TEAM DECISION THEORY OF SWITCHED STATIC AND DYNAMIC SYSTEMS

BY

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DISSERTATION

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Abstract

This dissertation considers the decentralized control of switched linear systems with parameter dependent cost and system matrices. This problem class is investigated under a number of different formulations of player information structure, performance criteria and switching architecture. Such decentralized switched systems can be encountered in various applications like network control, control in a changing environment, economic theory, power systems, decision making in organizations, resource allocation. The thesis is roughly divided into three parts.

The first part of the thesis focuses on the static quadratic team problem, where players observe partial observations of an underlying random state and generate actions with the objective of minimizing the expected value of a common quadratic cost function in the player actions. One of the motivations behind studying this problem is to solve a static stochastic-parameter problem useful in solving dynamic switched control problems encountered later. The problem however is studied in full generality and an operator theoretic framework is presented to analyze the same. We prove that a scheme where strategies are updated by sequentially applying the best responses of players, converges to the team optimal strategy. Such an update scheme provides a mechanism to numerically compute arbitrarily close approximations of the team optimal strategy. It also acts as a tool for validating structure of the team optimal strategy which can be beneficial in some cases for analytical computation of these strategies.

The second part of the thesis considers dynamic switched optimal control problems with quadratic cost and players having local parameter knowledge. One of these problems is studied under full state feedback and i.i.d. parameter; the remaining two problems are output feedback, distinguished by the type of information structure: partially nested and one-step delayed sharing. For the former output feedback problem, parameters and measurements follow a partially nested structure with the parameters possibly being correlated across all stages. For the latter case, parameters are assumed to be Markov processes, with their values along with measurements available instantaneously to local controllers, but with a one time step delay to others. The solution to all these problems rely on the optimal solution to a static (one-stage) stochastic-parameter problem with local parameter dependent Gaussian measurements, and for this purpose the static quadratic team problem, examined in first part is used. The strategies obtained in all these dynamic problems are affine in the measurements with the parameter dependent coefficients obtained by solving a set of

linear equations. These equations are immediately solvable when the total number of parameter values is finite. However, for the case of infinite parameter values, the update scheme examined in the first section also provides a mechanism to determine an approximation to the team optimal strategy.

In the final part of the thesis, we consider a setup with switched linear nested plant whose system matrices switch between a finite number of values, with transitions in time governed by a finite state automaton. A linear nested controller is sought with corresponding system matrices dependent on a finite path history of the plant's system matrices in order to stabilize the plant and achieve a desired level of ℓ_2-induced norm performance. The nested structures of both plant and controller are characterized by block lower-triangular system matrices with compatible dimensions. For this setup, exact conditions are provided for the existence of a finite path dependent synthesis. These include conditions for the completion of scaling matrices obtained through an extended matrix completion lemma. When individual controller dimensions are chosen at least as large as the plant, these conditions reduce to a set of linear matrix inequalities. The completion lemma also provides an algorithm to complete the closed loop scaling matrices leading to inequalities for controller synthesis.

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List of Abbreviations

i.i.d.	Independent and identically distributed
KYP	Kalman-Yakubovich-Popov
LEG	Linear exponential of quadratic Gaussian
LMI	Linear matrix inequality
LQG	Linear quadratic Gaussian
LQR	Linear quadratic regulator
LTI	Linear time invariant
LTV	Linear time varying
OSD	One step delayed sharing
PN	Partially nested
SDP	Semidefinite programming
SVD	Singular value decomposition

Notational Convention

$\mathbb{N}_0$	Natural numbers including 0
$\mathbb{Z}_+$	Positive integers
$\mathbb{R}$	Real numbers
$\mathbb{R}^n$	n-dimensional Euclidean space
$\mathbb{R}^{n\times m}$	Space of $n \times m$-dimensional real valued matrices
$\mathbb{S}^n$	Space of symmetric matrices of dimension n
$\mathbb{S}^n_+$	Space of symmetric positive definite matrices in $\mathbb{S}^n$
$\bar{\mathbb{S}}^n_+$	Space of symmetric positive semi-definite matrices in $\mathbb{S}^n$
ℓ^n (or ℓ)	Space of infinite sequences of vectors in $\mathbb{R}^n$
ℓ_2^n (or ℓ_2)	Hilbert space of infinite sequences of vectors in $\mathbb{R}^n$ with bounded norm
$\mathcal{L}(\mathcal{X},\mathcal{Y})$	Space of linear bounded operators mapping vector spaces $\mathcal{X}$ to $\mathcal{Y}$
$\lvert \cdot \rvert_2$	2-norm of vectors in Euclidean space
$\lVert \cdot \rVert_{\mathcal{X}}$	Norm corresponding to Banach space $\mathcal{X}$
$\lVert \cdot \rVert_{\mathcal{X}\to\mathcal{Y}}$	Induced norm of operators in $\mathcal{L}(\mathcal{X},\mathcal{Y})$
$\lVert \cdot \rVert$	Norm corresponding to ℓ_2 space, or induced ℓ_2 norm
$\mathcal{X} \oplus \mathcal{Y}$	Direct sum of linear spaces $\mathcal{X}$ and $\mathcal{Y}$
$\mathcal{X}^\perp$	Orthogonal complement of sub-space $\mathcal{X}$
W^T	Transpose of matrix W
$W^\dagger$	Pseudo-inverse of matrix W
$\mathrm{rank}(W)$	Rank of matrix W
$\mathrm{Im}(W)$	Image space of matrix W
$\mathrm{Ker}(W)$	Kernel space of matrix W
$\bar{\sigma}(W)$	Maximum singular value (equivalently induced 2-norm) of matrix W
$W_\perp$	Matrix with full column rank, satisfying $\mathrm{Im}(W_\perp) = \mathrm{Ker}(W)$ and $W_\perp^T W_\perp = I$
$W_{\vert\vert}$	Matrix with full column rank, satisfying $\mathrm{Im}(W_\perp) = \mathrm{Ker}(W)^\perp$ and $W_{\vert\vert}^T W_{\vert\vert} = I$

M	Number of players
$\mathcal{J}$	$\{1, \ldots, M\}$, index set of players
$\bar{\mathcal{J}}$	$\{0, \ldots, M\}$
$\mathbb{I}_i$	Information available to player i in static team problems
$\mathbb{I}_{it}$	Information available to player i at time t in dynamic team problems
$\mathcal{K}_i$	Strategy space for player i
$\mathcal{K}$	Decentralized strategy space, $\mathcal{K} = \mathcal{K}_1 \oplus \cdots \oplus \mathcal{K}_M$
γ_i	Strategy of player i
$\gamma_{\text{-}i}$	Strategy of all players except i
γ°	Team optimal strategy
$\mathbf{\Gamma}_i(\gamma_{\text{-}i})$	Best response of player i to other player strategies
θ_t (or $\theta(t)$)	Switching parameter at time t
Θ	Set consisting of all possible switching parameter values
$\mathcal{N}(\mu, \Sigma)$	Normal distribution with mean μ and covariance Σ
uniform(a, b)	Uniform distribution with support over $[a, b]$
Bernoulli(p)	Bernoulli distribution with probability of success p

Chapter 1

Introduction

Decentralized control has been a topic of interest in the controls community for at least half a century. However, the past decade has seen a huge escalation of efforts in advancing this field. This can be attributed to several factors like widespread adoption of large scale systems, vast improvements in communication networks, declining costs of computation power, advances in sensor technology, miniaturization. Despite several advances, decentralized control still remains a challenging field with a wealth of problems to be explored. One such class of problems is the control of switched systems which has been studied quite extensively in the context of centralized control, but has seen little attention in the decentralized setting. These directions are explored in this thesis with particular focus on achieving optimal or near-optimal costs.

In system dynamics, uncertainties are accounted for in two ways, either through a disturbance signal or through parametric uncertainties affecting the system model. Although the latter form of uncertainties is not well studied in the context of decentralized control, they do occur quite naturally in a number of applications. These include:

- Networked control systems [1–4]: It constitutes a broad class of applications where plant and controller subsystems are connected over a communication network. This introduces effects like bandwidth limitations, packet drops, sampling, discretization and delays.
- Power systems [5, 6]: Decentralization is inherent to power generation and distribution over a grid. Switching in the dynamics could be due to uncertainty in power generation (e.g., renewables) or variations in load demand.
- Building systems [7, 8]: An important application in this domain is the control of heating, ventilation and air conditioning systems to regulate indoor climate; i.e., temperature, air quality. Switching in such scenarios could represent variations in occupancy, environmental conditions, performance requirements.
- Economic models [9, 10].
- Formation flying and vehicular platoons [11–13]: Here even though individual subsystems may be dynamically decoupled, agents could have a common cost function or share measurements.

For such systems, switching could represent variations in shared environment or changes in command objectives.

- Resource allocation [14].

In this introductory chapter, we will present a brief literature review of decentralized and switched control relevant to this work and at the same time have a descriptive level understanding of some basic concepts. We will also give an overview of the class of problems we examine in this work and the organization of rest of the thesis.

1.1 Decentralized Control

1.1.1 Team Decision Theory

Much of literature in decentralized control can be traced back to the sixties when a number of studies appeared in team (decision) theory. Team theory was put forth by Marschak [15] and Radner [16] for static decision making and was originally intended for application in economics (see [17]). Team theory just like game theory involves the study of decision making process of a number of agents (also referred to as players or decision makers), collectively called a *team*, who take actions based on information available to them. However unlike game theory where players have individual costs representing possibly conflicting objectives, the agents here share a single cost function representing a common objective. The source of the decentralization lies in the dissimilar information held by the agents about the underlying state of the system. The goal of the team problem is then to synthesize individual player strategies (which map players' local information to their actions) in order to minimize the common cost function. In [16], the author considers a static team problem with a cost quadratic in the player actions while proving the existence and uniqueness of optimal solution and providing a necessary condition for optimality. For the case of non-stochastic cost matrices and Gaussian measurements, the team optimal strategy was shown to be affine in the player measurements, with coefficients solvable though a set of linear equations. Thereafter authors in [18] relaxed the conditions required for stationarity in [16], and in [19] they explored the the static linear exponential of Gaussian (LEG) problem showing that corresponding team optimal strategies are also affine.

In Chapter 3, we will look into static quadratic teams with particular focus on update schemes and their convergence. More background in this regard will be presented within Chapter 3. In general, for prior results in team theory, readers are directed to [20] which further focuses on static teams in Chapter 2.

1.1.2 Dynamic Teams and Information Structures

Team theory was subsequently expanded to a dynamic setting [21, 22], where agents take decisions repeatedly over a time horizon, based on dynamically evolving information. However this presented a significant complication in that the information of one agent at a particular time could depend on the strategy of another in the past, leading to difficult functional optimization. This complexity is best captured in the counterexample provided by Witsenhausen [23] for a simple two player, two stage problem with each agent acting at different stages. In [23] it was demonstrated that nonlinear strategies vastly outperform linear strategies and to this day, a clear solution to the problem does not exist. Thereafter studies [22, 24] have tried to characterize information patterns under which the problems still remain tractable. A detailed account on this topic can be found in [20, 25]. The two information structures of most relevance to this work are described below.

- *N-step delayed information sharing:* In this setting, each agent's information at a particular time step includes all its past information (perfect memory) and that of the other agents until N-steps prior to the current time. A special case of this is when $N = 1$ and is called one-step delayed sharing (OSD) information pattern.

- *Partially nested (PN):* Here, each agent has perfect memory of its own information. Further, if the action of one agent (say P1) at time t affects the information of another agent (say P2) at a future time $t + s$, then P2's information at time $t + s$ should contain P1's information at time t.

While the information pattern where all agents share their information instantaneously with other agents (equivalent to a centralized system) is called *classical*, the OSD and PN information sharing patterns are referred to as *quasi-classical*. All other information patterns are called non-classical, including the Witsenhausen counterexample.

1.1.3 Cost and Noise Structure

We now describe other important aspects of the decentralized system model. While several studies have focussed on discrete state space and action space models (e.g., [26, 27]), this work primarily focusses on the continuous counterpart with linear dynamics, to which we limit this discussion. We start with one of the most popular setups, the linear quadratic Gaussian (LQG) problem where the cost is the expected value of a quadratic function of the state and action variables, and where additive Gaussian noise affects both the state update and measurements. A generalization of the LQG problem is the $\mathcal{H}_2$ control problem, and their connection is discussed in Section 2.3. Another setup closely related to LQG is that of linear exponential of quadratic Gaussian (LEG) which assumes the same dynamical model as LQG, but where the quadratic cost is replaced by an exponential of the same quadratic function. The LEG cost has an associated risk parameter (chosen on the real line) which when negative, represents a risk-averse (or pessimistic) scenario and when positive represents a risk-seeking (or optimistic) scenario. In the limiting case of zero risk, the LEG objective coincides with that of LQG. For a detailed treatment of the centralized LEG problem see [28], [29]. Moving on to the $\mathcal{H}_\infty$ control problem, the corresponding cost criteria involves minimizing the $\ell_2 \to \ell_2$ induced norm from the disturbance to performance output. $\mathcal{H}_\infty$ problems can also be viewed as minimizing the performance criteria under the worst-case noise. Such a viewpoint can be best understood from a game theoretic formulation, presented in [29] which considers the control design problem as a minimax game with the controller being the minimizing player and the noise being the maximizing player. The corresponding minimax game is also closely related to the LEG problem.

The choice of cost and noise structures can play an significant role in the structure of the optimal strategies. This point is demonstrated by the references discussed next. For a two stage decentralized LQG problem, it was noted in [30] that when the cost function does not contain a product term between the decision variables, the resulting optimal strategies are linear. An example, which includes the cross terms is the Witsenhausen counterexample, where nonlinear strategies are known to outperform linear ones. This counterexample was also studied in [31], with an induced 2-norm cost instead of a quadratic cost, and it was shown that linear strategies are optimal under this setup. Recently in [32] it was noted that the choice of cost structure and noise covariance matrices in LQG problems can have a significant effect on the dimension of the optimal controller.

1.1.4 Tractable Problems in Optimal Decentralized Control

Having described some of the important information, cost and noise structures, we now list a few relevant studies which find tractable solutions under some combination of these structures. Decentralized LQG problems appeared prominently in the literature during the seventies. Explicit solutions were obtained for the OSD information pattern by several authors ([33–35]). The solution technique involved using dynamic programming, while solving a static team problem at each stage using the result in [16]. The solution structure involves separation between state estimation and control. While [24] conjectured the existence of such separation for general delayed structures, it was proved in [36] that separation holds only for OSD structures and not for general N-step delayed sharing information structures. A decentralized LEG problem with OSD information structures was solved in [37] using dynamic programming. The most notable result for PN information structures was provided by Ho and Chu [22]. For the decentralized LQG problem with such an information pattern, they proved that the finite horizon case has a linear optimal solution. However, unlike OSD problems, dynamic programming solutions are hard to obtain [38] and explicit solution for the strategies did not appear until recently. These results include the cases of partial state feedback ([39, 40]) and the two-player output feedback [41] for $\mathcal{H}_2$ control problems.

In a recent work [42], authors define an algebraic property called quadratic invariance to characterize the constraint sets (which captures the information structure) for the controller. It was shown that when this property holds, the constraint on the controller can be converted to an affine constraint on the corresponding Youla parameter. Subsequently, if the constraint set is convex, the resulting model matching problem for control design is convex. $\mathcal{H}_2$ control with sparsity constraints is one of the problems this result was used for in the same reference. In [43], it was shown that quadratic invariance and partial nestedness are equivalent concepts when they are well defined in LTI formulations.

1.2 Switched Systems

Systems with switched system matrices have been the focus of several studies within the centralized control literature in the past [2, 44–50]. We will limit our discussion here to linear, discrete time switched systems of form

$$\begin{aligned} x_{t+1} &= A(\theta_t)x_t + B(\theta_t)\, u_t \\ y_t &= C(\theta_t)x_t + D(\theta_t)\, u_t \end{aligned}$$

where the systems matrices vary in pre-defined sets. The exact nature of switching is captured by a parameter θ_t which takes values in a set Θ and is generated by a process assumed to be independent of the state x and input u. Switched systems of the above form have been studied under a variety of setups as described below.

Switching model: While the set Θ could be finite or infinite, the switching model relevant to us can be roughly classified as

- Unstructured: There is no structure in the switching and within the set Θ, parameters can switch from one value to another indiscriminately.
- Stochastic: The parameters are associated with a probability distribution over Θ. Some possibilities are the parameters being i.i.d. [51], Markov chain [45, 46] or even correlated over time.
- Language or automata based: This is a non-deterministic, non-stochastic setting where the sequences of switching parameter are restricted to a strict subset of all possible switching sequences [49, 52]. In particular, sequences could be generated by a finite state automata, in which case they are said to be generated by a regular switching language.

Controller access/memory of parameters: The controller may have a restrictive access to the parameter θ_t, which could be due to physical constraints of information availability or practical limitations regarding implementation of the controller. Different models of controller access typically studied are listed below

- Parameter independent: Controller has no access to switching parameter (e.g., [51]).
- Mode dependent: Controller has access only to parameter θ_t at time t implying that there is no parameter retention in memory. This is one of the most commonly studied setup due to its simplicity of analysis and implementation. (e.g., [53])
- Finite path dependent: Controller has access to parameters over a fixed window which may stretch to a future time i.e. there exist non-negative integers τ_1 and τ_2, such that controller at time t has access to $\theta_{t-\tau_1}, \ldots, \theta_{t+\tau_2}$ (e.g., [49, 52]).
- Complete past knowledge: Controller has perfect memory of all past parameters but does not have access to future parameters (e.g., [2, 50])
- Complete knowledge: This is same as the linear time varying (LTV) setup [54].

For the various switching models discussed above, a number of different stability and performance metrics are adopted in literature. For systems with stochastic switching models, the notions of stabilities generally considered include mean stability [51], mean-squared stability [51, 55] and almost sure stability [56], while in the non-stochastic setting uniform-exponential stability is generally sought. Performance metrics commonly used are quadratic costs [45, 46, 51] and induced ℓ_2 norm [49, 52, 53].

The above discussion mainly focuses on centralized control of switched systems; in comparison, literature dealing with decentralized control of switched systems is relatively sparse. In decentralized systems different controller agents could have different partial information about the switching parameters and this presents a rich set of possibilities and challenges in the control problem. The exact parameter availability to a particular controller agent would be captured by the information structure. Prior work include [57] which considers a robust stability problem and [58] which considers a system with parameter dependent A-matrix.

A related field where switched systems are encountered in a decentralized setting is networked control systems. Here plant and controller subsystems are connected over a communication network whose links can be thought of being switched. Problems well studied this domain include stabilization and estimation of linear system over noisy channels [2, 50, 59–62].

1.3 Overview of Problem Formulations

We now elaborate on the broad class of systems considered in this thesis. We typically consider a parameter dependent linear plant model as shown below

$$\begin{aligned} x_{t+1} &= A(\theta_t)x_t + B_u(\theta_t)u_t + B_w(\theta_t)w_t \\ z_t &= C_z(\theta_t)x_t + D_{zu}(\theta_t)u_t + D_{zw}(\theta_t)w_t \\ y_t &= C_y(\theta_t)x_t + D_{yw}(\theta_t)w_t. \end{aligned}$$

Chapter	4	5	6	7	8
Information structure	static	PN	OSD	full state	nested
Performance criteria	quadratic	quadratic	quadratic	quadratic	ℓ_2 induced norm
Parameter set, Θ	infinite	infinite	infinite	infinite	finite
Switching model	stochastic	stochastic	Markov process	i.i.d.	regular automata
Controller memory	NA	NA	perfect	perfect	finite history

NA: not applicable

Table 1.1: Summary of switched system models by chapter

Here x_t, u_t, y_t, w_t and z_t are the state, control input, measurement output, noise and performance output respectively. Further, the control input and measurement are partitioned into individual components as $u_t = \begin{bmatrix} u_{1t} \\ \vdots \\ u_{Mt} \end{bmatrix}$ and $y_t = \begin{bmatrix} y_{1t} \\ \vdots \\ y_{Mt} \end{bmatrix}$ respectively. The system matrices depend on a switched parameter θ_t generated by a process independent of the system. The overall interconnection diagram is depicted in Figure 1.1. We consider an M-agent decentralized controller, where each agent has private observations of both the switched process and measurement. For agent i at time t, these are denoted as θ_{it} and y_{it} respectively. Besides these private observations, agents could have access to others' observations based on the information structure. All observations private or shared available to player i will be called its information, and at time t this is denoted by $\mathbb{I}_{it}$. The objective then is to design strategies of individual players γ_{it} which map information $\mathbb{I}_{it}$ to control inputs u_{it}, in order to minimize the desired cost function.

We consider a number of different models throughout the thesis, these are summarized in the Table 1.1. While the relevant information structures were discussed in Section 1.1.2, we will describe the various cost structures in a little more detail in Section 2.3. Besides the models summarized in Table 1.1, in Chapter 3 we will consider a static team problem with a cost function quadratic in the control actions. Here we do not consider separate state and parameter, but consider a single random variable ξ to which players have partial observations constituting their informations.

1.4 Organization

Following this, in Chapter 2, we present some preliminary matter which includes mathematical notation used in the thesis and some useful background on linear operator theory, linear systems theory and linear matrix inequalties. In Chapter 3, we describe the static quadratic team problem (originally considered in [16]) and present an operator theoretic framework for its analysis. We show that the sequential update scheme converges exponentially to the team optimal strategy and provide bounds for the same. We further elaborate on how convergence of sequential update scheme helps in numerical and analytical computation of team optimal strategies. In Chapter 4, we solve a static

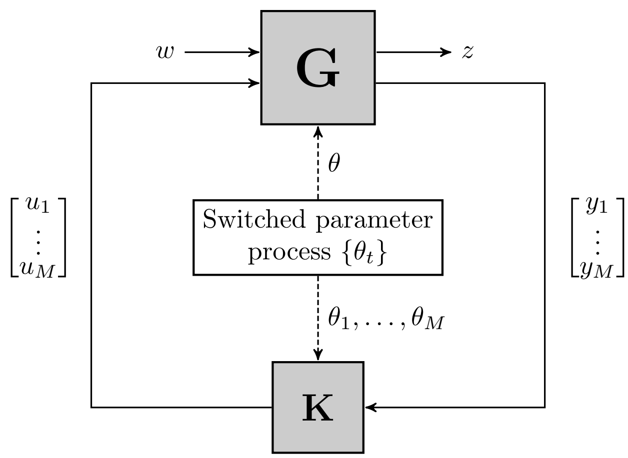


Figure 1.1: Interconnection diagram for general class of M-player control of switched systems

stochastic-parameter problem to illustrate the ideas developed in Chapter 3 and also to aid us in solving dynamic team problems in Chapters 5, 6 and 7 with switched cost function quadratic in the state and control inputs. In Chapter 5, a dynamic team problem with partially nested information structure is considered. Here we assume a hierarchical decision graph in which players act only once using information from above levels. The problem is modified to have a static information structure like that of Chapter 4 resulting in a static quadratic team problem. In Chapter 6, we consider a finite horizon dynamic team problem with one-step delayed information sharing with the parameter being a Markov process. The solution is obtained through dynamic programming while using the result of Chapter 4 at each stage. In Chapter 7, we consider a dynamic problem where controllers have access to full state feedback, however they have only partial knowledge about the parameter (assumed i.i.d.). Solutions to both finite and infinite horizon versions of this problem are presented. In Chapter 8, we consider a decentralized switched control problem with a discrete-time mode dependent switched linear plant which is nested and whose system matrices switch between a finite number of values according to a finite state automaton. The goal is to synthesize a finite-path dependent nested controller to achieve a desired level of ℓ_2-induced norm performance. For this setup, exact feasibility conditions for synthesis are provided along with an algebraic method for controller synthesis. Finally, in Chapter 9, we present the conclusions of this work and discuss some possible avenues of research which can be explored further.

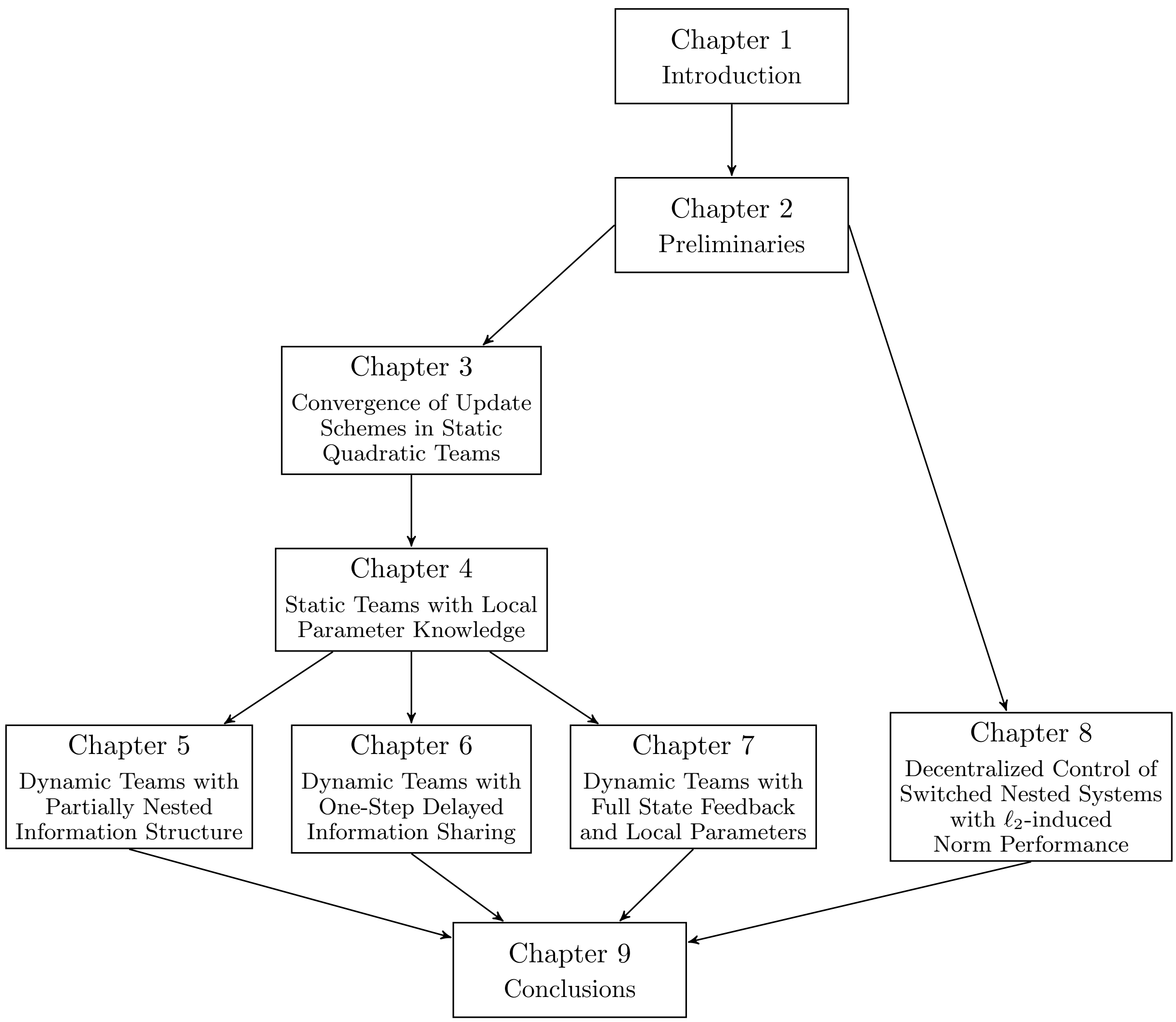


Figure 1.2: Chapter dependency

Chapter 2

Preliminaries

In this chapter we present some preliminary concepts and define notations used in the thesis.

2.1 Mathematical Preliminaries

We denote the set of real numbers, non-negative and positive integers by $\mathbb{R}$, $\mathbb{N}_0$ and $\mathbb{Z}_+$ respectively. The n-dimensional Euclidean space is denoted by $\mathbb{R}^n$ with the corresponding norm being $|\cdot|_2$. The space of $n \times m$ dimensional real valued matrices is denoted by $\mathbb{R}^{n\times m}$. The spaces of n-dimensional symmetric, positive-definite and positive-semidefinite matrices are denoted by $\mathbb{S}^n$, $\mathbb{S}^n_+$ and $\bar{\mathbb{S}}^n_+$ respectively. Elements (say X) of $\mathbb{S}^n_+$ and $\bar{\mathbb{S}}^n_+$ are also often indicated by $X \succ 0$ and $X \succeq 0$ respectively. For a matrix W: W^T, $W^\dagger$, $\mathrm{rank}(W)$, $\mathrm{Im}(W)$, $\mathrm{Ker}(W)$ and $\bar{\sigma}(W)$ represent its transpose, pseudo-inverse, rank, image space, kernel space and maximum singular value respectively. As a shorthand notation we represent a block diagonal matrix by $\mathrm{diag}(D_1, \ldots, D_k)$ with $\{D_i\}_{i=1}^k$ being its diagonal blocks. An identity matrix of dimension n is denoted by I_n or simply I. For a matrix W we will use $W_\perp$ and $W_{||}$ respectively to denote a matrix with full column rank satisfying $\mathrm{Im}(W_\perp) = \mathrm{Ker}(W)$ with $W_\perp^T W_\perp = I$, and $\mathrm{Im}(W_{||}) = \mathrm{Ker}(W)^\perp$ (the orthogonal complement of $\mathrm{Ker}(W)$) with $W_{||}^T W_{||} = I$.

For a matrix $W \in \mathbb{R}^{n\times m}$, its singular value decomposition (SVD), refers to the factorization $W = UDV^T$ where $D \in \mathbb{R}^{n\times m}$ is a diagonal matrix with non-negative diagonal entries called singular values, $U \in \mathbb{R}^{n\times n}$ and $V \in \mathbb{R}^{m\times m}$ are unitary matrices. The decomposition is done so that columns of U and V are also the eigenvectors of WW^T and W^TW respectively. Corresponding eigenvalues are same as the squares of the singular values of W.

Schur complement formula for positive-definite matrices describes the following equivalence

$$\begin{bmatrix} X_{11} & X_{12} & X_{13} \\ X_{12}^T & X_{22} & X_{23} \\ X_{13}^T & X_{23}^T & X_{33} \end{bmatrix} \succ 0 \;\Leftrightarrow\; X_{22} \succ 0 \text{ and } \begin{bmatrix} X_{11} & X_{13} \\ X_{13}^T & X_{33} \end{bmatrix} - \begin{bmatrix} X_{12} \\ X_{23}^T \end{bmatrix} X_{22}^{-1} \begin{bmatrix} X_{12}^T & X_{23} \end{bmatrix} \succ 0.$$

Following is an useful property of matrix inverse, when the constituent inverses exist

$$(Q + URV)^{-1} = Q^{-1} - Q^{-1}U(R^{-1} + VQ^{-1}U)^{-1}VQ^{-1}. \tag{2.1}$$

We will encounter several inequalities of the form $W^T HW \succ 0$ where H and W are matrices of compatible dimensions. To save space, we will sometimes write such inequalities as $[\bullet]^T HW \succ 0$. Also for partitioned symmetric matrices say $\begin{bmatrix} X_1 & X_2 \\ X_2^T & X_3 \end{bmatrix}$, we occasionally suppress repeated sub-blocks as $\begin{bmatrix} X_1 & X_2 \\ \cdot & X_3 \end{bmatrix}$. As an aid to identify compatible sub-blocks while multiplying partitioned matrices, we will sometimes use the notation $\left[\begin{array}{c:c} A & U \\ \hdashline V & P \end{array}\right]$ to separate out some parts of the partitioning.

We denote ℓ^n to be the space of infinite sequences in $\mathbb{R}^n$, namely an element is given by

$$x = (x_0, x_1, x_2, \dots) \text{ with } x_t \in \mathbb{R}^n \text{ for } t \in \mathbb{N}_0. \tag{2.2}$$

When the dimension n is clear from context, this space is simply denoted as ℓ. A subspace of ℓ is the Hilbert space ℓ_2^n (or simply ℓ_2) which is equipped with the inner-product $\langle x, y\rangle := \sum_{t=0}^{\infty} x_t^T y_t$ satisfying $\sum_{t=0}^{\infty} |x_t|_2^2 < \infty$. We denote the norm on ℓ_2 by $\|\cdot\|$. For a Hilbert space $\mathcal{X}$ (different from ℓ_2), the associated norm and inner product are denoted by $\|\cdot\|_{\mathcal{X}}$ and $\langle \cdot, \cdot \rangle_{\mathcal{X}}$ respectively.

For two vector spaces $\mathcal{X}$ and $\mathcal{Y}$, their external direct sum denoted by $\mathcal{X} \oplus \mathcal{Y}$ refers to the vector space

$$\{(x, y) | x \in \mathcal{X}, y \in \mathcal{Y}\}.$$

For a vector space $\mathcal{V}$, if $\mathcal{V}_1$ and $\mathcal{V}_2$ are its subspaces satisfying $\mathcal{V}_1 \cap \mathcal{V}_2 = \{0\}$ and $\mathcal{V} = \mathcal{V}_1 + \mathcal{V}_2$, then $\mathcal{V}$ is called the internal direct sum of $\mathcal{V}_1$ and $\mathcal{V}_2$ and is also denoted by $\mathcal{V}_1 \oplus \mathcal{V}_2$. Within the thesis, we will refer to both these kinds as simply 'direct sums' with them being internal or external clear from context.

We use $\text{Prob}\{E\}$ to denote the probability of an event E, $\mathbb{P}(\nu)$ to denote the distribution of a random variable ν and $\mathbb{P}(\nu_a|\nu_b)$ to denote the distribution of a random variable ν_a conditioned on another random variable ν_b. For a function g of a random variable ξ, the expected value of the function is written as $\mathbb{E}[g(\xi)]$, while its expectation conditioned on another random variable ν as $\mathbb{E}[g(\xi)|\nu]$. To keep the notation compact, for both conditional distribution and conditional expectation, we do not distinguish between the random variable and the value it takes. Following is a well known result

Lemma 1. $\mathbb{E}\big[\mathbb{E}[g(\xi)|\nu]\big] = \mathbb{E}[g(\xi)]$.

With a slight abuse of notation, we would sometimes condition the expectation on functions (say f) as $\mathbb{E}[g(f(\xi))|f]$ to stress the exact knowledge of the function (in this case) f.

A stochastic (or random) process is a collection of random variables indexed in time as $\{\nu_t\}_{t\in\mathbb{N}_0}$.

Such a process is called independent and identically distributed (i.i.d.) if random variables ν_t and ν_τ are mutually independent and have identical distribution for any $t \neq \tau \in \mathbb{N}_0$. A stochastic process is called a Markov process if it satisfies the Markov property $\mathbb{P}(\nu_t|\nu_0, \ldots, \nu_{t\text{-}1}) = \mathbb{P}(\nu_t|\nu_{t\text{-}1})$.

2.2 Operator Theory

We work with operators[1] which map one Hilbert space to another and satisfy the properties of linearity and boundedness, i.e. for an operator $\mathbf{Z}$ mapping Hilbert spaces $\mathcal{X}$ to $\mathcal{Y}$, it satisfies

- $\mathbf{Z}(ax + by) = a\mathbf{Z}(x) + b\mathbf{Z}(y)$ for all $x, y \in \mathcal{X}$ and scalars a, b
- There exists a positive constant α such that $\|\mathbf{Z}(x)\|_{\mathcal{Y}} \leq \alpha \|x\|_{\mathcal{X}}$ for all $x \in \mathcal{X}$

Typically we will write the operation $\mathbf{Z}(x)$ as $\mathbf{Z}x$ for simplicity. The space of linear bounded operators mapping Hilbert spaces $\mathcal{X}$ to $\mathcal{Y}$ is denoted by $\mathcal{L}(\mathcal{X}, \mathcal{Y})$ (or simply $\mathcal{L}(\mathcal{X})$ when the two spaces are same). The induced norm of an operator $\mathbf{Z}$ in such a space is defined by

$$\|\mathbf{Z}\|_{\mathcal{X}\to\mathcal{Y}} := \sup_{x\in\mathcal{X},\, x\neq 0} \frac{\|\mathbf{Z}x\|_{\mathcal{Y}}}{\|x\|_{\mathcal{X}}}.$$

For the special case when both these spaces are ℓ_2, the induced norm is denoted simply by $\|\cdot\|$. For operators $\mathbf{X} \in \mathcal{L}(\mathcal{V}, \mathcal{X})$ and $\mathbf{Y} \in \mathcal{L}(\mathcal{Y}, \mathcal{V})$, their composition $\mathbf{XY} \in \mathcal{L}(\mathcal{Y}, \mathcal{X})$ is defined by $(\mathbf{XY})(x) = \mathbf{X}(\mathbf{Y}(x))$ for all $x \in \mathcal{Y}$. The corresponding induced norms satisfy $\|\mathbf{XY}\|_{\mathcal{Y}\to\mathcal{X}} \leq \|\mathbf{X}\|_{\mathcal{V}\to\mathcal{X}} \|\mathbf{Y}\|_{\mathcal{Y}\to\mathcal{V}}$ referred to as submultiplicative property. The identity and zero operators will be denoted by $\mathbf{I}$ and $\mathbf{0}$ respectively.

For an operator $\mathbf{Z} \in \mathcal{L}(\mathcal{X})$, $\mathbf{Z}^* \in \mathcal{L}(\mathcal{X})$ represents its adjoint and satisfies $\langle \mathbf{Z}x, y\rangle_{\mathcal{X}} = \langle x, \mathbf{Z}^* y\rangle_{\mathcal{X}}$. An operator $\mathbf{Z} \in \mathcal{L}(\mathcal{X})$ is called self-adjoint if it satisfies $\mathbf{Z} = \mathbf{Z}^*$. Such an operator is said to be positive definite (written as $\mathbf{Z} \succ 0$) if there exists a constant $\epsilon > 0$ satisfying

$$\langle x, \mathbf{Z}x\rangle_{\mathcal{X}} \geq \epsilon \|x\|_{\mathcal{X}}^2 \quad \text{for all} \quad x \in \mathcal{X}.$$

However, if the previous inequality satisfies only with $\epsilon = 0$, operator $\mathbf{Z}$ is said to be positive semi-definite (written as $\mathbf{Z} \succeq \mathbf{0}$). Under the notation $\mathbf{Z} \succ \mathbf{0}$ or $\mathbf{Z} \succeq \mathbf{0}$, the operator $\mathbf{Z}$ is implicitly assumed to be self-adjoint. We use $\mathbf{Z} \prec \mathbf{0}$ to denote $-\mathbf{Z} \succ \mathbf{0}$ and similarly define $\mathbf{Z} \preceq \mathbf{0}$. For two self-adjoint operators $\mathbf{Z}$ and $\mathbf{Y}$, we use the notation $\mathbf{Z} \succ \mathbf{Y}$ and $\mathbf{Z} \succeq \mathbf{Y}$ to imply $\mathbf{Z} - \mathbf{Y} \succ \mathbf{0}$ and $\mathbf{Z} - \mathbf{Y} \succeq \mathbf{0}$ respectively.

An element in the Hilbert space $\mathcal{X}_1 \oplus \mathcal{X}_2$, constructed with elements $x_1 \in \mathcal{X}_1$ and $x_2 \in \mathcal{X}_2$ can be written in two equivalent ways: $(x_1,\, x_2)$ or $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$. A *partitioned* operator $\begin{bmatrix} \mathbf{Z}_{11} & \mathbf{Z}_{12} \\ \mathbf{Z}_{21} & \mathbf{Z}_{22} \end{bmatrix} \in$

[1]Throughout this thesis, the convention of using boldfaced alphabets for linear operators is adopted.

