AN INTERFERENCE ALIGNMENT SCHEME FOR MULTIPLE MULTICAST TRAFFIC

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THESIS

Submitted in partial fulfillment of the requirements
for the degree of Master of Science in Electrical and Computer Engineering
in the Graduate College of the
University of Illinois at Urbana-Champaign, 2014

Urbana, Illinois

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ABSTRACT

We propose a new coding scheme for interference alignment in a single hop fast fading wireless network with general message demands. For the X Channel, the Degrees of Freedom (DoF) region achievable by the scheme is shown to touch a known outer bound at several points. For multiple multicast demands we show that the achievable region is at least half of the cut-set bound region. We also recover previous results of the K-user interference channel, X channel and the multicast channel. The key innovation in our scheme is the reduction of the vector space alignment problem to a combinatorial arrangement problem.

To my family, for their love and support

ACKNOWLEDGMENTS

I would like to express my sincere gratitude to my advisor Prof. Pramod Viswanath for his guidance, support and engagement throughout this master's thesis. I also thank Sreeram Kannan for introducing me to and sharing his thoughts on the subject. I am deeply indebted to my labmates and friends in the Coordinated Sciences Laboratory for offering their kind advice whenever I was in need. Special thanks to my roommates Aditya and Aravind for making the entire experience very enjoyable. Finally I thank my brother and parents for their unconditional love, support and encouragement. I will be forever grateful for your love.

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CHAPTER 1

INTRODUCTION

Interference in wireless networks can severely constrain the capacity of the network. Some of the intereference management approaches traditionally used in practice are the following:

Orthogonalize In many systems the strength of the interference is comparable to the strength of the desired signals. In such instances, interference is avoided by scheduling the transmissions such that no receiver experiences interference. Such an orthogonalization – where spectrum is divided among co-existing users in a cake-cutting fashion – is the basis for time or frequency division medium access schemes. Information theoretically, a single-user AWGN channel has a capacity that can be expressed as $\log(\text{SNR}) + o(\log(\text{SNR}))$, so that the channel has 1 degree of freedom (DoF).[1] In general, for a K-user interference channel each user can get a sum rate of $\frac{1}{K}\log(\text{SNR}) + o(\log(\text{SNR}))$ or equivalently $\frac{1}{K}$ DoF by using orthogonal access schemes. Thus in this method, a cumulative DoF of 1 is divided among the users in a "cake-cutting" fashion.

Treat as Noise If interference is weak compared to the signal strength, then a natural approach is to treat interference as noise and use single-user coding methods. This approach has been used in the frequency-reuse of cellular systems for a long time. An information theoretic validation of this approach can be found in [1–4]. A surprising result of the study is that introducing structure into the interference signals does not improve the performance in this regime.

Successive Decoding If interference is strong, then the interfering signal can be decoded along with the desired signal. In the context of the

[1] A Gaussian channel has d degrees of freedom if the capacity can be expressed as $C(\text{SNR}) = d\log(\text{SNR}) + o(\log(\text{SNR}))$. See section 2.2.

two-user interference channel, this approach has been used in proving capacity results in [5–7]. A tradeoff in this method is that the decodability of the interfering signals limits the communication rates of the other users. This approach is less common in practice due to the complexity of multi-user detection.

However, such schemes can be suboptimal and limit the maximum throughput in the network. For over three decades information theorists have pursued the capacity characterizations of interference channels. A special case of the Han-Kobayashi scheme [6] has been shown in [1] to achieve the capacity of the two-user interference channel to within one bit. There is increasing interest in approximate capacity characterizations of wireless networks as a means to understanding their performance limits. In particular, the high SNR regime – where the local additive white Gaussian noise (AWGN) at each node can be neglected relative to the signal and interference powers – offers fundamental insights into optimal interference management schemes. The DoF approach provides a capacity approximation whose accuracy approaches 100% in the high signal-to-noise ratio (SNR) regime. A network has d DoF if and only if the capacity of the network can be expressed as $d\log(\text{SNR}) + o(\log(\text{SNR}))$. Since each orthogonal (noninterfering) signalling dimension contributes a rate of $\log(\text{SNR}) + o(\log(\text{SNR}))$, the DoF of a network may be interpreted as the number of resolvable signal space dimensions. Starting with the point-to-point mutliple-input and multiple-output (MIMO) channel [8,9], the DoF have been characterized for MIMO multiple access channel (MAC) [10], MIMO broadcast channel (BC) [11–13], two-user MIMO interference channel [14], various distributed relay networks [15–17], two-user MIMO X channel [18–22], and most recently the K-user interference channel [23]. Cadambe and Jafar [23] show that for a broad class of wireless networks, even when there are more than two interfering users, the sum capacity (per user) is $\frac{1}{2}\log(\text{SNR}) + o(\log(\text{SNR}))$. A key to this result is an achievable scheme called *interference alignment*.

Interference alignment (IA) is a relatively recent technique in which the transmitted signals are coded in such a way that the interference at each receiver is restricted to only a small number of dimensions, while keeping the desired signals separable from interference. This allows simultaneous communication of many interfering users over a small number of signaling dimensions (bandwidth). While it is currently in many sophisticated forms across a variety of applications, the earliest application of IA can be found in [24, 25] where the index coding problem was introduced. It was observed again in the context of the X channel by Maddah-Ali et al. [20]. The idea was crystallized by Jafar and Shamai in [22] before Cadambe and Jafar in [23] introduced a mechanism to align an arbitrarily large number of interferers, leading to the surprising conclusion that wireless networks are not essentially interference limited.

The vector interference alignment schemes of [23] are applicable to time-varying channels. Constant channels have been dealt with using the technique of real interference alignment [26–30]. The major difference between vector interference alignment and real interference alignment is that the former relies on the linear vector-space independence, while the latter relies on linear rational independence. Besides vector and real interference alignment schemes, it is also possible to utilize the ergodicity of the channel states in the so-called ergodic interference alignment scheme [31].

A majority of systems considered so far for interference alignment involve only multiple unicast traffic, where each transmitted message is demanded by only a single receiver. There are, however, wireless *multicast* applications where a common message may be demanded by multiple receivers, e.g., in a wireless video broadcasting. Such general message request sets have been considered in [32] where each message is assumed to be requested by an equal number of receivers. Ergodic IA was employed to derive an achievable sum rate. A different but related effort is the study of the compound multiple-input single-output broadcast channel [26, 33] where the channel between the base station and the mobile user is drawn from a known discrete set. However, the DoF region was not identified in [26]. A natural generalization of the multiple unicasts scenario of [23] to multicast messages is provided in [34]. However, they consider a model in which every transmitter has only a single multicast message. The DoF region for networks with mulitple multicast message demands have not yet been fully characterized.

In this thesis, we present an IA scheme that generalizes the scheme in [34] for (i) multiple unicasts (also referred to as the X Channel), in which each transmitter can have an independent message to each receiver and (ii) multiple multicasts, in which each transmitter can have an independent multicast message to each subset of receivers. For the case of multiple unicasts, we show that a region in the DoF outer bound given in [35] can be achieved. We also show that our scheme can achieve a DoF of within 1/2 of the cut-set bound in both cases, thus extending the result of [23] for multicast traffic. This thesis is based on the results of [36].

CHAPTER 2

SYSTEM MODEL

In this chapter, we explain our wireless system model followed by an introduction to DoF.

2.1 General Model

Consider a single hop wireless network in which there are K transmitters and K receivers, each having a single antenna. Assuming an ergodic and flat fading, the input-output relation at the jth receiver at time-slot t is described as

$$Y_j(t) = H_{j1}(t)X_1(t) + \ldots + H_{jK}(t)X_K(t) + Z_j(t), \quad j = 1, \ldots, K, \tag{2.1}$$

where $H_{ji}(t)$ denotes the channel fade coefficient between transmitter i and receiver j, $X_i(t)$ the symbol transmitted by transmitter i, $Y_j(t)$ the symbol received by receiver j and $Z_j(t)$ is the additive complex Gaussian noise (AWGN) of unit variance. To avoid degenerate channel conditions, such as all channel coefficients being equal, zero or infinity, we assume that the coefficients are drawn i.i.d. from a continuous distribution and the absolute value of all channel coefficients bounded between a nonzero minimum and a finite maximum value, $0 < H_{\min} \leq |H_{ji}(t)| \leq H_{\max} < \infty$. We also assume that the causal channel state information is known globally, i.e. at time slot τ each node knows all the elements of the set $\{H_{ji}(t) : (j,i) \in \{1,\ldots,K\} \times \{1,\ldots,K\}, t = 1,2,\ldots,\tau\}$. Now, if we use a symbol expansion of τ_{sym} time slots, then the input-output relation becomes

$$\mathbf{Y}_j(t') = \sum_{i=1}^{K} \mathbf{H}_{ji}(t')\mathbf{X}_i(t') + \mathbf{Z}_j(t'), \quad j = 1, 2, \ldots, K, \tag{2.2}$$

where $\mathbf{X}_i(t'), \mathbf{Y}_i(t'), \mathbf{Z}_j(t')$ are vectors of length τ_{sym} and $\mathbf{H}_{ji}(t')$ is a diagonal matrix of size $\tau_{\text{sym}} \times \tau_{\text{sym}}$. Here t' denotes the channel use index, for example the jth entry of $\mathbf{X}_i(t')$ is the same as $X_i((t'-1) \times \tau_{\text{sym}} + j)$ and so on. Throughout this thesis we restrict ourselves to the single antenna case.

Remark: For the purpose of our results there is no fundamental need for the signalling dimension to be time. A symbol expansion of τ_{sym} over time slots, frequency slots or a time-frequency tuple if coding is performed in both time and frequency, can all be thought of as being equivalent. Similarly it is enough if the channel matrices $\mathbf{H}_{ji}(t')$ commute with each other, without the need for being diagonal. However, the ergodic nature of the channel coefficients is an important assumption.

Depending on the traffic pattern, i.e., unicast, multicast etc., each transmitter in $1, \ldots, K$ can have one or many independent messages intended for receivers in $1, \ldots, K$. In general for independent messages $W_1, \ldots, W_m$ with the cardinality of ith message set being $|W_i(P)|$ (for a transmitter power P) and codewords of length τ_{sym}, the rates $R_i = \frac{\log |W_i(P)|}{\tau_{\text{sym}}}$ corresponding to the messages are said to be achievable if the probability of error for the message can be arbitrarily small by choosing an appropriately large τ_{sym}. The closure of the set of all achievable rate tuples is called the capacity region.

2.2 Degrees of Freedom (DoF)

The DoF approximates the capacity region to within $o(\log P)$ for transmission power P. Consider a point-to-point Gaussian channel

$$Y = HX + N,$$

where X and Y denote the input and the output symbols respectively, H denotes the channel coefficient and N is the additive Gaussian noise (AWGN) term. All symbols are in the complex field. Let us assume a power constraint of P on the input, i.e., $E[|X|^2] \leq P$ and let N be a circularly symmetric complex Gaussian $\mathcal{N}_c(0, \sigma^2)$. For such a channel, the capacity was shown by Shannon to be

$$C = \log \left(1 + P \frac{|H|^2}{\sigma^2} \right),$$

units of information per channel use. Formally, we define the DoF metric η for this channel as

$$\eta \triangleq \lim_{P\to\infty} \frac{C(P)}{\log P} = 1,$$

which can also be stated as $C(P) = \eta \log(P) + o(\log(P))$. Hence the DoF metric is often referred to as the *pre-log* factor. Notice that the strength of the channel gain H and the noise variance σ^2 do not figure in the DoF, since these quantities do not scale with the power P. For M parallel channels $Y_i = H_i X_i + N_i,\ i = 1, \ldots, M$ with an average power P per channel, it is easy to see that the total capacity is $M \log(P) + o(\log(P))$ implying a DoF of M. Note that the strength of the channels and the noise variance are irrelevant as before. In this sense, we can think of DoF as the number of available channels or signalling dimensions where 1 signalling dimension corresponds to an interference-free AWGN channel with an SNR that increases proportional to P as $P \to \infty$.

