# ON THE ANALYSIS OF STOCHASTIC OPTIMIZATION AND VARIATIONAL INEQUALITY PROBLEMS

BY

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DISSERTATION

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# ABSTRACT

Uncertainty has a tremendous impact on decision making. The more connected we get, it seems, the more sources of uncertainty we unfold. For example, uncertainty in the parameters of price and cost functions in power, transportation, communication and financial systems have stemmed from the way these networked systems operate and also how they interact with one another. Uncertainty influences the design, regulation and decisions of participants in several engineered systems like the financial markets, electricity markets, commodity markets, wired and wireless networks, all of which are ubiquitous. This poses many interesting questions in areas of understanding uncertainty (modeling) and dealing with uncertainty (decision making). This dissertation focuses on answering a set of fundamental questions that pertain to dealing with uncertainty arising in three major problem classes:

(1) Convex Nash games;

(2) Variational inequality problems and complementarity problems;

(3) Hierarchical risk management problems in financial networks.

Accordingly, this dissertation considers the analysis of a broad class of stochastic optimization and variational inequality problems complicated by uncertainty and nonsmoothness of objective functions.

Nash games and variational inequalities have assumed practical relevance in industry and business settings because they are natural models for many real-world applications. Nash games arise naturally in modeling a range of equilibrium problems in power markets, communication networks, market-based allocation of resources etc where as variational inequality problems allow for modeling frictional contact problems, traffic equilibrium problems etc. Incorporating uncertainty into convex Nash games leads us to *stochastic Nash games*. Despite the relevance of stochastic generalizations of Nash

games and variational inequalities, answering *fundamental questions* regarding existence of equilibria *in stochastic regimes has proved to be a challenge.* Amongst other reasons, the main challenge arises from the nonlinearity arising from the presence of the expectation operator. Despite the rich literature in deterministic settings, direct application of deterministic results to stochastic regimes is not straightforward.

The first part of this dissertation explores such fundamental questions in stochastic Nash games and variational inequality problems. Instead of directly using the deterministic results, by leveraging Lebesgue convergence theorems *we are able to develop a tractable framework for analyzing problems in stochastic regimes over a continuous probability space.* The benefit of this approach is that *the framework does not rely on evaluation of the expectation operator* to provide existence guarantees, *thus making it amenable to tractable use.* We extend the above framework *to incorporate nonsmoothness of payoff functions as well as allow for stochastic constraints in models,* all of which are important in practical settings.

The second part of this dissertation extends the above framework to *generalizations of variational inequality problems and complementarity problems.* In particular, we develop a set of almost-sure sufficiency conditions for stochastic variational inequality problems with single-valued and multi-valued mappings. We extend these statements to quasi-variational regimes as well as to stochastic complementarity problems. The applicability of these results is demonstrated in analysis of risk-averse stochastic Nash games used in Nash-Cournot production distribution models in power markets by recasting the problem as a stochastic quasi-variational inequality problem and in Nash-Cournot games with piecewise smooth price functions by modeling this problem as a stochastic complementarity problem.

The third part of this dissertation pertains to hierarchical problems in financial risk management. In the financial industry, risk has been traditionally managed by the imposition of value-at-risk or VaR constraints on portfolio risk exposure. Motivated by recent events in the financial industry, we examine the role that risk-seeking traders play in the accumulation of large and possibly infinite risk. We proceed to show that when traders employ a conditional value-at-risk (CVaR) metric, much can be said by studying the interaction between value at risk (VaR) (a non-coherent risk measure) and conditional value at risk CVaR (a coherent risk measure based on VaR).

Resolving this question requires characterizing the optimal value of the associated stochastic, and possibly nonconvex, optimization problem, often a challenging problem. Our study makes two sets of contributions. First, under general asset distributions on a compact support, traders accumulate finite risk with magnitude of the order of the upper bound of this support. Second, when the supports are unbounded, under relatively mild assumptions, such traders can take on an unbounded amount of risk despite abiding by this VaR threshold. In short, VaR thresholds may be inadequate in guarding against financial ruin.

*To my family for their love, support and sacrifices.*

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# LIST OF ABBREVIATIONS

| | |
|---|---|
| CP | Complementarity Problem |
| CVAR | Conditional Value at risk measure |
| NCP | Nonlinear Complementarity Problem |
| VIP | Variational Inequality Problem |
| VIPs | Variational Inequality Problems |
| SCP | Stochastic Complementarity Problem |
| SNE | Stochastic Nash equilibrium |
| SQVI | Stochastic Quasi-Variational Inequality Problem |
| SVIP | Stochastic Variational Inequality Problem |
| VAR | Value at risk measure |

| | |
|---|---|
| $x$ | A column vector |
| $x^T$ | The transpose of the vector $x$ |
| $x_{-i}$ | $\triangleq (x_j)_{j\neq i}$, components of $x$ other than the $i$th component |
| $\|x\|$ | $\triangleq \sum_j \|x_j\|$,1-norm of a vector $x$ |
| $(t)^+$ | $\triangleq \max(t, 0)$. |
| $\nabla g$ | $\triangleq \left(\nabla_{x_1} g^T, \ldots, \nabla_{x_N} g^T\right)^T$ |
| $\nabla_{ij} f(x)$ | the $j$th component of $\nabla_{x_i} f(x)$. |
| $\mathcal{P}$ | $\triangleq (\Omega, \mathcal{F}, \mathbb{P})$, a probability space |
| $\xi : \Omega \to \mathbb{R}^d$ | a random vector |

# CHAPTER 1

# AN OVERVIEW

## 1.1 Background

The concept of an equilibrium is central to the study of a range of economic and engineered systems like financial markets, electricity markets, commodity markets, internet commerce, wired and wireless networks etc, all of which are ubiquitous in today's world. An equilibrium refers to a state of the system in which competing influences are balanced. An equilibrium can be loosely viewed as a 'steady state' or an 'ideal state' of the system. The articulation of such systems might be complicated by a variety of factors. These include competition, nonlinearity and nonsmoothness of objective or utility functions and finally uncertainty in the specification of these functions. Designers, regulators and participants of such systems are interested in knowing the following:

- Characterization: Does an equilibrium exist and what can be said about the set of equilibria;
- Computation: What actions (*strategies for participants*) result in an equilibrium;
- Design: What incentives (*taxation, subsidization by designers*) will make the system function better?

The equilibria in such systems may be formulated (modeling), qualitatively analyzed (existence, uniqueness), and computed by representing the equilibria as one of the following: (i) system of equations; (ii) optimization problems; (iii) complementarity problems; or (iv) fixed point problems.[1]

[1] Readers not familiar with these terms may refer to Section 1.2.1 for an explanation of these techniques in the context of economic equilibrium problems.

The variational inequality problem (VIP) is a problem formulation that encompasses several classical mathematical problems, including (i)-(iv) above and is thus a tool to study several equilibrium problems. Thus, the VIP provides a unifying methodology to study several equilibrium problems in engineering and economics. The theory of VIPs can thus be looked upon as a tool/methodology that enables the modeling, analysis and computation of equilibria for a wide range of practical applications.

Variational inequalities were first introduced by Hartman and Stampacchia in 1966 [1] as a tool to study partial differential equations, with applications principally drawn from mechanics and defined on infinite-dimensional spaces. On the other hand, the finite-dimensional variational inequality problem (VIP), developed as a generalization of the finite-dimensional nonlinear complementarity problem (NCP) which was first identified in 1964 in the Ph.D thesis of Cottle [2]. Since then the field has developed into a fruitful subfield of mathematical programming with rich theory, solution algorithms, a multitude of connections to numerous disciplines and a wide range of important applications in engineering and economics. Practical application problems that can be formulated as finite-dimensional variational inequality problems include Nash equilibrium problems used to derive insights into strategic behavior in power markets [3, 4], resource sharing in wireless and wireline communication networks [5, 6], competitive interactions in cognitive radio networks [7], contact problems in structural engineering, traffic equilibrium problems [8], amongst others.[2]

Many of these applications often have an element of uncertainty, arising possibly from imperfect information or underlying variability in the problem data. For example, in the design of power markets, neither the parameters of the price function nor the cost of generation are known with complete certainty. In some problems, we may view the decision makers as solving an expected value problem and consequentially the resulting variational conditions contain expectations. A possible approach to incorporating uncertainty in classical models is to replace the deterministic map in the VIP by its expectation. This is often a natural consequence of modeling agents that are risk-neutral in nature. This approach is obviously more challenging to address due to the presence of the possibly highly nonlinear expectation

[2]Please see Section 1.2.2 for explanation of these and other equilibrium problems that have been modeled as variational inequalities.

operation. On the other hand, the avenue has the advantage of having uncertainty built into the model which leads to applications that are much more realistic than its deterministic counterpart. Thus, the study of stochastic generalizations of the deterministic VIP, known as *the stochastic variational inequality problem (SVIP)*, assumes relevance and has been an area of interest in recent years to researchers in the field of mathematical and stochastic programming and more generally to the operations research community. The efforts in this area of research have largely been addressed from a computational standpoint. (cf [9], a recent survey paper for developments in SVIPs). There are two basic formulations for the SVIP found in literature; the EV (expected value) formulation and the ERM (expected residual minimization) formulation. Sample-average approximation schemes [10] solve deterministic approximations of the EV problem where the expectation is replaced by the sample mean (cf. [11, 12, 13, 14, 15]). An alternate approach for solving the EV problem relies on using stochastic approximation schemes where past research has investigated almost-sure convergence of estimators and rate analysis [16, 17, 18, 19].When faced with the ERM formulation, much of the techniques have focused on minimizing the expected value of a residual given by a gap function [20, 21, 22, 23]. Despite these recent developments, the problem of existence and uniqueness of solutions to EV formulations when uncertainty is defined by a continuous distribution, though *a fundamental and important question, has been left unanswered.* This doctoral thesis began with a hope to fill in the void created by this unanswered question.

## 1.2 VIPs, SVIPs and their generalizations

Given a set $K$ in $\mathbb{R}^n$ and a mapping $F : \mathbb{R}^n \to \mathbb{R}^n$, the deterministic variational inequality problem, denoted by $\text{VI}(K, F)$, requires an $x \in K$ such that

$$(y - x)^T \ F(x) \ \geq \ 0, \qquad \forall y \in K.$$

Throughout the thesis, unless otherwise mentioned, we assume the set $K$ is closed and convex and the map $F$ is continuous. The quasi-variational generalization of $\text{VI}(K, F)$, referred to as a quasi-variational inequality and denoted by $\text{QVI}(K, F)$, emerges when $K$ is generalized from a constant map to a set-valued map $K : \mathbb{R}^n \to \mathbb{R}^n$ with closed and convex images. More formally, QVI$(K, F)$ requires an $x \in K(x)$ such that

$$(y - x)^T \; F(x) \;\geq\; 0, \qquad \forall y \in K(x).$$

If $K$ is a cone, then the variational inequality problem reduces to a complementarity problem, denoted by CP$(K, F)$, a problem that requires an $x \in K$ such that

$$K \;\ni\; x \;\perp\; F(x) \;\in\; K^*,$$

where $K^* \triangleq \{y : y^T d \geq 0, \forall d \in K\}$ and $y \perp w$ implies $y_i w_i = 0$ for $i = 1, \ldots, n$.

Thus, a VIP and its generalizations has as its inputs a single-valued map that can be generalized as a multi-valued map and a set that can be generalized to a set-valued map. Stochastic generalizations of the above problems may be formulated by simply replacing the mapping by the expected-value of the map. Thus, a SVIP has as its input a set and the expectation of a continuous map, referred to as *the integrand*. The SVIP can be generalized by either replacing the set by a set-valued map (leading to a *stochastic quasi-variational inequality abbreviated as SQVI*) or a generalization of the integrand to a multi-valued map, leading to a *multi-valued SVIP* or a generalization of both, leading to a *multi-valued SQVI*. If the underlying set of the SVIP is specialized to a cone, the SVIP reduces to a *stochastic complementarity problem (SCP)*. Needless to say, these generalizations of SVIPs are also widely applicable to important problems in economics and engineered systems. These stochastic extensions are defined formally in chapters 2 and 3.

### 1.2.1 Source problems

(i) **Systems of equations:** Many classical economic equilibrium problems have been formulated as a system of equations, since market clearing conditions necessarily equate the total supply with the total demand. Observe that the problem of solving a system of nonlinear equations $F(x) = 0$ can be viewed as a VI$(\mathbb{R}^n, F)$ . Clearly the zeros of $F$ correspond precisely with solutions to VI$(\mathbb{R}^n, F)$ .

(ii) **Optimization problems:** An optimization problem is characterized by its objective function that is to be maximized (profit) or minimized (loss) depending upon the problem and a set of constraints. Objective functions include expressions representing profits, costs, market share, risk etc. The constraint set include constraints that represent limited budgets or resources, non negativity constraints on the variables, conservation equations etc. A constrained optimization problem with objective function $f$ and constraint set $K$ can be represented as

$$\begin{array}{ll} \text{minimize} & f(x) \\ \text{subject to} & x \in K. \end{array}$$

When the objective function $f$ is continuously differentiable on an open subset of the closed, convex set $K$, by the minimum principle, the stationary points of the above optimization problem can be written as VI($K$, $\nabla f$). Further, when $f$ is a convex function, then every stationary point of the optimization problem is a global minimum and the optimization problem and VI($K$, $\nabla f$) are equivalent.

(iii) **Complementarity problems:** A complementarity condition $x.y = 0$ for $x, y \geq 0$ expresses the fact that if $x$ is positive then $y$ must be 0 and vice-versa. Often in economic systems, the balance of supply and demand is described by a complementary relation between the two sets of decision variables. As an example, price of a commodity and excess demand of a commodity will always obey a complementarity condition - if one is positive, the other must necessarily be zero and vice-versa. Complementarity problems (CP) attempt to ascertain if and when a complementarity condition is satisfied, whether multiple or unique $x$ and $y$ satisfy the condition etc. A simple example of a complementarity problem is the complementarity slackness condition of the Karush-Kuhn-Tucker (KKT) system for an optimization problem with inequality constraints. The VIP also contains the CP as a special case; when the underlying set $K$ of the VIP is a cone, then VIP can equivalently formulated as a CP.

(iv) **Fixed point problems:** A fixed point of a function is a point that is mapped to itself by the function. There is a close connection between the solutions of a VIP and an associated fixed point problem based on a projection mapping. In particular, all solutions to a VIP can be captured as fixed points of suitably defined projection map [24]. A Nash equilibrium of a game can also be formulated as a fixed point of the game's best response map and thus existence of an equilibrium reduces to determining fixed points of the best response map.

### 1.2.2 Application areas for stochastic equilibrium problems

(i) In Nash games [25, 26], the players compete in a noncooperative fashion and a stable point referred to as Nash equilibrium refers to a set of strategies from which unilateral deviation is unprofitable. An example is the Nash-Cournot production-distribution problem [27] in which several firms produce a homogeneous commodity (tablet pcs:- ipads, android tablets etc or cereal: Kashi, Kellogg's etc). The price of this commodity is given by a function of the aggregate quantity sold. Consequently, each agents' profit function is contingent on the decisions of his competitors. Thus each firm is interested in determining the quantity to be produced that will maximize their profits, given that profits are a function of aggregate quantity produced by all competing firms.

(ii) Spatial price equilibrium problems involve the computation of commodity prices, supplies and demand in a network of spatially separated markets. Such models have been used to study problems in agriculture, energy markets, mineral economics and finance [28] as well as the effects of taxation/subsidization in such markets.

(iii) Traffic equilibrium problems [29] seek a prediction of steady-state traffic flows in a congested network and are also used in traffic planning or to determine toll collection policy to alleviate traffic congestion. Wardrop equilibria have been used as a solution concept for network games when modeling transportation and telecommunication networks with congestion. The Wardrop user equilibrium principle [29] states that in equilibrium state, the total travel time (or cost) on all the routes actually used are equal, and routes with higher total travel time (or cost) will not be used (have no flow).

(iv) Oligopolistic market equilibrium problems capture market structures with a small number of firms and allow for strategic interactions amongst the firms. Examples include financial markets, electricity power markets, department stores, computer firms, automobile, chemical or mineral extraction industries. Such models are also used to study environmental networks for example the problem of environmental pollution and an economic-incentive (pollution permits) [30] based approach to pollution reduction.

## 1.3 Challenges

The presence of the highly nonlinear expectation operation in the SVIP is one of the main reasons that makes the problem difficult to tackle. A standard approach would focus on obtaining an analytical form of the expectation. Then by using existing theory available for deterministic VIPs, the analytical form of the expectation would be further examined to derive conclusions about the structural properties (existence, uniqueness) of the underlying SVIP. However, a direct application of deterministic results to stochastic regimes is challenging for several reasons:

(1) Applying existing deterministic characterization statements to the expected - value function, relies on having access to a tractable form of the integral (and its derivatives) for the nonlinear mapping. This is often unavailable when employing distributions over general probability spaces.

(2) If one does indeed obtain a characterization statement, such a notion is tied to the chosen distribution and has limited generalizability.

(3) If the integrands are multi-valued, then such an analysis is even harder to carry out.

## 1.4 Framework

To contend with these challenges, a different approach is needed. Consider the *scenario-based VIP* which refers to the deterministic VIP that results if a particular scenario from the uncertainty set were to occur. From existing theory of deterministic VIPs [24], we know that under *convexity assumptions* for the underlying set, if the mapping in the scenario-based VIP is single-valued and coercive then the scenario-based VIP admits a solution. Further, if the coercivity property of the scenario-based mapping holds in an almost-sure sense, by applying Fatou's lemma to this almost-sure coercivity property, we obtain a coercivity property for the expected-value map. Once the expected-value map satisfies the coercivity property, deterministic results imply that a solution to the SVIP is guaranteed to exist. For generalizations of the SVIP, a similar analysis is complicated by the nonsmoothness or set-valuedness of the mapping or the set in the VIP. However, by using tools from set-valued integration and nonsmooth analysis, we are able to develop a similar line of reasoning to guarantee the existence of solutions to SVIPs and their generalizations. The crucial fact that makes this novel approach succeed is *that it obviates the need for integration*, which was the precise obstacle in the direct application of deterministic results to stochastic regimes. Often the challenge lies in showing the boundedness of sets whose specification requires evaluating the integral. Again we consider the use of the scenario-based problems as a vehicle for showing that the required set is bounded. To summarize, by using Lebesgue convergence theorems, set-valued integration and nonsmooth analysis the framework developed is capable of accomplishing the task of providing existence guarantees to SVIPs *without requiring the evaluation of the expectation operation.* As a result, the question of existence of solution to the SVIP reduces to developing sufficiency conditions for the solvability of the scenario-based problem in an almost-sure sense. Yet, a direct application of this approach has to be established in a broad range of settings where it remains unclear that such an avenue indeed has merit.

## 1.5 Methods for generalizations

Once the framework described above for SVIPs was developed, a natural next attempt was to extend the framework to generalizations of SVIPs. The goal was to have results of the same flavor - *guarantee existence of solution to the generalized stochastic problem without requiring the evaluation of the expectation operator*. This would essentially be achieved by first developing almost-sure sufficient conditions for the scenario-based problem. Application of Lebesgue convergence theorems to these almost-sure scenario-based conditions would result in a sufficient condition for the stochastic problem and thereby guarantee existence of solution to the stochastic problem. To make all this work for the above generalizations of SVIPs, we used tools from variational analysis, nonsmooth analysis and set-valued analysis analysis [31].

## 1.6 Summary and key results

The chapter specific summary and contributions are described below:

- In Chapter 2, we model both smooth and nonsmooth stochastic Nash games as SVIs and provide almost-sure sufficient conditions for the scenario-based Nash game that guarantee existence of an equilibrium to the original stochastic Nash game.

  **(i) Smooth stochastic Nash games:** Our first set of results are obtained in Section 2.3 for smooth stochatic Nash games where the random player objectives are differentiable. Specifically, by leveraging Lebesgue convergence theorems, we develop conditions under which the satisfaction of a coercivity condition associated with a scenario-based Nash game in an almost-sure sense allows us to claim that the stochastic Nash game admits an equilibrium. The associated coercivity requirements can be further weakened when the mappings are monotone or the strategy sets are uncoupled. In a similar vein, we show that if the gradient maps associated with a scenario-based Nash game is strongly monotone over a set of positive measure, then the stochastic Nash game admits a unique equilibrium.

  **(ii) Nonsmooth stochastic Nash games:** When player payoffs are merely continuous, the variational conditions of scenario-based Nash games are given by multivalued variational inequalities. However, by utilizing a set-valued analog of Fatou's Lemma, in Section 2.4, we show that the existence relationship above for smooth stochastic Nash games may be recovered for general and monotone gradient maps.

**(iii) Stochastic Nash games with coupled stochastic constraints** Often a stochastic Nash game may be characterized by coupled stochastic constraints. We examine whether the equilibrium in primal variables and Lagrange multipliers (referred to as an equilibrium in the primal-dual space) may be characterized using the techniques from (i) above. Interestingly, we develop conditions in Section 2.5 for claiming the existence and uniqueness of the stochastic Nash game in the primal-dual space. Note that, in general, even when a mapping is strongly monotone in the primal space, the mapping in the full space is at best monotone; consequently uniqueness in the whole space is by no means immediate.

**(iv) Examples** In Section 2.2, we motivate the questions of interest by using a class of stochastic Nash-Cournot games where both nonsmoothness (a consequence of employing nonsmooth risk metrics) and shared stochastic constraints are introduced. We return to these examples in Section 2.6 where existence statements are provided.

- In Chapter 3, we examine and characterize solutions for the class of stochastic variational inequality problems and their generalizations when uncertainty is defined by a continuous distribution. We develop sufficient conditions for the solvability of stochastic variational inequalities and their generalizations that do *not necessitate evaluating expectations*. By leveraging Lebesgue convergence theorems and variational analysis, we provide a far more tractable and verifiable set of sufficiency conditions that guarantee the existence of solution. Our results can briefly be summarized as follows:

  **(i) Stochastic quasi-variational inequality problems (SQVIs):** In Section 3.3, we begin by recapping our past integration-free statements for stochastic VIs that required the use of Lebesgue convergence theorems and variational analysis. Additionally, we provide extensions to regimes with multi-valued maps and specialize the conditions for settings with monotone maps and Cartesian sets. We then extend these conditions to stochastic quasi-variational inequality problems where in addition to a coercivity-like property, the set-valued mapping needs to satisfy continuity, apart from other "well-behavedness" properties to allow for concluding solvability. Finally, we extend the sufficiency conditions to accommodate multi-valued maps.

**(ii) Stochastic complementarity problems (SCPs):** Solvability of complementarity problems over cones requires a significantly different tack. In Section 3.4, we show that analogous verifiable integration-free statements can be provided for stochastic complementarity problems. Refinements of such statements are also provided in the context of co-coercive maps.

**(iii) Applications:** Naturally, the utility of any sufficiency conditions is based on its level of applicability. In Section 3.5, we describe two application problems. Of these, the first is a nonsmooth stochastic Nash-Cournot game which leads to an SQVI while the second is a stochastic equilibrium problem in power markets which can be recast as a stochastic complementarity problem. Importantly, both application settings are modeled with a relatively high level of fidelity.

- In Chapter 4, we consider a risk management problem involving CVaR and VaR risk measure and their nonlinear interactions.

  In the financial industry, risk has been traditionally managed by the imposition of value-at-risk or VaR constraints on portfolio risk exposure. Motivated by recent events in the financial industry, we examine the role that risk-seeking traders play in the accumulation of large and possibly infinite risk. In Chapter 4, we proceed to show that when traders employ a conditional value-at-risk (CVaR) metric, much can be said by studying the interaction between value at risk (VaR) (a non-coherent risk measure) and conditional value at risk CVaR (a coherent risk measure based on VaR). Resolving this question requires characterizing the optimal value of the associated stochastic, and possibly nonconvex, optimization problem, often a challenging problem. Our study makes two sets of contributions. First, in Section 4.3, we show that for general asset distributions on a compact support, traders accumulate finite risk with magnitude of the order of the upper bound of this support. Second, in Section 4.4, we show that when the supports are unbounded (such as Gaussian, exponential or fat-tailed distributions), under relatively mild assumptions, such traders can take on an unbounded amount of risk despite abiding by this VaR threshold. In short, VaR thresholds may be inadequate in guarding against financial ruin. Apart from contributions to mathematical research, this work also has practical applications to risk management as practiced today in the financial industry.

## 1.7 Organization

This remainder of this thesis is organized as follows. In Chapter 2, we examine the question regarding existence and uniqueness of equilibria for continuous strategy stochastic Nash games. Such games may be analyzed by examining the variational conditions, specified compactly as finite-dimensional variational inequalities. We consider both smooth and nonsmooth payoffs and provide sufficiency conditions for existence and uniqueness of equilibria. The utility of this approach is demonstrated by using this framework to analyze Nash-Cournot games in risk-averse and coupled constraint settings. In Chapter 3, we build on results developed in Chapter 2 in the context of stochastic Nash games and focus on the more general stochastic variational inequality problems and their generalizations. We provide sufficiency conditions for existence of solutions to a range of stochastic variational inequality problems and generalizations when the maps are complicated by the presence of expectations, multi-valuedness and the presence of stochastic constraints in the underlying sets. In Chapter 4, we examine the role of VaR(Value-at-risk) constraints in managing financial risk accumulated by risk-seeking traders. Resolving this question requires characterizing the optimal value of the associated stochastic, and possibly nonconvex, optimization problem, often a challenging problem. This risk can be either finite or unbounded depending on the asset distributions. In either instance, VaR thresholds are shown to be inadequate in guarding against financial ruin. In Chapter 5, we provide a summary this dissertation and provide directions for future work.

Please note that material from Chapters 2 and 4 have been published as journal articles [32] and [33] while material from Chapter 3 is currently under review [34].

# CHAPTER 2

# ON THE CHARACTERIZATION OF SOLUTION SETS OF SMOOTH AND NONSMOOTH CONVEX STOCHASTIC NASH GAMES

## 2.1 Introduction

The theory of games has its origin in the work by Von Neumann and Morgenstern [25] while the notion of the Nash equilibrium was introduced by Nash in 1950 [26]. While finite-strategy games form an important class of game-theoretic problems in their own right, in this chapter, we concentrate primarily on continuous strategy noncooperative Nash games where the problem data is uncertain and agents solve expected-value problems. Accordingly, the resulting class of games of interest are termed as stochastic Nash games. These games emerge in a host of applications ranging from electricity markets [3, 4, 35], traffic equilibrium problems [36] and telecommunication networks [5] where designers are interested in the equilibrium properties of imperfectly competitive systems.

Continuous-strategy Nash games may be analyzed through an examination of the variational conditions, specified compactly as finite-dimensional variational inequalities (cf. [37]). Through such an avenue, a wealth of knowledge may be gained regarding the structural properties of solution sets of games, allowing for proving whether the set of equilibria is nonempty, a singleton or even whether equilibria are locally unique. Most noncooperative systems arising in practice are characterized by uncertainty regarding problem data. For instance, in the design of power markets, both availability of power as well as the cost of generation is not known with complete certainty. A possible model for capturing strategic behavior is one where players solve stochastic optimization problems, differentiated from their deterministic counterparts by an expected-value objective. In many settings, the differentiability assumptions on the objectives cannot be expected to hold. For instance, standard Cournot models may require differentiability of price functions, a property that is clearly violated when price caps are imposed [35]. Other instances where such a loss of differentiability may arise is when players are risk-averse with nonsmooth risk measures [38, 39] or when they face congestion costs that are piecewise smooth [40]. In such regimes, the resulting variational inequalities have multivalued mappings and their stochastic generalizations are far more difficult to analyze. Finally, many application settings dictate a need to impose constraints that require satisfaction in an average or expected-value sense. An instance arises when firms compete in networked power markets where their bidding decisions may be constrained by a joint set of transmission constraints [4, 41]. Accordingly, this chapter is motivated by the need to characterize the solution sets of stochastic Nash games particularly when such games are characterized by expected-value objectives, nonsmoothness and stochastic constraints.

Before proceeding, we discuss where this work resides in the larger context of noncooperative games. The Nash solution concept in the context of noncooperative games was introduced in the 50s [26, 42] and presented the notion of one-shot games where all the players make decisions simultaneously, without the knowledge of the strategies of their competitors. A key assumption of the model proposed by Nash was that of *complete information:* the parameters of the noncooperative game are known to all the players. However, in practice, players may be unclear about their payoffs, leading to the notion of imperfect information. Harsanyi's seminal work [43] introduced the notion of a Bayesian Nash game and proved the existence of a Bayesian equilibrium. Importantly, the framework provided by Harsanyi requires the availability of a prior distribution on the parameters, not unlike the need for distributions when employing a stochastic programming framework for modeling optimization problems under uncertainty. An alternative approach lies in using a distribution-free framework that precludes the need for such a distribution; of note, is the seminal work by Aghassi and Bertsimas [44] in which a distribution-free robust optimization framework is developed for examining incomplete information games.

We consider a direction, inspired by stochastic programming models, that is more aligned with Harsanyi's framework in which a prior distribution on uncertain parameters is assumed to be available. Consequently, under suitability convexity assumptions, equilibrium conditions of games with expected-value payoffs are given by variational inequalities. Recall that when strategy sets of the agents are merely closed and convex (and not bounded), characterizing the solution set of a continuous strategy Nash game requires appropriate properties on the mapping constructed from the gradients of the player-specific payoff functions. A standard approach would focus on obtaining an analytical form of the expectation which would then be examined further to obtain properties of the underlying variational problem. However, this may not prove sensible for several reasons: First, when the payoffs contain expectations, characterization statements rely on having access to a tractable integral of a nonlinear function, which is often unavailable. Second, if indeed one does obtain a characterization statement, such a notion is restricted to the chosen distribution and has limited generalizability. Third, if the player objectives are nonsmooth, then such an analysis is even harder to carry out.

### 2.1.1 Review of literature

Before proceeding, we briefly review preceding work in the area of stochastic programming and games. The origins of stochastic programming can be traced to the work of Dantzig [45] and Beale [46]. This subfield of mathematical programming has now grown to include linear, nonlinear and integer programming models (cf. [47, 48]). Yet, our interest lies in game-theoretic generalizations of stochastic programs, a less studied class of problems. Stochastic equilibrium problems appear to have been first investigated by Haurie, Zaccour and Smeers [49, 50] where a two-period adapted open-loop model is presented.

The use of variational inequalities for capturing the Nash equilibrium has relevance when considering Nash games in which players are faced by continuous convex optimization problems (cf. [37, 51, 52] for an excellent survey of variational inequalities). Such approaches have allowed for deriving insights into strategic behavior in power markets [3, 4], wireless and wireline communication networks [5, 6], cognitive radio networks [7], amongst others. These efforts have ranged from developing precise statements of existence/uniqueness for particular models to developing distributed algorithms (see [37, 52, 53, 17]). In many settings, nonsmooth player objectives are an essential specification, arising from certain problem-specific intricacies, such as nonsmooth price functions [35] (such as through the imposition of price-caps) or through the use of nonsmooth risk measures [38, 39]. In this setting, Facchinei and Pang [53] provide a detailed discussion of how equilibria may be characterized. Multi-stage generalizations have been less used in the past with some exceptions; in particular, Shanbhag et al. [4] use a two-stage model to capture strategic behavior in two-settlement power markets while Mookherjee et al. [54] consider dynamic oligopolistic competition using differential variational inequalities [55].

In stochastic regimes however, the efforts have been largely restricted to algorithmic schemes, a subset of these being matrix-splitting schemes [4] and approximation techniques [56]. Monte-Carlo sampling approaches, often referred to as sample-average approximation (SAA) methods [10], have proved useful in the solution of stochastic optimization problems [57, 58, 59, 60]. Of note is the recent work by Xu and his coauthors on the solution of stochastic Nash games. More specifically, Xu and Zhang [61], in what appears to be amongst the first papers to examine smooth and nonsmooth stochastic Nash games, demonstrate the convergence properties of SAA estimators to their true counterparts. A broader overview of such techniques in the context of stochastic generalized equations is provided by Xu [62]. More recently, Zhang et al. have examined related problems in the context of power markets [63]. However, variational approaches have been less useful in finite-strategy games where simulation-based schemes have assumed relevance (cf [64, 65]).

Yet, far less is known about characterization of solutions when the uncertainty is defined by a continuous distribution. In particular, when players solve expected-value problems with general probability measures, little is known about whether equilibria exist and are unique, particularly when strategy sets are unbounded. On the basis of precisely such a shortcoming, we develop a framework for claiming existence of equilibria in both smooth and nonsmooth regimes.

### 2.1.2 Contributions

**(1) Smooth stochastic Nash games** Our first set of results, provided in Section 2.3, is associated with stochastic Nash games where the random player objectives are differentiable. Specifically, by leveraging Lebesgue convergence theorems, we develop conditions under which the satisfaction of a coercivity condition associated with a scenario-based Nash game in an almost-sure sense allows us to claim that the stochastic Nash game admits an equilibrium. The associated coercivity requirements can be further weakened when the mappings are monotone or the strategy sets are uncoupled. In a similar vein, we show that if the gradient maps associated with a scenario-based Nash game is strongly monotone over a set of positive measure, then the stochastic Nash game admits a unique equilibrium.

**(2) Nonsmooth stochastic Nash games** When player payoffs are merely continuous, the variational conditions of scenario-based Nash games are given by multivalued variational inequalities. However, by utilizing a set-valued analog of Fatou's Lemma, in Section 2.4, we show that the existence relationship of (1.) may be recovered for general and monotone gradient maps.

**(3) Stochastic Nash games with coupled stochastic constraints** Often a stochastic Nash game may be characterized by coupled stochastic constraints. We examine whether the equilibrium in primal variables and Lagrange multipliers (referred to as an equilibrium in the primal-dual space) may be characterized using the techniques from (1). Interestingly, we develop conditions in Section 2.5 for claiming the existence and uniqueness of the stochastic Nash game in the primal-dual space. Note that, in general, even when a mapping is strongly monotone in the primal space, the mapping in the full space is at best monotone; consequently uniqueness in the whole space is by no means immediate.

**(4) Examples** In Section 2.2, we motivate the questions of interest by using a class of stochastic Nash-Cournot games where both nonsmoothness (a consequence of employing nonsmooth risk metrics) and shared stochastic constraints are introduced. We return to these examples in Section 2.6 where existence statements are provided.

We end the introduction with a roadmap to the rest of the chapter. Motivating examples and some background to variational inequalities and nonsmooth analysis are provided in Section 2.2. Smooth stochastic Nash games are considered in Section 2.3 while nonsmooth generalizations are examined in Section 2.4. Extensions to the regime with stochastic constraints are discussed in Section 2.5. Our framework is applied to a class of risk-neutral and risk-averse stochastic Nash-Cournot games in Section 2.6.

## 2.2 Motivating examples and background

### 2.2.1 Motivating examples

A motivating problem for pursuing our research agenda is the stochastic Nash-Cournot game. In a Cournot model, profit-maximizing agents compete in quantity levels while faced with a price function associated with aggregate output. Deterministic Nash-Cournot games have been studied extensively [3, 35, 66, 67] and have found applications in electricity markets [3, 35]. However, stochastic generalizations of such games have not been as well studied.