$\mathcal{L}(\mathcal{X}_1 \oplus \mathcal{X}_2, \mathcal{Y}_1 \oplus \mathcal{Y}_2)$ can be constructed from individual operators $\mathbf{Z}_{ij} \in \mathcal{L}(\mathcal{X}_j, \mathcal{Y}_i)$ for $i = 1, 2$ and $j = 1, 2$. Such an operator would correspond to the following operation for $x_1 \in \mathcal{X}_1$ and $x_2 \in \mathcal{X}_2$

$$\begin{bmatrix} \mathbf{Z}_{11} \, \mathbf{Z}_{12} \\ \mathbf{Z}_{21} \, \mathbf{Z}_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \mathbf{Z}_{11} x_1 + \mathbf{Z}_{12} x_2 \\ \mathbf{Z}_{21} x_1 + \mathbf{Z}_{22} x_2 \end{bmatrix}.$$

These definitions can be generalized to spaces having larger number of direct sum partitions.

Following is the operator version of Schur complement formula

Lemma 2. *Consider Hilbert spaces $\mathcal{V}_1$, $\mathcal{V}_2$ and $\mathcal{V} := \mathcal{V}_1 \oplus \mathcal{V}_2$, and operators $\mathbf{X}_1 \in \mathcal{L}(\mathcal{V}_1)$, $\mathbf{X}_2 \in \mathcal{L}(\mathcal{V}_1, \mathcal{V}_2)$ and $\mathbf{X}_3 \in \mathcal{L}(\mathcal{V}_2)$, then*

$$\mathbf{X} := \begin{bmatrix} \mathbf{X}_1 \, \mathbf{X}_2 \\ \mathbf{X}_2^* \, \mathbf{X}_3 \end{bmatrix} \succ \mathbf{0} \quad \Leftrightarrow \quad \mathbf{X}_3 \succ \mathbf{0} \;\; \textit{and} \;\; \mathbf{X}_1 - \mathbf{X}_2 \mathbf{X}_3^{-1} \mathbf{X}_2^* \succ \mathbf{0}.$$

Proof. To prove the above, we start by noting that $\mathbf{X} = \mathbf{L}^* \, \mathbf{M} \, \mathbf{L}$ with $\mathbf{M} = \mathrm{diag}(\mathbf{X}_1 - \mathbf{X}_2 \mathbf{X}_3^{-1} \mathbf{X}_2^*, \, \mathbf{X}_3)$ and $\mathbf{L} = \begin{bmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{X}_2 \mathbf{X}_3^{-1} & \mathbf{I} \end{bmatrix}$. $\mathbf{L}$ being invertible on $\mathcal{V}$, it is clear that positive-definiteness of either one of $\mathbf{X}$ or $\mathbf{M}$ implies the positive-definiteness of the other. □

2.3 Linear Systems Theory

We primarily work with linear discrete time systems and in this section we discuss some basic concepts of stability and performance associated with such systems in context of this work. Consider an LTV system described by

$$\begin{aligned} x_{t+1} &= A_t x_t + B_t w_t \\ z_t &= C_t x_t + D_t w_t \end{aligned} \tag{2.3}$$

with $x_0 = 0$ and where $x_t \in \mathbb{R}^n$, $w_t \in \mathbb{R}^{n^w}$ and $z_t \in \mathbb{R}^{n^z}$. Here x_t, w_t and z_t are respectively called the state, input and output of the system. These vectors, sequenced by t further define corresponding elements in ℓ similar to (2.2) and are denoted with the same name x, w and z. The above equations describe causal relationships, where given $w \in \ell$, unique solutions for $x \in \ell$ and $w \in \ell$ can be computed. In this thesis we explore such systems in both finite and infinite horizon settings. Unlike problems with finite horizon, in infinite horizon setting, it is important to achieve system stability, defined next.

Definition 3. *The system 2.3 is said to be* exponentially stable *if for $w \equiv 0$ and $x_0 \neq 0$, there exist constants $\alpha > 0$ and $0 < \beta < 1$ such that $|x_t|_2 \leq \alpha \beta^t |x_0|_2$ holds for all $t \in \mathbb{N}_0$.*

The following lemma describes the Lyapunov inequality condition for stability of LTV systems.

Lemma 4. *The system 2.3 is exponentially stable if and only if there exist positive constants a, b and ϵ, and a sequence of positive definite matrices $\{X_t\}_{t\in\mathbb{N}_0}$ satisfying*

$$aI \preceq X_t \preceq bI \quad \text{and} \quad X_t - A_t^T X_{t+1} A_t \succeq \epsilon I$$

for all $t \in \mathbb{N}_0$. Further, if the above condition is satisfied then with $w \equiv 0$, we have $|x_t|_2 \leq \sqrt{\frac{b}{a}}\left(1 - \frac{\epsilon}{b}\right)^{t/2} |x_0|_2$ for all $t \in \mathbb{N}_0$.

For the system 2.3, the input to output mapping is denoted by $w \mapsto z$. One of the cost criteria we examine is that of the ℓ_2 induced norm, which is defined by the ℓ_2 induced norm of the input w to output z i.e. $\|w \mapsto z\|$. In literature, this norm is also referred to as root mean square gain of the system. For LTI systems, this induced norm coincides with the $\mathcal{H}_\infty$ norm of the system, hence we refer to it as a $\mathcal{H}_\infty$-type norm.

Another cost criteria of interest is defined here for the case of finite horizon and when w is a white noise process. It is given by

$$\sum_{t=0}^{N} \mathbb{E}[|z_t|_2^2]. \tag{2.4}$$

For infinite horizon LTI case, the above cost also has a induced norm interpretation. If w is considered a signal in ℓ_2 instead, the above cost is equivalent to the $\mathcal{H}_2$ norm defined by the induced norm $\|w \mapsto z\|_{\ell_2 \to \ell_\infty}$.

The above cost criteria is also closely connected to the quadratic cost encountered in LQR problems. To see this connection we write the entire plant mode with control input $u_t \in \mathbb{R}^{n^u}$ and measurement output $y_t \in \mathbb{R}^{n^y}$ as below

$$\begin{aligned} x_{t+1} &= A_t x_t + B_t^w w_t + B_t^u u_t \\ z_t &= C_t^z x_t + D_t^{zw} w_t + D_t^{zu} u_t \\ y_t &= C_t^y x_t + D_t^{yw} w_t \end{aligned}$$

The quadratic cost defined by

$$\mathbb{E}\Big[\sum_{t=0}^{N-1} (x_t^T Q_t x_t + u_t^T R_t u_t) + x_N^T Q_N x_N\Big]$$

with $Q_t \in \bar{\mathbb{S}}_+^n$ and $R_t \in \mathbb{S}_+^{n^u}$ is then equivalent to the cost in (2.4) under the choice $C_t^z = \begin{bmatrix} Q_t^{\frac{1}{2}} \\ 0 \end{bmatrix}$, $D_t^{zu} = \begin{bmatrix} 0 \\ R_t^{\frac{1}{2}} \end{bmatrix}$ and $D_t^{zw} = 0$ for $t \in \{0, \ldots, N\}$ (while assuming $R_N = 0$).

While this subsection is a brief overview focussed mainly on non-stochastic problems, in this thesis we also encounter systems, where the system and cost matrices are functions of stochastic parameters. In such a setting we will be using the same cost criteria as in (2.4), but the expectation will be taken additionally with respect to the parameters. We will explain the setup in greater detail later when the problem is introduced.

2.4 Linear Matrix Inequalities

In chapter 8, the conditions for existence of a controller synthesis and the synthesis procedure itself are expressed in terms of linear matrix inequalities (LMI) which take the following form

$$F(X) \succ 0.$$

Here X is the unknown variable and takes values in a real vector space $\mathcal{X}$ and $F : \mathcal{X} \to \mathbb{S}$ is an affine mapping. The above LMI represents a feasibility problem, in which we seek an element $X \in \mathcal{X}$ satisfying the same inequality. LMIs form a special case of a broader class of convex optimization setup called semidefinite programming (SDP). In SDP, the goal is to minimize a linear objective function $c(X)$ under LMI and linear equality constraints in the variable $X \in \mathcal{X}$. Note that a finite sequence of LMIs can be written as a single LMI, where $F(X)$ corresponding to individual LMIs are arranged into a block-diagonal structure to form a singe affine function of X.

Several problems in control theory can be posed as LMIs [63], particularly in the context of $\mathcal{H}_\infty$ and $\mathcal{H}_2$ control [64] which are relevant to this thesis. The widespread adoption of LMIs as a synthesis tool can be attributed to efficient numerical techniques of interior point methods with suitably chosen barrier functions as presented in [65]. Reformulating problems as LMIs is an important goal in Chapter 8, however exploring any further numerical aspects of solving them is beyond the scope of this thesis. The examples presented in Chapter 8 were implemented using CVX tool [66, 67] run within MATLAB [68].

Chapter 3

Convergence of Update Schemes in Static Quadratic Teams

In this chapter, we focus on the static quadratic team problem originally considered in [16]. We adopt an operator theoretic framework to analyze the problem and explore the convergence of update schemes involving repeated application of best response mappings.

3.1 Background and Motivation

In [16], a class of problems with convex cost was considered and under specific conditions, uniqueness of person-by-person optimal (hence team optimal) solution was established. Further, a stationarity condition which also serves as the necessary condition of team optimality was provided. These conditions when applied to the quadratic team problem with non-stochastic cost matrices directly yields the corresponding team optimal strategies. The strategies thus obtained have been used in a number of dynamic LQG decentralized problems [22, 33–35] where no switching in the system matrices is involved. For dynamic switched problems (which can be seen as extensions of the decentralized LQG problems) discussed later, a corresponding static result is desirable. However, the stationary conditions provided in [16] do not directly provide any information or intuition about the structure of the controller. In specific cases, one may guess the structure of the optimal strategies and substitute them back into the stationary conditions to obtain equations in reduced dimensions; however to ensure that the structural guess is correct, one has to verify that these equations indeed have a solution. This is not always a straightforward task.

One method which can lead to the team optimal solution (or approximations of it), is iteratively applying the best response of the players at each stage while starting at some arbitrary strategy. Early uses of this idea include [35] and [69] where it was applied to the static problem encountered in their respective setups of decentralized LQG team and multi-criteria LQG game. The literature in game and team theory (e.g., [20, 70] and references therein) commonly use two schemes known as sequential (Gauss-Siedel) update and parallel (Cournot/Jacobi) update in this context. When these schemes converge, their limit is the team optimal solution (or Nash equilibrium in game problems); however, in general such convergence results are presented with additional conditions, usually in terms of contraction of certain operators. The main result of this chapter is showing that sequential update scheme converges to the team optimal solution for the M-player quadratic team problem with stochastic cost matrices. In the 2-player scenario, since the parallel update scheme coincides with sequential update, we can make a similar claim here as well; in the special case of a non-stochastic setting with two players this result was obtained in [20]. In our work here, we adopt an operator theoretic approach to show that the best response dynamics of the update scheme satisfy an operator Lyapunov inequality, and thereby prove its convergence and also provide a guaranteed rate. In order to demonstrate the effectiveness of this approach, we provide an example of a nonlinear static problem, where guessing the structure of the solution is generally impossible, and obtain its optimal strategy through numerically computing the best responses. We also demonstrate how the property of guaranteed convergence can be instrumental in isolating the structure of the team optimal strategy.

3.2 Static Team Theory

In this section, we provide a descriptive introduction to static team decision theory. Broadly speaking, multiple players are faced with the problem of finding feedback strategies in order to minimize a common cost function. These strategies are functions of local information available to each player. For a M-player static problem, consider a cost function $J(\xi, u_1, \ldots, u_M)$ where $u_i \in \mathbb{R}^{m_i}$ is the action of i-th player and ξ is a finite dimensional random variable with known probability distribution. Let the information available to player i be denoted by $\mathbb{I}_i$, which is a known function of ξ. The objective then is to find decentralized strategies $(\gamma_1, \ldots, \gamma_M)$, with γ_i assumed to be in a space $\mathcal{K}_i$ containing mappings from $\mathbb{I}_i$ to u_i, which minimize the expected cost

$$\bar{J}(\gamma_1, \ldots, \gamma_M) := \mathbb{E}[J(\xi, \gamma_1(\mathbb{I}_1), \ldots, \gamma_M(\mathbb{I}_M))]. \tag{3.1}$$

The minimizing solution $(\gamma_1^\circ, \ldots, \gamma_M^\circ)$, if it exists is called the *team optimal* strategy. The *best response* of a player is defined as a function of the other players strategies $\gamma_{\text{-}i} := (\gamma_1, \ldots, \gamma_{i\text{-}1}, \gamma_{i\text{+}1}, \ldots, \gamma_M)$ as[1]

$$\mathbf{\Gamma}_i(\gamma_{\text{-}i}) = \underset{\gamma_i}{\operatorname{argmin}}\, \mathbb{E}[J(\xi, \gamma_1(\mathbb{I}_1), \ldots, \gamma_M(\mathbb{I}_M))|\gamma_{\text{-}i}] \tag{3.2}$$

which can also be written point-wise as

$$(\mathbf{\Gamma}_i(\gamma_{\text{-}i}))(\mathbb{I}_i) = \underset{u_i}{\operatorname{argmin}}\, \mathbb{E}\Big[J\big(\xi, \gamma_1(\mathbb{I}_1), \ldots, \gamma_{i\text{-}1}(\mathbb{I}_{i\text{-}1}), u_i, \gamma_{i\text{+}1}(\mathbb{I}_{i\text{+}1}), \ldots, \gamma_M(\mathbb{I}_M)\big)\Big|\mathbb{I}_i, \gamma_{\text{-}i}\Big]. \tag{3.3}$$

When $\gamma_{\text{-}i} = \gamma_{\text{-}i}^\circ$, the above best response yields the optimal strategy γ_i°, i.e. $\mathbf{\Gamma}_i(\gamma_{\text{-}i}^\circ) = \gamma_i^\circ$. A detailed explanation of this fact can be found in [71].

[1] In general, the best response yields a set of strategies rather than a unique one. We however provide this definition in view of the problem defined later.

A tuple of strategies $\gamma^p = (\gamma_1^p, \ldots, \gamma_M^p)$ is said to be *person-by-person* optimal if they satisfy

$$\bar{J}(\gamma^p) \leq \bar{J}(\gamma_1^p, \ldots, \gamma_{i-1}^p, \gamma_i, \gamma_{i+1}^p, \ldots, \gamma_M^p) \;\text{ for all } \gamma_i \in \mathcal{K}_i$$

and for all $i \in \mathcal{J}$. The above definition also implies that γ^p satisfies $\gamma_i^p = \mathbf{\Gamma}_i(\gamma_{-i}^p)$ for $i \in \mathcal{J}$. Team decision problems can be viewed as game theoretic problems in which all players have the same cost function, with person-by-person optimality accordingly being equivalent to the concept of Nash equilibrium. Note that while a team optimal strategy is person-by-person optimal, the converse may not hold in general.

Under the assumption of J being convex and continuously differentiable in u_i for $i \in \mathcal{J}$, it was shown in [16] that a unique person-by-person optimal strategy exists, which is also the unique team optimal solution.

3.3 Static Quadratic Team Problem Setup

Consider the quadratic cost function

$$J(\xi, u_1, \ldots, u_M) = u^T Z(\xi) u + 2u^T d(\xi) + c(\xi) \tag{3.4}$$

where $u = \begin{bmatrix} u_1 \\ \vdots \\ u_M \end{bmatrix} \in \mathbb{R}^m$ constitutes the player actions with $m = \sum_{i=1}^M m_i$. The cost matrices

$$Z(\xi) = \begin{bmatrix} Z_{11}(\xi) & Z_{12}(\xi) & \ldots & Z_{1M}(\xi) \\ Z_{21}(\xi) & Z_{22}(\xi) & \ldots & Z_{2M}(\xi) \\ \vdots & & \ddots & \vdots \\ Z_{M1}(\xi) & Z_{M2}(\xi) & \ldots & Z_{MM}(\xi) \end{bmatrix} \quad \text{and} \quad d(\xi) = \begin{bmatrix} d_1(\xi) \\ \vdots \\ d_M(\xi) \end{bmatrix}$$

are, respectively, symmetric matrix valued and vector valued functions of ξ, with partitioning in compliance with that of u i.e. $Z_{ij}(\xi) \in \mathbb{R}^{m_i \times m_j}$ and $d_i(\xi) \in \mathbb{R}^{m_i}$. ξ is the random state which takes values in $\mathcal{X}$ and captures the underlying randomness of the system. The information available to player i takes values in the set[2] $\mathcal{I}_i$ and is a function of the state as $\mathbb{I}_i = \eta_i(\xi)$ where $\eta_i : \mathcal{X} \to \mathcal{I}_i$ is a Borel measurable function. The decentralized information can then be defined as $\mathbb{I}^d := (\mathbb{I}_1, \ldots, \mathbb{I}_M) \in \mathcal{I}_1 \times \cdots \times \mathcal{I}_M$. This also allows us to define the notation $\gamma(\mathbb{I}^d) := \begin{bmatrix} \gamma_1(\mathbb{I}_1) \\ \vdots \\ \gamma_M(\mathbb{I}_M) \end{bmatrix}$.

[2]We assume $\mathcal{I}_i$ to be a product of a finite set and a Euclidean space.

The goal is to design strategies $\gamma_i : \mathcal{I}_i \to \mathbb{R}^{m_i}$ for $i \in \mathcal{J}$ which minimize the following expected cost

$$\bar{J}(\gamma_1, \dots, \gamma_M) = \mathbb{E}[J(\xi, \gamma_1(\mathbb{I}_1), \dots, \gamma_M(\mathbb{I}_M))]. \tag{3.5}$$

We further have the following assumption on the structure of the cost matrices

Assumption 5. *(i) There exist positive constants $\underline{a}$ and $\bar{a}$ satisfying*

$$Prob\{\, \underline{a}I \preceq Z(\xi) \preceq \bar{a}I \,\} = 1$$

implying that the matrix valued function $Z(\cdot)$ is bounded from above and strictly positive.

(ii) $\mathbb{E}\left[|d(\xi)|_2^2\right] < \infty$ *and* $\mathbb{E}[|c(\xi)|] < \infty$.

Strategy Space The strategy for player i is a measurable function $\gamma_i : \mathcal{I}_i \to \mathbb{R}^{m_i}$ defined on the Hilbert space $\mathcal{K}_i$ equipped with the inner-product $\langle \alpha, \beta \rangle_{\mathcal{K}_i} := \mathbb{E}\left[\alpha^T(\mathbb{I}_i)\beta(\mathbb{I}_i)\right]$. Thus, a strategy $\gamma_i \in \mathcal{K}_i$ satisfies $\|\gamma_i\|_{\mathcal{K}_i} := \mathbb{E}\left[\,|\gamma_i(\mathbb{I}_i)|_2^2\,\right]^{\frac{1}{2}} < \infty$. Such a definition for the space of strategies was originally used for a static quadratic game problem in [72]. The total decentralized strategy is thus defined over the Hilbert space $\mathcal{K} = \mathcal{K}_1 \oplus \cdots \oplus \mathcal{K}_M$ with inner-product defined in the obvious way.

3.4 Operator Definitions

We define the operators $\mathbf{Z}_{ij} : \mathcal{K}_j \to \mathcal{K}_i$ for $i, j \in \mathcal{J}$ as

$$(\mathbf{Z}_{ij}(\gamma_j))(\mathbb{I}_i) = \mathbb{E}[Z_{ij}(\xi)\gamma_j(\mathbb{I}_j)|\mathbb{I}_i]. \tag{3.6}$$

For $i = j$, the above can be rewritten as

$$(\mathbf{Z}_{ii}(\gamma_i))\,(\mathbb{I}_i) = \mathbb{E}[Z_{ii}(\xi)|\mathbb{I}_i]\gamma_i(\mathbb{I}_i).$$

Further, the operator $\mathbf{Z}_{ii}$ is self-adjoint and positive definite as evident from the following

$$\begin{aligned} \langle \gamma_i, \mathbf{Z}_{ii}\gamma_i \rangle_{\mathcal{K}_i} = \langle \mathbf{Z}_{ii}\gamma_i, \gamma_i \rangle_{\mathcal{K}_i} &= \mathbb{E}\left[\gamma_i^T(\mathbb{I}_i)\mathbb{E}[Z_{ii}(\xi)|\mathbb{I}_i]\gamma_i(\mathbb{I}_i)\right] \\ &= \mathbb{E}\left[\gamma_i^T(\mathbb{I}_i)Z_{ii}(\xi)\gamma_i(\mathbb{I}_i)\right] \geq \underline{a}\|\gamma_i\|_{\mathcal{K}_i}^2 \end{aligned} \tag{3.7}$$

Clearly, operator $\mathbf{Z}_{ii}$ is invertible on $\mathcal{K}_i$ and we can define the following

$$\begin{aligned} (\mathbf{Z}_{ii}^{-1}\gamma_i)(\mathbb{I}_i) &= \left(\mathbb{E}[Z_{ii}(\xi)|\mathbb{I}_i]\right)^{-1}\gamma_i(\mathbb{I}_i) \text{ and} \\ (\mathbf{Z}_{ii}^{\frac{1}{2}}\gamma_i)(\mathbb{I}_i) &= \left(\mathbb{E}[Z_{ii}(\xi)|\mathbb{I}_i]\right)^{\frac{1}{2}}\gamma_i(\mathbb{I}_i). \end{aligned}$$

Further $\mathbf{Z}_{ij}^* = \mathbf{Z}_{ji}$ due to the following relation

$$\begin{aligned}\langle \gamma_i, \mathbf{Z}_{ij}\gamma_j \rangle_{\mathcal{K}_i} &= \mathbb{E}\left[\gamma_i^T(\mathbb{I}_i)\mathbb{E}[Z_{ij}(\xi)\gamma_j(\mathbb{I}_j)|\mathbb{I}_i]\right] = \mathbb{E}[\gamma_i^T(\mathbb{I}_i)Z_{ij}(\xi)\gamma_j(\mathbb{I}_j)] \\ &= \mathbb{E}\left[\mathbb{E}[Z_{ji}(\xi)\gamma_i(\mathbb{I}_i)|\mathbb{I}_j]^T\gamma_j(\mathbb{I}_j)\right] = \langle \mathbf{Z}_{ji}\gamma_i, \gamma_j \rangle_{\mathcal{K}_j}\end{aligned} \tag{3.8}$$

For the second equality, we take $\gamma_i(\mathbb{I}_i)$ inside the conditional expectation followed by using Lemma 1. We now define partitioned self-adjoint operator $\mathbf{Z} : \mathcal{K} \to \mathcal{K}$ and its operation as

$$\underbrace{\begin{bmatrix} \mathbf{Z}_{11} & \mathbf{Z}_{12} & \dots & \mathbf{Z}_{1M} \\ \mathbf{Z}_{21} & \mathbf{Z}_{22} & \dots & \mathbf{Z}_{2M} \\ \vdots & & \ddots & \vdots \\ \mathbf{Z}_{M1} & \mathbf{Z}_{M2} & \dots & \mathbf{Z}_{MM} \end{bmatrix}}_{=:\mathbf{Z}} \underbrace{\begin{bmatrix} \gamma_1 \\ \vdots \\ \gamma_M \end{bmatrix}}_{\gamma \in \mathcal{K}} = \underbrace{\begin{bmatrix} \sum_{i=1}^M \mathbf{Z}_{1i}\gamma_i \\ \vdots \\ \sum_{i=1}^M \mathbf{Z}_{Mi}\gamma_i \end{bmatrix}}_{\in \mathcal{K}}.$$

We define the mapping δ_i point-wise as $\delta_i(\mathbb{I}_i) = \mathbb{E}[d_i(\xi)|\mathbb{I}_i]$ for $i \in \mathcal{J}$. Due to Assumption 5(ii) it can be shown that $\delta_i \in \mathcal{K}_i$. We combine these mappings into $\delta := \begin{bmatrix} \delta_1 \\ \vdots \\ \delta_M \end{bmatrix} \in \mathcal{K}$. Further, we use $\mathbf{0}$ and $\mathbf{I}$ to denote the zero and identity operators in $\mathcal{K}_1, \dots, \mathcal{K}_M$ and $\mathcal{K}$.

Lemma 6. $\mathbf{Z}$ *satisfies* $\underline{a}\mathbf{I} \preceq \mathbf{Z} \preceq \bar{a}\mathbf{I}$.

Proof. Since $\mathbf{Z}_{ij}^* = \mathbf{Z}_{ji}$, it is clear that $\mathbf{Z}$ is self-adjoint. For positive-definiteness, we need to show that $\mathbf{Z}$ has a lower and an upper bound. For this we evaluate the following

$$\langle \gamma, \mathbf{Z}\gamma \rangle_{\mathcal{K}} = \sum_{i=1}^M \sum_{j=1}^M \langle \gamma_i, \mathbf{Z}_{ij}\gamma_j \rangle_{\mathcal{K}_i} = \mathbb{E}\left[\gamma(\mathbb{I}^d)^T Z(\xi)\gamma(\mathbb{I}^d)\right]$$

which uses the relations from (3.7) and (3.8). The above inner-product along with Assumption 5(i) leads to the result. □

3.5 Team Optimality

Using the operator framework described in the previous sections, we now motivate the use of update equations (in upcoming sections) for computing team optimal strategies. The necessary conditions for team optimality in [16], when applied to the quadratic team problem under consideration, results in

$$\mathbb{E}[Z_{ii}(\xi)|\mathbb{I}_i]\gamma_i^\circ(\mathbb{I}_i) + \sum_{j \neq i} \mathbb{E}[Z_{ij}(\xi)\gamma_j^\circ(\mathbb{I}_j)|\mathbb{I}_i] + \mathbb{E}[d_i(\xi)|\mathbb{I}_i] = 0 \tag{3.9}$$

for $i \in \mathcal{J}$ a.e. in ξ. Here $\gamma^\circ \in \mathcal{K}$ is the team optimal strategy which minimizes the expected cost function in (3.5). The above can be written compactly using the operator notation as

$$\mathbf{Z}\gamma^\circ + \delta = 0 \quad \text{or} \quad \gamma^\circ = -\mathbf{Z}^{-1}\delta. \tag{3.10}$$

Here, the invertibility of $\mathbf{Z}$ on $\mathcal{K}$ is immediate from its positive-definiteness. However, evaluating the above expression for team optimal strategy is hard in general due to difficulty in evaluating the inverse of operator $\mathbf{Z}$.

For a given $\gamma \in \mathcal{K}$, we obtain the following expression for the expected cost

$$\begin{aligned}\bar{J}(\gamma) &= \mathbb{E}\left[\gamma(\mathbb{I}^d)^T Z(\xi)\gamma(\mathbb{I}^d) + 2\gamma(\mathbb{I}^d)^T d(\xi) + c(\xi)\right] \\ &= \mathbb{E}\left[\sum_{i=1}^M \gamma_i(\mathbb{I}_i)^T \mathbb{E}\Big[\sum_{j=1}^M Z_{ij}(\xi)\gamma_j(\mathbb{I}_j) + 2d_i(\xi)\big|\mathbb{I}_i\Big]\right] + \mathbb{E}[c(\xi)] \\ &= \mathbb{E}\left[\sum_{i=1}^M \gamma_i(\mathbb{I}_i)^T \left(\sum_{j=1}^M (\mathbf{Z}_{ij}\gamma_j)(\mathbb{I}_i) + 2\delta_i(\mathbb{I}_i)\right)\right] + \mathbb{E}[c(\xi)] \\ &= \langle \gamma, \mathbf{Z}\gamma + 2\delta\rangle_{\mathcal{K}} + \mathbb{E}[c(\xi)].\end{aligned} \tag{3.11}$$

When $\gamma = \gamma^\circ$, the above along with (3.10) yields the following expression for optimal cost

$$\bar{J}(\gamma^\circ) = \langle \gamma^\circ, \delta\rangle_{\mathcal{K}} + \mathbb{E}[c(\xi)]. \tag{3.12}$$

3.6 Best Response and Update Equations

The best response of player i to the strategies of other players $\gamma_{\text{-}i} = (\gamma_1, \ldots, \gamma_{i\text{-}1}, \gamma_{i\text{+}1}, \ldots, \gamma_M)$ is defined using the operator $\mathbf{\Gamma}_i : \mathcal{K}_{\text{-}i} \to \mathcal{K}_i$ (where $\mathcal{K}_{\text{-}i} = \oplus_{j\neq i}\mathcal{K}_j$) as in (3.2). For the quadratic team setup considered here, this operator can be evaluated using the point-wise definition in (3.3), as

$$(\mathbf{\Gamma}_i(\gamma_{\text{-}i}))(\mathbb{I}_i) = \underset{u_i}{\operatorname{argmin}} \left(u_i^T \mathbb{E}[Z_{ii}(\xi)|\mathbb{I}_i]u_i + 2u_i^T \mathbb{E}\Big[\sum_{j\neq i} Z_{ij}(\xi)\gamma_j(\mathbb{I}_j) + d_i\big|\mathbb{I}_i\Big] + c_i(\mathbb{I}_i, \gamma_{\text{-}i})\right)$$

with $c_i(\mathbb{I}_i, \gamma_{\text{-}i}) = \mathbb{E}\Big[c(\xi) + \sum_{j\neq i}\sum_{l\neq i}\gamma_j(\mathbb{I}_j)^T Z_{jl}(\xi)\gamma_l(\mathbb{I}_l)|\mathbb{I}_i\Big]$. The cost function above as seen by player i is quadratic and continuously differentiable in u_i, with strict convexity being guaranteed by $\mathbb{E}[Z(\xi)|\mathbb{I}_i] \succ 0$ almost surely. Thus, the above minimization can be solved by setting the partial derivative of the quadratic cost with respect to u_i to zero. This leads to the following affine operator definition for the best response

$$\mathbf{\Gamma}_i(\gamma_{-i}) = -\mathbf{Z}_{ii}^{-1}\left(\sum_{j\neq i}\mathbf{Z}_{ij}\gamma_j + \delta_i\right). \tag{3.13}$$

Note that for any $\gamma_{-i} \in \mathcal{K}_{-i}$, the above best response exists and is unique in $\mathcal{K}_i$.

We now present the two update schemes: sequential and parallel, which serve as mechanisms to compute the team optimal solution when they converge. In both the schemes we start with an arbitrary initial strategy $\gamma^{(0)} \in \mathcal{K}$ and at every stage use the best response mapping to update the strategies. For both these schemes, we provide an operator description of the updates, which plays a crucial role in subsequent section for proving their convergence.

Parallel Update: In this scheme, all player strategies are updated simultaneously at each stage based on all strategies from the previous stage. The update equation for the strategy of i-th player is given by

$$\gamma_i^{(k+1)} = \mathbf{\Gamma}_i(\gamma_{-i}^{(k)}) = -\mathbf{Z}_{ii}^{-1}\left(\sum_{j\neq i}\mathbf{Z}_{ij}\gamma_j^{(k)} + \delta_i\right). \tag{3.14}$$

It is straightforward to see that the above can also be written as

$$\gamma^{(k+1)} = \mathbf{R}_p\gamma^{(k)} + r_p \tag{3.15}$$

with $\mathbf{R}_p := -\mathbf{D}^{-1}(\mathbf{Z}_l + \mathbf{Z}_u) \in \mathcal{L}(\mathcal{K})$ and $r_p := -\mathbf{D}^{-1}\delta \in \mathcal{K}$ while using the definitions

$$\mathbf{Z}_l := \begin{bmatrix} \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} \\ \mathbf{Z}_{21} & & & \mathbf{0} \\ \vdots & \ddots & & \vdots \\ \mathbf{Z}_{M1} & \dots & \mathbf{Z}_{M,M-1} & \mathbf{0} \end{bmatrix}, \quad \mathbf{Z}_u := \begin{bmatrix} \mathbf{0} & \mathbf{Z}_{12} & \dots & \mathbf{Z}_{1M} \\ \vdots & & \ddots & \vdots \\ \mathbf{0} & & & \mathbf{Z}_{M-1,M} \\ \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} \end{bmatrix}$$

and $\mathbf{D} := \mathrm{diag}(\mathbf{Z}_{11}, \mathbf{Z}_{22}, \dots, \mathbf{Z}_{MM})$. Note that $\mathbf{Z}_l = \mathbf{Z}_u^*$.

Sequential Update: At each step of the iteration, player strategies are updated sequentially in order from player 1 to M. The update equation for the i-th player is given by

$$\begin{aligned}\gamma_i^{(k+1)} &= \mathbf{\Gamma}_i(\gamma_1^{(k+1)}, \dots, \gamma_{i-1}^{(k+1)}, \gamma_{i+1}^{(k)}, \dots, \gamma_M^{(k)}) \\ &= -\mathbf{Z}_{ii}^{-1}\left(\sum_{j<i}\mathbf{Z}_{ij}\gamma_j^{(k+1)} + \sum_{j>i}\mathbf{Z}_{ij}\gamma_j^{(k)} + \delta_i\right)\end{aligned} \tag{3.16}$$

which is the best response to the most recent strategy of the other players. We can combine the update equations above for all the players into a single equation as

$$\gamma^{(k+1)} = \mathbf{R}_s \gamma^{(k)} + r_s \tag{3.17}$$

with $\mathbf{R}_s := -(\mathbf{D} + \mathbf{Z}_l)^{-1}\mathbf{Z}_u \in \mathcal{L}(\mathcal{K})$ and $r_s := -(\mathbf{D} + \mathbf{Z}_l)^{-1}\delta \in \mathcal{K}$. The steps involved in obtaining (3.17) from (3.16) are given below.

Derivation of Equation (3.17) for Sequential Update: We can rewrite the update equation in (3.16) as

$$\begin{bmatrix} \gamma_1^{(k+1)} \\ \vdots \\ \gamma_{i-1}^{(k+1)} \\ \hdashline \gamma_i^{(k+1)} \\ \hdashline \gamma_{i+1}^{(k)} \\ \vdots \\ \gamma_M^{(k)} \end{bmatrix} = \mathbf{D}^{-\frac{1}{2}} \hat{\mathbf{M}}_i \mathbf{D}^{\frac{1}{2}} \begin{bmatrix} \gamma_1^{(k+1)} \\ \vdots \\ \gamma_{i-1}^{(k+1)} \\ \hdashline \gamma_i^{(k)} \\ \hdashline \gamma_{i+1}^{(k)} \\ \vdots \\ \gamma_M^{(k)} \end{bmatrix} - \underbrace{\begin{bmatrix} 0 \\ \mathbf{Z}_{ii}^{-1}\delta_i \\ 0 \end{bmatrix}}_{r_i} \tag{3.18}$$

where

$$\hat{\mathbf{M}}_i = \left[\begin{array}{c:c:c} \mathbf{I} & \mathbf{0} & \mathbf{0} \\ \hdashline -\mathbf{M}_{i1} \dots -\mathbf{M}_{i,i-1} & \mathbf{0} & -\mathbf{M}_{i,i+1} \dots -\mathbf{M}_{i,M} \\ \hdashline \mathbf{0} & \mathbf{0} & \mathbf{I} \end{array}\right] \tag{3.19}$$

with $\mathbf{M}_{ij} = \mathbf{Z}_{ii}^{-\frac{1}{2}} \mathbf{Z}_{ij} \mathbf{Z}_{jj}^{-\frac{1}{2}}$ being the (i, j) block of the partitioned operator $\mathbf{M} = \mathbf{D}^{-\frac{1}{2}} \mathbf{Z} \mathbf{D}^{-\frac{1}{2}}$. We can combine the above update equations for all the players into the single equation in (3.17) with $\mathbf{R}_s = \mathbf{D}^{-\frac{1}{2}} \hat{\mathbf{M}}_M \dots \hat{\mathbf{M}}_1 \mathbf{D}^{\frac{1}{2}}$ and $r_s = -r_M - \sum_{i=0}^{M-1} \mathbf{D}^{-\frac{1}{2}} \hat{\mathbf{M}}_M \dots \hat{\mathbf{M}}_{i+1} \mathbf{D}^{\frac{1}{2}} r_i$. We define operators $\mathbf{M}_l = \mathbf{D}^{-\frac{1}{2}} \mathbf{Z}_l \mathbf{D}^{-\frac{1}{2}}$ and $\mathbf{M}_u = \mathbf{D}^{-\frac{1}{2}} \mathbf{Z}_u \mathbf{D}^{-\frac{1}{2}}$ consisting of the strict lower triangular and strict upper triangular parts of $\mathbf{M}$, respectively. Using a recursive argument starting from $i = M$ to $i = 1$, we can show that the first i-1 rows of $(\mathbf{I} + \mathbf{M}_l)\hat{\mathbf{M}}_M \dots \hat{\mathbf{M}}_i$ match that of $\mathbf{I} + \mathbf{M}_l$ while its last M-i+1 rows match that of $-\mathbf{M}_u$. This observation leads to the relations $\hat{\mathbf{M}}_M \dots \hat{\mathbf{M}}_1 = -(\mathbf{I} + \mathbf{M}_l)^{-1}\mathbf{M}_u$ and

$$\operatorname{diag}(\mathbf{0}_{(M-1)\times(M-1)}, \mathbf{I}) + \sum_{i=1}^{M-1} \hat{\mathbf{M}}_M \dots \hat{\mathbf{M}}_{i+1} \operatorname{diag}(\mathbf{0}_{(i-1)\times(i-1)}, \mathbf{I}, \mathbf{0}_{(M-i)\times(M-i)}) = (\mathbf{I}+\mathbf{M}_l)^{-1}$$

which then yield the simplified expressions $\mathbf{R}_s = -(\mathbf{D}+\mathbf{Z}_l)^{-1}\mathbf{Z}_u$ and $r_s = -(\mathbf{D}+\mathbf{Z}_l)^{-1}\delta$ corresponding to (3.17).

These update schemes along with appropriately chosen contraction conditions have been used for quadratic game problems (see [70]), to provide a mechanism that converges to a Nash equilibrium. The next remark says that the update equations presented here also apply to quadratic games (like those considered in [72, 73]).

Remark 7. *Consider a static quadratic game problem with the cost function corresponding to player i being $J_i(\xi, u_1, \dots, u_M) = u_i^T Z_{ii}(\xi) u_i + 2\sum_{j \neq i} u_i^T Z_{ij}(\xi) u_j + 2d_i^T(\xi) u_i$ with Prob$\{Z_{ii}(\xi) \succeq \underline{a} I\} = 1$ for $i \in \mathcal{J}$. For this setup, if we use the operator definition (3.6), we obtain the same expressions (3.17) and (3.15) for the sequential and parallel updates respectively. However note that unlike our team formulation, in games it may be that $\mathbf{Z}_l^* \neq \mathbf{Z}_u$.*

Some of the analysis performed in the subsequent subsections can also be applied to such game problems. However we will not pursue this direction any further because the stronger results that we obtain for team problems do not hold in general for game problems. In the next subsection, we will show that for the quadratic team setup, sequential update scheme always converges to the team optimal solution for general M-player scenario, while the parallel update scheme is guaranteed to converge only for $M = 2$. It was previously known (see [20]) that these convergence results hold for $M = 2$ under a non-stochastic setup.

3.7 Convergence of Update Schemes

In this subsection, we will examine the convergence of the two update schemes described earlier. However due to the the guaranteed convergence of sequential update scheme, we will focus mainly on this scheme. Before presenting the main result, we have a couple of useful lemmas. The following lemma is similar to Lemma 4 but for an operator setting.

