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On the non-vanishing of the first Betti number of hyperbolic three manifolds

C. S. Rajan

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Abstract. We show the non-vanishing of cohomology groups of sufficiently small congruence lattices in $SL(1, D)$, where D is a quaternion division algebra defined over a number field E contained inside a solvable extension of a totally real number field. As a corollary, we obtain new examples of compact, arithmetic, hyperbolic three manifolds, with non-torsion first homology group, confirming a conjecture of Waldhausen. The proof uses the characterisation of the image of solvable base change by the author, and the construction of cusp forms with non-zero cusp cohomology by Labesse and Schwermer.

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1. Introduction

In this paper, we are concerned with the following question in the context of arithmetic, hyperbolic three manifolds: suppose M is a manifold. Does there exist a finite cover M' of M with non-vanishing first Betti number? The class of arithmetic spaces that we consider arise as follows: let D be a quaternion division algebra over a number field E. Let G denote the connected, semisimple algebraic group $SL_1(D)$ over E. Denote by $G_\infty(E)$ the real Lie group $G(E \otimes \mathbb{R})$ and fix a maximal compact subgroup K_∞ of G_∞. Let s_1 (resp. $2r_2$) be the number of real (resp. complex) places of E at which D splits, and let $s = s_1 + r_2$. The quotient space $M := G_\infty/K_\infty$ with the natural $G_\infty(E)$-invariant metric, is a symmetric space isomorphic to $\mathcal{H}_2^{s_1} \times \mathcal{H}_3^{r_2}$, where for a natural number n, $\mathcal{H}_n$ denotes the simply connected hyperbolic space of dimension n.

Let $\mathbb{A}$ (resp. $\mathbb{A}_f$) denote the ring of adeles (resp. finite adeles) of $\mathbb{Q}$. Let K be a compact, open subgroup of $G(\mathbb{A}_f \otimes E)$, and denote by Γ_K the corresponding congruence arithmetic lattice in $G_\infty(E)$ defined by the projection to $G_\infty(E)$ of the group $G(E) \cap G_\infty(E)K$. For sufficiently small congruence subgroups K, Γ_K is a torsion-free lattice and $\Gamma_K\backslash M$ is a (compact) Riemannian manifold.

In this note, we prove

C. S. Rajan
Tata Institute of Fundamental Research, Homi Bhabha Road, Bombay 400 005, India
(e-mail: rajan@math.tifr.res.in)

Theorem 1. *With the above notation, assume further that E is a finite extension of a totally real number field F contained inside a solvable extension L of F. For sufficiently small congruence subgroups $\Gamma \subset G_\infty(E)$, the cohomology groups*

$$H^s(\Gamma\backslash M, \mathbb{C})$$

are non-zero.

A particular case of interest is the following:

Corollary 1. *With notation as in Theorem 1, assume further that E has exactly one pair of conjugate complex places, and the quaternion division algebra D is ramified at all the real places of E. For sufficiently small congruence subgroups $\Gamma \subset G_\infty(E)$, the first betti number of the compact, hyperbolic three manifold $\Gamma\backslash M$ is non-zero.*

A folklore conjecture (attributed to Waldhausen) is that the first betti number of a compact, hyperbolic three manifold becomes positive upon going to some finite cover. The first examples of compact, hyperbolic arithmetic three manifolds M_Γ with non-vanishing rational first homology group are due to Millson [M]. Using geometric arguments, Millson showed the non-vanishing of the first betti number for sufficiently small congruence subgroups, where the arithmetic structure arises from rank 4 quadratic forms over a totally real number field F, and of signature (3, 1) at one archimedean place and anisotropic at all other real places.

Theorem 1 was proved by Labesse and Schwermer [LS, Corollary 6.3], in the case when there exists a tower of field extensions

$$E = F_l \supset F_{l-1} \supset \cdots \supset F_0 = F,$$

such that F_{i+1}/F_i is either a cyclic extension of prime degree or a non-normal cubic extension. The theorem of Labesse and Schwermer generalizes the theorem of Millson, as Millson's theorem is the special case when E/F is quadratic and there exists a quaternion division algebra D_0 over F satisfying $D \simeq D_0 \otimes_F E$ [LM]. The proof of our theorem uses the theorem of Labesse and Schwermer and a criterion for the descent of an invariant cuspidal representation with respect to a solvable group of Galois automorphisms proved by the author in [R].