We motivate our line of questioning from three examples each of which introduces a complexity that is subsequently addressed. In section 2.6, we return to these examples with the intent of characterizing the solutions sets of equilibria arising in such settings. We begin with a stochastic generalization of a Nash-Cournot game. In such games, if at least one of the agents' production is uncapacitated then existence is not immediately available from standard fixed-point arguments. Furthermore, the firms may be risk-neutral, risk-averse or may have to contend with expected-value constraints. This leads to three classes of stochastic Nash-Cournot games that our research addresses.

Consider $N$ producers involved in production of a commodity. The quantity produced by firm $i$ is denoted by $x_i$, with the column vector $x = (x_i)_{i=1}^N$. Let $c_i(x_i;\omega)$ denote the random cost incurred by firm $i$ in production of the commodity. Let $p = p(x;\omega)$ denote the random price function associated with the good. In a stochastic Nash-Cournot game, the players are profit-maximizing and the expected loss for firm $i$ can be written as $\mathbb{E}[f_i(x;\omega)]$ where $f_i(x;\omega) \triangleq (c_i(x_i;\omega) - p(x;\omega)x_i)$. An expectation-based framework is inherently risk-neutral in that it does not impose higher cost on shortfall. This is the basis of the risk-neutral game which we define next. Through this chapter, we consider a probability space $\mathcal{P} \triangleq (\Omega, \mathcal{F}, \mathbb{P})$ and $\xi : \Omega \to \mathbb{R}^d$ a random vector. With a slight abuse of notation, we will use $\omega$ to denote

$\xi(\omega)$.

**Example :** (Risk-neutral stochastic Nash-Cournot game) Consider an $N$-player stochastic Nash-Cournot game, in which the $i$th player decision variable $x_i$ is constrained to lie in a closed convex set $K_i \subseteq \mathbb{R}^+$ that specifies production constraints and the $i$th player objective function is given by $\mathbb{E}[f_i(x_i; x_{-i}, \omega)]$. Therefore, for $i = 1, \ldots, N$, the $i$th agent solves the convex optimization problem

$$\boxed{\underset{x_i \in K_i}{\text{minimize}} \ \mathbb{E}\left[f_i(x; \omega)\right],}$$

where $f_i(x; \omega) = c_i(x_i; \omega) - p(x; \omega)x_i$. □

The risk-neutrality assumption can be relaxed to allow for risk-averse firms. While a utility-based approach can be used for capturing risk-preferences, we extend the risk-neutral stochastic Nash-Cournot framework to accommodate a conditional value-at-risk measure that captures the risk of lower profits.

**Example :** (Risk-averse stochastic Nash-Cournot game) Consider an $N-$player stochastic Nash-Cournot game, akin to that described in Example 1, except that

$$f_i(z; \omega) \triangleq r_i(x; \omega) + \kappa_i \rho_i(x, m_i; \omega),$$

where $\kappa_i \in [0, 1]$ is the player-specific risk-aversion parameter, $z_i = (x_i, m_i)$ and

$$\rho_i(z; \omega) \triangleq \min_{m_i \in \mathbb{R}} \left( m_i + \frac{1}{1 - \tau_i} (c_i(x_i) - p(x; \omega)x_i - m_i)^+ \right).$$

Therefore, for $i = 1, \ldots, N$, the $i$th agent solves the convex optimization problem

$$\boxed{\underset{x_i \in K_i, m_i \in \mathbb{R}}{\text{minimize}} \ \mathbb{E}\left[f_i(z; \omega)\right].}$$

Note that $\mathbb{E}[\rho_i(z_i; z_{-i}, \omega)]$ is the conditional value at risk (CVaR) measure at level $\tau_i \in [0, 1)$ associated with player's expected loss and $f_i(z; \omega)$ is a sum of the expected loss of the firm $i$ and its risk exposure. □

In many regimes, a firm may be faced by stochastic constraints. Accordingly, we allow for expected-value constraints.

**Example :** (Stochastic Nash-Cournot game with expected value constraints) Consider an $N$-player stochastic Nash-Cournot game in which the

$i$-th agent solves the convex optimization problem

$$\boxed{\begin{array}{ll} \text{minimize} & \mathbb{E}\left[f_i(x;\omega)\right] \\ \text{subject to} & \begin{array}{r} \mathbb{E}\left[d_i(x;\omega)\right] \leq 0, \\ x_i \geq 0. \end{array} \end{array}}$$

where $\mathbb{E}[f_i(x;\omega)]$ may be based on either a risk-neutral or a risk-averse model and $\mathbb{E}[d_i(x;\omega)] \leq 0$ represents a set of constraints that need to be satisfied in an expected-value sense. ∎ □

In all of the above examples, if the probability space $\mathcal{P}$ is discrete, one can establish existence and uniqueness of the stochastic games by using standard results from the analysis of variational inequalities (cf. [53]). However, when the probability space $\mathcal{P}$ is continuous, such an extension is difficult to employ since an analytic form is generally unavailable for the expectation. In such situations, we examine whether we can develop an avenue for claiming existence/uniqueness of the stochastic Nash equilibrium that relies on the satisfaction of a suitable requirement in an almost-sure sense. As a consequence, the need to integrate the expectation is obviated. Furthermore, we investigate whether these conditions may be extended to allow for nonsmoothness in the integrands of the expectations.

### 2.2.2 Background

In this section we provide a quick recap of several assumptions and concepts used throughout the chapter.

#### Nash games and variational inequalities

Consider a game in which the $i$th player's decisions, denoted by $x_i$, are no longer constrained to be in a fixed set $K_i$ but are allowed to depend on the strategies of the other players, namely $x_{-i}$, as well. The resulting game is a generalization of the classical Nash game in that in addition to the interaction of players through their objective functions, it allows for interaction through coupled strategy sets. The resulting game, referred to as a *generalized Nash game*, has received significant recent interest [53, 68]. In this chapter, we restrict our research to a particular class of generalized Nash games known as *generalized Nash game with shared constraints* and employ assumption (A1) throughout this chapter, unless mentioned otherwise.

**Assumption 2.1 (A1)** *(a) $\mathcal{P} \triangleq (\Omega, \mathcal{F}, \mathbb{P})$ denotes a non-atomic probability space and $\xi : \Omega \to \mathbb{R}^d$ is a random vector.*

*(b) $\mathcal{N} = \{1, 2, \ldots, N\}$ denotes a set of players, $n_1, \ldots, n_N$ are positive integers and $n := \sum_{i=1}^N n_i$. For each $i \in \mathcal{N}$, the player-specific strategy set is denoted by $K_i \subseteq \mathbb{R}^{n_i}$ and the strategy tuple $(x_i)_{i=1}^N$ is required to be feasible with respect to a constraint that couples strategy sets, denoted by $\mathcal{C} \subseteq \mathbb{R}^n$ . Unless otherwise specified, these sets are closed,convex and have a nonempty interior.*

*(c) In a Nash game with shared strategy sets, given $x_{-i}$, the feasibility set of the ith player is denoted by the continuous convex-valued set-valued map $K_i(x_{-i})$, defined as*

$$K_i(x_{-i}) \triangleq K_i \cap \mathcal{C}_i(x_{-i}) \text{ where } \mathcal{C}_i(x_{-i}) \triangleq \{x_i \in \mathbb{R}^{n_i}(x_i, x_{-i}) \in \mathcal{C}\}. \tag{2.1}$$

*(d) For each $\omega \in \Omega$, the function $f_i(x; \omega) : \mathbb{R}^n \times \mathbb{R}^d \to \mathbb{R}$ (denotes the objective function of player i) and is convex, Lipschitz continuous and continuously differentiable in $x_i$ for each $x_{-i} \in \prod_{j \neq i} K_j(x_{-j})$.*

*(e) The map $\omega \to f_i(x; \omega)$ is measurable.*

*(f) For each $x_{-i} \in \prod_{j \neq i} K_j(x_{-j})$, the Lipschitz constant $\alpha(\omega, x_{-i})$ is integrable in $\omega$.*

These games require a set-valued map $\mathbf{K}(x)$ and a set $K$ defined as

$$\mathbf{K}(x) \triangleq \prod_{i=1}^N K_i(x_{-i}); \qquad K \triangleq \left( \prod_{i=1}^N K_i \right) \cap \mathcal{C}. \tag{2.2}$$

**Definition 2.1 (Generalized Nash game with shared constraints)** *Let (A1) hold and let the ith player decisions $x_i$ be constrained to lie in $K_i(x_{-i})$ and the ith player objectives be $f_i : \mathbb{R}^n \to \mathbb{R}$. Then the resulting game is a generalized Nash game with shared constraints and is denoted by*

$\mathcal{G}(\mathbf{K}, \mathbf{f})$ *where* $\mathbf{K}$ *is given by* (2.2) *and* $\mathbf{f} = (f_i)_{i=1}^N$. *A tuple* $x^* \in \mathbf{K}(x^*)$ *is a Nash equilibrium (NE) of the game* $\mathcal{G}(\mathbf{K}, \mathbf{f})$ *if for every* $i = 1, \ldots, N$,

$$f_i(x^*) \leq f_i(x_i, x_{-i}^*), \quad \forall x_i \in K_i(x_{-i}^*).$$

In other words, given $x_{-i}^*$, $x_i^*$ is the global optimizer of player $i$'s optimization problem

$$\min_{x_i \in K_i(x_{-i}^*)} f_i(x_i; x_{-i}^*).$$

Note that if $\mathcal{C} = \mathbb{R}^n$, then the resulting game is just the classical Nash game. The game $\mathcal{G}(\mathbf{K}, \mathbf{f})$ is said to be a *smooth* game if each player objective $f_i$ is continuously differentiable in $x_i$ and the game is said to be a *nonsmooth* game if at least one player's objective is nonsmooth. In this chapter, we are concerned with characterizing solution sets of the stochastic extension of the generalized Nash game with shared constraints $\mathcal{G}(\mathbf{K}, \mathbf{f})$. A singular exception to this arises when we consider general coupled constraints in Section 2.5.

Given a pair of set-valued maps $\mathbf{K}$ and $\mathbf{F}$, the *generalized quasi-variational inequality,* (GQVI) [53] denoted by GQVI($\mathbf{K}, \mathbf{F}$), is the problem of finding an $x \in \mathbf{K}(x)$ and $u \in \mathbf{F}(x)$ such that

$$(y - x)^T u \geq 0, \quad \forall y \in \mathbf{K}(x).$$

If $\mathbf{K}$ is set-valued and $\mathbf{F}$ is a single or point-valued mapping $F$, then the GQVI($\mathbf{K}, \mathbf{F}$) reduces to the *quasi-variational inequality* QVI($\mathbf{K}, F$). Finally, when $\mathbf{K}$ is a constant map and $\mathbf{F}$ is single-valued, then GQVI($\mathbf{K}, \mathbf{F}$) reduces to the *variational inequality* VI($K, F$).

GQVI($\mathbf{K}, \mathbf{F}$) represents the most general type of variational inequalities. The major area of applications of GQVI are mathematical and equilibrium programming. In [69], a number of the applications of GQVIs are discussed, including minimization problems involving invex functions, generalized dual problems and saddle point problems, and equilibrium problems involving abstract economies. Next, we relate Nash games to variational inequalities [53].

**Smooth Nash games and quasi-variational inequalities** A standard approach for the analysis of Nash games in which agents have smooth objectives and continuous strategy sets is through variational analysis. Under a smoothness assumption on $\mathbf{f}$, $x^* \triangleq (x_i^*)_{i=1}^N$ is a Nash equilibrium of the game

$\mathcal{G}(\mathbf{K}, \mathbf{f})$ if and only if $x^*$ is a solution of QVI($\mathbf{K}, F$) where $\mathbf{K}$ is the set-valued mapping given by (2.2) and $F = \left(\nabla_{x_1} f_1^T, \dots, \nabla_{x_N} f_N^T\right)$.

Also, from Proposition 12.4 [53], every solution of VI($K, F$) with $K$ given by (2.2) and $F = \left(\nabla_{x_1} f_1^T, \dots, \nabla_{x_N} f_N^T\right)$ is a termed as a *variational equilibrium* or a VE, a terminology that finds its origin in [68]. In other words, solutions of VI($K, F$) capture a subset of Nash equilibria of the smooth game $\mathcal{G}(\mathbf{K}, \mathbf{f})$. The game may have other equilibria that are not captured by this variational inequality. The stochastic extension of smooth Nash games is explored in Section 2.3. Throughout this chapter, when the game has shared constraints, we focus primarily on VEs to avoid the challenge of contending with quasi-variational inequalities.

**Nonsmooth Nash games and generalized QVIs** For a nonsmooth Nash game $\mathcal{G}(\mathbf{K}, \mathbf{f})$, $x^* \triangleq (x_i^*)_{i=1}^N$ is a Nash equilibrium if and only if $x^*$ is a solution to the generalized quasi-variational inequality GQVI($\mathbf{K}, \partial F$) where the multifunction $\mathbf{K}(x)$ is given by (2.2) and the set valued mapping $\partial F(x) = \prod_{i=1}^N \partial_{x_i} f_i(x)$. Recall $\partial F(x) = \prod_{i=1}^N \partial_{x_i} f_i(x)$ where $\partial_{x_i} f_i(x)$ is the set of vectors $w_i \in \mathbb{R}^{n_i}$ such that

$$f_i(\hat{x}_i; x_{-i}) - f_i(x_i; x_{-i}) \geq (\hat{x}_i - x_i)^T w_i, \quad \forall \hat{x}_i \in K_i(x_{-i}).$$

It should be emphasized that $\partial F(x)$ is *not* the Clarke generalized gradient of a function but is a set given by the Cartesian product of the Clarke generalized gradients of the player payoffs. Again, from Proposition 12.4 [53], every solution of GVI($K, \partial F$) with $K$ given by (2.2) and $\partial_x F = \prod_{i=1}^N \partial_{x_i} f_i(x_i; x_{-i})$ is a Nash equilibrium. In other words, solutions of GVI($K, \partial F$) capture Nash equilibria of the nonsmooth game $\mathcal{G}(\mathbf{K}, \mathbf{f})$, the game may have other equilibria that are not captured by this generalized variational inequality. A stochastic extension of nonsmooth Nash games is explored in Section 2.4.

### Nonsmooth analysis

For purposes of completeness, we recall definitions of several concepts used in examining nonsmooth games. While a variety of avenues exist for defining generalized gradients, we look to Rademacher's theorem that asserts that a locally Lipschitz function is differentiable almost everywhere in a Lebesgue sense. We use a form of the generalized gradient given in [70] as

**Definition 2.2 (Clarke Generalized gradient)**

$$\partial f(x) = co\left\{\lim_{k\to\infty} \nabla f(x_k) : x_k \to x, x_k \notin S, x_k \notin \Omega_f\right\},$$

*where $co(.)$ denotes the convex hull, $\Omega_f$ is a set of points in $B(x,\epsilon)$ (an open ball of radius $\epsilon$ around $x$) at which $f$ is not differentiable and $S$ is any other set of measure zero.*

In this chapter, we restrict our interest to player objectives that are *regular*, in the sense of Clarke [70]. Note that $\partial_{ij} f(x)$ denotes the $j$th component of the $\partial_{x_i} f(x)$. In defining a regular function, we define the generalized directional derivative of $f$ when evaluated at $x$ in a direction $p$ as

$$f^\circ(x;p) := \limsup_{y\to x, \lambda\downarrow 0} \frac{f(y+\lambda p) - f(y)}{\lambda}.$$

**Definition 2.3 (Regular function)** *A function $f$ is said to be regular at $x$ if for all $v$, the directional derivative $f'(x;v)$ exists and is given by $f'(x;v) = f^\circ(x;v)$.*

In fact, if $f$ is locally Lipschitz near $x$ and convex, then $f$ is regular at $x$ (see Prop 2.3.6 [70]).

Finally, we recall the notion of a monotone set valued mapping from [37].

**Definition 2.4 (Monotone set valued map)** *A set-valued map $\phi : K \to \mathbb{R}^n$ is said to be monotone on $K$ if $(x-y)^T(u-v) \geq 0$ for all $x$ and $y$ in $K$, and all $u$ in $\phi(x)$ and $v$ in $\phi(y)$.*

### Risk measures

Suppose $Y$ denotes the random losses where $\mathbb{P}_Y$ denotes its distribution function, i.e. $\mathbb{P}_Y(u) \triangleq \mathbb{P}\{Y \leq u\}$. Then the value at risk (VaR) at the $\alpha$ level specifies the maximum loss with a confidence level $\alpha$ and is defined as

$$\mathrm{VaR}_\alpha(Y) \triangleq \inf\{u : \mathbb{P}_Y(u) \geq \alpha\}.$$

Its conditional variant, referred to as the conditional value at risk (CVaR), is the expected loss conditioned on the event that the loss exceeds the VaR level [71, Th. 10]:

$$\mathrm{CVAR}_\alpha(Y) = \inf_{m\in\mathbb{R}} \left\{ m + \frac{1}{1-\alpha}\mathbb{E}\left[(Y-m)^+\right] \right\}, \tag{2.3}$$

where $(t)^+ = \max\{t, 0\}$. Note that in (2.3), the least of the optimal $m$ represents the VaR at the $\alpha$-confidence level.

## 2.3 Smooth stochastic Nash games

In this section, we analyze the stochastic extension of a smooth Nash game as described in Section 2.2.2 and begin by defining a canonical smooth stochastic Nash game corresponding to the probability space $\mathcal{P}$.

**Definition 2.5 (Stochastic Nash game)** *Let (A1) hold. A stochastic Nash game, denoted by $\mathcal{G}(\mathbf{K}, \mathbf{f}, \mathcal{P})$ is an $N$-player game in which the $i$th player is faced with the stochastic optimization problem $S_i(x_{-i})$, defined as*

| | | |
|---|---|---|
| $S_i(x_{-i})$ | minimize | $\mathbb{E}\left[f_i(x;\omega)\right]$ |
| | subject to | $x_i \in K_i(x_{-i})$, |

*where for $i = 1, \ldots, N$, the strategy set $K_i(x_{-i}) \subseteq \mathbb{R}^{n_i}$ is defined by (2.1) and the random objective function $f_i$ (as in (A1)) is smooth. Then, $(x_i^*)_{i=1}^N \in \mathbf{K}(x^*)$ given by (2.2) is said to be a stochastic Nash equilibrium for $\mathcal{G}(\mathbf{K}, \mathbf{f}, \mathcal{P})$ if, for $i = 1, \ldots, N$, $x_i^*$ solves the convex optimization problem $S_i(x_{-i}^*)$, i.e. given $x_{-i}^*$,*

$$\mathbb{E}\left[f_i(x^*;\omega)\right] \leq \mathbb{E}\left[f_i(x_i; x_{-i}^*, \omega)\right], \quad \forall x_i \in K_i(x_{-i}^*).$$

Note that in the notation $\mathcal{G}(\mathbf{K}, \mathbf{f}, \mathcal{P})$, $\mathbf{K}$ is given by (2.2) and $\mathbf{f} \triangleq (\mathbb{E}[f_i])_{i=1}^N$. Also note that by (A1), for each $x$, $\mathbf{K}(x)$ is a closed convex set and for each $\omega$ and $x_{-i}$, the function $f_i$ is continuously differentiable and convex in $x_i$.

**Definition 2.6 (Scenario-based Nash game)** *Consider a stochastic Nash game $\mathcal{G}(\mathbf{K}, \mathbf{f}, \mathcal{P})$. For a fixed $\omega \in \Omega$, the related* scenario-based *Nash game, denoted by $\mathcal{G}(\mathbf{K}, \mathbf{f}, \omega)$, is the game where the $i$th agent solves the deterministic* *game* $S_i(x_{-i};\omega)$ *given by*

$$\boxed{\begin{array}{lll} S_i(x_{-i};\omega) & \text{minimize} & f_i(x;\omega) \\ & \text{subject to} & x_i \in K_i(x_{-i}). \end{array}}$$

*Then* $(x_i^*)_{i=1}^N \in \mathbf{K}(x^*)$ *is said to be a scenario-based Nash equilibrium for* $\mathcal{G}(\mathbf{K},\mathbf{f},\omega)$ *if, for* $i = 1,\ldots,N$, $x_i^*$ *solves the convex optimization problem* $S_i(x_{-i}^*;\omega)$ *or given* $x_{-i}^*$ *and* $\omega$,

$$f_i(x^*;\omega) \le f_i(x_i;x_{-i}^*,\omega), \quad \forall x_i \in K_i(x_{-i}^*).$$

Note that in the notation $\mathcal{G}(\mathbf{K},\mathbf{f},\omega)$, $\mathbf{K}$ is given by (2.2) and $\mathbf{f} \triangleq (f_i(.;\omega))_{i=1}^N$. Our goal in this chapter, is to articulate properties associated with the scenario-based Nash games that allow us to claim existence of a stochastic Nash equilibrium. We do this by relating the games to corresponding variational inequalities and then using known results from the theory of variational inequalities to draw conclusions about the original stochastic Nash game. Our next result extended from [68] relates the equilibria of a stochastic Nash game and its scenario-based counterpart to the solutions of corresponding variational inequalities. With respect to $\mathcal{G}(\mathbf{K},\mathbf{f},\mathcal{P})$ and $\mathcal{G}(\mathbf{K},\mathbf{f},\omega)$, we define the functions $F$ and $F(.;\omega)$ as

$$F(x) \triangleq \begin{pmatrix} \nabla_{x_1}\mathbb{E}\left[f_1(x;\omega)\right] \\ \vdots \\ \nabla_{x_N}\mathbb{E}\left[f_N(x;\omega)\right] \end{pmatrix} \text{ and } F(x;\omega) \triangleq \begin{pmatrix} \nabla_{x_1} f_1(x;\omega) \\ \vdots \\ \nabla_{x_N} f_N(x;\omega) \end{pmatrix}, \text{ respectively.} \tag{2.4}$$

**Lemma 2.1 (Variational equilibrium (VE))** *Consider a stochastic Nash game given by* $\mathcal{G}(\mathbf{K},\mathbf{f},\mathcal{P})$ *and suppose (A1) holds. Then the following hold:*

*(a) If* $x^*$ *is a solution of VI(*$K,F$*) where* $K$ *and* $F$ *are given by* (2.2) *and* (2.4) *respectively then* $x^*$ *is an equilibrium of* $\mathcal{G}(\mathbf{K},\mathbf{f},\mathcal{P})$.

*(b) Similarly if* $x^*$ *is a solution of VI(*$K,F(.;\omega)$*) where* $K$ *and* $F(.;\omega)$ *are given by* (2.2) *and* (2.4) *respectively then* $x^*$ *is an equilibrium of the scenario-based game* $\mathcal{G}(\mathbf{K},\mathbf{f},\omega)$.

**Proof :** (a) Suppose $x^* \in K$ is a solution of $\mathrm{VI}(K, F)$ where $K$ and $F$ are defined by (2.2) and (2.4), respectively. Then, given $x^*_{-i}$, it suffices to show that $x^*_i$ is a minimizer of $\mathrm{S}_i(x^*_{-i})$. Also, $x^* \in K$ gives that $x^* \in \mathbf{K}(x^*)$, or that $x^*_i \in K_i \cap \mathcal{C}(x^*_{-i})$. Consider a vector $y_i \in K_i \cap \mathcal{C}_i(x^*_{-i})$ and let $\mathbf{y}$ be defined as $(y_j)_{j=1}^N$ where $y_j = x^*_j$ for all $j \neq i$. Therefore $y \in K$. It follows from the fact that $x^*$ is a solution of $\mathrm{VI}(K, F)$,

$$0 \leq F(x^*)^T(y - x^*) = \nabla_i \mathbb{E}[f_i(x^*_i; x^*_{-i}, \omega)]^T (y_i - x^*_i), \quad \forall y_i \in K_i \cap \mathcal{C}_i(x^*_{-i})$$

which allows us to claim that given $x^*_{-i}$ and the convexity of $K_i$ and $\mathcal{C}_i(x^*_{-i})$ for all $i$, the vector $x^*_i$ minimizes $\mathbb{E}[f_i(x_i; x^*_{-i}, \omega)]$ over $K_i \cap \mathcal{C}_i(x^*_{-i})$.(b) This follows through a similar proof using the definition of $F(.; \omega)$ from (2.4). ∎

Lemma 2.1 shows that a variational equilibrium provides an equilibrium to the game $\mathcal{G}(\mathbf{K}, \mathbf{f}, \mathcal{P})$. Notably, the associated variational inequality $\mathrm{VI}(\mathbf{K}, F)$ is a stochastic variational inequality (see [61, 62] for more details on stochastic variational inequalities) in that the mapping $F(x)$ has expectation-valued components. However, characterizing solution sets would require deriving properties of $F(x)$, a task that is significantly complicated by the presence of an expectation. Instead, our research is motivated by building an avenue for characterizing solution sets through the analysis of the scenario-based Nash game. As a consequence of Lemma 2.1, a solution of $\mathrm{VI}(K, F(.; \omega))$ provides an equilibrium to the scenario-based game $\mathcal{G}(\mathbf{K}, \mathbf{f}, \omega)$. Thus, characterizing solution sets requires deriving properties of $F(.; \omega)$, a deterministic mapping.

In Section 2.3.1, we develop sufficient conditions for the existence of a stochastic Nash equilibrium, in cases where $K$ is a general (possibly unbounded) as well as when $K$ can be represented as a Cartesian product of (possibly unbounded) strategy sets. We examine similar questions in section 2.3.2, when addressing the uniqueness questions. Note that throughout, our analysis does not rely on the knowledge of the probability distribution $\mathbb{P}$.

### 2.3.1 Existence of stochastic Nash equilibria

Our first set of results allows us to claim that under suitable conditions, the satisfaction of a suitable growth condition in an almost-sure sense allows us to claim that the original stochastic Nash game admits an equilibrium. This result relies on convergence theorems that allow for interchange of expectations and limits [72] and require the use of an additional assumption that is necessary for the application of Fatou's Lemma.

**Assumption 2.2 (A2)** *There exists an $x^{ref} \in K$ and for each $i$ there exists a nonnegative integrable function $u_i(x; x^{ref}, \omega)$ such that $\nabla_{x_i} f_i(x;\omega)^T(x_i - x_i^{ref}) \geq -u_i(x; x^{ref}, \omega)$.*

Before proving our main result, we provide a result necessary for carrying out the interchange between integration and differentiation as well as for allowing the use of Fatou's Lemma.

**Lemma 2.2 (Interchange of integration and differentiation)**

*(1) Under assumptions (A1(d,e,f)) and (A2), we have*

$$\nabla_{x_i} \mathbb{E}\left[f_i(x;\omega)\right] = \mathbb{E}\left[\nabla_{x_i} f_i(x;\omega)\right]; \tag{2.5}$$

*(2) Given a sequence $\{x^k\} \in K$, Fatou's Lemma can be applied for the sequence*

$$H_k(x_k; x^{ref}, \omega) = \nabla_{x_i} f_i(x_i^k; x_{-i}^k, \omega)^T (x_i^k - x_i^{ref}),$$

*leading to*

$$\mathbb{E}\left(\liminf_{k\to\infty} H_k(x^k, x^{ref}, \omega)\right) \leq \liminf_{k\to\infty} \mathbb{E}\left(H_k(x^k, x^{ref}, \omega)\right). \tag{2.6}$$

**Proof :** From (A1(e,f)); the hypotheses 2.7.1 [70] are satisfied. Further we have, $\mathbb{R}^n$ is separable. From (A1(d)), we also have for each $\omega, x_{-i}$, $f_i$ is convex and Lipschitz and therefore regular. Thus, from Th. 2.7.2 [70] we get that (1) holds.

We observe that (2) follows from (A2). ∎

**Proposition 2.3 (Existence of stochastic Nash equilibrium)** *Consider a stochastic Nash game $\mathcal{G}(\mathbf{K}, \mathbf{f}, \mathcal{P})$ and suppose (A1) and (A2) hold. If there exists an $x^{ref} \in K$ such that*

$$\liminf_{\|x\|\to\infty, x\in K} F(x;\omega)^T(x - x^{ref}) > 0 \qquad \textit{almost surely,}$$

*then the stochastic Nash game $\mathcal{G}(\mathbf{K}, \mathbf{f}, \mathcal{P})$ admits an equilibrium.*

**Proof :** From Lemma 2.1, an equilibrium of VI$(K, F)$ with $F$ given by (2.4) is an equilibrium of $\mathcal{G}(\mathbf{K}, \mathbf{f}, \mathcal{P})$. Recall from [37, Ch. 2] that the solvability of VI$(K, F)$ requires showing that there exists an $x^{\text{ref}}$ such that

$$\liminf_{\|x\|\to\infty, x\in K} F(x)^T(x - x^{\text{ref}}) > 0.$$

Through the form of $F$ in (2.4) we get,

$$F(x)^T(x - x^{\text{ref}}) = \sum_{i\in\mathcal{N}} (\nabla_{x_i}\mathbb{E}\left[f_i(x;\omega)\right])^T(x_i - x_i^{\text{ref}}).$$

By (A2) and from Lemma 2.2(1), we may interchange the order of integration and differentiation, obtaining

$$\begin{aligned} F(x)^T(x - x^{\text{ref}}) &= \sum_{i\in\mathcal{N}} \mathbb{E}\left[\nabla_{x_i} f_i(x;\omega)\right]^T(x_i - x_i^{\text{ref}}) = \mathbb{E}\left[\sum_{i\in\mathcal{N}} \nabla_{x_i} f_i(x;\omega)^T(x_i - x_i^{\text{ref}})\right] \\ &= \mathbb{E}[F(x;\omega)^T(x - x^{\text{ref}})]. \end{aligned}$$

Thus, we have

$$\liminf_{\|x\|\to\infty, x\in\mathbf{K}} F(x)^T(x - x^{\text{ref}}) = \liminf_{\|x\|\to\infty, x\in\mathbf{K}} \mathbb{E}[F(x;\omega)^T(x - x^{\text{ref}})].$$

Now, by (A2) and from Lemma 2.2, we may use Fatou's lemma on the right-hand side to get,

$$\liminf_{\|x\|\to\infty, x\in\mathbf{K}} F(x)^T(x - x^{\text{ref}}) \geq \mathbb{E}\left[\liminf_{\|x\|\to\infty, x\in\mathbf{K}} [F(x;\omega)^T(x - x^{\text{ref}})]\right] > 0,$$

where the last inequality follows from the given hypothesis. ∎

In settings where $K$ is a Cartesian product (for example when $\mathcal{C} = \mathbb{R}^n$), VI$(K, F)$ is a partitioned VI as defined in [37, Ch. 3.5]). Accordingly, Prop. 2.3 can be weakened so that even if the coercivity property holds for just one index $\nu \in \{1, \ldots, N\}$, an equilibrium to $\mathcal{G}(\mathbf{K}, \mathbf{f}, \mathcal{P})$ exists.

**Proposition 2.4 (Existence over Cartesian strategy sets)** *Consider a stochastic Nash game $\mathcal{G}(\mathbf{K}, \mathbf{f}, \mathcal{P})$ and suppose (A1) and (A2) hold. Furthermore, $\mathcal{C} = \mathbb{R}^n$. If there exists an $x^{ref} \in K$ such that for every $x \in K$, there* *exists a $\nu \in \{1, \ldots, N\}$ such that if the coercivity property, given by*

$$\liminf_{\|x_\nu\| \to \infty, x_\nu \in K_\nu} F_\nu(x, \omega)^T (x_\nu - x_\nu^{ref}) > 0,$$

*in an almost sure sense, then $\mathcal{G}(\mathbf{K}, \mathbf{f}, \mathcal{P})$ admits an equilibrium.*

**Proof :** For the given $x^{\text{ref}} \in K$ and for any $x \in K$, there exists a $\nu \in \{1, \ldots, N\}$, such that

$$\liminf_{\|x_\nu\| \to \infty, x_\nu \in K_\nu} F_\nu(x, \omega)^T (x_\nu - x_\nu^{\text{ref}}) > 0$$

holds almost surely. Thus we obtain

$$\mathbb{E}\left( \liminf_{\|x_\nu\| \to \infty, x_\nu \in K_\nu} F_\nu(x, \omega)^T (x_\nu - x_\nu^{\text{ref}}) \right) > 0$$

Applying Fatou's lemma we get

$$\liminf_{\|x_\nu\| \to \infty, x_\nu \in K_\nu} \mathbb{E}\left[ F_\nu(x, \omega)^T (x_\nu - x_\nu^{\text{ref}}) \right] > 0.$$

This implies that $C_{\leq}$ is bounded where

$$C_{\leq} := \left\{ x \in K : \max_{1 \leq \nu \leq N} \mathbb{E}\left[ F_\nu(x, \omega)^T (x_\nu - x_\nu^{\text{ref}}) \right] \leq 0 \right\}.$$

From [37, Prop. 3.5.1], boundedness of $C_{\leq}$ allows us to conclude that VI$(K, F)$ is solvable. Thus, $\mathcal{G}(\mathbf{K}, \mathbf{f}, \mathcal{P})$ admits an equilibrium. ∎

We now present a weaker set of sufficient conditions for existence under the assumption that the mapping $F(x; \omega)$ is a monotone mapping over $K$ for almost every $\omega \in \Omega$.

**Corollary 2.5 (Existence of an SNE under monotonicity)** *Consider a stochastic Nash game $\mathcal{G}(\mathbf{K}, \mathbf{f}, \mathcal{P})$ and suppose (A1) and (A2) hold. Suppose $F(x, \omega)$ is a continuous monotone mapping on $K$ for almost every $\omega \in \Omega$. Then $\mathcal{G}(\mathbf{K}, \mathbf{f}, \mathcal{P})$ admits an equilibrium if there exists an $x^{ref}$ such that*

$$\liminf_{\|x\| \to \infty, x \in K} F(x^{ref}; \omega)^T (x - x^{ref}) > 0$$

*holds in almost sure sense.*

**Proof :** We begin with the observation that the monotonicity of $F(x;\omega)$ allows us to bound $F(x;\omega)^T(x-x^{\text{ref}})$ from below as follows:

$$\begin{aligned} F(x;\omega)^T(x-x^{\text{ref}}) &= (F(x;\omega)-F(x^{\text{ref}};\omega))^T(x-x^{\text{ref}}) + F(x^{\text{ref}};\omega)^T(x-x^{\text{ref}}) \\ &\geq F(x^{\text{ref}};\omega)^T(x-x^{\text{ref}}). \end{aligned}$$

Taking expectations on both sides gives us

$$\mathbb{E}\left[F(x;\omega)^T(x-x^{\text{ref}})\right] \geq \mathbb{E}\left[F(x^{\text{ref}};\omega)^T(x-x^{\text{ref}})\right].$$

This implies that

$$\liminf_{\|x\|\to\infty, x\in K} \mathbb{E}\left[F(x;\omega)^T(x-x^{\text{ref}})\right] \geq \liminf_{\|x\|\to\infty, x\in K} \mathbb{E}\left[F(x^{\text{ref}};\omega)^T(x-x^{\text{ref}})\right]. \tag{2.7}$$

By (A2), Fatou's Lemma can be employed in the last inequality to interchange limits and expectations leading to

$$\liminf_{\|x\|\to\infty, x\in K} \mathbb{E}\left[F(x^{\text{ref}};\omega)^T(x-x^{\text{ref}})\right] \geq \mathbb{E}\left(\liminf_{\|x\|\to\infty, x\in K} F(x^{\text{ref}};\omega)^T(x-x^{\text{ref}})\right).$$

But by assumption, we have that

$$\liminf_{\|x\|\to\infty, x\in K} F(x^{\text{ref}};\omega)^T(x-x^{\text{ref}}) > 0$$

holds in almost sure sense, implying that from (2.7), we have that

$$\liminf_{\|x\|\to\infty, x\in K} \mathbb{E}\left[F(x;\omega)^T(x-x^{\text{ref}})\right] > 0,$$

allowing us to conclude the existence of a stochastic Nash equilibrium. ∎

*Remark:* It is important to note the subtle difference in the assumptions made on $x^{\text{ref}}$ in Prop. 2.3 and Prop. 2.5. In Prop. 2.3, we have made an assumption on the behavior on $F(x;\omega)^T(x-x^{\text{ref}})$ while in Prop. 2.5, the almost sure monotonicity of $F(x;\omega)$ allows to derive existence through an assumption on $F(x^{\text{ref}};\omega)^T(x-x^{\text{ref}})$.

