathbb{Z}+\frac{1}{2}$. In this case instead of letting j run from 0 to $k-\frac{1}{2}$ in steps of $\frac{1}{2}$ we can let it run from 0 to $(2k-1)$ in steps of 1 by means of a further SL$(2,\mathbb{Z})$ duality transformation. For each of these terms the relevant (r,s) are $(1,0)$. Proceeding as in the $k,j\in\mathbb{Z}$ case we now see that the sum over j in (B.8) forces the condition $n=4n'k$ with $n'\in\mathbb{Z}$ and when this condition is satisfied we get a factor of $2k$.[7] The rest of the analysis proceeds as in the previous case and we obtain

$$\mathcal{I}_{2;k\in\mathbb{Z}+\frac{1}{2}} = -2\ln \prod_{\substack{n',b\in\mathbb{Z},k\in\mathbb{Z}+\frac{1}{2}\\ n'\geq 0,k>0}} \left\{ \left|1-\exp\{2\pi i(kT+n'U+bV)\}\right|^{2c^{(1,0)}(4n'k-b^2)} \right\} \tag{B.13}$$

[7]Note that in this case n is either an integer or a half integer, but the sum over j still forces n to be an integer multiple of k since the sum runs over $2k$ values instead of k values.

Thus the net contribution to the integral from the non-degenerate orbits take the form

$$\mathcal{I}_2 = -\ln \Bigg[\prod_{\substack{n',k,b\in \mathbb{Z}\\ n'\geq 0, k>0}} \left\{ \Big| 1 - \exp(2\pi i(kT + n'U + bV)) \Big|^{2c^{(0,0)}(4n'k-b^2)+2(-1)^{n'}c^{(0,1)}(4n'k-b^2)} \right.$$

$$\left. \prod_{\substack{n',b\in \mathbb{Z}, k\in \mathbb{Z}+\frac{1}{2}\\ n'\geq 0, k>0}} \left\{ \Big| 1 - \exp(2\pi i(kT + n'U + bV)) \Big|^{4c^{(1,0)}(4n'k-b^2)} \right\} \Bigg] \qquad \text{(B.14)}$$

3. Contribution $\mathcal{I}_3$ from the degenerate orbit

Here we consider A to be of the form

$$A = \begin{pmatrix} 0 & j \\ 0 & p \end{pmatrix}, \quad (j,p) \neq (0,0), \quad j \in \frac{1}{2}\mathbb{Z}, \quad p \in \mathbb{Z}\,. \qquad \text{(B.15)}$$

In this case the integration region $\mathcal{F}_A$ corresponds to the strip

$$-1/2 \leq \tau_1 \leq 1/2, \qquad \tau_2 \geq 0\,. \qquad \text{(B.16)}$$

Also we have

$$(r,s) = (0,0) \quad \text{for } j \in \mathbb{Z}, \qquad (r,s) = (0,1) \quad \text{for } j \in \mathbb{Z} + \tfrac{1}{2}\,. \qquad \text{(B.17)}$$

For A given in (B.15)

$$\mathcal{A} = j + pU, \qquad \tilde{\mathcal{A}} = j + p\bar{U}\,, \qquad \det A = 0\,, \qquad \text{(B.18)}$$

are independent of τ. Thus the exponential factor in (4.13) is independent of τ_1 and the only dependence on τ_1 of the integrand comes from the $\exp(2\pi i\tau n)$ term in the expansion of $F^{(r,s)}(\tau,z)$. The τ_1 integration now forces n to vanish and the coefficients $c^{(r,s)}(4n-b^2)$ multiplying the integrand reduces to $c^{(r,s)}(-b^2)$. It follows from the definition of $c^{(r,s)}(m)$ that these coefficients are non-zero only for $b=0$ and $b=\pm 1$.