Lemma 8. *Consider a Hilbert space $\mathcal{H}$ and an indexed sequence $\mathbf{A}_k \in \mathcal{L}(\mathcal{H})$, $k \in \mathbb{N}_0$. If there exists a sequence of positive-definite $\mathbf{X}_k$ for $k \in \mathbb{N}_0$ and positive constants a, b and ϵ satisfying*

$$a\mathbf{I} \preceq \mathbf{X}_k \preceq b\mathbf{I} \quad \textit{and} \quad \mathbf{X}_{k+1} - \mathbf{A}_k \mathbf{X}_k \mathbf{A}_k^* \succeq \epsilon \mathbf{I} \quad \textit{for } k \in \mathbb{N}_0$$

then $\|\mathbf{A}_{k\text{-}1} \dots \mathbf{A}_0\|_{\mathcal{H}\to\mathcal{H}} \leq \sqrt{\frac{b}{a}} \left(1 - \frac{\epsilon}{b}\right)^{k/2}$.

Proof. The proof uses a standard Lyapunov type argument which we provide here completeness. For any $q \in \mathcal{H}$, the given inequalities yield $a\|q\|_{\mathcal{H}}^2 \leq \langle q, \mathbf{X}_k q\rangle_{\mathcal{H}} \leq b\|q\|_{\mathcal{H}}^2$ and

$$\begin{aligned} \langle q, \mathbf{X}_{k+1} q\rangle_{\mathcal{H}} - \langle q, \mathbf{A}_k \mathbf{X}_k \mathbf{A}_k^* q\rangle_{\mathcal{H}} &\geq \epsilon \|q\|_{\mathcal{H}}^2 \geq \frac{\epsilon}{b} \langle q, \mathbf{X}_{k+1} q\rangle_{\mathcal{H}} \\ \Rightarrow \langle q, \mathbf{A}_k \mathbf{X}_k \mathbf{A}_k^* q\rangle_{\mathcal{H}} &\leq \left(1 - \frac{\epsilon}{b}\right) \langle q, \mathbf{X}_{k+1} q\rangle_{\mathcal{H}} \end{aligned}$$

Using $\tilde{\mathbf{A}}_{k\text{-}1} := \mathbf{A}_{k\text{-}1} \dots \mathbf{A}_0$, the above leads to

$$\left\langle q, \tilde{\mathbf{A}}_{k\text{-}1} \mathbf{X}_0 \tilde{\mathbf{A}}_{k-1}^* q \right\rangle_{\mathcal{H}} \leq \left(1 - \frac{\epsilon}{b}\right)^k \langle q, \mathbf{X}_k q\rangle_{\mathcal{H}} \Rightarrow \|\tilde{\mathbf{A}}_{k\text{-}1}^* q\|_{\mathcal{H}}^2 \leq \frac{b}{a} \left(1 - \frac{\epsilon}{b}\right)^k \|q\|_{\mathcal{H}}^2$$

□

Lemma 9. *For a self-adjoint operator $\mathbf{Z}$ satisfying $\underline{a}\mathbf{I} \preceq \mathbf{Z} \preceq \bar{a}\mathbf{I}$, its block lower triangular part satisfies $\|\mathbf{D} + \mathbf{Z}_l\|_{\mathcal{K}\to\mathcal{K}} \leq M\bar{a}$ and hence $\|(\mathbf{D} + \mathbf{Z}_l)^{-1}\|_{\mathcal{K}\to\mathcal{K}} \geq \frac{1}{M\bar{a}}$.*

Proof. We have

$$\begin{aligned}\|\mathbf{D} + \mathbf{Z}_l\|_{\mathcal{K}\to\mathcal{K}} &\leq \sum_{i=1}^{M} \left\| \begin{bmatrix} \mathbf{Z}_{ii} \\ \vdots \\ \mathbf{Z}_{Mi} \end{bmatrix} \right\|_{\mathcal{K}_i \to \mathcal{K}_i \oplus \cdots \oplus \mathcal{K}_M} \leq \sum_{i=1}^{M} \left\| \begin{bmatrix} \mathbf{Z}_{1i} \\ \vdots \\ \mathbf{Z}_{Mi} \end{bmatrix} \right\|_{\mathcal{K}_i \to \mathcal{K}} \\ &= \sum_{i=1}^{M} \sup_{\alpha=(0,\ldots,0,\alpha_i,0,\ldots,0)\neq 0} \frac{\|\mathbf{Z}\alpha\|_{\mathcal{K}}}{\|\alpha\|_{\mathcal{K}}} \leq M \sup_{\alpha \neq 0} \frac{\|\mathbf{Z}\alpha\|_{\mathcal{K}}}{\|\alpha\|_{\mathcal{K}}} = M\bar{a}.\end{aligned}$$

□

Following theorem is the main result of this chapter and proves that iterations in (3.17) converge to the team optimal strategy and provides explicit bounds for convergence.

Theorem 10. *Given $\gamma^{(0)} \in \mathcal{K}$ and $\gamma^{(k)}$ defined by sequential update in (3.17), the following hold*

(i) The sequence $\gamma^{(k)}$ converges in $\mathcal{K}$ to an element $\gamma^\star = -\mathbf{Z}^{-1}\delta$, with the following exponential bound on the rate

$$\|\gamma^{(k)} - \gamma^\star\|_{\mathcal{K}} \leq \sqrt{\frac{\bar{a}}{\underline{a}}} \left\{ 1 - \left(\frac{\underline{a}}{M\bar{a}}\right)^2 \right\}^{k/2} \|\gamma^{(0)} - \gamma^\star\|_{\mathcal{K}}. \tag{3.20}$$

(ii) Given any $\gamma \in \mathcal{K}$, the following inequality holds

$$\underline{a}\|\gamma - \gamma^\star\|_{\mathcal{K}}^2 \leq \bar{J}(\gamma) - \bar{J}(\gamma^\star) \leq \bar{a}\|\gamma - \gamma^\star\|_{\mathcal{K}}^2. \tag{3.21}$$

Thus, if $\gamma \in \mathcal{K}$ and $\bar{J}(\gamma) \leq \bar{J}(\gamma^\star)$ then $\gamma = \gamma^\star$.

(iii) The sequence $\bar{J}(\gamma^{(k)})$ is non-increasing and converges to $\bar{J}(\gamma^\star)$ exponentially.

(iv) The element $\gamma^\star$ is the unique solution in $\mathcal{K}$ of the equations

$$\gamma_i = \mathbf{\Gamma}_i \gamma_{-i} \quad \textit{for } i = 1, \ldots, M. \tag{3.22}$$

Proof. We start by proving (i). For this, we first note the following relationships

$$(\mathbf{D} + \mathbf{Z}_l)(\mathbf{Z}^{-1} - \mathbf{R}_s\mathbf{Z}^{-1}\mathbf{R}_s^*)(\mathbf{D} + \mathbf{Z}_l)^* = (\mathbf{Z} - \mathbf{Z}_u)\mathbf{Z}^{-1}(\mathbf{Z} - \mathbf{Z}_u)^* - \mathbf{Z}_u\mathbf{Z}^{-1}\mathbf{Z}_u^* = \mathbf{D}.$$

The first equality above is obtained by using the relations $\mathbf{D} + \mathbf{Z}_l = \mathbf{Z} - \mathbf{Z}_u$ and $(\mathbf{D} + \mathbf{Z}_l)\mathbf{R}_s = -\mathbf{Z}_u$. The above leads to

$$\mathbf{Z}^{-1} - \mathbf{R}_s\mathbf{Z}^{-1}\mathbf{R}_s^* = (\mathbf{D} + \mathbf{Z}_l)^{-1}\mathbf{D}(\mathbf{D} + \mathbf{Z}_l)^{-*} \succeq \frac{\underline{a}}{M^2\bar{a}^2}\mathbf{I} \tag{3.23}$$

The last inequality is due to Lemma 9 along with $\mathbf{D} \succeq \underline{a}\mathbf{I}$. Since $\frac{1}{\bar{a}}\mathbf{I} \preceq \mathbf{Z}^{-1} \preceq \frac{1}{\underline{a}}\mathbf{I}$, we obtain the following using Lemma 8

$$\|\mathbf{R}_s^k\|_{\mathcal{K}\to\mathcal{K}} \leq \sqrt{\frac{1/\underline{a}}{1/\bar{a}}}\left(1 - \frac{\underline{a}/(M^2\bar{a}^2)}{1/\underline{a}}\right)^{k/2}. \tag{3.24}$$

The inequality implies that $\|\mathbf{R}_s^k\|_{\mathcal{K}\to\mathcal{K}}$ is a contraction for sufficiently large k. Now using (3.17) we have

$$\gamma^{(k)} = \mathbf{R}_s^k\gamma^{(0)} + \left(\mathbf{I} + \mathbf{R}_s + \cdots + \mathbf{R}_s^{k-1}\right) r_s,$$

which along with the contraction of $\|\mathbf{R}_s^k\|_{\mathcal{K}\to\mathcal{K}}$ allows us to show that $\gamma^{(k)}$ is a Cauchy sequence and as $k \to \infty$ has a limit in $\mathcal{K}$, which we call $\gamma^\star$. We can further show that the optimal strategy has the following expression

$$\gamma^\star = \lim_{k\to\infty}\left(\mathbf{R}_s^l\gamma^{(0)} + \sum_{l=0}^{k-1}\mathbf{R}_s^l r_s\right) = (\mathbf{I} - \mathbf{R}_s)^{-1} r_s = -\left(\mathbf{I} + (\mathbf{D}+\mathbf{Z}_l)^{-1}\mathbf{Z}_u\right)^{-1}(\mathbf{D}+\mathbf{Z}_l)^{-1}\delta = -\mathbf{Z}^{-1}\delta.$$

The earlier expression for $\gamma^{(k)}$ leads to

$$\gamma^{(k)} - \gamma^\star = \mathbf{R}_s^k(\gamma^{(0)} - \gamma^\star).$$

By taking the norm of the above and using the bound (3.24), we obtain the inequality in (3.20).

To prove (ii), we use (3.11) to obtain the following for any $\gamma \in \mathcal{K}$

$$\bar{J}(\gamma) - \bar{J}(\gamma^\star) = \langle \gamma, \mathbf{Z}\gamma+2\delta\rangle_{\mathcal{K}} - \langle \mathbf{Z}\gamma^\star+2\delta, \delta\rangle_{\mathcal{K}} = \langle \gamma - \gamma^\star, \mathbf{Z}(\gamma - \gamma^\star)\rangle_{\mathcal{K}}. \tag{3.25}$$

The last equality uses $\gamma^\star = -\mathbf{Z}^{-1}\delta$ and the self-adjoint property of $\mathbf{Z}$ for intermediate steps. This along with Lemma 6 leads to (3.21). Now, due to the lower bound obtained in (3.21), the optimality of $\gamma^\star$ along with its uniqueness in achieving the cost is established.

We now prove (iii). Due to the construction (3.16), we have

$$\bar{J}(\gamma_1^{(k+1)}, \ldots, \gamma_i^{(k+1)}, \gamma_{i+1}^{(k)}, \ldots, \gamma_M^{(k)}) \quad \leq \quad \bar{J}(\gamma_1^{(k+1)}, \ldots, \gamma_{i-1}^{(k+1)}, \gamma_i^{(k)}, \ldots, \gamma_M^{(k)})$$

which translates to $\bar{J}(\gamma^{(k+1)}) \leq \bar{J}(\gamma^{(k)})$, showing that the cost is non-increasing with the stages. The upper bound in (3.21) allows us to show that $\bar{J}$ is continuous at $\gamma^\star$ with respect to $\|\cdot\|_{\mathcal{K}}$, since for any $\epsilon > 0$ we can set $\delta = 2\bar{a}\epsilon^2$ to have $\|\gamma - \gamma^\circ\|_{\mathcal{K}} < \epsilon \Rightarrow |\bar{J}(\gamma) - \bar{J}(\gamma^\circ)| < \delta$ for all $\gamma \in \mathcal{K}$. We already know that starting from any $\gamma^{(0)} \in \mathcal{K}$, the iterations converge to $\gamma^\star$ in $\mathcal{K}$. This along with the continuity of $\bar{J}$ proves that $\bar{J}(\gamma^{(k)})$ converges to $\bar{J}(\gamma^\star)$. To show that this convergence is exponential, we combine the right inequality in (3.21) along with (3.20).

To show (iv), we note that equation (3.22) is same as $\gamma = \mathbf{R}_s\gamma + r_s$, whose unique fixed point being $\gamma^\star$ is immediate from (i). $\square$

Note that the team optimal strategy $\gamma^\star$ obtained in the above theorem is same as γ° described in Section 3.5. Further it is straightforward to verify that the conditions of optimality in (3.10) is same as the equations in (3.22). Thus for static quadratic teams, the above theorem also provides an alternative proof to [16] for existence and uniqueness of this team optimal strategy and the necessary conditions associated with it. We point out that, for the case of $M = 2$, the contraction property in (3.24) can also be proved by applying Schur complement formula to $\mathbf{Z}$ instead of the using a Lyapunov argument as done here.

The next corollary shows that the players' order at each stage of the update (3.17) can be changed without affecting the exponential convergence.

Corollary 11. *Let $\sigma_k : \mathcal{J} \to \mathcal{J}$ be a sequence of permutations on the player index set. Then, a sequential update where player strategies are computed in the order $\sigma_k(1), \dots, \sigma_k(M)$ at stage k, converges to the team optimal strategy γ° and is bounded by the convergence rate in (3.20).* $\square$

Proof. We first define the permutation operator $\mathbf{\Pi}_{\sigma_k} : \mathcal{K} \to \mathcal{K}_{\sigma_k(1)} \oplus \cdots \oplus \mathcal{K}_{\sigma_k(M)}$ through $\begin{bmatrix} \gamma_1 \\ \vdots \\ \gamma_M \end{bmatrix} \mapsto \begin{bmatrix} \gamma_{\sigma_k(1)} \\ \vdots \\ \gamma_{\sigma_k(M)} \end{bmatrix}$. Then, we use the notation in (3.19) to define a stage-varying version of the operator $\mathbf{R}_s$ in (3.17) as $\mathbf{R}_{s,k} := \mathbf{D}^{-\frac{1}{2}}\mathbf{\Pi}^*_{\sigma_k}\hat{\mathbf{M}}_{\sigma_k(M)} \dots \hat{\mathbf{M}}_{\sigma_k(1)}\mathbf{\Pi}_{\sigma_k}\mathbf{D}^{\frac{1}{2}}$ at stage k. It can be shown that this operator satisfies $\mathbf{X}_{k+1} - \mathbf{R}_{s,k}\mathbf{X}_k\mathbf{R}^*_{s,k} \succeq \frac{a}{M^2\bar{a}^2}\mathbf{I}$ with $\mathbf{X}_k = \mathbf{Z}^{-1}$ resulting in $\|\mathbf{R}_{s,k-1} \dots \mathbf{R}_{s,0}\|_{\mathcal{K}\to\mathcal{K}} \leq \sqrt{\frac{\bar{a}}{\underline{a}}}\left(1 - \frac{a^2}{M^2\bar{a}^2}\right)^{k/2}$ using Lemma 8. Following steps similar to those in Theorem 10, the bound in (3.20) can be obtained. $\square$

Note that in contrast to the above corollary, players' ordering alters the convergence properties in a game setting, observed for example in [74].

The convergence of the parallel update scheme for $M = 2$ can be examined by converting it to a sequential update as summarized in the next remark. This connection between the convergence properties of the parallel and sequential updates for $M = 2$ has been observed in [75].

Remark 12. *For $M = 2$, the parallel update in (3.14) for player i at stages $2k$ and $2k+1$ can be combined as $\gamma_i^{(2k+2)} = \mathbf{\Gamma}_i\mathbf{\Gamma}_{-i}\gamma_i^{(2k)}$ which is same as the strategy after stage k of a sequential update starting with same initial strategy $\gamma_i^{(0)}$.*

For $M > 2$, as pointed out by [20], the parallel update scheme may fail to converge. This can be seen by examining the following simple 3-player test case with ξ being empty i.e. a matrix team problem where observations are irrelevant.

$$Z(\xi) = \begin{bmatrix} 1 & 0.9 & 0.9 \\ 0.9 & 1 & 0.9 \\ 0.9 & 0.9 & 1 \end{bmatrix}, \quad d(\xi) = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}.$$

3.8 Numerical Simulation

The global convergence property developed in the previous section provides us with a mechanism to compute the optimal strategy by sequentially applying the best response mappings. While such a mechanism has been suggested in the past for both team and game problems, the generality of the convergence property presented in the previous section allows us to apply this mechanism to a wide range of setups. In order to demonstrate that this scheme can be effective to compute strategies (which are otherwise hard to obtain), we present an example where strategies are computed numerically.

Let us consider the following one-step scalar dynamics with two players

$$x_+ = A(x) + u_1 + u_2, \quad y_i = x + v_i. \tag{3.26}$$

with $A(x) = -1^{\lfloor \frac{x}{10} \rfloor}$, $x \sim (0, X)$ and $v_i \sim \mathcal{N}(0, V_i)$. The information set for player i contains only y_i and the cost function is given by $J(x, u_1, u_2) = Qx_+^2 + R_1u_1^2 + R_2u_2^2$. With $\xi = (x, v_1, v_2)$, the above results in $d_i(\xi) = QA(x)$, $c(\xi) = A(x)^T QA(x)$ and

$$Z_{ij}(\xi) = \begin{cases} R_i + Q & \text{for } i = j \\ Q & \text{otherwise} \end{cases}$$

under the notation of previous section. The cost and noise parameters are chosen as below

$$Q = 1, R_1 = 0.5, R_2 = 0.1, V_1 = 0.01, V_2 = 0.5.$$

Since $V_1 < V_2$, Player 1 has more reliable observations than Player 2. However Player 1's action is penalized more than Player 2's (R_1 is larger than R_2). With these parameters we find the (approximate)r team optimal strategies by computing the following for sufficiently large k

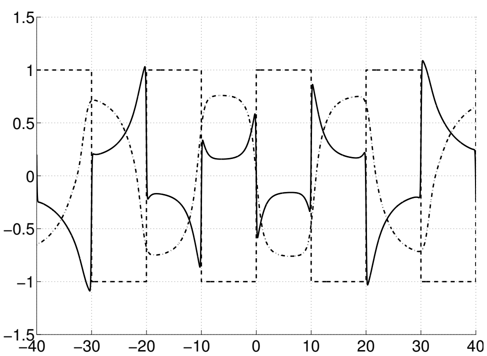
$$\gamma_i^{(k)} = (\mathbf{\Gamma}_i\mathbf{\Gamma}_{\text{-}i})^k(0) = (\mathbf{I} + \hat{\mathbf{\Gamma}}_i + \hat{\mathbf{\Gamma}}_i^2 + \cdots + \hat{\mathbf{\Gamma}}_i^{k\text{-}1})(\tilde{\mathbf{\Gamma}}_i\delta_{\text{-}i} - \mathbf{Z}_{ii}^{\text{-}1}\delta_i). \tag{3.27}$$

The above expression uses

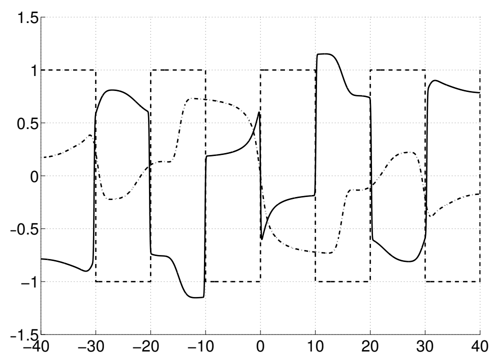
$$(\hat{\mathbf{\Gamma}}_i\gamma_i)(y_i) = (\mathbf{Z}_{ii}^{-1}\mathbf{Z}_{i,\text{-}i}\mathbf{Z}_{\text{-}i,\text{-}i}^{-1}\mathbf{Z}_{\text{-}i,i}\gamma_i)(y_i) = q_i\,\mathbb{E}[\mathbb{E}[\gamma_i(y_i)|y_{\text{-}i}]|y_i]$$
$$(\tilde{\mathbf{\Gamma}}_i\delta_{\text{-}i})(y_i) = (\mathbf{Z}_{ii}^{-1}\mathbf{Z}_{i,\text{-}i}\mathbf{Z}_{\text{-}i,\text{-}i}^{-1}\delta_{\text{-}i})(y_i) = q_i\,\mathbb{E}[\mathbb{E}[A(x)|y_{\text{-}i}]|y_i]$$

with $q_i = \frac{Q^2}{(R_i+Q)(R_{\text{-}i}+Q)}$. Each term in the summation (3.27) can be computed by numerically integrating $A(x)$ with the appropriate conditional distributions. The distribution associated with each of these conditional expectations is a Gaussian with the mean being an affine function of y_i. Further details regarding this calculation are skipped.

We plot the strategies for two different prior distributions of x in Figure 3.1 and try to explain the behavior qualitatively. One can expect that the players would try to cancel out as much of $A(x)$ as possible with $-(u_1 + u_2)$ and thus the strategy of player i at $y_i = a$ would possibly be of opposite sign as $A(a)$. Though the exact values of the strategies would be in accordance with the cost coefficients Q, R_1 and R_2 and how reliable the observations are. Due to the cost structure, Player 2 applies a larger control action, while Player 1 applies a smaller action and tries to correct Player 2's action when possible. This correction is evident when there is a jump in $A(x)$, which Player 1 can detect more reliably and hence is more aggressive than Player 2. This reasoning seems to hold well for values of x close to the mean (zero). However when x is far away from zero, the prior distribution of x seems to have a strong effect. This is evident from the fact that Player 2's strategy holds the same sign as $A(x)$ even though one moves away from the mean. For example, the effect can be seen in the plot of $\gamma_2(y_2)$ in Figure 3.1(b) for values of y_2 around 10 to 13. This implies that, Player 2 believes that the actual x is smaller than y_2 (in absolute value) owing to the prior distribution. In such scenarios, Player 1 tries to compensate for Player 2's behaviour by applying a larger control.



(a) $x \sim \mathcal{N}(0, 10)$



(b) $x \sim \mathcal{N}(0, 1)$

Figure 3.1: Plots showing $A(x)$ vs x (dashed), $\gamma_1(y_1)$ vs y_1 (solid) and $\gamma_2(y_2)$ vs y_2 (dot-dashed) for two different prior distributions of x

3.9 Tools for Analytical Computation of Strategies

In this section we present some results using Theorem 10, which can be helpful in obtaining an analytical expression for the team optimal strategy or approximations to it. These results will be used in the forthcoming chapters to solve the static problems encountered in dynamic team setting. First, we have the following result.

Corollary 13. *Suppose subsets $\mathcal{S}_1 \subset \mathcal{K}_1$, $\mathcal{S}_2 \subset \mathcal{K}_2$, ..., $\mathcal{S}_M \subset \mathcal{K}_M$ are closed and let $\mathcal{S} := \oplus_{i=1}^{M} \mathcal{S}_i$. If for each $\gamma \in \mathcal{S}$, the condition $\mathbf{\Gamma}_i \gamma_{\text{-}i} \in \mathcal{S}_i$ holds for all $i \in \mathcal{J}$, then $\gamma^\circ \in \mathcal{S}$.* □

The proof of the above is straightforward, since the sequential update with a starting point $\gamma^{(0)} \in \mathcal{S}$ stays in $\mathcal{S}$ and ultimately converges to the team optimal solution. The above corollary can be useful to isolate the structure of the strategy e.g. if one has a guess for the structure of the optimal strategy. One could possibly gain intuition about such a structure by evaluating the steps of sequential update for a few iterations with $\gamma^{(0)} = 0$. If we have a structural description of the strategy described by a subspace $\mathcal{S}$, we may be able to write a set of linear equations using (3.10). The unique solution to this set of equations is the team optimal solution. In particular, when $\mathcal{S}$ is finite dimensional, we can obtain the team optimal solution directly as explained in the following result.

Corollary 14. *Suppose sets $\mathcal{S}_i$ in Corollary 13 are finite dimensional subspaces with basis $\{\psi_l^i\}_{l=1}^{b_i}$ for each $i \in \mathcal{J}$. Then the optimal strategy can be obtained by solving the following set of linear equations in the coefficients $\{a_l^i\}_{l=1}^{b_i}$ corresponding to the optimal strategy $\gamma_i^\circ = \sum_{l=1}^{b_i} a_l^i \psi_l^i$*

$$\sum_{j=1}^{M} \sum_{p=1}^{b_j} a_p^j \left\langle \psi_l^i, \mathbf{Z}_{ii}^{\text{-}1} \mathbf{Z}_{ij} \psi_p^j \right\rangle_{\mathcal{K}_i} + \left\langle \psi_l^i, \mathbf{Z}_{ii}^{\text{-}1} \delta_i \right\rangle_{\mathcal{K}_i} = 0$$

for $l = 1, \ldots, b_i$ and $i = 1, \ldots, M$. □

When $\mathcal{S}$ is infinite dimensional, the linear equations obtained through (3.10) may not be easily solvable. However an approximation of the team optimal strategy may still be computable by using a finite truncation of the sequential update iterations. We will apply all the ideas presented in this subsection to a specific team application in the next section.

Chapter 4

Static Teams with Local Parameter Knowledge

In this chapter we introduce a static problem where the underlying random variable comprises of independent components x (state) and θ (parameter). Players have partial knowledge about the parameter θ. We will consider the cases of both partial and full observations of x and obtain corresponding team optimal strategies using techniques developed in previous chapter. The solution developed here for the static problem will be helpful in later chapters to solve dynamic team problems with local model information.

4.1 Setup

Consider a special case of M-player static quadratic team problem with random variable $\xi = (x, \theta, v_1, \ldots, v_M)$. Here x is the random state assuming values in $\mathbb{R}^n$ and $\theta = (\theta_1, \ldots, \theta_M)$ constitutes the players' local *types* or parameters with θ_i taking values in Θ_i which is assumed to be a product of a Euclidean space and a finite set. The local types θ_i can be viewed as partial observations of the global parameter θ, which collectively determine θ. The measurements available to players are of the form

$$y_i = C_i(\theta_i)x + v_i \quad \text{for } i \in \mathcal{J} \tag{4.1}$$

with v_i being the measurement noise and C_i dependent only on the local type θ_i. Thus information available to player i consists of the local types and measurements as

$$\mathbb{I}_i = (\theta_i, y_i). \tag{4.2}$$

We consider a quadratic cost function of the form

$$\begin{aligned} J(\xi, u_1, \ldots, u_M) &= u^T Z(\theta) u + 2u^T Y(\theta) x + c(x, \theta) \\ &= \sum_{i=1}^{M} \sum_{j=1}^{M} u_i^T Z_{ij}(\theta) u_j + \sum_{i=1}^{M} 2u_i^T Y_i(\theta) x + c(x, \theta) \end{aligned} \tag{4.3}$$

which results in $Z_{ij}(\xi)$, $d_i(\xi)$ and $c(\xi)$ of Chapter 3 to be written as $Z_{ij}(\theta)$, $Y_i(\theta)x$ and $c(x,\theta)$ respectively.

We further make the following assumptions for this problem

Assumption 15. *(i) There exist positive constants $\underline{a}$ and $\bar{a}$ such that $\underline{a}I \preceq Z(\theta) \preceq \bar{a}I$ for all $\theta \in \Theta = \Theta_1 \times \cdots \times \Theta_M$*

(ii) $x \sim \mathcal{N}(\bar{x}, X)$ and $v_i \sim \mathcal{N}(0, V_i)$ for $i = 1, \ldots, M$

(iii) x, v_1, …, v_M and θ are independent of each other

(iv) All players have complete knowledge of the maps $Z(\cdot)$, $Y(\cdot)$, $c(\cdot)$, $\{C_i(\cdot)\}_{i=1}^M$ and the underlying statistics.

Strictly speaking, the matrix valued functions of types defined here are deterministic functions and we should be using them with their arguments as $Y(\theta)$ or $C_i(\theta_i)$. In order to keep notation compact, however we will sometimes treat them as random matrices without explicitly writing the type arguments.

The following lemma lists some useful definitions and properties making use of linear estimation theory.

Lemma 16. *Consider the observation model in (4.1) and information structure (4.2).*

(a) The distribution of the random vector x conditioned on local information of player i is given by $\mathcal{N}(\hat{x}_i, \hat{X}_i)$ with

$$\hat{x}_i := \mathbb{E}[x|\mathbb{I}_i] = \bar{x} + L_i(y_i - C_i\bar{x}) \text{ and}$$
$$\hat{X}_i := \mathbb{E}[(x - \hat{x}_i)(x - \hat{x}_i)^T|\mathbb{I}_i] = (I - L_iC_i)X$$

where $L_i(\theta_i) := XC_i^T(V_i + C_iXC_i^T)^{-1}$ is the local Kalman gain.

(b) Define $e_i := y_i - C_i\bar{x} = C_i(x - \bar{x}) + v_i$, then $\mathbb{E}[e_i] = 0$,

$$\mathbb{E}[e_ie_j^T] = \begin{cases} \mathbb{E}[V_i + C_iXC_i^T] \text{ for } i = j \\ \mathbb{E}[C_iXC_j^T] \quad \text{for } i \neq j \end{cases}$$

and $\mathbb{E}[xe_i^T] = X\mathbb{E}_{\theta_i}[C_i^T]$. As a result for G, a matrix valued function of θ of appropriate dimension, we have

- $\mathbb{E}[Ge_i] = 0$
- $\mathbb{E}[e_i^TGe_i] = \mathbb{E}[Tr(Ge_ie_i^T)] = Tr\big(\mathbb{E}[G(V_i + C_iXC_i^T)]\big)$
- $\mathbb{E}[Ge_j|\mathbb{I}_i] = \mathbb{E}[GC_j|\theta_i]L_ie_i$ *for $j \neq i$*

$\square$

Note that L_i being a function of C_i, is also dependent on θ_i. However for simplicity, we choose to suppress this dependence. The above lemma uses standard properties (see for example [76]), and the proof is skipped. Note that the definitions above and those to follow use the same variable to represent a random variable and the value it takes. Again, this is done to keep the notation compact.

The previous lemma leads to

$$\delta_i(\mathbb{I}_i) = \mathbb{E}[d_i(\xi)|\mathbb{I}_i] = \hat{Y}_i(\theta_i)\hat{x}_i = \hat{Y}_i(\theta_i)(\bar{x} + L_i(\theta_i)e_i)$$

where we have used $\hat{Y}_i(\theta_i) = \mathbb{E}[Y_i(\theta)|\theta_i]$. The best response in (3.13) then evaluates point-wise to the following

$$\big(\mathbf{\Gamma}_i(\gamma_{\text{-}i})\big)(\mathbb{I}_i) = -\hat{Z}_i(\theta_i)^{-1}\Big\{\mathbb{E}\Big[\sum_{j\neq i} Z_{ij}(\theta)\gamma_j(\mathbb{I}_j)|\mathbb{I}_i\Big] + \hat{Y}_i(\theta_i)\hat{x}_i\Big\} \tag{4.4}$$

where $\hat{Z}_i(\theta_i) := \mathbb{E}\left[Z_{ii}(\theta)\middle|\theta_i\right]$.

4.2 Team Optimal Solution

We define the following subspaces of $\mathcal{K}_i$

$$\mathcal{Z}_i := \{\gamma_i \in \mathcal{K}_i : \gamma_i = K_i(\theta_i)\bar{x},\ K_i \in \mathcal{R}_i^{m_i \times n}\}$$
$$\mathcal{W}_i := \{\gamma_i \in \mathcal{K}_i : \gamma_i = K_i(\theta_i)e_i,\ K_i \in \mathcal{R}_i^{m_i \times n}\}$$

with $\mathcal{R}_i^{a\times b}$ being the space of $a \times b$ dimensional matrix valued functions of local type θ_i. Using the above, we define $\mathcal{Z} = \mathcal{Z}_1 \times \cdots \times \mathcal{Z}_M$ and $\mathcal{W} = \mathcal{W}_1 \times \cdots \times \mathcal{W}_M$ which are subspaces of $\mathcal{K}$. It can be verified that $\mathcal{Z}$ and $\mathcal{W}$ are orthogonal with respect to the inner-product $\langle \cdot , \cdot \rangle_{\mathcal{K}}$ due to $\mathbb{E}[\bar{x}^T M_i(\theta_i)e_i] = 0$ for $M_i \in \mathcal{R}_i^{n\times n}$. We can thus define the (internal) direct sum $\mathcal{Z} \oplus \mathcal{W}$.

Theorem 17. *For the static team problem described by information and cost structures in (4.2) and (4.3) respectively, the team optimal strategy γ° lies in the subspace $\mathcal{Z} \oplus \mathcal{W}$. More specifically*

$$\gamma_i^\circ(\mathbb{I}_i) = K_i^s(\theta_i)\bar{x} + K_i^o(\theta_i)e_i \tag{4.5}$$

where K_i^s and K_i^o are obtained by solving the following equations

$$\hat{Z}_i K_i^s \bar{x} + \sum_{j\neq i} \mathbb{E}[Z_{ij} K_j^s | \theta_i]\bar{x} + \hat{Y}_i \bar{x} = 0 \tag{4.6a}$$

$$\hat{Z}_i K_i^o e_i + \sum_{j\neq i} \mathbb{E}[Z_{ij} K_j^o C_j | \theta_i] L_i e_i + \hat{Y}_i L_i e_i = 0 \tag{4.6b}$$

for $i = 1, \dots, M$. The resulting optimal expected cost is given by

$$\bar{J}(\gamma^\circ) = \bar{x}^T \left(\sum_{i=1}^{M} \mathbb{E}[(K_i^s)^T Y_i] \right) \bar{x} + \mathbb{E}[c(x,\theta)] + \sum_{i=1}^{M} Tr\big(\mathbb{E}[(K_i^o)^T Y_i L_i (V_i + C_i X C_i^T)]\big). \tag{4.7}$$

Proof. For any strategy $\gamma \in \mathcal{Z} \oplus \mathcal{W}$, we can use (4.4) and Lemma 16 to verify that $\mathbf{\Gamma}_i(\gamma_{\text{-}i}) \in \mathcal{Z}_i \oplus \mathcal{W}_i$ holds for $i = 1, \dots, M$. We can then use Corollary 13, to assert that the team optimal strategy γ° also lies in $\mathcal{Z} \oplus \mathcal{W}$. Since the optimal solution satisfies (3.10), we can project the corresponding equation onto the two orthogonal subspaces as

$$\mathcal{P}_{\mathcal{Z}}(\mathbf{Z}\gamma^\circ + \delta) = 0, \quad \mathcal{P}_{\mathcal{W}}(\mathbf{Z}\gamma^\circ + \delta) = 0.$$

where $\mathcal{P}_{\mathcal{Z}}$ and $\mathcal{P}_{\mathcal{W}}$ are projection operators on $\mathcal{Z}$ and $\mathcal{W}$ respectively. γ° being in $\mathcal{Z} \oplus \mathcal{W}$ has the structure (4.5) and upon substitution into the above equations we obtain (4.6). Note that these equations may not have unique solutions for K_i^s and K_i^o, but the strategies that they describe in spaces $\mathcal{Z}$ and $\mathcal{W}$ are unique in $\mathcal{K}$.

We can evaluate the optimal cost using (3.12), with

$$\begin{aligned}\langle \gamma^\circ, \delta \rangle_{\mathcal{K}} &= \sum_{i=1}^{M} \mathbb{E}\Big[\big(K_i^s(\theta_i)\bar{x} + K_i^o(\theta_i)e_i\big)^T \hat{Y}_i(\theta_i)\big(\bar{x} + L_i(\theta_i)e_i\big)\Big] \\ &= \sum_{i=1}^{M} \Big(\bar{x}^T \mathbb{E}[K_i^s(\theta_i)^T \hat{Y}_i(\theta_i)]\bar{x} + \mathbb{E}[e_i^T K_i^o(\theta_i)^T \hat{Y}_i(\theta_i) L_i(\theta_i) e_i]\Big).\end{aligned}$$

Using Lemmas 1 and 16(b), the above leads to (4.7). □

We now have the following remark on the structure of optimal strategy.

Remark 18. *The optimal strategy (4.5) consists of two components $K_i^s(\theta_i)\bar{x}$ and $K_i^o(\theta_i)e_i$ added together. The first component is the optimal decentralized full-state feedback strategy applied to the expected value of the state and the second component is a corrective term based on local measurement.*

4.3 Computing Strategies Through Sequential Update

When we have a finite number of types, the space $\mathcal{Z} \oplus \mathcal{W}$ is finite dimensional. So we can follow the description in Corollary 14 to reduce (4.6) to standard linear equations. For infinite types, although the optimal solution (4.5) is guaranteed to exist, finding the exact strategy by solving (4.6) may not always be possible. For such scenarios, computing an approximate solution through a finite number of sequential updates could still be viable and we present the details in the following theorem.

Theorem 19. *For the static team problem described by cost and information structures in (4.3) and (4.2) respectively, the following is an approximation to the team optimal strategy*

$$\gamma_i^{(k)}(\mathbb{I}_i) = K_i^{s,(k)}(\theta_i)\bar{x} + K_i^{o,(k)}(\theta_i)e_i \tag{4.8}$$

obtained through recursions

$$K_i^{s,(k+1)}\bar{x} = -\hat{Z}_i^{-1}\left(\mathbb{E}\Big[\sum_{j<i} Z_{ij}K_j^{s,(k+1)} + \sum_{j>i} Z_{ij}K_j^{s,(k)}\big|\theta_i\Big] + \hat{Y}_i\right)\bar{x}$$

$$K_i^{o,(k+1)}e_i = -\hat{Z}_i^{-1}\left(\mathbb{E}\Big[\sum_{j<i} Z_{ij}K_j^{o,(k+1)}C_j + \sum_{j>i} Z_{ij}K_j^{o,(k)}C_j\big|\theta_i\Big] + \hat{Y}_i\right)L_i e_i \tag{4.9}$$

$$K_i^{s,(0)} = 0, \qquad K_i^{o,(0)} = 0$$

computed in the order $i = 1, \ldots, M$ at each stage. The resulting expected cost is bounded by

$$\bar{J}(\gamma^{(k)}) \leq \bar{J}(\gamma^{\circ}) + \frac{\bar{a}^2}{\underline{a}}\left(1 - \frac{\underline{a}^2}{M^2\bar{a}^2}\right)^k \|\gamma^{\circ}\|_{\mathcal{K}}^2 \tag{4.10}$$

Proof. The strategy in (4.8) is obtained by using the definition of sequential update in (3.16) applied to the current setup, starting with $\gamma^{(0)} = 0$. Theorem 10 being applicable here, the bound for expected cost is obtained using (3.20) and (3.21). $\square$

4.4 Full State Knowledge

For the setup discussed in 4.1, we now look at the special case when the players observe the state exactly i.e. $y_i = x$ for $i \in \mathcal{J}$. The following theorem summarizes the result.

Theorem 20. *For the static team problem described by cost structure in (4.3) and information structure $\mathbb{I}_i = (\theta_i, x)$ for $i \in \mathcal{J}$, the team optimal strategy γ° is given by*

$$\gamma_i^\circ(\mathbb{I}_i) = K_i^s(\theta_i)x \tag{4.11}$$

where coefficients $\{K_i^s\}_{i\in\mathcal{J}}$ are obtained by solving the following equations

$$\hat{Z}_i K_i^s x + \sum_{j\neq i} \mathbb{E}[Z_{ij}K_j^s|\theta_i]x + \hat{Y}_i x = 0 \tag{4.12}$$

for $i \in \mathcal{J}$. The resulting optimal expected cost is given by

$$\bar{J}(\gamma^\circ) = x^T \left(\sum_{i=1}^{M} \mathbb{E}[(K_i^s)^T Y_i] \right) x + \mathbb{E}[c(x,\theta)]. \tag{4.13}$$

Further, an approximation to the team optimal strategy $\gamma_i^{(k)}(\mathbb{I}_i) = K_i^{s,(k)}(\theta_i)x$ can be obtained through recursions

$$K_i^{s,(k+1)}x = -\hat{Z}_i^{-1}\left(\mathbb{E}\Big[\sum_{j<i} Z_{ij}K_j^{s,(k+1)} + \sum_{j>i} Z_{ij}K_j^{s,(k)}\Big|\theta_i\Big] + \hat{Y}_i\right)x, \qquad K_i^{s,(0)} = 0 \tag{4.14}$$

computed in the order $i = 1, \ldots, M$ at each stage. The resulting expected cost is bounded by (4.10).