### 2.3.2 Uniqueness of stochastic Nash equilibria

Next, we examine whether uniqueness statements may be available by examining scenario-based games. Recall that if $F(x;\omega)$ is a monotone mapping in an almost sure sense, then $\mathbb{E}[F(x;\omega)]$ is also monotone. However, mere monotonicity on the set does not yield any immediate results as far as existence or uniqueness of the original stochastic Nash game are concerned. However, if in addition to almost-sure monotonicity, the mapping $F(x;\omega)$ is assumed to be strongly monotone on a set of arbitrarily small but positive measure then the stochastic Nash game not only admits a solution but the solution is also unique. Before proceeding we define $\epsilon$-strongly and $\epsilon$-strictly monotone mappings

**Definition 2.7 ($\epsilon$-strongly (strictly) monotone mapping )**
*The mapping $F(x;\omega)$ of the scenario-based game is said to be an $\epsilon$-strongly (strictly) monotone mapping, if the following hold:*

*(i) It is monotone in an almost sure sense on $\Omega$;*

*(ii) Additionally, if there is a subset $U \subseteq \Omega$ with $\mathbb{P}(U) \geq \epsilon > 0$ such that $F(x;\omega)$ is strongly (strictly) monotone when $\omega \in U$.*

The next proposition shows that under $\epsilon$-strong monotonicity both existence and uniqueness of a stochastic Nash equilibrium can be guaranteed.

**Proposition 2.6 (Existence and uniqueness of SNE)**
*Consider the stochastic Nash game $\mathcal{G}(\mathbf{K}, \mathbf{f}, \mathcal{P})$ and suppose (A1) holds. We further assume the mapping $F(x,\omega)$ is an $\epsilon$-strongly monotone mapping. Then $\mathcal{G}(\mathbf{K}, \mathbf{f}, \mathcal{P})$ admits a unique equilibrium.*

**Proof :** Our result rests on showing that $F(x)$ is strongly monotone map under the specified assumptions. On the set $U \subseteq \Omega$, by strong monotonicity, there exists a constant $c > 0$ such that

$$(F(x,\omega) - F(y;\omega))^T (x-y) \geq c\|x-y\|^2, \forall x, y \in K, \quad \text{for almost every} \quad \omega \in U.$$

It follows that $\mathbb{E}[F(x;\omega) - F(y;\omega)^T (x - x^{\text{ref}})]$ can be bounded as

$$\int_U (F(x;\omega) - F(y;\omega))^T (x-y) d\mathbb{P} + \int_{\Omega \setminus U} (F(x;\omega) - F(y;\omega))^T (x-y) d\mathbb{P}$$
$$\geq \int_U c\|x-y\|^2 d\mathbb{P} = c\|x-y\|^2 \mathbb{P}(U).$$

Therefore, $F(x)$ is a strongly monotone map and $\mathcal{G}(\mathbf{K},\mathbf{f},\mathcal{P})$ admits a unique equilibrium [37, Th. 2.3.3]. Note that (ii) follows in an analogous fashion. ∎

This result suggests that if we have a scenario-based game characterized by an $\epsilon-$strongly monotone map, then the expected value game *does* have a unique equilibrium. Notably, the former implies that the scenario-based Nash game admits a unique equilibrium with positive probability. Next, we examine the consequences of a weaker requirement on the scenario-based mapping.

**Proposition 2.7 (Uniqueness of stochastic Nash equilibrium)** *Consider the stochastic Nash game $\mathcal{G}(\mathbf{K},\mathbf{f},\mathcal{P})$ and suppose (A1) holds. Furthermore, suppose $\mathcal{G}(\mathbf{K},\mathbf{f},\mathcal{P})$ admits a Nash equilibrium. If we further assume the mapping $F(x,\omega)$ is an $\epsilon$-strongly monotone mapping.Then the original stochastic Nash game $\mathcal{G}(\mathbf{K},\mathbf{f},\mathcal{P})$ admits a unique equilibrium.*

**Proof :** Our result rests on showing that $F(x)$ is strictly monotone map under the specified assumptions. This allows us to claim that at most one solution to VI$(K, F(x))$ exists. Together with an assumption of existence, uniqueness follows readily. On the set $U \subseteq \Omega$, by strict monotonicity, we have that

$$(F(x,\omega) - F(y;\omega))^T(x-y) > 0, \forall x,y \in K, \quad \text{for all} \quad \omega \in U.$$

It follows that $\mathbb{E}[(F(x;\omega) - F(y;\omega))^T(x - x^{\text{ref}})]$ can be expressed as

$$\int_U (F(x;\omega) - F(y;\omega))^T(x-y)d\mathbb{P} + \int_{\Omega\setminus U} (F(x;\omega) - F(y;\omega))^T(x-y)d\mathbb{P} > 0.$$

Therefore, $F(x)$ is a strictly monotone map. ∎

## 2.4 Nonsmooth stochastic Nash games

A crucial restriction in the discussion in the earlier section pertains to the differentiability of the functions $f_i \in \mathbf{f}$. This ensures that the gradients are single-valued as opposed to being multivalued. Yet, in many settings complicated by nonsmooth cost and price functions, such as those arising from risk measures or the imposition of price caps, the need to examine nonsmooth generalizations of stochastic Nash games remains paramount.

We begin by defining the multivalued variational inequality that represents the equilibrium conditions of a nonsmooth stochastic Nash game. Through this section, we employ a modified form of assumption (A1).

**Assumption 2.3 (A1$^\dagger$)** *Define (A1$^\dagger$) to be (A1) except instead of (A1(d)), we assume the following:*

*(A1d$^\dagger$) For each $\omega \in \Omega$, the functions $f_i(x;\omega) : \mathbb{R}^n \times \mathbb{R}^d \to \mathbb{R}$ are convex and Lipschitz continuous in $x_i$ for each $x_{-i} \in \prod_{j\neq i} K_j(x_{-j})$.*

In contrast with Section 2.3, in this section, we concentrate on player-specific functions $f(x_i; x_{-i}, \omega)$ that are not necessarily smooth but are convex and Lipschitz everywhere, implying that they are regular.

### 2.4.1 Existence of nonsmooth stochastic Nash equilibrium

In the same vein as before, our intent is to derive conditions under which the existence of an equilibrium to the nonsmooth stochastic Nash can be obtained from the almost-sure satisfaction of a coercivity result pertaining to the *scenario-based* nonsmooth Nash game. We begin by stating two existence results for nonsmooth Nash games, of which the former allows for convex shared constraints while the latter insists on Cartesian strategy sets that preclude coupling. Akin to the results on smooth stochastic Nash games, our efforts rely on the analysis of the associated stochastic variational inequalities with multi-valued mappings [61, 62].

**Proposition 2.8 (Nonsmooth Nash games)**
*Consider a Nash game $\mathcal{G}(\mathbf{K}, \mathbf{f})$.*

**(a) Nash games with shared constraints [53, Th. 12.3]** *Suppose $\mathcal{C} \subseteq \mathbb{R}^n$. If there exists an $x^{ref} \in K$ such that $L_<$ is bounded where*

$$L_< \triangleq \left\{ y \in K : \exists w \in \partial F(x) \text{ such that } (y - x^{ref})^T w < 0 \right\}, \tag{2.8}$$

*then the Nash game $\mathcal{G}(\mathbf{K}, \mathbf{f})$ admits an equilibrium.*

**(b) Nash games with Cartesian strategy sets [53, Cor. 12.1]** *Suppose $\mathcal{C} = \mathbb{R}^n$. If there exists an $x^{ref} \in K$ such that $L_<$ is bounded where*

$$L_< \triangleq \left\{ \begin{array}{l} (y_i)_{i=1}^N \in K : \textit{for every } i \textit{ such that } y_i \neq x_i^{ref}, \\ \qquad\qquad (y_i - x_i^{ref})^T w_i < 0 \textit{ for some } w_i \in \partial_{x_i} f_i(x) \end{array} \right\}, \quad (2.9)$$

*then the Nash game $\mathcal{G}(\mathbf{K}, \mathbf{f})$ admits an equilibrium.*

These conditions are implied by the following results, similar to [53, Cor. 12.1].

**Proposition 2.9** *Consider a Nash game $\mathcal{G}(\mathbf{K}, \mathbf{f})$.*

*(a) Suppose $\mathcal{C} = \mathbb{R}^n$. If there exists an $x^{ref} \in K$ such that*

$$\liminf_{x_k \in K, \|x_k\| \to \infty, k \to \infty} \left( \min_{w \in \partial F(x_k)} (x_k - x^{ref})^T w \right) > 0, \quad (2.10)$$

*then the Nash game $\mathcal{G}(\mathbf{K}, \mathbf{f})$ admits an equilibrium.*

*(b) Suppose $\mathcal{C} \subseteq \mathbb{R}^n$. If there exists an $x^{ref} \in K$ such that*

$$\liminf_{x_k \in K, \|x_k\| \to \infty, k \to \infty} \max_{i \in \{1,\ldots,N\}} \left( \min_{w_i \in \partial_{x_{k,i}} f_i(x_k)} (x_{k,i} - x_i^{ref})^T w_i \right) > 0, \quad (2.11)$$

*then the Nash game $\mathcal{G}(\mathbf{K}, \mathbf{f})$ admits an equilibrium.*

**Proof :**

**(a)** We show that if (2.10) holds then the set $L_<$, defined in (2.8), is bounded. If $L_<$ is empty, it is trivially bounded. We proceed by a contradiction and assume that $L_<$ is nonempty and unbounded. Then there exists a sequence $\{x_k\}$ of elements belonging to $L_<$ such that $\|x_k\|$ goes to $\infty$. But from definition of $L_<$ we have that for each $k$, there exists a $w_k \in \partial F(x_k)$ such that $(x_k - x^{\rm ref})^T w_k < 0$. This clearly implies that for each $k$, $\min_{w \in \partial F(x_k)} (x_k - x^{\rm ref})^T w < 0$. This further implies for the sequence $\{x_k\}$ we have that

$$\liminf_{x_k \in K, \|x_k\| \to \infty, k \to \infty} \min_{w \in \partial F(x_k)} (x_k - x^{\rm ref})^T w \leq 0.$$

But this contradicts the assumption (2.10). This contradiction implies that $L_<$ must be bounded. Thus by Prop. 2.9, we conclude that the game $\mathcal{G}(\mathbf{K}, \mathbf{f})$ admits an equilibrium.

**(b)** Omitted. ∎

In developing a relationship between the solvability of a *scenario-based nonsmooth* Nash game and its expected-value counterpart, we first need to ensure that an interchange between the expectation and differentiation operator remains valid.

**Lemma 2.10** *Suppose (A1†) holds. Then the sets* $\partial_{ij}\mathbb{E}[f_i(x_i; x_{-i}, \omega)]$ *and* $\mathbb{E}[\partial_{ij} f_i(x_i; x_{-i}, \omega)]$ *are identical; specifically*

$$\partial_{ij}\mathbb{E}\left[f_i(x_i; x_{-i}, \omega)\right] = \mathbb{E}\left[\partial_{ij} f_i(x_i; x_{-i}, \omega)\right].$$

**Proof :** The proof follows by noting that $K \subseteq \mathbb{R}^n$, a separable metric space, and by invoking Th.2.7.2 [70] . ∎

One can therefore conclude that if $w \in \partial F(x)$ implies that for all $j = 1, \ldots, n_i$ and $i = 1, \ldots, N$, $w_{ij} \in \partial_{ij}\mathbb{E}[f_i(x_i; x_{-i}, \omega)]$. It follows that $w_{ij}$ lies in $\mathbb{E}[\partial_{ij} f_i(x_i; x_{-i}, \omega)]$ or $w_{ij} \in \mathbb{E}[\partial_{ij} f_i(x_i; x_{-i}, \omega)]$. To obtain a precise relationship between the solvability of a scenario-based nonsmooth Nash game and its stochastic counterpart, we need to be able to express $w$ in terms of the *scenario-based* generalized gradients. To facilitate this, we analyze the set-valued map $\mathbb{E}[\partial_{ij} f_i(x_i; x_{-i}, \omega)]$ further. This analysis requires some definitions:

**Definition 2.8 (Def. 8.1.2 [31])** *Let* $(\Omega, \mathcal{F}, \mathbb{P})$ *be a probability space. Consider a set-valued map* $H : \Omega \to \mathbb{R}^n$. *A measurable map* $h : \Omega \to \mathbb{R}^n$ *satisfying* $\forall \omega \in \Omega, h(\omega) \in H(\omega)$, *is called a measurable selection of* $H$.

As a consequence, for all $\omega \in \Omega, j = 1, \ldots, n_i$ and $i = 1, \ldots, N$, the measurable map $w_{ij}(x; \omega)$ satisfying

$$w_{ij}(x; \omega) \in \partial_{ij} f_i(x; \omega)$$

is a measurable selection of $\partial_{ij} f_i(x; \omega)$. In fact, Aumann [73] defined the integral of a set-valued map using the set of all integrable selections; in particular if the set of all integrable selections of $H(\omega)$ is denoted by $\mathcal{H}$ and given by

$$\mathcal{H} \triangleq \left\{ f \in L^1(\Omega, \mathcal{F}, \mathbb{P}) : h(\omega) \in H(\omega) \text{ for almost all } \omega \in \Omega \right\},$$

then the expectation of $H(\omega)$ is given by

$$\int_\Omega H(\omega) d\mathbb{P} := \left\{ \int_\Omega h(\omega) d\mathbb{P} \mid h(\omega) \in \mathcal{H} \right\}.$$

If the images of $H(\omega)$ are convex then this set-valued integral is convex [31, Definition 8.6.1]. In this chapter, we are concerned with the integral of Clarke generalized gradients. Since the Clarke generalized gradient map has a convex image, the convexity of the set-valued integral in this case is immediate. Note that when the assumption of convexity of images of $H$ does not hold, then the convexity of this integral follows from Th. 8.6.3 [31] provided that the probability measure is non-atomic.

Given the convexity of the set, we may define an *extremal selection.* A point $\bar{z}$ of a convex set $K$ is said to be extremal if there are no two points $x, y \in K$ such that $\lambda x + (1 - \lambda) y = \bar{z}$ for $\lambda \in (0, 1)$ and is denoted by $\bar{z} \in \text{ext}(K)$. Similarly, as per Def. 8.6.5 [31], we say $h \in \mathcal{H}$ is an extremal selection of $H$ if

$$\int_\Omega h(\omega) d\mathbb{P} \text{ is an extremal point of } \int_\Omega H(\omega) d\mathbb{P}.$$

Then the set $\mathcal{H}_e$ is the set of extremal selections and is defined as

$$\mathcal{H}_e \triangleq \left\{ h \in \mathcal{H} \mid \int_\Omega h(\omega) d\mathbb{P} \in \text{ext} \left( \int_\Omega H(\omega) d\mathbb{P} \right) \right\}.$$

Before proceeding to prove our main existence result of this section, we make an assumption pertaining to $w_{ij}(x; x^{\text{ref}}, \omega)$ which is defined as

$$w_{ij}(x; x^{\text{ref}}, \omega) \triangleq ([\nabla f_i(x)]_j ([x_i]_j - [x_i^{\text{ref}}]_j)),$$

for any $x$ and $x^{\text{ref}}$. 4

**Assumption 2.4 (A2$^\dagger$)** *Assume that $w_{ij}(x; x^{ref}, \omega)$ is a sequence of extended real-valued measurable functions on $\mathcal{P}$ for all $j \in \{1, \dots, n_i\}$ and for all $i \in$*

$\mathcal{N}$. *Furthermore, there exists a nonnegative integrable function* $\bar{u}(x; x^{ref}, \omega)$ *such that* $w_{ij}(x; x^{ref}, \omega) \geq -\bar{u}(x; x^{ref}, \omega)$ *for all* $x, i$ *and* $j$.

**Theorem 2.11 (Existence of nonsmooth stochastic Nash equilibrium)** *Consider a stochastic Nash game* $\mathcal{G}(\mathbf{K}, \mathbf{f}, \mathcal{P})$ *and suppose (A1†) and (A2†) hold. If there exists an* $x^{ref} \in K$ *such that*

$$\liminf_{k\to\infty, x_k\in K, \|x_k\|\to\infty} \left( \min_{w\in\partial F(x_k;\omega)} w^T(x_k - x^{ref}) \right) > 0 \text{ almost surely,}$$

*then* $\mathcal{G}(\mathbf{K}, \mathbf{f}, \mathcal{P})$ *admits an equilibrium.*

**Proof :** Recall that the solvability of the stochastic VI$(K, \partial F)$ requires showing that there exists an $x^{\text{ref}}$ such that

$$\liminf_{k\to\infty, x_k\in K, \|x_k\|\to\infty} \left( \min_{w\in\partial F(x_k)} w^T(x_k - x^{\text{ref}}) \right) > 0.$$

We proceed by contradiction and assume that for any $x^{\text{ref}} \in K$, we have that

$$\liminf_{k\to\infty, x_k\in K, \|x_k\|\to\infty} \left( \min_{w\in\partial F(x_k)} w^T(x_k - x^{\text{ref}}) \right) \leq 0. \tag{2.12}$$

For a given $x_k$, suppose we denote the selection that minimizes $w^T(x_k - x^{\text{ref}})$ by $w_k$. By Prop. 7.1.4 [37], $\partial F(x_k)$ is a closed and convex set, implying that $w_k \in \partial F(x_k)$. Therefore, the following holds:

$$\liminf_{k\to\infty} \; w_k^T(x_k - x^{\text{ref}}) \leq 0. \tag{2.13}$$

But $w_k^T(x_k - x^{\text{ref}})$ can be expressed as $w_k^T(x_k - x^{\text{ref}}) = \sum_{i=1}^N w_{k,i}^T(x_{k,i} - x_i^{\text{ref}})$, where

$$w_{k,i} \in \partial_{x_{k,i}} \mathbb{E}\left[f_i(x_k;\omega)\right] = \mathbb{E}\left[\partial_{x_{k,i}} f_i(x_k;\omega)\right]. \tag{2.14}$$

Note that the right-hand side of (2.14) is an integral of a set-valued map $\partial f_i(x_k;\omega)$ with range $\mathbb{R}^{n_i}$ and is a convex set. Thus by Carathéodory's theorem for convex sets, there exist $\lambda_{i,l}(x_k) \geq 0$ and $y_{i,l}(x_k) \in \text{ext}\left(\int_\Omega \partial_{x_i} f_i(x_k;\omega) d\mathbb{P}\right)$ such that

$$w_{k,i}(x) = \sum_{l=0}^{n_i} \lambda_{i,l}(x_k) y_{i,l}(x), \quad \sum_{l=0}^{n_i} \lambda_{i,l}(x_k) = 1.$$

By Th. 8.6.3 [31], for each $l$, there exists an extremal selection $g_{i,l}(x_k;\omega)$ of $\partial_{x_i} f_i(x_k;\omega)$ such that $y_{i,l}(x_k) = \int_\Omega g_{i,l}(x_k;\omega)d\mathbb{P}$. Consequently, $w_{k,i}$ may be expressed as

$$w_{k,i} = \sum_{l=0}^{n_i} \lambda_{i,l}(x_k) \int_\Omega g_{i,l}(x_k;\omega)d\mathbb{P}, \quad \sum_{l=0}^{n_i} \lambda_{i,l}(x_k) = 1. \tag{2.15}$$

Since $g_{i,l}(x_k;\omega)$ is a selection from $\partial_{x_i} f_i(x_k;\omega)$ for each $l$, we have that for any $\omega \in \Omega$, $g_{i,l}(x_k;\omega) \in \partial_{x_i} f_i(x_k;\omega)$. From the convexity of the set $\partial f_i(x_k;\omega)$, we get that for each $\omega$, the convex combination $\sum_{l=0}^{n_i} \lambda_{i,l}(x) g_{i,l}(x_k;\omega) \in \partial_{x_i} f_i(x_k;\omega)$. Therefore, we have that $\sum_{l=0}^{n_i} \lambda_{i,l}(x_k) g_{i,l}(x_k;\omega)$ is also an integrable selection of $\partial_{x_i} f_i(x_k;\omega)$ which we denote by $g_i(x_k;\omega)$ or

$$g_i(x_k;\omega) \triangleq \sum_{l=0}^{n_i} \lambda_{i,l}(x_k) g_{i,l}(x_k;\omega) \in \partial_{x_i} f_i(x_k;\omega).$$

From (2.15) and by interchanging the order of integration and summation, the following holds

$$\begin{aligned} w_{k,i} = \sum_{l=0}^{n_i} \lambda_{i,l}(x_k) \int_\Omega g_{i,l}(x_k;\omega)d\mathbb{P} &= \int_\Omega \sum_{l=0}^{n_i} \lambda_{i,l}(x_k) g_{i,l}(x_k;\omega)d\mathbb{P} \\ &= \int_\Omega g_i(x_k;\omega)d\mathbb{P}. \end{aligned}$$

Substituting $w_{k,i} = \int_\Omega g_i(x_k;\omega)d\mathbb{P}$ into $w_k^T(x_k - x^{\text{ref}})$, we obtain

$$w_k^T(x_k - x^{\text{ref}}) = \int_\Omega \sum_{i=1}^{N} g_i(x_k;\omega)^T (x_{k,i} - x_{k,i}^{\text{ref}})d\mathbb{P}.$$

Letting $g(x_k;\omega) = (g_i(x_k;\omega))_{i=1}^N \in \partial F(x_k;\omega)$, this expression may be rewritten as

$$w_k^T(x_k - x^{\text{ref}}) = \int_\Omega g(x_k;\omega)^T (x_k - x^{\text{ref}})d\mathbb{P}.$$

By substituting the expression on the right into (2.13), we obtain that

$$\begin{aligned}
0 &\geq \liminf_{k\to\infty, x_k\in K, \|x_k\|\to\infty} \left(w_k^T(x_k - x^{\text{ref}})\right) && (2.16)\\
&= \liminf_{k\to\infty, x_k\in K, \|x_k\|\to\infty} \left(\int_\Omega g(x_k;\omega)^T(x_k - x^{\text{ref}})d\mathbb{P}\right) && (2.17)\\
&\geq \int_\Omega \left(\liminf_{k\to\infty, x_k\in K, \|x_k\|\to\infty} g(x_k;\omega)^T(x_k - x^{\text{ref}})d\mathbb{P}\right), && (2.18)
\end{aligned}$$

where the second inequality is a consequence of employing Fatou's Lemma. It follows that there exists a set $U \subseteq \Omega$ with $\mathbb{P}(U) > 0$ such that for $\omega \in U \subseteq \Omega$, we have that

$$\liminf_k g(x_k;\omega)^T(x_k - x^{\text{ref}}) \leq 0. \tag{2.19}$$

But this implies that for $\omega \in U$, a set of positive measure,

$$\liminf_{k\to\infty, x_k\in K, \|x_k\|\to\infty} \min_{w\in\partial F(x_k;\omega)} w^T(x_k - x^{\text{ref}}) \leq 0, \tag{2.20}$$

which further implies that

$$\liminf_{k\to\infty, x_k\in K, \|x_k\|\to\infty} \min_{w\in\partial F(x;\omega)} w^T(x - x^{\text{ref}}) \leq 0. \tag{2.21}$$

Since this holds for all $x^{\text{ref}} \in K$, it contradicts the hypothesis that there exists an $x^{\text{ref}} \in K$ such that

$$\liminf_{\|x\|\to\infty, x_k\in K} \min_{w\in\partial F(x;\omega)} w^T(x - x^{\text{ref}}) > 0 \text{ almost surely.}$$

Consequently, the nonsmooth stochastic Nash game admits an equilibrium. ■

Under an assumption of Cartesian strategy sets, a corollary of the previous result is now provided without a proof.

**Corollary 2.12 (Existence under Cartesian strategy sets)** *Consider a stochastic Nash game $\mathcal{G}(\mathbf{K}, \mathbf{f}, \mathcal{P})$ and suppose (A1$^\dagger$) and (A2$^\dagger$) hold. Furthermore, suppose $\mathcal{C} = \mathbb{R}^n$. If there exists an $x^{ref} \in K$ such that*

$$\liminf_{k\to\infty, x_k\in K, \|x_k\|\to\infty} \max_{i\in\{1,\ldots,N\}} \left(\min_{w_i\in\partial_{x_{k,i}} f_i(x_k;\omega)} w_i^T(x_{k,i} - x_i^{ref})\right) > 0 \ a.s.,$$

*then* $\mathcal{G}(\mathbf{K}, \mathbf{f}, \mathcal{P})$ *admits an equilibrium.*

Having proved the existence of a nonsmooth stochastic Nash equilibrium, we investigate if the sufficiency conditions can be weakened if the mapping $\partial F(x, \omega)$ is a monotone set-valued mapping in an almost-sure sense.

**Proposition 2.13 (Existence of a SNE under monotonicity)** *Consider a nonsmooth stochastic Nash game* $\mathcal{G}(\mathbf{K}, \mathbf{f}, \mathcal{P})$ *and suppose (A1$^\dagger$) and (A2$^\dagger$) hold. If there exists an* $x^{ref}$ *such that*

$$\liminf_{x_k \in K, \|x_k\| \to \infty, k \to \infty} \left( \min_{w \in \partial F(x^{ref}; \omega)} w^T (x_k - x^{ref}) \right) > 0 \text{ almost surely,}$$

*then* $\mathcal{G}(\mathbf{K}, \mathbf{f}, \mathcal{P})$ *admits an equilibrium.*

**Proof :** We proceed along the same avenue as in Th. 2.11. We begin by noting that the solvability of stochastic VI$(K, \partial F)$ requires showing that

$$\liminf_{x_k \in K, \|x_k\| \to \infty, k \to \infty} \left( \min_{w \in \partial F(x_k)} w^T (x_k - x^{\text{ref}}) \right) > 0.$$

Proceeding by contradiction, this requires that for all $x^{\text{ref}} \in K$, we have that

$$\liminf_{x_k \in K, \|x_k\| \to \infty, k \to \infty} \left( \min_{w \in \partial F(x^{\text{ref}}; \omega)} w^T (x_k - x^{\text{ref}}) \right) \leq 0. \tag{2.22}$$

By the closedness of $\partial F(x_k)$, we have that $w_k$, a minimizer of $w^T(x_k - x^{\text{ref}})$, lies in $\partial F(x_k)$. Therefore, we have the following:

$$\liminf_{x_k \in K, \|x_k\| \to \infty, k \to \infty} \left( w_k^T (x_k - x^{\text{ref}}) \right) \leq 0. \tag{2.23}$$

By adding and subtracting $w^{\text{ref}}$, a selection from $\partial F(x^{\text{ref}})$, we have that

$$0 \geq \liminf_{x_k \in K, \|x_k\| \to \infty, k \to \infty} \left( (w_k - w^{\text{ref}})^T (x_k - x^{\text{ref}}) + (w^{\text{ref}})^T (x_k - x^{\text{ref}}) \right) \tag{2.24}$$

$$\geq \lim_{x_k \in K, \|x_k\| \to \infty, k \to \infty} \left( (w^{\text{ref}})^T (x_k - x^{\text{ref}}) \right), \tag{2.25}$$

the second inequality being a consequence of the monotonicity of $\partial F$. By proceeding in the same fashion as in Th. 2.11, we can express $(w^{\text{ref}})^T(x_k - x^{\text{ref}})$ as an integral $\int_\Omega g(x^{\text{ref}}; \omega)^T (x_k - x^{\text{ref}}) d\mathbb{P}$. The use of Fatou's lemma then allows us to conclude as before that there exists a set $U \subseteq \Omega$ of positive measure on which the hypothesis

$$\liminf_{\|x\| \to \infty, x \in K} \min_{w \in \partial F(x^{\mathrm{ref}};\omega)} w^T(x - x^{\mathrm{ref}}) > 0 \text{ almost surely.}$$

is contradicted. Consequently, the nonsmooth monotone stochastic Nash game admits an equilibrium. ∎

## 2.5 Stochastic Nash games with shared stochastic constraints

Our general approach in the two preceding sections has been a largely primal one in that the equilibrium conditions in the primal space are analyzed. Often, there may be an interest in equilibria in the primal-dual space, allowing us to make statements about strategies as well as the associated Lagrange multipliers (prices) corresponding to the constraints. In this section, we assume that the shared constraint game of the form specified by (A1), where the set $\mathcal{C}$ is given by

$$\mathcal{C} \triangleq \{x : \mathbb{E}\left[c(x;\omega)\right] \geq 0\}, \tag{2.26}$$

and refer to it as $\mathcal{G}^E$. Our interest lies in the variational equilibrium of $\mathcal{G}^E$. Recall that that the variational equilibrium (VE) in the primal-dual space, under a suitable regularity condition, or constraint qualification (cf. [37]), is given by the solution to a CP$(\mathbb{R}^{m+n}_+, H)$ defined as[1]

$$\boxed{0 \leq z \perp H(z) \geq 0,}$$

[1]Recall that given a closed and convex cone $K$, the complementarity problem CP$(K, F)$ requires an $x \in K$ such that $K \ni x \perp F(x) \in K^*$, the dual of $K$.

where the mapping $H(z)$ takes on a form given by

$$H(x,\lambda) \triangleq \begin{pmatrix} \mathbb{E}\left[\nabla_{x_1}(f_1 - c^T\lambda_1)\right] \\ \vdots \\ \mathbb{E}\left[\nabla_{x_N}(f_N - c^T\lambda_N)\right] \\ \mathbb{E}\left[c(x;\omega)\right] \end{pmatrix} \quad \text{and} \quad z \triangleq \begin{pmatrix} x_1 \\ \vdots \\ x_N \\ \lambda \end{pmatrix}. \tag{2.27}$$

Note that we require that $\lambda$ is a common multiplier for the shared constraint. In particular, we use a strict version of the Mangasarian-Fromovitiz constraint qualification [37] which we define next.

**Definition 2.9** *(SMFCQ) For a pair $z = (x, \lambda)$ that solves CP($\mathbb{R}^+_{m+n}, H$), we may define an index set $\gamma$ and $\beta$ given by*

$$\gamma = \{i : \lambda_i > 0\}, \qquad \beta = \{i : \lambda_i = 0\}. \tag{2.28}$$

*Then the strict Mangasarian-Fromovitz constraint qualification (SMFCQ) is said to hold at $(x,\lambda)$ if*

*(a) The gradients $\{\nabla\mathbb{E}[c_i(x;\omega)]\}_{i\in\gamma}$ are linearly independent;*

*(b) There exists a vector $v \in \mathbb{R}^n$ such that*

$$\begin{aligned} \nabla\mathbb{E}\left[c_i(x;\omega)\right]^T v &= 0, \quad \forall i \in \gamma \\ \nabla\mathbb{E}\left[c_i(x;\omega)\right]^T v &> 0, \quad \forall i \in \beta. \end{aligned}$$

Karamardian [74] showed the equivalence between the solutions of the variational inequality VI($K, F$) and a complementarity problem CP($K, F$) when $K$ is a closed convex cone. Therefore, one may utilize the results from the earlier sections to analyze the complementarity problem in the larger space. The associated Jacobian of this map is given by

$$\nabla H(x,\lambda) \triangleq \begin{pmatrix} \nabla_x \mathbf{L}(x,\lambda) & -A(x)^T \\ A(x) & \mathbf{0} \end{pmatrix} \succeq 0,$$

$$\begin{aligned} \text{where} \quad & \mathbf{L}(x,\lambda) := F(x) - A(x)^T\lambda \\ \text{and} \quad & A(x) := \begin{pmatrix} \nabla_{x_1}\mathbb{E}\left[c(x;\omega)\right] & \ldots & \nabla_{x_N}\mathbb{E}\left[c(x;\omega)\right] \end{pmatrix}. \end{aligned}$$

The uniqueness of the equilibria associated with $\mathcal{G}$ is less easy to guarantee in the primal-dual space even if the associated mapping $F(x)$ is strongly monotone. In particular, if $F(x)$ is a strongly monotone map and $c(x)$ is a set of concave constraints, then $H(x, \lambda)$ is merely a monotone map over the entire space. Uniqueness, unfortunately, requires a stronger requirement in general. Next, we demonstrate that even in this constrained regime without strong monotonicity, the game admits a unique Nash equilibrium in the primal-dual space under a suitable regularity condition.

**Theorem 2.14 (Existence and uniqueness in primal-dual space)** *Consider the Nash game given by $\mathcal{G}^E$. Suppose assumptions (A1) and (A2) hold, the (SMFCQ) holds at any solution of $\mathcal{G}^E$ and suppose $H(x, \lambda; \omega)$ is monotone in an almost sure sense. Furthermore, suppose $F(x;\omega)$ is strictly monotone map for $\omega \in U \subseteq \Omega$ where $\mathbb{P}(U) \geq \epsilon > 0$. Then $\mathcal{G}$ admits a unique Nash equilibrium in the primal-dual space.*

**Proof :** We prove the result in two parts. First, we show that $\mathcal{G}^E$ admits a nonempty compact set of equilibria in the primal-dual space. We proceed to show that any equilibrium to this game is necessarily locally unique, facilitating a global uniqueness result.