We first consider the case $j \in \mathbb{Z}$. We begin with the contribution from the $n=0, b=0$ term and proceed as in [28]. We multiply the integrand with the regulating factor $(1-\exp(-\Lambda/\tau_2))$, then integrate over τ_2 and finally perform the sum over j and p. Integrating over τ_2 we obtain

$$\mathcal{I}_{3,b=0;j\in\mathbb{Z}} = c^{(0,0)}(0) \Bigg[\frac{U_2}{\pi} \sum_{\substack{(j,p)\neq(0,0)\\ j,p\in\mathbb{Z}}} \left(\frac{1}{|j+Up|^2} - \frac{1}{|j+Up|^2 + \Lambda U_2^2/\pi Y} \right)$$

$$- \int_{\mathcal{F}} d^2\tau \frac{1-\exp(-\frac{\Lambda}{\tau_2})}{\tau_2} \Bigg]\,. \qquad \text{(B.19)}$$

Note that we have introduced a subtraction term proportional to $\int_{\mathcal{F}} d^2\tau \frac{1-\exp(-\Lambda/\tau_2)}{\tau_2}$ in eq.(B.19), – this is one of the two terms appearing in (B.2). This is necessary in order to get a finite value of the integral in the $\Lambda \to \infty$ limit. The result of the integration in the second terms inside the square brackets is $\ln\Lambda+\gamma_E+1+\ln(2/3\sqrt{3})$. To evaluate the summation we use[29]

$$\begin{aligned}
\sum_{j\in\mathbb{Z}} \frac{\exp(i\theta j)}{(j+B)^2+C^2} &= \frac{\pi}{C}\exp(-i\theta(B-iC))\frac{1}{1-\exp(-2\pi i(B-iC))} \\
&\quad + \frac{\pi}{C}\exp(-i\theta(B+iC))\frac{\exp(2\pi i(B+iC))}{1-\exp(2\pi i(B+iC))} \\
&\qquad \text{for} \quad C>0, \quad 0\le\theta\le 2\pi \\
\sum_{\substack{j\in\mathbb{Z}\\ j>0}} \frac{\cos\theta j}{j^2} &= \frac{\theta(\theta-2\pi)}{4}+\frac{\pi^2}{6}\,.
\end{aligned} \tag{B.20}$$

We now regroup the summation in (B.19) as $\sum_{p=0,j\neq 0} + \sum_{j=-\infty,p\neq 0}^{j=+\infty}$ and use (B.20) at $\theta=0$ to obtain

$$\begin{aligned}
\mathcal{I}_{3,b=0;j\in\mathbb{Z}} &= c^{(0,0)}(0)\left[\frac{\pi}{3}U_2 + \sum_{\substack{p>0\\ p\in\mathbb{Z}}}\left\{\frac{2}{p}\frac{e^{-2\pi ip\bar{U}}}{1-e^{-2\pi ip\bar{U}}} + \frac{2}{p}\frac{e^{2\pi ipU}}{1-e^{2\pi ipU}}\right.\right. \\
&\left.\left.\quad + \left(\frac{2}{p}-\frac{2}{\sqrt{p^2+\Lambda/\pi Y}}\right)\right\} - \left(\ln\Lambda+\gamma_E+1+\ln(2/3\sqrt{3})\right)\right]\,.
\end{aligned} \tag{B.21}$$

Next we expand

$$\frac{x}{1-x} = \sum_{l=1}^{\infty} x^l\,, \tag{B.22}$$

for $x=e^{-2\pi ipU}$ and $x=e^{2\pi ip\bar{U}}$ in (B.21) and perform the sum over p in the first two terms. Finally we use

$$\sum_{\substack{p>0\\ p\in\mathbb{Z}}}\left(\frac{2}{p}-\frac{2}{\sqrt{p^2+\Lambda/\pi Y}}\right) = -\ln\frac{\pi Y}{\Lambda}+2\gamma_E-\ln 4 \quad \text{as } \Lambda\to\infty\,, \tag{B.23}$$

to obtain

$$\mathcal{I}_{3,b=0;j\in\mathbb{Z}} = c^{(0,0)}(0)\,\left(\frac{\pi}{3}U_2-\ln Y+\kappa'\right) - \ln\prod_{l\in\mathbb{Z},l>0}\left\{|1-\exp(2\pi ilU)|^{4c^{(0,0)}(0)}\right\} \tag{B.24}$$

where

$$\kappa' = \gamma_E - 1 - \ln(8\pi/3\sqrt{3})\,. \tag{B.25}$$

We now evaluate the contribution of $n = 0$, $b = \pm 1$. The corresponding coefficient is $c^{(0,0)}(-1)$. Integrating over τ_2 we obtain

$$\mathcal{I}_{3,b=\pm1;j\in\mathbb{Z}} = c^{(0,0)}(-1)\frac{U_2}{\pi}\sum_{\substack{(j,p)\neq(0,0)\\ j,p\in\mathbb{Z}}}\frac{1}{|j+pU|^2}\exp\left(\frac{2\pi ib}{U_2}(jV_2 + p(V_2U_1 - V_1U_2))\right) \tag{B.26}$$