In general, for a multi-user Gaussian network with m messages $W_1, \ldots, W_m$ and a power constraint of P per user, we can associate a DoF to each message that is sent. Denoting the rate of the ith message by $R_i(P)$, the DoF region is given by

$$\begin{aligned} \mathcal{D} \triangleq \{\mathbf{d} = (d_1, \ldots, d_m) \in \mathbb{R}_+^m : \exists (R_1(P), \ldots, R_m(P)) \in C(P), \\ \text{such that } d_i = \lim_{P\to\infty} \frac{R_i(P)}{\log(P)}, 1 \le i \le m\}, \end{aligned} \tag{2.3}$$

where $C(P)$ is the capacity region of the network. A limitation of the DoF metric is that it is defined for the scenario wherein all the users have the same power constraint in dB scale up to an additive constant. To incorporate the diversity in the signal strengths, one can modify the definition in equation (2.3) to create a metric called the *generalized* DoF [37], but for our purpose the definition in equation (2.3) suffices.

2.3 Notation

For any $k \in \mathbb{N}$, let $[k]$ denote the set $\{1, \ldots, k\}$. For any two matrices $\mathbf{U}$ and $\mathbf{V}$ we let $\mathbf{U} \subset \mathbf{V}$ mean the (column) subspace spanned by $\mathbf{U}$ is contained in the subspace spanned by $\mathbf{V}$. Similarly $\mathbf{U} \equiv \mathbf{V}$ means the subspace spanned by both the matrices coincide.

CHAPTER 3

PRELIMINARIES

In this chapter, we illustrate the IA principle through an example of the K-user interference channel and also describe the mechanics of an asymptotic alignment scheme adopted by [23].

3.1 K-User Interference Channel

The K-user interference channel consists of K transmitter nodes and K receiver nodes, where each transmitter can have a single message for the corresponding receiver (formal definition in section 3.2.1). Consider the example of $K = 3$. If at any time only one of the transmitters is on with the others turned off, it can achieve a DoF of 1. Therefore by time-sharing we can achieve a DoF of 1/3 for each user (more generally, we can achieve $1/K$ per user if there are K users). Now, for illustration let us consider a symbol extension of 3, i.e., each codeword is of length 3. Suppose that user 1 transmits over a space of dimension 2. Let the beamforming vectors corresponding to the two-dimensional space be $\mathbf{v}_1^1$ and $\mathbf{v}_1^2$. Analogously let users 2 and 3 beamform along the one-dimensional spaces $\mathbf{v}_2$ and $\mathbf{v}_3$ respectively. We arbitrarily set $\mathbf{v}_2$ to be the all-one's vector $\mathbf{1}_{3\times1}$. Let $\mathbf{H}_{ji}$ denote the channel matrix between transmitter i and receiver j. Then we are able to succesfully decode at all the receivers if:

- At receiver 1, the interfering signals from users 2 and 3 are aligned:

$$\mathbf{H}_{12}\mathbf{v}_2 = \mathbf{H}_{13}\mathbf{v}_3 \Rightarrow \mathbf{v}_3 = (\mathbf{H}_{13})^{-1}\mathbf{H}_{12}\mathbf{1}_{3\times1}.$$

  Since now, we can zero-force the interference by projecting along vectors orthogonal to $\mathbf{H}_{13}\mathbf{v}_3$ to recover the messages from user 1.

- At receiver 2, the interfering signals from users 1 and 3 are aligned.

Since interference from user 1 is two-dimensional, suppose we are able to align the signal from user 3 to one of the signals from user 1:

$$\mathbf{H}_{23}\mathbf{v}_3 = \mathbf{H}_{21}\mathbf{v}_1^1 \Rightarrow \mathbf{v}_1^1 = (\mathbf{H}_{21})^{-1}\mathbf{H}_{23}(\mathbf{H}_{13})^{-1}\mathbf{H}_{12}\mathbf{1}_{3\times 1},$$

(in general the signal from user 3 could be aligned to any vector in the two-dimensional subspace of user 1 at receiver 2).

- At receiver 3, the interfering signals from users 1 and 2 are aligned. This is similar to the previous case:

$$\mathbf{H}_{32}\mathbf{v}_2 = \mathbf{H}_{31}\mathbf{v}_1^2 \Rightarrow \mathbf{v}_1^2 = (\mathbf{H}_{31})^{-1}\mathbf{H}_{32}\mathbf{1}_{3\times 1}.$$

Thus if we choose the beamforming vectors as above, we are guaranteed a per symbol DoF of 2/3 for user 1 and 1/3 each for users 2 and 3. This technique in which the transmitted signals are beamformed in such a way that all the interfering signals overlap and are contained within a fixed subspace at each receiver is called interference alignment.

The above example guarantees a sum-DoF of 4/3 for the three-user interference channel. While it is an improvement over time-sharing, it is still suboptimal. In the following section we discuss the optimal scheme.

3.2 Alignment Scheme

In this section we will see an interference alignment scheme that can asymptotically achieve a sum-DoF of $K/2$ for the K-user interference channel. In order to achieve a per user DoF of 1/2, we require the signal subspace and the interference subspace to be of the same dimension at each receiver. However this problem is overconstrained and hence does not have a solution [23]. What is possible, though, is a scheme where most of the interfering signals are aligned within a subspace while the receivers allow for an "overflow" space for interfering signals that do not align perfectly. Under such a scheme, one can show that the fraction of the overflow space in comparison to the signal space diminishes as the symbol extension grows. Thus asymptotically we are able to achieve a DoF of 1/2 per user. Equivalently, for any $\epsilon > 0$, we can find a sufficiently large symbol extension such that the DoF for each user is within ϵ of 1/2. This scheme was proposed by Cadambe and Jafar in [23] and has since gathered considerable attention. We explain this scheme in the following sections as we will use it as a foundation for our alignment scheme in chapter 4.

3.2.1 Channel Model

The K-user interference channel consists of K transmitters $\{1, \dots, K\}$ and K receivers $\{1, \dots, K\}$ where each transmitter can have a message for the corresponding receiver. As discussed in section 2.1, the channel output at the jth receiver at time t is given by

$$\mathbf{Y}_j(t) = \sum_{i=1}^{K} \mathbf{H}_{ji}(t)\mathbf{X}_i(t) + \mathbf{Z}_j(t), \quad j = 1, 2, \dots, K, \tag{3.1}$$

where $\mathbf{Y}_j(t)$ is the output vector at the jth receiver, $\mathbf{X}_i(t)$ is the input vector at transmitter i and $\mathbf{H}_{ji}(t)$ is the channel fading matrix between transmitter i and receiver j at tth channel use. $\mathbf{Z}_j(t)$ is the additive noise assumed to be independent and identically distributed (i.i.d) with zero mean and unit variance at the jth receiver. The channel coefficients are assumed to be drawn from a continuous distribution with their absolute values bounded between a nonzero minimum value and a finite maximum. We also assume that the channel knowledge is known globally at each time slot, i.e., $\mathbf{H}_{ji}(t), \forall i, j \in [K]$ is known to all the users at time t.

3.2.2 Interference Alignment Scheme

For the sake of simplicity we only consider the case of a three-user intereference channel as before and show that the DoF tuple $(d_1, d_2, d_3) = (\frac{n+1}{2n+1}, \frac{n}{2n+1}, \frac{n}{2n+1})$ is achievable $\forall n \in \mathbb{N}$. This implies that

$$d_1 + d_2 + d_3 = \sup_n \frac{3n+1}{2n+1} = \frac{3}{2}$$

lies in the DoF region since it is a closed set. To show this, let us consider a symbol extension of $2n+1$. Let $\mathbf{X}_k(t)$ be the vector transmitted by user k at time slot t. The signal received by receiver j is given by equation (3.1). Since we are interested only in the DoF, we can assume that we are operating in the high SNR regime and ignore the noise term. To prove that (d_1, d_2, d_3) is achievable in the original channel, we show that $(\tilde{d}_1, \tilde{d}_2, \tilde{d}_3) = (n+1, n, n)$ is achievable over the extended channel. Let $\mathbf{v}_k^{[m]}, m = 1, \ldots, M$, be M linearly independent beamforming vectors of user k, i.e.,

$$\mathbf{X}_1(t) = \sum_{m=1}^{n+1} x_1^{[m]}(t)\mathbf{v}_1^{[m]} = \mathbf{V}_1\tilde{\mathbf{X}}_1(t) \tag{3.2}$$

$$\mathbf{X}_j(t) = \sum_{m=1}^{n} x_j^{[m]}(t)\mathbf{v}_j^{[m]} = \mathbf{V}_j\tilde{\mathbf{X}}_j(t), \quad j = 1, 2, \tag{3.3}$$

where $\tilde{\mathbf{X}}_j(t)$ corresponds to the message vector of user j. Now, ignoring noise, the received signal at node j is

$$\mathbf{Y}_j(t) = \mathbf{H}_{j1}\mathbf{V}_1\tilde{\mathbf{X}}_1(t) + \mathbf{H}_{j2}\mathbf{V}_2\tilde{\mathbf{X}}_2(t) + \mathbf{H}_{j3}\mathbf{V}_3\tilde{\mathbf{X}}_3(t). \tag{3.4}$$

In order to be able to recover $\mathbf{X}_1(t)$ (and hence $\tilde{\mathbf{X}}_1(t)$) at receiver 1, we would like two things to happen: (i) the interfering signals $\mathbf{X}_2$ and $\mathbf{X}_3$ are aligned in an n-dimensional subspace and (ii) the subspace spanned by $\mathbf{V}_1$ is linearly independent of the above interference subspace. We can then zero-force the interference and recover $\mathbf{X}_1$ by projecting along the $n+1$ vectors that are orthogonal to the interference subspace. Analogous conditions apply at receiver 2 (respectively 3), but as the interference subspace from user 1 is larger than that from user 3 (respectively 2) the subspace from user 3 (respectively 2) must be entirely contained within the subspace from user 1. Mathematically we would like:

$$\mathbf{H}_{12}\mathbf{V}_2 \equiv \mathbf{H}_{13}\mathbf{V}_3 \tag{3.5}$$

$$\mathbf{H}_{23}\mathbf{V}_3 \subset \mathbf{H}_{21}\mathbf{V}_1 \tag{3.6}$$

$$\mathbf{H}_{32}\mathbf{V}_2 \subset \mathbf{H}_{31}\mathbf{V}_1. \tag{3.7}$$

Since the channel matrices $\mathbf{H}_{ji}$ are modeled as generic, they have a full-rank almost surely. Let

$$\mathbf{A} = \mathbf{V}_1 \tag{3.8}$$
$$\mathbf{B} = (\mathbf{H}_{21})^{-1}\mathbf{H}_{23}\mathbf{V}_3 \tag{3.9}$$
$$\mathbf{C} = (\mathbf{H}_{31})^{-1}\mathbf{H}_{32}\mathbf{V}_2 \tag{3.10}$$
$$\mathbf{T} = \mathbf{H}_{12}(\mathbf{H}_{21})^{-1}\mathbf{H}_{23}(\mathbf{H}_{32})^{-1}\mathbf{H}_{31}(\mathbf{H}_{13})^{-1}, \tag{3.11}$$

then the conditions in equations (3.5)-(3.7) can equivalently be expressed as

$$\mathbf{B} \equiv \mathbf{TC} \tag{3.12}$$
$$\mathbf{B} \subset \mathbf{A} \tag{3.13}$$
$$\mathbf{C} \subset \mathbf{A}. \tag{3.14}$$

In the following, we will construct the matrices $\mathbf{A}, \mathbf{B}$ and $\mathbf{C}$. Let $\mathbf{w}$ be the $(2n+1)$ length column vector of all 1's. Construct $\mathbf{A}, \mathbf{B}$ and $\mathbf{C}$ as

$$\mathbf{A} = \begin{bmatrix}\mathbf{w} & \mathbf{Tw} & \mathbf{T}^2\mathbf{w} & \dots & \mathbf{T}^n\mathbf{w}\end{bmatrix}, \tag{3.15}$$
$$\mathbf{B} = \begin{bmatrix}\mathbf{Tw} & \mathbf{T}^2\mathbf{w} & \dots & \mathbf{T}^n\mathbf{w}\end{bmatrix}, \tag{3.16}$$
$$\mathbf{C} = \begin{bmatrix}\mathbf{w} & \mathbf{Tw} & \dots & \mathbf{T}^{n-1}\mathbf{w}\end{bmatrix}. \tag{3.17}$$

It can then be easily verified that equations (3.15)-(3.17) satisfy the required conditions in equations (3.12)-(3.14). The beamforming matrices $\mathbf{V}_1, \mathbf{V}_2$ and $\mathbf{V}_3$ can now be constructed from $\mathbf{A}, \mathbf{B}, \mathbf{C}$ and satisfy the alignment constraints.