*Existence:* By noting that $F(x;\omega)$ is monotone in an almost sure sense, it follows that $H(x;\omega)$ also satisfies a similar property, a consequence of observing that

$$\nabla H(x;\omega) = \begin{pmatrix} \nabla \mathbf{L}(x, \lambda; \omega) & -\nabla c(x;\omega)^T \\ \nabla c(x;\omega) & \mathbf{0} \end{pmatrix}$$

is a positive semidefinite matrix for almost all $\omega \in \Omega$ since $\lambda_i \geq 0$ and

$$\nabla_x \mathbf{L}(x, \lambda) = \nabla_x F(x;\omega) - \sum_{i=1}^{m} \lambda_i \underbrace{\nabla^2_{xx} c_i(x;\omega)}_{-\nabla^2_{xx} c_i(x) \succeq 0} \succeq 0.$$

By specifying a vector $z^{\text{ref}} \triangleq (x^{\text{ref}}, \mathbf{0}) \in K \times \mathbb{R}^+_m$, we have the following:

$$
\begin{aligned}
&\liminf_{\|z\|\to\infty, z\in K\times\mathbb{R}^+_n} H(z;\omega)^T(z - z^{\text{ref}}) \\
&= \liminf_{\|z\|\to\infty, z\in K\times\mathbb{R}^+_n} \big((F(x;\omega) - \nabla c(x;\omega)^T\lambda)^T(x - x^{\text{ref}}) + c(x;\omega)^T\lambda\big) \\
&= \liminf_{\|z\|\to\infty, z\in K\times\mathbb{R}^+_n} \left( F(x;\omega)^T(x - x^{\text{ref}}) + \lambda^T \underbrace{(\nabla c(x;\omega)(x - x^{\text{ref}}) + c(x;\omega))}_{\textbf{Term a}} \right).
\end{aligned}
$$

The concavity of $c_j(x;\omega)$ in $x$ for $j = 1, \ldots, m$ allows one to claim that term (a), defined above, can be bounded from below by

$$
\nabla c_j(x;\omega)(x_j - x_j^{\text{ref}}) + c_j(x;\omega)) \geq c_j(x^{\text{ref}};\omega), \quad j = 1, \ldots, m.
$$

Moreover, by assumption we have that there exists an $x^{\text{ref}}$ such that $c_j(x^{\text{ref}};\omega) > 0$, allowing us to conclude that

$$
\begin{aligned}
&\liminf_{\|z\|\to\infty, z\in K\times\mathbb{R}^+_n} \big(F(x;\omega)^T(x - x^{\text{ref}}) + \lambda^T(\nabla c(x;\omega)(x - x^{\text{ref}}) + c(x;\omega))\big) \\
&\geq \liminf_{\|z\|\to\infty, z\in K\times\mathbb{R}^+_n} \big(F(x;\omega)^T(x - x^{\text{ref}}) + \lambda^T c(x^{\text{ref}};\omega)\big) \\
&\geq \liminf_{\|x\|\to\infty, x\in K} F(x;\omega)^T(x - x^{\text{ref}}).
\end{aligned}
$$

But by assumption, we have that there exists an $x^{\text{ref}}$ such that

$$
\liminf_{\|x\|\to\infty, x\in K} F(x;\omega)^T(x - x^{\text{ref}}) > 0
$$

in an almost sure sense. This allows us to conclude that

$$
\mathbb{E}\left( \liminf_{\|z\|\to\infty, z\in K\times\mathbb{R}^+_m} H(z;\omega)^T(z - z^{\text{ref}}) \right) \geq \mathbb{E}\left( \liminf_{\|x\|\to\infty, x\in K} F(x;\omega)^T(x - x^{\text{ref}}) \right) > 0.
$$

By invoking Fatou's Lemma, we have that

$$
\begin{aligned}
&\liminf_{\|z\|\to\infty, z\in K\times\mathbb{R}^+_m} \mathbb{E}\left[H(z;\omega)^T(z - z^{\text{ref}})\right] \\
&\geq \mathbb{E}\left[ \liminf_{\|z\|\to\infty, z\in K\times\mathbb{R}^+_m} H(z;\omega)^T(z - z^{\text{ref}}) \right] > 0,
\end{aligned}
$$

which is a sufficient condition for the existence of a stochastic Nash equilibrium in the primal-dual space.

*Uniqueness:* A uniqueness statement in the presence of monotonicity of the mapping can be derived from showing that an equilibrium in the primal-dual space satisfies local uniqueness. Recall that local uniqueness of $x$ over the set $K$ follows if the only solution to the linear complementarity problem

$$\mathcal{D}(x; K, F) \ni v \perp \nabla L(x; \lambda) v \in \mathcal{D}(x; K, F)^* \tag{2.29}$$

is $v \equiv 0$, where $\mathcal{D}(x; K, F)$ is the critical cone at $x$, defined as

$$\mathcal{D}(x; K, F) \triangleq \mathcal{T}(x; K, F) \cap F(x)^{\perp},$$

and $\mathcal{T}(x; K, F)$ is the tangent cone at $x$, associated with $K$ and $F$. But this holds if $\nabla L(x; \lambda)$ is a positive definite matrix. By assumption, we have that

$$\begin{aligned} \nabla L(x; \lambda, \omega) &= \nabla F(x; \omega) - \sum_{i=1}^{m} \nabla^2 c_i(x; \omega) \\ \implies \nabla L(x; \lambda, \omega) \succeq 0 \text{ a.s since } & \nabla F(x; \omega) \succeq 0 \text{ and } -\nabla^2 c_i(x; \omega) \succeq 0 \text{ a.s.}, \end{aligned}$$

the latter a consequence of the concavity of $c_i(x; \omega)$ in an almost-sure sense. But by assumption, there exists a measurable set $U \subseteq \Omega$ over which $F(x; \omega)$ is strictly monotone. It follows that

$$\begin{aligned} \int_{\omega} v^T (\nabla L(x; \lambda, \omega) v d\mathbb{P} &= \int_{U} v^T (\nabla L(x; \lambda, \omega) v d\mathbb{P} + \int_{\Omega / U} v^T (\nabla L(x; \lambda, \omega) v d\mathbb{P} \\ &\geq \int_{U} v^T (\nabla L(x; \lambda, \omega) v d\mathbb{P} > 0. \end{aligned}$$

Therefore $\mathbb{E}[\nabla L(x, \lambda; \omega)]$ is positive definite and $v = 0$ is the only solution of (2.29). This implies that $x$ is a locally unique solution of $\mathrm{VI}(K, F)$.

By Prop. 3.3.12. [37], the tuple $(x, \lambda)$ is a locally unique solution of $\mathrm{VI}(K \times \mathbb{R}^+_m, H)$ if $\mathcal{M}(x)$ is a singleton. But the latter holds from SMFCQ and by employing Prop. 3.2.1(a) [37]. Finally, by Th. 3.6.6 [37], if a complementarity problem with a $\mathbf{P}_0$ mapping admits a locally unique solution, then the solution is globally unique. But $\mathrm{CP}(\mathbb{R}^+_{m+n}, H)$ is a monotone complementarity problem, implying that $H \in \mathbf{P}_0$ and the required global uniqueness result holds. ∎

## 2.6 Stochastic Nash-Cournot games and their extensions

In this section, our principal goal lies in applying the framework developed over the previous three sections. This requires ascertaining whether the almost-sure requirements can be expected to hold in practical settings. For this purpose, we consider two extensions to a canonical stochastic Nash-Cournot games, the first into the realm of nonsmoothness while the second is in the regime of stochastic coupled constraints (See [3, 75] for a discussion of Cournot games). In a risk-neutral Nash-Cournot game, firms make production decisions prior to the revelation of the uncertainty. In the first extension, we relax the risk-neutrality assumption by allowing firms to be risk-averse; this requires the use of a conditional value-at-risk (CVaR) measure [38]. This measure belongs to the larger class of coherent risk measures and has gained significant applicability in a variety of settings (see [38, 71, 76, 77] for an introduction to the value at risk (VAR), conditional value at risk (CVAR) and coherent risk measures). Subsequently, we examine the regime with shared stochastic constraints.

Throughout this section, we consider a stochastic Nash-Cournot game in which the players make quantity bids, denoted by $x_1, \ldots, x_N$. The players compete for profit which is given by the expected revenue less cost. In the standard Cournot framework, the $i$th player's revenue function is given by $p(x;\omega)x_i$ where $p(x;\omega)$ is the random price function while his cost function is denoted by $c_i(x_i)$. In Section 2.6.1, we examine a risk-averse stochastic Nash-Cournot framework to accommodate a conditional value-at-risk measure that captures the risk of lower profits. Expectation-based shared constraints are introduced in Section 2.6.2.

### 2.6.1 Risk-averse Nash-Cournot game

The model we consider is a modification of a risk-neutral framework that employs an expectation-based framework. This can be generalized to allow for risk preferences by using the conditional value at risk (CVaR) measure that captures the risk of low profits.

**Definition 2.10 (Risk-averse Nash-Cournot Game)**

*Consider an $N-$player game in which the $i$th player has decision variable $z_i = (x_i, m_i)$, strategy set $K_i \times \mathbb{R}$ and objective*

$$f_i(z;\omega) \triangleq r_i(x;\omega) + \kappa_i \rho_i(z;\omega)$$

*where $\mathbb{E}[r_i(x;\omega)] = c_i(x_i) - \mathbb{E}[p(x;\omega)]x_i$ is the negative of the expected profits, $\kappa_i \in [0,1]$ is the player-specific risk-aversion parameter and $\mathbb{E}[\rho_i(z;\omega)]$ is the CVaR measure at level $\tau_i \in [0,1]$ associated with player's expected loss. The player-specific risk is defined as*

$$\mathbb{E}\left[\rho_i(z;\omega)\right] \triangleq m_i + \frac{1}{1-\tau_i}\mathbb{E}\left[(c_i(x_i) - p(x;\omega)x_i - m_i)^+\right].$$

*Then $z_i^* = (x_i^*, m_i^*)_{i=1}^N \in K \times \mathbb{R}^N$ is a risk-averse Nash-Cournot equilibrium of the Nash game, denoted by $\mathcal{G}^{NCR}(\mathbf{K}, \mathbf{f}, \mathcal{P})$, if $z_i^* = (x_i^*, m_i^*)$ solves the convex optimization problem $G_i(z_{-i}^*)$, defined as*

$$\boxed{\underset{(x_i,m_i)\in K_i\times\mathbb{R}}{\text{minimize}} \ \mathbb{E}\left[f_i(z_i; z_{-i}^*, \omega)\right].}$$

Consider the objective function of the $i$-th agent corresponding to a scenario $\omega$ for some $\omega \in \Omega$:

$$f_i(z;\omega) = c_i(x_i) - p(x,\omega)x_i + \kappa_i(m_i + \frac{1}{1-\tau_i}[c_i(x_i) - p(x,\omega)x_i - m_i]^+).$$

The nonsmoothness of the second term implies that the gradient map is multivalued in nature. However, the argument of the loss function is convex and Lipschitz (implying that it is regular in the Clarke sense). Since one of the summands within $f_i$ are continuously differentiable, it follows that $\partial f_i$ is given by

$$\begin{aligned}\partial_{z_i} f_i(z;\omega) = {} & \nabla_{z_i}(c_i(x_i) - p(x,\omega)x_i) \\ & + \kappa_i \partial_{z_i}(m_i + \frac{1}{1-\tau_i}[c_i(x_i) - p(x,\omega)x_i - m_i]^+)\end{aligned}$$

where $\nabla_{z_i}(c_i(x_i) - p(x,\omega)x_i)$ is given by

$$\begin{pmatrix} c_i'(x_i) - p'(x,\omega)x_i - p(x,\omega) \\ 0 \end{pmatrix}.$$

Additionally $\kappa_i \partial_{x_i}(m_i + \frac{1}{1-\tau_i}[c_i(x_i) - p(x,\omega)x_i - m_i]^+)$ is given by

$$\begin{cases} \kappa_i \frac{1}{1-\tau_i}(c_i'(x_i) - (p(x,\omega)x_i)'), \\ \kappa_i \frac{1}{1-\tau_i}\partial_{x_i} \max(c_i(x_i) - p(x,\omega)x_i - m_i, 0), \\ 0, \end{cases}$$

if $c_i(x_i) - p(x,\omega)x_i - m_i$ is positive, zero and negative respectively. Similarly, we can write $\kappa_i \partial_{m_i}(m_i + \frac{1}{1-\tau_i}[c_i(x_i) - p(x,\omega)x_i - m_i]^+)$ as

$$\begin{cases} \kappa_i \left(1 - \frac{1}{1-\tau_i}\right), \\ \kappa_i \left(1 + \frac{1}{1-\tau_i}\partial_{m_i} \max(c_i(x_i) - p(x,\omega)x_i - m_i, 0)\right), \\ \kappa_i, \end{cases}$$

if $c_i(x_i) - p(x,\omega)x_i - m_i$ is positive, zero and negative respectively. Since, the function $c_i(x_i) - p(x,\omega)x_i - m_i$ is regular in the sense of Clarke ( [70]), when $c_i(x_i) - p(x,\omega)x_i - m_i$ is zero, we have

$$\begin{aligned} \partial_{x_i} \max(c_i(x_i)p(x,\omega)x_i - m_i, 0) &= \text{co}((c_i(x_i) - p(x,\omega)x_i - m_i)', 0) \\ &= \{\alpha_i(c_i'(x_i) - p'(x,\omega)x_i - p(x,\omega))|\alpha_i \in [0,1]\}, \end{aligned}$$

where co(.) represents the convex hull. Similarly, when $c_i(x_i) - p(x,\omega)x_i - m_i$ is zero, we have

$$\begin{aligned} \partial_{m_i} \max(c_i(x_i)p(x,\omega)x_i - m_i, 0) &= \text{co}(\partial_{m_i}(c_i(x_i)p(x,\omega)x_i - m_i), 0) \\ &= \{-\beta_i|\beta_i \in [0,1]\}, \end{aligned}$$

It follows that any element $w_i \in \partial f_i(z_i; z_{-i}, \omega)$ is given by

$$w_i \triangleq \begin{pmatrix} w_i^x \\ w_i^m \end{pmatrix} = \begin{pmatrix} c_i'(x_i) - p'(x,\omega)x_i - p(x,\omega) \\ 1 \end{pmatrix} + w_d$$

where $w_i^d$ is given by

$$w_i^d = \kappa_i \begin{pmatrix} \frac{1}{1-\tau_i}\alpha_i(c_i'(x_i) - p'(x,\omega)x_i - p(x,\omega)) \\ 1 - \frac{\beta_i}{1-\tau_i} \end{pmatrix},$$

and $\alpha_i, \beta_i$ are defined as

$$\begin{cases} (\alpha_i, \beta_i) = (1,1) & c_i(x_i) - p(x,\omega)x_i - m_i > 0, \\ (\alpha_i, \beta_i) \in [0,1] \times [0,1] & c_i(x_i) - p(x,\omega)x_i - m_i = 0, \\ (\alpha_i, \beta_i) = (0,0) & c_i(x_i) - p(x,\omega)x_i - m_i < 0. \end{cases}$$

The variational inequality corresponding to the scenario-based game is denoted by $\text{VI}(K \times \mathbb{R}^N, \partial F(z;\omega))$, where

$$\partial F(z;\omega) = \prod_{i=1}^{N} \partial f_i(z_i; z_{-i}, \omega).$$

The term $w^T(z - z^{\text{ref}})$ may be expressed as

$$w^T(z - z^{\text{ref}}) = \sum_{i\in\mathcal{N}} w_i^T(z_i - z_i^{\text{ref}}) = \sum_{i\in\mathcal{N}} \left( \underbrace{w_i^x(x_i - x_i^{\text{ref}})}_{\textbf{Term(a)}} \right) + \sum_{i\in\mathcal{N}} \left( \underbrace{w_i^m(m_i - m_i^{\text{ref}})}_{\textbf{Term(b)}} \right).$$

By (A3), $c_i(x_i)$ are strictly increasing strictly convex cost functions. Thus, $c_i'$ is a strictly increasing function of $x_i$. We require the following assumption on the cost functions.

**Assumption 2.5 (A3)** *Suppose the cost function $c_i(x_i)$ is a strictly increasing strictly convex twice continuously differentiable function for all $i \in \mathcal{N}$. Furthermore, suppose the price function is given by a random affine function*

$$p(x;\omega) \triangleq a(\omega) - b(\omega) \sum_{i\in\mathcal{N}} x_i, \tag{2.30}$$

*where $a(\omega)$ and $b(\omega)$ are positive in an almost-sure sense and integrable.*

**Assumption 2.6 (A4)** *Suppose that for large $\|x_i\|$, $c_i(x_i)' \geq a(\omega)$ in an almost-sure sense for all $i \in \mathcal{N}$.*

**Proposition 2.15** *Consider the game $\mathcal{G}^{NCR}(\mathbf{K}, \mathbf{f}, \mathcal{P})$ where the ith player's objective is given by*

$$\begin{aligned} \mathbb{E}\left[f_i(z;\omega)\right] &= c_i(x_i) - \mathbb{E}\left[p(x;\omega)\right] x_i \\ &\quad + \kappa_i(m_i + \frac{1}{1-\tau_i}\mathbb{E}\left[(c_i(x_i) - p(x;\omega)x_i - m_i)^+\right]), \end{aligned}$$

*for all $i \in \mathcal{N}$. Suppose (A3) and (A4) hold. Then the game $\mathcal{G}^{NCR}(\mathbf{K}, \mathbf{f}, \mathcal{P})$ admits an equilibrium.*

**Proof :** Recall from the definition of $\partial F(z;\omega)$, we have that if $w \in \partial F(z;\omega)$, then $w$ is given by

$$w = \begin{pmatrix} (1+\frac{\kappa_i\alpha_i}{1-\tau_i})(c_i'(x_i) - p'(x,\omega)x_i - p(x,\omega)) \\ \kappa_i(1-\frac{\beta_i}{1-\tau_i}) \end{pmatrix},$$

where $(\alpha_i, \beta_i)$ for $i \in \mathcal{N}$ are specified by

$$\begin{cases} (\alpha_i,\beta_i) = (1,1) & c_i(x_i) - p(x,\omega)x_i - m_i > 0, \\ (\alpha_i,\beta_i) \in [0,1]\times[0,1] & c_i(x_i) - p(x,\omega)x_i - m_i = 0, \\ (\alpha_i,\beta_i) = (0,0) & c_i(x_i) - p(x,\omega)x_i - m_i < 0. \end{cases}$$

It follows that

$$\begin{aligned} w^T(z - z^{\text{ref}}) &= \sum_{i\in\mathcal{N}} w_i^T(z_i - z_i^{\text{ref}}) \\ = \sum_{i\in\mathcal{N}} &\underbrace{(1+\frac{\kappa_i\alpha_i}{1-\tau_i})(c_i'(x_i) - p'(x,\omega)x_i - p(x,\omega))(x_i - x_i^{\text{ref}})}_{\textbf{Term(a)}} \\ + &\underbrace{\kappa_i(1-\frac{\beta_i}{1-\tau_i})(m_i - m_i^{\text{ref}})}_{\textbf{Term(b)}}. \end{aligned}$$

It can be seen that for sufficiently large $x$, term (a) tends to infinity at a quadratic rate for $\alpha_i \geq 0$. This follows by noting from assumption 2.6 that for sufficiently large $x$, we have $c_i'(x_i) \geq a(\omega)$ for all $i \in \mathcal{N}$ in an almost-sure sense. Term (a) above can be bounded from below as follows

$$\begin{aligned} (c_i'(x_i) - p'(x,\omega)x_i - p(x,\omega)) &= (c_i'(x_i) + b(\omega)(\sum_{j\in\mathcal{N}} x_j + x_i) - a(\omega)) \\ &\geq (c_i'(x_i) - a(\omega). \end{aligned}$$

where the first inequality follows from the nonnegativity of $x_i$ and $b(\omega)$ (almost-surely). From assumption 2.6, it follows that for sufficiently large $\|x\|$, we have $(c_i'(x_i) - a(\omega) > 0$ in an almost-sure sense. Since $K_i \subseteq \mathbb{R}_+$, we have that if $x \in \mathbf{K}$, $\|x\| \to \infty$, we have that term (a) tends to $+\infty$.

Term (b) can be written as $\kappa_i(1-\frac{\beta_i}{1-\tau_i})(m_i - m_i^{\text{ref}})$. There are several cases possible:

(i) Suppose $m_i \to \infty$ and $x_i$ stays bounded, implying that $(c_i(x_i) - p(X)x_i - m_i) < 0$. Therefore $\beta_i = 0$ and we have term (b) tending to $+\infty$.

(ii) Suppose $m_i \to -\infty$ and $x_i$ stays bounded, implying that $(c_i(x_i) - p(X)x_i - m_i) > 0$. Therefore $\beta_i = 1$ and we have term (b) tending to $+\infty$, yet again since $1 - \frac{1}{1-\tau_i} < 0$.

(iii) Suppose $(c_i(x_i) - p(X)x_i - m_i) = 0$ , implying that $x_i, m_i \to +\infty$. Therefore $\beta_i \in [0,1]$ and we have term (b) tending to $+\infty$ or $-\infty$, based on the value of $\beta_i$. However, term(a) tends to $+\infty$ at a quadratic rate, giving us the required result.

It follows that along any sequence $z_k = (x_k, m_k) \in K \times \mathbb{R}^n$ with $\|z_k\| \to \infty$, we have $w^T(z_k - z^{\text{ref}}) \to +\infty$. Therefore, there exists a $z^{\text{ref}}$ such that

$$\liminf_{\|z\|\to\infty, z\in\mathbf{K}} w^T(z - z^{\text{ref}}) > 0,$$

almost-surely for all $w \in \partial F(x;\omega)$, implying that

$$\liminf_{\|z\|\to\infty, z\in\mathbf{K}} \left( \min_{w\in\partial F(x;\omega)} w^T(z - z^{\text{ref}}) \right) > 0,$$

almost surely. From Proposition 2.11, we may conclude that $\mathcal{G}^{NCR}(\mathbf{K}, \mathbf{f}, \mathcal{P})$ admits an equilibrium if (A2$^\dagger$) holds. The remainder of the proof proves that the latter is indeed the case. If $w_i$ is any element of $\partial_{z_i} f_i(z;\omega)$, then $w_i$ is given by

$$w_i = \begin{pmatrix} (1 + \frac{\kappa_i\alpha_i}{1-\tau_i})(c_i'(x_i) - p'(x;\omega)x_i - p(x;\omega)) \\ \kappa_i(1 - \frac{\beta_i}{1-\tau_i}) \end{pmatrix},$$

where $(\alpha_i, \beta_i)$ as given as above. Thus, for a fixed $z^{\text{ref}}$ we have

$$\begin{aligned}
&w_i^T(z_i - z_i^{\text{ref}}) \\
&= (1 + \frac{\kappa_i \alpha_i}{1-\tau_i})(c_i'(x_i) - p'(x;\omega)x_i - p(x;\omega))(x_i - x_i^{\text{ref}}) + \kappa_i(1 - \frac{\beta_i}{1-\tau_i})(m_i - m_i^{\text{ref}}) \\
&\geq (b(\omega)x_i + b(\omega)X - a(\omega))(x_i - x_i^{\text{ref}}) + \kappa_i(1 - \frac{\beta_i}{1-\tau_i})(m_i - m_i^{\text{ref}}) \\
&\geq -a(\omega)(x_i - x_i^{\text{ref}}) + \kappa_i(1 - \frac{\beta_i}{1-\tau_i})(m_i - m_i^{\text{ref}}).
\end{aligned}$$

If $\eta_i = (1 - \frac{\beta_i}{1-\tau_i})$, then by defining $\bar{u}_i(z; z_i^{\text{ref}}, \omega)$ as

$$\triangleq \begin{cases} -a(\omega)(x_i - x_i^{\text{ref}}) + \kappa_i\eta_i(m_i - m_i^{\text{ref}}), & (x_i - x_i^{\text{ref}}), \eta_i(m_i - m_i^{\text{ref}}) < 0, \\ a(\omega)(x_i - x_i^{\text{ref}}) + \kappa_i\eta_i(m_i - m_i^{\text{ref}}), & (x_i - x_i^{\text{ref}}) > 0, \eta_i(m_i - m_i^{\text{ref}}) < 0, \\ -a(\omega)(x_i - x_i^{\text{ref}}) - \kappa_i\eta_i(m_i - m_i^{\text{ref}}), & (x_i - x_i^{\text{ref}}) < 0, \eta_i(m_i - m_i^{\text{ref}}) > 0, \\ a(\omega)(x_i - x_i^{\text{ref}}) - \kappa_i\eta_i(m_i - m_i^{\text{ref}}), & (x_i - x_i^{\text{ref}}), \eta_i(m_i - m_i^{\text{ref}}) > 0, \end{cases}$$

the integrability of $\bar{u}_i(z; z_i^{\text{ref}}, \omega)$ follows from the integrability of $a(\omega)$ and (A2$^\dagger$) holds. ∎

### 2.6.2 Stochastic Nash-Cournot game with expected value constraints

We now consider whether the framework developed earlier can be used for claiming existence of solution for stochastic Nash games with convex expected value constraints. Consider the game with expected value constraints where the $i$th agent solves

$$\boxed{\begin{array}{ll} \text{minimize} & \mathbb{E}\left[f_i(x;\omega)\right] \\ \text{subject to} & \begin{array}{r} \mathbb{E}\left[d_i(x;\omega)\right] \leq 0, \\ x_i \geq 0. \end{array} \end{array}}$$

where $f_i(x_i; x_{-i}, \omega) = c_i(x_i) - p(x, \omega)x_i$, assumption (2.5) holds and $d_i : \mathbb{R}^n \times \mathbb{R}^d \to \mathbb{R}^{m_i}$ are convex constraints in $x_i$, given $x_{-i}$. Before proceeding, its worth examining the possible avenues that can be taken:

**Complementarity approach** One option lies in constructing an appropriate complementarity problem by assuming a regularity condition. Then under suitable conditions, this problem can be seen to be solvable if a related scenario-based problem admits a solution in an almost-sure sense. This approach is described in Section 2.5. If the constraints satisfy an additional requirement of shared constraints, even stronger statements are possible as Section 2.5 demonstrates.

**Nonsmooth penalty approach** An alternate avenue that has not been considered here is an exact penalty approach introduced in [78]. Here, the coupled constraints are penalized via an exact penalty function. This leads to a nonsmooth Nash game whose existence is analyzed by leveraging the properties of the associated multivalued variational problem. Importantly, any equilibrium of the penalized game is an equilibrium of the original problem only if it satisfies an appropriate feasibility requirement.

We proceed to transform this problem into a nonsmooth Nash game and use the methodology developed in the previous section to claim existence of solution for the penalized game. Consider the penalized game where each agent solves

$$\boxed{\begin{array}{ll}\text{minimize} & \mathbb{E}\left[f_i(x;\omega) + \rho d_i(x;\omega)^+\right] \\ \text{subject to} & x_i \geq 0.\end{array}}$$

This is a stochastic nonsmooth game. Again, in the same vein as before we analyze the properties of the scenario-based game to show existence of an equilibrium to the stochastic game. We require the following assumption on $d_i$.

**Assumption 2.7 (A5)** *For all $i \in \mathcal{N}$, the constraint function $d_i(x;\omega)$ has bounded derivatives in an almost sure sense.*

**Proposition 2.16** *Consider the penalized stochastic Nash-Cournot game with expected value constraints $\mathcal{G}_\rho^{NCE}(\mathbf{K}, \mathbf{f}, \mathcal{P})$ where the ith player's objective is given by $\mathbb{E}[h_i(x;\omega)]$ for all $i \in \mathcal{N}$, where $h_i(x;\omega) \triangleq f_i(x;\omega) + \rho d_i(x;\omega)^+$. Suppose (A3), (A4) and (A5) hold. Then the game $\mathcal{G}_\rho^{NCE}(\mathbf{K}, \mathbf{f}, \mathcal{P})$ admits an equilibrium.*

**Proof :** Consider the penalized scenario-based game of $\mathcal{G}_\rho^{NCE}(\mathbf{K}, \mathbf{f}, \mathcal{P})$ corresponding to the point $\omega \in \Omega$. Then the objective function for the $i$th agent in the scenario-based game is given by $h_i(x;\omega) = f_i(x;\omega) + \rho d_i(x;\omega)^+$, where $f_i(x;\omega) = c_i(x_i) - p(x,\omega)x_i$. Now, $\partial_{x_i} h_i(x;\omega)$ can be written as

$$\begin{aligned}\partial_{x_i} h_i(x;\omega) &= \nabla_{x_i} f_i(x;\omega) + \rho \sum_{j=1}^{m_i} \partial_{x_i} \max(d_{i,j}(x;\omega), 0) \\ &= c_i'(x_i) - p'(x,\omega)x_i - p(x,\omega) + \rho \sum_{j=1}^{m_i} \beta_{ij} d_{ij}'(x_i, x_{-i};\omega),\end{aligned}$$

where $\beta_{ij}$ is derived similar to the risk-averse setting and is given by

$$\beta_{ij} \begin{cases} = 1 & c_{ij}(x_i, x_{-i}, \omega) > 0, \\ \in [0,1] & c_{ij}(x_i, x_{-i}, \omega) = 0, \\ = 0 & c_{ij}(x_i, x_{-i}, \omega) < 0. \end{cases}$$

This scenario-based game admits an equilibrium, if $\mathrm{VI}(\mathbf{K} \times \mathbb{R}^N, \partial h(x;\omega))$ is solvable, where $\partial_x h(x;\omega) = \prod_{i=1}^{N} \partial_{x_i} h_i(x;\omega)$, For $w \in \partial_x h(x;\omega)$, consider

$$\begin{aligned} w^T(x - x^{\text{ref}}) &= \sum_{i \in \mathcal{N}} w_i^T (x_i - x_i^{\text{ref}}) \\ &= \sum_{i \in \mathcal{N}} \underbrace{c_i'(x_i) - p'(x,\omega)x_i - p(x,\omega)^T(x_i - x_i^{\text{ref}})}_{\textbf{Term(a)}} + \underbrace{\rho \sum_{j=1}^{m_i} \beta_{ij} d_{ij}'(x_i, x_{-i};\omega)^T (x_i - x_i^{\text{ref}})}_{\textbf{Term(b)}}. \end{aligned}$$

By (A3) and (A4), it can be seen that for sufficiently large $x$, term (a) tends to infinity at a quadratic rate. By (A5), the derivatives $d_{ij}'$ are bounded in an almost-sure sense. Thus, we have that Term(b) grows at a linear rate. This gives that for sufficiently large $x$, we have $w^T(x - x^{\text{ref}}) \to +\infty$. Therefore, there exists a $x^{\text{ref}}$ such that

$$\liminf_{\|x\| \to \infty, x \in K} w^T(x - x^{\text{ref}}) > 0,$$

almost-surely for all $w \in \partial h(x;\omega)$. It follows that

$$\liminf_{\|x\| \to \infty, x \in K} \left( \min_{w \in \partial F(x;\omega)} w^T(x - x^{\text{ref}}) \right) > 0$$

almost surely. From Proposition 2.11, we may conclude that $\mathcal{G}_\rho^{NCE}(\mathbf{K}, \mathbf{f}, \mathcal{P})$ admits an equilibrium if (A2$^\dagger$) holds. The latter can be shown to hold in a fashion similar to Prop. 2.15 and we choose to omit this exercise for purposes of brevity. ∎

If this equilibrium additionally satisfies feasibility with respect to the penalized constraints, we may claim that the original game admits an equilibrium.

**Proposition 2.17** *Suppose (A3), (A4) and (A5) hold and suppose for a* $\rho > 0$*, an equilibrium* $x^*$ *of the penalized stochastic Nash-Cournot game with expected value constraints* $\mathcal{G}_\rho^{NCE}(\mathbf{K}, \mathbf{f}, \mathcal{P})$ *is additionally feasible to the original stochastic Nash-Cournot game with expected value constraints* $\mathcal{G}^{NCE}(\mathbf{K}, \mathbf{f}, \mathcal{P})$*. Then* $x^*$ *is an equilibrium of* $\mathcal{G}^{NCE}(\mathbf{K}, \mathbf{f}, \mathcal{P})$*.*

**Proof :** Follows from Theorem 1 [78]. ∎

# CHAPTER 3

# ON THE EXISTENCE OF SOLUTIONS TO STOCHASTIC QUASI-VARIATIONAL INEQUALITY AND COMPLEMENTARITY PROBLEMS

## 3.1 Introduction

**Motivation:** In deterministic regimes, a wealth of conditions exist for characterizing the solution sets of variational inequality, quasi-variational inequality and complementarity problems (cf. [79, 37, 52]), including sufficiency statements of existence and uniqueness of solutions as well as more refined conditions regarding the compactness and connectedness of solution sets and a breadth of sensitivity and stability questions. Importantly, the analytical verifiability of such conditions from problem primitives (such as the underlying map and the set) is essential to ensure the applicability of such schemes, as evidenced by the use of such conditions in analyzing a range of problems arising in power markets [3, 35, 80], communication networks [53, 81, 7, 6], structural analysis [82, 83], amongst others. The first instance of a stochastic variational inequality problem was presented by King and Rockafellar [84] in 1993 and the resulting stochastic variational inequality problem requires an $x \in X$ such that

$$(y - x)^T \mathbb{E}[F(x, \xi(\omega))] \geq 0, \quad \forall y \in X,$$

where $X \subseteq \mathbb{R}^n$, $\xi : \Omega \to \mathbb{R}^d$, $F : X \times \mathbb{R}^d \to \mathbb{R}^n$, $\mathbb{E}[.]$ denotes the expectation, and $(\Omega, \mathcal{F}, \mathbb{P})$ represents the probability space. In the decade that followed, there was relatively little effort on addressing analytical and computational challenges arising from such problems. But in the last ten years, there has been an immense interest in the solution of such *stochastic* variational inequality problems via Monte-Carlo sampling methods [16, 85, 18, 10]. But **verifiable** conditions for characterization of solution sets have proved to be relatively elusive given the presence of an integral (arising from the expectation) in the map. Despite the rich literature in deterministic settings, direct application of deterministic results to stochastic regimes is not straightforward and is complicated by several challenges: First, a direct application of such techniques on stochastic problems requires the availability of closed-form expressions of the expectations. Analytical expressions for expectation are not easy to derive even for single-valued problems with relatively simple continuous distributions. Second, any statement is closely tied to the distribution. Together, these barriers severely limit the generalizability of such an approach. To illustrate the complexity of the problem class under consideration, we consider a simple stochastic linear complementarity problem.[1]

**Example :** Consider the following stochastic linear complementarity problem:

$$0 \leq x \perp \mathbb{E}\left[\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} x + \begin{pmatrix} -2+\omega_1 \\ -4+\omega_2 \end{pmatrix}\right] \geq 0.$$

Specifically, this can be cast as an affine stochastic variational inequality problem VI$(K, F)$ where

$$K \triangleq \mathbb{R}_2^+ \text{ and } F(x) \triangleq \mathbb{E}[F(x;\omega)], \text{ where } F(x;\omega) \triangleq \left[\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} x + \begin{pmatrix} -2+\omega_1 \\ -4+\omega_2 \end{pmatrix}\right].$$

□

Consider two cases that pertain to either when the expectation is available in closed-form (a); or not (b):

**(a) Expectation $\mathbb{E}[.]$ available:** Suppose in this instance, $\omega$ is a random variable that takes values $\omega^1$ of $\omega^2$, given by the following:

$$\omega^1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \text{ or } \omega^2 = \begin{pmatrix} -1 \\ -1 \end{pmatrix} \text{ with probability } 0.5.$$

Consequently, the stochastic variational inequality problem can be expressed as

$$0 \leq x \perp \left[\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} x + \begin{pmatrix} -2 \\ -4 \end{pmatrix}\right] \geq 0.$$

In fact, this problem is a strongly monotone linear complementarity problem and admits a unique solution given by $x^* = (0, 2)$. If one cannot ascertain

[1]Formal definitions for these problems are provided in Section 3.1.1

monotonicity, a common approach lies in examining coercivity properties of the map; specifically, VI$(K, F)$ is solvable since there exists an $x^{\rm ref} \in K$ such that (cf. [37, Ch. 2])

$$\liminf_{\|x\|\to\infty, x\in K} F(x)^T(x - x^{\rm ref}) > 0.$$

**(b) Expectation $\mathbb{E}[.]$ not available in closed-form:** However, in many practical settings, closed-form expressions of the expectation are unavailable. Two possible avenues are available:

(i) If $K$ is compact, under continuity of the expected value map, VI$(K, F)$ is solvable.