We split this summation as before $\sum_{p=0,j\neq0} + \sum_{p\neq0,j}$. We shall assume, for definiteness, that

$$V_2 < 0\,. \tag{B.27}$$

For the $p = 0$ one can apply the second formula in (B.20) to obtain

$$4\pi\, c^{(0,0)}(-1)\left(\frac{V_2^2}{U_2} + V_2 + \frac{U_2}{6}\right) \tag{B.28}$$

Let us now turn to the contribution from the $p \neq 0$ terms. Since (B.26) contains the contribution for both $b = 1$ and $b = -1$, care should be taken so that the θ in $(B.20)$ is between $0 \leq \theta \leq 2\pi$. Here $\theta = -2\pi V_2/U_2 \leq 1$. For the $p \neq 0$ case one splits the summation for $p > 0, b = \pm1$ and $p < 0, b = \pm1$, then one changes $j \to -j$ or $p \to -p$ so that one can always apply the formula in (B.20). Carefully taking all these contributions into account one obtains, after using (B.20), the total contribution from the $p \neq 0$ terms to be

$$-\ln\prod_{l\in\mathbb{Z},l>0,b=\pm1}|1-\exp(2\pi i(lU+bV)|^{4c^{(0,0)}(-1)} - \ln|1-\exp(-2\pi iV)|^{4c^{(0,0)}(-1)} \tag{B.29}$$

Thus the net contribution from the $b = \pm1$, $j \in \mathbb{Z}$ terms are

$$\begin{aligned}\mathcal{I}_{3,b=\pm1;j\in\mathbb{Z}} = 4\pi\, c^{(0,0)}(-1)&\left(\frac{V_2^2}{U_2} + V_2 + \frac{U_2}{6}\right)\\ &- \ln\prod_{\substack{l\in\mathbb{Z},l>0\\ b=\pm1}}\left\{|1-\exp(2\pi i(lU+bV))|^{4c^{(0,0)}(-1)}\right\}\\ &- \ln|1-\exp(-2\pi iV)|^{4c^{(0,0)}(-1)}\end{aligned} \tag{B.30}$$

Note that the last term in the above equation is singular as $V \to 0$.

Next we turn to the contribution from the $j \in \mathbb{Z} + \frac{1}{2}$ terms. In this case $(r, s) = (0, 1)$. The analog of (B.20) is obtained by replacing $B \to B + \frac{1}{2}$ in this formula and multiplying the resulting equation by a factor of $e^{i\theta/2}$ on both sides:

$$\begin{aligned}\sum_{j\in\mathbb{Z}+\frac{1}{2}} \frac{\exp(i\theta j)}{(j+B)^2+C^2} &= \frac{\pi}{C}\exp(-i\theta(B-iC))\frac{1}{1+\exp(-2\pi i(B-iC))} \\ &\quad - \frac{\pi}{C}\exp(-i\theta(B+iC))\frac{\exp(2\pi i(B+iC))}{1+\exp(2\pi i(B+iC))} \\ &\qquad \text{for} \quad C>0, \quad 0\le\theta\le 2\pi \end{aligned} \tag{B.31}$$

Using this result we can get the analogs of (B.24) and (B.30):

$$\mathcal{I}_{3,b=0;j\in\mathbb{Z}+\frac{1}{2}} = c^{(0,1)}(0)\ (\pi U_2 - \ln Y + \kappa') - \ln \prod_{l\in\mathbb{Z},l>0} \left\{|1-\exp(2\pi i l U)|^{4(-1)^l c^{(0,1)}(0)}\right\} \tag{B.32}$$

$$\begin{aligned}\mathcal{I}_{3,b=\pm1;j\in\mathbb{Z}+\frac{1}{2}} &= 4\pi\, c^{(0,1)}(-1)\ \left(V_2 + \frac{U_2}{2}\right) \\ &\quad - \ln \prod_{\substack{l\in\mathbb{Z},l>0\\ b=\pm1}} \left\{|1-\exp(2\pi i(lU+bV)|^{4(-1)^l c^{(0,1)}(-1)}\right\} \\ &\quad - \ln|1-\exp(-2\pi i V)|^{4c^{(0,1)}(-1)} \end{aligned} \tag{B.33}$$