Proof. The best response in this case is given point-wise by the following

$$\big(\mathbf{\Gamma}_i(\gamma_{-i})\big)(\mathbb{I}_i) = -\hat{Z}_i^{-1}\Big\{\mathbb{E}\Big[\sum_{j\neq i} Z_{ij}\gamma_j(\mathbb{I}_j)|\mathbb{I}_i\Big] + \hat{Y}_i x\Big\}$$

Using Corollary 13, it can be verified that the optimal strategies have the structure (4.11). Upon substitution into the optimality conditions (3.10), equations (4.12) are obtained. The optimal cost can be computed as done in Theorem 17. The update equations (4.14) that converge to the optimal strategy can be obtained in the same way as Theorem 19. □

Note that since Equation (4.12) has to hold for all possible $x \in \mathbb{R}^n$, the term x can be dropped from the equation.

4.5 One-Stage Problem

We now further specialize the results of Theorem 17 to a problem with one-stage dynamics and quadratic cost, while the information structure remaining the same.

Corollary 21. *Consider the following one stage dynamics*

$$x_+ = A(\theta)x + \sum_{i=1}^{M} B_i(\theta)u_i + w \tag{4.15}$$
$$y_i = C_i(\theta_i)x + v_i, \text{ for } i = 1, \dots, M$$

with $w \sim (0, W)$ being independent of other random variables. The associated cost function is given by

$$J(\xi, u_1, \dots, u_M) = u^T R(\theta)u + x_+^T S(\theta)x_+ \tag{4.16}$$

with $R(\theta) \succeq \underline{a}I$ and $R(\theta) + B_i(\theta)^T S(\theta)B_i(\theta) \preceq \bar{a}I$ for all $\theta \in \Theta$ for positive constants $\underline{a}$ and $\bar{a}$. The information structure here is same as (4.2) with identical assumptions on x, v_i and θ as used in Theorem 17. We then have the following expressions when adapted to the notation used in Chapter 3

$$\xi = (x, \theta, w, v_1, \dots, v_M), \qquad Z_{ij}(\xi) = R_{ij}(\theta) + B_i^T(\theta)S(\theta)B_j(\theta),$$
$$d_i(\xi) = B_i^T(\theta)S(\theta)\left(A(\theta)x + w\right), \qquad c(\xi) = \big(A(\theta)x + w\big)^T S(\theta)\big(A(\theta)x + w\big).$$

Consequently, the team optimal solution is given by (4.5) and (4.6) with $Y_i(\theta) = B_i^T(\theta)S(\theta)A(\theta)$ and

$$\mathbb{E}[c(x,\theta)] = \mathbb{E}[(Ax + w)^T S(Ax + w)] = \bar{x}^T \mathbb{E}[A^T SA]\bar{x} + Tr(\mathbb{E}[A^T SA]X) + Tr(\mathbb{E}[S]W) \tag{4.17}$$

□

Example We end this section with an example demonstrating the use of the previous theorem. Consider the following one-stage scalar dynamics

$$x_+ = Ax + \theta_1 u_1 + u_2$$
$$y_1 = x + v_1, \quad y_2 = \theta_2 x + v_2$$

and a quadratic cost $x_+^2 + u^T Ru$ which can be expanded as

$$\begin{bmatrix} u_1 \\ u_2 \end{bmatrix}^T \begin{bmatrix} R_{11}+\theta_1^2 & R_{12}+\theta_1 \\ R_{21}+\theta_1 & R_{22}+1 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} + 2\begin{bmatrix} u_1 \\ u_2 \end{bmatrix}^T \begin{bmatrix} \theta_1 Ax \\ Ax \end{bmatrix} + A^2x^2$$

with R being independent of types. We assume that $\theta_1 \sim \text{uniform}(a, b)$ and $\theta_2 \sim \text{Bernoulli}(p)$ are independent of each other. For this setup, we have $\hat{Z}_1 = R_{11} + \theta_1^2$, $\hat{Z}_2 = R_{22}$, $\hat{Y}_1 = \theta_1 A$, $\hat{Y}_2 = A$,

$C_1 = 1$, $C_2 = \theta_2$ leading to the following

$$K_1^{s,(k+1)}(\theta_1) = -\frac{(R_{12}+\theta_1)K_2^{s,(k)} + \theta_1 A}{R_{11}+\theta_1^2}, \quad \theta_1 \in [a,b] \tag{4.18a}$$

$$K_2^{s,(k+1)}(\theta_2) = -\int_a^b \frac{(R_{21}+\theta_1)K_1^{s,(k+1)}}{(b-a)(R_{22}+1)}\, d\theta_1 - \frac{A}{R_{22}+1}, \quad \theta_2 \in \{0,1\} \tag{4.18b}$$

$$K_1^{o,(k+1)}(\theta_1) = -\frac{p(R_{12}+\theta_1)K_2^{o,(k)}(1) + \theta_1 A}{R_{11}+\theta_1^2} L_1, \quad \theta_1 \in [a,b] \tag{4.18c}$$

$$K_2^{o,(k+1)}(1) = -\left\{ \int_a^b \frac{(R_{21}+\theta_1)K_1^{o,(k+1)}}{(b-a)(R_{22}+1)}\, d\theta_1 + \frac{A}{R_{22}+1} \right\} L_2(1) \tag{4.18d}$$

Note that both $K_1^{s,(k)}$ and $K_2^{s,(k)}$ are not affected by p (also $K_2^{s,(k)}$ doesn't depend on θ_2). This is because both these terms correspond to the state feedback law (see Remark 18) whereas parameter θ_2 (hence p) only affects the observation structure. Further, since $L_2(0) = 0$ we have $K_2^{o,(k)}(0) = 0$ from (4.9). In fact, from Equation (4.6b) we can deduce that $K_2^o(0) = 0$. This is understandable (again with regards to Remark 18) because for $\theta_2 = 0$ the observation contains no additional information.

With the following choice of parameters

$$a = -1,\ b = 0,\ X = 0.5,\ V_1 = 0.2,\ V_2 = 0.2$$
$$A = 2,\ R_{11} = R_{22} = 1,\ R_{12} = R_{21} = 0.7$$

we performed two simulations with $p = 0.25$ and $p = 0.75$ for 10 stages of the updates. The strategy gains thus computed for Player 1 is plotted in Figure 4.1, and for Player 2 is given by $K_2^s(\theta_2) = -1.0903$ (for all values of p and θ_2), $K_2^o(1) = -0.73387$ for $p = 0.25$ and $K_2^o(1) = -0.74482$ for $p = 0.75$. □

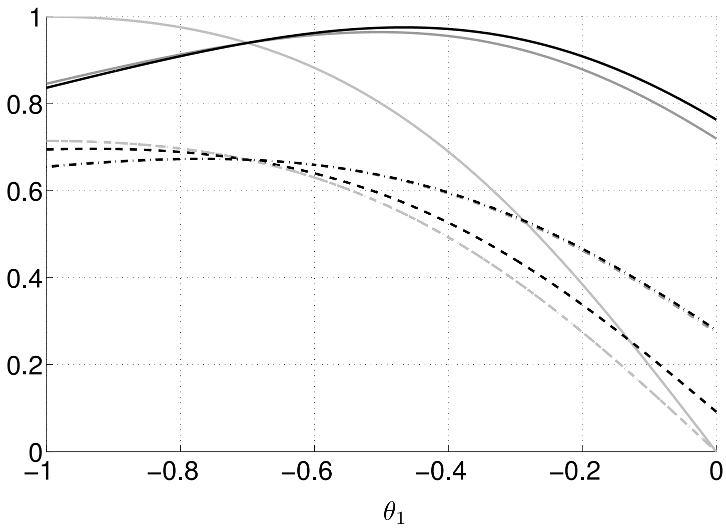


Figure 4.1: Strategy coefficients K_1^s (solid), K_1^o for $p = 0.25$ (dashed) and K_1^o for $p = 0.75$ (dot-dashed) of player 1 plotted against θ_1. Black lines indicate final strategies after $k = 10$ iterations, while lighter shades indicate strategies at intermediate steps of $k = 1$ and $k = 2$.

Chapter 5

Dynamic Teams with Partially Nested Information Structure

In this section we will consider a partially nested information structure. A non-switched version of this problem was introduced in [22], where the authors showed that under a partially nested information structure, the decentralized LQG problem has affine team optimal strategies. The approach there involved converting the corresponding dynamic problem to a static one and then applying the result of [16] for static team problems.

Here we consider a M-player problem similar to that introduced in Section 4 with cost function (4.3). However the observation model and information structure are different. But before we explain them, let us define the set consisting of indices of all players' whose actions affect the information of player i as

$$\hat{\phi}^i = \{\phi^i_1, \ldots, \phi^i_{p_i}\}$$

where p_i is the count of such players. Also define $\phi^i = \hat{\phi}^i \cup \{i\}$ and $\bar{\phi}^i = \mathcal{J} \backslash \phi^i$ (set containing indices of players not affecting the information of player i). To be able to enforce the partially nested structure, we will assume following conditions on the index sets

(i) If $j \in \hat{\phi}^i$ then $\hat{\phi}^j \subset \hat{\phi}^i$,

(ii) If $j \in \hat{\phi}^i$ then $i \in \bar{\phi}^j$.

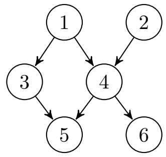
The local measurement corresponding to player i is assumed to be

$$y_i = C_i(\theta_i)x + \sum_{j \in \hat{\phi}^i} D_{ij}(\theta_i)u_j + v_i \tag{5.1}$$

whose coefficients depend on the local type θ_i. The above then describes the information available to player i as

$$\mathbb{I}_i = (\hat{y}_i, \tilde{\theta}_i) \tag{5.2}$$

with $\hat{y}_i := \left[y^T_{\phi^i_1} \cdots y^T_{\phi^i_{p_i}} \; y^T_i \right]^T$ and $\tilde{\theta}_i := (\theta_{\phi^i_1}, \ldots, \theta_{\phi^i_{p_i}}, \theta_i)$. To have a better understanding of the above notation, let us consider an example with the following decision graph.



$$\hat{\phi}^1 = \emptyset,\ \hat{\phi}^2 = \emptyset,\ \hat{\phi}^3 = \{1\},$$
$$\hat{\phi}^4 = \{1,2\},\ \hat{\phi}^6 = \{1,2,4\},$$
$$\hat{\phi}^5 = \{1,2,3,4\}.$$

For this setup, the information available to each player is given by

$$\mathbb{I}_1 = (y_1, \theta_1),\ \mathbb{I}_2 = (y_2, \theta_2),\ \mathbb{I}_3 = \left(\begin{bmatrix} y_1 \\ y_3 \end{bmatrix},\ \theta_1, \theta_3 \right),$$
$$\mathbb{I}_4 = \left(\begin{bmatrix} y_1 \\ y_2 \\ y_4 \end{bmatrix}, \theta_1, \theta_2, \theta_4 \right),\ \mathbb{I}_5 = \left(\begin{bmatrix} y_1 \\ \vdots \\ y_5 \end{bmatrix} \theta_1, \dots, \theta_5 \right),\ \mathbb{I}_6 = \left(\begin{bmatrix} y_1 \\ y_2 \\ y_4 \\ y_6 \end{bmatrix} \theta_1, \theta_2, \theta_4, \theta_6 \right).$$

Note that the action of player j affecting the information of player i, could happen either because $\text{Prob}\{D_{ij} \neq 0\} > 0$ or through a series of players $i_1, \dots, i_r$ (each from the set $\hat{\phi}^i$) such that

$$\text{Prob}\{D_{i_1,j} \neq 0, D_{i_2,i_1} \neq 0, \dots, D_{i,i_r} \neq 0\} > 0. \tag{5.3}$$

The information structure described above in (5.2) is partially nested because $\mathbb{I}_{\phi^i_j}$, the information corresponding to player ϕ^i_j is also available to player i. Since in a team problem as this, we can assume that players have knowledge of all other players' strategies, this also means that player i can compute $u_{\phi^i_j}$. As a result, the information structure in (5.2) is equivalent to

$$\hat{\mathbb{I}}_i = (\hat{y}_i, \tilde{\theta}_i, \hat{u}_i) \tag{5.4}$$

with $\hat{u}_i := \begin{bmatrix} u^T_{\phi^i_1} & \dots & u^T_{\phi^i_{p_i}} & u^T_i \end{bmatrix}^T$.

Before presenting the main theorem of the section, we introduce some additional notation. We define $e_i = y_i - \sum_{j \in \phi^i} D_{ij}(\theta_i) u_j = C_i(\theta_i)(x - \bar{x}) + v_i$ which can be computed by player i based on its information. We also define $\hat{Y}_i(\tilde{\theta}_i) = \mathbb{E}[Y_i | \tilde{\theta}_i]$,

$$\tilde{C}_i(\tilde{\theta}_i) = \begin{bmatrix} C_{\phi^i_1}(\theta_{\phi^i_1}) \\ \vdots \\ C_{\phi^i_{p_i}}(\theta_{\phi^i_{p_i}}) \\ C_i(\theta_i) \end{bmatrix}, \tilde{D}_{ij}(\tilde{\theta}_i) = \begin{bmatrix} D_{\phi^i_1,j}(\theta_{\phi^i_1}) \\ \vdots \\ D_{\phi^i_{p_i},j}(\theta_{\phi^i_{p_i}}) \\ D_{i,j}(\theta_i) \end{bmatrix}, \tilde{e}_i = \begin{bmatrix} e_{\phi^i_1} \\ \vdots \\ e_{\phi^i_{p_i}} \\ e_i \end{bmatrix}.$$

$\tilde{v}_i := \begin{bmatrix} v^T_{\phi^i_1} & \dots & v^T_{\phi^i_{p_i}} & v^T_i \end{bmatrix}^T$ whose covariance matrix is given by $\tilde{V}_i = \text{diag}(V_{\phi^i_1}, \dots, V_{\phi^i_{p_i}}, V_i)$.

Theorem 22. *For a decentralized partially nested problem described by information structure (5.2) and cost structure (4.3) along with Assumption 15, the team strategy which minimizes the expected* *value of the cost function is given by*

$$\gamma_i^\circ(\mathbb{I}_i) = K_i^s(\tilde{\theta}_i)\bar{x} + K_i^o(\tilde{\theta}_i)\tilde{e}_j = K_i^s(\tilde{\theta}_i)\bar{x} + \sum_{j\in\phi^i} K_{ij}^o(\tilde{\theta}_i)e_j \tag{5.5}$$

where $K_i^o(\tilde{\theta}_i) = \left[K^o_{i,\phi_1^i}(\tilde{\theta}_i)\dots K^o_{i,\phi^i_{p_i}}(\tilde{\theta}_i)\ K^o_{ii}(\tilde{\theta}_i)\right]$. $\{K_i^s\}_{i\in\mathcal{J}}$ *and* $\{K_i^o\}_{i\in\mathcal{J}}$ *are obtained by solving the following equations*

$$\sum_{j\in\mathcal{J}} \mathbb{E}[Z_{ij}K_j^s|\theta_i]\bar{x} + \hat{Y}_i\bar{x} = 0 \tag{5.6a}$$

$$\sum_{\substack{j\in\mathcal{J}\\ l\in\phi^j\cap\phi^i}} \mathbb{E}[Z_{ij}|\tilde{\theta}_i]K_{jl}^o e_l + \sum_{\substack{j\in\bar{\phi}^i\\ l\in\phi^j\cap\bar{\phi}^i}} \mathbb{E}[Z_{ij}K_{jl}^o C_l|\tilde{\theta}_i]\tilde{L}_i\tilde{e}_i + \hat{Y}_i\tilde{L}_i\tilde{e}_i = 0 \tag{5.6b}$$

for $i\in\mathcal{J}$ *with* $\tilde{L}_i(\tilde{\theta}_i) = X\tilde{C}_i^T(\tilde{V}_i + \tilde{C}_iX\tilde{C}_i)^{-1}$. *The resulting optimal expected cost is given by*

$$\bar{J}(\gamma^\circ) = \bar{x}^T\left(\sum_{i=1}^M \mathbb{E}[(K_i^s)^TY_i]\right)\bar{x} + \mathbb{E}[c(x,\theta)] + \sum_{i=1}^M Tr\Big(\mathbb{E}[(K_i^o)^TY_i\tilde{L}_i(\tilde{V}_i + \tilde{C}_iX\tilde{C}_i^T)]\Big) \tag{5.7}$$

Proof. We combine the measurements available to player i as

$$\hat{y}_i = \tilde{C}_i(\tilde{\theta}_i)x + \sum_{j\in\hat{\phi}^i}\tilde{D}_{ij}(\tilde{\theta}_i)u_j + \tilde{v}_i$$

Note that due to the explanation given in equation (5.3) and prior to it, we have $\tilde{D}_{ij}(\tilde{\theta}_i) \neq 0$ for some $\tilde{\theta}_i$, for all $j\in\hat{\phi}^i$. As explained earlier that information structures (5.2) and (5.4) are equivalent, player i has access to $\{u_j\}_{j\in\hat{\phi}^i}$ and hence can compute the following exactly

$$\tilde{y}_i := \hat{y}_i - \sum_{j\in\hat{\phi}^i}\tilde{D}_{ij}(\tilde{\theta}_i)u_j = \tilde{C}_i(\tilde{\theta}_i)x + \tilde{v}_i.$$

This allows us to rewrite the dynamic information structure described in (5.2) by the equivalent static information structure $\tilde{\mathbb{I}}_i = (\tilde{y}_i, \tilde{\theta}_i)$. As a result we have a static problem very similar to the one presented in Section 4. The main difference being that $\tilde{v}_i$ here is correlated among players. So although we cannot use the result in Theorem 17, we can follow similar steps to obtain the optimal control. For the above information structure, we can write relations similar to those in Lemma 16. In particular we have $\mathbb{E}[x|\mathbb{I}_i] = \tilde{L}_i\tilde{e}_i$ and for any matrix valued function G

$$\mathbb{E}[Ge_j|\tilde{\mathbb{I}}_i] = \begin{cases} \mathbb{E}[G|\tilde{\theta}_i]e_j & \text{for } j\in\phi^i \\ \mathbb{E}[GC_j|\tilde{\theta}_i]\tilde{L}_i\tilde{e}_i & \text{for } j\in\bar{\phi}^i \end{cases}$$

We can use the above to show that for strategies having a structure similar to (5.5), each player's best response to others' strategies retains the same structure. We can thereby use Corollary 13 to obtain (5.5) and (5.6). Note that we could split (5.6) into two independent equations for the same reasons presented in Theorem 17. Further, the optimal cost can be computed using

$$\langle \gamma^\circ, \delta \rangle_{\mathcal{K}} = \sum_{i=1}^{M} \mathbb{E}\Big[\big(K_i^s \bar{x} + K_i^o \tilde{e}_i\big)^T \hat{Y}_i(\bar{x} + \tilde{L}_i \tilde{e}_i)\Big]$$

added to $\mathbb{E}[c(x, \theta)]$. $\square$

We point out that when the above problem is setup with players acting repeatedly over time, obtaining the optimal controller involves solving linear equations over the entire time horizon. However with additional structure on the problem, it may be possible to obtain a recursive solution similar to [39, 40, 77].

Chapter 6

Dynamic Teams with One-Step Delayed Information Sharing

In this section, we consider the M-player decentralized control of a discrete-time switched system where individual controllers share their information with others after a delay of one time step. The 3-player case is pictured in Figure 6.1. Such a delayed information sharing structure can be applied in several decentralized control scenarios where controllers are connected by fast communication network so that they have access to local parameters instantaneously but can access the parameters of the entire system after a small but non-zero delay.

6.1 Problem Description

We consider a linear time varying system controlled by M players having the following dynamics

$$\begin{aligned} x_{t+1} &= A_t(\theta_t)\, x_t + \sum_{i=1}^{M} B_{it}(\theta_t)\, u_{it} + w_t \\ y_{it} &= C_{it}(\theta_{it})\, x_t + v_{it}, \quad \text{for } i = 1, \ldots, M \end{aligned} \tag{6.1}$$

Here $x_t \in \mathbb{R}^n$ is the state of the system, $u_{it} \in \mathbb{R}^{m_i}$ and $y_{it} \in \mathbb{R}^{l_i}$ are respectively the control input and measurement of the i-th player at time t. We define $y_t = \begin{bmatrix} y_{1t}^T \ldots\, y_{Mt}^T \end{bmatrix}^T$ and similarly introduce u_t and v_t. The system matrices are functions of time varying random type $\theta_t = (\theta_{1t}, \ldots, \theta_{Mt})$ which takes value in $\Theta = \Theta_1 \times \cdots \times \Theta_M$ as in Chapter 4. Note that the single subscripts on θ_t, y_t, u_t and v_t correspond to time and is different from the notation used in prior chapters. We assume that the player types are taken from a Markov process with known transitions $\mathbb{P}(\theta_t|\theta_{t\text{-}1})$ and initial distribution $\mathbb{P}(\theta_0)$. The initial state $x_0 \sim \mathcal{N}(\bar{x}_0, X_0)$, (i.i.d.) process noise $w_t \sim \mathcal{N}(0, W)$, players' (i.i.d.) measurement noise $v_{it} \sim \mathcal{N}(0, V_i)$ and θ_t are assumed independent across time. Further, all player have complete knowledge of the bounded mappings $A_t(\cdot)$, $B_{it}(\cdot)$ and $C_{it}(\cdot)$ for $i = 1, \ldots, M$ and all t, and the distributions of all underlying random variables. The information available to player i at time t is given by

$$\mathbb{I}_{it} = (\mathbb{I}_t^c, y_{it}, \theta_{it}) \tag{6.2}$$

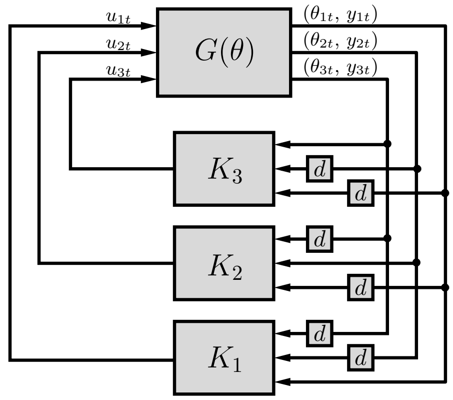


Figure 6.1: System under consideration, shown here for three players. Parameter θ_{it} and measurement y_{it} are instantaneously available locally but with a delay (identified here with block d) of one time step to the other players.

where $\mathbb{I}_t^c = (y_0, \ldots, y_{t\text{-}1}, \theta_0, \ldots, \theta_{t\text{-}1})$ for $t > 0$ and $\mathbb{I}_0^c = \emptyset$ (a 0-tuple). Let us denote the decentralized information at time t as $\mathbb{I}_t^d = (\mathbb{I}_{1t}, \ldots, \mathbb{I}_{Mt})$.

Suppose the information available to player i at time t is $\mathbb{I}_{it}$ and takes values in the space $\mathcal{I}_{it}$. The strategy γ_{it} maps the information set of player i at time t to its control input as $u_{it} = \gamma_{it}(\mathbb{I}_{it})$ and is considered on a Hilbert space $\mathcal{K}_{it}$ consisting of measurable functions satisfying $\|\gamma_{it}\|_{\mathcal{K}_{it}} := \mathbb{E}\left[\, |\gamma_{it}(\mathbb{I}_{it})|_2^2 \,\middle|\, \gamma^{t\text{-}1} \,\right]^{\frac{1}{2}} < \infty$. The probability measure associated with the above expectation depends on the choice of past strategies $\gamma^{t\text{-}1} = (\gamma_0, \ldots, \gamma_{t\text{-}1})$, assumed to be known. Similar definitions of the strategy space has been used in [78]. As in the static case, the inner-product associated with $\mathcal{K}_{it}$ is defined as $\langle \alpha, \beta \rangle_{\mathcal{K}_{it}} := \mathbb{E}[\alpha(\mathbb{I}_{it})^T \beta(\mathbb{I}_{it}) | \gamma^{t\text{-}1}]$ and the decentralized strategy at time t is defined on the Hilbert space $\mathcal{K}_t = \mathcal{K}_{1t} \oplus \cdots \oplus \mathcal{K}_{Mt}$. For a decentralized strategy $\gamma_t \in \mathcal{K}_t$ at time t, we use the notation $\gamma_t(\mathbb{I}_t^d) = \begin{bmatrix} \gamma_{1t}(\mathbb{I}_{1t}) \\ \vdots \\ \gamma_{Mt}(\mathbb{I}_{Mt}) \end{bmatrix}$.

We will consider the following finite N-step horizon quadratic cost function

$$J_N\left(x_0, u_0, \ldots, u_{N\text{-}1}\right) = x_N^T Q_N(\theta_N) x_N + \sum_{t=0}^{N-1} \left\{ x_t^T Q_t(\theta_t) x_t + \sum_{i=1}^{M} \sum_{j=1}^{M} u_{it}^T R_{ij,t}(\theta_t) u_{jt} \right\} \tag{6.3}$$

For all t, we assume that $Q_t(\theta_t) \in \bar{\mathbb{S}}_+$ and $R_t(\theta_t) \in \mathbb{S}_+$ with $R_{ij,t}(\theta_t)$ being its (i,j)-th block. In particular, we assume that R_t has a lower bound as $R_t(\theta) \succeq \underline{a}_t I$ for all $\theta \in \Theta$, and in similar sense both R_t and Q_t are bounded from above.

The main objective of the decentralized control problem is to find the decentralized control strategy $\gamma = (\gamma_0, \ldots, \gamma_{N\text{-}1})$ which minimizes the expected cost

$$\bar{J}(\gamma) = \mathbb{E}\left[J_N\left(x_0, \gamma_0(\mathbb{I}_0^d), \ldots, \gamma_{N\text{-}1}(\mathbb{I}_{N\text{-}1}^d) \right) \right]$$

with the expectation taken over $\theta_0, \dots, \theta_{N\text{-}1}$, $w_0, \dots, w_{N\text{-}1}$, $v_0, \dots, v_{N\text{-}1}$ and x_0.

6.2 Multistage Solution

We now solve the multistep problem described in Section 6.1 using a dynamic programming approach. We denote the expected cost-to-go at time step t as $\mathcal{V}_t(\mathbb{I}^c_t)$ which is a function of the common information $\mathbb{I}^c_t$. We will see in the forthcoming discussion that the optimal expected cost-to-go at time t has the quadratic form

$$\mathcal{V}^\circ_t(\mathbb{I}^c_t) = \mathbb{E}[x_t^T \Pi_t(\theta_{t\text{-}1}) x_t | \mathbb{I}^c_t] + c_t(\theta^{t\text{-}1}) = \bar{x}_t^T \Pi_t(\theta_{t\text{-}1}) \bar{x}_t + \mathrm{Tr}(\Pi_t X_t) + c_t(\theta^{t\text{-}1}) \tag{6.4}$$

with Π_t being a function of player types at $t-1$ and c_t dependent on the past types $\theta^{t\text{-}1} := (\theta_0, \dots, \theta_{t\text{-}1})$ which is a part of the common information $\mathbb{I}^c_t$. The quadratic structure allows us to use the one-step result (obtained in Theorem 17) to solve the minimization problem at each step of dynamic programming. Further the information structure (6.2) implies that the controllers have access to all measurements, inputs and types until the previous step. This allows each controller to use a centralized Kalman filter and have a common estimate $\bar{x}_t$ of the state for the current time step. The state covariance matrix X_t of the corresponding Kalman filter is obtained through a forward propagating Riccati equation and depends on the past history of player types $\theta^{t\text{-}1}$. The controller which is linear in the measurements uses coefficient matrices obtained by solving a set of linear equations dependent on both Π_{t+1} and X_t. We now present the main result of the section.

Theorem 23. *For the system described by (6.1) and information structure (6.2), the optimal control policy which minimizes the expected value of (6.3) is given by*

$$\gamma^\circ_{it}(\mathbb{I}_{it}) = K^s_{it}(\tilde{\theta}_{it})\bar{x}_t + K^o_{it}(\tilde{\theta}_{it}) e_{it} \tag{6.5}$$

with $e_{it} = y_{it} - C_{it}(\theta_{it})\bar{x}_t$, $\tilde{\theta}_{it} = (\theta_{it}, \theta_{t\text{-}1})$ *and* K^s_{it}, K^o_{it} *given by the solution of following equations*

$$\mathbb{E}[Z_{ii,t}|\tilde{\theta}_{it}] K^s_{it} \bar{x}_t + \mathbb{E}\Big[\sum_{j \neq i} Z_{ij,t} K^s_{jt} + B^T_{it} \Pi_{t+1} A_t \big| \tilde{\theta}_{it}\Big] \bar{x}_t = 0$$

$$\mathbb{E}[Z_{ii,t}|\tilde{\theta}_{it}] K^o_{it} e_{it} + \mathbb{E}\Big[\sum_{j \neq i} Z_{ij,t} K^o_{jt} C_{jt} + B^T_{it} \Pi_{t+1} A_t \big| \tilde{\theta}_{it}\Big] L_{it} e_{it} = 0$$

for $i = 1, \dots, M$, *with* $L_{it}(\theta_{it}, \theta^{t\text{-}1}) = X_t C^T_{it}(V_i + C_{it} X_t C^T_{it})^{-1}$ *and* $Z_{ij,t}(\theta_t) = R_{ij,t} + B^T_{it}\Pi_{t+1} B_{jt}$. *Further,* $\bar{x}_t$ *is obtained from a centralized Kalman filter estimate of the current state computable by both players based on common information*

$$\bar{x}_t = A_{t\text{-}1}\bar{x}_{t\text{-}1} + B_{t\text{-}1} u_{t\text{-}1} + A_{t\text{-}1} L^c_{t\text{-}1}(y_{t\text{-}1} - C_{t\text{-}1}\bar{x}_{t\text{-}1}) \tag{6.6}$$

and Π_t is obtained by a backward recursion[1] *as below*

$$\Pi_t(\theta_{t\text{-}1}) \quad := \quad \mathbb{E}[Q_t \;+\; A_t^T \Pi_{t+1} A_t | \theta_{t\text{-}1}] \;+\; \sum_{i=1}^{M} \mathbb{E}[(K_{it}^s)^T B_{it}^T \Pi_{t+1} A_t | \theta_{t\text{-}1}] \tag{6.7}$$

with terminal condition $\Pi_N(\theta_{N\text{-}1}) = \mathbb{E}[Q_N|\theta_{N\text{-}1}]$ and $L_t^c(\theta^t) = X_t C_t^T (V + C_t X_t C_t^T)^{\text{-}1}$ being the centralized Kalman gain. The state error covariance X_t is obtained by a forward recursion

$$X_t(\theta^{t\text{-}1}) = A_{t\text{-}1}^T \big(X_{t\text{-}1} - L_{t\text{-}1}^c C_{t\text{-}1} X_{t\text{-}1}\big) A_{t\text{-}1} + W \tag{6.8}$$

initialized with X_0, the covariance of x_0. The resulting optimal expected cost is given by

$$\bar{J}(\gamma^\circ) = \bar{x}_0^T \Pi_0 \bar{x}_0 + Tr(\Pi_0 X_0) + c_0 \tag{6.9}$$

where c_t is obtained by the backwards recursion

$$\begin{aligned} c_t(\theta^{t\text{-}1}) = \mathbb{E}[c_{t+1}(\theta^t)|\mathbb{I}_t^c] + Tr(\mathbb{E}[\Pi_{t+1}|\theta_{t\text{-}1}]W) - Tr\big(\Pi_t X_t\big) + \\ \sum_{i=1}^{M} Tr\big(\mathbb{E}[(K_{it}^o)^T B_{it}^T \Pi_{t+1} A_t L_{it} (C_{it} X_t C_{it}^T + V_i)^T | \theta_{t\text{-}1}]\big) \end{aligned} \tag{6.10}$$

with $c_N(\theta^{N\text{-}1}) = 0$.

Proof. Due to the one-step delayed information sharing, players have knowledge of all past system matrices and inputs. Thus the distribution of x_t conditioned on the common information $\mathbb{I}_t^c$ is Gaussian and can be obtained through a centralized Kalman filter (after prediction but before update step). The filter (associated with a Kalman gain L_t^c) has a mean $\bar{x}_t$ obtained through the update equation (6.6) and error covariance X_t obtained through a forward Riccati equation (6.8). This serves as the prior distribution of the state for the corresponding time step of the dynamic program. For a given choice of strategies $\gamma = (\gamma_0, \ldots, \gamma_{N\text{-}1})$ the expected cost-to-go is defined as

$$\begin{aligned} \mathcal{V}_t^\gamma(\mathbb{I}_t^c) &= \mathbb{E}\left[\sum_{k=t}^{N-1} \Big(x_k^T Q_k x_k + \gamma_k(\mathbb{I}_k^d)^T R_k \gamma_k(\mathbb{I}_k^d)\Big) + x_N^T Q_N x_N \Big| \mathbb{I}_t^c\right] \\ &= \mathbb{E}[x_t^T Q_t x_t + \gamma_t(\mathbb{I}_t^d)^T R_t \gamma_t(\mathbb{I}_t^d) + \mathcal{V}_{t+1}^\gamma(\mathbb{I}_{t+1}^c)|\mathbb{I}_t^c] \end{aligned} \tag{6.11}$$

with $\mathcal{V}_N^\gamma(\mathbb{I}_N^c) = \mathbb{E}[x_N^T Q_N(\theta_N) x_N|\mathbb{I}_N^c]$. The cost-to-go achieved by the optimal strategy is denoted by $\mathcal{V}_t^\circ$ which has the form (6.4) as we show next. We start with the terminal time $t = N$, the cost-to-go here is $\mathcal{V}_N^\circ(\mathbb{I}_N^c) = \mathbb{E}[x_N^T Q_N(\theta_N) x_N|\mathbb{I}_N^c]$ which can be written as $\mathbb{E}[x_N^T \Pi_N(\theta_{N\text{-}1}) x_N|\mathbb{I}_N^c] + c_N(\theta^{N\text{-}1})$ with $\Pi_N(\theta_{N\text{-}1}) = \mathbb{E}[Q_N(\theta_N)|\mathbb{I}_N^c] = \mathbb{E}[Q_N(\theta_N)|\theta_{N\text{-}1}]$ and $c_N(\theta^{N\text{-}1}) = 0$. Now assuming structure

[1] At time $t = 0$, the conditioning is over an empty set of variables. Hence the expectation is taken with respect to the initial distribution p_{i0}. For the same reason Π_0 and c_0 are constants and don't take any arguments.

(6.4) for $\mathcal{V}^\circ_{t+1}(\mathbb{I}^c_{t+1})$, we write the Bellman equation for dynamic programming

$$\begin{aligned}\mathcal{V}^\circ_t(\mathbb{I}^c_t) &= \inf_{\gamma_t \in \mathcal{K}_t} \mathbb{E}\left[x_t^T Q_t x_t + \gamma_t(\mathbb{I}^d_t)^T R_t \gamma_t(\mathbb{I}^d_t) + \mathcal{V}^\circ_{t+1}(\mathbb{I}^c_{t+1}) | \mathbb{I}^c_t\right] \\ &= \mathbb{E}[x_t^T Q_t x_t + c_{t+1}(\theta^t) | \mathbb{I}^c_t] + \inf_{\gamma_t \in \mathcal{K}_t} \mathbb{E}\Big[\gamma_t(\mathbb{I}^d_t)^T R_t \gamma_t(\mathbb{I}^d_t) + x_{t+1}^T \Pi_{t+1} x_{t+1} | \mathbb{I}^c_t\Big] \\ &= \mathbb{E}[x_t^T \Pi_t(\theta_{t-1}) x_t | \mathbb{I}^c_t] + c_t(\theta^{t-1}). \end{aligned} \tag{6.12}$$

In the second line above, the term c_{t+1} from $\mathcal{V}^\circ_{t+1}$ can be taken out of the infimization because it doesn't depend on the strategy γ_t. Now as described in [26,36], an important consequence of the one-step sharing information structure is that the state (which would be (x_t, θ_t) in their context) of the system conditioned on the common information can be used as the information state for the dynamic program. This conditional distribution has the simple structure $\mathbb{P}(x_t, \theta_t | \mathbb{I}^c_t) = \mathbb{P}(\theta_t | \theta_{t-1})\, \mathbb{P}(x_t | \mathbb{I}^c_t)$ where the last term corresponds to a Gaussian distribution $\mathcal{N}(\bar{x}_t, X_t)$. Further the local information of the players depend on the current state (x_t) and parameters (θ_t) just as in the one stage case. Thus for a given common past information, the minimization problem encountered above is same as the one in the one-stage problem and can be solved by applying Corollary 21. This results in strategies (6.5). The associated cost is obtained using (4.7) and (4.17), leading to the last expression in (6.12) with expansions of Π_t and c_t given in (6.7) and (6.10) respectively. Thus starting with the form (6.4) for $\mathcal{V}^\circ_{t+1}$, we recover same for $\mathcal{V}^\circ_t$, thus verifying the structure through an inductive argument. Continuing in this manner until $t = 0$, we obtain the optimal cost for the entire horizon. □

In general $c_t(\theta^{t-1})$ (and hence c_0) is hard to compute as it involves evaluating X_t at each time for every possible sequence of past types. However it does not play any role in the computation of the optimal strategies.

Remark 24. *When the player types are independent in time, this assumption simplifies the solution obtained in the previous theorem. The cost-to-go still has the same quadratic structure, but Π_t is a constant and does not depend on player types. The expression of the strategies remain the same as (6.5). However Π_t being independent of types, (6.7) reduces to a backwards recursion in matrices rather than one in functions.*

We end this section with a two-stage extension of the example presented in Section 4.5.

Example Consider the following scalar dynamics corresponding to system shown in Figure 6.2

$$\begin{aligned} x_{t+1} &= A x_t + \theta_{1t}\, u_{1t} + u_{2t} + w_t, \qquad \text{for } \ t = 0, 1 \\ y_{1t} &= x_t + v_{1t}, \quad y_2 = \theta_{2t}\, x_t + v_{2t} \end{aligned}$$

with OSD information sharing and a two-stage quadratic cost function given by

$$J_2(x_0, u_0, u_1) = x_0^2 + x_1^2 + x_2^2 + u_0^T R u_0 + u_1^T R u_1$$

with R being independent of types. We assume that $x_0 \sim \mathcal{N}(\bar{x}_0, X_0)$, $w_t \sim \mathcal{N}(0, W)$, $v_{it} \sim \mathcal{N}(0, V_i)$, $\theta_{1t} \sim \text{uniform}(a, b)$ and $\theta_{2t} \sim \text{Bernoulli}(p)$ are independent of each other and across all time stages.