(ii) If there exists a single $x \in K$ that solves VI$(K, F(.;\omega))$ for almost every $\omega \in \Omega$, VI$(K, F)$ is solvable.

Clearly, $K$ is a cone and (i) does not hold. Furthermore (ii) appears to be possible only for pathological examples and in this case, there does not exist a single $x$ that solves the scenario-based VI$(K, F(.;\omega))$ for every $\omega \in \Omega$. Specifically, the unique solutions to VI$(K, F(.;\omega^1))$ and VI$(K, F(.;\omega^2))$ are $x(\omega^1) = (0, 3/2)$ and $x(\omega^2) = (1/3, 7/3)$, respectively and since $x(\omega^1) \neq x(\omega^2)$, avenue (ii) cannot be traversed. Consequently, neither of the obvious approaches can be adopted yet VI$(K, F)$ is indeed solvable with a solution given by $x^* = (0, 2)$.

Consequently, unless the set is compact or the scenario-based VI is solvable by the **same vector** in an almost sure sense, ascertaining solvability of stochastic variational inequality problems for which the expectation is unavailable in closed form does not appear to be immediately possible through known techniques. In what could be seen as the first step in this direction, our prior work [32] examined the solvability of convex stochastic Nash games by analyzing the equilibrium conditions, compactly stated as a stochastic variational inequality problem. Specifically, this work relies on utilizing Lebesgue convergence theorems to develop integration-free sufficiency conditions that could effectively address stochastic variational inequality problems, with both single-valued and a subclass of multi-valued maps arising from nonsmooth Nash games. As a simple illustration of the avenue adopted, consider Example 3.1 again and assume that the expectation is unavailable in closed form, we examine whether the a.s. coercivity property holds (as presented in the next section). It can be easily seen that there exists an $x^{\rm ref} \in K$, namely $x^{\rm ref} \triangleq \mathbf{0}$, such that

$$\liminf_{\|x\|\to\infty, x\in K} F(x;\omega)^T(x - x^{\rm ref}) > 0, \text{ for } \omega = \omega^1 \text{ and } \omega = \omega^2.$$

It will be shown that satisfaction of this coercivity property in an almost-sure sense is sufficient for solvability. But such statements, as is natural with any first step, do not accommodate stochastic quasi-variational problems and can be refined significantly to accommodate complementarity problems. Moreover, they cannot accommodate multi-valued variational maps. The present work focuses on extending such sufficiency statements to quasi-variational inequality problems and complementarity problems and accommodate settings where the maps are multi-valued.

**Contributions:** This chapter provides amongst the first attempts to examine and characterize solutions for the class of stochastic quasi-variational inequality and complementarity problems when expectations are unavailable in closed form. Our contributions can briefly be summarized as follows:

**(i) Stochastic quasi-variational inequality problems (SQVIs):** We begin by recapping our past integration-free statements for stochastic VIs that required the use of Lebesgue convergence theorems and variational analysis. Additionally, we provide extensions to regimes with multi-valued maps and specialize the conditions for settings with monotone maps and Cartesian sets.We then extend these conditions to stochastic quasi-variational inequality problems where in addition to a coercivity-like property, the set-valued mapping needs to satisfy continuity, apart from other "well-behavedness" properties to allow for concluding solvability. Finally, we extend the sufficiency conditions to accommodate multi-valued maps.

**(ii) Stochastic complementarity problems (SCPs):** Solvability of complementarity problems over cones requires a significantly different tack. We show that analogous verifiable integration-free statements can be provided for stochastic complementarity problems. Refinements of such statements are also provided in the context of co-coercive maps.

**(iii) Applications:** Naturally, the utility of any sufficiency conditions is based on its level of applicability. We describe two application problems. Of these, the first is a nonsmooth stochastic Nash-Cournot game which leads to an SQVI while the second is a stochastic equilibrium problem in power markets which can be recast as a stochastic complementarity problem. Importantly, both application settings are modeled with a relatively high level of fidelity.

**Remark:** Finally, we emphasize three points regarding the relevance and utility of the provided statements: (i) First, such techniques are of relevance when integration cannot be carried out easily and have less utility when sample spaces are finite; (ii) There are settings where alternate models for incorporating uncertainty have been developed [20, 21, 86]. Such models assume relevance when the interest lies in *robust* solutions. Naturally, an expected-value formulation has less merit in such settings and correspondingly, such robust approaches cannot capture risk-neutral decision-making.[2] Consequently, the challenge of analyzing existence of this problem cannot be done away with by merely changing the formulation, since an alternate formulation may be inappropriate. (iii) Third, we present sufficiency conditions for solvability. Still, there are simple examples which can be constructed in finite (and more general) sample spaces where such conditions will not hold and yet solvability does hold. We believe that this does not diminish the importance of our results; in fact, this is not unlike other sufficiency conditions for variational inequality problems. For instance, we may construct examples where the coercivity of a map may not hold over the given set [37] but the variational inequality problem may be solvable. However, in our estimation, in some of the more practically occurring engineering-economic systems, such conditions do appear to hold, reinforcing the relevance of such techniques. In particular, we show such conditions find applicability in a class of risk-neutral and risk-averse stochastic Nash games in [32]. In the present work, we show that such conditions can be employed for analyzing a class of stochastic generalized Nash-Cournot games with nonsmooth price functions as well as for a relatively more intricate networked power market in uncertain settings. Before proceeding to our results we provide a brief

[2] A similarly loose dichotomy exists between stochastic programming and robust optimization.

history of deterministic and stochastic variational inequalities.

**Background and literature review:** The variational inequality problem provides a broad and unifying framework for the study of a range of mathematical problems including convex optimization problems, Nash games, fixed point problems, economic equilibrium problems and traffic equilibrium problems [37]. More generally, the concept of an equilibrium is central to the study of economic phenomena and engineered systems, prompting the use of the variational inequality problem [87]. Harker and Pang [51] provide an excellent survey of the rich mathematical theory, solution algorithms, and important applications in engineering and economics while a more comprehensive review of the analytical and algorithmic tools is provided in the recent two volume monograph by Facchinei and Pang [37].

Increasingly, the deterministic model proves inadequate when contending with models complicated by risk and uncertainty. Uncertainty in variational inequality problems has been considered in a breadth of application regimes, ranging from traffic equilibrium problems [20, 88], cognitive radio networks [89, 90], Nash games [32, 86], amongst others. Compared to the field of optimization, where stochastic programming [91, 92] and robust optimization [93] have provided but two avenues for accommodating uncertainty in static optimization problems, far less is currently available either from a theoretical or an algorithmic standpoint in the context of stochastic variational inequality problems. Much of the efforts in this regime have been largely restricted to Monte-Carlo sampling schemes [12, 16, 84, 17, 94, 18, 95, 13], and a recent broader survey paper on stochastic variational inequality problems and stochastic complementarity problems [96].

### 3.1.1 Formulations

To help explain the two main formulations for stochastic variational inequality problems found in literature - the expected value formulation and almost-sure formulation; we now define variational inequality problems and generalizations and their stochastic counterparts. Given a set $K \subseteq \mathbb{R}^n$ and a mapping $F : \mathbb{R}^n \to \mathbb{R}^n$, the deterministic variational inequality problem, denoted by VI$(K, F)$, requires an $x \in K$ such that

$$(y-x)^T \; F(x) \;\geq\; 0, \qquad \forall y \in K. \tag{VI(K,F)}$$

The quasi-variational generalization of VI$(K, F)$ referred to as a quasi-variational inequality, emerges when $K$ is generalized from a constant map to a set-valued map $K : \mathbb{R}^n \to \mathbb{R}^n$ with closed and convex images. More formally, QVI$(K, F)$, requires an $x \in K(x)$ such that

$$(y-x)^T \; F(x) \;\geq\; 0, \qquad \forall y \in K(x). \tag{QVI(K,F)}$$

If $K$ is a cone, then the variational inequality problem reduces to a complementarity problem, denoted by CP$(K, F)$, a problem that requires an $x \in K$ such that

$$K \;\ni\; x \;\perp\; F(x) \;\in\; K^* \tag{CP(K,F)}$$

where $K^* \triangleq \{y : y^T d \geq 0, \forall d \in K\}$ and $y \perp w$ implies $y_i w_i = 0$ for $i = 1, \ldots, n$. In settings complicated by uncertainty, stochastic generalizations of such problems are of particular relevance. Given a continuous probability space $(\Omega, \mathcal{F}, \mathbb{P})$, let $\mathbb{E}[\bullet]$ denote the expectation operator with respect to the measure $\mathbb{P}$. Throughout this chapter, we often denote the expectation $\mathbb{E}\left[F(x; \xi(\omega))\right]$ by $F(x)$, where $F(x; \xi(\omega))$ denotes the scenario-based map, $F_i(x) \triangleq \mathbb{E}[F_i(x; \xi(\omega))]$ and $\xi : \Omega \to \mathbb{R}^d$ is a $d-$dimensional random variable. For notational ease, we refer to $F(x; \xi(\omega))$ as $F(x; \omega)$ through the entirety of this chapter.

**The expected-value formulation** We may now formally define the *stochastic variational inequality problem* as an expected-value formulation. We consider this formulation in the analysis in this chapter.

**Definition 3.1 (Stochastic variational inequality problem (SVI$(K, F)$))** Let $K \subseteq \mathbb{R}^n$ be a closed and convex set, $F : \mathbb{R}^n \times \Omega \to \mathbb{R}^n$ be a single-valued map and $F(x) \triangleq \mathbb{E}[F(x; \omega)]$. Then the stochastic variational inequality problem, denoted by SVI$(K, F)$, requires an $x \in K$ such that the following holds:

$$(y-x)^T \; \mathbb{E}\left[F(x;\omega)\right] \;\geq\; 0, \quad \forall y \in K. \tag{SVI(K,F)}$$

Figure 3.1 provides a schematic of the stochastic variational inequality problem. Note that when $x$ solves SVI$(K, F)$, there may exist $\omega \in \Omega$ for which there exist $y \in K$ such that $(y - x)^T F(x; \omega) < 0$. Naturally, in instances where the expectation is simple to evaluate, as seen in Example 3.1 earlier, the resulting SVI$(K, F)$ is no harder than its deterministic counterpart. For instance, if the sample space $\Omega$ is finite, then the expectation reduces to a finite summation of deterministic maps which is itself a deterministic map. Consequently, the **analysis** of this problem is as challenging as a deterministic variational inequality problem. Unfortunately, in most stochastic regimes, this evaluation relies on a multi-dimensional integration and is not a straightforward task. In fact, a more general risk functional can be introduced instead of the expectation leading to a risk-based variational inequality problem that requires an $x$ such that

$$(y - x)^T \rho[F(x; \omega)] \geq 0, \quad \forall y \in K,$$

where $\rho[F(x; \omega)] \triangleq \mathbb{E}[F(x; \omega)] + \mathbb{D}[F(x; \omega)]$ and $\mathbb{D}[\bullet]$ is a map incorporating dispersion measures such as standard deviation, upper semi-deviation, or the conditional value at risk (CVaR) (cf. [97, 98, 92] for recent advances in the optimization of these risk measures).

Extensions to set-valued and conic regimes follow naturally. For instance, if $K$ is a point-to-set mapping defined as $K : \mathbb{R}^n \to \mathbb{R}^n$, then the resulting problem is a stochastic quasi-variational inequality, and is denoted by SQVI$(K, F)$. When $K$ is a cone, then VI$(K, F)$ is equivalent to a complementarity problem CP$(K, F)$ and its stochastic generalization is given next.

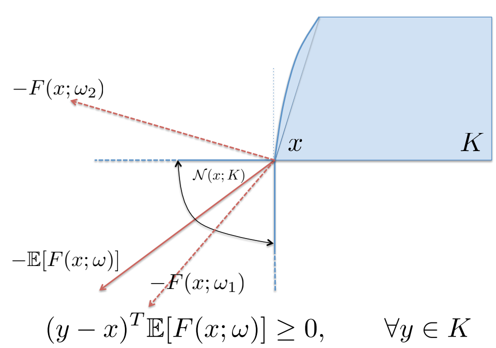


Figure 3.1: The stochastic variational inequality (SVI$(K, F)$): The expected-value formulation

**Definition 3.2 (Stochastic complementarity problem (SCP$(K, F)$))** Let $K$ be a closed and convex cone in $\mathbb{R}^n$, $F : \mathbb{R}^n \times \Omega \to \mathbb{R}^n$ be a single-valued mapping and $F(x) \triangleq \mathbb{E}[F(x; \omega)]$. Then the stochastic complementarity problem, denoted by SCP$(K, F)$, requires an $x \in K$ such that

$$K \ni x \perp \mathbb{E}[F(x;\omega)] \in K^*.$$

∎

If the integrands of the expectation $(F(x;\omega))$ are multi-valued instead of single-valued, then we denote the mapping by $\Phi$. Accordingly, the associated variational problems are denoted by SVI$(K, \Phi)$, SQVI$(K, \Phi)$ and SCP$(K, \Phi)$.

The origin of the expected-value formulation can be traced to a paper by King and Rockafellar [84] where the authors considered a *generalized equation* [99, 100] with an expectation-valued mapping. Notably, the analysis and computation of the associated solutions are hindered significantly when the expectation is over a general measure space. Evaluating this integral is challenging, at best, and it is essential that specialized analytical and computational techniques be developed for this class of variational problems. From an analytical standpoint, Ravat and Shanbhag have developed existence statements for equilibria of stochastic Nash games that obviate the need to evaluate the expectation by combining Lebesgue convergence theorems with standard existence statements [101, 32]. Our earlier worked focused on Nash games and examined such settings with nonsmooth payoff functions and stochastic constraints. It represents a starting point for our current study where we focus on more general variational inequality and complementarity problems and their generalizations and refinements. Accordingly, this chapter is motivated by the need to provide sufficiency conditions for stochastic variational inequality problems, stochastic quasi-variational inequality problems, and stochastic complementarity problems. In addition, we consider variants when the variational maps are complicated by multi-valuednes. Stability statements for stochastic generalized equations have been provided by Liu, Römisch, and Xu [102].

From a computational standpoint, sampling approaches have addressed analogous stochastic optimization problems effectively [103, 10], but have focused relatively less on variational problems. In the latter context, there have been two distinct threads of research effort. Of these, the first employs sample-average approximation schemes [10]. In such an approach, the expectation is replaced by a sample-mean and the effort is on the asymptotic and rate analysis of the resulting estimators, which are obtainable by solving a deterministic variational inequality problem (cf. [12, 13, 14, 15]). A rather different track is adopted by Jiang and Xu [16] where a stochastic approximation scheme is developed for solving such stochastic variational inequality problems. Two regularized counterparts were presented by Koshal, Nedić, and Shanbhag [17, 18] where two distinct regularization schemes were overlaid on a standard stochastic approximation scheme (a Tikhonov regularization and a proximal-point scheme), both of which allow for almost-sure convergence. Importantly, this work also presents distributed schemes that can be implemented in networked regimes. A key shortcomings of standard stochastic approximation schemes is the relatively ad-hoc nature of the choice of steplength sequences. In [19], Yousefian, Nedić, and Shanbhag develop distributed stochastic approximation schemes where users can independently choose a steplength rule. Importantly, these rules collectively allow for minimizing a suitably defined error bound and are equipped with almost-sure convergence guarantees.

Finally, Wang et al. [104] focus on developing a sample-average approximation method for expected-value formulations of the stochastic variational inequality problems while Lu and Budhiraja [105] examine the confidence statements associated with estimators associated with a sample-average approximation scheme for stochastic variational inequality problem, again with expectation-based maps.

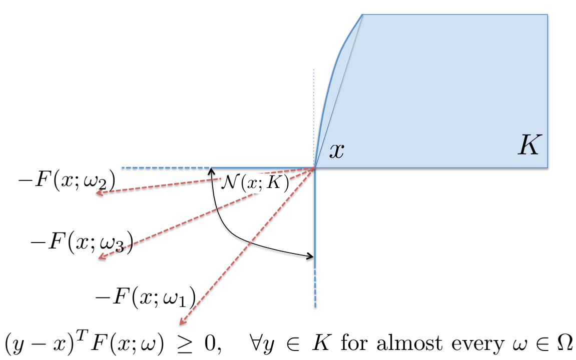


Figure 3.2: The stochastic variational inequality problem SVI$(K,F)$: The almost-sure formulation

**The almost-sure formulation** While there are many problem settings where expected-value formulations are appropriate (such as when modeling risk-neutral decision-making in a competitive regime) but there are also instances where the expected-value formulation proves inappropriate. A case in point arises when attempting to obtain solutions to a variational inequality problem that are robust to parametric uncertainty; such problems might arise when faced with traffic equilibrium or structural design problems. In this setting, the *almost-sure* formulation of the stochastic variational inequality (see Figure 3.2) is more natural.

Given a random mapping $F$, the almost-sure formulation of the stochastic variational inequality problem requires a (deterministic) vector $x \in K$ such that for almost every $\omega \in \Omega$,

$$(y - x)^T F(x; \omega) \geq 0, \quad \forall\, y \in K. \tag{3.1}$$

Naturally, if $K$ is an $n-$dimensional cone, then (3.1) reduces to CP$(K, F)$, a problem that requires an $x$ such that for almost all $\omega \in \Omega$,

$$K \ni x \perp F(x; \omega) \in K^*. \tag{3.2}$$

A natural question is how one may relate solutions of the almost sure formulation of the SVI to that of the expected value formulation. The following result provided without a proof clarifies the relationship.

**Proposition 3.1** Suppose there exists a single $x \in K$ such that $x$ solves VI$(K, F(.; \omega))$ (CP$(K, F(.; \omega))$) for almost every $\omega \in \Omega$. Then $x \in K$ is a solution of VI$(K, F)$ (CP$(K, F)$).

Yet, we believe that obtaining such an $x$ is possible only in pathological settings and this condition is relatively useless in deriving solvability statements for SVI$(K, F)$ and its variants. Furthermore, this is a **stronger condition** than the ones we develop as evidenced by Example 3.1. In this instance, there is no such $x$ that solves VI$(K, F(.; \omega))$ for every $\omega$ but the problem is indeed solvable.

If $K \triangleq \mathbb{R}^n_+$, this problem is a nonlinear complementarity problem (NCP) and for a fixed but arbitrary realization $\omega \in \Omega$, the residual of this system can be minimized as follows:

$$\underset{x \geq 0}{\text{minimize}} \; \| \mathbf{\Phi}(x; \omega) \|,$$

where $\mathbf{\Phi}(x; \omega)$ denotes the equation reformulation of the NCP (See [37]). Consequently, a solution to the almost-sure formulation is obtainable by considering the following minimization problem:

$$\underset{x\geq 0}{\text{minimize}}\ \mathbb{E}\left[\|\,\mathbf{\Phi}(x;\omega)\|\right]. \tag{3.3}$$

More precisely, if $x$ is a solution of the almost-sure formulation of NCP $(K, F)$ if and only if $x$ is a minimizer of (3.3) with $\mathbb{E}\left[\|\,\mathbf{\Phi}(x;\omega)\|\right] = 0$. If the Fischer-Burmeister $\phi_{\rm FB}$ is chosen as the C-function, the *expected residual minimization* (ERM) problem in [106, 107] solves the following stochastic program to compute a solution of the stochastic NCP (3.2):

$$\begin{aligned} &\underset{x\geq 0}{\text{minimize}} && \mathbb{E}\left[\|\,\Phi_{\rm FB}(x;\omega)\,\|\right], \\ &\text{where} && \Phi_{\rm FB}(x;\omega) \triangleq \left(\sqrt{x_i^2 + F_i(x;\omega)^2} - (\,x_i + F_i(x;\omega)\,)\right)_{i=1}^{n}; \end{aligned} \tag{3.4}$$

see [96, equation (3.8)]. In [108], Luo and Lin consider the **almost-sure formulation** of a stochastic complementarity problem and minimize the expected residual. Convergence analysis of the expected residual minimization (ERM) technique has been carried out in the context of stochastic Nash games [86] and stochastic variational inequality problems [21]. In more recent work, Chen, Wets, and Zhang [20] revisit this problem and present an alternate ERM formulation, with the intent of developing a smoothed sample average approximation scheme. In contrast with the expected-value formulation and the almost-sure formulation, Gwinner and Raciti [109, 88] consider an infinite-dimensional formulation of the variational inequality for capturing randomness and provide discretization-based approximation procedures for such problems.

The remainder of the chapter is organized as follows. In section 3.2, we outline our assumptions used, motivate our study by considering two application instances, and provide the relevant background on integrating set-valued maps. In section 3.3, we recap our sufficiency conditions for the solvability of stochastic variational inequality problems with single and multi-valued mappings and we provide results for the stochastic quasi-variational inequality problems with single and multi-valued mappings. Refinements of the statements for SVIs are provided for the complementarity regime in Section 3.4 under varying assumptions on the map. Finally, in section 3.5, we revisit the motivating examples of section 3.2 and verify that the results developed in this chapter are indeed applicable.

## 3.2 Assumptions, examples, and background

In Section 3.2.1, we provide a summary of the main assumptions employed in the chapter. The utility of such models is demonstrated by discussing some motivating examples in Section 3.2.2. Finally, some background is provided on the integrals of set-valued maps in Section 3.2.3.

### 3.2.1 Assumptions

We now state the main assumptions used throughout the chapter and refer to them when appropriate. The first of these pertains to the probability space.

**Assumption 3.1 (Nonatomicity of measure $\mathbb{P}$)** The probability space $\mathcal{P} = (\Omega, \mathcal{F}, \mathbb{P})$ is nonatomic.

The next assumption focuses on the properties of the single-valued map, referred to as $F$.

**Assumption 3.2 (Continuity and integrability of $F$)**

(i) $F : \mathbb{R}^n \times \Omega \to \mathbb{R}^n$ is a single-valued map. Furthermore, $F(x;\omega)$ is continuous in $x$ for almost every $\omega \in \Omega$ and is integrable in $\omega$, for every $x$.

(ii) $F(x)$ is continuous in $x$.

Note that, the assumption of Lipschitz continuity of $F(x;\omega)$ with an integrable Lipschitz constant implies that $\mathbb{E}[F(x;\omega)]$ is also Lipschitz continuous. The next two assumptions pertain to the set-valued maps employed in this chapter. When the map is multi-valued, to avoid confusion, we employ the notation $\Phi(x)$, defined as $\Phi(x) \triangleq \mathbb{E}[\Phi(x;\omega)]$, and impose the following assumptions.

**Assumption 3.3 (Lower semicontinuity and integrability of $\Phi$)** $\Phi : \mathbb{R}^n \times \Omega \to 2^{\mathbb{R}^n}$ is a set-valued map satisfying the following:

(i) $\Phi(x;\omega)$ has nonempty and closed images for every $x$ and every $\omega \in \Omega$.

(ii) $\Phi(x;\omega)$ is lower semicontinuous in $x$ for almost all $\omega \in \Omega$ and integrably bounded for every $x$.

Finally, when considering quasi-variational inequality problems, $K$ is a set-valued map, rather than a constant map.

**Assumption 3.4** The set-valued map $K : \mathbb{R}^n \to 2^{\mathbb{R}^n}$ is deterministic, closed-valued, convex-valued.

We conclude this subsection with some notation. $\mathrm{cl}(U)$ denotes the closure of a set $U \subset \mathbb{R}^n$, $\mathrm{bd}(U)$ denotes the boundary of the set $U$ and $\mathrm{dom}(K)$ denotes the domain of the mapping $K$.

### 3.2.2 Examples

We now provide two instances of where stochastic variational problems arise in practice.

**Nonsmooth stochastic Nash-Cournot equilibrium problems:**

Cournot's oligopolistic model is amongst the most widely used models for modeling strategic interactions between a set of noncooperative firms [110]. Under an assumption that firms produce a homogeneous product, each firm attempts to maximize profits by making a quantity decision while taking as given, the quantity of its rivals. Under the Cournot assumption, the price of the good is assumed to be dependent on the aggregate output in the market. The resulting Nash equilibrium, qualified as the Nash-Cournot equilibrium, represents a set of quantity decisions at which no firm can increase profit through a unilateral change in quantity decisions. unilaterally changing the quantity of the product it produces. To accommodate uncertainty in costs and prices in Nash-Cournot models and loss of differentiability of price functions which can occur for example by introduction of price caps [35] we consider a stochastic generalization of the classical deterministic Nash-Cournot model and allow for piecewise smooth price functions, as captured by the following assumption on costs and prices.

**Assumption 3.5** Suppose the cost function $c_i(x_i)$ is an increasing convex twice continuously differentiable function for all $i = 1, \ldots, N$. Let $X \triangleq \sum_{i=1}^n x_i$. Since $x_i$ denotes the quantity produced, $x_i \geq 0$. The price function

$p(X;\omega)$ is a decreasing piecewise smooth convex function where $p(X;\omega)$ is given by

$$p(X;\omega) = \begin{cases} p^1(X;\omega), & 0 \leq X \leq \beta^1 \\ p^j(X;\omega), & \beta^{j-1} \leq X \leq \beta^j, j = 2, \ldots, s \\ p^s(X;\omega), & \beta^s \leq X \end{cases} \tag{3.5}$$

where $p^j(X;\omega) = a^j(\omega) - b^j(\omega)X$ is a strictly decreasing affine function of $X$ for $j = 1, \ldots, s$. Finally, $\beta^1, \ldots, \beta^s$ are a set of increasing positive scalars and $(a^j(\omega), b^j(\omega))$ are positive in an almost-sure sense and integrable for $j = 1, \ldots, s$.

Consider an $N-$player generalized Nash-Cournot game. Given the tuple of rival strategies $x_{-i}$, the $i$th player's strategy set is given by $K_i(x_{-i})$ while his objective function is given by $\mathbb{E}[f_i(x;\omega)] \triangleq c_i(x_i) - \mathbb{E}[p(x;\omega)x_i]$. Then $\{x_i^*\}_{i=1}^N$ denotes a stochastic Nash-Cournot equilibrium if $x_i^*$ solves the convex optimization problem $\mathrm{G}_i(x_{-i}^*)$, defined as

$$\begin{aligned} \underset{x_i}{\text{minimize}} \quad & \mathbb{E}[f_i(x;\omega)] \\ \text{subject to} \quad & x_i \in K_i(x_{-i}). \end{aligned}$$

The equilibrium conditions of this problem are given a stochastic QVI with multi-valued mappings. In section 3.5, we revisit this problem with the intent of developing existence statements.

**Strategic behavior in power markets:** Consider a power market model in which a collection of generation firms compete in a single-settlement market. Economic equilibria in power markets has been extensively studied using a complementarity framework; see [35, 111, 112]. Our model below is based on the model of Hobbs and Pang [35] which we modify to account for uncertainty in prices and costs.

Consider a set of nodes $\mathcal{N}$ of a network. The set of generation firms is indexed by $f$, where $f$ belongs to the finite set $\mathcal{F}$. At a node $i$ in the network, a firm $f$ may generate $g_{fi}$ units at node $i$ and sell $s_{fi}$ units to node $i$. The total amount of power sold to node $i$ by all generating firms is $S_i$. The generator firms' profits are revenue less costs. If the nodal power price at node $i$th is a random function given by $p_i(S_i;\omega)$ where $p_i(S_i;\omega)$ is a decreasing function of $S_i$, then the firms' revenue is just the price times sales $s_{fi}$. The costs incurred by the firm $f$ at node $i$ are the costs of generating $g_{fi}$ and transmitting the excess $(s_{fi} - g_{fi})$. Let the cost of generation associated with firm $f$ at node $i$ be given by $c_{fi}(g_{fi};\omega)$ and the cost of transmitting power from an arbitrary node (referred to as the hub) to node $i$ be given by $w_i$. The constraint set ensures a balance between sales and generation at all nodes, nonnegative sales and generation and generation subject to a capacity limit. The price and cost functions are assumed to satisfy the following requirement.

**Assumption 3.6** For $i \in \mathcal{N}$, the price functions $p_i(S_i;\omega)$ is a decreasing function, bounded above by a nonnegative integrable function $\bar{p}_i(\omega)$. Furthermore, the cost functions $c_{fi}(g_{fi};\omega)$ are nonnegative and increasing.

The resulting problem faced by the $f$th generating firm $f$ is to determine sales $s_{fi}$ and generation $g_{fi}$ at all nodes $i$ such that

$$\begin{array}{ll}
\underset{s_{fi},\,g_{fi}}{\text{maximize}} & \mathbb{E}\left[\sum_{i\in\mathcal{N}} \left( p_i(S_i;\omega)s_{fi} - c_{fi}(g_{fi};\omega) - (s_{fi} - g_{fi})w_i \right)\right] \\
\text{subject to} & \left\{ \begin{array}{lcl} 0 & \leq & g_{fi} \leq \text{cap}_{fi} \\ 0 & \leq & s_{fi} \end{array} \right\}, \quad \forall\, i \in \mathcal{N} \\
\text{and} & \sum_{i\in\mathcal{N}} \left( s_{fi} - g_{fi} \right) \; = \; 0.
\end{array}$$

Note that, the generating firm sees the transmission fee $w_i$ and the rival firms' sales $s_{-fi} \equiv \{s_{hi} \;:\; h \neq f\}$ as exogenous parameters to its optimization problem even though they are endogenous to the overall equilibrium model as we will see shortly.

The ISO sees the transmission fees $w = (w_i)_{i\in\mathcal{N}}$ as exogenous and prescribes flows $y = (y_i)_{i\in\mathcal{N}}$ as per the following linear program

$$\begin{array}{ll}
\text{maximize} & \sum_{i\in\mathcal{N}} y_i w_i \\
\text{subject to} & \sum_{i\in\mathcal{N}} \text{PDF}_{ij} y_i \;\leq\; T_j, \qquad \forall\, j \in \mathcal{K},
\end{array}$$

where $\mathcal{K}$ is the set of all arcs or links in the network with node set $\mathcal{N}$, $T_j$

denotes the transmission capacity of link $j$, $y_i$ represents the transfer of power (in MW) by the system operator from a hub node to node node $i$ and $\text{PDF}_{ij}$ denotes the power transfer distribution factor, which specifies the MW flow through link $j$ as a consequence of unit MW injection at an arbitrary hub node and a unit withdrawal at node $i$.

Finally, to clear the market, the transmission flows $y_i$ must must balance the net sales at each node:

$$y_i = \sum_{f \in \mathcal{F}} ( s_{fi} - g_{fi} ), \qquad \forall\, i \in \mathcal{N}.$$

The above market equilibrium problem which comprises of each firm's problem, the ISO's problem and market-clearing condition, can be expressed as a stochastic complementarity problem by following the technique from [35]. This equivalent formulation of the above market equilibrium problem is described in the last section 3.5. We also illustrate the solvability of such problems using the framework developed in this chapter.

### 3.2.3 Background on integrals of set-valued mappings

Recall that by Assumption 3.1, $(\Omega, \mathcal{F}, \mathbb{P})$ is a nonatomic continuous probability space. Consider a set-valued map $H$ that maps from $\Omega$ into nonempty, closed subsets of $\mathbb{R}^n$ . We recall three definitions from [31, Ch. 8]

**Definition 3.3 (Measurable set-valued map)** A map $H$ is measurable if the inverse image of each open set in $\mathbb{R}^n$ is a measurable set: for all open sets $O \subseteq \mathbb{R}^n$, we have

$$H^{-1}(O) = \{\omega \in \Omega \mid H(\omega) \cap O \neq \emptyset\} \in \mathcal{F}.$$

**Definition 3.4 (Integrably bounded set-valued map)** A map $H$ is integrably bounded if there exists a nonnegative integrable function $k \in L^1(\Omega, \mathbb{R}, \mathbb{P})$ such that

$$H(\omega) \subseteq k(w)B(0,1) \text{ almost everywhere.}$$

**Definition 3.5 (Measurable selection)** Suppose a measurable map $h : \Omega \to \mathbb{R}^n$ satisfies $h(\omega) \in H(\omega)$ for almost all $\omega \in \Omega$. Then $h$ is called a measurable selection of $H$.

The existence of a measurable selection is proved in [31, Th. 8.1.3].

**Definition 3.6 (Integrable selection)** A measurable selection $h : \Omega \to \mathbb{R}^n$ is an integrable selection if $\mathbb{E}[h(\omega)] < \infty$ where

$$\mathbb{E}[h(\omega)] = \int_\Omega h d\mathbb{P} < \infty.$$

The set of all integrable selections of $H$ is denoted by $\mathcal{H}$ and is defined as follows:

$$\mathcal{H} \triangleq \left\{ h \in L^1(\Omega, \mathcal{F}, \mathbb{P}) : h(\omega) \in H(\omega) \text{ for almost all } \omega \in \Omega \right\},$$

**Definition 3.7 (Expectation of a set-valued map)** The expectation of the set-valued map $H$, denoted by $\mathbb{E}[H(\omega)]$, is the set of integrals of integrable selections of $H$:

$$\mathbb{E}[H(\omega)] \triangleq \int_\Omega H d\mathbb{P} \triangleq \left\{ \int_\Omega h d\mathbb{P} \mid h \in \mathcal{H} \right\}.$$

If the images of $H(\omega)$ are convex then this set-valued integral is convex [31, Definition 8.6.1]. If the assumption of convexity of images of $H$ does not hold, then the convexity of this integral follows from Th. 8.6.3 [31], provided that the probability measure is nonatomic. We make precisely such an assumption (See Assumption 3.1) and are therefore guaranteed that the integral of the set-valued map $H$ is a convex set [31, Th. 8.6.3] .

Recall that, a point $\bar{z}$ of a convex set $K$ is said to be extremal if there are no two points $x, y \in K$ such that $\lambda x + (1 - \lambda) y = \bar{z}$ for $\lambda \in (0, 1)$ and is denoted by $\bar{z} \in \text{ext}(K)$. Similarly, as per Def. 8.6.5 [31], we define an extremal selection as follows:

**Definition 3.8 (Extremal selection)** Given the convex set $\int_\Omega H d\mathbb{P}$, an integrable selection $h \in \mathcal{H}$ is an extremal selection of $H$ if

$$\int_\Omega h d\mathbb{P} \text{ is an extremal point of the closure of the convex set } \int_\Omega H d\mathbb{P}.$$

The set of all extremal selections is denoted by $\mathcal{H}_e$ and is defined as follows:

$$\mathcal{H}_e \triangleq \left\{ h \in \mathcal{H} \mid \int_\Omega h d\mathbb{P} \in \text{ext}\left( \text{cl}\left( \int_\Omega H d\mathbb{P} \right) \right) \right\}.$$

By Theorem [31, Th. 8.6.3], we have the following Lemma for the representation of extremal points of closure of $\int_\Omega H d\mathbb{P}$.