Using a construction of algebraic Hecke characters due to Weil, and the automorphic induction of suitable such characters, Clozel proved novanishing results for the cohomology groups as in the conclusion of Theorem 1 under the following assumption: if v is a finite place of E where D is ramified, then the completion E_v of E at v should not contain any quadratic extension of $\mathbb{Q}_p$, where v divides the rational prime v. In particular, this is the case if either D is unramified at all finite places of E, or if the Galois closure of E over $\mathbb{Q}$ is of odd degree over $\mathbb{Q}$.

A different proof of Corollary 1 (and generalizations to higher dimensional arithmetical hyperbolic manifolds), using theta functions and a Siegel-Weil type formula in the case when E/F is quadratic extension was obtained by Li and Millson [LM].

Example. We give an example of a lattice satisfying the hypothesis of the corollary and not covered by the results of Labesse-Schwermer. To achieve this, we need to produce a quartic, primitive extension E (i.e., not containing any quadratic extension) of $\mathbb{Q}$ with exactly one pair of conjugate complex places. By class field theory, for any even number S of places of E containing the real places and not containing the complex place, there exists a unique quaternion division algebra D which is ramified precisely at the places belonging to S. For such D, we obtain new examples of compact, hyperbolic three manifolds with non-vanishing first betti number as in the above corollary.

Let $P(x)$ be an irreducible quartic polynomial over the rationals, and let E be the quartic field defined by $P(x)$. From the definition of the discriminant $D(P)$ of $P(x)$ in terms of the roots, it follows that E has exactly one pair of conjugate complex embeddings if and only if $D(P) < 0$. The field E is primitive precisely when the Galois group G of the splitting field defined by $P(x)$ over the rationals is either A_4 or S_4.

For a positive prime a, let $P_a(x) = x^4 + ax - a$. The discriminant of $P_a(x)$ is $-27a^4 - 256a^3$, and it is irreducible by Eisenstein's criterion. The resolvent polynomial is $x^3 + 4ax + a^2$, and is irreducible. Hence G contains S_3, and it follows that $G \simeq S_4$. The quartic fields defined by $P_a(x)$ have the required properties.

2. General coefficients

Theorem 1 can be generalized for suitable non-trivial coefficient systems also. Let F and E be as in the hypothesis of the theorem. Given a finite dimensional complex representation V of $SL_2(\mathbb{R} \otimes F)$, we now define the base change representation $\Psi(V)$ of the group $G_\infty(E)$ [LS]. We define it first when V is irreducible and extend it additively. If V is irreducible, then V can be written as,

$$V \simeq \otimes_{v \in P_\infty(F)} V_v,$$

where $P_\infty(F)$ is the collection of the archimedean places of F, and the component V_v of V at the place v is an irreducible representation of $SL_2(F_v) \simeq SL_2(\mathbb{R})$, say of dimension $k(v)$.

Let V_k (resp. $\bar{V}_k$) denote the irreducible, holomorphic (resp. anti-holomorphic) representation of $SL_2(\mathbb{C})$ of dimension k. Restricted to $SU(2)$ they give raise to isomorphic representations, which we continue to denote by V_k. Define the representation W_k of $SL_2(\mathbb{C})$ by $W_k = V_k \otimes \bar{V}_k$.

Suppose D is a quaternion algebra over E. We define the base change coefficients $\Psi(V)$ of $G_\infty(E)$, as a tensor product of the representations $\Psi(V)_w$ of the component groups $G(E_w)$, as w runs over the collection of archimedean places of E. Suppose w lies over a place v of F. Define,

$$\Psi(V)_w \simeq \begin{cases} V_{k(v)} & \text{if } w \text{ is real,} \\ W_{k(v)} & \text{if } w \text{ is complex.} \end{cases}$$

Restricting the representation $\Psi(V)$ to a torsion-free lattice Γ gives rise to a well defined local system $\mathcal{L}_{\Psi(V)}$ on the manifold $\Gamma\backslash M$. The extension of Theorem 1 to non-trivial coefficients is the following:

Theorem 2. *Let F be a totally real number field, and L be a solvable finite extension of F. Let E be a finite extension of F contained in L, and D be a quaternion division algebra over E. Let V be a finite dimensional complex representation of $SL_2(\mathbb{R} \otimes F)$. Then for sufficiently small congruence subgroups $\Gamma \subset G_\infty(E)$,*

$$H^s(\Gamma\backslash M, \mathcal{L}_{\Psi(V)}) \neq 0.$$

3. Proof

Let Γ be a co-compact torsion-free lattice in a connected, real semisimple Lie group H, and let M be a maximal compact subgroup of H. The space $L^2(\Gamma\backslash H)$ consisting of square integrable functions on $\Gamma\backslash H$ decomposes as a direct sum of irreducible admissible representations η of H with finite multiplicity $m(\eta)$:

$$L^2(\Gamma\backslash H) \simeq \oplus_\eta m(\eta)\eta.$$

Let U be a finite dimensional representation of H. By the Matsushima formula [BW],

$$H^*(\Gamma, U) \simeq \oplus_\eta m(\eta) H^*(\mathfrak{h}, M, \eta \otimes U), \tag{1}$$

where $\mathfrak{h}$ denotes the Lie algebra of H, and the cohomology groups on the right are the relative Lie algebra cohomology groups defined as in [BW].

We restrict now to the case when $H = G_\infty(E)$, and take for U the representation $\Psi(V)$ as defined above. Let ρ denote the representation of $G(\mathbb{A} \otimes E)$ acting by right translations on the space $L^2(G(E)\backslash G(\mathbb{A}\otimes E))$ consisting of square integrable functions on $G(E)\backslash G(\mathbb{A} \otimes E)$. This decomposes as a direct sum of irreducible admissible representations π of $G(\mathbb{A} \otimes E)$ with finite multiplicity $m(\pi)$:

$$\rho = \oplus_\pi m(\pi)\pi.$$

With respect to the decomposition $G(\mathbb{A} \otimes E) = G_\infty(E)G(\mathbb{A}_f \otimes E)$, write $\pi = \pi_\infty \otimes \pi_f$, where π_∞ (resp. π_f) is a representation of $G_\infty(E)$ (resp. $G(\mathbb{A}_f \otimes E)$). Let Γ_K be a lattice as defined above corresponding to a compact open subgroup $K \subset G(\mathbb{A}_f \otimes E)$. Since G is simply connected, we obtain from equation (1) and strong approximation, the adelic version of Matsushima's formula:

$$H^*(\Gamma_K, \Psi(V)) \simeq \oplus_\pi m(\pi) H^*(\mathfrak{g}, K_\infty, \pi_\infty \otimes \Psi(V)) \otimes \pi_f^K, \tag{2}$$

where π_f^K denotes the space of K invariants of the representation space of π_f. Taking a direct limit indexed by the compact open subgroups $K \subset G(\mathbb{A}_f \otimes E)$, we define and obtain,

$$\begin{aligned} H^*(G, E; \Psi(V)) := \varinjlim_K H^*(\Gamma_K, \Psi(V)) &\simeq \varinjlim_K H^*(\Gamma_K \backslash M, \mathcal{L}_{\Psi(V)}) \\ &\simeq \oplus_\pi m(\pi) H^*(\mathfrak{g}, K_\infty, \pi_\infty \otimes \Psi(V)) \otimes \pi_f. \end{aligned} \tag{3}$$

Hence in order to prove Theorem 2, it is enough to construct an irreducible representation π of $G(\mathbb{A}_E)$ with $m(\pi)$ positive and such that $H^s(\mathfrak{g}, K_\infty, \pi_\infty \otimes \Psi(V))$ is non-zero.

We can assume that V is irreducible of the form $V \simeq \otimes_{v \in P_\infty(F)} V_{k(v)}$. Let D_k^+ (resp. D_k^-) be the holomorphic (resp. antiholomorphic) discrete series of $SL_2(\mathbb{R})$ of weight $k+1$. We have,

$$H^q(\mathfrak{sl}_2(\mathbb{R}), SO(2), D_k^\pm \otimes V_k) = \begin{cases} \mathbb{C} & \text{if } q = 1, \\ 0 & \text{otherwise,} \end{cases} \tag{4}$$

where $\mathfrak{sl}_2(\mathbb{R})$ and $\mathfrak{sl}_2(\mathbb{C})$ denotes respectively the Lie algebras of $SL_2(\mathbb{R})$ and $SL_2(\mathbb{C})$.