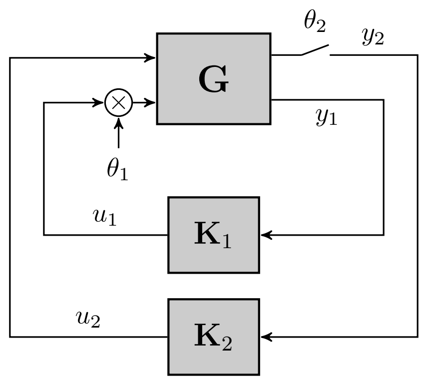


Figure 6.2: Block diagram of a dynamic team problem with multiplicative uncertainties

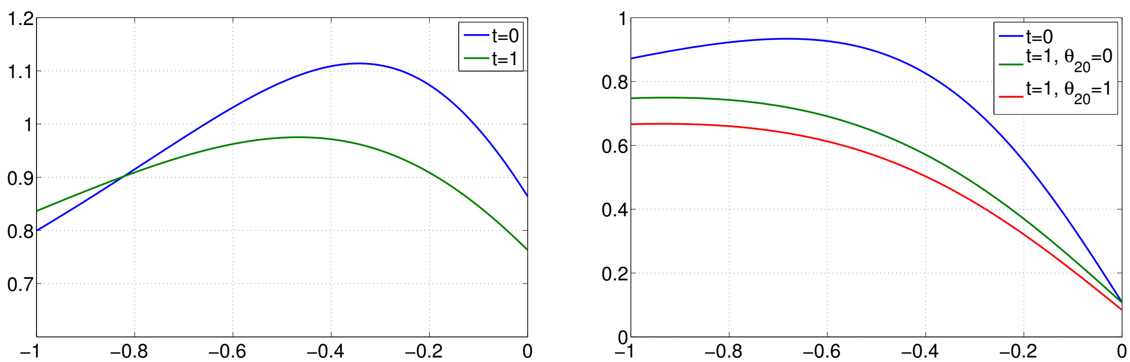
With the following choice of the system variables

$$a = -1,\ b = 0,\ p = 0.25,\ X = 0.5,\ V_1 = 0.2,\ V_2 = 0.2$$
$$A = 2,\ R_{11} = R_{22} = 1,\ R_{12} = R_{21} = 0.7,\ W = 0.1$$

the strategies were computed using sequential update. For the backwards recursion, variables Π_2, $\{K^s_{11}, K^s_{21}\}$, Π_1, $\{K^s_{10}, K^s_{20}\}$, Π_0 were computed in the same order, with coefficients K^s_{it} obtained using update equations similar to (4.18a),(4.18b). Using the forward recursion, state covariances $X_1(0)$ and $X_1(1)$ are computed. Finally, K^o_{10}, K^o_{20}, K^o_{11} and K^o_{21} are computed using update equations similar to (4.18c),(4.18d). The team optimal strategies thus computed are given below and in Figure 6.3.

$$K^s_{20}(\theta_{20}) = -1.235 \quad \text{for } \theta_{20} = 0, 1, \qquad K^s_{21}(\theta_{21}) = -1.09 \quad \text{for } \theta_{21} = 0, 1,$$

$$K^o_{20}(1) = -0.863, \qquad K^o_{21}(1) = \begin{cases} -0.794 & \text{for } \theta_{20} = 0 \\ -0.702 & \text{for } \theta_{20} = 1 \end{cases}$$



(a) $K^s_{1t}(\theta_{1t})$ (b) $K^o_{1t}(\theta_{1t})$

Figure 6.3: Plots showing player 1's strategy coefficients as a function of local parameter θ_{1t} at different time instances.

Chapter 7

Dynamic Teams with Full State Feedback and Local Parameters

In this chapter, we present a dynamic team problem with full state feedback, in order to minimize a switched quadratic cost as in previous chapters. We assume that the parameters are independent in time and solve both finite and infinite horizon versions of the problem. The finite horizon case can be seen as a special case of the output feedback problem of Chapter 6. However, the controllers, due to the availability of complete state information don't find each other's information about past parameters helpful. So we do not consider any sharing of parameters in this chapter. For the infinite horizon case, we additionally assume no process noise and obtain the optimal solution as a limit of the finite horizon case. The steps involved in obtaining this result uses ideas developed in [51]. In [51], a centralized control problem is considered with i.i.d. system/cost matrices and the controller's only knowledge about the underlying stochasticity being its statistics. Special cases of the problem considered in this chapter were solved in [79] and [58].

7.1 Finite Horizon

Here we consider the following dynamics for a finite horizon of N time steps

$$x_{t+1} = A_t(\theta_t)\, x_t + \sum_{i=1}^{M} B_{it}(\theta_t)\, u_{it} + w_t \tag{7.1}$$

The information available to player i, common-information and decentralized information at time t are given by

$$\mathbb{I}_{it} = (\mathbb{I}^c_t, \theta_{it}), \quad \mathbb{I}^c_t = (x_0, \dots, x_t), \quad \mathbb{I}^d_t = (\mathbb{I}_{1t}, \dots, \mathbb{I}_{Mt}). \tag{7.2}$$

We assume that mappings A_t and B_{it} are bounded, and parameters θ_t are independent in time. The strategy spaces $\mathcal{K}_{it}$ and $\mathcal{K}_t$ for $i \in \mathcal{J}$, $t \in \mathbb{N}_0$ are defined from the above information variables in the same way as done in Section 6.1. The next theorem presents the team optimal strategies for the above setup, in order to minimize the expected value of cost function (6.3) with same boundedness assumptions on Q_t and R_t as before.

Theorem 25. *For the system described by (7.1) and information structure (7.2), the optimal control policy which minimizes the expected value of (6.3) is given by*

$$\gamma^{\circ}_{it}(\mathbb{I}_{it}) = K^s_{it}(\theta_{it})x_t \tag{7.3}$$

with K^s_{it} given by the solution of following equations

$$\mathbb{E}[Z_{ii,t}|\theta_{it}]K^s_{it}x_t + \mathbb{E}\Big[\sum_{j\neq i} Z_{ij,t}K^s_{jt} + B^T_i \Pi_{t+1}A_t\Big|\theta_{it}\Big]x_t = 0$$

for $i \in \mathcal{J}$, with $Z_{ij,t}(\theta_t) = R_{ij,t} + B^T_{it}\Pi_{t+1}B_{jt}$. Π_t is obtained by a backward recursion as below

$$\Pi_t = \mathbb{E}[Q_t + A^T_t \Pi_{t+1}A_t] + \sum_{i=1}^{M} \mathbb{E}[(K^s_{it})^T B^T_{it}\Pi_{t+1}A_t] \tag{7.4}$$

with terminal condition $\Pi_N = \mathbb{E}[Q_N]$. The resulting optimal cost is given by

$$\bar{J}(\gamma^{\circ}) = x^T_0 \Pi_0 x_0. \tag{7.5}$$

The proof of the above theorem can be obtained by dynamic programming following steps similar to the proof of Theorem 23. However the optimal expected cost-to-go for this setup at time t is chosen as

$$\mathcal{V}^{\circ}_t(x_t) = x^T_t \Pi_t x_t. \tag{7.6}$$

Proof. For a given choice of strategies $\gamma = (\gamma_0, \ldots, \gamma_{N-1})$ the expected cost-to-go function is defined as the following conditional expectation

$$\begin{aligned}\mathcal{V}^{\gamma}_t(x_t) &= \mathbb{E}\left[\sum_{k=t}^{N-1}\Big(x^T_k Q_k x_k + \gamma_k(\mathbb{I}^d_k)^T R_k \gamma_k(\mathbb{I}^d_k)\Big) + x^T_N Q_N x_N \Big| x_t\right] \\ &= \mathbb{E}[x^T_t Q_t x_t + \gamma_t(\mathbb{I}^d_t)^T R_t \gamma_t(\mathbb{I}^d_t) + \mathcal{V}^{\gamma}_{t+1}(x_{t+1})|x_t] \end{aligned} \tag{7.7}$$

with $\mathcal{V}^{\gamma}_N(x_N) = x^T_N \mathbb{E}[Q_N(\theta_N)]x_N$. The cost-to-go achieved by the optimal strategy is denoted by $\mathcal{V}^{\circ}_t$ which has the form (7.6) and is shown next. Let us assume the structure (7.6) for $\mathcal{V}^{\circ}_{t+1}(x_{t+1})$ and write the Bellman equation for dynamic programming as below

$$\begin{aligned}\mathcal{V}^{\circ}_t(x_t) &= \inf_{\gamma_t \in \mathcal{K}_t} \mathbb{E}\left[x^T_t Q_t x_t + \gamma_t(\mathbb{I}^d_t)^T R_t \gamma_t(\mathbb{I}^d_t) + \mathcal{V}^{\circ}_{t+1}(x_{t+1})\big|x_t\right] \text{ with } x_{t+1} = A_t x_t + B_t\gamma_t(\mathbb{I}^d_t) \\ &= x^T_t \mathbb{E}[Q_t]x_t + \inf_{\gamma_t \in \mathcal{K}_t} \mathbb{E}\Big[\gamma_t(\mathbb{I}^d_t)^T R_t \gamma_t(\mathbb{I}^d_t) + x^T_{t+1}\Pi_{t+1}x_{t+1}\big|x_t\Big] = x^T_t \Pi_t x_t. \end{aligned} \tag{7.8}$$

We start with the terminal time $t = N$, where the cost-to-go is $\mathcal{V}_N^\circ(x_N) = x_N^T \Pi_N x_N$ with $\Pi_N = \mathbb{E}[Q_N(\theta_N)]$ and no player actions are involved. Then at time $t = N-1$, the quadratic hypothesis for $\mathcal{V}_N^\circ$ is known to be true and the term within the conditional expectation above is not affected by past player action and types owing to the conditioning on x_t and independence of parameters. Thus the minimization problem in the second line above is of the form encountered in Theorem 20. Consequently for $t = N-1$, the optimal strategy can be obtained to be (7.3) using Theorem 20, while the quadratic structure for $\mathcal{V}_{N\text{-}1}(x_{N\text{-}1})$ is also recovered with $\Pi_{N\text{-}1}$ defined as in (7.4). This process can be continued in a backward recursive manner until $t = 0$, while at each stage $t \in \{0, \ldots, N-2\}$, the optimal strategy can be evaluated in the same manner as described above for $t = N-1$. Finally at $t = 0$, the optimal cost for the entire horizon is obtained as in (7.5) with backward recursions for Π_t given in (7.4). $\square$

7.2 Infinite Horizon

We now consider the same dynamics as in (7.1), but with no process noise i.e. $w_t = 0$.

$$x_{t+1} = A(\theta_t)\, x_t + \sum_{i=1}^{M} B_i(\theta_t)\, u_{it}. \tag{7.9}$$

Here, parameters $\theta_t = (\theta_{1t}, \ldots, \theta_{Mt}) \in \Theta$, are generated by an i.i.d. process, with joint distribution of θ_t being $\mathcal{F}$ for each $t \in \mathbb{N}_0$ and known to all players. Further, mappings $A(\cdot)$ and $B_i(\cdot)$ are time invariant and bounded.

The information set of the players is same as (7.2), but the cost function is now described by the following limit

$$\bar{J}(\gamma) = \lim_{N\to\infty} \mathbb{E}\left[J_N \left(x_0, \gamma_0(\mathbb{I}_0^d), \ldots, \gamma_{N\text{-}1}(\mathbb{I}_{N\text{-}1}^d) \right) \right] \tag{7.10}$$

where J_N is defined below in the same way as (6.3)

$$J_N(x_0, u_0, \ldots, u_{N\text{-}1}) = x_N^T Q(\theta_N) x_N + \sum_{t=0}^{N-1} \left\{ x_t^T Q(\theta_t) x_t + \sum_{i=1}^{M} \sum_{j=1}^{M} u_{it}^T R_{ij}(\theta_t) u_{jt} \right\}$$

but now the bounded mappings $Q(\cdot)$ and $R_{ij}(\cdot)$ are also time invariant. We assume that $Q(\theta) \succeq 0$ and $R(\theta) \succ \underline{a} I$ for all $\theta \in \Theta$ and some $\underline{a} > 0$. Define the corresponding means $\bar{Q} := \mathbb{E}[Q(\theta)]$ and $\bar{R} := \mathbb{E}[R(\theta)]$ for $\theta \sim \mathcal{F}$. Further, we have an assumption of mean square observability on Q described later in (32).

The solution methodology closely follows that of [51] which considers a centralized optimal control problem with stochastic system matrices. However, the differences between the problem considered here and in [51] are large enough to warrant us retracing the steps of the proof.

Before we proceed to solve the infinite horizon problem, we first introduce some definitions and notations in the lines of [51]. Let us consider the autonomous system (for definition purposes) shown below

$$x_{t+1} = H(\theta_t)x_t \tag{7.11}$$

where $H : \Theta \to \mathbb{R}^{n\times n}$ is a time invariant mapping.

Definition 26. *The system in (7.11) is said to be* mean square stable *when* $\mathbb{E}[|x_t|_2^2] \to 0$ *as* $t \to 0$ *for any initial condition* $x_0 \in \mathbb{R}^n$.

We denote $\mathcal{D}$ as the space coefficients $L : \Theta \to \mathbb{R}^{m\times n}$ of the form $L(\theta) = \begin{bmatrix} L_1(\theta_1) \\ \vdots \\ L_M(\theta_M) \end{bmatrix}$ with $L_i : \Theta_i \to \mathbb{R}^{m_i\times n}$ being functions of local parameters. Such an element $L \in \mathcal{D}$ describes linear decentralized strategies dependent on current local parameters as

$$\gamma_{it}(\mathbb{I}_{it}) = L_i(\theta_{it})x_t \quad i \in \mathcal{J} \tag{7.12}$$

In this section, we will use $L \in \mathcal{D}$ to also refer to decentralized strategies of the above form. For such a strategy, let us denote the closed loop system mapping as $A_L(\cdot)$ which is defined as $A_L(\theta) := A(\theta) + B(\theta)L(\theta)$ for $\theta \sim \mathcal{F}$.

Definition 27. *The system (7.9) is said to be* mean square stabilizable *by a decentralized control if there exists feedback policy in $\mathcal{D}$ which renders the closed loop system mean square stable.*

Since we seek the optimal decentralized controller which stabilizes the plant in mean square sense, it is natural to have the following assumption on the plant.

Assumption 28. *We assume that the system (7.9) is mean square stabilizable by a decentralized control of the form (7.12).*

One possible scenario where the above assumption holds is when the system (7.9) can be made mean square stable by a single agent (possibly setting the input of the other to zero).

We now present a few notational descriptions, useful for the upcoming result. Let $\mathbf{A} : \bar{\mathbb{S}}^n_+ \to \bar{\mathbb{S}}^n_+$ be the linear transformation defined by

$$\mathbf{A}(X) := \mathbb{E}[H(\theta)^T X\, H(\theta)] \quad \text{for} \quad \theta \sim \mathcal{F}.$$

$\mathbf{A}^i(X)$ denotes the above operation repeated i times with the convention $\mathbf{A}^0(X) = X$. For a given decentralized control $L \in \mathcal{D}$, we define a similar map for the closed loop system as

$$\mathbf{A}_L(X) := \mathbb{E}[A_L(\theta)^T X A_L(\theta)] \quad \text{for} \quad \theta \sim \mathcal{F}.$$

We also define the linear transformation $\mathbf{R}_L : \bar{\mathbb{S}}^n_+ \to \bar{\mathbb{S}}^n_+$ as follows

$$\begin{aligned}\mathbf{R}_L(X) &:= \mathbb{E}\left[A_L(\theta)^T X A_L(\theta) + Q(\theta) + L(\theta)^T R(\theta) L(\theta)\right] \\ &= \mathbf{A}_L(X) + \bar{Q} + \mathbb{E}\left[L(\theta)^T R(\theta) L(\theta)\right] \quad \text{for} \quad \theta \sim \mathcal{F}\end{aligned} \tag{7.13}$$

Let $\mathbf{R}^i_L(X)$ denote the above transformation being applied recursively i times. On comparing (7.13) with (4.3), one can verify that $x^T \mathbf{R}_L(X) x$ has a structure similar to the expected value of the cost in (4.3) with the following choice

$$\begin{aligned}&Z_{ij}(\theta) = R_{ij}(\theta) + B_i(\theta)^T X B_j(\theta), \quad Y_i(\theta) = B_i(\theta)^T X A(\theta) \\ \text{and} \quad &c(x,\theta) = x^T \left(Q(\theta) + A(\theta)^T X A(\theta)\right) x\end{aligned}$$

when a control policy of $L(\theta)x$ is applied. Moreover from Theorem 20, we know that for a full state feedback with local parameter knowledge, the corresponding static cost function can be minimized by applying the decentralized control policy $L^X \in \mathcal{D}$ which satisfies

$$\mathbb{E}\Big[\sum_{j=1}^{M} \left(R_{ij}(\theta) + B_i(\theta)^T X B_j(\theta)\right) L^X_j(\theta_j) + B_i(\theta)^T X A(\theta) \Big| \theta_i\Big] = 0. \tag{7.14}$$

Here we have used the notation $L^X(\theta) = \begin{bmatrix} L^X_1(\theta_1) \\ \vdots \\ L^X_M(\theta_M) \end{bmatrix}$. The above equation above is obtained from (4.12). The corresponding cost (optimal for the static problem) can be obtained using (4.13) and is given by $x^T \mathbf{R}^\circ(X) x$, where $\mathbf{R}^\circ : \bar{\mathbb{S}}^n_+ \to \bar{\mathbb{S}}^n_+$ is defined below

$$\mathbf{R}^\circ(X) := \mathbf{R}_{L^X}(X) = \mathbb{E}[Q + A^T X A + (L^X)^T B^T X A]. \tag{7.15}$$

Thus we have $\mathbf{R}^\circ(X) \preceq \mathbf{R}_L(X)$ for all $X \in \bar{\mathbb{S}}_+$ and $L \in \mathcal{D}$. Note that $\mathbf{R}^\circ$ is a non-linear operator.

Remark 29. *For the N-step finite horizon problem in Theorem 23 under a i.i.d. parameter setting, $x_0^T (\mathbf{R}^\circ)^N (\bar{Q}_N) x_0$ represents its optimal cost.*

Although in this section we assume a full-state feedback, for the purpose of the next definition, consider the observation model

$$y_t = C(\theta_t)\, x_t \tag{7.16}$$

with $y_t \in \mathbb{R}^l$ and C being a mapping from Θ to $\mathbb{R}^{l \times n}$.

Definition 30. *A pair (C, H) corresponding to (7.11) and (7.16) is said to be* mean square observable *if there exists $p > 0$ such that $\mathbb{E}[|y_t|_2^2] = \mathbb{E}[|C(\theta_t) x_t|_2^2] = 0$ for all $t = 0, \ldots, p-1$, implies $x_0 = 0$.*

The above definition results in the following equivalence as proved in [51, Theorem 3.5].

Lemma 31. *(C,H) is mean square observable if and only if there exists a $k \in \mathbb{Z}_+$ such that $\sum_{\tau=0}^{k-1} \mathbf{A}_0^{\tau}(\overline{C^TC}) \succ 0$ (using definition $\overline{C^TC} := \mathbb{E}[C(\theta)^T C(\theta)]$ for $\theta \sim \mathcal{F}$)*

With respect to the system (7.9) and cost (7.10), we now have the following assumption

Assumption 32. *For $Q^{\frac{1}{2}}(\theta) := Q(\theta)^{\frac{1}{2}}$, it is assumed that $(Q^{\frac{1}{2}}, A)$ is mean square observable.*

We now present the main result for the infinite horizon case.

Theorem 33. *Consider the system (7.9) with information set (7.2) and cost function (7.10). Under Assumption 28 and $R \succ 0$, the optimal decentralized control of form (7.12) is given by*

$$\gamma_i^{\circ}(\mathbb{I}_{it}) = L_i^P(\theta_{it})\, x_t \tag{7.17}$$

where $P \in \bar{\mathbb{S}}_+$ and L^P are the solution to equations

$$P = \mathbb{E}[Q + A^T P A + (L^P)^T B^T P A] \tag{7.18}$$

$$\mathbb{E}\Big[\sum_{j=1}^{M}(R_{ij} + B_i^T P B_j) L_j^P + B_i^T P A \Big| \theta_i\Big] = 0 \quad \text{for} \;\; i \in \mathcal{J}. \tag{7.19}$$

Further with $(Q^{\frac{1}{2}}, A)$ being mean square observable, such a solution exists and is unique, and the closed loop system is mean square stable. The corresponding optimal cost is given by $x_0^T P x_0$. □

The proof of the above result relies on a series of lemmas which are developed next. The following lemma is proved in [51] (by combining Lemma 2.2 and Theorem 3.2 of the reference).

Lemma 34. *Consider the system in (7.11) and the equation*

$$X = \mathbf{A}(X) + W, \quad W \succeq 0 \tag{7.20}$$

(a) If (7.11) is mean square stable then there exists a solution $X \succeq 0$ to (7.20)

(b) Conversely if there exists a $X \succeq 0$ and a $W \succ 0$ satisfying (7.20) then (7.11) is mean square stable and $X \succ 0$ □

The following lemma enumerates some properties of the mapping defined in (7.15).

Lemma 35. *(a) $\mathbf{R}^{\circ}(X) \succeq 0$ for all $X \in \bar{\mathbb{S}}_+$*

(b) For $X, Y \in \bar{\mathbb{S}}_+$ with $X \preceq Y$, we have $\mathbf{R}^{\circ}(X) \preceq \mathbf{R}^{\circ}(Y)$

(c) For $X, Y \in \bar{\mathbb{S}}_+$ with $X \preceq Y$, we have $(\mathbf{R}^{\circ})^i(X) \preceq (\mathbf{R}^{\circ})^i(Y)$ for $i \in \mathbb{Z}_+$

Proof. (a) This is straightforward from the definition of $\mathbf{R}_L$ in (7.13), since $Q(\theta)$, $R(\theta) \in \bar{\mathbb{S}}_+$ and that $\mathbf{R}^\circ(X)$ is obtained by setting $L = L^X$.

(b) We have

$$\mathbf{R}^\circ(X) = \mathbf{R}_{L^X}(X) \preceq \mathbf{R}_{L^Y}(X) \preceq \mathbf{R}_{L^Y}(Y) = \mathbf{R}^\circ(Y).$$

The first inequality uses the fact that L^X is the optimal decentralized feedback gain for the cost function corresponding to $\mathbf{R}_L(X)$ and the second inequality follows from the linearity of the mapping $\mathbf{R}_{L^Y}(\cdot)$.

(c) This is obtained by repeated application of part (b).

□

The following is adapted from [51] (by combining Theorem 3.2 and Lemma 3.2).

Lemma 36. *Consider the system in (7.11) and the equation*

$$X = \mathbf{A}(X) + \mathbb{E}[W(\theta)] \quad \textit{for} \quad \theta \sim \mathcal{F} \tag{7.21}$$

with $W : \Theta \to \bar{\mathbb{S}}^n_+$*. Then the existence of a solution* $X \in \bar{\mathbb{S}}^n_+$ *and* $(W^{\frac{1}{2}}, H)$ *being mean square observable implies that (7.11) is mean square stable and* $X \succ 0$*.*

Lemma 37. *Under assumptions* $R \succ 0$ *and* $(Q^{\frac{1}{2}}, A)$ *being mean square observable, we have* $((\bar{Q} + \mathbb{E}[L^T RL])^{\frac{1}{2}}, A_L)$ *to be mean square observable for all* $L \in \mathcal{D}$*.*

Proof. Using Lemma 31, we know that $(Q^{\frac{1}{2}}, A)$ being mean square observable implies the existence of $k \in \mathbb{Z}_+$ such that

$$\sum_{\tau=0}^{k-1} \mathbf{A}_0^\tau(\bar{Q}) = \mathbf{R}_0^k(0) \succ 0$$

Again using Lemma 31, if $\left((\bar{Q} + \mathbb{E}[L^T RL])^{\frac{1}{2}}, A_L\right)$ is not mean square observable, then there exists a non-zero initial condition x_0 such that $x_0 \left(\sum_{\tau=0}^{k-1} \mathbf{A}_0^\tau(\bar{Q} + \mathbb{E}[L^T RL])\right) x_0 = x_0^T \mathbf{R}_L^k(0) x_0 = 0$ for all $k \in \mathbb{Z}_+$. This means that the corresponding expected cost is exactly zero. This under the assumption of $R \succ 0$ implies that $u_t = L(\theta_t) x_t = 0$ almost surely which means that $x_0^T \mathbf{R}_L^k(0) x_0 = x_0^T \mathbf{R}_0^k(0) x_0 = 0$. This is a contradiction. □

Lemma 38. *Under Assumption 28 and* $R \succ 0$ *, the limit* $S = \lim_{k\to\infty} (\mathbf{R}^\circ)^k(0)$ *exists and satisfies* $S = \mathbf{R}^\circ(S)$*. Further with* $(Q^{\frac{1}{2}}, A)$ *being mean square observable, this limit satisfies* $S \succ 0$ *and is unique, and the closed loop system with control feedback* L^S *is mean square stable.*

Proof. By Lemma 35, part (a) we have $\mathbf{R}^\circ(0) \succeq 0$. Then by part (c) we have $(\mathbf{R}^\circ)^{k+1}(0) \succeq (\mathbf{R}^\circ)^k(0) \succeq 0$ for $k \in \mathbb{N}_0$, implying that $\{(\mathbf{R}^\circ)^k(0)\}_{k\in\mathbb{N}_0}$ is a non-decreasing sequence. By Assumption 28 we know that there exists a decentralized control, say $\tilde{L} \in \mathcal{D}$ which makes the closed loop mean square stable. Noting that $\bar{Q} + \mathbb{E}[\tilde{L}^T R \tilde{L}] \succeq 0$ we can apply Lemma 34 to the corresponding closed loop system to say that there exists a solution $\tilde{X} \succeq 0$ of the following

$$\tilde{X} = \mathbf{A}_{\tilde{L}}(\tilde{X}) + \bar{Q} + \mathbb{E}[\tilde{L}^T R \tilde{L}]$$

Thus $\tilde{X} = \mathbf{R}^k_{\tilde{L}}(\tilde{X}) \succeq (\mathbf{R}^\circ)^k(\tilde{X}) \succeq (\mathbf{R}^\circ)^k(0)$. $\{(\mathbf{R}^\circ)^k(0)\}_{k\in\mathbb{N}_0}$ being a non-decreasing sequence bounded by $\tilde{X}$, by a monotone convergence theorem argument, it has a limit which we call S. Since we find S by taking the limit of $(\mathbf{R}^\circ)^k(0)$, it must satisfy $S = \mathbf{R}^\circ(S) \succeq 0$. This however doesn't guarantee a unique fixed point of $S = \mathbf{R}^\circ(S)$.

By Lemma 37, we know that $\left((\bar{Q} + \mathbb{E}[L^T R L])^{\frac{1}{2}}, A_L\right)$ is mean square observable for any $L \in \mathcal{D}$. In particular this is true when $L = L^S$ (defined using (7.14)). Now using Lemma 36 we know that the closed loop system with control L^S is mean square stable and $S \succ 0$. To show the uniqueness of the solution S, we construct a sequence $(\mathbf{R}^\circ)^k(S')$ which starts at some $S' \succeq 0$ instead of 0. Now

$$(\mathbf{R}^\circ)^k(0) \preceq (\mathbf{R}^\circ)^k(S') \preceq \mathbf{R}^k_{L^S}(S') = \mathbf{R}^k_{L^S}(0) + \mathbf{A}^k_{L^S}(S') \preceq \mathbf{R}^k_{L^S}(S) + \mathbf{A}^k_{L^S}(S') = S + \mathbf{A}^k_{L^S}(S')$$

The first equality above is obtained by expanding out the terms in $\mathbf{R}^k_{L^S}(S')$ using its definition. As $k \to \infty$, the leftmost expression converges to S while the term $\mathbf{A}^k_{L^S}(S')$ in the rightmost expression converges to 0 due to the mean square stability of closed loop system with control L^S. This establishes the fact that a sequence $(\mathbf{R}^\circ)^k(S')$ constructed with any initial S' converges to S, precluding the possibility of another fixed point. □

Proof of Theorem 33. Having solved the finite horizon problem and noting that the corresponding expected cost is given by $(\mathbf{R}^\circ)^k(\bar{Q}_k)$, extension to the infinite horizon case involves setting $k \to \infty$. However one needs to ensure that the corresponding limit exists and is independent of the terminal cost matrix. This along with the mean square stability of the system was proved in Lemma 38. We thus have optimal control policy (7.17) using the finite horizon result of Theorem 23 and the corresponding optimal cost as $x_0^T P x_0$. □

7.3 Computation of Team Optimal Strategy

In Theorem 33, equations (7.18) and (7.19) in variables P and L^P are coupled and may be hard to solve in general. However if (7.19) can be solved efficiently or the two equations can be decoupled, then we can use (7.18) to find an approximate P by starting at an arbitrary guess $P^{(0)}$ and following the recursions

$$P^{(k+1)} = \mathbf{R}^\circ(P^{(k)}) \tag{7.22}$$

for sufficiently large k. The convergence of such a scheme was already established in the proof of Lemma 38.

Equation (7.18) if used in its current form may lead to asymmetric P when L^P is not computed accurately. So we may use the relation $\mathbb{E}\left[(L^P)^T(R+B^TPB)L^P+(L^P)^TB^TPA\right]=0$ obtained from (7.19) to have the following equivalent form of (7.18)

$$P=\mathbb{E}[Q+A^TPA-(L^P)^T(R+B^TPB)L^P] \tag{7.23}$$

which guarantees P to be symmetric.

We next discuss a couple of special cases, where the iterative scheme suggested in (7.22) can be easily applied.

Case I (Finite number of parameter values): If the parameters take a finite number of values, Equation (7.19) can be written in the form of a standard finite dimensional linear equation. We demonstrate this using the following example.

We consider a two player problem with each player having two possible local parameters described by: $\Theta_1=\{a,b\}$, $\Theta_2=\{c,d\}$, $\Theta=\{(a,c),(a,d),(b,c),(b,d)\}$
Since the parameters used for the two players use different alphabets, we use can follow following the shorthand notations

$$\begin{aligned}
p_{\alpha_1\alpha_2} &:= \text{Prob}\{\theta_1=\alpha_1,\theta_2=\alpha_2\} \quad \text{for} \quad \alpha_1\in\Theta_1,\ \alpha_2\in\Theta_2,\\
p_{\alpha_i} &:= \text{Prob}\{\theta_i=\alpha_i\}=\sum_{\alpha_{-i}\in\Theta_{-i}} p_{\alpha_1\alpha_2} \quad \text{for} \quad \alpha_i\in\Theta_i,\ i=1,2,\\
p_{\alpha_i|\alpha_j} &:= \text{Prob}\{\theta_i=\alpha_i|\theta_j=\alpha_j\}=\frac{p_{\alpha_1\alpha_2}}{p_{\alpha_j}} \quad \text{for} \quad \alpha_i\in\Theta_i,\ \alpha_j\in\Theta_j,\ i\neq j.
\end{aligned}$$

For a given $X\in\mathbb{S}^n_+$, we denote $\tilde{R}^X_{ij}(\theta)=R_{ij}(\theta)+B_i(\theta)^TXB_j(\theta)$. This allows us to write equation (7.19) as

$$\begin{bmatrix}
p_{c|a}\tilde{R}^X_{11}(ac)+p_{d|a}\tilde{R}^X_{11}(ad) & 0 & p_{c|a}\tilde{R}^X_{12}(ac) & p_{d|a}\tilde{R}^X_{12}(ad)\\
0 & p_{c|b}\tilde{R}^X_{11}(bc)+p_{d|b}\tilde{R}^X_{11}(bd) & p_{c|b}\tilde{R}^X_{12}(bc) & p_{d|b}\tilde{R}^X_{12}(bd)\\
p_{a|c}\tilde{R}^X_{21}(ac) & p_{b|c}\tilde{R}^X_{21}(bc) & p_{a|c}\tilde{R}^X_{22}(ac)+p_{b|c}\tilde{R}^X_{22}(bc) & 0\\
p_{a|d}\tilde{R}^X_{21}(ad) & p_{b|d}\tilde{R}^X_{21}(bd) & 0 & p_{a|d}\tilde{R}^X_{22}(ad)+p_{b|d}\tilde{R}^X_{22}(bd)
\end{bmatrix}
\begin{bmatrix} L^X_1(a)\\ L^X_1(b)\\ L^X_2(c)\\ L^X_2(d)\end{bmatrix}
+\begin{bmatrix}
p_{c|a}B_1(ac)^TXA(ac)+p_{d|a}B_1(ad)^TXA(ad)\\
p_{c|b}B_1(bc)^TXA(bc)+p_{d|b}B_1(bd)^TXA(bd)\\
p_{a|c}B_2(ac)^TXA(ac)+p_{b|c}B_2(bc)^TXA(bc)\\
p_{a|d}B_2(ad)^TXA(ad)+p_{b|d}B_2(bd)^TXA(bd)
\end{bmatrix}=0$$

which can be solved for the controller coefficients L^X for all possible parameters by a simple matrix inversion. Now starting with an arbitrary $P^{(0)}\in\bar{\mathbb{S}}^n_+$, we use the following update equations, while using the above linear equations to obtain $L^{P^{(k)}}$ from $P^{(k)}$.

$$P^{(k+1)} = \sum_{\alpha_1\alpha_2 \in \Theta} p_{\alpha_1\alpha_2} \Bigg(Q(\alpha_1\alpha_2) + A(\alpha_1\alpha_2)^T P^{(k)} A(\alpha_1\alpha_2) \\ - \begin{bmatrix} L_1^{P^{(k)}} \\ L_2^{P^{(k)}} \end{bmatrix}^T \Big(R(\alpha_1\alpha_2) + B(\alpha_1\alpha_2)^T P^{(k)} B(\alpha_1\alpha_2) \Big) \begin{bmatrix} L_1^{P^{(k)}} \\ L_2^{P^{(k)}} \end{bmatrix} \Bigg).$$

This process would converge to P corresponding to the optimal controller. This process described for a two player, four parameter setting can easily be generalized to M players and any finite number of parameters.

As a test case for this setup, we choose the following values

$$p_{ac} = 0.5,\ p_{ad} = 0.1,\ p_{bc} = 0.1,\ p_{bd} = 0.3,$$
$$Q(\theta) = R(\theta) = I \quad \text{for} \quad \theta \in \Theta$$

$$A(ac) = \begin{bmatrix} 0.2 & -0.5 \\ 0 & 0 \end{bmatrix}, \quad A(ad) = \begin{bmatrix} 0 & 0 \\ 0.2 & 1.1 \end{bmatrix}, \quad A(bc) = \begin{bmatrix} -0.1 & -0.4 \\ -0.5 & 0 \end{bmatrix}, \quad A(bd) = \begin{bmatrix} 1 & -0.3 \\ 0.5 & 0.2 \end{bmatrix}$$
$$B(ac) = \begin{bmatrix} 0 & 0 \\ 0 & -0.9 \end{bmatrix}, \quad B(ad) = \begin{bmatrix} 0 & 0.4 \\ 0.4 & 0 \end{bmatrix}, \quad B(bc) = \begin{bmatrix} 0 & 0 \\ -1.7 & 0 \end{bmatrix}, \quad B(bd) = \begin{bmatrix} 0 & 1.7 \\ 0.4 & 0.1 \end{bmatrix}.$$

We then obtain the following optimal controller

$$L_1^P(a) = \begin{bmatrix} -0.0269 & -0.119 \end{bmatrix} \qquad L_1^P(b) = \begin{bmatrix} -0.2262 & -0.0171 \end{bmatrix}$$
$$L_2^P(c) = \begin{bmatrix} -0.0295 & 0.0738 \end{bmatrix} \qquad L_2^P(d) = \begin{bmatrix} -0.4574 & 0.1664 \end{bmatrix}$$

with $P = \begin{bmatrix} 2.349 & -0.4256 \\ -0.4256 & 1.7235 \end{bmatrix}$.

Case II: Here we make the following assumptions

- local parameters $\{\theta_i\}_{i\in\mathcal{J}}$ are independent of each other
- A does not dependent on parameters
- R is block diagonal
- B_i and R_{ii} depend only on local parameter θ_i for each $i \in \mathcal{J}$.

A similar setup was also considered in [79]. In this scenario, Equation (7.19) leads to

$$(R_{ii} + B_i^T P B_i) L_i^P + B_i^T P\, \mathbb{E}\Big[\sum_{j\neq i} B_j L_j^P \Big] + B_i^T P A = 0 \quad \text{for} \quad i \in \mathcal{J}.$$

If we define $\bar{L}_i^P := \mathbb{E}[B_i L_i^P]$ and $\hat{B}_i := \mathbb{E}[B_i(R_{ii} + B_i^T P B_i)^{-1} B_i^T]$ then the above equations lead to

$$\underbrace{\begin{bmatrix} I & \hat{B}_2 P \dots & \hat{B}_M P \\ \hat{B}_1 P & I & \dots \hat{B}_M P \\ \vdots & \ddots & \vdots \\ \hat{B}_1 P & \hat{B}_2 P \dots & I \end{bmatrix}}_{\tilde{B}} \begin{bmatrix} \bar{L}_1^P \\ \bar{L}_2^P \\ \vdots \\ \bar{L}_M^P \end{bmatrix} = -\underbrace{\begin{bmatrix} \hat{B}_1 P \\ \hat{B}_2 P \\ \vdots \\ \hat{B}_M P \end{bmatrix}}_{\tilde{P}} A \tag{7.24}$$

The above can be solved for $\bar{L}_i^P$ for $i \in \mathcal{J}$, which then gives

$$L_i^P(\theta_i) = -\big(R_{ii} + B_i(\theta_i)^T P B_i(\theta_i)\big)^{-1} B_i(\theta_i)^T P\Big(\sum_{j \neq i} \bar{L}_j^P + A\Big)$$

With $V := \begin{bmatrix} I \\ \vdots \\ I \end{bmatrix}$, $\tilde{P}_d := \mathrm{diag}(\hat{B}_1 P, \dots, \hat{B}_M P)$ and $\tilde{B}$, $\tilde{P}$ defined in (7.24), we have $\tilde{B} = I - \tilde{P}_d + \tilde{P}V^T$, which leads to the following corresponding to (7.18)

$$\begin{aligned} P &= Q + A^T P A + \sum_{i=1}^{M} (\bar{L}_i^P)^T P A = Q + A^T P A - A^T \tilde{P}^T \tilde{B}^{-T} V P A \\ &= Q + A^T\big(I - \tilde{P}^T (I - \tilde{P}_d + \tilde{P}V^T)^{-T} V\big) P A = Q + A^T\big(I + V^T (I - \tilde{P}_d)^{-1}\tilde{P}\big)^{-T} P A \\ &= Q + A^T P\Bigg(I + \sum_{i=1}^{M} \hat{B}_i P (I - \hat{B}_i P)^{-1}\Bigg)^{-1} A. \end{aligned}$$

The equality in the second line above uses the identity in (2.1). The final expression above does not depend on the player strategy and hence P can be easily computed by iterations similar to (7.22).

An example of a networked control system where this framework can be applied is shown in Figure 7.1. Here the local parameter θ_{it} affects the input matrix B_i multiplicatively.

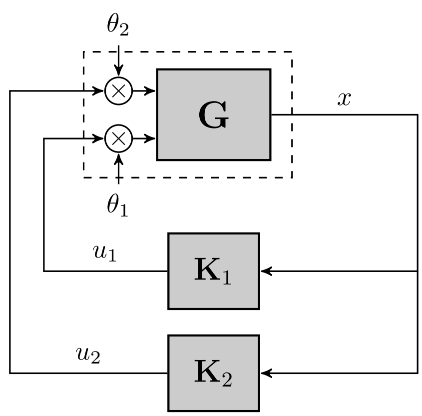


Figure 7.1: Example of a networked control system with full state feedback.