Adding all the contributions we obtain.

$$\begin{aligned}\mathcal{I}_3 &= \mathcal{I}_{3,b=0;j\in\mathbb{Z}} + \mathcal{I}_{3,b=\pm1;j\in\mathbb{Z}} + \mathcal{I}_{3,b=0;j\in\mathbb{Z}+\frac{1}{2}} + \mathcal{I}_{3,b=\pm1;j\in\mathbb{Z}+\frac{1}{2}} \\ &= c^{(0,0)}(0)\ \left(\frac{\pi}{3}U_2 - \ln Y + \kappa'\right) + 4\pi\, c^{(0,0)}(-1)\ \left(\frac{V_2^2}{U_2} + V_2 + \frac{U_2}{6}\right) \\ &\quad + c^{(0,1)}(0)\ (\pi U_2 - \ln Y + \kappa') + 4\pi\, c^{(0,1)}(-1)\ \left(V_2 + \frac{U_2}{2}\right) \\ &\quad - \ln \prod_{l\in\mathbb{Z},l>0} \left\{|1-\exp(2\pi i l U)|^{4c^{(0,0)}(0)}\right\} - \ln\left\{|1-\exp(-2\pi i V)|^{4c^{(0,0)}(-1)}\right\} \\ &\quad - \ln \prod_{\substack{l\in\mathbb{Z},l>0\\ b=\pm1}} \left\{|1-\exp(2\pi i(lU+bV)|^{4c^{(0,0)}(-1)}\right\} \\ &\quad - \ln \prod_{l\in\mathbb{Z},l>0} \left\{|1-\exp(2\pi i l U)|^{4(-1)^l c^{(0,1)}(0)}\right\} - \ln|1-\exp(-2\pi i V)|^{4c^{(0,1)}(-1)} \\ &\quad - \ln \prod_{\substack{l\in\mathbb{Z},l>0\\ b=\pm1}} \left\{|1-\exp(2\pi i(lU+bV)|^{4(-1)^l c^{(0,1)}(-1)}\right\} \end{aligned} \tag{B.34}$$

Combining the contribution from all the orbits and noting that

$$c^{(0,0)}(0) = 10, \quad c^{(0,0)}(-1) = 1, \quad c^{(0,1)}(0) = 2, \quad c^{(0,1)}(-1) = 1,$$

$$c^{(1,0)}(0) = 4, \quad c^{(1,0)}(-1) = 0, \quad c^{(1,1)}(0) = 4, \quad c^{(1,1)}(-1) = 0\,, \tag{B.35}$$

we can now express the full integral as

$$\begin{aligned}
\mathcal{I} &= \mathcal{I}_1 + 2\mathcal{I}_2 + \mathcal{I}_3, \\
&= -2\ln\Bigg[\kappa\,(\det \mathrm{Im}\Omega)^6 \Bigg| \exp(2\pi i(\frac{1}{2}T + U + V)) \\
&\quad \prod_{\substack{(k,l.b)\in \mathbb{Z}\\(k,l,b)>0}} (1-\exp(2\pi i(kT+lU+bV))^{c^{(0,0)}(4kl-b^2)+(-1)^l c^{(0,1)}(4kl-b^2)} \\
&\quad \prod_{\substack{l,b\in \mathbb{Z}, k\in \mathbb{Z}+\frac{1}{2}\\ l\geq 0, k>0}} \left\{ |1-\exp(2\pi i(kT+lU+bV))|^{2c^{(1,0)}(4lk-b^2)} \right\}\Bigg|^2\Bigg]
\end{aligned} \tag{B.36}$$

where

$$\kappa = \left(\frac{8\pi}{3\sqrt{3}}e^{1-\gamma_E}\right)^6 \tag{B.37}$$

and $(k,l,b) > 0$ means $k > 0, l \geq 0, b \in \mathbb{Z}$ or $k = 0, l > 0, b \in \mathbb{Z}$ or $k = 0, l = 0, b < 0$. Note that we have $2\mathcal{I}_2$ because of the two copies of the upper half plane.