Let S be a finite set of finite places of F, containing all the finite places v of F dividing a finite place of E at which D ramifies. By [LS, Proposition 2.5], there exists an irreducible, admissible representation of $SL_2(\mathbb{A} \otimes F)$ satisfying the following properties:

- The multiplicity $m_0(\pi)$ of π occuring in the cuspidal spectrum $L_0^2(SL_2(F)\backslash SL_2(\mathbb{A} \otimes F))$ consisting of square integrable cuspidal functions on $SL_2(F)\backslash SL_2(\mathbb{A} \otimes F)$ is nonzero. Further π is stable in the sense of [LL].
- The local component π_v of π at an archimedean place v of F is a discrete series representation, with $\pi_v \in \{D_{k(v)}^+, D_{k(v)}^-\}$.
- For any $v \in S$, the local component π_v of π is isomorphic to the Steinberg representation of $SL_2(F_v)$.

Let Π be a cuspidal, automorphic representation of $GL_2(\mathbb{A} \otimes F)$, such that π occurs in the restriction of Π to $SL_2(\mathbb{A} \otimes F)$. Let Π_L be the base change of Π to $GL_2(\mathbb{A} \otimes L)$ defined by Langlands in [L]. Since π is stable, i.e., Π is not automorphically induced from a character of a quadratic extension of F, the base change Π_L is a *cuspidal* automorphic representation of $GL_2(\mathbb{A} \otimes L)$. We now quote the following descent theorem for invariant cuspidal representations [R]:

Theorem 3. *Let K/k be a solvable extension of number fields, and let Θ be a unitary, cuspidal automorphic representation of $GL_2(\mathbf{A}_K)$ which is* Gal(K/k)*-invariant. Then there exists a $G(K/k)$-invariant Hecke character ψ of K, and a cuspidal automorphic representation θ of $GL_2(\mathbf{A}_K)$ such that*

$$\theta_K \simeq \Theta \otimes \psi,$$

where θ_K is the base change lift of θ to $GL_2(\mathbb{A} \otimes K)$ defined by Langlands in [L].

Let H be the Galois group of L over E. Since Π_L is H-invariant and cuspidal, by the above descent theorem, there exists an idele class character χ of L, such that the representation $\Pi_L \otimes \chi$ is the base change from E to L of a cuspidal representation Π_E of $GL_2(\mathbb{A} \otimes E)$. Let π_E be a constituent of the restriction of Π_E to $SL_2(\mathbb{A} \otimes E)$, and occuring in the automorphic spectrum of $G(\mathbb{A}_E)$ with non-zero multiplicity $m(\pi_E)$.

Base change makes sense at the level of L-packets (see [LS]), and let $\pi_{k,\mathbb{C}}$ denote the representation of $SL_2(\mathbb{C})$ obtained as base change of the L-packet $\{D_k^+, D_k^-\}$ (L-packets for complex groups consist of only one element). It is known that (see [LS]),

$$H^1(\mathfrak{sl}_2(\mathbb{C}), SU(2), \pi_{k,\mathbb{C}} \otimes W_k) \neq 0. \tag{5}$$

Let w be an archimedean place of E lying over a real place v of F. Now twisting by a character does not alter the restriction of an automorphic representation of GL_2 to SL_2. Hence if w is a real place of E, then the local component $\pi_{E,w}$ of π_E at w belongs to $\{D^+_{k(v)}, D^-_{k(v)}\}$, and if w is a complex place of E, then $\pi_{E,w}$ is isomorphic to $\pi_{k(v),\mathbb{C}}$.

The local components of the base change to E of π continues to be the Steinberg representation of $SL_2(E_w)$, at the places of E where D ramifies. By the theorem of Jacquet-Langlands ([JL], [LS]) applied to L-packets of SL_2 and it's inner forms, we get an automorphic representation $JL(\pi_E)$ of G over E. At a place w where D is ramified, the local component $JL(\pi_E)_w$ is isomorphic to the restriction of the representation $V_{k(v)}$ to $SU(2)$, where v is a place of F dividing w. In particular, the 0-th relative Lie cohomology group

$$H^0(\mathfrak{su}_2, SU_2, V_k \otimes V_k) = (V_k \otimes V_k)^{SU(2)} \neq 0. \tag{6}$$

At a place w of E where D splits, $JL(\pi_E)_w \simeq \pi_{E,w}$, and hence the first relative Lie algebra cohomology with coefficients in the component of $\Psi(V)$ at w is non-zero. It follows from equations (4), (5), (6) and by the Kunneth formula for the relative Lie algebra cohomology that

$$H^s(\mathfrak{g}, K_\infty, JL(\pi_E)_\infty \otimes \Psi(V)) \neq 0.$$

By Equation (3), this proves Theorem 2.

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