It only remains to be seen that condition (ii) is also satisfied at the receivers, i.e., the subspace spanned by the messages are linearly independent of the interference subspaces. Let us consider receiver 1. The desired message from transmitter 1 is received along $\mathbf{H}_{11}\mathbf{V}_1$ while the interference vectors are aligned in the subspace $\mathbf{H}_{12}\mathbf{V}_2$. Therefore, in order to show that there are $n+1$ dimensions available in $\mathbf{X}_1$, we must show that the $2n+1$ by $2n+1$ matrix

$$\begin{bmatrix}\mathbf{H}_{11}\mathbf{V}_1 & \mathbf{H}_{12}\mathbf{V}_2\end{bmatrix} \tag{3.18}$$

is full rank almost surely. We can show this using the random nature of the channel matrices, but we refer to [23] for the proof.

So far we have discussed the concept of interference alignment and have seen an application of it to the K-user interference channel that can achieve the optimal DoF. In the following chapters we extend the aforementioned ideas and present a combinatorial alignment scheme for more general channel models.

3.3 Asymptotic Alignment

The key problem in vector interference alignment is given an arbitrary number of linear transformations $\mathbf{T}_1, \ldots, \mathbf{T}_N$ construct a matrix $\mathbf{V}$ such that the spaces $\mathbf{T}_1\mathbf{V}, \ldots, \mathbf{T}_N\mathbf{V}$ are all aligned, i.e., $\mathbf{T}_1\mathbf{V} \equiv \mathbf{T}_2\mathbf{V} \equiv \ldots \equiv \mathbf{T}_N\mathbf{V}$. In the interference channel problem, the $\mathbf{T}_i$'s corresponded to the channel matrices and $\mathbf{V}$ represented the beamforming matrix. As was the case in the K-user interference channel, this problem is overconstrained [37] and one can only construct a $\mathbf{V}$ such that $\mathbf{T}_1\mathbf{V} \approx \mathbf{T}_2\mathbf{V} \approx \ldots \approx \mathbf{T}_n\mathbf{V}$. This is done as follows [37, section 4.6.2]. Let $\mathcal{I} \triangleq \mathbf{V} \cup \mathbf{T}_1\mathbf{V} \cup \ldots \cup \mathbf{T}_N\mathbf{V}$ and $\mathbf{w}$ be an arbitrary vector in $\mathbb{R}^n$. Construct $\mathbf{V}$ as

$$\mathbf{V}_n = \left\{ (\mathbf{T}_1)^{\alpha_1}(\mathbf{T}_2)^{\alpha_2} \ldots (\mathbf{T}_N)^{\alpha_N}\mathbf{w}, \text{s.t. } \sum_{i=1}^{N} \alpha_i \leq n-1, \alpha_1, \ldots, \alpha_N \in \mathbb{Z}_+ \right\}.$$

It is easy to verify that $\mathbf{T}_i\mathbf{V}_n \subset \mathcal{I}_n$ and

$$\frac{|\mathcal{I}_n|}{|\mathbf{T}_i\mathbf{V}_n|} = \frac{|\mathcal{I}_n|}{|\mathbf{V}_n|} = \frac{n+N}{n} \to 1 \quad \text{as} \quad n \to \infty.$$

That is we can construct the beamforming matrix such that alignment happens asymptotically. A crucial assumption we need here is that the transformation $\mathbf{T}_i$'s should be commutative. This holds if we assume our channel matrices to be diagonal. Note that the dimension of the space should grow at least as fast as $\binom{n+N}{N} = O(n^N)$. The arbitrary vector $\mathbf{w}$ which was used to evolve the space $\mathbf{V}$ is termed a *base vector*. Thus we are able to construct a space $\mathbf{V}$ such that $\mathbf{V} \approx \mathbf{T}_i\mathbf{V}$ $\forall i \in [N]$.

CHAPTER 4

X CHANNEL

The X channel (also called X network) represents the most general class of non-multicast communication scenario possible in a single-hop wireless network. A K-user X channel comprises of K transmitter-receiver pairs wherein each transmitter can have an independent message to each receiver. The input-output relation is described as in the K-user interference channel, equation (3.1). A complete DoF region characterisation for this channel has not yet been done. In [35] the authors present an outer bound for the DoF region of the X channel as

$$\mathcal{D}^{out} = \left\{ [(d_j^i)] \; : \; \forall (m,n) \; \in \; [K]^2, \sum_{q=1}^{K} d_q^m + \sum_{p=1}^{K} d_n^p - d_n^m \; \leq \; 1 \right\}, \quad (4.1)$$

where d_j^i refers to the DoF of the message from transmitter i to receiver j. One approach towards understanding the region better is to think of the channel as being comprised of an union of several interference channels and time-share over them.

In the approach that we follow, we show a stronger achievable region than can be done by time-sharing. Further our scheme easily generalizes to multicast channels where time-sharing is suboptimal (chapter 5). We first illustrate the key ideas of our scheme by means of an example in section 4.1.

4.1 Relative Arrangement Problem

Let $\{\mathbf{X}_i^{[1]}, \ldots, \mathbf{X}_i^{[N]}\}$ denote a set of N message vectors (of equal dimension) from transmitter i for each transmitter in a K-user X channel. Each message is meant for a particular receiver in $[K]$ denoted by $label(\mathbf{X}_i^{[l]})$. Suppose we use N base vectors (section 3.3) $\mathbf{w}_1, \ldots, \mathbf{w}_N$ to evolve the beamforming matrices $\mathbf{V}_i^{[1]}, \ldots, \mathbf{V}_i^{[N]}$ respectively for each transmitter $i \in [K]$. Now, let $(\tilde{\mathbf{X}}_i^{[1]}, \ldots, \tilde{\mathbf{X}}_i^{[N]})$ denote a permutation of the set of messages of transmitter i, such that the message $\tilde{\mathbf{X}}_i^{[l]}$ is beamformed using $\mathbf{V}_i^{[l]}$, i.e.,

$$\mathbf{X}_i(t) = \sum_{l=1}^{N} \mathbf{V}_i^{[l]} \tilde{\mathbf{X}}_i^{[l]}(t), \tag{4.2}$$

is the signal transmitted by the ith user. A beamforming matrix $\mathbf{V}_i^{[l]}$ undergoes the linear transformation $\mathbf{V}_i^{[l]} \to \mathbf{H}_{ji}\mathbf{V}_i^{[l]}$ when subjected to the channel matrix $\mathbf{H}_{ji}$ between transmitter i and receiver j. Now, suppose further that $\mathbf{V}_i^{[l]}$'s have the following property.

Property 1. *At each receiver j, the interference subspaces in the set $\{\mathbf{H}_{ji}\mathbf{V}_i^{[l]} : i \in [K],\ label(\tilde{\mathbf{X}}_i^{[l]}) \neq j\}$ align with each other $\forall\ l \in [N]$, while all the message subspaces in $\{\mathbf{H}_{ji}\mathbf{V}_i^{[l]} : i \in [K], l \in [N]\ label(\tilde{\mathbf{X}}_i^{[l]}) = j\}$ become linearly independent to the interference spaces and to each other.*

With this property the permutation of the messages become important. To see this, consider a two-user X channel where transmitter 1 has the messages $\{\mathbf{X}_1^{[1]}, \mathbf{X}_1^{[2]}\}$ for receiver 1, the message $\{\mathbf{X}_1^{[3]}\}$ for receiver 2, while transmitter 2 has the message $\{\mathbf{X}_2^{[1]}\}$ for receiver 1 and the messages $\{\mathbf{X}_2^{[2]}, \mathbf{X}_2^{[3]}\}$ for receiver 2 as shown in Figure 4.1(a). Suppose user 1 transmits its messages as

$$\mathbf{X}_1(t) = \mathbf{V}_1^{[1]}\mathbf{X}_1^{[1]}(t) + \mathbf{V}_1^{[2]}\mathbf{X}_1^{[2]}(t) + \mathbf{V}_1^{[3]}\mathbf{X}_1^{[3]}(t).$$

Now, consider the following two ways by which user 2 can transmit:

$$\begin{aligned}
\text{(i)}\quad \mathbf{X}_2(t) =& \mathbf{V}_2^{[1]}\mathbf{X}_2^{[1]}(t) + \mathbf{V}_2^{[2]}\mathbf{X}_2^{[2]}(t) + \mathbf{V}_2^{[3]}\mathbf{X}_2^{[3]}(t) \\
\text{(ii)}\quad \mathbf{X}_2(t) =& \mathbf{V}_2^{[1]}\mathbf{X}_2^{[3]}(t) + \mathbf{V}_2^{[2]}\mathbf{X}_2^{[2]}(t) + \mathbf{V}_2^{[3]}\mathbf{X}_2^{[1]}(t).
\end{aligned}$$

This is shown in Figures 4.1(b) and 4.1(c) respectively. Now assuming property 1 holds, at receiver 1 the arrangement in Figure 4.1(b) shows that all of $\mathbf{H}_{12}\mathbf{V}_2^{[1]}$, $\mathbf{H}_{12}\mathbf{V}_2^{[2]}$ and $\mathbf{H}_{11}\mathbf{V}_1^{[3]}$ contain an interfering signal (denoted by φ in the figure) corresponding to a total of three interference dimensions. The signals themselves occupy three dimensions (corresponding to the subspaces $\mathbf{H}_{11}\mathbf{V}_1^{[1]}, \mathbf{H}_{11}\mathbf{V}_1^{[2]}$ and $\mathbf{H}_{12}\mathbf{V}_2^{[3]}$). Therefore the DoF of the signal received at 1 is 3/6. Similarly receiver 2 receives 3/6 DoF. Hence the sum-DoF of this arrangement is 1. In the second arrangement however, Figure 4.1(c), at

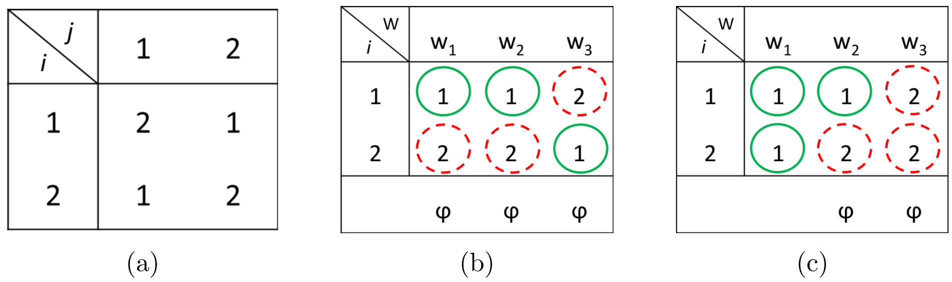

Figure 4.1: An example showing a message demand for a two-user X channel and two possible permutations of the messages. (a) Message demand matrix. (b) Arrangement i. (c) Arrangement ii. Here i denotes the transmitter, j the receiver and w_l's denote the base vectors corresponding to $\mathbf{V}^{[l]}$. The interfering messages for receiver 1 are shown in dotted-red while the intended messages are shown in solid-green.

receiver 1 $\mathbf{H}_{12}\mathbf{V}_2^{[2]}$ forms an interference subspace but interference subspaces $\mathbf{H}_{11}\mathbf{V}_1^{[3]}$ and $\mathbf{H}_{12}\mathbf{V}_2^{[3]}$ align with each other (by property 1). Hence there are only two interfering dimensions at receiver 1 yielding a receive DoF of 3/5. Likewise for receiver 2. Therefore we get a sum-DoF of 6/5 in this case.

Clearly the way we start out with message arrangements dictates the efficiency of the communication scheme. This is precisely the problem we address. In the sections that follow we describe how best to permute the message tuples at each transmitter in order to achieve the optimal DoF.