Chapter 8

Decentralized Control of Switched Nested Systems with ℓ_2-induced Norm Performance

8.1 Introduction

In this chapter, we are interested in decentralized control of nested systems with switched dynamics as depicted in Figure 8.1. The system matrices of the linear plant switch within a finite set, with the switching being governed by a parameter $\theta(t)$ generated by a finite-state automaton. The controller has access to recent values of this parameter with a finite memory. Further the plant and controller dynamics are restricted to be nested, representing a hierarchy of subsystems with a unidirectional flow of information amongst them. Such a nested structure also corresponds to the system matrices having a block lower triangular sparsity structure, which further translates to an input-output mapping of the same sparsity structure as depicted in blocks of Figure 8.1. For this setup, our goal is to stabilize the closed loop system while achieving a contractive induced ℓ_2 norm performance.

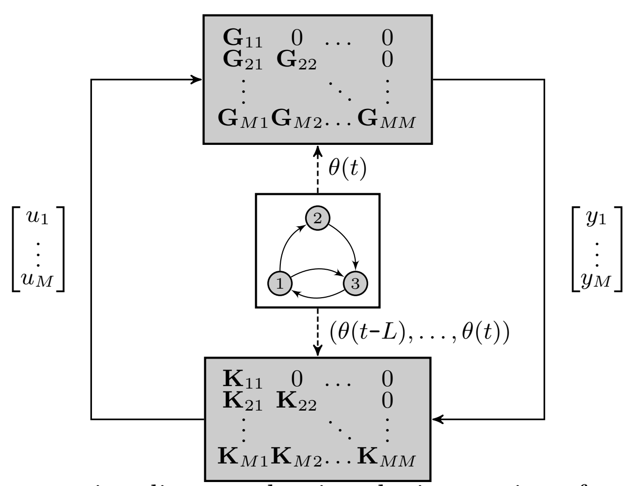


Figure 8.1: Interconnection diagram showing the interaction of controller with plant

The $\mathcal{H}_\infty$-type cost criteria of induced ℓ_2 norm, is also referred to as disturbance attenuation or root-mean square gain in literature. In general, decentralized control of systems under $\mathcal{H}_\infty$-type cost criteria has been a challenging problem with a few notable results for the non-switched setup. Some of the prior work include [80–82] where authors have considered systems distributed over lattices/graphs and synthesize controllers which assume the same topology as the plant. To be able to extend the centralized synthesis scheme, these studies restrict the scaling matrices to be of block diagonal structure with separate blocks corresponding to time and spatial updates. This however leads only to sufficient conditions for existence of controllers. Recently, [83] considered the decentralized control of continuous-time time-invariant systems with nested interconnection structure. Although the interconnection topology is more restrictive than those considered in [80–82], the conditions for existence of controllers are tight. Motivated by results in [83], a discrete-time time-varying version of the corresponding result was solved in [84] using an operator theoretic approach similar to [54]. Other recent studies which consider decentralized control of nested systems include [39–41, 85], however for different performance criteria. In this chapter, we further develop the ideas in [83, 84] and apply them towards control of a mode-dependent switched system using a finite path dependent controller. The centralized version of these results were presented in [49], which were further extended in [52] to also allow controller access to a finite number of future parameters as well. To the author's knowledge, the results presented here form the first such exploration involving decentralized control of a switched system under an $\mathcal{H}_\infty$ type performance criteria.

This chapter is organized as follows. In Section 8.2, we describe the switched problem under consideration while laying out necessary background and results regarding switched systems. In Section 8.3, necessary conditions for existence of path-dependent controllers are developed which upon use of a new result on completion of scaling matrices in Section 8.4, leads to the exact conditions presented in Section 8.5. In Section 8.6, the controller synthesis procedure is described. In Section 8.7, some possible variations of the switched result is presented. Finally, a numerical example is provided in Section 8.8.

8.2 Switched Decentralized Control Problem

In this section, we describe the decentralized switched problem under consideration in this paper. In the process, we also introduce some background and useful notations for switched systems. For the class of systems encountered here, existing analysis results in the form of conditions for achieving stability and performance are also provided.

8.2.1 Mode Dependent Switched Systems

Let us consider a switched system

$$\begin{aligned} x(t+1) &= A_{\theta(t)}x(t) + B_{\theta(t)}w(t) \\ z(t) &= C_{\theta(t)}x(t) + D_{\theta(t)}w(t) \end{aligned} \tag{8.1}$$

where the system matrices depend on switching parameters $\theta(t)$ sequenced in time. Such a system whose dynamics depend only on the current value of the switched parameter is called a *mode-dependent* system. We assume that the switching parameters take values from a finite set $\Theta = \{1, \ldots, n_s\}$ and the switching between these values in time is governed by a finite-state automata. The parameter sequences generated by such an automaton will be referred to as *admissible* sequences. We denote the set of admissible sequences of length $r \in \mathbb{N}_0$ as $\mathcal{A}_r$. Let us consider an example where switching parameter $\theta(t)$ is governed by an automaton with 3 states $\Theta = \{1, 2, 3\}$ as shown in Figure 8.2a. Here the directed edges indicate allowed switching transitions which occur exactly once every time step. Thus[1]

$$\begin{aligned} &\mathcal{A}_1 = \Theta, \quad \mathcal{A}_2 = \{12,\, 13,\, 23,\, 33,\, 31\}, \\ &\mathcal{A}_3 = \{123,\, 131,\, 133,\, 231,\, 233, 312,\, 313,\, 331,\, 333\}. \end{aligned}$$

We denote a sequence of zero length as $\emptyset$ and adopt the convention $\mathcal{A}_0 = \{\emptyset\}$.

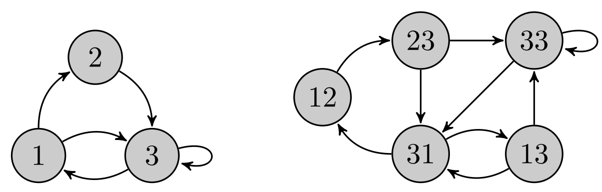


Figure 8.2: (a) Example of switching automata, (b) Corresponding induced automata for memory $L = 1$

8.2.2 Plant Description

For the decentralized control problem, we consider the following mode-dependent switched plant

$$\begin{aligned} x(t{+}1) &= A_{\theta(t)}\, x(t) + B^w_{\theta(t)}\, w(t) + B^u_{\theta(t)}\, u(t) \\ z(t) &= C^z_{\theta(t)}\, x(t) + D^{zw}_{\theta(t)}\, w(t) + D^{zu}_{\theta(t)}\, u(t) \\ y(t) &= C^y_{\theta(t)}\, x(t) + D^{yw}_{\theta(t)}\, w(t) \end{aligned} \tag{8.2}$$

with $x(0) = 0$. Here $w(t) \in \mathbb{R}^{n^w}$ is the disturbance input, $z(t) \in \mathbb{R}^{n^z}$ is the performance output, $u(t) \in \mathbb{R}^{n^u}$ is the control input and $y(t) \in \mathbb{R}^{n^y}$ is the measurement available to the controller. These vectors, sequenced by t further define corresponding elements in ℓ similar to (2.2) and are denoted

[1] Ideally we should write the sequences as $(1, 2, 3)$. However to save space we instead write such a sequence as 123 for path-dependent case.

with the same name x, w and z. The states, inputs and outputs are partitioned as

$$x(t) = \begin{bmatrix} x_1(t) \\ \vdots \\ x_M(t) \end{bmatrix}, \; u(t) = \begin{bmatrix} u_1(t) \\ \vdots \\ u_M(t) \end{bmatrix}, \; y(t) = \begin{bmatrix} y_1(t) \\ \vdots \\ y_M(t) \end{bmatrix}$$

where $x_i(t) \in \mathbb{R}^{n_i}$, $u_i(t) \in \mathbb{R}^{n_i^u}$ and $y_i(t) \in \mathbb{R}^{n_i^y}$. Corresponding sequences x_i, u_i, y_i in ℓ for $i \in \mathcal{J}$ are also defined. The dimensions satisfy $n = \sum_{i=1}^{M} n_i$, $n^u = \sum_{i=1}^{M} n_i^u$ and $n^y = \sum_{i=1}^{M} n_i^y$. We introduce the tuple $\bar{n} = (n_1, \ldots, n_M)$ and similarly define $\bar{n}^u$ and $\bar{n}^y$. As described in Section 8.2.1, the switching sequence $(\theta(0), \theta(1), \ldots)$ is governed by a finite state automaton with $\theta(t)$ taking values in a finite set Θ.

We define the space of block-lower triangular matrices of the form

$$\begin{bmatrix} H_{11} & 0 & \ldots & 0 \\ H_{21} & H_{22} & & 0 \\ \vdots & & \ddots & \vdots \\ H_{M1} & H_{M2} & \ldots & H_{MM} \end{bmatrix}$$

by $\mathcal{S}\big((m_1, \ldots, m_M), (k_1, \ldots, k_M)\big)$ so that $H_{ij} \in \mathbb{R}^{m_i \times k_j}$ and $H_{ij} = 0$ for $i < j$. For the system (8.2), we have the following assumption which enforces the nested structure.

Assumption 39. *We assume that* $A_\phi \in \mathcal{S}(\bar{n}, \bar{n})$, $B_\phi^w \in \mathcal{S}(\bar{n}, \bar{n}^u)$ *and* $C_\phi^y \in \mathcal{S}(\bar{n}^y, \bar{n})$ *for all* $\phi \in \Theta$.

As a result, it is clear that the mappings $x_j \mapsto x_i$, $u_j \mapsto x_i$, $x_j \mapsto y_i$ and $u_j \mapsto y_i$ are all zero operators for $i < j$ and $i, j \in \mathcal{J}$.

8.2.3 Path Dependent Systems and Induced Switching Sequence

Consider the switched system

$$\begin{aligned} x(t+1) &= A_{\Omega(t)} x(t) + B_{\Omega(t)} w(t) \\ z(t) &= C_{\Omega(t)} x(t) + D_{\Omega(t)} w(t) \end{aligned} \tag{8.3}$$

whose system matrices at time t depend on a switching path $\Omega(t) = (\theta(t-L), \ldots, \theta(t)) \in \mathcal{A}_{L+1}$ consisting of $L + 1$ recent values of the switching parameters. Such a system is referred to as a *finite-path dependent* system with *memory* of length L. We can modify such systems to be mode-dependent by introducing *induced* automata to reflect the path dependence (as previously suggested in [56], [52]). This is done by assuming the induced automata state-space to be $\tilde{\Theta} = \mathcal{A}_{L+1}$ with transitions governed by the original automata. Admissible sequences of length r in the induced automata is denoted by $\tilde{\mathcal{A}}_r^L$. It is not hard to verify that elements in $\tilde{\mathcal{A}}_r^L$ are equivalent to those of $\mathcal{A}_{r+L}$ for $r > 0$. To explain this we consider a finite-path dependent system with memory 1, governed by the same switching automaton as in Figure 8.2a. The induced automaton shown in

Figure 8.2b has 5 states $\tilde{\Theta} = \mathcal{A}_2$, so the set containing admissible sequences of length 2 is given by

$$\tilde{\mathcal{A}}_2^1 = \{(12,23),(13,31),(13,33),(23,31),(23,33),(31,12),(31,13),(33,31),(33,33)\}.$$

This is equivalent to $\mathcal{A}_3$ (denoted $\tilde{\mathcal{A}}_2^1 \simeq \mathcal{A}_3$).

For a sequence $\Phi = (\alpha_0, \ldots, \alpha_r) \in \tilde{\mathcal{A}}_{r+1}^L$, there exists an equivalent sequence $(\beta_0, \ldots, \beta_{r+L}) \in \mathcal{A}_{r+L+1}$. Correspondingly, for $r > 0$, we define $\bar{\Phi}, \underline{\Phi} \in \mathcal{A}_{r+L} \simeq \tilde{\mathcal{A}}_r^L$, $\Phi_\dagger \in \tilde{\Theta} = \mathcal{A}_{L+1}$ and $\Phi_\star \in \Theta$ as

$$\begin{aligned}
\bar{\Phi} &:= (\beta_1, \ldots, \beta_{r+L}) \simeq (\alpha_1, \ldots, \alpha_r), \\
\underline{\Phi} &:= (\beta_0, \ldots, \beta_{r+L-1}) \simeq (\alpha_0, \ldots, \alpha_{r-1}), \\
\Phi_\dagger &:= (\beta_r, \ldots, \beta_{r+L}) = \alpha_r, \qquad \Phi_\star := \beta_{r+L}.
\end{aligned}$$

For $r = 0$, these definitions reduce to

$$\begin{aligned}
\bar{\Phi} &:= (\beta_1, \ldots, \beta_L), \quad \underline{\Phi} := (\beta_0, \ldots, \beta_{L-1}), \\
\Phi_\dagger &:= (\beta_0, \ldots, \beta_L), \quad \Phi_\star := \beta_L.
\end{aligned}$$

However unlike $r > 0$, in this case $\mathcal{A}_L$ is not equivalent to $\tilde{\mathcal{A}}_0^L = \{\emptyset\}$.

When memory $L = 0$, which also corresponds to mode-dependent systems, $\tilde{\mathcal{A}}_r^0$ coincides with $\mathcal{A}_r$. For a sequence $\Phi = (\beta_0, \ldots, \beta_r) \in \mathcal{A}_{r+1}$ with $r > 0$, earlier definitions give

$$\bar{\Phi} := (\beta_1, \ldots, \beta_r), \quad \underline{\Phi} := (\beta_0, \ldots, \beta_{r-1}), \; \Phi_\dagger = \Phi_\star = \beta_r.$$

For $r = 0$, $\bar{\Phi} = \underline{\Phi} = \emptyset$ and $\Phi_\dagger = \Phi_\star = \beta_0$.

The above definitions depend on the type of the sequence determined by length r and memory L. To keep the notation simple, symbols used for sequences (eg. Φ, Ψ used later) do not carry this information. However the exact set on which they are defined will be clearly specified.

8.2.4 System Analysis

Consider the following Linear Time Varying (LTV) system dynamics

$$\begin{aligned}
x(t+1) &= A_t x(t) + B_t w(t) \\
z(t) &= C_t x(t) + D_t w(t)
\end{aligned} \tag{8.4}$$

with $x(0) = 0$ and where $x(t) \in \mathbb{R}^n$, $w(t) \in \mathbb{R}^{n^w}$ and $z(t) \in \mathbb{R}^{n^z}$. Note that given $w \in \ell$, there is a unique solution $x \in \ell$. The input to output mapping from ℓ^{n^w} to ℓ^{n^z} is denoted by $w \mapsto z$. In this chapter, we consider the performance criteria of a contractive induced ℓ_2 norm or $\|w \mapsto z\| < 1$. In this regard, we present the LTV version of the well known Kalman-Yakubovich-Popov (KYP) lemma (see [54]).

Lemma 40. *The system (8.4) is exponentially stable and satisfies the performance criteria of* $\|w \mapsto z\| < 1$ *if and only if there exist positive constants* a, b *and* ϵ, *and positive definite matrices* $\{X_t\}_{t\in\mathbb{N}_0}$ *satisfying* $a \preceq X_t \preceq bI$ *and*

$$\begin{bmatrix} X_t & 0 \\ 0 & I \end{bmatrix} - \begin{bmatrix} A_t & B_t \\ C_t & D_t \end{bmatrix}^T \begin{bmatrix} X_{t+1} & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} A_t & B_t \\ C_t & D_t \end{bmatrix} \succeq \epsilon I$$

for each $t \in \mathbb{N}_0$. □

We refer to X_t as scaling matrix.

Since switched systems introduced earlier are special cases of LTV systems, the definitions of stability and performance criteria defined above apply to such systems as well. Analogous to above KYP lemma, the next lemma presents conditions for stability and performance for switched systems. This result was proved in [49] and extended to incorporate a look-ahead horizon in [52].

Lemma 41. *The mode-dependent system (8.1) is exponentially stable and satisfies* $\|w \mapsto z\| < 1$ *if and only if there exists an* $r \in \mathbb{N}_0$ *and a set of positive-definite matrices* $\{X_\Psi\}_{\Psi\in\mathcal{A}_r}$ *satisfying*

$$\begin{bmatrix} X_{\underline{\Phi}} & 0 \\ 0 & I \end{bmatrix} - \begin{bmatrix} A_{\Phi_\star} & B_{\Phi_\star} \\ C_{\Phi_\star} & D_{\Phi_\star} \end{bmatrix}^T \begin{bmatrix} X_{\bar{\Phi}} & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} A_{\Phi_\star} & B_{\Phi_\star} \\ C_{\Phi_\star} & D_{\Phi_\star} \end{bmatrix} \succ 0$$

for all $\Phi \in \mathcal{A}_{r+1}$. □

For the case of $r = 0$, the notation of $\bar{\Phi} = \underline{\Phi} = \emptyset$ implies that $X_{\bar{\Phi}} = X_{\underline{\Phi}} = X_\emptyset$, or that the scaling matrices take the same value $X_\emptyset$ for any choice of switching path $\Phi \in \mathcal{A}_1$. Also note that, the number of inequalities in the above lemma is finite, unlike in Lemma 40; hence we don't need to explicitly specify the uniform bounds for the inequalities. Following lemma is an extension of the above lemma to path-dependent systems.

Lemma 42. *The finite-path dependent system (8.3) with a memory* $L \in \mathbb{N}_0$ *is exponentially stable and satisfies* $\|w \mapsto z\| < 1$ *if and only if there exists an* $r \in \mathbb{N}_0$ *and a set of positive-definite matrices* $\{X_\Psi\}_{\Psi\in\mathcal{A}_{r+L}}$ *satisfying*

$$\begin{bmatrix} X_{\underline{\Phi}} & 0 \\ 0 & I \end{bmatrix} - \begin{bmatrix} A_{\Phi_\dagger} & B_{\Phi_\dagger} \\ C_{\Phi_\dagger} & D_{\Phi_\dagger} \end{bmatrix}^T \begin{bmatrix} X_{\bar{\Phi}} & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} A_{\Phi_\dagger} & B_{\Phi_\dagger} \\ C_{\Phi_\dagger} & D_{\Phi_\dagger} \end{bmatrix} \succ 0 \tag{8.5}$$

for all $\Phi \in \tilde{\mathcal{A}}^L_{r+1}$. □

Note that the above lemma is immediate from Lemma 41 for $r > 0$, through the use of induced switching automata. For the case of $r = 0$, the sufficiency part of the proof requires retracing the proof of Lemma 41 (see [49, 52]). For the necessity part, one can increase r to be greater than 0, so that the above inequalities are satisfied for a large enough r.

Remark 43. *Finite-path dependent systems with memory $L_1 \in \mathbb{N}_0$ are also contained in the set of finite-path dependent systems with memory $L_2 > L_1$. Also, suppose the system in (8.3) with a memory L_1 has positive-definite scaling matrices $\{X_\Psi\}_{\Psi \in \mathcal{A}_{r_1+L_1}}$ satisfying (8.5) for some $r_1 > 0$. Then, we can alternatively choose a memory $L_2 = L_1 + r'$ and $r_2 = r_1 - r'$ for some non-negative integer $r' \leq r_1$ and use the same scaling matrices $\{X_\Psi\}_{\Psi \in \mathcal{A}_{r_2+L_2}}$ to describe the same set of inequalities, and hence the same stability and performance properties.*

8.2.5 Synthesis Problem

For the plant (8.2), our goal is to design finite-dimensional finite-path dependent linear controller with block lower triangular sparsity structure in order to stabilize the closed loop system and achieve a performance of contractive induced ℓ_2 norm from disturbance w to performance output z. We use the following state space representation for a finite-path dependent controller

$$\begin{aligned} x^K(t+1) &= A^K_{\Omega(t)} x^K(t) + B^K_{\Omega(t)} y(t) \\ u(t) &= C^K_{\Omega(t)} x^K(t) + D^K_{\Omega(t)} y(t). \end{aligned} \tag{8.6}$$

For a controller with memory L, the above system matrices at time t depend on a switching path given by $\Omega(t) = (\theta(t-L), \ldots, \theta(t)) \in \mathcal{A}_{L+1}$ consisting of $L+1$ recent values of the plant's switching parameter. The controller state $x^K(t) \in \mathbb{R}^{n^K}$ is partitioned as $\left[(x^K_1(t))^T \ldots (x^K_M(t))^T\right]^T$ with $x^K_i(t) \in \mathbb{R}^{n^K_i}$ thus satisfying $n^K = n^K_1 + \ldots + n^K_M$. For given controller dimensions $\{n^K_i\}_{i=1}^M$, our objective is to design the above controller by determining a finite memory L and associated structured controller matrices

$$\begin{aligned} &A^K_\Psi \in \mathcal{S}(\bar{n}^K, \bar{n}^K), && B^K_\Psi \in \mathcal{S}(\bar{n}^K, \bar{n}^y), \\ &C^K_\Psi \in \mathcal{S}(\bar{n}^u, \bar{n}^K), && D^K_\Psi \in \mathcal{S}(\bar{n}^u, \bar{n}^y) \end{aligned} \tag{8.7}$$

for every admissible sequence $\Psi \in \mathcal{A}_{L+1}$. Here we have used $\bar{n}^K = (n^K_1, \ldots, n^K_M)$. The resulting controller has a y to u mapping with a lower triangular sparsity structure as depicted in Figure 8.1.

8.3 Necessary Conditions for Existence of Controller

This section is devoted to developing necessary conditions for existence of a finite-path dependent synthesis. But first we describe the closed loop system and define notations associated with it. We also present a lemma useful for eliminating controller matrices from the closed loop KYP inequality.

8.3.1 Closed Loop System

While using a path-dependent controller of memory L (as described in (8.6)) with system (8.2), it is clear that the closed-loop system is also path-dependent with memory L. In particular, the closed loop has the following dynamics

$$\begin{aligned} x^C(t+1) &= A^C_{\Omega(t)} x^C(t) + B^C_{\Omega(t)} w(t) \\ z(t) &= C^C_{\Omega(t)} x^C(t) + D^C_{\Omega(t)} w(t) \end{aligned} \tag{8.8}$$

with $x^C(t) = \begin{bmatrix} x(t) \\ x^K(t) \end{bmatrix}$. At time t, the closed-loop system matrices $A^C_{\Omega(t)}$, $B^C_{\Omega(t)}$, $C^C_{\Omega(t)}$ and $D^C_{\Omega(t)}$ depend on the same switching sequence $\Omega(t) = (\theta(t-L), \ldots, \theta(t)) \in \mathcal{A}_{L+1}$ as the controller in (8.6). For all possible $\Psi \in \mathcal{A}_{L+1}$, we can write the closed-loop system matrices as an affine combination of the controller matrices as $Q^C_\Psi :=$

$$\begin{bmatrix} A^C_\Psi & B^C_\Psi \\ C^C_\Psi & D^C_\Psi \end{bmatrix} = \left[\begin{array}{cc|c} A_{\Psi_\star}+B^u_{\Psi_\star} D^K_\Psi C^y_{\Psi_\star} & B^u_{\Psi_\star} C^K_\Psi & B^w_{\Psi_\star}+B^u_{\Psi_\star} D^K_\Psi D^{yw}_{\Psi_\star} \\ B^K_\Psi C^y_\Psi & A^K_\Psi & B^K_\Psi D^{yw}_{\Psi_\star} \\ \hline C^z_{\Psi_\star}+D^{zu}_{\Psi_\star} D^K_\Psi C^y_{\Psi_\star} & D^{zu}_{\Psi_\star} C^K_\Psi & D^{zw}_{\Psi_\star}+D^{zu}_{\Psi_\star} D^K_\Psi D^{yw}_{\Psi_\star} \end{array}\right].$$

It is well-known that the above can be written as

$$Q^C_\Psi = R_{\Psi_\star} + (U^C_{\Psi_\star})^T Q^K_\Psi V^C_{\Psi_\star} \tag{8.9}$$

with $Q^K_\Psi = \begin{bmatrix} A^K_\Psi & B^K_\Psi \\ C^K_\Psi & D^K_\Psi \end{bmatrix}$ representing the unknown controller matrices, and the following defined for $\phi \in \Theta$

$$R_\phi = \left[\begin{array}{cc|c} A_\phi & 0 & B^w_\phi \\ 0 & 0 & 0 \\ \hline C^z_\phi & 0 & D^{zw}_\phi \end{array}\right], \; (U^C_\phi)^T = \left[\begin{array}{cc} 0 & B^u_\phi \\ I & 0 \\ \hline 0 & D^{zu}_\phi \end{array}\right], \; V^C_\phi = \left[\begin{array}{cc|c} 0 & I & 0 \\ C^y_\phi & 0 & D^{yw}_\phi \end{array}\right].$$

The matrix Q^K_Ψ being structured, can be written as a linear combination of unstructured ones as described by the following relation[2]

$$Q^K_\Psi = \sum_{i=1}^{M} \begin{bmatrix} \bar{E}^K_{i-1} & \\ 0 & \bar{E}^u_{i-1} \end{bmatrix} Q_{i,\Psi} \begin{bmatrix} E^K_i & 0 \\ 0 & E^y_i \end{bmatrix}^T \tag{8.10}$$

[2]Note that the decomposition (8.10) is not unique. However existence of Q^K_Ψ implies the existence of $\{Q_{i,\Psi}\}_{i\in\mathcal{J}}$ and vice-versa.

where $Q_{i,\Psi} \in \mathbb{R}^{\left((n_i^K+n_i^u)+\ldots+(n_M^K+n_M^u)\right)\times\left((n_1^K+n_1^y)+\ldots+(n_i^K+n_i^y)\right)}$ are unstructured and matrices $E_i^\bullet$ and $\bar{E}_i^\bullet$ ($\bullet$ can be replaced with K, u or y) are defined using

$$E_i^\bullet = \begin{bmatrix} I_{n_1^\bullet+\ldots+n_i^\bullet} \\ 0 \end{bmatrix}, \quad \bar{E}_i^\bullet = \begin{bmatrix} 0 \\ I_{n_{i+1}^\bullet+\ldots+n_M^\bullet} \end{bmatrix}$$

for $i \in \bar{\mathcal{J}}$. Note that these satisfy $(\bar{E}_i^{\bullet T})_\perp = E_i^\bullet$ and $(E_i^{\bullet T})_\perp = \bar{E}_i^\bullet$. We can thus write (8.9) as

$$Q_\Psi^C = R_{\Psi_\star} + \sum_{i=1}^{M} (U_{i,\Psi_\star}^C)^T Q_{i,\Psi} V_{i,\Psi_\star}^C \tag{8.11}$$

with

$$U_{i,\phi}^C := \begin{bmatrix} \bar{E}_{i-1}^K & \\ 0 & \bar{E}_{i-1}^u \end{bmatrix}^T U_\phi^C = \left[\begin{array}{cc|c} 0 & (\bar{E}_{i-1}^K)^T & 0 \\ (\bar{E}_{i-1}^u)^T (B_\phi^u)^T & 0 & (\bar{E}_{i-1}^u)^T (D_\phi^{zu})^T \end{array}\right],$$
$$V_{i,\phi}^C := \begin{bmatrix} E_i^K & 0 \\ 0 & E_i^y \end{bmatrix}^T V_\phi^C = \left[\begin{array}{cc|c} 0 & (E_i^K)^T & 0 \\ (E_i^y)^T C_\phi^y & 0 & (E_i^y)^T D_\phi^{yw} \end{array}\right]$$

for $\phi \in \Theta$ and $i \in \mathcal{J}$. Note that $U_{1,\phi}^C = U_\phi^C$ and $V_{M,\phi}^C = V_\phi^C$.

The following matrices (through their image spaces) describe the kernels of the above matrices

$$(U_{i,\phi}^C)_\perp = \left[\begin{array}{cc} N_{i-1,\phi}^{u,x} & 0 \\ 0 & E_{i-1}^K \\ \hline N_{i-1,\phi}^{u,z} & 0 \end{array}\right], \quad (V_{i,\phi}^C)_\perp = \left[\begin{array}{cc} N_{i,\phi}^{y,x} & 0 \\ 0 & \bar{E}_i^K \\ \hline N_{i,\phi}^{y,w} & 0 \end{array}\right].$$

which further use the following definitions

$$N_{i,\phi}^y = \begin{bmatrix} N_{i,\phi}^{y,x} \\ N_{i,\phi}^{y,w} \end{bmatrix} = \left[(E_i^y)^T C_\phi^y \;\; (E_i^y)^T D_\phi^{yw}\right]_\perp,$$
$$N_{i,\phi}^u = \begin{bmatrix} N_{i,\phi}^{u,x} \\ N_{i,\phi}^{u,z} \end{bmatrix} = \left[(\bar{E}_i^u)^T (B_\phi^u)^T \;\; (\bar{E}_i^u)^T (D_\phi^{zu})^T\right]_\perp$$

with the row-dimensions of $N_{i,\phi}^{y,x}$, $N_{i,\phi}^{y,w}$, $N_{i,\phi}^{u,x}$, $N_{i,\phi}^{u,z}$, $N_{i,\phi}^y$ and $N_{i,\phi}^u$ being n, n^w, n, n^z, $n+n^w$ and $n+n^z$ respectively. Also $N_{0,\phi}^y = I$ and $N_{M,\phi}^u = I$.

With respect to Lemma 42, the closed loop scaling matrices are denoted by $X_\Psi^C \in \mathbb{S}_+^{n+n^K}$, defined for each $\Psi \in \mathcal{A}_{r+L}$ and some appropriately chosen $r \in \mathbb{N}_0$. These matrices are partitioned into plant and controller sections as

$$X_\Psi^C = \begin{bmatrix} X_\Psi & X_\Psi^{GK} \\ (X_\Psi^{GK})^T & X_\Psi^K \end{bmatrix}, \quad (X_\Psi^C)^{-1} = \begin{bmatrix} Y_\Psi & Y_\Psi^{GK} \\ (Y_\Psi^{GK})^T & Y_\Psi^K \end{bmatrix} \tag{8.12}$$

with $X_\Psi, Y_\Psi \in \mathbb{S}^n_+$, $X^{GK}_\Psi, Y^{GK}_\Psi \in \mathbb{R}^{n\times n^K}$ and $X^K_\Psi, Y^K_\Psi \in \mathbb{S}^{n^K}_+$. We further define the following for $i \in \bar{\mathcal{J}}$

$$\begin{aligned} Z_{i,\Psi} :=& \left\{ X_\Psi - X^{GK}_\Psi \bar{E}^K_i \left((\bar{E}^K_i)^T X^K_\Psi \bar{E}^K_i \right)^{-1} (X^{GK}_\Psi \bar{E}^K_i)^T \right\}^{-1} \\ =& Y_\Psi - Y^{GK}_\Psi E^K_i \left((E^K_i)^T Y^K_\Psi E^K_i \right)^{-1} (Y^{GK}_\Psi E^K_i)^T \end{aligned} \tag{8.13}$$

while noting that $Z_{0,\Psi} = Y_\Psi$ and $Z_{N,\Psi} = X^{-1}_\Psi$. The equality above is as a result of Lemma 62 in appendix. Also, note that $Z_{i,\Psi}$ is the $(1,1)$ block of the inverse of $X^C_{i,\Psi}$, or alternatively $Z^{-1}_{i,\Psi}$ is the $(1,1)$ block of the inverse of $Y^C_{i,\Psi}$, which are defined below

$$X^C_{i,\Psi} := \begin{bmatrix} X_\Psi & X^{GK}_\Psi \bar{E}^K_i \\ \bar{E}^K_i)^T (X^{GK}_\Psi)^T & (\bar{E}^K_i)^T X^K_\Psi \bar{E}^K_i \end{bmatrix}, \; Y^C_{i,\Psi} := \begin{bmatrix} Y_\Psi & Y^{GK}_\Psi E^K_i \\ (E^K_i)^T (Y^{GK}_\Psi)^T & (E^K_i)^T Y^K_\Psi E^K_i \end{bmatrix}. \tag{8.14}$$

8.3.2 Elimination Lemma

Before we proceed to the controller synthesis, we develop a lemma to eliminate structured matrices in this subsection.

Lemma 44. *Consider $W \in \mathbb{S}^n$, matrices $S_0 = 0$, $P_{M+1} = 0$, $\{P_i\}^M_{i=1}$ and $\{S_i\}^M_{i=1}$ each with column dimension n, satisfying*

$$\mathrm{Ker}(P_1) \subset \mathrm{Ker}(P_2) \subset \cdots \subset \mathrm{Ker}(P_M),$$

and full column rank matrices $\{N_i\}^M_{i=0}$ satisfying $\mathrm{Im}(N_i) = \mathrm{Ker}(S_0) \cap \cdots \cap \mathrm{Ker}(S_{i-1})$ for $i \in \mathcal{J}$. Then the following hold.

(i) The inequality

$$W + \sum_{i=1}^{M} \left(P_i^T Q_i S_i + S_i^T Q_i^T P_i \right) \succ 0 \tag{8.15}$$

in the unstructured variables $\{Q_i\}^M_{i=1}$ has a solution if and only if W is positive-definite on the subspaces $\mathrm{Ker}(S_0) \cap \cdots \cap \mathrm{Ker}(S_i) \cap \mathrm{Ker}(P_{i+1})$ for $i \in \bar{\mathcal{J}}$.

(ii) Further, if a solution exists $\{Q_i\}^M_{i=1}$ can be constructed by recursively solving the following inequalities

$$N_i^T \left(W + \sum_{j=i}^{M} \left(P_j^T Q_j S_j + S_j^T Q_j^T P_j \right) \right) N_i \succ 0 \tag{8.16}$$

in the order $i = M, \ldots, 1$. □

The above lemma is from [86, Theorem 2], which was further used in [83] to solve a decentralized control problem in continuous-time. For the discrete-time setting, we present the following lemma while making use of Lemma 44(i).

Lemma 45. *Given $Z \in \mathbb{S}_+^n$, $H \in \mathbb{S}_+^m$, $R \in \mathbb{R}^{n\times m}$, and matrices $\{U_i\}_{i=1}^M$ and $\{V_i\}_{i=1}^M$ with column dimensions n and m respectively, satisfying*

$$\begin{aligned} &\mathrm{Ker}(U_1) \subset \mathrm{Ker}(U_2) \subset \cdots \subset \mathrm{Ker}(U_M) \\ \textit{and} \quad &\mathrm{Ker}(V_1) \supset \mathrm{Ker}(V_2) \supset \cdots \supset \mathrm{Ker}(V_M), \end{aligned}$$

the inequality

$$Z - \Big(R + \sum_{i=1}^{M} U_i^T Q_i V_i\Big)^T H \Big(R + \sum_{i=1}^{M} U_i^T Q_i V_i\Big) \succ 0 \tag{8.17}$$

in the unstructured variables $\{Q_i\}_{i=1}^M$ has a solution if and only if the following hold

$$\begin{bmatrix} U_{i+1\perp} & 0 \\ 0 & V_{i\perp} \end{bmatrix}^T \begin{bmatrix} H^{-1} & R \\ R^T & Z \end{bmatrix} \begin{bmatrix} U_{i+1\perp} & 0 \\ 0 & V_{i\perp} \end{bmatrix} \succ 0 \tag{8.18}$$

for $i = 0, \ldots, M$. Here we have additional definitions of $V_{0\perp} = I$ and $U_{M+1\perp} = I$.

Proof. Using the Schur compliment formula, we can write (8.17) equivalently in the form of (8.15) with

$$W = \begin{bmatrix} H^{-1} & R \\ R^T & Z \end{bmatrix}, \ P_i = \begin{bmatrix} U_i & 0 \end{bmatrix} \text{ and } S_i = \begin{bmatrix} 0 & V_i \end{bmatrix}.$$

Further $P_{i\perp} = \begin{bmatrix} U_{i\perp} & 0 \\ 0 & I \end{bmatrix}$, $S_{i\perp} = \begin{bmatrix} I & 0 \\ 0 & V_{i\perp} \end{bmatrix}$ and $\begin{bmatrix} P_{i+1} \\ S_i \end{bmatrix}_\perp = \begin{bmatrix} U_{i+1} & 0 \\ 0 & V_i \end{bmatrix}_\perp = \begin{bmatrix} U_{i+1\perp} & 0 \\ 0 & V_{i\perp} \end{bmatrix}$ whose columns also form the basis of the space $\mathrm{Ker}(S_i) \cap \mathrm{Ker}(P_{i+1})$. Having the above definitions in place, we can use Lemma 44 to show the equivalence between (8.18) and (8.17). $\square$

8.3.3 Necessary Conditions

The next lemma develops a necessary condition for existence of the controller by using Lemmas 42 and 45.

Lemma 46. *Consider the system (8.2) along with the structural description in Assumption 39. There exists a finite path dependent controller (8.6) structured as (8.7) which stabilizes this system if and only if there exist an $L \in \mathbb{N}_0$ and positive-definite $\{X_\Psi^C\}_{\Psi\in\mathcal{A}_L}$ such that corresponding $\{Z_{i,\Psi}\}_{i=0}^M$*

(defined by (8.12) and (8.13)) satisfy

$$[\bullet]^T \left[\begin{array}{cc|cc} Z_{i,\bar{\Phi}} & 0 & A_{\Phi_\star} & B^w_{\Phi_\star} \\ 0 & I & C^z_{\Phi_\star} & D^{zw}_{\Phi_\star} \\ \hline A^T_{\Phi_\star} & (C^z_{\Phi_\star})^T & Z^{-1}_{i,\underline{\Phi}} & 0 \\ (B^w_{\Phi_\star})^T & (D^{zw}_{\Phi_\star})^T & 0 & I \end{array}\right] \begin{bmatrix} N^u_{i,\Phi_\star} & 0 \\ 0 & N^y_{i,\Phi_\star} \end{bmatrix} \succ 0 \textit{ for all } i \in \bar{\mathcal{J}} \textit{ and } \Phi \in \mathcal{A}_{L+1}. \tag{8.19}$$

Proof. ($\Longrightarrow$) Given a finite path dependent controller with memory L' as $Q^K_\Psi = \begin{bmatrix} A^K_\Psi & B^K_\Psi \\ C^K_\Psi & D^K_\Psi \end{bmatrix}$ for $\Psi \in \mathcal{A}_{L'+1}$ that stabilizes the plant and achieves contractive performance, we can construct the closed loop system with memory L' using (8.8) and (8.9). Since the closed loop is stable and contractive, using Lemma 42, we know that there exist an $r \in \mathbb{N}_0$ and positive-definite scaling matrices $\{X^C_\Psi\}_{\Psi \in \mathcal{A}_{r+L'}}$ satisfying

$$\begin{bmatrix} X^C_{\underline{\Phi}} & 0 \\ 0 & I \end{bmatrix} - (Q^C_{\Phi_\dagger})^T \begin{bmatrix} X^C_{\bar{\Phi}} & 0 \\ 0 & I \end{bmatrix} Q^C_{\Phi_\dagger} \succ 0 \tag{8.20}$$

for all $\Phi \in \tilde{\mathcal{A}}^{L'}_{r+1}$. Substituting expansion (8.11) into the above, we get a set of inequalities in unstructured controller variables $\{Q_{i,\Psi}\}_{i \in \mathcal{J}, \Psi \in \mathcal{A}_{L'+1}}$ in addition to the scaling matrices. We next eliminate these controller matrices from the above inequalities using Lemma 45. This however can be done only[3] for the case of $r = 0$. So we extend the controller/closed loop memory to $L = L' + r$ (see Remark 43) and make the sequence length of scaling matrices in (8.20) to be zero. Now applying Lemma 45, we know that (8.20) implies the existence of $L \in \mathbb{N}_0$ and positive-definite $\{X^C_\Psi\}_{\Psi \in \mathcal{A}_L}$ such that the following is satisfied for all $i \in \bar{\mathcal{J}}$ and $\Phi \in \tilde{\mathcal{A}}^L_1 \simeq \mathcal{A}_{L+1}$

$$\begin{bmatrix} (U^C_{i+1,\Phi_\star})_\perp & 0 \\ 0 & (V^C_{i,\Phi_\star})_\perp \end{bmatrix}^T \left[\begin{array}{c|c} \begin{matrix} (X^C_{\bar{\Phi}})^{-1} & 0 \\ 0 & I \end{matrix} & R_{\Phi_\star} \\ \hline R^T_{\Phi_\star} & \begin{matrix} X^C_{\underline{\Phi}} & 0 \\ 0 & I \end{matrix} \end{array}\right] \begin{bmatrix} (U^C_{i+1,\Phi_\star})_\perp & 0 \\ 0 & (V^C_{i,\Phi_\star})_\perp \end{bmatrix} \succ 0. \tag{8.21}$$

Upon use of definitions in Section 8.3.1, we note that (8.21) is same as

$$[\bullet]^T \begin{bmatrix} Y_{\bar{\Phi}} & Y^{GK}_{\bar{\Phi}} & 0 & A_{\Phi_\star} & 0 & B^w_{\Phi_\star} \\ (Y^{GK}_{\bar{\Phi}})^T & Y^K_{\bar{\Phi}} & 0 & 0 & 0 & 0 \\ 0 & 0 & I & C^z_{\Phi_\star} & 0 & D^{zw}_{\Phi_\star} \\ A^T_{\Phi_\star} & 0 & (C^z_{\Phi_\star})^T & X_{\underline{\Phi}} & X^{GK}_{\underline{\Phi}} & 0 \\ 0 & 0 & 0 & (X^{GK}_{\underline{\Phi}})^T & X^K_{\underline{\Phi}} & 0 \\ (B^w_{\Phi_\star})^T & 0 & (D^{zw}_{\Phi_\star})^T & 0 & 0 & I \end{bmatrix} \begin{bmatrix} N^{u,x}_{i,\Phi_\star} & 0 & 0 & 0 \\ 0 & E^K_i & 0 & 0 \\ N^{u,z}_{i,\Phi_\star} & 0 & 0 & 0 \\ 0 & 0 & N^{y,x}_{i,\Phi_\star} & 0 \\ 0 & 0 & 0 & \bar{E}^K_i \\ 0 & 0 & N^{y,w}_{i,\Phi_\star} & 0 \end{bmatrix} \succ 0.$$

[3]For $r > 0$, direct elimination using Lemma 45 is not feasible because the unknown matrix $Q_{i,\Psi}$ for a particular $\Psi \in \mathcal{A}_{L'+1}$ appears in multiple inequalities of (8.20).