**Theorem 3.2 (Representation of extreme points of set-valued integral)** Suppose Assumption 3.1 holds and let $H$ be a measurable set-valued map from $\Omega$ to subsets of $\mathbb{R}^n$ with nonempty closed images. Then the following hold:

(a) $\int_\Omega H d\mathbb{P}$ is convex and extremal points of $\mathrm{cl}(\int_\Omega H d\mathbb{P})$ are contained in $\int_\Omega H d\mathbb{P}$.

(b) If $x \in \mathrm{ext}\left(\mathrm{cl}\left(\int_\Omega H d\mathbb{P}\right)\right)$, then there exists a unique $h \in \mathcal{H}_e$ with $x = \int_\Omega h d\mathbb{P}$.

(c) If $H$ is integrably bounded, then the integral $\int_\Omega H d\mathbb{P}$ is also compact.

As a corollary to the above theorem, we have a representation of points in a set-valued integral from [31, Th. 8.6.6]

**Corollary 3.3 (Representation of points in a set-valued integral)** Let $H$ be a measurable integrably bounded set-valued map from $\Omega$ to subsets of $\mathbb{R}^n$ with nonempty closed images. If $\mathbb{P}$ is nonatomic, then for every $x \in \int_\Omega H d\mathbb{P}$, there exist $n+1$ extremal selections $h_k \in \mathcal{H}_e$ and $n+1$ measurable sets $A_k \in \mathcal{F}$, $k = 0, \cdots, n$, such that

$$x = \int_\Omega \left( \sum_{k=1}^{n} \chi_{A_k} h_k \right) d\mathbb{P}$$

where $\chi_{A_k}$ is the characteristic function of $A_k$.

## 3.3 Stochastic quasi-variational inequality problems

In this section, we develop sufficiency conditions for the solvability of stochastic quasi-variational inequality problems under a diversity of assumptions on the map. More specifically, we begin by recapping sufficiency conditions for the solvability of stochastic variational inequality problems with single-valued and multi-valued maps in Section 3.3.1. In many settings, the variational inequality problems may prove incapable of capturing the problem in question.

For instance, the equilibrium conditions of convex generalized Nash games are given by a quasi-variational inequality problem. As mentioned earlier, when the constant map $K$ is replaced by a set-valued map $K : \mathbb{R}^n \to \mathbb{R}^n$, the resulting problem is an SQVI. In this section, we extend the sufficiency conditions presented in the earlier section to accommodate the SQVI$(K, F)$ (Section 3.3.2) and SQVI$(K, \Phi)$ (Section 3.3.3), respectively. Throughout this section, Assumption 3.4 holds for the set-valued map $K$.

### 3.3.1 SVIs with single-valued and multi-valued mappings

In this section, we begin by assuming that the scenario-based mappings $F(\bullet;\omega)$ are single-valued for each $\omega \in \Omega$. With this assumption, we provide sufficient conditions that the scenario-based VI$(K, F(\bullet;\omega))$ must satisfy in order to conclude the existence of solution to the stochastic SVI$(K, F)$ without requiring the evaluation of expectation operator. Recall that in SVI$(K, F)$, $F(x) = \mathbb{E}[F(x;\omega)]$. In particular, in the next proposition, we show that if a certain coercivity condition holds for the scenario-based map $F(\bullet;\omega)$ in an almost-sure sense then existence of the solution to the above SVI may be concluded without resorting to formal evaluation of the expectation.

**Proposition 3.4 (Solvability of SVI$(K, F)$)** Consider a stochastic variational inequality SVI$(K, F)$. Suppose Assumption 3.2 holds and $G(x;\omega) \triangleq F(x;\omega)^T(x - x^{\text{ref}})$. Suppose there exists an $x^{\text{ref}} \in K$ such that the following hold:

(i) $\liminf_{\|x\|\to\infty, x\in K} \left[F(x;\omega)^T(x - x^{\text{ref}})\right] > 0$ almost surely;

(ii) Suppose there exists a nonnegative integrable function $u(\omega)$ such that $G(x;\omega) \geq -u(\omega)$ holds almost surely for any $x$.

Then the stochastic variational inequality SVI$(K, F)$ has a solution.

**Proof :** Recall from [37, Ch. 2] that the solvability of SVI$(K, F)$ requires showing that there exists an $x^{\text{ref}}$ such that

$$\liminf_{\|x\|\to\infty, x\in K} \left[F(x)^T(x - x^{\text{ref}})\right] > 0. \tag{3.6}$$

But we have that

$$\liminf_{\|x\|\to\infty, x\in K} \left[F(x)^T(x - x^{\text{ref}})\right] = \liminf_{\|x\|\to\infty, x\in K} \left[\int_\Omega F(x;\omega)^T(x - x^{\text{ref}})d\mathbb{P}\right].$$

By hypothesis (ii), we may apply Fatou's lemma to obtain the following inequality:

$$\liminf_{\|x\|\to\infty, x\in K} \left[F(x)^T(x - x^{\text{ref}})\right] \geq \int_\Omega \liminf_{\|x\|\to\infty, x\in K} \left[F(x;\omega)^T(x - x^{\text{ref}})\right] d\mathbb{P} > 0,$$

where the last inequality follows from the given hypothesis. Thus (3.6) holds and therefore SVI$(K, F)$ has a solution. ∎

In settings where $K$ is a Cartesian product, defined as

$$K \triangleq \prod_{\nu=1}^{N} K_\nu, \tag{3.7}$$

VI$(K, F)$ is a partitioned variational inequality probem, as defined in [37, Ch. 3.5]. Accordingly, Proposition 3.4 can be weakened so that even if the coercivity property holds for just one index $\nu \in \{1, \ldots, N\}$, the stochastic variational inequality is solvable.

**Proposition 3.5 (Solvability of SVI($K, F$) for Cartesian $K$)** Consider a stochastic variational inequality SVI$(K, F)$ where $K$ is a Cartesian product of closed and convex sets as specified in (3.7). Suppose that Assumption 3.2 and the following hold:

(i) There exists an $x^{\text{ref}} \in K$ and an index $\nu \in \{1, \ldots, N\}$ such that for any $x \in K$,

$$\liminf_{\|x_\nu\|\to\infty, x_\nu\in K_\nu} \left[F_\nu(x;\omega)^T(x_\nu - x_\nu^{\text{ref}})\right] > 0,$$

holds in an almost sure sense; and

(ii) For the above $\nu$ and for any $x$, let $G(x;\omega) = F_\nu(x;\omega)^T(x_\nu - x_\nu^{\text{ref}})$. Suppose there exists a nonnegative integrable function $u(\omega)$ such that $G(x;\omega) \geq -u(\omega)$ holds almost surely for any $x$.

Then SVI$(K, F)$ admits a solution.

**Proof :** For the given $x^{\text{ref}} \in K$ and for any $x \in K$, there exists a $\nu \in \{1, \ldots, N\}$, such that

$$\liminf_{\|x_\nu\| \to \infty, x_\nu \in K_\nu} \left[ F_\nu(x;\omega)^T (x_\nu - x_\nu^{\text{ref}}) \right] > 0$$

holds almost surely. Thus we obtain

$$\mathbb{E}\left[ \liminf_{\|x_\nu\| \to \infty, x_\nu \in K_\nu} F_\nu(x;\omega)^T (x_\nu - x_\nu^{\text{ref}}) \right] > 0.$$

By hypothesis (ii) above we may apply Fatou's lemma to get

$$\liminf_{\|x_\nu\| \to \infty, x_\nu \in K_\nu} \mathbb{E}\left[ F_\nu(x;\omega)^T (x_\nu - x_\nu^{\text{ref}}) \right] > 0.$$

This implies that $C_{\leq}$ is bounded where

$$C_{\leq} := \left\{ x \in K : \max_{1 \leq \nu \leq N} \mathbb{E}\left[ F_\nu(x;\omega)^T (x_\nu - x_\nu^{\text{ref}}) \right] \leq 0 \right\}.$$

From [37, Prop. 3.5.1], boundedness of $C_{\leq}$ allows us to conclude that SVI$(K, F)$ is solvable. ∎

We now present a weaker set of sufficient conditions for existence under the assumption that the mapping $F(x;\omega)$ is a monotone mapping over $K$ for almost every $\omega \in \Omega$.

**Corollary 3.6 (Solvability of SVI$(K, F)$ under monotonicity)** Consider SVI$(K, F)$ and suppose that Assumption 3.2 holds. Further, let $F(x;\omega)$ be a monotone mapping on $K$ for almost every $\omega \in \Omega$. Suppose there exists an $x^{\text{ref}} \in K$ such that $G(x;\omega) = F(x^{\text{ref}};\omega)^T (x - x^{\text{ref}})$ and the following hold:

(i) $\liminf_{\|x\| \to \infty, x \in K} \left[ F(x^{\text{ref}};\omega)^T (x - x^{\text{ref}}) \right] > 0$ holds in almost sure sense;

(ii) Suppose there exists a nonnegative integrable function $u(\omega)$ such that $G(x;\omega) \geq -u(\omega)$ holds almost surely for any $x$.

Then SVI$(K, F)$ is solvable.

**Proof :** We begin with the observation that the monotonicity of $F(x;\omega)$ allows us to bound $F(x;\omega)^T(x - x^{\rm ref})$ from below as follows:

$$\begin{aligned} F(x;\omega)^T(x - x^{\rm ref}) &= \left[F(x;\omega) - F(x^{\rm ref};\omega)\right]^T (x - x^{\rm ref}) + F(x^{\rm ref};\omega)^T(x - x^{\rm ref}) \\ &\geq F(x^{\rm ref};\omega)^T(x - x^{\rm ref}). \end{aligned}$$

Taking expectations on both sides gives us

$$\mathbb{E}\left[F(x;\omega)^T(x - x^{\rm ref})\right] \geq \mathbb{E}\left[F(x^{\rm ref};\omega)^T(x - x^{\rm ref})\right].$$

This implies that

$$\liminf_{\|x\|\to\infty, x\in K} \mathbb{E}\left[F(x;\omega)^T(x - x^{\rm ref})\right] \geq \liminf_{\|x\|\to\infty, x\in K} \mathbb{E}\left[F(x^{\rm ref};\omega)^T(x - x^{\rm ref})\right]. \tag{3.8}$$

By hypothesis (ii) above, Fatou's Lemma can be employed in the last inequality to interchange limits and expectations leading to

$$\liminf_{\|x\|\to\infty, x\in K} \mathbb{E}\left[F(x^{\rm ref};\omega)^T(x - x^{\rm ref})\right] \geq \mathbb{E}\left[\liminf_{\|x\|\to\infty, x\in K} F(x^{\rm ref};\omega)^T(x - x^{\rm ref})\right].$$

But by assumption, we have that

$$\liminf_{\|x\|\to\infty, x\in K} \left[F(x^{\rm ref};\omega)^T(x - x^{\rm ref})\right] > 0$$

holds in almost sure sense, implying that from (3.8), we have that

$$\liminf_{\|x\|\to\infty, x\in K} \mathbb{E}\left[F(x;\omega)^T(x - x^{\rm ref})\right] > 0.$$

Now, Proposition 3.4 allows us to conclude the solvability of the stochastic variational inequality SVI$(K, F)$ ∎

Next, we consider a stochastic variational inequality SVI$(K, \Phi)$ where $\Phi(x) \triangleq \mathbb{E}[\Phi(x;\omega)]$ and $\Phi(x;\omega)$ is a multi-valued mapping. Before proceeding to prove existence of solutions to SVIs with multi-valued maps, we restate Corollary 3.3 for the case of the set-valued integral $\Phi(x) = \mathbb{E}[\Phi(x;\omega)]$ of interest. This lemma helps represent elements of set-valued integral an integral of convex combination of extremal selections

**Lemma 3.7 (Representation of elements of set-valued integral)** Suppose Assumption 3.1 holds. Let $\Phi$ be a measurable integrably bounded set-valued map from $\mathbb{R}^n \times \Omega$ to subsets of $\mathbb{R}^n$ with closed nonempty images. Then any $w \in \mathbb{E}[\Phi(x;\omega)]$ can be expressed as

$$w = \int_{\Omega} g(x;\omega) d\mathbb{P}$$

where $g(x;\omega) = \sum_{k=1}^{n} \lambda_k(x) f_k(x;\omega)$ and $\lambda_k(x) \geq 0, \sum_{k=0}^{n} \lambda_k(x) = 1$ and each $f_k(x;\omega)$ is an extremal selection of $\Phi(x;\omega)$

**Proof :** Since $w \in \mathbb{E}[\Phi(x;\omega)]$ and $\mathbb{E}[\Phi(x;\omega)]$ is a convex set, thus by Carathéodory's theorem for convex sets, there exists $\lambda_k(x) \geq 0$, $w_k \in \text{ext}\,(\text{cl}\,(\mathbb{E}[\Phi(x;\omega)]))$ such that

$$w = \sum_{k=0}^{n} \lambda_k(x) w_k, \quad \sum_{k=0}^{n} \lambda_k(x) = 1$$

Now, since $w_k \in \text{ext}\,(\text{cl}\,(\mathbb{E}[\Phi(x;\omega)]))$, by [31, Th. 8.6.3], for each index $k$, there exists an extremal selection $f_k(x;\omega)$ from $\Phi(x;\omega)$ such that $\int_{\Omega} f_k(x;\omega) d\mathbb{P} = w_k$. Thus, we obtain

$$w = \sum_{k=0}^{n} \lambda_k(x) \int_{\Omega} f_k(x;\omega) d\mathbb{P},$$

which can be rewritten as

$$w = \int_{\Omega} g(x;\omega) d\mathbb{P}$$

where $g(x;\omega) = \sum_{k=0}^{n} \lambda_k(x) f_k(x;\omega)$. The required representation result follows. ∎

We begin by providing a coercivity-based sufficiency condition for deterministic multi-valued variational inequalities [113].

**Theorem 3.8 (Solvability of VI $(K, \Phi)$)** Suppose $K$ is a closed and convex set in $\mathbb{R}^n$ and let $\Phi : K \rightrightarrows \mathbb{R}^n$ be a lower semicontinuous multifunction with nonempty closed and convex images. Consider the following statements:

(a) Suppose there exists an $x^{\text{ref}} \in K$ such that $L_{<}(K, \Phi)$ is bounded (possibly empty) where

$$L_<(K,\Phi) \triangleq \left\{ x \in K : \inf_{y\in\Phi(x)} (x - x^{\text{ref}})^T y < 0 \right\}.$$

(b) The variational inequality $\text{VI}(K,\Phi)$ is solvable

Then, (a) implies (b). Furthermore, if $\Phi(x)$ is a pseudomonotone mapping over $K$, then (a) is equivalent to (b).

Using this condition, we proceed to develop sufficiency conditions for the existence of solutions to $\text{SVI}(K,\Phi)$.

**Proposition 3.9 (Solvability of SVI$(K,\Phi)$)** Consider $\text{SVI}(K,\Phi)$ and suppose assumptions 3.1 and 3.3 hold. Further, suppose the following hold:

(i) Suppose there exists an $x^{\text{ref}} \in K$ such that

$$\liminf_{x\in K, \|x\|\to\infty} \left( \inf_{w\in\Phi(x;\omega)} w^T(x - x^{\text{ref}}) \right) > 0 \text{ almost surely.}$$

(ii) For the above $x^{\text{ref}}$, suppose there exists a nonnegative integrable function $U(\omega)$ such that $g(x;\omega)^T(x - x^{\text{ref}}) \geq -U(\omega)$ holds almost surely for any integrable selection $g(x;\omega)$ of $\Phi(x;\omega)$ and for any $x$.

Then $\text{SVI}(K,\Phi)$ is solvable.

**Proof :** The proof proceeds in two parts.

(a) We first show that the following coercivity condition holds for the expected value map: there exists an $x^{\text{ref}} \in K$ such that

$$\liminf_{x\in K, \|x\|\to\infty} \left( \inf_{w\in\Phi(x)} w^T(x - x^{\text{ref}}) \right) > 0. \tag{3.9}$$

(b) If (a) holds, then we show that for the given $x^{\text{ref}} \in K$, the set $L_<(K,\Phi)$ is bounded (possibly empty) where

$$L_<(K,\Phi) \triangleq \left\{ x \in K : \inf_{y\in\Phi(x)} (x - x^{\text{ref}})^T y < 0 \right\}. \tag{3.10}$$

**Proof of (a):** We proceed by a contradiction and assume that (3.9) does not hold for the expected value map. Thus, for any $x^{\text{ref}}$, there exists a sequence $x_k \in K$ with $\|x_k\| \to \infty$ such that

$$\liminf_{k\to\infty} \left( \inf_{w\in\Phi(x_k)} w^T(x_k - x^{\text{ref}}) \right) \le 0.$$

Since $\Phi(x_k)$ is a closed set, the infimum above is attained at $y_k \in \Phi(x_k)$. Thus, we have

$$\liminf_{k\to\infty} \left[ y_k^T(x_k - x^{\text{ref}}) \right] \le 0 \tag{3.11}$$

Now, $y_k \in \Phi(x_k) = \mathbb{E}[\Phi(x_k;\omega)]$. By the representation Lemma (Lemma 3.7), since $y_k \in \Phi(x_k)$, we have that

$$y_k = \int_\Omega g_k(x_k;\omega) d\mathbb{P}$$

for some $g_k(x_k;\omega) = \sum_{l=1}^n \lambda_l(x_k) f_l(x_k;\omega)$ where $\lambda_l(x_k) \ge 0, \sum_{l=0}^n \lambda_l(x) = 1$ and each $f_l(x_k;\omega)$ is an extremal selection of $\Phi(x_k;\omega)$. With this substitution, (3.11) becomes

$$\liminf_{k\to\infty} \left[ \int_\Omega g_k(x_k;\omega)^T(x_k - x^{\text{ref}}) d\mathbb{P} \right] \le 0.$$

By hypothesis (ii), we may use Fatou's Lemma to interchange the order of integration and limit infimum, as shown next:

$$\int_\Omega \liminf_{k\to\infty} \left[ \ g_k(x_k;\omega)^T(x_k - x^{\text{ref}}) \right] d\mathbb{P} \le 0.$$

Consequently, there is a set of positive measure $U \subseteq \Omega$, over which the integrand is nonpositive or

$$\liminf_{k\to\infty} \ \left[ g_k(x_k;\omega)^T(x_k - x^{\text{ref}}) \right] \le 0, \quad \forall \omega \in U.$$

Substituting the expression for $g_k$, we obtain the following inequality.

$$\liminf_{k\to\infty} \ \left[ (x_k - x^{\text{ref}})^T \left( \sum_{l=1}^n \lambda_l(x_k) f_l(x_k;\omega) \right) \right] \le 0, \quad \forall \omega \in U.$$

As a result, for at least one index $l \in \{1, ..., n\}$, we have that

$$\liminf_{k\to\infty} \left[\lambda_l(x_k) f_l(x_k;\omega)^T(x_k - x^{\text{ref}})\right] \leq 0, \quad \forall \omega \in U.$$

Since $0 \leq \lambda_l(x_k) \leq 1$, the following must be true for the above $l$:

$$\liminf_{k\to\infty} \left[f_l(x_k;\omega)^T(x_k - x^{\text{ref}})\right] \leq 0, \quad \forall \omega \in U.$$

Moreover, $f_l(x_k;\omega) \in \Phi(x_k;\omega)$ since it is an extremal selection and we have that

$$\liminf_{k\to\infty} \left[\inf_{w\in\Phi(x_k;\omega)} w^T(x_k - x^{\text{ref}})\right] \leq 0, \quad \forall \omega \in U.$$

Since, this holds for any $x^{\text{ref}}$, it holds for the vector $x^{\text{ref}}$ in the hypothesis and for a set of positive measure $U$, we have that

$$\liminf_{x\in K, \|x\|\to\infty} \left[\inf_{w\in\Phi(x;\omega)} w^T(x - x^{\text{ref}})\right] \leq 0, \forall \omega \in U.$$

This contradicts the hypothesis and condition (3.9) must hold for the expected value map.

**Proof of (b)** Next, we show that when the condition (3.9) holds for the expected value map, then for the given $x^{\text{ref}} \in K$ the set $L_<(K,\Phi)$ is bounded (possibly empty) where

$$L_<(K,\Phi) \triangleq \left\{x \in K : \inf_{y\in\Phi(x)} \left[(x - x^{\text{ref}})^T y\right] < 0\right\}.$$

If the set $L_<(K,\Phi)$ is empty, then the result follows by Theorem 3.8. Suppose $L_<(K,\Phi)$ is nonempty and unbounded. Then, there exists a sequence $\{x_k\} \in L_<(K,\Phi)$ with $\|x_k\| \to \infty$. Since $x_k \in L_<(K,\Phi)$, we have for each $k$,

$$\inf_{y\in\Phi(x_k)} \left[(x_k - x^{\text{ref}})^T y\right] < 0.$$

This implies that for the sequence $\{x_k\}$, we have that

$$\liminf_{x_k\in K, \|x_k\|\to\infty} \left[\inf_{w\in\Phi(x_k)} w^T(x_k - x^{\text{ref}})\right] \leq 0.$$

But this contradicts the coercivity property of the expected value map proved earlier:

$$\liminf_{x \in K, \|x\| \to \infty} \left[ \inf_{w \in \Phi(x)} w^T (x - x^{\text{ref}}) \right] > 0.$$

This contradiction implies that $L_<(K, \Phi)$ is bounded and by Theorem 3.8, $\text{SVI}(K, \Phi)$ is solvable. ∎

### 3.3.2 SQVIs with single-valued mappings

Our first result is an extension of [37, Cor. 2.8.4] to the stochastic regime. In particular, we assume that the mapping $\mathbb{E}[F(x;\omega)]$ cannot be directly obtained; instead, we provide an existence statement that relies on the scenario-based map $F(x;\omega)$.

**Proposition 3.10 (Solvability of SQVI$(K, F)$)** Suppose Assumptions 3.2 and 3.4 hold. Furthermore, suppose there exists a bounded open set $U \subset \mathbb{R}^n$ and a vector $x^{\text{ref}} \in U$ such that the following hold:

(a) For every $\bar{x} \in \text{cl}(U)$, the image $K(\bar{x})$ is nonempty and $\lim_{x \to \bar{x}} K(x) = K(\bar{x})$;

(b) $x^{\text{ref}} \in K(x)$ for every $x \in \text{cl}(U)$;

(c) $L_<(K, F; \omega) \cap \text{bd}(U) = \emptyset$ holds almost surely, where

$$L_<(K, F; \omega) \triangleq \left\{ x \in K(x) \mid (x - x^{\text{ref}})^T F(x;\omega) < 0 \right\}.$$

Then, SQVI$(K, F)$ has a solution.

**Proof :** Recall that by [37, Cor. 2.8.4], the stochastic SQVI$(K, F)$ is solvable if $L_<(K, F) \cap \text{bd}(U) = \emptyset$, where

$$L_<(K, F) \triangleq \left\{ x \in K(x) \mid (x - x^{\text{ref}})^T \mathbb{E}\left[F(x;\omega)\right] < 0 \right\}.$$

We proceed by contradiction and assume that there exists an $x \in L_<(K, F)$ and $x \in \text{bd}(U)$. By assumption, $x \notin L_<(K, F; \omega)$ for any $\omega$ implying that $(x - x^{\text{ref}})^T F(x;\omega) \geq 0$ for all $\omega \in \Omega$. It follows that $(x - x^{\text{ref}})^T \mathbb{E}[F(x;\omega)] \geq 0$, implying that $x \notin L_<(K, F)$. This contradicts our assertion that $x \in$

$L_<(K,F)$. Therefore, we must have that $L_<(K,F) \cap \mathrm{bd}(U) = \emptyset$ and by [37, Cor. 2.8.4], the stochastic SQVI$(K,F)$ has a solution. ∎

Another avenue for ascertaining existence of equilibrium in stochastic regimes is an extension of Harker's result [114, Th. 2] which we present next.

**Theorem 3.11 (Solvability of SQVI($K,F$) under compactness)**
Suppose Assumptions 3.2 and 3.4 hold and there exists a nonempty compact convex set $\Gamma$ such that the following hold:

(i) $K(x) \subseteq \Gamma$, $\forall x \in \Gamma$;

(ii) $K$ is a nonempty, continuous, closed and convex-valued mapping on $\Gamma$.

Then the SQVI$(K,F)$ has a solution.

**Proof :** Since $F$ is continuous by Assumption 3.2(ii), all conditions of [114, Th. 2] hold. Thus, the SQVI$(K,F)$ has a solution. ∎

The above theorem relies on properties of the map $K$ and the continuity of the map $F$ to ascertain existence of solution. By Assumption 3.2, continuity of the map $F$ holds in the settings we consider and thus the solvability of SQVI$(K,F)$ follows readily. This theorem has a slightly different flavor compared to other results in this chapter in the sense that we do not look at properties of the scenario-based map (other than continuity) that then guarantee existence of solution. We have listed this theorem here for completeness as it presents an alternate perspective of looking at the question of solvability of SQVI$(K,F)$.

### 3.3.3 SQVIs with multi-valued mappings

In this section, we relax the assumption of single-valuedness of the scenario-based mappings $F(\bullet;\omega)$ and instead allow for the map $\Phi(\bullet;\omega)$ to be multi-valued. In the spirit of the rest of this chapter, our interest lies in deriving results that do not rely on evaluation of expectation. We use the concepts of set-valued integrals discussed in the previous section 3.2.3 and require that Assumption 3.4 holds throughout this subsection. Our first existence result relies on a sufficiency condition for generalized quasi-variational inequalities [115, Cor. 3.1]. We recall [115, Cor. 3.1] which can be applied to the multi-valued SQVI$(K,\Phi)$.

**Proposition 3.12** Consider the SQVI$(K, \Phi)$. Suppose that there exists a nonempty compact convex set $C$ such that the following hold:

(a) $K(C) \subseteq C$;

(b) $\mathbb{E}[\Phi(x; \omega)]$ is a nonempty contractible-valued and compact-valued upper semicontinuous mapping on $C$;

(c) $K$ is nonempty continuous convex-valued mapping on $C$.

Then the stochastic SQVI$(K, \Phi)$ admits a solution

However, this result requires evaluating $\mathbb{E}[\Phi(x; \omega)]$, an object that admits far less tractability; instead, we develop almost-sure sufficiency conditions that imply the requirements of Proposition 3.12.

**Proposition 3.13 (Solvability of SQVI($K, \Phi$)))** Suppose Assumptions 3.4 and 3.3 hold. Furthermore, suppose there exists a nonempty compact convex set $C$ such that the following hold:

(a) $K(C) \subseteq C$;

(b) $\Phi(x; \omega)$ is a nonempty upper semicontinuous mapping for $x \in C$ in an almost-sure sense;

(c) $K$ is nonempty, continuous and convex-valued mapping on $C$.

Then the stochastic SQVI$(K, \Phi)$ admits a solution.

**Proof :** From Proposition 3.12, it suffices to show that under the above assumptions, $\mathbb{E}[\Phi(x; \omega)]$ is a nonempty contractible-valued, compact-valued, upper semicontinuous mapping on $C$.

(i) $\mathbb{E}[\Phi(x; \omega)]$ is nonempty: Since $\Phi(x; \omega)$ is lower semicontinuous, it is a measurable set-valued map. Since it is a measurable set-valued map with nonempty closed images, by Aumann's measurable selection theorem [31, Th. 8.1.3], there exists a measurable selection $h$ from $\Phi(x; \omega)$. Since $\Phi(x; \omega)$ is integrably bounded, every measurable selection is integrable. Thus, $\int_\Omega h d\mathbb{P} \in \int_\Omega \Phi(x; \omega)$, implying that $\mathbb{E}[\Phi(x; \omega)]$ is nonempty.

(ii) $\mathbb{E}[\Phi(x;\omega)]$ is contractible-valued: Since the probability space is nonatomic by definition, we have that $\mathbb{E}[\Phi(x;\omega)]$ is a convex set. Since a convex set is contractible, we have that $\mathbb{E}[\Phi(x;\omega)]$ is contractible.

(iii) $\mathbb{E}[\Phi(x;\omega)]$ is compact-valued: Since $\Phi(x;\omega)$ is integrably bounded, by [31, Th. 8.6.3], we get that $\mathbb{E}[\Phi(x;\omega)]$ is compact.

(iv) $\mathbb{E}[\Phi(x;\omega)]$ is upper semicontinuous: By hypothesis, we have that $\Phi(x;\omega)$ is a measurable, integrably bounded and upper-semicontinuous $x \in C$. Thus, from [73, Cor. 5.2], it follows that $\mathbb{E}[\Phi(x;\omega)]$ is upper semicontinuous.

∎

The previous result relies on the compact-valuedness of $K$ with respect to a compact set $C$, a property that cannot be universally guaranteed. An alternate result for deterministic generalized QVI problems [115, Cor. 4.1] leverages coercivity properties of the map $\Phi(x)$ to claim existence of a solution. We state this result next.

**Proposition 3.14** Let $K$ be a set-valued map from $\mathbb{R}^n$ to subsets of $\mathbb{R}^n$ and $\Phi$ from $\mathbb{R}^n$ to subsets of $\mathbb{R}^n$ be a measurable integrably bounded set-valued map with closed nonempty images. Suppose that there exists a vector $x^{\text{ref}}$ such that

$$x^{\text{ref}} \in \bigcap_{x \in \text{dom}(K)} K(x)$$

$$\text{and} \quad \lim_{\|x\| \to \infty, x \in K(x)} \left[ \inf_{y \in \Phi(x)} \frac{(x - x^{\text{ref}})^T y}{\|x\|} \right] = \infty. \tag{3.12}$$

Suppose the following hold:

(i) $\Phi(x)$ is a nonempty, contractible-valued, compact-valued, upper semicontinuous map on $\mathbb{R}^n$;

(ii) $K$ is convex-valued;

(iii) There exists a $\rho_0 > 0$ such that $K(x) \cap B_\rho$ is a continuous mapping for all $\rho > \rho_0$ where $B_\rho$ is a ball of radius $\rho$ centered at the origin.

Then for each vector $q$, SQVI$(K, \Phi + q)$ has a solution. Moreover, there exists an $r > 0$ such that $\|x^*\| < r$ for each solution $(x^*, y^*)$.

In the next proposition, by using the properties of $\Phi(x;\omega)$, we develop an integration-free analog of this result for multi-valued SQVI$(K,\Phi)$.

**Proposition 3.15 (Solvability of SQVI$(K,\Phi)$)** Suppose Assumptions 3.3 and 3.4 hold. Suppose that there exists a vector $x^{\text{ref}}$ such that

(i) $x^{\text{ref}} \in \bigcap_{x \in \text{dom}(K)} K(x)$;

(ii)

$$\lim_{\|x\|\to\infty, x\in K(x)} \left[ \inf_{y\in\Phi(x;\omega)} \frac{(x-x^{\text{ref}})^T y}{\|x\|} \right] = \infty, \quad \text{a.s.} \tag{3.13}$$

(iii) For the above $x^{\text{ref}}$, suppose there exists a nonnegative integrable function $U(\omega)$ such that $g(x;\omega)^T(x-x^{\text{ref}}) \geq -U(\omega)$ holds almost surely for any integrable selection $g(x;\omega)$ of $\Phi(x;\omega)$ and for any $x$.

(iv) $\Phi(x;\omega)$ is an upper semicontinuous mapping on $\mathbb{R}^n$ in an almost-sure sense;

(v) There exists a $\rho_0 > 0$ such that $K(x) \cap B_\rho$ is a continuous mapping for all $\rho > \rho_0$.

Then for each vector $q$ the stochastic SQVI$(K, \Phi + q)$ has a solution. Moreover, there exists an $r > 0$ such that $\|x^*\| < r$ for each solution $(x^*, y^*)$.