4.2 Alignment Scheme

Let $\mathbf{d} = [d_j^i]_{K\times K} \in [0,1]^{K\times K}$ denote the DoF matrix, where the (i,j)th entry d_j^i refers to the DoF of the message from transmitter i to receiver j. Since the vertices of the DoF region given in equation (4.1) are rational, we consider only the achievability of points with rational coordinates. Let us first consider those points $\mathbf{d}$ on the outer bound for which $\sum_{j=1}^{K} d_j^i = D\,\forall i \in [K]$ for some $D \in \mathbb{R}$, i.e., the sum-DoF of the messages from each transmitter are equal. We later consider the scheme for a general DoF point. Now for any rational point $\mathbf{d}$, let $\kappa \in \mathbb{Z}^+$ be such that $\mathbf{n} \triangleq \kappa\mathbf{d}$ has all integral entries, i.e., $n_j^i \in \mathbb{Z}\,\forall i,j \in [K]$ where $n_j^i = \kappa d_j^i$. We interpret n_j^i as the number of messages from transmitter i to receiver j. Therefore, every transmitter has a total of $N = \kappa D$ messages. In the scheme we propose, we show that such

i \ j	1	2	3
1	0.1	0.25	0.15
2	0.35	0.05	0.1
3	0.15	0.15	0.2

(a)

i \ j	1	2	3
1	2	5	3
2	7	1	2
3	3	3	4

(b)

i \ w	w_1	w_2	w_3	w_4	w_5	w_6	w_7	w_8	w_9	w_{10}
1	1	1	2	2	2	2	3	2	3	3
2	1	1	1	1	1	1	1	2	3	3
3	1	1	1	2	2	3	3	2	3	3
Block	1	1	φ	φ	φ	φ	φ	2	3	3

(c)

Figure 4.2: An example showing an optimal arrangement of messages for the X channel with the given symmetric message demand. (a) DoF matrix $\mathbf{d}$. (b) Message demand matrix $\mathbf{n}$. (c) Arrangement of the messages as an array B. The red dotted lines show the interfering messages at receiver 1.

points on the boundary of the outer bound region given by equation (4.1) are achievable. The following sections 4.2.1 and 4.2.2 are the two key steps involved in the scheme. As before we ignore the additive noise in our model and focus only on the interference since we are interested only in the DoF characteristics.

4.2.1 Step 1: Combinatorial Message Alignment

Since each transmitter has N messages in total, let us use a set of N base vectors $W = \{\mathbf{w}_i, 1 \leq i \leq N\}$ (where $\mathbf{w}_i$'s are generic) in order to evolve the beamforming matrices. Notice that the same set of base vectors is used in all the transmitters. As discussed previously in section 4.1, the ultimate aim here is to optimally assign each message to one of the beamforming matrices.

We view such an assignment of messages to beamforming matrices or equivalently, to base vectors, as an array B where $B(i, j)$ denotes the message from transmitter i listed under base vector j. Figure 4.2(c) shows an example assignment for $K = 3, N = 10$. We have listed only the message labels (intended recipients of the message) and not the actual messages itself in the assignment B because for any assignment of messages, exchanging of messages having the same label does not change the performance.

Let us call a column a j-block (for $j \in [K]$) if all the entries of that column have the label j. A column which is not a j-block for any j is called a ϕ-block. Let N_j denote the number of j-blocks in B for $j \in [K]$ and let $I_j \triangleq N - N_j$.

Proposition 1. *For any point* $\mathbf{d}$ *in the outer bound region, equation* (4.1)*,* *such that the sum-DoF of messages from all transmitters are equal, there exists an arrangement B such that*

$$\sum_{i'=1}^{K} n_j^{i'} + I_j \leq \kappa \qquad \forall\, j \in [K], \tag{4.3}$$

where $\mathbf{n}, \kappa, I_j$ are as defined above.

Proof. Since $\mathbf{d}$ is within the outer bound region, we must have

$$\sum_{i'=1}^{K} n_j^{i'} + \sum_{j' \in [K]: j' \neq j} n_{j'}^{i} \leq \kappa \qquad \forall\, j \in [K]. \tag{4.4}$$

Let $\delta_1 = \min_i \{n_1^i : 1 \leq i \leq K\}$ be the smallest message to receiver 1 from a transmitter. We first arrange δ_1 of the n_1^i messages from each transmitter i together in δ_1 columns. This implies $I_1 = N - \delta_1$. In fact, $I_1 = \sum_{j' \in [K]: j' \neq 1} n_{j'}^{i^*}$ where i^* is a transmitter having the largest number of interfering messages or equivalently the smallest number of non-interfering messages for 1. As such, we have $I_1 \leq \sum_{j' \in [K]: j' \neq 1} n_{j'}^{i}\ \forall i \in [K]$. Therefore, from equation (4.4) we get

$$\sum_{i'=1}^{K} n_1^{i'} + I_1 \leq \kappa. \tag{4.5}$$

Now, we sequentially perform this operation for each receiver so that equation (4.3) holds. This is possible because, $\sum_{j=1}^{K} \delta_j \leq \sum_{j=1}^{K} n_j^i \leq N$ since $\delta_j \leq n_j^i\ \forall i \in [K]$ and hence the above operations require no more than $N = |W|$ columns. The remaining messages can be arbitrarily assigned to any unassigned columns. The proposition follows. □

Figure 4.2 shows an illustration of a $K = 3$ case. We scale the DoF demand $\mathbf{d}$ with $D = 0.5$ shown in Figure 4.2(a) by $\kappa = 20$ to get the integral message demand matrix $\mathbf{n}$ shown in Figure 4.2(b) and $N = 10$. As discussed in the proof above, we arrange the label 1 messages to form the 1-blocks in columns 1 and 2. Similarly we arrange the label 2 messages to form the 2-block in column 8 and the label 3 messages to form the 3-blocks in columns 9 and 10. The remaining messages are arbitrarily arranged. In the resulting arrangement array B, we have $N_1 = 2, N_2 = 1, N_3 = 2 \Rightarrow I_1 = 8, I_2 =$

$9, I_3 = 8$. As claimed by the proposition, we have $\sum_{i'=1}^{K} n_j^{i'} + I_j \leq 20 = \kappa$ for $j = 1, 2, 3$.

4.2.2 Step 2: Evolution of Beamforming Matrices

The next step is to generate the beamforming matrices from the base vectors. The subspaces are created such that they satisfy property 1, i.e., an interfering message from a subspace remains within the subspace while a non-interfering message becomes linearly independent to the subspace at the receiver. This property, together with the previous combinatorial arrangement of the messages allows us to achieve interference alignment. Let us use a symbol expansion of $\tau = \kappa_t(\lambda + 1)^{K^2 - K}$ with $\kappa_t = \kappa$. Let $\mathbf{V}_i^{[l]}$ denote the beamforming matrix associated with base vector $\mathbf{w}_l$ for $l \in [N]$ at transmitter i. Now, the evolution is done as

$$\mathbf{V}_i^{[l]} = \left\{ \left(\prod_{\substack{(m,n)\in[K]^2 \\ B(m,l)\neq n}} \mathbf{H}_{nm}^{\alpha_{nm}} \right) \mathbf{w}_l : \begin{array}{c} \alpha_{nm} \in \{0, \ldots, \lambda\} \text{ if } m \neq i, \\ \alpha_{nm} \in \{0, \ldots, \lambda - 1\} \text{ if } m = i \end{array} \right\}, \quad (4.6)$$

i.e., for each $\mathbf{V}_i^{[l]}$ we use all the K^2 channel matrices except those associated with the messages listed under the base vector $\mathbf{w}_l$ in B. In the following proposition we show that constructing the beamforming matrices this way guarantees alignment. The arguments we use in this section are based on the results in [35] and [34].

Proposition 2. *The set of beamforming matrices generated according to equation* (4.18) *satisfies property 1.*

Proof. The proof is as follows.

(1) Alignment of Interfering Messages: We first show that at every receiver all the interfering vectors are aligned within a subspace. Let

$$\hat{\mathbf{V}}^{[l]} \triangleq \left\{ \left(\prod_{\substack{(m,n)\in[K]^2 \\ B(m,l)\neq n}} \mathbf{H}_{nm}^{\alpha_{nm}} \right) \mathbf{w}_l : \alpha_{nm} \in \{0, \ldots, \lambda\} \right\}, \quad (4.7)$$

and

$$\hat{\mathbf{V}}(j) \triangleq [\hat{\mathbf{V}}^{[l]} : l \in [N], \exists\, i \text{ s.t. } B(i,l) \neq j], \tag{4.8}$$

where for matrices $\mathbf{A}, \mathbf{B}$, $[\mathbf{A}\ \mathbf{B}]$ stands for the augmented matrix. Now for any interference message coded via the matrix $\mathbf{V}_i^{[l]}$ it is easy to see that the received signal space obeys

$$\mathbf{H}_{ji}\mathbf{V}_i^{[l]} \subset \hat{\mathbf{V}}^{[l]} \subset \hat{\mathbf{V}}(j),$$

since left multiplication by the channel matrix only increases the exponent α_{ji} in the vectors of $\mathbf{V}_i^{[l]}$ by 1, and such vectors are already included in $\hat{\mathbf{V}}^{[l]}$. In other words, our construction ensures that all the interfering messages in column l of B are aligned at the receivers for each l. Such a columnwise alignment implies a global alignment of all the interfering messages within a subspace. Hence we conclude that for any receiver $j \in [K]$, the interference component is contained within the subspace $\hat{\mathbf{V}}(j)$. We now proceed to show linear independence between the signal and interference spaces at the receiver.

(2) Linear Independence of Signal and Interference: Let $\hat{\mathbf{S}}(j)$ denote the space formed by the beamforming matrices corresponding to the message signals at receiver j

$$\hat{\mathbf{S}}(j) = [\mathbf{H}_{ji}\mathbf{V}_i^{[l]} : B(i,l) = j], \tag{4.9}$$

so that if $\hat{\mathbf{S}}(j)$ and $\hat{\mathbf{V}}(j)$ are linearly independent, then $\hat{\mathbf{V}}(j)$ can be zero-forced to retrieve the information of the messages in $\hat{\mathbf{S}}(j)$. Let $\mathbf{\Lambda}(j)$ denote the combined space

$$\mathbf{\Lambda}(j) = [\hat{\mathbf{S}}(j)\ \hat{\mathbf{V}}(j)].$$

For example, $\mathbf{\Lambda}(1)$ in Figure 4.2(c) above is given by

$$\begin{aligned}\mathbf{\Lambda}(1) &= [\hat{\mathbf{S}}(1)\ \hat{\mathbf{V}}(1)] \\ &= [\mathbf{H}_{11}\mathbf{V}_1^{[1]}\ \mathbf{H}_{11}\mathbf{V}_1^{[2]}\ \mathbf{H}_{12}\mathbf{V}_2^{[1]}\ \ldots\ \mathbf{H}_{12}\mathbf{V}_2^{[7]} \\ &\quad\ \mathbf{H}_{13}\mathbf{V}_3^{[1]}\ \mathbf{H}_{13}\mathbf{V}_3^{[2]}\ \mathbf{H}_{13}\mathbf{V}_3^{[3]}\ \hat{\mathbf{V}}^{[3]}\ \ldots\ \hat{\mathbf{V}}^{[10]}].\end{aligned}$$

Note that $\mathbf{V}_i^{[l]}$ is of dimension $\kappa(\lambda+1)^{K^2-K}$ by $\kappa\lambda^{K-1}(\lambda+1)^{K^2-2K+1}$ while $\hat{\mathbf{V}}^{[l]}$ is $\kappa(\lambda+1)^{K^2-K}$ by $\kappa(\lambda+1)^{K^2-K}$. Therefore $\mathbf{\Lambda}(j)$ is a tall matrix. Since all channel matrices are assumed to be diagonal, each element of $\mathbf{\Lambda}(j)$ is a monomial term of several random variables. For instance, the (r,c)th element, for any (r,c), of any matrix block of $\hat{\mathbf{S}}(j)$ (equation (4.9)) is given by

$$[\mathbf{H}_{ji}\mathbf{V}_i^{[l]}]_{r,c} = \mathbf{H}_{ji}(r) \prod_{\substack{(m,n)\in[K]^2 \\ B(m,l)\neq n}} (\mathbf{H}_{nm}(r))^{\alpha_{nm}}\mathbf{w}_l(r),$$

where the exponents α_{nm} are as given in equation (4.18) and $\mathbf{H}_{nm}(r)$ refers to the rth element of the diagonal matrix $\mathbf{H}_{nm}$. Similarly the (r,c)th element of any matrix block comprising $\hat{\mathbf{V}}(j)$ (equation (4.8)) is also a monomial term of a similar form. Hence by [35, Lemma 1], $\mathbf{\Lambda}(j)$ will be of full-rank if we can show that the different monomial terms either involve statistically independent random variables or they involve the same random variables with different exponents. We show this in several steps. For any row in $\mathbf{\Lambda}(j)$:

1. Monomials within the same matrix block $\mathbf{H}_{ji}\mathbf{V}_i^{[l]}$ have different exponents (by construction).