From the modular transformation laws (3.11) and the series expansion (3.22) it follows that

$$c^{(1,1)}(4lk-b^2) = (-1)^l\, c^{(1,0)}(4lk-b^2) \qquad \text{for } k \in \mathbb{Z}+\tfrac{1}{2},\, l \in \mathbb{Z}\,. \tag{B.38}$$

Using this we can reexpress (B.36) in a more symmetric fashion:

$$\begin{aligned}
\mathcal{I} &= -2\ln\Bigg[\kappa\,(\det \mathrm{Im}\Omega)^6 \Bigg| \exp\left(2\pi i\left(\frac{1}{2}T + U + V\right)\right) \\
&\quad \prod_{r,s=0}^{1} \prod_{\substack{(l,b)\in \mathbb{Z}, k\in \mathbb{Z}+\frac{r}{2}\\(k,l,b)>0}} \left\{(1-\exp(2\pi i(kT+lU+bV))^{(-1)^{ls}\, c^{(r,s)}(4kl-b^2)}\right\}\Bigg|^2\Bigg]\,.
\end{aligned} \tag{B.39}$$

C. Explicit Results for $a(n,m,r)$

In this appendix we present the results of explicit computation of the coefficients $a(n,m,r)$ for Φ_k. These were calculated using the expression given in [7] as well as the expression found in the present paper and found to be the same. To write the expansion of Φ_k in a convenient way we define $t = \exp(2\pi i T)$, $u = \exp(2\pi i U)$, $v = \exp(2\pi i V)$. Then for $N = 2$

$$\Phi_6 = \left[(2-\frac{1}{v}-v)u + (-4+\frac{2}{v^2}+2v^2)u^2 + (-16-\frac{1}{v^3}-\frac{4}{v^2}+\frac{13}{v}+13v-4v^2-v^3)u^3\right] t$$

$$
\begin{aligned}
&+ \left[(-4+\frac{2}{v^2}+2v^2)u+(32-\frac{16}{v^2}-16v^2)u^2+(-72-\frac{4}{v^4}+\frac{40}{v^2}+40v^2-4v^4)u^3\right]t^2 \\
&+ \Big[(-16-\frac{1}{v^3}-\frac{4}{v^2}+\frac{13}{v}+13v-4v^2-v^3)u \\
&+ (-72-\frac{4}{v^4}+\frac{40}{v^2}+40v^2-4v^4)u^2 \\
&+ \left(336+\frac{13}{v^5}+\frac{40}{v^4}-\frac{87}{v^3}-\frac{64}{v^2}-\frac{70}{v}-70v-64v^2-87v^3+40v^4+13v^5\right)u^3\Big]t^3 \\
&+ \cdots
\end{aligned}
\tag{C.1}
$$

For $N=3$

$$
\begin{aligned}
\Phi_4 =& \left((2-\frac{1}{v}-v)u+(\frac{2}{v^2}-\frac{2}{v}-2v+2v^2)u^2\right)t \\
&+\left((\frac{2}{v^2}-\frac{2}{v}-2v+2v^2)u+(4-\frac{2}{v^3}-\frac{6}{v^2}+\frac{6}{v}+6v-6v^2-2v^3)u^2\right)t^2+\cdots
\end{aligned}
\tag{C.2}
$$

For $N=5$

$$
\begin{aligned}
\Phi_2 =& \left((2-\frac{1}{v}-v)u+(4+\frac{2}{v^2}-\frac{4}{v}-4v+2v^2)u^2\right)t \\
&+\left((4+\frac{2}{v^2}-\frac{4}{v}-4v+2v^2)u+(28-\frac{4}{v^3}+\frac{10}{v^2}-\frac{20}{v}-20v+10v^2-4v^3)u^2\right)t^2+\cdots
\end{aligned}
\tag{C.3}
$$

For $N=7$

$$
\begin{aligned}
\Phi_1 =& \left((2-\frac{1}{v}-v)u+(6+\frac{2}{v^2}-\frac{5}{v}-5v+2v^2)u^2\right)t \\
&+\left((6+\frac{2}{v^2}-\frac{5}{v}-5v+2v^2)u+(52-\frac{5}{v^3}+\frac{19}{v^2}-\frac{40}{v}-40v+19v^2-5v^3)u^2\right)t^2+\cdots
\end{aligned}
\tag{C.4}
$$

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