**Proof :** First we show that (3.13) implies that the coercivity property (3.12) holds for the expected-value map $\Phi(x) = \mathbb{E}[\Phi(x;\omega)]$. Suppose $\mathbb{E}[\Phi(x;\omega)]$ does not satisfy (3.12). Then, there exists a sequence $\{x_k\}$ such that

$$\lim_{\|x_k\|\to\infty, x_k\in K(x_k)} \left[ \inf_{y\in\mathbb{E}[\Phi(x_k;\omega)]} \frac{(x_k-x^{\text{ref}})^T y}{\|x_k\|} \right] < \infty.$$

Since $\Phi(x_k;\omega)$ is integrably bounded, we have that $\mathbb{E}[\Phi(x_k;\omega)]$ is compact (by [31, Th. 8.6.3]) and therefore it is a closed set. Thus, we may conclude that there exists a $y_k \in \mathbb{E}[\Phi(x_k;\omega)]$ for which the infimum is attained and the above statement can be rewritten as follows:

$$\lim_{\|x_k\|\to\infty, x_k\in K(x_k)} \left[ \frac{(x_k-x^{\text{ref}})^T y_k}{\|x_k\|} \right] < \infty, \text{ where } y_k \in \mathbb{E}[\Phi(x_k;\omega)].$$

By lemma 3.7, $y_k = \int_\Omega g_k(x_k;\omega)d\mathbb{P}$ where $g_k(x_k;\omega) = \sum_{j=0}^n \lambda_j(x_k) f_j(x_k;\omega)$ where $\lambda_j(x_k) \geq 0, \sum_{j=0}^n \lambda_j(x_k) = 1$ and each $f_j(x_k;\omega)$ is an extremal selections of $\Phi(x_k;\omega)$. Thus, we can write the above limit as

$$\lim_{\|x_k\|\to\infty, x_k\in K(x_k)} \left[\int_\Omega \frac{(x_k - x^{\text{ref}})^T g_k(x_k;\omega)}{\|x_k\|} d\mathbb{P}\right] < \infty.$$

Since, $f_j(x_k;\omega)$ is integrable for every $j = 0,\ldots,n$, each $g_k(x_k;\omega)$ is integrable. Hypothesis (iii) allows for the application of Fatou's lemma, through which we have that the following sequence of inequalities:

$$\begin{aligned}
&\int_\Omega \lim_{\|x_k\|\to\infty, x_k\in K(x_k)} \left[\frac{(x_k - x^{\text{ref}})^T g_k(x_k;\omega)}{\|x_k\|}\right] d\mathbb{P} \\
&\leq \lim_{\|x_k\|\to\infty, x_k\in K(x_k)} \left[\int_\Omega \frac{(x_k - x^{\text{ref}})^T g_k(x_k;\omega)}{\|x_k\|} d\mathbb{P}\right] < \infty.
\end{aligned}$$

But this implies that the integrand must be finite almost surely. In other words, for the sequence $\{x_k\}$, we obtain that

$$\lim_{\|x_k\|\to\infty, x_k\in K(x_k)} \left[\frac{(x_k - x^{\text{ref}})^T g_k(x_k;\omega)}{\|x_k\|}\right] < \infty, \text{ a.s.}$$

Substituting the expression for $g_k(x_k;\omega)$ in terms of extremal selections $f_j(x_k,\omega)$ from $\Phi(x;\omega)$, it follows that

$$\lim_{\|x_k\|\to\infty, x_k\in K(x_k)} \left[\frac{(x_k - x^{\text{ref}})^T \sum_{j=0}^n \lambda_j(x_k) f_j(x_k;\omega)}{\|x_k\|}\right] < \infty, \text{ a.s.}$$

Consequently, for $j \in \{0,\cdots,n\}$, the following holds

$$\lim_{\|x_k\|\to\infty, x_k\in K(x_k)} \left[\frac{(x_k - x^{\text{ref}})^T \lambda_j(x_k) f_j(x_k;\omega)}{\|x_k\|}\right] < \infty, \text{ a.s.} \tag{3.14}$$

We may now claim that for some index $j$,

$$\lim_{\|x_k\|\to\infty, x_k\in K(x_k)} \left[\frac{(x_k - x^{\text{ref}})^T f_j(x_k;\omega)}{\|x_k\|}\right] < \infty, \text{ a.s.} \tag{3.15}$$

Suppose this claim is false, then for $j \in \{0, \ldots, n\}$, for $\omega \in U_j$, a set of positive measure, we have the following:

$$\lim_{\|x_k\| \to \infty, x_k \in K(x_k)} \left[ \frac{(x_k - x^{\text{ref}})^T f_j(x_k; \omega)}{\|x_k\|} \right] = \infty. \tag{3.16}$$

Since the denominator in (3.16) goes to $+\infty$ as $k \to \infty$, this implies that numerator goes to $+\infty$ at a faster rate than the denominator $\|x_k\|$; in effect, the numerator is $\mathbf{\Omega}(\|x_k\|^{n_j})$ where $n_j > 1$ and $v_k \in \mathbf{\Omega}(u_k)$ implies that $v_k \geq u_k$ for sufficiently large $k$. Thus, for $j \in \{0, \ldots, n\}$ and $\omega \in U_j \subset \Omega$, where $U_j$ has positive measure, we have the following:

$$(x_k - x^{\text{ref}})^T f_j(x_k; \omega) = \mathbf{\Omega} \left( \|x_k\|^{n_j} \right), \text{ where } n_j > 1. \tag{3.17}$$

But from (3.14), for $j \in \{0, \ldots, n\}$ and for $\omega \in U_j$,

$$\lim_{\|x_k\| \to \infty, x_k \in K(x_k)} \left[ \frac{\lambda_j(x_k)(x_k - x^{\text{ref}})^T f_j(x_k; \omega)}{\|x_k\|} \right] < \infty, \tag{3.18}$$

whereby (3.17) implies that we must have that for each $j$, as $k \to \infty$, $\lambda_j(x_k) \to 0$ faster than $(x_k - x^{\text{ref}})^T f_j(x_k, \omega) \to +\infty$ to ensure that the limit given by (3.18) remains finite. But this leads to a contradiction since for each $k$, by construction, we have that for $j \in \{0, \ldots, n\}$, $\lambda_j(x_k) \geq 0$ and $\sum_{j=0}^{n} \lambda_j(x_k) = 1$. The resulting contradiction proves that for some $j$, we must have (3.15) holds, where $f_j(x_k; \omega)$ is an extremal selection of $\Phi(x_k; \omega)$. This implies that

$$\lim_{\|x_k\| \to \infty, x_k \in K(x_k)} \left[ \inf_{y \in \Phi(x_k; \omega)} \frac{(x_k - x^{\text{ref}})^T y}{\|x_k\|} \right] < \infty, \text{ a.s.}$$

which is in contradiction to hypothesis (3.13). It follows that the coercivity requirement (3.12) holds for $\Phi(x) = \mathbb{E}[\Phi(x; \omega)]$.

Further, from the proof of the Proposition 3.13, we may claim that $\mathbb{E}[\Phi(x; \omega)]$ is a nonempty contractible-valued, compact-valued, upper semicontinuous mapping on $\mathbb{R}^n$ and from Assumption 3.4, the map $K$ is convex-valued. Thus, all the hypotheses of Proposition 3.14 are satisfied and the multi-valued SQVI$(K, \Phi)$ admits a solution. ∎

## 3.4 Stochastic complementarity problems

When the set $K$ in a VI$(K, F)$ is a cone in $\mathbb{R}^n$, then the VI$(K, F)$ is equivalent to a CP$(K, F)$ [74]. Our approach in the previous sections required us to assume that the map $K$ was deterministic. In practical settings, however, the map $K$ may take on a variety of forms. For instance, $K$ may be defined by a set of algebraic resource or budget constraints in financial applications, capacity constraints in network settings or supply and demand constraints in economic equilibrium settings. Naturally, these constraints may often be expectation or risk-based constraints. In such an instance, a complementarity approach assumes relevance; specifically, in such a case, this problem is defined in the joint space of primal variables and the Lagrange multipliers corresponding to the stochastic constraints. Such a transformation yields an SCP$(K, H)$ where the map $H$ may be expectation-valued while the set $K$ is a deterministic cone. However, such complementarity problems may also arise naturally, as is the case when modeling frictional contact problems [37] and stochastic counterparts of such problems emerge from attempting to model risk and uncertainty. In the remainder of this section, we consider complementarity problems with single-valued maps.

Before proceeding, we provide a set of definitions.

**Definition 3.9 (CP$(K, q, M)$)** Given a cone $K$ in $\mathbb{R}^n$, an $n \times n$ matrix $M$ and a vector $q \in \mathbb{R}^n$, the complementarity problem CP$(K, q, M)$ requires an $x \in K, Mx + q \in K^*$ such that $x^T(Mx + q) = 0$.

Recall, from section 3.1.1, $K^* \triangleq \{y : y^T d \geq 0, \forall d \in K\}$. The recession cone, denoted by $K_\infty$, is defined as follows.

**Definition 3.10 (Recession cone $K_\infty$)** The recession cone associated with a set $K$ (not necessarily a cone) is defined as $K_\infty \triangleq \{d : \text{for some } x \in K, \{x + \tau d : \tau \geq 0\} \in K\}$.

Note that when $K$ is a closed and convex cone, $K_\infty = K$. Next, we define the CP kernel of a pair $(K, M)$ and define its $\mathbf{R}_0$ variant.

**Definition 3.11 (CP kernel of the pair $(K, M)$ ($\mathcal{K}(K, M)$))** The CP kernel of the pair $(K, M)$ denoted by $\mathcal{K}(K, M)$ is given by
$\mathcal{K}(K, M) = \text{SOL}(K_\infty, 0, M)$.

**Definition 3.12 ($\mathbf{R}_0$ pair $(K, M)$)** $(K, M)$ is said to be an $\mathbf{R}_0$ pair if $\mathcal{K}(K, M) = \{0\}$.

From [37, Th. 2.5.6], when $K$ is a closed and convex cone, $(K, M)$ is an $\mathbf{R}_0$ pair if and only if the solutions of the CP$(K, q, M)$ are uniformly bounded for all $q$ belonging to a bounded set.

**Definition 3.13** Let $K$ be a cone in $\mathbb{R}^n$ and $M$ be an $n \times n$ matrix. Then $M$ is said to be

(a) copositive on $K$ if $x^T Mx \ \geq 0, \quad \forall\, x \in K$;

(b) strictly copositive on $K$ if $x^T Mx \ > 0, \quad \forall\, x \in K \backslash \{0\}$.

The main result in this section is an almost-sure sufficiency condition for the solvability of a stochastic complementarity problem SCP$(K, H)$. The next Lemma is a simple result that shows that $(K, M)$ being an $\mathbf{R}_0$ pair scales in a certain sense. It is seen that this result is useful in guaranteeing integrability, a necessity when conducting the analysis in stochastic regimes.

**Lemma 3.16** Consider SCP$(K, H)$ and suppose $\omega$ denotes an arbitary, but fixed, element in $\Omega$. Suppose there exists a nonzero matrix $M \in \mathbb{R}^{n \times n}$ which is copositive on $K$ such that $(K, M)$ is an $\mathbf{R}_0$ pair and $T$ is bounded where

$$T \triangleq \bigcup_{\tau > 0} \mathrm{SOL}(K, H(\bullet; \omega) + \tau M). \tag{3.19}$$

If $\beta \triangleq \frac{1}{\|M\|}$, then the matrix $\bar{M} \triangleq \beta M$ is copositive on $K$, $(K, \bar{M})$ is an $\mathbf{R}_0$ pair and $\bar{T}$ is bounded where

$$\bar{T} \triangleq \bigcup_{\tau > 0} \mathrm{SOL}(K, H(\bullet; \omega) + \tau \bar{M}). \tag{3.20}$$

**Proof :** Clearly, when $M$ is copositive, $\bar{M}$ is also copositive. Further, since $(K, M)$ is an $\mathbf{R}_0$ pair, we have that if $x \in K$ with $x^T Mx = 0$ then we must have $x = 0$. Since $\beta > 0$, this implies that for $x \in K$, with $x^T(\beta M)x = 0$ then it follows that $x = 0$. Thus, $(K, \beta M)$ or $\left(K, \bar{M}\right)$ is also an $\mathbf{R}_0$ pair. Finally, observe that by scaling $\tau$ in (3.20) by using $\bar{\tau} = \tau\beta$; we get that

$$\bar{T} = \bigcup_{\bar{\tau} > 0} \mathrm{SOL}\left(K, H(\bullet; \omega) + \bar{\tau} M\right).$$

By invoking the boundedness of $T$, it follows that the set $\bar{T}$ is bounded and the result follows. ∎

We now state the following sufficiency condition [37, Th. 2.6.1] for the solvability of deterministic complementarity problems which is subsequently used in analyzing the stochastic generalizations.

**Theorem 3.17** Let $K$ be a closed convex cone in $\mathbb{R}^n$ and let $F$ be a continuous map from $K$ into $\mathbb{R}^n$. If there exists a copositive matrix $E \in \mathbb{R}^{n\times n}$ on $K$ such that $(K, E)$ is an $\mathbf{R}_0$ pair and the union

$$\bigcup_{\tau>0} \mathrm{SOL}(K, F + \tau E)$$

is bounded, then the $\mathrm{CP}(K, F)$ has a solution.

Recall that, in our notation for $\mathrm{SCP}(K, H)$, $H(x) = \mathbb{E}[H(x;\omega)]$. We now present an intermediate result required in deriving an integration-free sufficiency condition.

**Lemma 3.18** Let $K = \mathbb{R}^n_+$ and let $H(x;\omega)$ be a mapping that satisfies Assumption 3.2. Given an $\omega \in \Omega$, suppose the following holds:

$$\liminf_{\|x\|\to\infty, x\in K} H(x;\omega) > 0. \tag{3.21}$$

Then, there exists a copositive matrix $M_\omega \in \mathbb{R}^{n\times n}$ on $K$ such that $(K, M_\omega)$ is an $\mathbf{R}_0$ pair and $T_\omega$ is bounded where

$$T_\omega \triangleq \bigcup_{\tau>0} \mathrm{SOL}(K, H(\bullet;\omega) + \tau M_\omega). \tag{3.22}$$

Further, if $M_\omega \not\equiv 0$, without loss of generality we may assume that $\|M_\omega\| = 1$.

**Proof :** We proceed by a contradiction and assume that the there is no copositive matrix $M_\omega$ where $(K, M_\omega)$ is an $\mathbf{R}_0$ pair and the set $T_\omega$ is bounded where $T_\omega$ is given by (3.22). Therefore for any copositive matrix $M$ with $(K, M)$ an $\mathbf{R}_0$ pair, the set $T$ is always unbounded where

$$T \triangleq \bigcup_{\tau>0} \mathrm{SOL}(K, H(\bullet;\omega) + \tau M).$$

Since $T$ is unbounded, by definition, there exists a sequence $\{x_k\} \in T$ and a sequence $\{\tau_k\} > 0$, with $\lim_{k\to\infty} \|x_k\| = \infty$, $x_k \geq 0$, $H(x_k;\omega) + \tau_k x_k^M x_k \geq 0$ and $x_k^T H(x_k;\omega) + \tau_k x_k^T M x_k = 0$.

Now, $M$ is copositive on $K$ and $\|x_k\| \to \infty$ implies that that the sequence $\{x_k^T M x_k\}$ goes to $\infty$. Since $\tau_k > 0$ and $\{x_k^T M x_k\}$ goes to $\infty$, the sequence $\{\tau_k x_k^T M x_k\}$ either goes to $\infty$ or 0. This together with $x_k^T H(x_k;\omega) + \tau_k x_k^T M x_k = 0$ implies that $x_k^T H(x_k;\omega)$ goes to $-\infty$ or 0. From (3.21) and since $\|x_k\| \to \infty$, we get that $x_k^T H(x_k;\omega) > 0$ for large $k$, which contradicts the assertion that $x_k^T H(x_k;\omega)$ goes to $-\infty$ or 0. The boundedness of $T$ follows.

Further, if $M_\omega \not\equiv 0$, taking $\beta = \frac{1}{\|M_\omega\|}$, by Lemma 3.16, we may use $\beta M_\omega$ instead of $M_\omega$ in (3.22). Therefore, without loss of generality, we may assume that $\|M_\omega\| = 1$. ∎

We are now prepared to prove our main result.

**Theorem 3.19 (Solvability of SCP($K, H$))** Consider the stochastic complementarity problem SCP($K, H$) where $K$ is the nonnegative orthant. Suppose Assumption 3.2 holds for the mapping $H$ and $G(x;\omega) \triangleq x^T H(x;\omega)$. Further suppose the following hold:

(i)

$$\liminf_{\|x\|\to\infty, x\in K} H(x;\omega) > 0, \text{ a.s.} \tag{3.23}$$

(ii) Suppose there exists a nonnegative integrable function $u(\omega)$ such that $G(x;\omega) \geq -u(\omega)$ holds almost surely for any $x$.

Then the stochastic complementarity problem SCP($K, H$) admits a solution.

**Proof :** Note that if (3.23) holds, since $K$ is the nonnegative orthant and $\|x_k\| \to \infty$ as $k \to \infty$, we must have $\|x_k\| > 0$ for sufficiently large $k$. This allows us to conclude that

$$\liminf_{\|x\|\to\infty, x\in K} \left[x^T H(x;\omega)\right] > 0, \text{ a.s.} \tag{3.24}$$

In other words, if hypothesis (3.23) in Lemma 3.18 holds, then (3.24) holds.

From hypothesis (3.23) in Lemma 3.18 , we may conclude that for almost every $\omega \in \Omega$, there exists a copositive matrix $M_\omega \in \mathbb{R}^{n \times n}$ on $K$ such that $(K, M_\omega)$ is an $\mathbf{R}_0$ pair and the union $T_\omega$ is bounded where

$$T_\omega \triangleq \bigcup_{\tau > 0} \text{SOL}(K, H(\bullet; \omega) + \tau M_\omega). \tag{3.25}$$

Observe that, by Lemma 3.18, for each $\omega$ for which $M_\omega \not\equiv 0$, we may assume that $\|M_\omega\| = 1$. Consequently, $\mathbb{E}\left[\|M_\omega\|\right] < +\infty$ and the integrability of $M_\omega$ follows.

We will prove our main result by using Theorem 3.17. In particular, we show that there exists a copositive matrix $M$, defined as $M \triangleq \mathbb{E}[M_\omega] \in \mathbb{R}^{n \times n}$, where $M$ is copositive on $K$, $(K, M)$ is an $\mathbf{R}_0$ pair, and the set $T$ is bounded, where

$$T \triangleq \bigcup_{\tau > 0} \text{SOL}(K, H + \tau M).$$

**(i)** $M = \mathbb{E}[M_\omega]$ **is copositive on** $K$**:** Consider the matrix $\mathbb{E}[M_\omega]$. Since $M_\omega$ is copositive on $K$ in an almost-sure sense, it follows that

$$x^T \mathbb{E}[M_\omega] x = \mathbb{E}\left[x^T M_\omega x\right] \geq 0, \quad \forall\, x \in K,$$

implying that $M = \mathbb{E}[M_\omega]$ is a copositive matrix on $K$.

**(ii)** $(K, M)$ **is an** $\mathbf{R}_0$ **pair:** We need to show that $\text{SOL}(K_\infty, 0, \mathbb{E}\left[M_\omega\right]) = \{0\}$. Since $K$ is a closed and convex cone, $K_\infty = K$ and it suffices to show that $\text{SOL}(K, 0, \mathbb{E}\left[M_\omega\right]) = \{0\}$. Clearly, $0 \in \text{SOL}(K, 0, \mathbb{E}\left[M_\omega\right])$. It remains to show that $\text{SOL}(K, 0, \mathbb{E}\left[M_\omega\right]) \in 0$. Suppose this claim is false and there exists a $d \in K$ such that $d \neq 0$ and $d \in \text{SOL}(K, 0, \mathbb{E}\left[M_\omega\right])$. It follows that $d^T \mathbb{E}[M_\omega] d = 0$. This can be written as $\mathbb{E}\left[d^T M_\omega d\right] = 0$ and there are sets $U_>, U_<$ and $U_=$ such that

$$d^T M_\omega d \begin{cases} > 0, & \omega \in U_> \\ = 0, & \omega \in U_= \\ < 0. & \omega \in U_< \end{cases}$$

We consider each of these possibilities next.

(a) Suppose $\mathbb{P}(U_=) > 0$. We have $d^T M_\omega d = 0$ for $\omega \in U_=$. Since each $(K, M_\omega)$ is an $\mathbf{R}_0$ pair, we obtain that $d = 0$. But this contradicts our assumption that $d \neq 0$, implying that $\mathbb{P}(U_=) = 0$.

(b) Suppose $\mathbb{P}(U_<) > 0$. For $\omega \in U_<$, we have $d^T M_\omega d < 0$. Since $M_\omega$ is copositive, for $d \in K$ we must have that $d^T M_\omega d \geq 0$. This contradiction implies that $\mathbb{P}(U_<) = 0$.

(a) and (b) above imply that $\mathbb{P}(U_>) = 1$ or in other words, we have $d^T M_\omega d > 0$ for $\omega \in \Omega$. This implies that $\mathbb{E}\left[d^T M_\omega d\right] > 0$. This contradicts our assertion that $\mathbb{E}\left[d^T M_\omega d\right] = 0$. Thus, we must have $d = 0$ and $\mathrm{SOL}(K, 0, \mathbb{E}[M_\omega]) = \{0\}$. Therefore, $(K, M)$ is indeed an $\mathbf{R}_0$ pair.

**(iii) The set $T$ is bounded:** We proceed to show that the set $T$ is bounded where

$$T \triangleq \bigcup_{\tau > 0} \mathrm{SOL}(K, H + \tau M). \tag{3.26}$$

It suffices to show that there exists an $m > 0$ such that for all $x \in K$, $\|x\| > m$ implies $x \notin T$. Suppose there is no such finite $m$, implying that

$$\text{for any } m > 0,\ \exists\, x \in K, \|x\| > m \text{ and } x \in T. \tag{3.27}$$

For each $k > 0$, choose $x_k \in K$ such that $\|x_k\| > k$ and $x_k \in T$. For this sequence $\|x_k\| \to \infty$. Since $\|x_k\| > k$ observe that for any $k$, $x_k \neq 0$. Now, for each $k$, since $x_k \in T$, it follows that $x_k \in \mathrm{SOL}(K, H + \tau_k M)$ for some $\tau_k > 0$. Thus, for each $k$ we have $x_k^T H(x_k) + \tau_k x_k^T M x_k = 0$. Since $x_k \neq 0$ this means that for each $k$,

$$x_k^T H(x_k) = \mathbb{E}[x_k^T H(x_k; \omega)] = -\tau_k x_k^T M x_k. \tag{3.28}$$

Observe that, since $x_k \in K$ and $\|x_k\| \to \infty$, we have $x_k \neq 0$. Further, since $M$ is copositive we have $x_k^T M x_k \geq 0$. Since $\|x_k\| \to \infty$, we have that $x_k^T M x_k \geq 0$. Since $\tau_k > 0$, there are two possibilities for the sequence $\tau_k x_k^T M x_k$: either it $\tau_k x_k^T M x_k \to +\infty$ or $\tau_k x_k^T M x_k \to 0$ as $k \to \infty$. In either case, as $k \to \infty$ from (3.28) we can conclude

$$\liminf_{k\to\infty}\ x_k^T H(x_k) = \liminf_{k\to\infty}\ \left[\mathbb{E}[x_k^T H(x_k;\omega)]\right] = \liminf_{k\to\infty} \left[-\tau_k x_k^T M x_k\right] \leq 0. \tag{3.29}$$

On the other hand, by (3.24) we have that

$$\liminf_{k\to\infty}\ x_k^T H(x_k;\omega) > 0 \ \text{ a.s.} \tag{3.30}$$

By hypothesis (ii), Fatou's lemma can be applied, giving us

$$\liminf_{k\to\infty}\ x_k^T H(x_k) = \liminf_{k\to\infty}\ \left[\mathbb{E}\left[x_k^T H(x_k;\omega)\right]\right] \geq \mathbb{E}\left[\liminf_{k\to\infty} x_k^T H(x_k;\omega)\right] > 0, \tag{3.31}$$

where the last inequality follows from (3.30). But this contradicts (3.29) and implies that there is a scalar $m$ such that $x \in K, \|x\| > m$ implies $x \notin T$. In other words, $T$ is bounded. We have shown that all the conditions of Theorem 3.17 are satisfied and we may conclude that the stochastic complementarity problem SCP$(K, H)$ has a solution. ∎

In the above proposition, hypothesis (3.23) guaranteed the existence of a copositive matrix $M$, so that all of the conditions of Theorem 3.17 are satisfied for the SCP. In certain applications, it may be possible to show the existence of a copositive matrix $M$ such that $(K, M)$ is an $\mathbf{R}_0$ pair, for example, choosing $M$ as the identity matrix may suffice. In fact, if we assume existence of a copositive matrix $M$ such that $(K, M)$ is an $\mathbf{R}_0$ pair, then we do not require hypothesis (3.23) above but merely equation (3.24) suffices. This is demonstrated in the proposition below.

**Proposition 3.20** Consider the stochastic complementarity problem SCP$(K, H)$ where $K$ is the nonnegative orthant. Suppose Assumption 3.2 holds for the mapping $H$, $G(x;\omega) = x^T H(x;\omega)$, and there exists a copositive matrix $M$ on $K$ such that $(K, M)$ is an $\mathbf{R}_0$ pair and the following hold :

(i)

$$\liminf_{x\in K, \|x\|\to\infty} \left[x^T H(x;\omega)\right] > 0, \quad \text{almost surely.} \tag{3.32}$$

(ii) Suppose there exists a nonnegative integrable function $u(\omega)$ such that $G(x;\omega) \geq -u(\omega)$ holds almost surely for any $x$.

Then SCP$(K, H)$ has a solution.

**Proof :** We proceed to show that the set $T$ is bounded where

$$T \triangleq \bigcup_{\tau>0} \text{SOL}(K, H + \tau M). \tag{3.33}$$

As earlier, it suffices to show that there exists an $m > 0$ such that for all $x \in K, \|x\| > m$ implies $x \notin T$. Suppose there is no such $m$ implying that

$$\text{for any } m > 0\ \exists x \in K, \|x\| > m \text{ and } x \in T. \tag{3.34}$$

Construct a sequence $\{x_k\}$ as follows: For each $m = k > 0$, choose $x_k \in K$ such that $x_k \in K, \|x_k\| > k$ and $x_k \in T$. For this sequence $\|x_k\| \to \infty$. Since $\|x_k\| > k$ observe that for any $k$, $x_k \neq 0$. Now, for each $k$ since $x_k \in T$, it follows that $x_k \in \text{SOL}(K, H + \tau_k M)$ for some $\tau_k > 0$. Thus, for each $k$, we have $x_k > 0$, $H(x_k) + \tau_k M x_k \geq 0, \tau_k > 0$ and $x_k^T H(x_k) + \tau_k x_k^T M x_k = 0$. Since $x_k \neq 0$ and $M$ is copositive and $(K, M)$ is an $\mathbf{R}_0$ pair we have that $x_k^T M x_K \geq 0$. This together with $\tau_k > 0$ means that for each $k$,

$$\mathbb{E}[x_k^T H(x_k;\omega)] = x_k^T H(x_k) = -\tau_k x_k^T M x_k \leq 0.$$

This gives us that

$$\liminf_{k\to\infty}\ \mathbb{E}[x_k^T H(x_k;\omega)] = \liminf_{k\to\infty}\ \left[-\tau_k x_k^T M x_k\right] \leq 0. \tag{3.35}$$

On the other hand, since $x_k \in K$ and $\|x_k\| \to \infty$, by hypothesis (3.32) we have that

$$\liminf_{k\to\infty}\ x_k^T H(x_k;\omega) > 0 \ \text{ a.s.}$$

This means that

$$\mathbb{E}\left[\liminf_{k\to\infty}\ x_k^T H(x_k;\omega)\right] > 0. \tag{3.36}$$

Now, by hypothesis (ii), Fatou's lemma is applicable, implying that

$$\liminf_{k\to\infty} \mathbb{E}\left[x_k^T H(x_k;\omega)\right] \geq \mathbb{E}\left[\liminf_{k\to\infty}\ x_k^T H(x_k;\omega)\right] > 0,$$

where the last inequality follows from (3.36). This contradicts (3.35). This contradiction implies that there exists an $m$ such that $x \in K, \|x\| > m$ implies $x \notin T$. In other words, $T$ is bounded. By hypothesis, we have that there exists a copositive matrix $M$ on $K$ such that $(K, M)$ is an $\mathbf{R}_0$ pair and we have shown that $T$ is bounded. Thus, all conditions of Theorem 3.17 are satisfied and we may conclude that the stochastic complementarity problem SCP$(K, H)$ has a solution. ∎

We now consider several corollaries, the first of which requires defining a *co-coercive mapping.*

**Definition 3.14 (Co-coercive function)** A mapping $F : K \subseteq \mathbb{R}^n \to \mathbb{R}^n$ is said to be co-coercive on $K$ if there exists a constant $c > 0$ such that

$$(F(x) - F(y))^T(x - y) \geq c\|F(x) - F(y)\|^2, \ \forall x, y \in K.$$

We state Cor. [37, Cor. 2.6.3], which is used in the proof of the next proposition.

**Corollary 3.21** Let $K$ be a pointed, closed, convex cone in $\mathbb{R}^n$ and let $F : \mathbb{R}^n \to \mathbb{R}^n$ be a continuous map. If $F$ is co-coercive on $\mathbb{R}^n$, then CP$(K, F)$ has a nonempty compact solution set if and only if there exists a vector $u \in \mathbb{R}^n$ satisfying $F(u) \in \text{int}(K^*)$.

Our next result provides sufficiency conditions for the existence of a solution to an SCP when an additional co-coercivity assumption is imposed on the mapping. In particular, we assume that the mapping $H(z; \omega)$ is co-coercive for almost every $\omega \in \Omega$.

**Proposition 3.22 (Solvability under co-coercivity)** Let $K$ be a pointed, closed and convex cone in $\mathbb{R}^n$. Suppose Assumption 3.2 holds for the mapping $H(x; \omega)$ and $H(x; \omega)$ is co-coercive on $K$ with constant $\eta(\omega) > 0$. Suppose $\eta(\omega) \geq \bar{\eta} > 0$ for all $\omega \in \bar{\Omega}$ where $\mathbb{P}(\bar{\Omega}) = 1$ and there exists a deterministic vector $u \in \mathbb{R}^n$ satisfying $H(u; \omega) \in \text{int}(K^*)$ in an almost sure sense. Then the solution set of the SCP$(K, H)$ is a nonempty and compact set.

**Proof :** First we show that, $\mathbb{E}\left[H(x; \omega)\right]$ is co-coercive in $x$. We begin by noting that

$$(x-y)^T\left(\mathbb{E}\left[H(x;\omega)\right]-\mathbb{E}\left[H(y;\omega)\right]\right)=\int_{\Omega}(x-y)^T(H(x;\omega)-H(y;\omega))d\mathbb{P}$$
$$=\int_{\bar{\Omega}}(x-y)^T(H(x;\omega)-H(y;\omega))d\mathbb{P},$$

where the second equality follows by noting that $\mathbb{P}(\bar{\Omega})=1$. This allows us to conclude that

$$\int_{\bar{\Omega}}(x-y)^T(H(x;\omega)-H(y;\omega))d\mathbb{P}\geq\int_{\bar{\Omega}}\eta(\omega)\|H(x;\omega)-H(y;\omega)\|^2d\mathbb{P}$$
$$\geq\bar{\eta}\int_{\bar{\Omega}}\|H(x;\omega)-H(y;\omega)\|^2\,d\mathbb{P},$$

where the first inequality follows from the co-coercivity of $H(x;\omega)$, the second inequality follows from noting that $\eta(\omega)\geq\bar{\eta}$ for $\omega\in\bar{\Omega}$, a set of unitary measure. Finally, by again recalling that $\bar{\Omega}$ has measure one and by leveraging Jensen's inequality since $\|.\|^2$ is a convex function, the required result follows:

$$\bar{\eta}\int_{\bar{\Omega}}\|H(x;\omega)-H(y;\omega)\|^2\,d\mathbb{P}=\bar{\eta}\int_{\Omega}\|H(x;\omega)-H(y;\omega)\|^2\,d\mathbb{P}$$
$$\geq\bar{\eta}\left\|\mathbb{E}\left[H(x;\omega)\right]-\mathbb{E}\left[H(y;\omega)\right]\right\|^2.$$

Further, since $H(u;\omega)\in\text{int}(K^*)$ holds almost surely for a deterministic vector $u$, we have that for all $x\in K$, $H(u;\omega)^Tx\geq 0$ holds almost surely. This implies that for all $x\in K,\mathbb{E}\left[H(u;\omega)\right]^T x\geq 0$ holds. Thus, there exists a $u\in\mathbb{R}^n$ such that $\mathbb{E}\left[H(u;\omega)\right]\in K^*$.

It remains to show that $\mathbb{E}\left[H(u;\omega)\right]$ lies in $\text{int}(K^*)$. If $\mathbb{E}\left[H(u;\omega)\right]\notin\text{int}(K^*)$, then there exists an $x\in K$ such that $\mathbb{E}\left[H(u;\omega)\right]^T x=0$. Since $x\in K$ and by assumption, $H(u;\omega)\in\text{int}(K^*)$ almost surely, it follows that $H(u;\omega)^Tx>0$ almost surely, implying that $\mathbb{E}\left[H(u;\omega)\right]^T x>0$. Thus, we arrive at a contradiction, proving that $\mathbb{E}\left[H(u;\omega)\right]\in\text{int}(K^*)$. Thus, by Corollary 3.21, since $\mathbb{E}\left[H(z;\omega)\right]$ is co-coercive and there is a vector $u\in\mathbb{R}^n$ such that $\mathbb{E}\left[H(u;\omega)\right]\in\text{int}(K^*)$, it follows that $\text{SCP}(K,H)$ has a nonempty compact solution set. ∎

The next corollary is a direct application of Theorem 3.17 to $\text{SCP}(K,H)$ when $E$ is the identity matrix and can be viewed as a *theorem of the alter-*

*native* for CPs.

**Corollary 3.23 (Cor. 2.6.2 [37])** Let $K$ be a closed convex cone in $\mathbb{R}^n$ and let $H(x;\omega)$ satisfy Assumption 3.2. Either SCP$(K,H)$ has a solution or there exists an unbounded sequence of vectors $\{x_k\}$ and a sequence of positive scalars $\{\tau_k\}$ such that for every $k$, the following complementarity condition holds:

$$K \ni x_k \ \perp\ \mathbb{E}\left[H(x_k;\omega)\right] + \tau_k x_k \in K^*.$$

We may leverage this result in deriving a stochastic generalization.

**Proposition 3.24 (Theorem of the alternative)** Let $K$ be a closed convex cone in $\mathbb{R}^n$ and let $H(x;\omega)$ be a mapping that satisfies Assumption 3.2. Either SCP$(K,H)$ has a solution or there exists an unbounded sequence of vectors $\{x_k\}$ and a sequence of positive scalars $\{\tau_k\}$ such that for every $k$, the following complementarity condition holds almost-surely:

$$K \ni x_k \ \perp\ H(x_k;\omega) + \tau_k x_k \in K^*. \tag{3.37}$$

**Proof :** Suppose (3.37) holds almost surely. Consequently, it also holds in expectation or

$$K \ni x_k \ \perp\ \mathbb{E}[H(x_k;\omega)] + \tau_k x_k \in K^*. \tag{3.38}$$

Therefore by Cor. 3.23, SCP$(K,H)$ does not admit a solution. ∎

## 3.5 Examples revisited

We now revisit the motivating examples presented in Section 3.2 and show the applicability of the developed sufficiency conditions in the context of such problems.

### 3.5.1 Stochastic Nash-Cournot games with nonsmooth price functions

In Section 3.2.1, we described a stochastic Nash-Cournot game in which the price functions were nonsmooth. We revisit this example in showing the associated stochastic quasi-variational inequality problem is solvable.

Before proceeding, we recall that $f_i(x;\omega)$ is a convex function of $x_i$, given $x_{-i}$ (see [35, Lemma 1]).

**Lemma 3.25** Consider the function $f_i(x;\omega) = c_i(x_i) - x_i p(X;\omega)$ where $p(X;\omega)$ is given by (3.5). Then $f_i(x_i;x_{-i})$ is a convex function in $x_i$ for all $x_{-i}$.

The convexity of $f_i$ and $K_i(x_{-i})$ allows us to claim that the first-order optimality conditions are sufficient; these conditions are given by a multi-valued quasi-variational inequality SQVI$(K,\Phi)$ where $\Phi$, the Clarke generalized gradient, is defined as

$$\Phi(x) \triangleq \mathbb{E}\left[\prod_{i=1}^{N} \partial_{x_i} f_i(x;\omega)\right],$$

and $\Phi(x;\omega)$ is defined as $\prod_{i=1}^{N} \partial_{x_i} f_i(x;\omega)$. The subdifferential set of $f_i(x;\omega)$ is defined as

$$\partial_{x_i} f_i(x;\omega) = c_i'(x_i) - \partial_{x_i}(x_i p(X;\omega)) = c_i'(x_i) - p(X;\omega) - x_i \partial_{x_i} p(X;\omega).$$

Thus, if $w \in \Phi(x;\omega)$, then $w = \prod_{i=1}^{n} w_i$ where $w_i \in \partial_{x_i} f_i(x;\omega)$. Based on the piecewise smooth nature of $p(X;\omega)$, the Clarke generalized gradient of $p$ is defined as follows:

$$\partial_{x_i} p(X;\omega) \in \begin{cases} \{-b_1(\omega)\}, & 0 \le X < \beta^1 \\ -[b^{j-1}(\omega), b^j(\omega)], & \beta^{j-1} = X,\ j = 2,\ldots,s \\ \{-b^s(\omega)\}, & \beta^s < X \end{cases} \tag{3.39}$$

Since our interest lies in showing the applicability of our sufficiency conditions when the map $\Phi$ is expectation valued, we impose the required assumptions on the map $\mathbf{K}$ as captured by Prop. 3.15 (i) and (v). Existence of a nonsmooth stochastic Nash-Cournot equilibrium follows from showing that hypotheses (ii) – (iv) of Prop. 3.15 do indeed hold.