2. Between two matrix blocks corresponding to messages $\mathbf{H}_{ji}\mathbf{V}_i^{[l]}$ and $\mathbf{H}_{ji'}\mathbf{V}_{i'}^{[l']}$:

   - If $l = l'$ then $i \neq i'$, i.e., the premultiplying channel matrices $\mathbf{H}_{ji}$ and $\mathbf{H}_{ji'}$ are different. Also for such matrix blocks, the channel matrices are not involved in the construction of the beamforming matrix.

   - If $i = i'$ then $l \neq l'$, i.e., the beamforming matrices used are different. Recall that the beamforming matrices use statistically independent base vectors.

3. Between two matrix blocks corresponding to interference $\hat{\mathbf{V}}^{[l]}$ and $\hat{\mathbf{V}}^{[l']}$, $l \neq l'$:

   - The base vectors are different.

4. Between a message matrix block and an interference matrix block $\mathbf{H}_{ji}\mathbf{V}_i^{[l]}$ and $\hat{\mathbf{V}}^{[l']}$):

- If $l \neq l'$ then they have to use different base vectors.
- If $l = l'$ then the monomials contain terms from the channel matrix $\mathbf{H}_{ji}$ in the message block but not in the interference block.

And between any two rows the associated random variables are different. Thus in $\mathbf{\Lambda}(j)$ (i) each term is a monomial in a set of random variables (we can take this set to be all the random variables involved in the construction: all channel matrix elements and also base vector elements); (ii) the random variables involved in different rows are different; (iii) monomials in the same row do not have the same exponent tuple. Hence by the argument used in [34] and [35, Lemma 1] we conclude that $\mathbf{\Lambda}(j)$ has full-rank almost surely. □

With property 1 being true, looking back at the message assignment algorithm of section 4.2.1, we see that a j-block is an interference block for all receivers except j, a ϕ-block is an interference block for all the receivers and I_j counts the number of interference blocks for receiver j. What we have ensured is that all the interfering messages at receiver j are aligned under the largest interfering message for receiver j.

Since the message blocks in $\mathbf{\Lambda}(j), j \in [K]$ with a dimension of $\kappa * (\lambda + 1)^{K^2-K}$ by $(\lambda)^{K-1}(\lambda+1)^{K^2-2K+1}$ have full-column rank, asymptotically [37] we can expect a DoF of

$$\frac{|\mathbf{H}_{ji}\mathbf{V}_i^{[l]}|}{\tau} = \lim_{\lambda \to \infty} \frac{(\lambda)^{K-1}(\lambda+1)^{K^2-2K+1}}{\kappa * (\lambda+1)^{K^2-K}} = \frac{1}{\kappa},$$

for each message $\tilde{\mathbf{X}}_i^{[l]}(t)$ or equivalently a sum-DoF of $n_j^i/\kappa = d_j^i$ for each source-destination pair. It is crucial to note that we are able to use a time expansion with a scaling of $\kappa_t = \kappa$ because our arrangement satisfied proposition 1. For any other arrangement of B, it is possible that $\max_j\{\sum_{i'=1}^{K} n_j^{i'} + I_j\}$ is larger than κ in which case $\mathbf{\Lambda}(j)$ would no longer be full-rank for all j. Therefore any DoF point in the outer bound given by equation (4.1) with equal sum-DoF is achievable. We now discuss more general DoF points followed by a converse theorem.

i \ j	1	2	3
1	1/13	2/13	3/13
2	3/13	0	1/13
3	2/13	3/13	4/13

(a)

i \ j	1	2	3
1	1	2	3
2	3	0	1
3	2	3	4

(b)

i \ W	w_1	w_2	w_3	w_4	w_5	w_6	w_7	w_8	w_9
1	1		2	2		3	3	3	
2	1	1			1	3			
3	1	1	2	2	2	3	3	3	3
Block	1	1	2	2	φ	3	3	3	3

(c)

Figure 4.3: An example showing the arrangement for a general message demand in the X channel. (a) Message DoF matrix $\mathbf{d}$. (b) Message demand matrix $\mathbf{n}$. (c) Arrangement of the messages as an array B.

4.3 Achievable Region

So far we have been looking at the simple case where $\sum_{j=1}^{K} d_j^i = D\ \forall i \in [K]$ for some $D \in \mathbb{R}$. We now explain the scheme for a general message demand. Consider any rational point $\mathbf{d} \in [0,1]^{K\times K}$. As before, we scale it by a positive integer κ such that

$$\mathbf{n} = \kappa \mathbf{d} \in \mathbb{Z}^{K\times K}. \tag{4.10}$$

Let $N = \max_i \sum_{j=1}^{K} n_j^i$. We follow the two steps outlined in sections 4.2.1 and 4.2.2.

Combinatorial Arrangement: It is clear that we need at least N base vectors since there are N messages from the transmitter having the largest row-sum. The main problem now is to efficiently arrange the messages. In section 4.2, we solved this problem for the case where the row-sums are all equal. This meant all the rows in $\mathbf{n}$ had exactly N messages. But in general there could be some transmitters which have fewer (i.e., lower rate) messages than others. This translates to the presence of blanks in the array B, i.e., there could exist base vectors that are not used by any message from a transmitter. Our aim here is to minimize (or bound) the maximum signal plus interference dimension at the receivers as in proposition 1. We have illustrated this problem for the example of a $K = 3$ user X network in Figure 4.3.

Consider the array B with N columns and K rows. As before, we call a column a j-block, $j \in [K]$ if all the entries in that column either have the label j or are blank. Such a block would be an interference block for all receivers except j. A column in which there are at least two messages with different labels is called a ϕ-block, i.e., it is an interference block for all the receivers $j \in [K]$. Let N_s denote the number of s-blocks, $s \in [K] \cup \{\phi\}$, in B and let $I_j \triangleq N - N_j, j \in [K]$. For the illustration in Figure 4.3(c) we have $N_1 = 2, N_2 = 2, N_3 = 4, N_\phi = 1$, i.e., two 1-blocks, four 3-blocks etc.

We first show that for any feasible arrangement B the number of interference blocks I_j's is bounded from below.

Proposition 3. *Let B be any arrangement of messages corresponding to a DoF demand of $\mathbf{d}$. Then for any $s \subseteq [K], s \neq \{\}$ it is necessary that*

$$\sum_{j:j\in s} I_j \geq (|s|-1)N + \max_i \left\{ \sum_{j:j\in s^c} n_j^i \right\}. \tag{4.11}$$

Further, for any $(I_1, \ldots, I_K)$ satisfying the above there exists an arrangement of messages B with those many interference blocks.

Proof. Consider any message assignment array B for DoF demand $\mathbf{d}$ (and the corresponding $\mathbf{n} = \kappa\mathbf{d}$). Note that B is of dimension K rows by $N = \kappa \max_i\{\sum_j d_j^i\}$ columns. Then in any s-block of B the entry for transmitter i (ith row), $\forall i$, is either a message labeled i or a blank space, since placing any other message in that position would no longer make it an s-block. As such, we can lower bound the number of blank entries in each row as follows:

$$\sum_{j=1}^{K} (N_j - n_j^i)_+ \leq N - \sum_{j=1}^{K} n_j^i, \ \forall i \in [K], \tag{4.12}$$

where $(x)_+$ denotes $\max\{0, x\}$. Now removing the $(.)_+$ gives the following $2^K - 1$ equations for each transmitter i and each set $s \subseteq [K], s \neq \{\}$,

$$\begin{aligned} &\sum_{j:j\in s} (N_j - n_j^i) \leq N - \sum_{j=1}^{K} n_j^i \\ &\quad \Rightarrow \sum_{j:j\in s} N_j \leq N - \sum_{j=1}^{K} n_j^i + \sum_{j:j\in s} n_j^i. \end{aligned} \tag{4.13}$$

Combining all such equations from each transmitter i, we get for every s

$$\begin{aligned} \sum_{j:j\in s} N_j &\leq \min_i \left\{ N - \sum_{j=1}^{K} n_j^i + \sum_{j:j\in s} n_j^i \right\} \\ \Rightarrow \sum_{j:j\in s} N_j &\leq \min_i \left\{ N - \sum_{j:j\in s^c} n_j^i \right\} \\ \Rightarrow \sum_{j:j\in s} N_j &\leq N - \max_i \left\{ \sum_{j:j\in s^c} n_j^i \right\}. \end{aligned} \tag{4.14}$$

Since $I_j = N - N_j$ substituting for N_j above yields for all s

$$\begin{aligned} \sum_{j:j\in s} (N - I_j) &\leq N - \max_i \left\{ \sum_{j:j\in s^c} n_j^i \right\} \\ \Rightarrow \sum_{j:j\in s} I_j &\geq (|s| - 1)N + \max_i \left\{ \sum_{j:j\in s^c} n_j^i \right\}, \end{aligned} \tag{4.15}$$

where $|s|$ above refers to the cardinality of the set s. In other words, the number of interference blocks for every receiver is lower bounded.

Now, for DoF demand $\mathbf{d}$ consider any $(I_1, \ldots, I_K)$ satisfying equation (4.15). It is easy to see that this implies the N_j's $(= N - I_j)$ satisfy equation (4.14). In particular, setting $s = [K]$ in equation (4.14) we get

$$\sum_{j\in[K]} N_j \leq N. \tag{4.16}$$

Therefore, we can align N_1 blocks of label 1 messages (or blanks), N_2 blocks of label 2 messages and so on in a greedy fashion without exceeding the total number of columns N (similar to the symmetric case in proposition 1). Hence we conclude that for every $(I_1, \ldots, I_K)$ satisfying equation (4.11) there exists an arrangement array B with those parameters. This completes the proof. □

Further discussion on the optimal arrangement possible has been deferred to sections 4.3.1 and 4.3.2. We now briefly discuss the evolution of the beamforming matrices for any arrangement B (with possible blank entries).

Evolution of Beamforming Matrices: Consider an arbitrary arrangement of messages B, with K rows and N' columns, corresponding to a message demand matrix $\mathbf{n}$ $(= \kappa\mathbf{d})$. Since the arrangement is arbitrary N' need not be equal to $N = \kappa \max_i \sum_j d_j^i$. We would like to show that evolution of the beamforming matrices such that property 1 (with N' instead of N) holds can be done even in this case. In order to do that, we first fill in the blank entries in B with arbitrary labels to create an augmented array B'. Then, for a symbol extenstion of $\tau_{\text{sym}} = \kappa_t(\lambda+1)^{K^2-K}$ where we set

$$\kappa_t = \max_{j\in[K]}\{\sum_{i\in[K]} n_j^i + I_j\}, \tag{4.17}$$

(where I_j's are with respect to B and not B') the beamforming matrix $\mathbf{V}_i^{[l]}$ associated with base vector $\mathbf{w}_l$ for $l \in [N']$ at transmitter i is generated as in section 4.2.2

$$\mathbf{V}_i^{[l]} = \left\{ \left(\prod_{\substack{(m,n)\in[K]^2 \\ B(m,l)\neq n}} \mathbf{H}_{nm}^{\alpha_{nm}} \right) \mathbf{w}_l : \begin{array}{c} \alpha_{nm} \in \{0,\ldots,\lambda\} \text{ if } m \neq i, \\ \alpha_{nm} \in \{0,\ldots,\lambda-1\} \text{ if } m = i \end{array} \right\}. \tag{4.18}$$

We treated the blank entries as arbitrary message labels only to ensure equal dimension among the generated subspaces. Equivalently we can decrease the λ used in the evolution of each $\mathbf{V}_i^{[l]}$ depending on the number of blank entries in column i to achieve the same effect. Notice that, by populating the blanks, the resulting array B' looks exactly like that obtained for the symmetric case of section 4.2. As such if we set all the message vectors corresponding to the extra labels added to B as all-zero vectors, then we can essentially reuse the proof of proposition 2 to show that property 1 holds for the augmented array B. Thus we have the following proposition.