Further, the above is equivalent to

$$[\bullet]^T \begin{bmatrix} Y_{\bar{\Phi}} & 0 & Y_{\bar{\Phi}}^{GK} & A_{\Phi_\star} & B_{\Phi_\star}^w & 0 \\ 0 & I & 0 & C_{\Phi_\star}^z & D_{\Phi_\star}^{zw} & 0 \\ (Y_{\bar{\Phi}}^{GK})^T & 0 & Y_{\bar{\Phi}}^K & 0 & 0 & 0 \\ (A_{\Phi_\star})^T & (C_{\Phi_\star}^z)^T & 0 & X_{\underline{\Phi}} & 0 & X_{\underline{\Phi}}^{GK} \\ (B_{\Phi_\star}^w)^T & (D_{\Phi_\star}^{zw})^T & 0 & 0 & I & 0 \\ 0 & 0 & 0 & (X_{\underline{\Phi}}^{GK})^T & 0 & X_{\underline{\Phi}}^K \end{bmatrix} \begin{bmatrix} N_{i,\Phi_\star}^{u,x} & 0 & 0 & 0 \\ N_{i,\Phi_\star}^{u,z} & 0 & 0 & 0 \\ 0 & E_i^K & 0 & 0 \\ 0 & 0 & N_{i,\Phi_\star}^{y,x} & 0 \\ 0 & 0 & N_{i,\Phi_\star}^{y,w} & 0 \\ 0 & 0 & 0 & \bar{E}_i^K \end{bmatrix} \succ 0,$$

which can be seen by multiplying permutation matrices $\mathrm{diag}\left(\left[\begin{smallmatrix} I & 0 & 0 \\ 0 & 0 & I \\ 0 & I & 0 \end{smallmatrix}\right], \left[\begin{smallmatrix} I & 0 & 0 \\ 0 & 0 & I \\ 0 & I & 0 \end{smallmatrix}\right]\right)$ between the products. Using Schur complement formula twice, followed by relations defined in (8.13), the above inequality (hence also (8.21)) can be shown to be equivalent to (8.19).

($\Longleftarrow$) The proof follows the same steps as in the converse direction but in the reverse order. Note that the step involving the use of elimination lemma leading to inequality (8.20) proves the existence a controller of memory L. $\square$

In the previous lemma, we obtained inequalities which are necessary for the existence of the controller. However, they are not sufficient because for some L, the existence of $\{Z_{i,\Psi}\}_{i\in\mathcal{J}}$ doesn't directly imply the existence of a X_Ψ^C for each $\Psi \in \mathcal{A}_L$. Additional conditions that ensure sufficiency will be developed in the next section. Also, the inequalities (8.19) are not linear in $\{Z_{i,\Psi}\}_{i\in\mathcal{J},\Psi\in\mathcal{A}_L}$. Towards the goal of obtaining linear inequalities, the next lemma defines a factorization which was originally performed in [83] for a similar context.

Lemma 47. *For a symmetric matrix* $X = \begin{bmatrix} X_1 & X_2 \\ X_2^T & X_3 \end{bmatrix}$ *with invertible* $X_1 \in \mathbb{S}^{m_1}$*,* $X_2 \in \mathbb{R}^{m_1 \times m_2}$ *and* $X_3 \in \mathbb{S}^{m_2}$*, we can define the triple* $\{Z^a, Z^b, Z^c\}$ *with* $Z^a \in \mathbb{S}^{m_1}$*,* $Z^b \in \mathbb{R}^{m_1\times m_2}$ *and* $Z^c \in \mathbb{S}^{m_2}$*, related to* X *by the following bijective mapping*

$$Z^a = X_1^{-1}, \quad Z^b = -X_1^{-1}X_2, \quad Z^c = X_3 - X_2^T X_1^{-1} X_2.$$

The triple then defines the following unique factorization

$$X = \begin{bmatrix} I & 0 \\ -(Z^b)^T & Z^c \end{bmatrix} \begin{bmatrix} Z^a & Z^b \\ 0 & I \end{bmatrix}^{-1}. \tag{8.22}$$

Further $X \succ 0$ *if and only if* $Z^a \succ 0$ *and* $Z^c \succ 0$. $\square$

In view of this lemma, for positive-definite $\{Z_{i,\Psi}\}_{i\in\bar{\mathcal{J}},\Psi\in\mathcal{A}_L}$ we define the following associated matrices

$$
\begin{aligned}
Z^a_{i,\Psi} &:= (E_i^T Z_{i,\Psi} E_i)^{-1}, \quad Z^b_{i,\Psi} := -Z^a_{i,\Psi}(E_i^T Z_{i,\Psi} \bar{E}_i),\\
Z^c_{i,\Psi} &:= \bar{E}_i^T Z_{i,\Psi} \bar{E}_i - (E_i^T Z_{i,\Psi} \bar{E}_i)^T (E_i^T Z_{i,\Psi} E_i)^{-1} E_i^T Z_{i,\Psi} \bar{E}_i.
\end{aligned}
\tag{8.23}
$$

for $i \in \bar{\mathcal{J}}$ and $\Psi \in \mathcal{A}_L$. Note that $Z^c_{0,\Psi} = Y_\Psi$ and $Z^a_{M,\Psi} = X_\Psi$, while $Z^a_{0,\Psi}$, $Z^b_{0,\Psi}$, $Z^b_{M,\Psi}$ and $Z^c_{M,\Psi}$ have at least one of their dimensions as zero. Since $Z_{i,\Psi} \in \mathbb{S}^n_+$, using the above relations it can be verified that $Z^a_{i,\Psi} \in \mathbb{S}^{n_1+\ldots+n_i}_+$, $Z^b_{i,\Psi} \in \mathbb{R}^{(n_1+\ldots+n_i)\times(n_{i+1}+\ldots+n_M)}$, and $Z^c_{i,\Psi} \in \mathbb{S}^{n_{i+1}+\ldots+n_M}_+$ for $i \in \bar{\mathcal{J}}$. These matrices define the following factorization similar to (8.22)

$$
Z_{i,\Psi} = Z^l_{i,\Psi}(Z^u_{i,\Psi})^{-1} = (Z^u_{i,\Psi})^{-T}(Z^l_{i,\Psi})^T \tag{8.24}
$$

$$
\text{with} \quad Z^l_{i,\Psi} = \begin{bmatrix} I & 0 \\ -(Z^b_{i,\Psi})^T & Z^c_{i,\Psi} \end{bmatrix} \text{ and } Z^u_{i,\Psi} = \begin{bmatrix} Z^a_{i,\Psi} & Z^b_{i,\Psi} \\ 0 & I \end{bmatrix}. \tag{8.25}
$$

Note that $Z^l_{i,\Psi}$ and $Z^u_{i,\Psi}$ are invertible due to positive-definiteness of $Z^c_{i,\Psi}$ and $Z^a_{i,\Psi}$ respectively.

We now use the factorization in (8.24) and corresponding change of variables to convert the inequalities in Lemma 46 to be linear in the new variables.

Lemma 48. *Given positive-definite matrices $\{X^C_\Psi\}_{\Psi\in\mathcal{A}_L}$, define associated $\{Z_{i,\Psi}\}_{i\in\bar{\mathcal{J}},\Psi\in\mathcal{A}_L}$ and $\{Z^a_{i,\Psi},\, Z^b_{i,\Psi},\, Z^c_{i,\Psi}\}_{i\in\bar{\mathcal{J}},\Psi\in\mathcal{A}_L}$ using (8.13) and (8.23). Then the inequality (8.19) is equivalent to the following inequalities linear in variables $\{Z^a_{i,\Psi},\, Z^b_{i,\Psi},\, Z^c_{i,\Psi}\}_{i\in\bar{\mathcal{J}},\Psi\in\mathcal{A}_L}$*

$$
\begin{bmatrix} N^u_{i,\Phi_\star} & 0 \\ 0 & N^y_{i,\Phi_\star} \end{bmatrix}^T
\left[\begin{array}{cc|cc}
(Z^u_{i\bar{\Phi}})^T Z^l_{i\bar{\Phi}} & 0 & (Z^u_{i\bar{\Phi}})^T A_{\Phi_\star} Z^l_{i\underline{\Phi}} & (Z^u_{i\bar{\Phi}})^T B^w_{\Phi_\star} \\
0 & I & C^z_{\Phi_\star} Z^l_{i\underline{\Phi}} & D^{zw}_{\Phi_\star} \\ \hline
\bullet & \bullet & (Z^u_{i\underline{\Phi}})^T Z^l_{i\underline{\Phi}} & 0 \\
\bullet & \bullet & 0 & I
\end{array}\right]
\begin{bmatrix} N^u_{i,\Phi_\star} & 0 \\ 0 & N^y_{i,\Phi_\star} \end{bmatrix} \succ 0 \tag{8.26}
$$

for all $i \in \bar{\mathcal{J}}$ and $\Phi \in \mathcal{A}_{L+1}$. □

Remark 49. *The above inequalities are linear in the variables due to the following simplifications:*

$$
(Z^u_{i,\Psi})^T Z^l_{i,\Psi} = (Z^l_{i,\Psi})^T Z^u_{i,\Psi} = \begin{bmatrix} Z^a_{i,\Psi} & 0 \\ 0 & Z^c_{i,\Psi} \end{bmatrix} \text{ for } \Psi \in \mathcal{A}_L \text{ and}
$$

$$
(Z^u_{i\bar{\Phi}})^T A_{\Phi_\star} Z^l_{i\underline{\Phi}} = \begin{bmatrix} Z^a_{i\bar{\Phi}} \tilde{A}^{11}_{i,\Phi_\star} & 0 \\ (Z^b_{i\bar{\Phi}})^T \tilde{A}^{11}_{i,\Phi_\star} + \tilde{A}^{21}_{i,\Phi_\star} - \tilde{A}^{22}_{i,\Phi_\star}(Z^b_{i\underline{\Phi}})^T & \tilde{A}^{22}_{i,\Phi_\star} Z^c_{i\underline{\Phi}} \end{bmatrix} \text{ for } \Phi \in \mathcal{A}_{L+1}
$$

with $\tilde{A}^{11}_{i,\Phi_\star} = E_i^T A_{\Phi_\star} E_i$, $\tilde{A}^{21}_{i,\Phi_\star} = \bar{E}_i^T A_{\Phi_\star} E_i$ and $\tilde{A}^{22}_{i,\Phi_\star} = \bar{E}_i^T A_{\Phi_\star} \bar{E}_i$.

Proof of Lemma 48. Using Lemma 61 in appendix we have the inequality (8.19) being equivalent to the following for all $i \in \bar{\mathcal{J}}$ and $\Phi \in \mathcal{A}_{L+1}$

$$\big((W_{i,\Phi_\star} S_{i,\Phi})_\perp\big)^T S_{i,\Phi}^T H_{i,\Phi} S_{i,\Phi} (W_{i,\Phi_\star} S_{i,\Phi})_\perp \succ 0$$

with
$$H_{i,\Phi} = \left[\begin{array}{cc|cc} Z_{i,\bar{\Phi}} & 0 & A_{\Phi_\star} & B^w_{\Phi_\star} \\ 0 & I & C^z_{\Phi_\star} & D^{zw}_{\Phi_\star} \\ \hline \bullet & \bullet & Z^{-1}_{i,\underline{\Phi}} & 0 \\ \bullet & \bullet & 0 & I \end{array}\right], \quad S_{i,\Phi} = \operatorname{diag}(Z^u_{i,\bar{\Phi}}, I, Z^l_{i,\underline{\Phi}}, I)$$

and
$$W_{i,\Phi_\star} = \begin{bmatrix} (\bar{E}^u_i)^T (B^u_{\Phi_\star})^T & (\bar{E}^u_i)^T (D^{zu}_{\Phi_\star})^T & 0 & 0 \\ 0 & 0 & (E^y_i)^T C^y_{\Phi_\star} & (E^y_i)^T D^{yw}_{\Phi_\star} \end{bmatrix}.$$

Using the relations $(E^y_i)^T C^y_{\Phi_\star} Z^l_{i,\underline{\Phi}} = (E^y_i)^T C^y_{\Phi_\star}$ and $(\bar{E}^u_i)^T (B^u_{\Phi_\star})^T Z^u_{i,\bar{\Phi}} = (\bar{E}^u_i)^T (B^{u\,T}_{\Phi_\star}$, we have $W_{i,\Phi_\star} S_{i,\Phi} = W_{i,\Phi_\star}$. Further along with (8.24), the above inequality leads to inequality (8.26). $\square$

8.4 Completion of Scaling Matrices

First we have the following well known result for completing matrices.

Lemma 50. *Given matrices $R_1, S_1 \in \mathbb{S}^n_+$ and a positive integer n^K, there exists matrices $R_2, S_2 \in \mathbb{R}^{n \times n^K}$ and $R_3, S_3 \in \mathbb{S}^{n^K}_+$ satisfying*

$$R := \begin{bmatrix} R_1 & R_2 \\ R_2^T & R_3 \end{bmatrix} \succ 0 \quad \text{and} \quad \begin{bmatrix} S_1 & S_2 \\ S_2^T & S_3 \end{bmatrix} = \begin{bmatrix} R_1 & R_2 \\ R_2^T & R_3 \end{bmatrix}^{-1}$$

if and only if
$$\begin{bmatrix} R_1 & I \\ I & S_1 \end{bmatrix} \succeq 0 \quad \text{and} \quad \operatorname{rank} \begin{bmatrix} R_1 & I \\ I & S_1 \end{bmatrix} \leq n + n^K.$$

The above rank condition is always satisfied for $n^K \geq n$. Further if the above conditions are satisfied, the unknown matrices can be constructed such that $\bar{\sigma}(R) \leq \bar{\sigma}(R_1) + \bar{\sigma}^{\frac{1}{2}}(R_1 - S_1^{-1}) + 1$ and $\bar{\sigma}(S) \leq \bar{\sigma}(S_1)\big(1 + \bar{\sigma}^{\frac{1}{2}}(R_1 - S_1^{-1})\big)^2 + 1$. $\square$

For the proof and a possible construction, see for example [64, Lemma 7.9]. For the norm bounds see [81].

In the previous lemma, the known subsections R_1 and S_1, of the larger matrix R and its inverse were of the same dimensions. The next lemma extends this result for the case when these dimensions are not the same.

Lemma 51. *Given matrices $R_{11} \in \mathbb{S}^n_+$ and $S_1 = \begin{bmatrix} S_{11} & S_{12} \\ S_{12}^T & S_{22} \end{bmatrix} \in \mathbb{S}^{n+m}_+$ such that $S_{11} \in \mathbb{S}^n_+$. Then for a positive integer n^K, there exists matrices $R_{12} \in \mathbb{R}^{n \times m}$, $R_{22} \in \mathbb{S}^m_+$, $R_{13}, S_{13} \in \mathbb{R}^{n \times n^K}$, $R_{23}, S_{23} \in$*

$\mathbb{R}^{m\times n^K}$ *and* $R_{33}, S_{33} \in \mathbb{S}_+^{n^K}$ *satisfying*

$$R := \begin{bmatrix} R_{11} & R_{12} & R_{13} \\ R_{12}^T & R_{22} & R_{23} \\ R_{13}^T & R_{23}^T & R_{33} \end{bmatrix} \succ 0 \quad \textit{and} \quad S := R^{-1} = \begin{bmatrix} S_{11} & S_{12} & S_{13} \\ S_{12}^T & S_{22} & S_{23} \\ S_{13}^T & S_{23}^T & S_{33} \end{bmatrix}$$

if and only if

$$\begin{bmatrix} R_{11} & I \\ I & \bar{S}_{11}^{-1} \end{bmatrix} \succeq 0 \quad \textit{and} \quad \operatorname{rank} \begin{bmatrix} R_{11} & I \\ I & \bar{S}_{11}^{-1} \end{bmatrix} \leq n + n^K \tag{8.27}$$

where $\bar{S}_{11} = (S_{11} - S_{12}S_{22}^{-1}S_{12}^T)^{-1}$*. The above rank condition is always satisfied for* $n^K \geq n$*. Further if the above conditions are satisfied, the unknown matrices can be constructed such that* $\bar{\sigma}(R) \leq \bar{\sigma}(S_1^{-1}) + \big(1 + \bar{\sigma}^{\frac{1}{2}}(R_{11} - \bar{S}_{11})\big)^2$ *and* $\bar{\sigma}(S) \leq \bar{\sigma}(S_1)\big(1 + \bar{\sigma}^{\frac{1}{2}}(R_{11} - \bar{S}_{11})\big)^2 + 1$*.*

Proof. Let us define the matrices

$$R_1 = \begin{bmatrix} R_{11} & R_{12} \\ R_{12}^T & R_{22} \end{bmatrix}, \quad R_2 = \begin{bmatrix} R_{13} \\ R_{23} \end{bmatrix}, \quad R_3 = R_{33},$$

$\bar{S}_{12} \in \mathbb{R}^{n\times m}$ and $\bar{S}_{22} \in \mathbb{S}_+^m$ so that $\begin{bmatrix} \bar{S}_{11} & \bar{S}_{12} \\ \bar{S}_{12}^T & \bar{S}_{22} \end{bmatrix} = S_1^{-1}$.

Since $S_1 \succ 0$ and due to the following relation

$$\begin{bmatrix} R_1 & I \\ I & S_1 \end{bmatrix} = \begin{bmatrix} I & S_1^{-1} \\ 0 & I \end{bmatrix} \begin{bmatrix} R_1 - S_1^{-1} & 0 \\ 0 & S_1 \end{bmatrix} \begin{bmatrix} I & 0 \\ S_1^{-1} & I \end{bmatrix}$$

we have

$$\begin{bmatrix} R_1 & I \\ I & S_1 \end{bmatrix} \succeq 0 \quad \Leftrightarrow \quad R_1 - S_1^{-1} \succeq 0 \tag{8.28}$$

and

$$\operatorname{rank} \begin{bmatrix} R_1 & I \\ I & S_1 \end{bmatrix} = n + m + \operatorname{rank}(R_1 - S_1^{-1}). \tag{8.29}$$

Also note the following expansion

$$R_1 - S_1^{-1} = \begin{bmatrix} R_{11} - (S_{11} - S_{12}S_{22}^{-1}S_{12}^T)^{-1} & R_{12} - \bar{S}_{12} \\ R_{12}^T - \bar{S}_{12}^T & R_{22} - \bar{S}_{22} \end{bmatrix}. \tag{8.30}$$

We now provide the main arguments of the proof.

($\Longrightarrow$) From Lemma 50, we know that $R \succ 0$ and $S \succ 0$ implies $\begin{bmatrix} R_1 & I \\ I & S_1 \end{bmatrix} \succeq 0$ and $\operatorname{rank} \begin{bmatrix} R_1 & I \\ I & S_1 \end{bmatrix} \leq n + m + n^K$. Using (8.28) and (8.30), this further implies

$$R_1 - S_1^{-1} \succeq 0 \quad \Rightarrow R_{11} - (S_{11} - S_{12}S_{22}^{-1}S_{12}^T)^{-1} \succeq 0$$

Using a Schur complement argument again, the above is same as the matrix inequality in (8.27). Using (8.29) and (8.30), we have rank $\left(R_{11} - (S_{11}\text{-}S_{12}S_{22}^{-1}S_{12}^{T})^{-1}\right) \leq \text{rank}(R_1 - S_1^{-1}) \leq n^K$. Using a property similar to (8.29), we can arrive at the rank condition in (8.27) from above.

($\Longleftarrow$) This part of the proof is constructive. If we assume that R_1 is known completely, then by combining Lemma 50 with (8.28)-(8.30) we know that the matrices R and S can be completed iff

$$R_1 - S_1^{-1} \succeq 0 \quad \text{and} \quad \text{rank}(R_1 - S_1^{-1}) \leq n^K.$$

So we can instead focus on the problem of completing the matrix R_1 which satisfy the above conditions. The expansion (8.30) suggests that choosing $R_{12} = \bar{S}_{12}$ and $R_{22} = \bar{S}_{22}$ would result in $R_1 - S_1^{-1} = \text{diag}\left(R_{11}\text{-}\bar{S}_{11}, 0\right)$. From (8.27), it is clear that the above conditions are satisfied. Thereafter, we can complete the remaining blocks of R by following steps in [64, Lemma 7.9]: choose $R_3 = I$ and R_2 such that $R_1 - S_1^{-1} = R_2 R_2^T$. This is same as setting $R_{33} = I$, $R_{23} = 0$ and choosing R_{13} such that $R_{11}\text{-}\bar{S}_{11} = R_{13}R_{13}^T$. Finally obtain the unknown blocks of S by inverting the constructed R.

The norm bounds can be found for the above construction by separating the sub-blocks of R and S followed by using triangular and sub-multiplicative inequalities. In the process, we make use of relations $\bar{\sigma}(R_2) = \bar{\sigma}^{\frac{1}{2}}(R_1\text{-}S_1^{-1})$, $\bar{\sigma}(R_1\text{-}S_1^{-1}) = \bar{\sigma}(R_{11}\text{-}\bar{S}_{11})$ and $\bar{\sigma}(R_1) \leq \bar{\sigma}(S_1^{-1}) + \bar{\sigma}(R_{11}\text{-}\bar{S}_{11})$. $\square$

We now utilize the previous lemma to provide necessary and sufficient conditions for completion of the closed loop scaling operator $\{X_\Psi^C\}_{\Psi\in\mathcal{A}_L}$ given partial information about it.

Lemma 52. *Given positive-definite matrices $\{Z_{i,\Psi}\}_{i\in\bar{\mathcal{J}},\Psi\in\mathcal{A}_L}$, we can construct positive-definite $\{X_\Psi^C\}_{\Psi\in\mathcal{A}_L}$ satisfying (8.12) and (8.13) iff*

$$\begin{bmatrix} Z_{i,\Psi}^{-1} & I \\ I & Z_{i\text{-}1,\Psi} \end{bmatrix} \succeq 0, \quad \text{rank}\begin{bmatrix} Z_{i,\Psi}^{-1} & I \\ I & Z_{i\text{-}1,\Psi} \end{bmatrix} \leq n\text{+}n_i^K \tag{8.31}$$

for all $i \in \mathcal{J}$ and $\Psi \in \mathcal{A}_L$. Further the above rank conditions are always satisfied for $n_i^K \geq n$.

Proof. We will use matrices $\{Z_{i,\Psi}\}_{i\in\bar{\mathcal{J}}}$ to construct X_Ψ^C for each $\Psi \in \mathcal{A}_L$, as shown in the following steps

- First, we construct $Y_{1,\Psi}^C$ (defined in (8.14)) using $Z_{0,\Psi} = Y_\Psi$ and $Z_{1,\Psi}$. We do this pointwise using Lemma 50, which yields the following condition for completion

$$\begin{bmatrix} Z_{1,\Psi}^{-1} & I \\ I & Z_{0,\Psi} \end{bmatrix} \succeq 0, \ \text{rank}\begin{bmatrix} Z_{1,\Psi}^{-1} & I \\ I & Z_{0,\Psi} \end{bmatrix} \leq n\text{+}n_1^K \quad \forall \Psi \in \mathcal{A}_L.$$

  A possible construction can be found in [64, Lemma 7.9].

- We construct $Y^C_{i,\Psi}$ in a recursive manner in the order $i = 2, \ldots, N$. For this, we use Lemma 51 with $R_{11} = Z^{-1}_{i,\Psi}$ and $S_1 = Y^C_{i-1,\Psi}$ to complete the matrix $S = Y^C_{i,\Psi} \succ 0$. This can be done iff following conditions are satisfied

  $$\begin{bmatrix} Z^{-1}_{i,\Psi} & I \\ I & Z_{i-1,\Psi} \end{bmatrix} \succeq 0, \ \text{rank} \begin{bmatrix} Z^{-1}_{i,\Psi} & I \\ I & Z_{i-1,\Psi} \end{bmatrix} \leq n + n^K_i$$

  Note the use of relation (8.13) for index $i-1$ while using Lemma 51. A possible construction is given in the proof of Lemma 51.

After performing the above steps, we are left with $Y^C_{N,\Psi}$ which is same as Y^C_Ψ for $\Psi \in \mathcal{A}_L$. Since the above steps use 'if and only if' arguments, the converse direction of the proof also holds. □

Remark 53. *In the previous lemma, the completed matrices satisfy the norm bounds*

$$\bar{\sigma}(Y^C_{i,\Psi}) \leq \bar{\sigma}(Y^C_{i-1,\Psi})\big(1 + \bar{\sigma}(Z^{-1}_{i,\Psi} - Z^{-1}_{i-1,\Psi})^{\frac{1}{2}}\big)^2 + 1$$
$$\bar{\sigma}\left((Y^C_{i,\Psi})^{-1}\right) \leq \bar{\sigma}\left((Y^C_{i-1,\Psi})^{-1}\right) + \big(1 + \bar{\sigma}(Z^{-1}_{i,\Psi} - Z^{-1}_{i-1,\Psi})^{\frac{1}{2}}\big)^2$$

for $i = 2, \ldots, M$ *with* $\bar{\sigma}(Y^C_{1,\Psi}) \leq \bar{\sigma}(Y_\Psi) + \bar{\sigma}(Y_\Psi - Z_{1,\Psi})^{\frac{1}{2}} + 1$ *and* $\bar{\sigma}\left((Y^C_{1,\Psi})^{-1}\right) \leq \bar{\sigma}(Z^{-1}_{1,\Psi})\big(1 + \bar{\sigma}(Y_\Psi - Z_{1,\Psi})^{\frac{1}{2}}\big)^2 + 1.$

8.5 Exact Conditions for Existence of Controller Synthesis

We now present the main result of the paper.

Theorem 54. *Consider the mode-dependent system (8.2) along with the structural description in Assumption 39. There exists a synthesis of a finite-path dependent controller (8.6) which*

(i) is structured as (8.7),

(ii) has dimensions $\{n^K_i\}^M_{i=1}$,

(iii) stabilizes the plant, and

(iv) achieves closed loop performance $\|w \mapsto z\| < 1$

iff there exist an $L \in \mathbb{N}_0$ *and block-diagonal operators* $\{Z^a_{i,\Psi}, Z^b_{i,\Psi}, Z^c_{i,\Psi}\}_{i \in \bar{\mathcal{J}}, \Psi \in \mathcal{A}_L}$ *satisfying the following*

$$Z^a_{i,\Psi} \succ 0,\ Z^c_{i,\Psi} \succ 0 \ \textit{for all}\ i \in \bar{\mathcal{J}}, \Psi \in \mathcal{A}_L, \tag{8.32a}$$

$$[\bullet]^T \left[\begin{array}{cc|cc} (Z^u_{i,\bar{\Phi}})^T Z^l_{i,\bar{\Phi}} & 0 & (Z^u_{i,\bar{\Phi}})^T A_{\Phi_\star} Z^l_{i,\underline{\Phi}} & (Z^u_{i,\bar{\Phi}})^T B^w_{\Phi_\star} \\ 0 & I & C^z_{\Phi_\star} Z^l_{i,\underline{\Phi}} & D^{zw}_{\Phi_\star} \\ \hline \bullet & \bullet & (Z^u_{i,\Psi})^T Z^l_{i,\Psi} & 0 \\ \bullet & \bullet & 0 & I \end{array}\right] \begin{bmatrix} N^u_{i,\Phi_\star} & 0 \\ 0 & N^y_{i,\Phi_\star} \end{bmatrix} \succ 0 \textit{ for all } i \in \bar{\mathcal{J}}, \Phi \in \mathcal{A}_{L+1}, \tag{8.32b}$$

$$\begin{bmatrix} (Z^u_{i,\Psi})^T Z^l_{i,\Psi} & (Z^l_{i,\Psi})^T Z^u_{i\text{-}1,\Psi} \\ (Z^u_{i\text{-}1,\Psi})^T Z^l_{i,\Psi} & (Z^u_{i\text{-}1,\Psi})^T Z^l_{i\text{-}1,\Psi} \end{bmatrix} \succeq 0 \qquad \textit{and} \tag{8.32c}$$

$$\text{rank} \begin{bmatrix} (Z^u_{i,\Psi})^T Z^l_{i,\Psi} & (Z^l_{i,\Psi})^T Z^u_{i\text{-}1,\Psi} \\ (Z^u_{i\text{-}1,\Psi})^T Z^l_{i,\Psi} & (Z^u_{i\text{-}1,\Psi})^T Z^l_{i\text{-}1,\Psi} \end{bmatrix} \leq n + n^K_i \textit{ for all } \Psi \in \mathcal{A}_L \textit{ and } i \in \mathcal{J}. \tag{8.32d}$$

where $Z^l_{i,\Psi}$ and $Z^u_{i,\Psi}$ are defined using $Z^a_{i,\Psi}$, $Z^b_{i,\Psi}$ and $Z^c_{i,\Psi}$ as in (8.25). Further, rank conditions above are always satisfied when $n^K_i \geq n$, leaving us with LMIs (8.32a)-(8.32c). □

We have already verified in Remark 49 that all elements in the above inequalities are affine in constituent variables. The only additional term encountered here is

$$(Z^l_{i,\Psi})^T Z^u_{i\text{-}1,\Psi} = \text{diag}(Z^a_{i\text{-}1,\Psi}, I_{n_i}, Z^c_{i,\Psi}) + \begin{bmatrix} 0 & Z^b_{i\text{-}1,\Psi} \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} 0 & Z^b_{i,\Psi} \\ 0 & 0 \end{bmatrix}$$

which is also affine.

Proof of Theorem 54. ($\Longleftarrow$) Let there be $\{Z^a_{i,\Psi}, Z^b_{i,\Psi}, Z^c_{i,\Psi}\}_{i\in\bar{\mathcal{J}},\Psi\in\mathcal{A}_L}$ satisfying (8.32a)-(8.32d). We can then use relations in (8.23) to obtain corresponding positive-definite matrices $\{Z_{i,\Psi}\}_{i\in\bar{\mathcal{J}},\Psi\in\mathcal{A}_L}$. Now using definitions (8.25) and the transformation

$$\begin{bmatrix} Z^l_{i,\Psi} & 0 \\ 0 & Z^u_{i\text{-}1,\Psi} \end{bmatrix}^T \begin{bmatrix} Z^{-1}_{i,\Psi} & I \\ I & Z_{i\text{-}1,\Psi} \end{bmatrix} \begin{bmatrix} Z^l_{i,\Psi} & 0 \\ 0 & Z^u_{i\text{-}1,\Psi} \end{bmatrix} = \begin{bmatrix} (Z^u_{i,\Psi})^T Z^l_{i,\Psi} & (Z^l_{i,\Psi})^T Z^u_{i\text{-}1,\Psi} \\ (Z^u_{i\text{-}1,\Psi})^T Z^l_{i,\Psi} & (Z^u_{i\text{-}1,\Psi})^T Z^l_{i\text{-}1,\Psi} \end{bmatrix}$$

it is clear that (8.32c) and (8.32d) imply that $Z_{i,\Psi}$ and $Z_{i\text{-}1,\Psi}$ satisfy (8.31) for each $i \in \mathcal{J}$ and $\Psi \in \mathcal{A}_L$. Thus Lemma 52 can be applied to construct X^C_Ψ satisfying (8.12) and (8.13). Now using Lemma 48 along with (8.32b), we know that inequalities in (8.19) are satisfied. Finally we can apply Lemma 46 to argue the existence of a desired controller.

($\Longrightarrow$) This part of the proof retraces the above steps in the backwards direction. However Lemma 52 is not applied directly, but through steps contained in it. □

8.6 Controller Synthesis

In this section, we discuss the decentralized controller synthesis using the scaling matrices obtained earlier. We first start with the following lemma motivated by [87], [88, Lemma 5.2] applicable for centralized controller synthesis.

Lemma 55. *Given $Z \in \mathbb{S}_{+}^{n}$, $H \in \mathbb{S}_{+}^{m}$, $R \in \mathbb{R}^{n\times m}$, and matrices U and V with column dimensions n and m respectively, satisfying*

$$\begin{aligned} V_{\perp}^{T}(Z - R^{T}HR)V_{\perp} &\succ 0 \quad \textit{and} \\ U_{\perp}^{T}(H^{-1} - RZ^{-1}R^{T})U_{\perp} &\succ 0, \end{aligned} \tag{8.33}$$

we can construct Q satisfying the inequality

$$\begin{bmatrix} H^{-1} & R + U^{T}QV \\ \bullet & Z \end{bmatrix} \succ 0 \tag{8.34}$$

as

$$Q = \left(U_{||}^{T}U^{T}\right)^{\dagger}\left(-W_{23}^{T} + W_{13}^{T}W_{11}^{-1}W_{12}\right)\left(VV_{||}\right)^{\dagger} \tag{8.35}$$

where $W_{ij} := H_i^{T}WH_j$ for $i,j \in \{1,2,3\}$ with following commensurately partitioned matrices

$$W = \begin{bmatrix} H^{-1} & R \\ R^{T} & Z \end{bmatrix},\ H_1 = \begin{bmatrix} U_{\perp} & 0 \\ 0 & V_{\perp} \end{bmatrix},\ H_2 = \begin{bmatrix} 0 \\ V_{||} \end{bmatrix},\ H_3 = \begin{bmatrix} U_{||} \\ 0 \end{bmatrix}.$$

□

Proof of the above lemma uses ideas similar to [87, Lemma 3.1], and is given here for completeness.

Proof. Using $P = \begin{bmatrix} U & 0 \end{bmatrix}$ and $S = \begin{bmatrix} 0 & V \end{bmatrix}$ each with column dimension $n+m$, we write (8.34) as

$$W + P^{T}QS + S^{T}QP \succ 0.$$

Note that H_1, H_2 and H_3 are full column rank matrices satisfyig $\mathrm{Im}(H_1) = \mathrm{Ker}(P) \cap \mathrm{Ker}(S)$, $\mathrm{Im}\begin{bmatrix} H_1 & H_2 \end{bmatrix} = \mathrm{Ker}(P)$ and $\mathrm{Im}\begin{bmatrix} H_1 & H_3 \end{bmatrix} = \mathrm{Ker}(S)$. Also $H := \begin{bmatrix} H_1 & H_2 & H_3 \end{bmatrix}$ is square and non-singular. Since we are free to choose Q and, $PH_2 = UU_{||}$ and $SH_3 = VV_{||}$ being of full column rank, it can be seen that $J := H_3^{T}P^{T}QSH_2$ is freely assignable. Post- and pre-multiplying inequality

(8.34) with H and its transpose, we have

$$\begin{bmatrix} W_{11} & W_{12} & W_{13} \\ \cdot & W_{22} & W_{23} + J^T \\ \cdot & \cdot & W_{33} \end{bmatrix} \succ 0.$$

Using Schur complement formula, the above is further equivalent to

$$\begin{bmatrix} W_{22} - W_{12}^T W_{11}^{-1} W_{12} & W_{23} + J^T - W_{12}^T W_{11}^{-1} W_{13} \\ \cdot & W_{33} - W_{13}^T W_{11}^{-1} W_{13} \end{bmatrix} \succ 0.$$

Since the diagonal blocks of the above matrix are equivalent to (8.33) (upon use of Schur complement formula and invertibility of $\begin{bmatrix} U_\perp & U_{||} \end{bmatrix}$ and $\begin{bmatrix} V_\perp & V_{||} \end{bmatrix}$), we can choose $J = -W_{32} + W_{13}^T W_{11}^{-1} W_{12}$ to satisfy the above inequality. From this, the choice of Q in (8.35) is immediate. □

Note that in above lemma, the matrices $U_\perp$, $U_{||}$, $V_\perp$, $V_{||}$ can be computed using the SVD of U and V. In this case, the pseudo-inverses in (8.35) can be written directly by inspection.

The next theorem presents an algebraic method for construction of a finite-path dependent controller with memory L and dimensions $n_i^K = n$ for $i \in \mathcal{J}$. But first let us consider the following alternative expansion of the closed loop matrices similar to (8.9):

$$Q_\Psi^C = R_{\Psi_\star} + \sum_{i=1}^{M} (U_{i,\Psi_\star}^C)^T \tilde{Q}_{i,\Psi}^K \tilde{V}_{i,\Psi_\star}^C \tag{8.36}$$

with
$$\tilde{V}_{i,\Psi_\star}^C = \left[\begin{array}{cc|c} 0 & (e_i^K)^T & 0 \\ (e_i^y)^T C_{\Psi_\star}^y & 0 & (e_i^y)^T D_{\Psi_\star}^{yw} \end{array}\right],$$

$$\tilde{Q}_{i,\Psi}^K = \begin{bmatrix} \bar{E}_{i-1}^K & \\ 0 & \bar{E}_{i-1}^u \end{bmatrix}^T Q_\Psi^K \begin{bmatrix} e_i^K & 0 \\ 0 & e_i^y \end{bmatrix} \quad \text{and} \quad e_i^\bullet = \begin{bmatrix} 0_{(n_1^\bullet + \ldots + n_{i-1}^\bullet) \times n_i^\bullet} \\ I_{n_i^\bullet} \\ 0_{(n_{i+1}^\bullet + \ldots + n_M^\bullet) \times n_i^\bullet} \end{bmatrix}.$$

Note that $\tilde{Q}_{i,\Psi}^K$ consists of the i-th block columns of lower-triangular parts of A_Ψ^K, B_Ψ^K, C_Ψ^K, D_Ψ^K. In (8.9), the controller was decomposed into $\{Q_{i,\Psi}\}_{i=1}^M$ containing redundancies, which did not affect the existence conditions as we eliminated these controller matrices. However for synthesis, we choose the above decomposition, which eliminates such redundancies and keeps the number of variables to the minimum. For $\tilde{V}_{,}^C$ defined above, we can verify that

$$\mathrm{Ker}(\tilde{V}_{1,\Psi_\star}^C) \cap \cdots \cap \mathrm{Ker}(\tilde{V}_{i,\Psi_\star}^C) = \mathrm{Ker}(V_{i,\Psi_\star}^C) \tag{8.37}$$

for $i \in \mathcal{J}$.