Proposition 4. *For an arbitrary message arrangement B, with N' columns, corresponding to a general message demand $\mathbf{n}$, generating the beamforming matrices as above ensures that:*

1. *At each receiver j, the interference subspaces in the set $\{\mathbf{H}_{ji}\mathbf{V}_i^{[l]} : i \in [K],\ label(\tilde{\mathbf{X}}_i^{[l]}) \neq \{j,\phi\}\}$ align with each other $\forall\ l \in [N']$.*
2. *All the message subspaces in $\{\mathbf{H}_{ji}\mathbf{V}_i^{[l]} : i \in [K], l \in [N], label(\tilde{\mathbf{X}}_i^{[l]}) = j\}$ become linearly independent to the interference spaces and to each*

other.

Notice that property 4 alone is not sufficient to prove achievability of any DoF point $\mathbf{d}$. In section 4.2.2 we saw that the dimension of each subspace corresponding to a base vector was $(\lambda+1)^{K^2-K}$ while the total dimension of the signal space was given by $\kappa_t(\lambda+1)^{K^2-K}$ with $\kappa_t = \kappa$. This yielded a DoF of $1/\kappa$ per message in a subspace. In general, we have a per message DoF of $1/\kappa_t$ resulting in the DoF point $\kappa\mathbf{d}/\kappa_t$. Hence on the one hand we want the space to have a small enough dimension (i.e., small κ_t) to offer a large DoF for the messages; on the other hand the space should be large enough to accommodate all the linearly independent messages and interference at each receiver, i.e.,

$$\kappa_t \geq \max_{j\in[K]}\{\sum_{i\in[K]} n_j^i + I_j\}. \tag{4.19}$$

For example, if we started out with the integer message assignment $\mathbf{n}$ from Figure 4.3(b) then we would need to have a $\kappa_t \geq 13$ in order for the messages and interference to be linearly independent. This yields the DoF shown in Figure 4.3(a). Hence for a given DoF demand $\mathbf{d}$ an optimal arrangement B is one that minimizes κ_t

$$B = \arg\min_{B\in\mathcal{B}} \max_{j\in[K]}\{\sum_{i\in[K]} n_j^i + I_j\}, \tag{4.20}$$

where $\mathcal{B}$ denotes the set of all possible array arrangements. In the following sections we address this issue of optimal arrangement. We first derive and outer bound for our scheme in section 4.3.1 and then later in section 4.3.2 we show that every point in this outer bound is achivable, thus completely characterizing the achievable region of our scheme.

4.3.1 Outer Bound

We now derive an outer bound for our scheme. Note that because proposition 4 holds for arbitrary arrangements we take the evolution of the beamforming matrices for granted. The following proposition establishes an outer bound based on the message arrangement.

Proposition 5. *The DoF region specified by*

$$(|s|-1)D + \max_i \left\{ \sum_{j:j\in s^c} d_j^i \right\} + \sum_{\substack{i,j \\ j\in s \\ i\in[K]}} d_j^i \leq |s|, \tag{4.21}$$

$\forall s \subseteq [K], s \neq \{\}$ *where* $D = \max_i \sum_{j\in[K]} d_j^i$ *constitutes an outer bound for the achievable region of our scheme.*

Proof. For any DoF demand $\mathbf{d}$ achievable by our scheme, consider an optimal arrangement B. We retain the usual meanings for κ, n_j^i, I_j, N etc. Then using proposition 3 and equation (4.19) we have $\forall s \subseteq [K], s \neq \{\}$

$$(|s|-1)N + \max_i \left\{ \sum_{j:j\in s^c} n_j^i \right\} + \sum_{\substack{i,j \\ j\in s \\ i\in[K]}} n_j^i \leq \sum_{j:j\in s} I_j + \sum_{\substack{i,j \\ j\in s \\ i\in[K]}} n_j^i \tag{4.22}$$

$$= \sum_{j:j\in s} \left\{ I_j + \sum_{i\in[K]} n_j^i \right\} \tag{4.23}$$

$$\leq |s|\kappa_t. \tag{4.24}$$

Since $\mathbf{d}$ in the achievable region of our scheme, in the optimal arrangement B we must have $\kappa_t \leq \kappa$, where κ_t is given by equation (4.17), for otherwise the achieved DoF $\kappa\mathbf{d}/\kappa_t$ is stricly less than $\mathbf{d}$. The case of $\kappa_t < \kappa$ corresponds to $\mathbf{d}$ being in the interior of the achievable region. For any $\mathbf{d}$ in the boundary of the achievable region, we must have $\kappa_t = \kappa$. Hence, dividing both sides of equation (4.24) by $\kappa_t = \kappa$, we get the required result. □

Next, we show that any point in this outer bound region is in fact achievable.

4.3.2 Converse

Let $\mathbf{d}$ be any point in the DoF outer bound region in equation (4.21) and $\mathbf{n} = \kappa\mathbf{d}$ be the corresponding message demand matrix. From the discussion previously in this section we know that $\mathbf{d}$ is achievable by our scheme if there exists an arrangement B such that $\kappa_t \leq \kappa$ for that arrangement. From equation (4.17) we need to show the existence of an arrangement B such that

$$\kappa_t = \max_{j\in[K]} \left\{ I_j + \sum_{\substack{i \\ i\in[K]}} n_j^i \right\} \leq \kappa. \tag{4.25}$$

If we show such an existence, then together with proposition 5 we get the following theorem.

Theorem 1. *The DoF region of the X channel achievable by our IA scheme is given by*

$$\mathcal{R} = \left\{ \mathbf{d} \; : \; (|s|-1)D + \max_i \left\{ \sum_{j:j\in s^c} d_j^i \right\} + \sum_{\substack{i,j \\ j\in s \\ i\in[K]}} d_j^i \leq |s|, \; \forall s \subseteq [K], s \neq \{\} \right\}, \tag{4.26}$$

where D denotes $\max_i \sum_{j\in[K]} d_j^i$.

Proof. We now prove the existence claim of equation (4.25). For any $\mathbf{d}$ ($\mathbf{n} = \kappa\mathbf{d}$) in the outer bound in equation (4.21) consider the set

$$\mathcal{S} = \left\{ (I_1, \ldots, I_K) \in \mathbb{Z}^K : \max_i \left\{ \sum_{j':j'\neq j} n_{j'}^i \right\} \leq I_j \leq \kappa - \sum_{i:i\in[K]} n_j^i \right\}. \tag{4.27}$$

Since $\mathbf{d}$ is in the outer bound, setting s to be singleton sets in equation (4.21) we get that the K-dimensional cuboidal region $\mathcal{S}$ is non-empty. Consider the corner point of the cuboid $(I_1^*, \ldots, I_K^*)$ where

$$I_j^* = \kappa - \sum_{i:i\in[K]} n_j^i, \quad j \in [K]. \tag{4.28}$$

Clearly for this choice of I_j's equation (4.25) is met. Now, for any $s \subseteq [K]$, we have

$$\sum_{j:j\in s} I_j^* = |s|\kappa - \sum_{j:j\in s}\sum_{i:i\in[K]} n_j^i \tag{4.29}$$

$$= |s|\kappa - \sum_{\substack{i,j \\ j\in s \\ i\in[K]}} n_j^i \tag{4.30}$$

$$\geq (|s|-1)N + \max_i \left\{ \sum_{j:j\in s^c} n_j^i \right\}, \tag{4.31}$$

where the last inequality follows from equation (4.21). Thus $(I_1^*, \ldots, I_K^*)$ satisfies equation (4.11). Hence by proposition 3 we conclude that there exists an arrangement B with the interference blocks satisfying equation (4.25). This proves that every point in our outer bound region, equation (4.21), is achievable. The theorem follows. □

We have presented here an inner bound for the achievable DoF region for the X network. We see that the inner bound touches the outer bound at points where the row-sums are equal. In fact, for the two-user case both the regions coincide. In chapter 5 we discuss alignment with multicast message demands.

CHAPTER 5

MULTIPLE MULTICAST CHANNEL

5.1 Channel Model

In chapter 4 we presented a scheme for the most general non-multicast communication scenario. We now extend it to the most general multicast model. Consider a single hop wireless network with K transmitters and K receivers, each having a single antenna. Each transmitter i in $1, \ldots, K$ can have independent multicast messages to one or many subsets of receivers. We call such a network a generalized multiple multicast network. An interference alignment scheme for a network in which each transmitter has a single multicast message has been presented in [34]. They show an achievable region

$$\mathcal{D} = \left\{ \mathbf{d} \in \mathbb{R}_+^K : \sum_{k \in \mathcal{M}_j} d_k + \max_{i \in \mathcal{M}_j^c} (d_i) \leq M, \quad \forall 1 \leq j \leq K \right\}, \tag{5.1}$$

where M denotes the number of antennas and $\mathcal{M}_j$ denotes the set of users whose multicast receiver set includes receiver j. The scheme that we present extends this result to include multiple multicasts from each user. The channel input-output relation is as in the interference channel case equation (3.1). We retain the assumptions made on the channel matrices and channel state information as before.

Let $s(j), j = 1, \ldots, 2^K - 1$ be an ordering of $\{s : s \in 2^{[K]}, s \neq \phi\}$, the set of all possible receiver sets for a multicast message from transmitter i, into sets of increasing cardinality (for sets with same cardinality we order them arbitrarily). Let $\mathbf{d} = [d^i_{s(j)}]$ denote a point in the DoF region of this network. In section 5.2, we first present an algorithm for interference alignment in this network. We will then see some performance guarantees for the scheme.

5.2 Alignment Scheme

The key ideas involved here are similar to the X channel case of chapter 4. Let, $\mathbf{n} = [n^i_{s(j)}] = [d^i_{s(j)}] \times \kappa$, where $\kappa \in \mathbb{N}$ is chosen such that $n^i_{s(j)} \in \mathbb{Z}\, \forall j \in \{1, \dots, 2^K - 1\}$. As discussed in section 4.1, the problem here is to efficiently assign messages to beamforming matrices at each transmitter such that the DoF is maximized. Letting $D = \max_i \sum_{s \in 2^{[K]}} d^i_s$, we use $N = \kappa D$ number of randomly generated (and generic) beamforming matrices at each transmitter.

5.2.1 Combinatorial Message Alignment

We have N base vectors and at most N messages from each transmitter $i = 1, \dots, K$ to be arranged. Let us label the messages by their recepient set, i.e., all the n^i_s messages will have the label s and so on. We view the arrangement of messages under the base vectors as an array B where $B(i, j)$ denotes the message from transmitter i listed under base-vector j. Let us suppose we arrange multicast messages labeled $s_1, s_2, \dots, s_K$ from transmitters $1, 2, \dots, K$ respectively, in column l of B. Now, let us assume that the evolution of the beamforming matrices can be done such that at any receiver the interfering messages in a column of B are all aligned while the message signals become linearly independent, analogous to proposition 4 and property 1. This means that at the jth receiver, for any $j \in [K]$, the subspace corresponding to the lth beamforming matrix $\hat{\mathbf{V}}^{[l]}$ would contain interference if j is not in at least one of the $s_i, i \in [K]$. We call such a column in the arrangement of messages B as a "$\bigcap_i s_i$ *block*". An s-block is not an interference block only for receivers $j \in s$, i.e., all the messages in an s-block would be desired messages at every receiver $j \in s$. Notice that, this implies an s-block is also an r-block if $r \subset s$. Hence maximizing the number of s-blocks with $j \in s$ in our arrangement would reduce the number of interference blocks I_j for receiver j. Letting N_s denote the number of s-blocks in an arrangement of messages, we have

$$I_j = N - \sum_{\substack{s \subseteq [K] \\ j \in s}} N_s. \tag{5.2}$$

An example arrangement for the message demand shown in Figure 5.1 can

i \ $s(j)$	{1}	{2}	{3}	{1,2}	{1,3}	{2,3}	{1,2,3}
1	1	0	2	0	4	1	4
2	1	1	1	2	3	3	1
3	4	3	1	0	2	0	2

Figure 5.1: Message demand matrix $\mathbf{n}$.

i \ W	w_1	w_2	w_3	w_4	w_5	w_6	w_7	w_8	w_9	w_{10}	w_{11}	w_{12}
1	1	13	13	123	123	123	3	13	13	23	123	3
2	1	12	12	2	23	23	3	13	13	23	123	13
3	1	1	1	2	2	2	3	13	13	123	123	1
Block	1	1	1	2	2	2	3	13	13	23	123	φ

Figure 5.2: Message arrangement as an array B. The block labels are also shown at the bottom of each column.

be seen in Figure 5.2. As before, in equation (4.17), the symbol extension parameter κ_t is given by

$$\kappa_t = \max_{j\in[K]} \left\{ I_j + \sum_{i\in[K]} \sum_{\substack{s\subseteq[K] \\ j\in s}} n_s^i \right\} \tag{5.3}$$

$$= N + \max_{j\in[K]} \left\{ \sum_{i\in[K]} \sum_{\substack{s\subseteq[K] \\ j\in s}} n_s^i - \sum_{\substack{s\subseteq[K] \\ j\in s}} N_s \right\}, \tag{5.4}$$

and a DoF point $\mathbf{d}$ is achievable by our scheme if and only if there exists an arrangement B having a $\kappa_t \leq \kappa$. It is also clear that for $s \subseteq [K]$, we must have

$$N_s \leq \min_i \left\{ \sum_{\substack{r\in[K] \\ r\supseteq s}} n_r^i \right\}. \tag{5.5}$$

Using equations (5.4) and (5.5) it is possible to derive an analytic outer bound for the scheme similar to proposition 5. We do not discuss this, instead we present a weaker but more interesting inner bound for the scheme in section 5.2.4. The best possible arrangement for a given $\mathbf{d}$ can, however, be posed as an integer optimization problem. In the following we illustrate

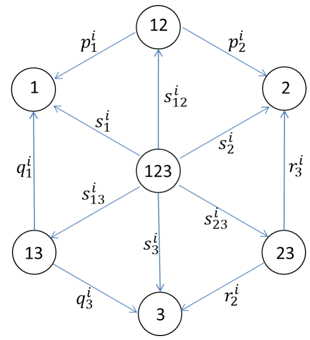

Figure 5.3: Schematic diagram showing the transfer of messages in array B.

this for a $K = 3$ user case.

5.2.2 Optimal Arrangement for $K = 3$

In this section, we present the combinatorial arrangement for the generalized multiple multicast as an integer optimization problem for the $K = 3$ user case. This can easily be extended to the many user case. From equation (5.3) the best arrangement is

$$B = \arg\min_{B \in \mathcal{B}} \max_{j \in [K]} \left\{ \sum_{i \in [K]} \sum_{\substack{s \subseteq [K] \\ j \in s}} n_s^i - \sum_{\substack{s \subseteq [K] \\ j \in s}} N_s \right\}, \tag{5.6}$$

where $\mathcal{B}$ denotes the set of all arrangement arrays. Consider any DoF point $\mathbf{d}$ and $\mathbf{n} = \kappa \mathbf{d}$ for a three-user multicast channel. The message labels in this case belong to the set $\{1, 2, 3, 12, 13, 23, 123\}$. Now, since a message with label s can be used to form an s' block for any $s' \subseteq s$, we let the variables $s_{s'}^i$ denote the number of label 123 messages of transmitter i that are used to form s' blocks, $s' \subset \{1, 2, 3\}$, in B. Similarly, let $p_{s'}^i, q_{s'}^i$ and $r_{s'}^i$ denote the number of label 12, 13 and 23 messages respectively that are transferred to their subsets. This is schematically represented in Figure 5.3. The circles denote the labels of the messages, while the arrows indicate the number of messages that have been transferred. For example, in the arrangement shown in Figure 5.2 we have $s_2^1 = 3$ since out of the four messages labeled 123 for user 1 ($n_{123}^1 = 4$ in Figure 5.1) we have used three of them to create 2-blocks. Similarly $q_1^1 = 2$. The rest of the transfer variables for user 1 are zero.

Under such a scenario, once we have transferred the required number of messages, we cannot have a greater number of s-blocks than the total number of messages in circle s. Specifically, for every node $s \in \{1, 2, 3, 12, 13, 23, 123\}$ and for every transmitter $i \in \{1, 2, 3\}$, we must have

$$n_s^i + \{\text{number of messages received}\} - \{\text{number of messages sent}\} \geq N_s.$$

Hence for the seven nodes, we get the following for $i = 1, 2, 3$:

$$\begin{aligned}
n_{123}^i - \sum_{r \subset \{1,2,3\}} s_r^i - N_{123} &\geq 0 \\
n_{12}^i + s_{12}^i - p_1^i - p_2^i - N_{12} &\geq 0 \\
n_{13}^i + s_{13}^i - q_1^i - q_3^i - N_{13} &\geq 0 \\
n_{23}^i + s_{23}^i - r_2^i - r_3^i - N_{23} &\geq 0 \\
n_1^i + s_1^i + p_1^i + q_1^i - N_1 &\geq 0 \\
n_2^i + s_2^i + p_2^i + r_2^i - N_2 &\geq 0 \\
n_3^i + s_3^i + q_3^i + r_3^i - N_3 &\geq 0,
\end{aligned} \tag{5.7}$$

where the number of blocks are given by

$$\begin{aligned}
N_{123} &= \min_i \{n_{123}^i\} \\
N_{12} &= \min_i \{n_{12}^i + s_{12}^i\} \\
N_{23} &= \min_i \{n_{23}^i + s_{23}^i\} \\
N_{13} &= \min_i \{n_{13}^i + s_{13}^i\} \\
N_1 &= \min_i \{n_1^i + s_1^i + p_1^i + q_1^i\} \\
N_2 &= \min_i \{n_2^i + s_2^i + p_2^i + r_2^i\} \\
N_3 &= \min_i \{n_3^i + s_3^i + q_3^i + r_3^i\}.
\end{aligned} \tag{5.8}$$

With this we can pose our problem as

$$\text{minimize} \quad \max_{j\in[K]} \left\{ \sum_{i\in[K]} \sum_{\substack{s\subseteq[K] \\ j\in s}} n_s^i - \sum_{\substack{s\subseteq[K] \\ j\in s}} N_s \right\}, \tag{5.9}$$

subject to the contraints given by equation (5.7) and all variables $p_s^i, q_s^i, r_s^i, \forall i, s$ belong to non-negative integers.

5.2.3 Evolution of the Signal Subspaces

Once we have fixed an arrangement of messages B, the evolution of the beamforming matrices are done in a way similar to the X channel case. Let us first consider the case of equal sum-DoF from the transmitters, i.e., $\sum_{s\subseteq[K]} d_s^i = D, \forall i$ for some constant D. This means B with K rows and $N = \kappa D$ columns does not have any blank entries. For a symbol expansion of $\tau = \kappa_t(\lambda+1)^{(K^2-K)}$ where κ_t is as in equation (5.3), let us denote the set of base vectors by $W = \{\mathbf{w}_l, 1 \le l \le N\}$. The entries of $\mathbf{w}_l$ are independent and identically drawn from some continuous distribution. We also assume that the absolute value of the entries are bounded between a positive minimum and a finite maximum value. As before we let the base vector $\mathbf{w}_l$ be associated with a beamforming matrix $\mathbf{V}_i^{[l]}$ for user i. For each $\mathbf{V}_i^{[l]}$ we use all the K^2 channel matrices except those associated with the messages listed under column l in B, i.e.,

$$\mathbf{V}_i^{[l]} = \left\{ \left(\prod_{\substack{(m,n)\in[K]^2 \\ n\notin B(m,l)}} \mathbf{H}_{nm}^{\alpha_{nm}} \right) \mathbf{w}_l : \begin{array}{c} \alpha_{nm} \in \{0,\dots,\lambda_l\} \text{ if } m \neq i, \\ \alpha_{nm} \in \{0,\dots,\lambda_l - 1\} \text{ if } m = i \end{array} \right\}. \tag{5.10}$$

Notice that here the number of channel matrices associated with the messages in the lth column of B (i.e., $|\{(m,n) \in [K]^2 : n \in B(m,l)\}|$) can be anywhere from K (all message labels are singleton sets in the column) to K^2 (all messages have the label $[K]$). Since the number of matrices available to do the evolution can be fewer than $K^2 - K$, we increase the maximum exponent λ_l in equation (5.10) in order to maintain the rank of $\mathbf{V}^{[l]}$. Suppose the message $B(i,l)$ from transmitter i under base-vector $\mathbf{w}_l$ is an interference to $K - |B(i,l)|$ receivers. Then there will be $K^2 - \sum_{i=1}^{K} |B(i,l)|$ number of channel matrices associated with that base vector. Since we want a total subspace of dimension $(\lambda+1)^{(K^2-K)}$ we require,

$$\begin{aligned}(\lambda+1)^{(K^2-K)} &= (\lambda_l+1)^{K^2-\sum_{i=1}^{K}|B(i,l)|} \\ \Rightarrow \quad \lambda_l &= (\lambda+1)^{\left(\frac{K^2-K}{K^2-\sum_{i=1}^{K}|B(i,l)|}\right)} - 1,\end{aligned} \tag{5.11}$$

for $K^2 - \sum_{i=1}^{K} |B(i,l)| \neq 0$. In the case when $K^2 - \sum_{i=1}^{K} |B(i,l)| = 0$, we can generate a generic rank $(\lambda+1)^{(K^2-K)}$ matrix. As before, for any interfering message from transmitter i to receiver j, the message vector is left-multiplied by the channel matrix between i and j. Since we have included such matrices in our beamforming design, the interfering messages are aligned column wise. Further, any desired messages originating from the transmitters become linearly independent to the interference subspace and to each other at the receiver, since the (generic) channel matrix corresponding to the message is not involved in the subspace creation. Hence the signal and interference spaces are linearly independent at the receivers. By the same argument we used in section 4.2.2 we can conclude that the space of received vectors $\mathbf{\Lambda}(j)$ is full-rank at every receiver j.

The message blocks in $\mathbf{\Lambda}(j)$ have a dimension of $\kappa * (\lambda_l+1)^{K^2-\sum_{i'=1}^{K}|B(i',l)|}$ by $(\lambda_l)^{K-|B(i,l)|}(\lambda_l+1)^{K^2-\sum_{i'=1}^{K}|B(i',l)|-K+|B(i,l)|}$ and have full-column rank for $j \in [K]$. Hence asymptotically [37] we can expect a DoF of

$$\frac{|\mathbf{H}_{ji}\mathbf{V}_i^{[l]}|}{\tau} = \lim_{\lambda\to\infty} \frac{(\lambda_l)^{K-|B(i,l)|}(\lambda_l+1)^{K^2-\sum_{i'=1}^{K}|B(i',l)|-K+|B(i,l)|}}{\kappa * (\lambda_l+1)^{K^2-\sum_{i'=1}^{K}|B(i',l)|}} = \frac{1}{\kappa},$$

corresponding to each message $\tilde{\mathbf{X}}_i^{[l]}(t)$ or equivalently a sum-DoF of $n_j^i/\kappa = d_j^i$ for each source-destination pair. Therefore in the asymptotic case the interfering messages lie within $\mathbf{V}^{[l]}$ at the receiver with probabilty 1. Thus we achieve interference alignment.

We have given an explicit beamforming matrix evolution algorithm for the case when the row-sums are equal. In the case where the row-sums are not equal, we can adopt a similar strategy as in section 4.3. That is, we first fill in all the blank spots (if any) of B with arbitrary message labels to form B'. We then evolve the beamforming matrices as in the equal sum-DoF case given by equation (5.10). A crucial difference is that we can now use the κ_t

(equation (5.3)) corresponding to B rather than that of B'. Hence we can conclude the following.