Theorem 56. *Given matrices* $\{Z^a_{i,\Psi}, Z^b_{i,\Psi}, Z^c_{i,\Psi}\}_{i\in\mathcal{J},\Psi\in\mathcal{A}_L}$ *satisfying (8.32a)-(8.32c), corresponding* $\{X^C_\Psi\}_{\Psi\in\mathcal{A}_L}$ *obtained using (8.12) and (8.13) can be used to obtain the following LMI*

$$\begin{bmatrix} diag((X^C_{\bar{\Phi}})^{-1}, I) & \left(R_{\Phi_\star} + \sum_{j=i}^{M} (U^C_{j,\Phi_\star})^T \tilde{Q}^K_{j,\Phi} \tilde{V}^C_{j,\Phi_\star}\right)(V^C_{i-1,\Phi_\star})_\perp \\ \bullet & (V^C_{i-1,\Phi_\star})^T_\perp diag(X^C_{\underline{\Phi}}, I)(V^C_{i-1,\Phi_\star})_\perp \end{bmatrix} \succ 0 \tag{8.38}$$

in variable $\tilde{Q}^K_{i,\Phi}$ *for each* $\Phi \in \mathcal{A}_{L+1}$*, and solved in the order* $i = M, \ldots, 1$*. Further this can be done point-wise for each* $i \in \mathcal{J}$ *and* $\Phi \in \mathcal{A}_{L+1}$ *using (8.35) in Lemma 55 by choosing*

$$\begin{aligned} &Q = \tilde{Q}^K_{i,\Phi}, \quad U = U^C_{i,\Phi_\star}, \quad V = \tilde{V}^C_{i,\Phi_\star}(V^C_{i-1,\Phi_\star})_\perp \\ &H = diag(X^C_{\bar{\Phi}}, I),\; Z = (V^C_{i-1,\Phi_\star})^T_\perp diag(X^C_{\underline{\Phi}}, I)(V^C_{i-1,\Phi_\star})_\perp \\ &R = \left(R_{\Phi_\star} + \sum_{j=i+1}^{M} (U^C_{j,\Phi_\star})^T \tilde{Q}^K_{j,\Phi} \tilde{V}^C_{j,\Phi_\star}\right)(V^C_{i-1,\Phi_\star})_\perp. \end{aligned} \tag{8.39}$$

Proof. With $\{Z^a_{i,\Psi}, Z^b_{i,\Psi}, Z^c_{i,\Psi}\}_{i\in\mathcal{J},\Psi\in\mathcal{A}_L}$ we can construct scaling matrices $\{X^C_\Psi\}_{\Psi\in\mathcal{A}_L}$ using steps in proof of Theorem 54. Now use Lemma 44 with the following choice

$$\begin{aligned} W &= \begin{bmatrix} \mathrm{diag}((X^C_{\bar{\Phi}})^{-1}, I) & R_{\Phi_\star} \\ R^T_{\Phi_\star} & \mathrm{diag}(X^C_{\underline{\Phi}}, I) \end{bmatrix}, \\ Q_i &= \tilde{Q}^K_{i,\Phi},\; P_i = \begin{bmatrix} U^C_{i,\Phi_\star} & 0 \end{bmatrix} \text{ and } S_i = \begin{bmatrix} 0 & \tilde{V}^C_{i,\Phi_\star} \end{bmatrix}. \end{aligned}$$

so that corresponding inequality (8.15) is same as the KYP type inequality (8.5) for the closed loop and is already known to hold for some choice of controller from Theorem 54. Thus Lemma 44(i) along with (8.37) implies that (8.21) holds. The above definitions along with (8.37) yield

$$N_i = \begin{bmatrix} S_1 \\ \vdots \\ S_i \end{bmatrix}_\perp = \begin{bmatrix} 0 & \tilde{V}^C_{1,\Phi_\star} \\ \vdots & \vdots \\ 0 & \tilde{V}^C_{i,\Phi_\star} \end{bmatrix}_\perp = \begin{bmatrix} I & 0 \\ 0 & (V^C_{i,\Phi_\star})_\perp \end{bmatrix}.$$

corresponding to Lemma 44, which further leads to (8.38) using inequality (8.16) in Lemma 44(ii).

The use of Lemma 55 with the choice (8.39) to solve for $\tilde{Q}^K_{i,\Phi}$ in (8.38) requires us to show that corresponding inequalities (8.33) are satisfied. This is indeed true, because the inequalities in (8.33) for $i = M$ correspond to (8.21) with $i = M$ and $i = M - 1$. For any other $i = k$, inequalities in (8.33) correspond to (8.21) with $i = k-1$ and (8.38) with $i = k+1$. Note that in intermediate steps we use Schur complement formula, the following property obtained using Lemma 60 and relation (8.37)

$$\mathrm{Im}\left(V^C_{k-1,\Phi_\star}\left(\tilde{V}^C_{k,\Phi_\star}(V^C_{k-1,\Phi_\star})_\perp\right)_\perp\right) = \mathrm{Im}\left((V^C_{k,\Phi_\star})_\perp\right).$$

□

The controller $Q^K_\Phi = \begin{bmatrix} A^K_\Phi & B^K_\Phi \\ C^K_\Phi & D^K_\Phi \end{bmatrix}$ structured as (8.7) can be can be constructed from $\{\tilde{Q}^K_{i,\Phi}\}_{i\in\mathcal{J}}$ as

$$Q^K_\Phi = \sum_{i=1}^{M} \begin{bmatrix} \bar{E}^K_{i-1} & \\ 0 & \bar{E}^u_{i-1} \end{bmatrix} \tilde{Q}^K_{i,\Phi} \begin{bmatrix} e^K_i & 0 \\ 0 & e^y_i \end{bmatrix}^T \quad \text{for all } \Phi \in \mathcal{A}_{L+1}. \tag{8.40}$$

Remark 57. *An alternative to the synthesis procedure described above is to solve the following LMI in the structured controller matrices Q^K_Φ point-wise for $\Phi \in \mathcal{A}_{L+1}$*

$$\begin{bmatrix} diag((X^C_{\bar{\Phi}})^{-1}, I) & R_{\Phi_\star} + (U^C_{\Phi_\star})^T Q^K_\Phi V^C_{\Phi_\star} \\ (R_{\Phi_\star} + (U^C_{\Phi_\star})^T Q^K_\Phi V^C_{\Phi_\star})^T & diag(X^C_{\underline{\Phi}}, I) \end{bmatrix} \succ 0. \tag{8.41}$$

Remark 58. *If a closed loop performance of $\|w \mapsto z\| < \gamma$ is sought, Theorem 54 can be updated to have C^z_ϕ, C^y_ϕ, D^{zw}_ϕ, D^{zu}_ϕ and D^{yw}_ϕ scaled by $\frac{1}{\gamma}$ for all $\phi \in \Theta$. The controller obtained for this modified system using the procedure above, can be used to find the desired controller by scaling B^K_Ψ and D^K_Ψ with $\frac{1}{\gamma}$ for all Ψ.*

In order to find a controller having a near optimal performance, we can use a bisection algorithm. The performance level γ generated at each step of the bisection algorithm can be used to check the feasibility LMIs (8.32a)-(8.32c) for a system obtained by making the substitutions described in Remark 58. Thereafter, for the smallest γ which solves the feasibility LMIs, corresponding scaling matrices can be used to synthesize the structured controller using Theorem 56 and (8.40).

8.7 Possible Variations in the Setup

8.7.1 Nested LTI Systems

For a linear time invariant (LTI) formulation, we can obtain existence and synthesis results similar to Theorem 54 and Theorem 56 by simply choosing an automaton with one element ($n_s = 1$) having a self-loop. As a result, for any memory length L, there exists only one sequence in $\mathcal{A}_{L+1}$ implying that the controller is time-invariant and there is a single scaling matrix. Since the size L doesn't play any role, we can simply choose $L = 0$ in Theorem 54 and adopt the same conditions. We would then arrive at the following result.

Theorem 59. *Consider a time-invariant system[4] (8.2) along with the structural description in Assumption 39. There exists a synthesis of a structured controller (8.6)-(8.7) with dimensions*

[4]This is done by choosing $n_s = 1$ which implies a single possibility for $\theta(t)$ and induced sequences constructed with it. So we can ignore the switching subscripts altogether in this theorem.

$\{n_i^K\}_{i=1}^M$ *which stabilizes the plant and achieves closed loop performance* $\|w \mapsto z\| < 1$ *if and only if there exist matrices* $\{Z_i^a, Z_i^b, Z_i^c\}_{i=1}^M$ *satisfying the following*

$$Z_i^a \succ 0,\ Z_i^c \succ 0 \qquad \textit{and} \tag{8.42}$$

$$\begin{bmatrix} N_i^u & 0 \\ 0 & N_i^y \end{bmatrix}^T \left[\begin{array}{cc|cc} (Z_i^u)^T Z_i^l & 0 & (Z_i^u)^T A Z_i^l & (Z_i^u)^T B_w \\ 0 & I & C_z Z_i^l & D_{zw} \\ \hline (Z_i^l)^T A^T Z_i^u & (Z_i^l)^T C_z^T & (Z_i^u)^T Z_i^l & 0 \\ B_w^T Z_i^u & D_{zw}^T & 0 & I \end{array}\right] \begin{bmatrix} N_i^u & 0 \\ 0 & N_i^y \end{bmatrix} \succ 0 \quad \textit{for } i \in \bar{\mathcal{J}}, \tag{8.43}$$

$$\begin{bmatrix} (Z_i^u)^T Z_i^l & (Z_i^l)^T Z_{i\text{-}1}^u \\ (Z_{i\text{-}1}^u)^T Z_i^l & (Z_{i\text{-}1}^u)^T Z_{i\text{-}1}^l \end{bmatrix} \succeq 0 \qquad \textit{and} \tag{8.44}$$

$$\text{rank} \begin{bmatrix} (Z_i^u)^T Z_i^l & (Z_i^l)^T Z_{i\text{-}1}^u \\ (Z_{i\text{-}1}^u)^T Z_i^l & (Z_{i\text{-}1}^u)^T Z_{i\text{-}1}^l \end{bmatrix} \leq n + n_i^K \quad \textit{for} \ \ i \in \mathcal{J}. \tag{8.45}$$

Further, the above rank conditions are always satisfied when $n_i^K \geq n$*, leaving us with LMIs (8.42)-(8.44).* □

Note that the above LTI result was also presented in [84, Thoerem 10]. In that reference, an LTV version of the results described here was solved using operator theoretic representations. The solution to the LTI problem as described in Theorem 59 was then obtained by an averaging argument.

8.7.2 Extensions

We point out that the synthesis conditions and procedure presented in this chapter can be extended to more general setting of non-regular switching sequences and include a finite look-ahead horizon i.e. controller has knowledge of future modes of pre-defined length. For background on these topics, see [52] in the context of centralized control.

Extensions to control of Markovian jump linear systems where the switching sequence is generated by a Markov chain instead of an automaton, to achieve almost sure performance (as described in [49, Section 4]) can also be achieved.

8.8 Example

Let us consider a two player example with dynamics (8.2) and 3-mode switching automaton as shown in Figure 8.2a. The corresponding system matrices are chosen as

$$A_1 = A_2 = \begin{bmatrix} 1.4 & 0 \\ 0.2 & 1.4 \end{bmatrix},\ A_3 = \begin{bmatrix} 0.7 & 0 \\ 0.2 & 0.7 \end{bmatrix},\ B_1^u = B_2^u = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix},\ B_3^u = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix},$$
$$C_1^y = C_2^y = \operatorname{diag}(1,0),\ C_3^y = I_2,\ D_1^{zu} = D_2^{zu} = \begin{bmatrix} 0 & 1 \end{bmatrix},\ D_3^{zu} = \begin{bmatrix} 4 & 0 \end{bmatrix},$$

and the following defined for $\phi \in \{1, 2, 3\}$

$$B_\phi^w = \begin{bmatrix} 1 \\ 1 \end{bmatrix},\ D_\phi^{yw} = \begin{bmatrix} 0 \\ 1 \end{bmatrix},\ C_\phi^z = \begin{bmatrix} 0.5 & 2 \end{bmatrix},\ D_\phi^{zw} = 0.5.$$

Here we have chosen dimensions $n_1 = n_2 = n_1^u = n_2^u = n_1^y = n_2^y = n^z = n^w = 1$.

For different memory lengths, the above system was examined with $n_i^K = 2$ for $i \in \mathcal{J}$. Using a bisection algorithm, the smallest performance level γ (see Remark 58) satisfying conditions in in Theorem 54 was found. The values thus obtained are tabulated below along with corresponding performance obtained for a centralized controller.

Memory	0	1	2	3	4	5
Decentralized	∞	5.468	3.663	3.606	3.604	3.604
Centralized	∞	5.461	3.634	3.561	3.561	3.561

For zero memory length, the system is not stabilizable, resulting in infinite induced norm. For the above example, there is very little difference in the performance of centralized and decentralized control.

Here, changing C_3^y to $\operatorname{diag}(0,1)$ doesn't affect the centralized performance. However the decentralized performance gain increases to 13.278 for $L = 1, \ldots, 5$.

8.9 Appendix

We present a few useful lemmas here.

Lemma 60. *Consider matrices W and P with identical column dimensions. Define V such that* $\operatorname{Im}(V) = \operatorname{Ker}(W) \cap \operatorname{Ker}(P)$. *Then* $\operatorname{Im}(V) = \operatorname{Im}(P_\perp (W P_\perp)_\perp)$.

Proof. First we prove $\operatorname{Im}(V) \subset \operatorname{Im}(P_\perp (W P_\perp)_\perp)$. Consider non-zero $x \in \operatorname{Im}(V)$, this implies $x \in \operatorname{Ker}(W)$ and $x \in \operatorname{Ker}(P) = \operatorname{Im}(P_\perp)$. Thus there exists a non-zero z such that $x = P_\perp z$. Since $Wx = W P_\perp z = 0$ we must have $z \in \operatorname{Ker}(W P_\perp)$. Thus $x \in \operatorname{Im}(P_\perp (W P_\perp)_\perp)$.

Now we prove $\mathrm{Im}(P_\perp(WP_\perp)_\perp) \subset \mathrm{Im}(V)$. Consider non-zero $x \in \mathrm{Im}(P_\perp(WP_\perp)_\perp)$. Clearly $x \in \mathrm{Im}(P_\perp) = \mathrm{Ker}(P)$. Also there exists non-zero z such that $x = P_\perp(WP_\perp)_\perp z$. Clearly $Wx = WP_\perp(WP_\perp)_\perp z = 0$. We have thus proved that x is an element of both $\mathrm{Ker}(W)$ and $\mathrm{Ker}(P)$ implying that $x \in \mathrm{Im}(V)$. □

Lemma 61. *Consider $W \in \mathbb{R}^{m\times k}$, $H \in \mathbb{R}^{k\times k}$ and $S \in \mathbb{R}^{k\times k}$ with H being symmetric and S being invertible. Then, we have*

$$W_\perp^T H W_\perp \succ 0 \;\; \textit{if and only if} \;\; (WS)_\perp^T (S^T H S)(WS)_\perp \succ 0$$

□

The proof is immediate from Lemma 60 by setting $P = 0$, $V = W_\perp$ and $P_\perp = S$.

Lemma 62. *Consider the following symmetric positive-definite matrices partitioned with identical block dimensions*

$$R = \begin{bmatrix} R_{11} & R_{12} & R_{13} \\ R_{12}^T & R_{22} & R_{23} \\ R_{13}^T & R_{23}^T & R_{33} \end{bmatrix} \qquad S = \begin{bmatrix} S_{11} & S_{12} & S_{13} \\ S_{12}^T & S_{22} & S_{23} \\ S_{13}^T & S_{23}^T & S_{33} \end{bmatrix}$$

and satisfying $R = S^{-1}$. Then we have

$$R_{11} - R_{13}R_{33}^{-1}R_{13}^T = (S_{11} - S_{12}S_{22}^{-1}S_{12}^T)^{-1}. \tag{8.46}$$

Proof. We use block matrix inversion formula. First consider the inverse of a sub-block of $S = R^{-1}$ as

$$\begin{bmatrix} S_{11} & S_{12} \\ S_{12}^T & S_{22} \end{bmatrix}^{-1} = \begin{bmatrix} R_{11} & R_{12} \\ R_{12}^T & R_{22} \end{bmatrix} - \begin{bmatrix} R_{13} \\ R_{23} \end{bmatrix} R_{33}^{-1} \begin{bmatrix} R_{13}^T & R_{23}^T \end{bmatrix}.$$

The $(1,1)$ block of the above leads to (8.46). □

Chapter 9

Conclusions

This dissertation explored the decentralized control of linear switched parameter systems and presented the controller design techniques for a few different setups varying in performance criteria, switching architecture and information/parameter availability to controllers.

We first considered the static quadratic team problem in Chapter 3 and using an operator theoretic framework showed that the sequential update scheme convergences exponentially to the team optimal solution and provided bounds for the convergence rate. Consequently, such an update scheme can be used as a mechanism for obtaining the team optimal strategies, while the rate bounds can be helpful in choosing a cost function with suitable convergence properties. An example of a static team problem leading to nonlinear strategies is presented to demonstrate this. The convergence result also provides us with tools which may help us to comment on the structure of the team optimal strategies. The use of these tools was demonstrated by solving a static stochastic-parameter decision problem in Chapter 4.

In Chapters 5-7, we presented three separate dynamic switched decentralized control problems with players having partial access to the stochastic parameters. The problem in Chapter 5 involved a partially nested information structure where the optimal control was obtained by converting the dynamic problem into the static stochastic-parameter problem solve in Chapter 4. In Chapter 6, the setup comprised of a one-step delayed information sharing pattern where the stochastic parameters form a Markov process. The optimal strategies were obtained through dynamic programming while invoking the results of the one-step stochastic-parameter problem at each stage. In Chapter 7, we looked at a full state feedback problem with parameters being independent in time and obtained the optimal control for both finite and infinite horizon cases. For all these dynamic problems, the resulting optimal strategies were found to be affine in locally available measurements with parameter dependent coefficients. In general, these chapters have demonstrated how a certain class of decentralized problems can be extended to their switched counterparts, and there is a good scope of broadening the class of problems where similar techniques can be applied to obtain optimal control for switched versions of decentralized systems.

Finally in Chapter 8, we considered a dynamic switched problem with nested information structure under ℓ_2-induced norm performance criteria. For a mode dependent nested mode-dependent plant, we presented the exact conditions for existence of a nested finite-path dependent controller synthesis. These are in the form of coupled LMIs and rank conditions (which can be dropped for large enough controller dimensions). Once these conditions are solved for some memory length, we can construct the closed loop scaling matrices by a matrix completion developed here, followed by synthesis of controller using efficiently solvable algebraic expressions. It can be noted that solving a decentralized control problem with controllers having access to nested parameters, instead of a common parameter as studied here, appears challenging.

9.1 Possible Future Directions

We now list some directions where the ideas presented in this dissertation can be extended towards.

1. In Chapter 3, the study of convergence properties of update schemes was geared towards static team problems with quadratic cost function. The next obvious question is whether the techniques developed here can be extended to more general convex cost functions. In particular, we may note that in [18], authors have shown that stationarity conditions in [16] hold under more relaxed conditions. Even for the case of quadratic cost, [18, Example 1] presents a scenario where the cost coefficients are unbounded. Thus, it would be interesting to explore whether the assumption of bounded cost can be relaxed in a similar way and still maintain convergence of update schemes (possibly starting with initial strategies yielding finite expected cost).

2. In the examples presented in Chapters 3 and 4, the effects of discretizing the parameter space was assumed to be negligible. Studying the effects of such discretization on the convergence and the error in cost can be helpful in determining a desired level of discretization especially in numerically intensive problems.

3. In Chapter 5, the optimal controller obtained for the dynamic team problem with PN information structure involves solving a set of linear equations in the strategy coefficients for all the players. For PN problems, with dynamics evolving over time, such a solution may be computationally intractable for large time horizons. One may assume additional structure on the problem, for example consider switched versions of the problems considered in [38–40,77], and investigate whether a Riccati type recursive solution analogous to their non-switched counterparts can be obtained to compute the optimal strategy.

4. An iterative scheme was suggested in Theorem 19 for computation of arbitrarily close approximations of team optimal strategies. This scheme can also be applied towards the dynamic problem with OSD information pattern in Chapter 6. However, finding bounds on the convergence rate and cost error is a challenging problem and it is desirable to be able to obtain such bounds based on the cost and system matrices.

5. In Chapter 7, we examined an infinite horizon problem under full-state feedback and i.i.d. parameters; from this a number of possible explorations arise naturally. These include obtaining exact conditions for decentralized stabilizability under this information structure. One would also like to consider infinite horizon problems with output feedback, either with OSD or PN information structure. The techniques used here do not extent directly to these scenarios. Also the numerical scheme presented in Section 7.3 for infinite parameters requires further exploration. Since it involves alternating between iterations in the strategies and updating P, it will be interesting to see what effects the errors in strategy iterations have on the convergence of this process.

6. In Theorem 56, we provided a direct algebraic method for synthesis of decentralized controller. Such a synthesis can also be performed by solving LMIs in either (8.41) or (8.38) through SDP. We may note (through a simple complexity analysis counting the number of variables and constraints) that these synthesis LMIs are are computationally more demanding than the feasibility LMIs (8.32a)-(8.32c). Although the computational time of the algebraic synthesis method (in Theorem 56) seems to be much faster than the LMI schemes, a more rigorous analysis is required to compare their numerical robustness. For example, one might expect the single stage LMI in (8.41) to achieve numerical convergence more often than the stage-wise LMIs in (8.38); as in the later, it is possible that some initial stage solutions may lead to ill-conditioned problems in the later stages. One may also implement other synthesis schemes in literature as [89] (which claim better numerical stability for centralized control) applied at each stage of Theorem 56.

References

[1] P. Seiler and R. Sengupta, "An $\mathcal{H}_\infty$ approach to networked control," *IEEE Transactions on Automatic Control*, vol. 50, no. 3, pp. 356–364, March 2005.

[2] O. C. Imer, S. Yüksel, and T. Başar, "Optimal control of lti systems over unreliable communication links," *Automatica*, vol. 42, no. 9, pp. 1429–1439, 2006.

[3] L. Zhang, Y. Shi, T. Chen, and B. Huang, "A new method for stabilization of networked control systems with random delays," *IEEE Transactions on Automatic Control*, vol. 50, no. 8, pp. 1177–1181, Aug 2005.

[4] V. Gupta, B. Hassibi, and R. M. Murray, "Optimal LQG control across packet-dropping links," *System & Control Letters*, vol. 56, no. 6, pp. 439 – 446, 2007.

[5] A. Dimeas and N. Hatziargyriou, "Operation of a multiagent system for microgrid control," *Power Systems, IEEE Transactions on*, vol. 20, no. 3, pp. 1447–1455, Aug 2005.

[6] Y. Wang, D. Hill, and G. Guo, "Robust decentralized control for multimachine power systems," *Circuits and Systems I: Fundamental Theory and Applications, IEEE Transactions on*, vol. 45, no. 3, pp. 271–279, Mar 1998.

[7] V. Chandan and A. Alleyne, "Decentralized architectures for thermal control of buildings," in *Proceedings of American Control Conference*, June 2012, pp. 3657–3662.

[8] J. P. Lynch, K. H. Law et al., "Decentralized control techniques for large-scale civil structural systems," in *Proc. of the 20th Int. Modal Analysis Conference*, 2002.

[9] J. B. do Val and T. Başar, "Receding horizon control of jump linear systems and a macroeconomic policy problem," *Journal of Economic Dynamics and Control*, vol. 23, no. 8, pp. 1099 – 1131, 1999.

[10] F. Zampolli, "Optimal monetary policy in a regime-switching economy: The response to abrupt shifts in exchange rate dynamics," *Journal of Economic Dynamics and Control*, vol. 30, no. 910, pp. 1527 – 1567, 2006.

[11] J. Wolfe, D. Chichka, and J. Speyer, "Decentralized controllers for unmanned aerial vehicle formation flight," *AIAA Paper*, pp. 96–3833, 1996.

[12] A. Jadbabaie, J. Lin, and A. S. Morse, "Coordination of groups of mobile autonomous agents using nearest neighbor rules," *IEEE Transactions on Automatic Control*, vol. 48, no. 6, pp. 988–1001, 2003.

[13] S. Stankovic, M. Stanojevic, and D. Siljak, "Decentralized overlapping control of a platoon of vehicles," *Control Systems Technology, IEEE Transactions on*, vol. 8, no. 5, pp. 816–832, Sep 2000.

[14] V. Gupta and F. Luo, "On a control algorithm for time-varying processor availability," *IEEE Transactions on Automatic Control*, vol. 58, no. 3, pp. 743–748, March 2013.

[15] J. Marschak, "Elements for a theory of teams," *Management Science*, vol. 1, no. 2, pp. 127–137, 1955.

[16] R. Radner, "Team decision problems," *The Annals of Mathematical Statistics*, vol. 33, no. 3, pp. 857–881, 1962.

[17] J. Marschak and R. Radner, "Economic theory of teams," 1972.

[18] J. Krainak, J. Speyer, and S. Marcus, "Static team problems–part i: Sufficient conditions and the exponential cost criterion," *IEEE Transactions on Automatic Control*, vol. 27, no. 4, pp. 839–848, 1982.

[19] J. Krainak, J. Speyer, and S. Marcus, "Static team problems–part ii: Affine control laws, projections, algorithms, and the legt problem," *IEEE Transactions on Automatic Control*, vol. 27, no. 4, pp. 848–859, 1982.

[20] S. Yüksel and T. Başar, "Stochastic networked control systems: Stabilization and optimization under information constraints," 2013.

[21] H. Witsenhausen, "On information structures, feedback and causality," *SIAM Journal on Control*, vol. 9, no. 2, pp. 149–160, 1971.

[22] Y. C. Ho and K. C. Chu, "Team decision theory and information structures in optimal control problems–part i," *IEEE Transactions on Automatic Control*, vol. 17, no. 1, pp. 15–22, 1972.

[23] H. S. Witsenhausen, "A counterexample in stochastic optimum control," *SIAM Journal on Control*, vol. 6, no. 1, pp. 131–147, 1968.

[24] H. Witsenhausen, "Separation of estimation and control for discrete time systems," *Proceedings of the IEEE*, vol. 59, no. 11, pp. 1557–1566, 1971.

[25] A. Mahajan, N. C. Martins, M. C. Rotkowitz, and S. Yuksel, "Information structures in optimal decentralized control," in *Proceedings of IEEE Conference on Decision and Control*, 2012, pp. 1291–1306.

[26] A. Nayyar, A. Mahajan, and D. Teneketzis, "Optimal control strategies in delayed sharing information structures," *IEEE Transactions on Automatic Control*, vol. 56, no. 7, pp. 1606–1620, 2011.

[27] M. Aicardi, F. Davoli, and R. Minciardi, "Decentralized optimal control of markov chains with a common past information set," *IEEE Transactions on Automatic Control*, vol. 32, no. 11, pp. 1028–1031, 1987.

[28] P. Whittle, "Risk-sensitive linear/quadratic/gaussian control," *Advances in Applied Probability*, vol. 13, no. 4, pp. pp. 764–777, 1981.

[29] T. Başar and P. Bernhard, *$\mathcal{H}_\infty$-optimal Control and Related Minimax Design Problems: A Dynamic Game Approach.* Birkhauser, 2008.

[30] R. Bansal and T. Başar, "Stochastic teams with nonclassical information revisited: When is an affine law optimal?" *IEEE Transactions on Automatic Control*, vol. 32, no. 6, pp. 554–559, 1987.

[31] M. Rotkowitz, "Linear controllers are uniformly optimal for the witsenhausen counterexample," in *Proceedings of IEEE Conference on Decision and Control*, Dec 2006, pp. 553–558.

[32] L. Lessard, "Optimal control of a fully decentralized quadratic regulator," in *Proceedings of the Allerton Conference*, 2012, pp. 48–54.

[33] B. Kurtaran and R. Sivan, "Linear-quadratic-gaussian control with one-step-delay sharing pattern," *IEEE Transactions on Automatic Control*, vol. 19, no. 5, pp. 571–574, 1974.

[34] J. Sandell, N. and M. Athans, "Solution of some nonclassical LQG stochastic decision problems," *IEEE Transactions on Automatic Control*, vol. 19, no. 2, pp. 108 – 116, Apr 1974.

[35] M. Toda and M. Aoki, "Second-guessing technique for stochastic linear regulator problems with delayed information sharing," *IEEE Transactions on Automatic Control*, vol. 20, no. 2, pp. 260 – 262, Apr 1975.

[36] P. Varaiya and J. Walrand, "On delayed sharing patterns," *IEEE Transactions on Automatic Control*, vol. 23, no. 3, pp. 443–445, 1978.

[37] C.-H. Fan, J. Speyer, and C. Jaensch, "Centralized and decentralized solutions of the linear-exponential-gaussian problem," *IEEE Transactions on Automatic Control*, vol. 39, no. 10, pp. 1986–2003, 1994.

[38] J. Swigart and S. Lall, "An explicit dynamic programming solution for a decentralized two-player optimal linear-quadratic regulator," in *Proceedings of mathematical theory of networks and systems*, 2010.

[39] P. Shah and P. A. Parrilo, "$\mathcal{H}_2$-optimal decentralized control over posets: A state space solution for state-feedback," in *Proceedings of IEEE Conference on Decision and Control*, 2010, pp. 6722–6727.

[40] J. Swigart, "Optimal controller synthesis for decentralized systems," Ph.D. dissertation, Stanford University, 2010.

[41] L. Lessard and S. Lall, "Optimal controller synthesis for the decentralized two-player problem with output feedback," in *Proceedings of American Control Conference*, 2012, pp. 6314–6321.

[42] M. Rotkowitz and S. Lall, "A characterization of convex problems in decentralized control*," *IEEE Transactions on Automatic Control*, vol. 51, no. 2, pp. 274–286, Feb. 2006.

[43] M. Rotkowitz, "On information structures, convexity, and linear optimality," in *Proceedings of IEEE Conference on Decision and Control*, Dec 2008, pp. 1642–1647.

[44] D. Liberzon and A. Morse, "Basic problems in stability and design of switched systems," *Control Systems, IEEE*, vol. 19, no. 5, pp. 59–70, Oct 1999.

[45] O. L. V. Costa and M. Fragoso, "Discrete-time LQ-optimal control problems for infinite markov jump parameter systems," *IEEE Transactions on Automatic Control*, vol. 40, no. 12, pp. 2076–2088, 1995.

[46] H. Chizeck and Y. Ji, "Optimal quadratic control of jump linear systems with gaussian noise in discrete-time," in *Proceedings of IEEE Conference on Decision and Control*, 1988, pp. 1989–1993 vol.3.

[47] J. Nilsson, B. Bernhardsson, and B. Wittenmark, "Stochastic analysis and control of real-time systems with random time delays," *Automatica*, vol. 34, no. 1, pp. 57 – 64, 1998.

[48] H. Chan and U. Ozguner, "Optimal control of systems over a communication network with queues via a jump system approach," in *Control Applications, 1995., Proceedings of the 4th IEEE Conference on*, 1995, pp. 1148–1153.

[49] J. Lee and G. Dullerud, "Optimal disturbance attenuation for discrete time switched and markovian jump linear systems," *SIAM Journal on Control and Optimization*, vol. 45, no. 4, pp. 1329–1358, 2006.

[50] B. Sinopoli, L. Schenato, M. Franceschetti, K. Poolla, M. I. Jordan, and S. S. Sastry, "Kalman filtering with intermittent observations," *IEEE Transactions on Automatic Control*, vol. 49, no. 9, pp. 1453–1464, 2004.

[51] W. De Koning, "Infinite horizon optimal control of linear discrete time systems with stochastic parameters," *Automatica*, vol. 18, no. 4, pp. 443–453, 1982.

[52] R. Essick, J. W. Lee, and G. Dullerud, "Control of linear switched systems with receding horizon modal information," *IEEE Transactions on Automatic Control*, to appear.

[53] P. Seiler and R. Sengupta, "A bounded real lemma for jump systems," *IEEE Transactions on Automatic Control*, vol. 48, no. 9, pp. 1651–1654, Sept 2003.

[54] G. E. Dullerud and S. Lall, "A new approach for analysis and synthesis of time-varying systems," *IEEE Transactions on Automatic Control*, vol. 44, no. 8, pp. 1486–1497, 1999.

[55] O. Costa and M. Fragoso, "Stability results for discrete-time linear systems with markovian jumping parameters," *Journal of Mathematical Analysis and Applications*, vol. 179, no. 1, pp. 154 – 178, 1993.

[56] J. W. Lee and G. E. Dullerud, "Uniform stabilization of discrete-time switched and markovian jump linear systems," *Automatica*, vol. 42, no. 2, pp. 205 – 218, 2006.

[57] J. Xiong, V. A. Ugrinovskii, and I. R. Petersen, "Local mode dependent decentralized stabilization of uncertain markovian jump large-scale systems," *IEEE Transactions on Automatic Control*, vol. 54, no. 11, pp. 2632–2637, 2009.

[58] F. Farokhi and K. H. Johansson, "Limited model information control design for linear discrete-time systems with stochastic parameters," in *Proceedings of IEEE Conference on Decision and Control*, 2012, pp. 855–861.

[59] N. Elia, "Remote stabilization over fading channels," *System & Control Letters*, vol. 54, no. 3, pp. 237 – 249, 2005.

[60] S. Tatikonda and S. Mitter, "Control over noisy channels," *IEEE Transactions on Automatic Control*, vol. 49, no. 7, pp. 1196–1201, July 2004.

[61] A. Sahai, "Anytime information theory," Ph.D. dissertation, Massachusetts Institute of Technology, 2001.

[62] C. Hadjicostis and R. Touri, "Feedback control utilizing packet dropping network links," in *Proceedings of IEEE Conference on Decision and Control*, vol. 2, Dec 2002, pp. 1205–1210 vol.2.

[63] S. Boyd, L. El Ghaoui, E. Feron, and V. Balakrishnan, *Linear Matrix Inequalities in System and Control Theory*, ser. Studies in Applied Mathematics. Philadelphia, PA: SIAM, June 1994, vol. 15.

[64] G. E. Dullerud and F. Paganini, *A course in robust control theory.* Springer New York, 2000, vol. 6.

[65] Y. Nesterov, A. Nemirovskii, and Y. Ye, *Interior-point polynomial algorithms in convex programming.* SIAM, 1994, vol. 13.

[66] "http://cvxr.com/cvx/."

[67] M. C. Grant and S. P. Boyd, "The CVX users guide release 2.0," 2014.

[68] "http://www.mathworks.com/products/matlab/."

[69] T. Başar, "Decentralized multicriteria optimization of linear stochastic systems," *IEEE Transactions on Automatic Control*, vol. 23, no. 2, pp. 233–243, Apr 1978.

[70] T. Başar and G. J. Olsder, *Dynamic noncooperative game theory.* SIAM, 1999.

[71] Y. C. Ho, "Team decision theory and information structures," *Proceedings of the IEEE*, vol. 68, no. 6, pp. 644–654, 1980.

[72] T. Başar, "Equilibrium solutions in two-person quadratic decision problems with static information structures," *IEEE Transactions on Automatic Control*, vol. 20, no. 3, pp. 320–328, 1975.

[73] T. Başar, "Equilibrium solutions in static decision problems with random coefficients in the quadratic cost," *IEEE Transactions on Automatic Control*, vol. 23, no. 5, pp. 960–962, 1978.

[74] R. Srikant and T. Başar, "Sequential decomposition and policy iteration schemes for m-player games with partial weak coupling," *Automatica*, vol. 28, no. 1, pp. 95 – 105, 1992.

[75] T. Başar, "Relaxation techniques and asynchronous algorithms for on-line computation of non-cooperative equilibria," *Journal of Economic Dynamics and Control*, vol. 11, no. 4, pp. 531–549, 1987.

[76] B. Anderson and J. Moore, "Optimal filtering," *Prentice-Hall Information and System Sciences Series, Englewood Cliffs: Prentice-Hall*, vol. 1, 1979.

[77] L. Lessard and A. Nayyar, "Structural results and explicit solution for two-player LQG systems on a finite time horizon," in *Proceedings of IEEE Conference on Decision and Control*, 2013, pp. 6542–6549.

[78] T. Başar, "Two-criteria LQG decision problems with one-step delay observation sharing pattern," *Information and Control*, vol. 38, no. 1, pp. 21–50, 1978.

[79] A. Mishra, C. Langbort, and G. E. Dullerud, "A team theoretic approach to decentralized control of systems with stochastic parameters," in *Proceedings of IEEE Conference on Decision and Control*, 2012, pp. 2116–2121.

[80] R. D'Andrea and G. E. Dullerud, "Distributed control design for spatially interconnected systems," *IEEE Transactions on Automatic Control*, vol. 48, no. 9, pp. 1478–1495, sept. 2003.

[81] G. E. Dullerud and R. D'Andrea, "Distributed control of heterogeneous systems," *IEEE Transactions on Automatic Control*, vol. 49, no. 12, pp. 2113–2128, Dec. 2004.

[82] C. Langbort, R. Chandra, and R. D'Andrea, "Distributed control design for systems interconnected over an arbitrary graph," *IEEE Transactions on Automatic Control*, vol. 49, no. 9, pp. 1502 – 1519, Sept. 2004.

[83] C. W. Scherer, "Structured $\mathcal{H}_\infty$-optimal control for nested interconnections: A state-space solution," *System & Control Letters*, vol. 62, no. 12, pp. 1105 – 1113, 2013.

[84] A. Mishra, C. Langbort, and G. Dullerud, "Decentralized control of linear time-varying nested systems with $\mathcal{H}_\infty$-type performance," in *Proceedings of American Control Conference*, to appear, 2014.

[85] P. G. Voulgaris, "Control of nested systems," in *Proceedings of American Control Conference*, vol. 6, 2000, pp. 4442–4445.

[86] C. W. Scherer, "A complete algebraic solvability test for the nonstrict lyapunov inequality," *System & Control Letters*, vol. 25, no. 5, pp. 327–335, Aug. 1995.

[87] P. Gahinet and P. Apkarian, "A linear matrix inequality approach to $\mathcal{H}_\infty$ control," *International Journal of Robust and Nonlinear Control*, vol. 4, no. 4, pp. 421–448, 1994.

[88] A. Packard, "Gain scheduling via linear fractional transformations," *System & Control Letters*, vol. 22, no. 2, pp. 79 – 92, 1994.

[89] P. Gahinet, "Explicit controller formulas for LMI-based $\mathcal{H}_\infty$ synthesis," *Automatica*, vol. 32, no. 7, pp. 1007–1014, July 1996.