Proposition 6. *For any message arrangement B with N columns, corresponding to a general multiple multicast message demand $\mathbf{n}$, generating the beamforming matrices as above ensures that:*

1. *At each receiver j, the interference subspaces in the set $\{\mathbf{H}_{ji}\mathbf{V}_i^{[l]} : i \in [K],\ label(\tilde{\mathbf{X}}_i^{[l]}) \neq \{j, \phi\}\}$ align with each other $\forall l \in [N]$.*
2. *All the message subspaces in $\{\mathbf{H}_{ji}\mathbf{V}_i^{[l]} : i \in [K], l \in [N], j \in\ label(\tilde{\mathbf{X}}_i^{[l]})\}$ become linearly independent to the interference spaces and to each other $\forall l \in [N]$.*

For an achievable DoF demand $\mathbf{d}$, there is an arrangement B where κ_t is less than or equal to κ. In section 5.2.4 we give an inner bound for the scheme.

5.2.4 Inner Bound

One of the interesting results of [23] is that they show a constant DoF of 1/2 per user is achievable in a K-user interference channel irrespective of the number of users K. So far the validity of such a result for multicast channels has not yet been established. In this section we show that 1/2 of the cut-set bound is achievable even for general message demands.

Proposition 7. *For a K-user multiple multicast network as defined in section 5.1, the following DoF region is achievable,*

$$\sum_{\substack{s \subseteq [K] \\ j \in s}} \sum_{i=1}^{K} d_s^i \leq \frac{1}{2} \quad \forall j \in [K] \tag{5.12}$$

$$\sum_{s \subseteq [K]} d_s^i \leq \frac{1}{2} \quad \forall i \in [K]. \tag{5.13}$$

Proof. For a DoF demand $\mathbf{d} \in [0,1]^{K \times K}$ with a scaling of κ, we saw in section 5.2.1 that we use $\max_i(\sum_j d_{s(j)}^i) * \kappa = N$ number of base vectors. Consider an arrangement of messages B with I_j number of interfering blocks at receiver j. Then $\mathbf{d}$ is achievable if (equation (5.3)),

$$\sum_{\substack{s\subseteq[K]\\ j\in s}} \sum_{i=1}^{K} n_s^i + I_j \leq \kappa \quad \forall j \in [K]. \tag{5.14}$$

However, by our scheme it is clear that $I_j \leq N$, no matter how we do the message arrangement. Hence if,

$$\sum_{\substack{s\subseteq[K]\\ j\in s}} \sum_{i=1}^{K} n_s^i + N \leq \kappa \quad \forall j \in [K], \tag{5.15}$$

or equivalently,

$$\sum_{\substack{s\subseteq[K]\\ j\in s}} \sum_{i=1}^{K} d_s^i + \max_i (\sum_{s\subseteq[K]} d_s^i) \leq 1 \quad \forall j \in [K], \tag{5.16}$$

then $\mathbf{d}$ is achievable. Specifically, if

$$\sum_{\substack{s\subseteq[K]\\ j\in s}} \sum_{i=1}^{K} d_s^i \leq \frac{1}{2} \quad \forall j \in [K] \tag{5.17}$$

$$\sum_{s\subseteq[K]} d_s^i \leq \frac{1}{2} \quad \forall i \in [K], \tag{5.18}$$

then $\mathbf{d}$ is achievable. □

This shows that our scheme comes within $\frac{1}{2}$ of the cut-set bound. Since the X channel and multiple multicast channels are only a special case of the above generalized multiple multicast channel, this inner bound holds even in those cases.

CHAPTER 6

CONCLUSION

We have presented a novel interference alignment scheme that simplifies the problem of interference alignment to a combinatorial problem in networks with general message demands. This framework allows us to easily see the DoF region for simpler networks like the K-user interference network. We have focused on two main message demand scenarios: the X network and the generalized multiple multicast network. For the X network we have shown that the achievable region of our scheme touches the previously known outer bound. For a general multiple multicast network, we have presented the alignment problem as an integer optimization problem. We have also extended the result of achievability of half of the cut-set bound to general multicast networks. A key challenge that still remains is to devise alignment schemes that are practical to implement. Given the tremendous scope for interference alignment, this is perhaps the most important problem to be solved.

REFERENCES

[1] R. Etkin, D. Tse, and H. Wang, "Gaussian interference channel capacity to within one bit," *Information Theory, IEEE Transactions on*, vol. 54, no. 12, pp. 5534–5562, Dec 2008.

[2] A. Motahari and A. Khandani, "Capacity bounds for the Gaussian interference channel," *Information Theory, IEEE Transactions on*, vol. 55, no. 2, pp. 620–643, Feb 2009.

[3] X. Shang, G. Kramer, and B. Chen, "A new outer bound and the noisy-interference sum-rate capacity for Gaussian interference channels," *Information Theory, IEEE Transactions on*, vol. 55, no. 2, pp. 689–699, Feb 2009.

[4] V. Annapureddy and V. Veeravalli, "Gaussian interference networks: Sum capacity in the low-interference regime and new outer bounds on the capacity region," *Information Theory, IEEE Transactions on*, vol. 55, no. 7, pp. 3032–3050, July 2009.

[5] A. Carleial, "A case where interference does not reduce capacity (corresp.)," *Information Theory, IEEE Transactions on*, vol. 21, no. 5, pp. 569–570, Sep 1975.

[6] T. Han and K. Kobayashi, "A new achievable rate region for the interference channel," *Information Theory, IEEE Transactions on*, vol. 27, no. 1, pp. 49–60, Jan 1981.

[7] H. Sato, "The capacity of the Gaussian interference channel under strong interference (corresp.)," *Information Theory, IEEE Transactions on*, vol. 27, no. 6, pp. 786–788, Nov 1981.

[8] G. J. Foschini and M. J. Gans, "On limits of wireless communications in a fading environment when using multiple antennas," *Wireless Personal Communications*, vol. 6, pp. 311–335, 1998.

[9] E. Telatar, "Capacity of multi-antenna Gaussian channels," *European Transactions on Telecommunications*, vol. 10, no. 6, pp. 585–595, 1999. [Online]. Available: http://dx.doi.org/10.1002/ett.4460100604

[10] D. Tse, P. Viswanath, and L. Zheng, "Diversity-multiplexing tradeoff in multiple-access channels," *Information Theory, IEEE Transactions on*, vol. 50, no. 9, pp. 1859–1874, Sept 2004.

[11] W. Yu and J. Cioffi, "Sum capacity of Gaussian vector broadcast channels," *Information Theory, IEEE Transactions on*, vol. 50, no. 9, pp. 1875–1892, Sept 2004.

[12] D. Tse and P. Viswanath, "On the capacity of the multiple antenna broadcast channel," *DIMACS Series in Discrete Mathematics and Theoretical Computer Science*, vol. 62, pp. 87–106, 2004.

[13] S. Vishwanath, N. Jindal, and A. Goldsmith, "Duality, achievable rates, and sum-rate capacity of Gaussian MIMO broadcast channels," *Information Theory, IEEE Transactions on*, vol. 49, no. 10, pp. 2658–2668, 2003.

[14] S. A. Jafar and M. J. Fakhereddin, "Degrees of freedom for the MIMO interference channel," *Information Theory, IEEE Transactions on*, vol. 53, no. 7, pp. 2637–2642, 2007.

[15] H. Bolcskei, R. U. Nabar, O. Oyman, and A. J. Paulraj, "Capacity scaling laws in MIMO relay networks," *Wireless Communications, IEEE Transactions on*, vol. 5, no. 6, pp. 1433–1444, 2006.

[16] V. I. Morgenshtern, H. Bolcskei, and R. U. Nabar, "Distributed orthogonalization in large interference relay networks," in *Information Theory, 2005. ISIT 2005. Proceedings. International Symposium on.* IEEE, 2005, pp. 1211–1215.

[17] S. Borade, L. Zheng, and R. Gallager, "Maximizing degrees of freedom in wireless networks," pp. 561–570, 2003.

[18] M. A. Maddah-Ali, A. S. Motahari, and A. K. Khandani, "Communication over MIMO X channels: Interference alignment, decomposition, and performance analysis," *Information Theory, IEEE Transactions on*, vol. 54, no. 8, pp. 3457–3470, 2008.

[19] S. A. Jafar and S. Shamai, "Degrees of freedom region for the MIMO X channel," *CoRR*, vol. abs/cs/0607099, 2006.

[20] M. A. Maddah-Ali, A. K. Khandani, and A. S. Motahari, *Communication over X channel: signalling and multiplexing gain.* Citeseer, 2006.

[21] M. A. Maddah-Ali, A. S. Motahari, and A. K. Khandani, "Communication over MIMO X channels: Interference alignment, decomposition, and performance analysis," *Information Theory, IEEE Transactions on*, vol. 54, no. 8, pp. 3457–3470, 2008.

[22] S. A. Jafar and S. Shamai, "Degrees of freedom region of the MIMO X channel," *Information Theory, IEEE Transactions on*, vol. 54, no. 1, pp. 151–170, 2008.

[23] V. R. Cadambe and S. A. Jafar, "Interference alignment and degrees of freedom of the K-user interference channel," *Information Theory, IEEE Transactions on*, vol. 54, no. 8, pp. 3425–3441, 2008.

[24] Y. Birk and T. Kol, "Informed-source coding-on-demand (ISCOD) over broadcast channels," in *INFOCOM'98. Seventeenth Annual Joint Conference of the IEEE Computer and Communications Societies. Proceedings. IEEE*, vol. 3. IEEE, 1998, pp. 1257–1264.

[25] Y. Birk and T. Kol, "Coding on demand by an informed source (ISCOD) for efficient broadcast of different supplemental data to caching clients," *IEEE/ACM Transactions on Networking (TON)*, vol. 14, no. SI, pp. 2825–2830, 2006.

[26] T. Gou, S. A. Jafar, and C. Wang, "On the degrees of freedom of finite state compound wireless networks," *Information Theory, IEEE Transactions on*, vol. 57, no. 6, pp. 3286–3308, 2011.

[27] G. Bresler, A. Parekh, and D. N. Tse, "The approximate capacity of the many-to-one and one-to-many Gaussian interference channels," *Information Theory, IEEE Transactions on*, vol. 56, no. 9, pp. 4566–4592, 2010.

[28] S. Sridharan, A. Jafarian, S. Vishwanath, S. A. Jafar, and S. Shamai, "A layered lattice coding scheme for a class of three user Gaussian interference channels," in *Communication, Control, and Computing, 2008 46th Annual Allerton Conference on*. IEEE, 2008, pp. 531–538.

[29] R. Etkin and E. Ordentlich, "On the degrees-of-freedom of the K-user Gaussian interference channel," in *Information Theory, 2009. ISIT 2009. IEEE International Symposium on*. IEEE, 2009, pp. 1919–1923.

[30] A. S. Motahari, S. O. Gharan, M.-A. Maddah-Ali, and A. K. Khandani, "Real interference alignment: Exploiting the potential of single antenna systems," *arXiv preprint arXiv:0908.2282*, 2009.

[31] B. Nazer, M. Gastpar, S. A. Jafar, and S. Vishwanath, "Ergodic interference alignment," *Information Theory, IEEE Transactions on*, vol. 58, no. 10, pp. 6355–6371, 2012.

[32] B. Nazer, M. Gastpar, S. A. Jafar, and S. Vishwanath, "Interference alignment at finite SNR: General message sets," in *Communication, Control, and Computing, 2009. Allerton 2009. 47th Annual Allerton Conference on*. IEEE, 2009, pp. 843–848.

[33] M. A. Maddah-Ali, "The degrees of freedom of the compound MIMO broadcast channels with finite states," *CoRR*, vol. abs/0909.5006, 2009.

[34] L. Ke, A. Ramamoorthy, Z. Wang, and H. Yin, "Degrees of freedom region for an interference network with general message demands," *Information Theory, IEEE Transactions on*, vol. 58, no. 6, pp. 3787–3797, 2012.

[35] V. R. Cadambe and S. A. Jafar, "Interference alignment and the degrees of freedom of wireless networks," *Information Theory, IEEE Transactions on*, vol. 55, no. 9, pp. 3893–3908, 2009.

[36] S. Bojja Venkatakrishnan, P. Viswanath, and S. Kannan, "Degrees of freedom for multiple-multicast traffic," 2014, manuscript submitted for publication.

[37] S. A. Jafar, "Interference alignment - a new look at signal dimensions in a communication network," *Foundations and Trends in Communications and Information Theory*, vol. 7, no. 1, pp. 1–134, 2011. [Online]. Available: http://dx.doi.org/10.1561/0100000047