**Theorem 3.26 (Existence of stochastic Nash-Cournot equilibrium)** Consider the stochastic generalized Nash-Cournot game and suppose Assumptions 3.3, 3.4 and 3.5 hold. Further, assume that conditions (i) and (v) of Prop. 3.15 hold. Then, this game admits an equilibrium.

**Proof :** Since $\partial_{x_i} f_i(x;\omega)$ is a Clarke generalized gradient, it is a nonempty upper semicontinuous mapping at $x_i$, given $x_{-i}$. Furthermore, the integrability of $(a^j(\omega), b^j(\omega))$ for $j = 1, \ldots, s$ allows us to claim that $\partial_{x_i} f_i(x;\omega)$ is integrably bounded. Consequently, hypothesis (iv) in Proposition 3.15 holds.

By Assumption 3.5, since $a_i(\omega)$ and $b_i(\omega)$ are positive, we have that they are bounded below by the nonnegative constant (integrable) function 0. From, this and the description of $\Phi$ derived above, we see that hypothesis (iii) in Prop. 3.15 holds. Thus Fatou's lemma can be applied.

We now proceed to show that hypothesis (ii) in Proposition 3.15 holds. It suffices to show that there exists an $x^{\rm ref} \in \mathbf{K}(x)$ such that

$$\lim_{\|x\| \to \infty, x \in \mathbf{K}(x)} \left( \inf_{w \in \Phi(x;\omega)} \frac{(x - x^{\rm ref})^T w}{\|x\|} \right) = \infty.$$

Consider a vector $x^{\rm ref}$ such that

$$x^{\rm ref} \in \bigcap_{x \,\in\, \mathrm{dom}(\mathbf{K})} \mathbf{K}(x).$$

Then $w^T(x - x^{\rm ref})$ can be expressed as the sum of several terms:

$$w^T(x - x^{\rm ref}) = \sum_{i=1}^{N} c_i'(x_i)(x_i - x_i^{\rm ref}) - p(X;\omega)(x - x^{\rm ref}) \\ - \sum_{i=1}^{N} x_i(x_i - x_i^{\rm ref}) \partial_{x_i} p(X;\omega).$$

When $\|x\| \to \infty$, from the nonnegativity of $x$, it follows that $X \to \infty$ and for sufficiently large $X$, we have that $\partial_{x_i} p(X;\omega) = -b^s(\omega)$. Consequently, for almost every $\omega \in \Omega$, we have that

$$\begin{aligned}
&\lim_{\|x\|\to\infty, x\in\mathbf{K}(x)} \inf_{w\in\Phi(x;\omega)} \frac{(x - x^{\text{ref}})^T w}{\|x\|} \\
&= \lim_{\|x\|\to\infty, x\in\mathbf{K}(x)} \left( \frac{\sum_{i=1}^N (c_i'(x_i) + b^s(\omega)x_i)(x_i - x_i^{\text{ref}})}{\|x\|} \right) \\
&\qquad - \lim_{\|x\|\to\infty, x\in\mathbf{K}(x)} \left( \frac{p(X;\omega)(x - x^{\text{ref}})}{\|x\|} \right) \\
&= \lim_{\|x\|\to\infty, x\in\mathbf{K}(x)} \overbrace{\left( \frac{\sum_{i=1}^N (c_i'(x_i) + b^s(\omega)(X + x_i))(x_i - x_i^{\text{ref}})}{\|x\|} \right)}^{Term\,(a)} \\
&\qquad - \lim_{\|x\|\to\infty, x\in\mathbf{K}(x)} \overbrace{\left( \frac{a^s(\omega)(x - x^{\text{ref}})}{\|x\|} \right)}^{Term\,(b)} = \infty,
\end{aligned}$$

where the last equality is a consequence of noting that the numerator of Term (a) tends to $+\infty$ at a quadratic rate while the numerator of Term (b) tends to $+\infty$ at a linear rate. The existence of an equilibrium follows from the application of Prop. 3.15. ∎

### 3.5.2 Strategic behavior in power markets

In Section 3.2.1, we have presented a model for strategic behavior in imperfectly competitive electricity markets. We will now develop a stochastic complementarity-based formulation of such a problem. The developed sufficiency conditions will then be applied to this problem.

Recall that, the resulting problem faced by firm $f$ can be stated as follows:

$$\begin{array}{llcll}
\underset{s_{fi},\, g_{fi}}{\text{maximize}} & \mathbb{E}\left[ \sum_{i\in\mathcal{N}} (\, p_i(S_i;\omega)s_{fi} - c_{fi}(g_{fi};\omega) - (s_{fi} - g_{fi})w_i \,) \right] & & & \\
\text{subject to} & g_{fi} & \le & \text{cap}_{fi} \quad (\,\mu_{fi}\,), & \forall\, i \in \mathcal{N} \\
& 0 & \le & g_{fi}, & \forall\, i \in \mathcal{N} \\
& 0 & \le & s_{fi}, & \forall\, i \in \mathcal{N} \\
\text{and} & \sum_{i\in\mathcal{N}} (\, s_{fi} - g_{fi} \,) & = & 0. \quad (\,\lambda_f\,) &
\end{array}$$

The equilibrium conditions of this problem are given by the following complementarity problem.

$$\begin{aligned}
0 \leq s_{fi} &\perp \mathbb{E}\left[-p_i'(S_i;\omega)s_{fi} - p_i(S_i;\omega) + w_i\right] - \lambda_f \geq 0, && \forall\, i \in \mathcal{N}\\
0 \leq g_{fi} &\perp \mathbb{E}\left[c_{fi}'(g_{fi};\omega) - w_i\right] + \mu_{fi} + \lambda_f \geq 0, && \forall\, i \in \mathcal{N}\\
0 \leq \mu_{fi} &\perp \text{cap}_{fi} - g_{fi} \geq 0, && \forall\, i \in \mathcal{N}\\
\lambda_f &\perp \sum_{i\in\mathcal{N}} (\, s_{fi} - g_{fi}\,) = 0. && \forall\, f \in \mathcal{F}
\end{aligned}$$

The ISO's optimization problem is given by

$$\begin{aligned}
\underset{y_i}{\text{maximize}} \quad & \sum_{i\in\mathcal{N}} y_i w_i\\
\text{subject to} \quad & \sum_{i\in\mathcal{N}} \text{PDF}_{ij} y_i \,\leq\, T_j, \quad (\eta_j) \qquad \forall\, j \in \mathcal{K}
\end{aligned}$$

and its optimality conditions are as follows:

$$\begin{aligned}
w_i \;=\; & \sum_{j\in\mathcal{K}} \eta_j \text{PDF}_{ij} && \forall\, i \in \mathcal{N},\\
0 \leq \eta_j \perp \;& T_j - \sum_{i\in\mathcal{N}} \text{PDF}_{ij} y_i \geq 0 && \forall\, j \in \mathcal{K}.
\end{aligned} \tag{3.40}$$

The market clearing conditions are given by the following.

$$y_i \;=\; \sum_{h\in\mathcal{F}} (\, s_{hi} - g_{hi}\,)\,, \qquad \forall\, i \in \mathcal{N}.$$

Next, we define $\ell_i(\omega)$ and $h_i(\omega)$ as follows:

$$\ell_i(\omega) = -p_i'(S_i;\omega)s_{fi} - p_i(S_i;\omega) + w_i = -p_i'(S_i;\omega)s_{fi} - p_i(S_i;\omega) + \sum_{j\in\mathcal{K}} \eta_j \text{PDF}_{ij} \tag{3.41}$$

$$h_i(\omega) = c_{fi}'(g_{fi};\omega) - w_i = c_{fi}'(g_{fi};\omega) - \sum_{j\in\mathcal{K}} \eta_j \text{PDF}_{ij}. \tag{3.42}$$

Then, by aggregating all the equilibrium conditions together and eliminating $w_i$ and $y_i$ based on the equality constraints (3.40), we get the equilibrium conditions in $s_{fi}, g_{fi}, \mu_{fi}, , \lambda_f$, and $\eta_j$ are as follows

$$\left\{ \begin{array}{llll} 0 \leq \, s_{fi} & \perp \;\; \mathbb{E}\left[\ell_i(\omega)\right] - \lambda_f & \geq 0, & \forall\, i \in \mathcal{N} \\ 0 \leq \, g_{fi} & \perp \;\; \mathbb{E}\left[h_i(\omega)\right] + \mu_{fi} + \lambda_f & \geq 0, & \forall\, i \in \mathcal{N} \\ 0 \leq \, \mu_{fi} & \perp \;\; \mathrm{cap}_{fi} - g_{fi} & \geq 0, & \forall\, i \in \mathcal{N} \\ \qquad \lambda_f & \perp \;\; \sum_{i \in \mathcal{N}} (\, s_{fi} - g_{fi}\,) & = 0, & \end{array} \right\}, \quad \forall\, f \in \mathcal{F}$$

$$\text{and } \; 0 \leq \eta_j \;\; \perp \;\; T_j - \sum_{i \in \mathcal{N}} \mathrm{PDF}_{ij} \sum_{h \in \mathcal{F}} (\, s_{hi} - g_{hi}\,) \;\; \geq 0. \qquad \forall\, j \in \mathcal{K}$$

This can be viewed as the following stochastic (mixed)-complementarity problem where $x$, $B$, and $H(x;\omega)$ are appropriately defined:

$$\begin{aligned} 0 \;\leq\; x \;\perp\; \mathbb{E}[H(x;\omega)] - B^T\lambda \;\geq\; 0 \\ \lambda \;\perp\; Bx \;=\; 0. \end{aligned}$$

It follows that the inner product in the coercivity condition (3.32) reduces to $x^T\mathbb{E}[H(x;\omega)]$ as observed by this simplification:

$$\begin{pmatrix} x \\ \lambda \end{pmatrix}^T \begin{pmatrix} \mathbb{E}[H(x;\omega)] - B^T\lambda \\ Bx \end{pmatrix} = x^T\mathbb{E}[H(x;\omega)].$$

Next, we show that this inner product is bounded from below by $-u(\omega)$ where $u(\omega)$ is a nonnegative integrable function.

**Lemma 3.27** For the stochastic complementarity problem $\mathrm{SCP}(K, H)$ above that represents the strategic behavior in power markets, there exists a nonnegative integrable function $u(\omega)$ such that have that the following holds:

$$G(x;\omega) = x^T H(x;\omega) \geq -u(\omega) \text{ almost surely for all } x \;\in\; K.$$

**Proof :** The product $x^T H(x;\omega)$ can be expressed as follows:

$$\sum_{f,i}\left(-p_i'(S_i;\omega)s_{fi}^2 - p_i(S_i;\omega)s_{fi} + \left(\sum_{j\in\mathcal{K}}\eta_j \mathrm{PDF}_{ij}\right)s_{fi}\right)$$
$$+\sum_{f,i}\left(c_{fi}'(g_{fi};\omega)g_{fi} - \left(\sum_{j\in\mathcal{K}}\eta_j \mathrm{PDF}_{ij}\right)g_{fi} + \mu_{fi}g_{fi}\right)$$
$$+\sum_{f,i}\left(\mu_{fi}\mathrm{cap}_{fi} - \mu_{fi}g_{fi}\right) + \sum_j \eta_j\left(T_j - \sum_{i\in\mathcal{N}}\mathrm{PDF}_{ij}\sum_{h\in\mathcal{F}}(s_{hi} - g_{hi})\right).$$

After appropriate cancellations, this reduces to

$$\sum_{f,i}\left(-p_i'(S_i;\omega)s_{fi}^2 - p_i(S_i;\omega)s_{fi}\right) + \sum_{f,i}\left(c_{fi}'(g_{fi};\omega)g_{fi}\right) + \sum_{f,i}\left(\mu_{fi}\mathrm{cap}_{fi}\right) + \sum_j \eta_j T_j.$$

By Assumption 3.6, the price functions are decreasing functions bounded above by an integrable function and the cost functions are non-decreasing. Furthermore, $K$ is the nonnegative orthant, $\mu_{fi}, \eta_j$ are nonnegative, and $\mathrm{cap}_{fi}, T_j$ denote nonnegative capacities. Consequently, we have the following sequence of inequalities.

$$\sum_{f,i}\left(-p_i'(S_i;\omega)s_{fi}^2 - p_i(S_i;\omega)s_{fi}\right) + \sum_{f,i}\left(c_{fi}'(g_{fi};\omega)g_{fi}\right) + \sum_{f,i}\left(\mu_{fi}\mathrm{cap}_{fi}\right) + \sum_j \eta_j T_j$$
$$\geq \sum_{f,i}\left(-p_i(S_i;\omega)s_{fi}\right) \geq -\left(\max_i \bar{p}_i(\omega)\right)\sum_{f,i}\mathrm{cap}_{fi} \triangleq -u(\omega),$$

where $p_i(S_i;\omega) \leq \bar{p}_i(\omega)$ for all nonnegative $S_i$ and $\sum_{f,i} s_{fi} \leq \sum_{f,i}\mathrm{cap}_{fi}$. Integrability of $u(\omega)$ follows immediately by its definition. ∎

Having presented the supporting results, we now prove the existence of an equilibrium.

**Proposition 3.28 (Existence of an imperfectly competitive equilibrium)** Consider the imperfectly competitive model in power markets. Under Assumption 3.6, this problem admits a solution.

**Proof :** The result follows by showing that Proposition 3.20 can be applied. Lemma 3.27 shows that hypothesis (ii) of Proposition 3.20 holds. We proceed to show that hypothesis (i) of Proposition 3.20 also holds. We show that the following property holds almost surely:

$$\liminf_{\|x\|\to\infty,\ x\geq 0} x^T H(x;\omega) > 0. \tag{3.43}$$

Consider the expression for $x^T H(x;\omega)$ derived in Lemma 3.27.

$$x^T H(x;\omega) = \sum_{f,i} \left(-p_i'(S_i;\omega)s_{fi}^2 - p_i(S_i;\omega)s_{fi}\right) + \sum_{f,i} \left(c_{fi}'(g_{fi};\omega)g_{fi}\right)$$
$$+ \sum_{f,i} \left(\mu_{fi}\text{cap}_{fi}\right) + \sum_j \eta_j T_j.$$

For large $\|x\|$, the first summation is dominated by its first term and by Assumption 3.6, as $\|x\|$ goes to $\infty$, this term goes to $\infty$. The other terms are all nonnegative by Assumption 3.6. Thus, the entire expression can only increase to $\infty$ as $\|x\|$ goes to $\infty$. This proves that (3.43) holds and the required result follows.

∎

# CHAPTER 4

# ON THE INADEQUACY OF VAR-BASED RISK MANAGEMENT: VAR, CVAR, AND NONLINEAR INTERACTIONS

## 4.1 Introduction

Risk is a complex notion and can take on varied forms with diverse applications. Managing risk is one of the many problems faced by firms in the financial industry. In the context of trading firms, such management has been traditionally achieved by the introducing value-at-risk thresholds on the portfolio risk accumulated by the traders. This work is motivated by the question of adequacy of such thresholds when traders are risk-seeking. The following example clarifies the context.

**Example 4.1 (A motivating example)** *Consider a typical day at $\infty-$Alpha Asset Management, a fictitious risk management firm, where Mike manages a trader, Theresa. Mike's training is in classical finance while Theresa has a Ph.D. in mathematics. Mike imposes a VaR constraint on the portfolio risk assumed by Theresa while she trades on a coherent risk measure. Since she is seeking higher returns, she maximizes her risk; however, she is constrained by Mike's VaR threshold. In other words, she will trade so as to maximize her coherent risk measure subject to Mike's VaR constraints.*

*What can we learn about Mike's ability to manage Theresa's risk?*

Widely accepted by the financial industry and regulators alike and extensively used by practitioners, the value at risk (VaR) is essentially a measurement of quantiles. However, the academic field of mathematical finance has long realized that VaR has certain noticeable shortcomings; for example, it does not properly capture diversification. Alternate measures have been suggested in the form of *coherent* risk measures which do, in fact, have many desirable properties. While the risk preference of a trader can take on a continuum of possibilities, given our interest in showing the inadequacy of a VaR threshold, it suffices to consider any reasonable risk measure. Consequently, we assume that traders employ a conditional value-at-risk measure or a CVaR measure. It should be noted that in general this problem is challenging in that it leads to a stochastic nonconvex optimization problem, whose optimal value is often hard to determine either analytically or computationally. It is worth emphasizing that the choice of the risk measure, while relevant in proving our result, does not limit the main claim of inadequacy. More precisely, *it suffices to show that the VaR threshold is inadequate for risk management, if a risk-seeking trader employing any reasonable risk measure can accumulate large or infinite risk.*

Given our choice of CVaR as the trader's risk measure, our study reduces to an examination of the interactions between VaR and CVaR. Furthermore, we believe that this interaction poses a new source of information. For instance, the interaction between different nonlinear phenomena has long been a source of new insight. In physics, for example, properties of subatomic particles are typically measured by how the particles are deflected by magnetic fields [116]. We consider a similar line of argument concerning the relation between VaR and a particular coherent risk measure, namely the conditional VaR or CVaR.

### 4.1.1 Relevance

In the financial industry, the regulators are primarily concerned with controlling risk and avoiding default, thereby protecting the firm and their customers. As the events of the last decade have revealed, the implications of such risk management can be felt on the entire economy. The regulators achieve this by laying down rules and regulations that participating firms must abide by. In the financial industry, the Basel Accords (currently Basel III) provide banking regulations issued by the Basel Committee on Banking Supervision (BCBS) [117]. Amongst other guidelines, these regulations provide an international standard for the reserve requirements that banks should abide by. Naturally, these requirements grow with the level of risk exposure. In contrast with banking regulators, banks and security firms see the capital adequacy requirements as a trade-off between the risk of default and the potential revenues from operating with higher levels of capital.

Banks, securities firms, and regulators use models to quantify risk and set capital requirements accordingly (see [118] for explanation of different types of models used by regulators and firms). Value-at-risk models are popular in financial institutions. For a given portfolio, these models are designed to estimate the maximum amount that a bank could lose over a specified period with a given probability [119]. The resulting measures provide a metric of the risk exposure of the given portfolio. Risk managers may then decide if the firm is comfortable with this level of exposure. Value-at-risk (VaR) models are extensively used for reporting (both internal and regulatory requirements) and limiting risk, allocating capital, and measuring performance [120]. Despite its popularity and broad acceptance as an industry standard in financial industry, VaR models may have played a role in many financial losses like the failure of Long Term Capital Management (LTCM) hedge fund in 1998 (cf. [121, 122]). More recently a VaR model masked JP Morgan's $2 billion loss in 2012 [123]. Yet, VaR continues to be used for measuring and managing risk. In the wake of the recent financial crisis, the issue of risk quantification, management, and mitigation has been of interest both from a firm perspective and a regulatory standpoint.

In recent years, yet another concern of regulators is that the increasing reliance on high bonuses results in reduced profits in the investment banking sector. This may actually expose trading firms to high levels of operational risks. This concern has also supported raising the capital requirements of firms [124]. In [125], Rajan has argued that firm managers are given incentives to take risk that generate severely adverse consequences with small probability; however, such firms offer generous compensation. This motivates a question as to whether risk-seeking behavior at the trader's level may have severe consequences from a firm standpoint. To the best of our knowledge, no simple models exist to provide formal support. The inadequacy of VaR was shown numerically through the study of the portfolio management of bonds, a model employed by several banks [126]. Specifically, it was observed that a 15% reduction in VaR resulted in a 15% increase in CVaR, a somewhat unexpected outcome given the overall goal of risk management. Our study is different in that it considers the management of risk in a general setting with an emphasis on examining the analytical problem from the standpoint of a trader. More specifically, this study constructs a model that shows that despite imposing a VaR threshold, risk-seeking traders may still assume an extremely high level of CVaR risk. This could lead the financial firm to ruin, a consequence that both firms and regulators want to avoid.

### 4.1.2 Summary of contributions

The main contributions of our work are as follows:

(i) First, when the asset returns are defined through a uniform distribution, we determine the maximum CVaR that a trader may assume while constrained by a VaR threshold. We generalize this result in showing that for general distributions with compact support, the maximum CVaR risk is finite. More important, the maximum risk level is bounded below by the upper bound of the support of the distribution and could possibly be large.

(ii) Second, motivated by the result in (i), we we show that risk-seeking traders can assume unbounded risk (captured via a CVaR metric) despite a VaR threshold when employing a portfolio of assets with Gaussian returns. This requires proving the unboundedness of the optimal value of a stochastic nonconvex optimization problem, a relatively challenging class of problems. We further extend this result to a regime where the asset returns have distributions with unbounded support.

It must be noted that when the distribution has bounded or unbounded support, even though there is a finite probability that the loss of the random variable is greater than a given VaR threshold and CVAR at a given threshold may be finite, it does not immediately follow that the CVAR function is unbounded as the risk-aversion parameter tends to its maximum value, namely 1. Our results provide the required mathematical support for this conjecture.

This chapter is organized as follows. In section 4.2, we provide a setting for the problem, provide a brief review of VaR and CVaR risk measures, and define the problem. In section 4.3, we show that for portfolios with independent asset returns with compact support, firms accumulate finite risk whose value is of the order of the upper bound of the support. In section 4.4, we extend this result to accommodate portfolios consisting of assets with independent Gaussian returns and more generally for distributions with unbounded support. In such an instance, it is seen that the traders can accumulate unbounded CVaR risk.

## 4.2 Problem setting

In this study, we consider a setting where a financial firm uses Value-At-Risk (VaR) as a means of managing risk. A trader employed at such a firm is allowed to choose any position as long as the VaR of his/her preferred position does not exceed a threshold imposed by the trading firm. Given this VaR constraint, the trader chooses a particular position based on her risk preferences. We model this problem from the trader's standpoint and consider settings under diverse assumptions on the distribution of asset returns.

In this problem, we assume the trader employs a Conditional Value-at-Risk (CVaR) measure for choosing her preferred position and hence refer to her as a CVaR trader. Given the VaR constraint imposed by the firm, the trader seeks to take on as much risk as possible as measured by CVaR. However, the trader must simultaneously ensure that the VaR constraint is not violated. Thus, a CVaR traders' preferred position is chosen based on it being the best (the one with most risk) possible position with respect to the CVaR risk measure while also abiding by the VaR constraint imposed by the firm. This VaR constrained CVaR maximization problem is referred to as the *Trader's problem*. Before discussing the Trader's problem statement in detail, we briefly review VaR and CVaR risk measures.

The Value-At-Risk (VaR) has been widely used as a measure of risk and is considered a de-facto standard for managing risk in the financial industry. Despite its wide acceptance and popularity, VaR as a risk measure has certain serious shortcomings; it ignores losses beyond VaR regardless of how large these losses might be. Furthermore, it is nonconvex unless derived from a Gaussian distribution and, consequently, difficult to optimize [38].

In 2002, Artzner et al [127] introduced an axiomatic methodology to characterize desirable properties of risk measures such as translation invariance, subadditivity, positive homogeneity and monotonicity. Risk measures satisfying these four axioms were said to be *coherent*. VaR is not a coherent risk measure because it lacks sub-additivity. However, the conditional value-at-risk (CVaR) measure [38, 128] belongs to this class of coherent risk measures. CVaR as a risk measure for financial applications has been quite popular in academic literature due to its superior mathematical properties.

We refer the reader to [38, 76, 77] for an introduction to the value at risk (VAR), conditional value at risk (CVAR) and coherent risk measures and define each formally before proceeding. The value at risk at the $\alpha$ level specifies the maximum loss with a specified confidence level $\alpha$ while its conditional variant is the conditional expected loss under the condition that the loss exceeds the VaR level.

Let $(\Omega, \mathscr{F}, \mathbb{P})$ be a probability space, where $\Omega \triangleq \mathbb{R}$ and $\mathscr{F} \triangleq \mathscr{B}(\mathbb{R})$. Let $X_i$ be an independent random variable defined on this probability space where $i \in \{1, 2, ..., n\}$ and $X_i$ denotes the loss incurred by stock $i$ in a specified time interval. For $w = (w_1, w_2, \ldots, w_n) \in \mathbb{R}^n_+$, consider the portfolio $Z(w) = w_1X_1 + w_2X_2 + \ldots + w_nX_n$. The Value-At-Risk (VaR) at level $\alpha$ of the portfolio $Z(w)$ is denoted by $V_\alpha(w)$ and the Conditional-VaR (CVaR) at level $\beta$ of the portfolio $Z(w)$ is denoted by $C(\beta, w)$.

Consider a trading firm that measures risk using a VaR risk measure. A trader in such a firm may have his own set of preferred positions and risk levels that arise from an optimization problem the trader solves. Suppose the trading firm uses a VaR confidence level $\alpha \in [0, 1]$ (typically $\alpha \geq 0.95$) and a VaR threshold $v \in \mathbb{R}_+$. The trading firm imposes a constraint that the positions $w$ satisfy $V_\alpha(w) \leq v$. This constraint restricts the portfolios of the trader and the set of VaR-admissible positions $w = (w_1, w_2, \ldots, w_n)$ for the portfolio $Z(w)$ is denoted by $\mathcal{A}$.

$$\mathcal{A} \stackrel{\text{def}}{=} \{w \in \mathbb{R}^n | \quad V_\alpha(w) \leq v\}.$$

Consider the problem from the standpoint of a trader. We referred to this problem as the *Trader's problem.* The traders' goal is to take on as much CVaR risk as possible while maintaining the VaR constraint imposed by the trading firm. Thus, it is natural to assume that the trader chooses $\bar{w}$ as his preferred position based on it being the maximizer of a CVaR-risk-maximization problem problem over the set of VaR - admissible positions denoted by $\mathcal{A}$. Also, the CVaR function depends on the confidence level chosen by the trader. However, since the traders' goal is to maximize risk, the trader will want to maximize the CVaR function over all possible confidence levels. Mathematically, this problem can be viewed as the following optimization problem:

**Definition 4.1 (Trader's problem)** Given $\alpha, v$ such that $\alpha \in (0,1)$ and $v > 0$. Then the Trader's problem is given by

$$\begin{aligned} \text{(Trad(D))} \qquad \max_{\beta, w \geq 0} \quad & C(\beta, w) \\ \text{subject to} \quad & V_\alpha(w) \leq v, \\ & \beta \in [0,1]. \end{aligned}$$

where in the notation (Trad(D)), D is a vector denoting the distributions of the random variables $X_1, X_2, \ldots, X_n$ that make up the portfolio $Z(w) = w_1X_1 + w_2X_2 + \ldots + w_nX_n$. ∎

VaR is nonconvex unless it is derived from a Gaussian distribution. Thus, the set of VaR-admissible positions or the constraint set in the Trader's problem (Trad(D)) is not convex in general. Furthermore, if $Z(w)$ is convex in $w$ then $C(\beta, w)$ is convex in $w$ for a given $\beta$ (cf. [38]), the resulting optimization problem requires maximizing a convex function over a possibly nonconvex constraint. In effect, global optima of the Trader's problem (Trad(D)) are not easily solvable. We may also refer to this Trader's problem as the Trader's CVaR problem to emphasize the fact that the risk measure being maximized in the Trader's problem (Trad(D)) is the CVaR risk measure. In the next two sections, we provide a mathematical solution for the Trader's problem, under differing assumptions on the distribution.

Before proceeding, we review some notation. Let $Z$ denote a random variable (as in definition 4.1) where $Z = w_1X_1 + w_2X_2 + \ldots + w_nX_n$. We assume that $Z$ has density $f(x)$ and distribution function $F(x)$. Furthermore, we denote by $z_\beta$ the quantile of $Z$ at level $\beta$.

## 4.3 Asset distributions with bounded support

We first consider the simplest distribution with bounded support, viz. the Uniform distribution on $[0, 1]$. In this case, it is possible to articulate the solution of the Trader's problem precisely as shown in Section 4.3.1. It is natural to wonder what happens when the asset returns have a general distribution with bounded support. We analyze this question in section 4.3.2 and show that for general distributions with bounded support, the maximum CVaR risk remains finite. We also show that the maximum risk level is bounded below by the upper bound of the support of the distribution and could possibly be large.

### 4.3.1 Returns with uniform distibutions

Assume $X_i$ are independent uniform random variables on $[0, 1]$, where $i \in \{1, 2\}$. Let $Z(w_1, w_2) \triangleq w_1X_1 + w_2X_2$. By symmetry and without loss of generality, we may assume $0 \leq w_1 \leq w_2$. Then, the density of Z is given by

$$f_{Z(w_1,w_2)}(a) = \begin{cases} \frac{a}{w_1w_2} & 0 \leq a \leq w_1, \\ \frac{1}{w_2} & w_1 \leq a \leq w_2, \\ \frac{w_1+w_2-a}{w_1w_2} & w_2 \leq a \leq w_1 + w_2, \\ 0 & \text{otherwise.} \end{cases}$$

This gives us the distribution function of Z as

$$F_{Z(w_1,w_2)}(a) = \begin{cases} 0 & a \leq 0 \\ \frac{a^2}{2w_1w_2} & 0 \leq a \leq w_1, \\ \frac{a}{w_2} - \frac{w_1}{2w_2} & w_1 \leq a \leq w_2, \\ 1 - \frac{(w_1+w_2-a)^2}{2w_1w_2} & w_2 \leq a \leq w_1 + w_2, \\ 1, & a \geq w_1 + w_2. \end{cases}$$

Figure 1 shows the density and distribution functions for the sum of two uniform random variables with $a = 20$.

The Trader's problem requires articulating the solution set of the problem given by (Trad(D)). Here, we address this problem when the returns are cap-

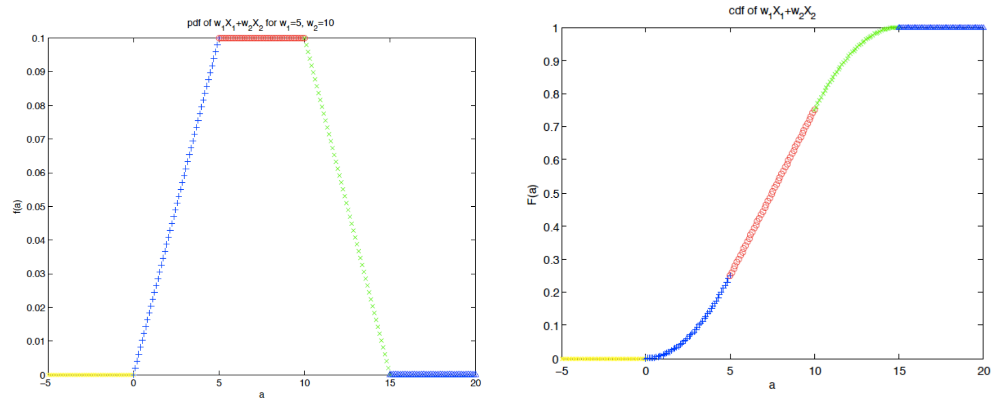


Figure 4.1: Density and distribution functions for the sum of two uniform random variables

tured by independent and identically distributed uniform random variables. More specifically, we derive explicit analytical expressions for VaR and CVaR functions in the case of two independent and identically distributed uniform random variables.

For a fixed $\alpha \in [0, 1]$, let $w = (w_1, w_2) \in \mathbb{R}^2_+$ and suppose the VaR at level $\alpha$ at the point $w$ is denoted by $V_\alpha(w)$. Since $Z(w)$ is a continuous random variable, $V_\alpha(w)$ is the inverse of the distribution function $F$ at the $\alpha$ level and is given by

$$V_\alpha(w_1, w_2) = \begin{cases} \sqrt{2\alpha w_1 w_2}, & 0 \le \alpha \le \frac{w_1}{2w_2}, \\ w_2\left(\alpha + \frac{w_1}{2w_2}\right), & \frac{w_1}{2w_2} \le \alpha \le 1 - \frac{w_1}{2w_2}, \\ w_1 + w_2 - \sqrt{2w_1 w_2(1-\alpha)}, & 1 - \frac{w_1}{2w_2} \le \alpha \le 1. \end{cases} \tag{4.1}$$

This can be rewritten compactly as

$$\begin{aligned} V_\alpha(w_1, w_2) = & \sqrt{2\alpha w_1 w_2}\mathbf{1}_{[0, \frac{w_1}{2w_2}]}(\alpha) + w_2\left(\alpha + \frac{w_1}{2w_2}\right)\mathbf{1}_{(\frac{w_1}{2w_2}, 1-\frac{w_1}{2w_2}]}(\alpha) \\ & + w_1 + w_2 - \sqrt{2w_1 w_2(1-\alpha)}\mathbf{1}_{(1-\frac{w_1}{2w_2}, 1]}(\alpha). \end{aligned}$$

Further, for a given $\beta \in [0, 1]$, the CVaR at level $\beta$ at the point $w$ is denoted by $C(\beta, w)$. Since $Z(w)$ is a continuous random variable, $C(\beta, w)$ represents the conditional tail expectation and is given by

$$C(\beta, w) \triangleq \mathbb{E}[Z(w) | Z(w) \ge V_\beta(w)].$$

Thus, $C(\beta, w)$ is given by

$$C(\beta,w) = \begin{cases} \frac{1}{1-\beta}\left[\frac{-2\beta\sqrt{2w_1w_2\beta}}{3} + \frac{w_1+w_2}{2} + \frac{w_1^2}{8w_2}\right] & 0 \le \beta \le \frac{w_1}{2w_2}, \\ \frac{w_1}{2} + \frac{w_2(1+\beta)}{2} + \frac{w_1^2}{6w_2(1-\beta)} & \frac{w_1}{2w_2} \le \beta \le 1 - \frac{w_1}{2w_2}, \\ (w_1+w_2) - \frac{2}{3}\sqrt{2w_1w_2(1-\beta)} & 1 - \frac{w_1}{2w_2} \le \beta \le 1. \end{cases} \quad (4.2)$$

It can also be viewed as

$$C(\beta,w) = \begin{cases} V_\beta(w) + \frac{1}{1-\beta}\left[\sqrt{2w_1w_2\beta}(\frac{\beta-3}{3}) + \frac{w_1+w_2}{2} + \frac{w_1^2}{8w_2}\right] & 0 \le \beta \le \frac{w_1}{2w_2}, \\ V_\beta(w) + \frac{w_2(1-\beta)}{2} + \frac{w_1^2}{6w_2(1-\beta)} & \frac{w_1}{2w_2} \le \beta \le 1 - \frac{w_1}{2w_2}, \\ V_\beta(w) + \frac{1}{3}\sqrt{2w_1w_2(1-\beta)} & 1 - \frac{w_1}{2w_2} \le \beta \le 1. \end{cases}$$

By substituting $r = \frac{w_1}{w_2}$, it follows that $0 \le r \le 1$, $V_\alpha(w)$ may be written as

$$V_\alpha(w) = V_\alpha(w_2, r) = \begin{cases} w_2\sqrt{2\alpha r} & 0 \le \alpha \le \frac{r}{2}, \\ w_2\left(\alpha + \frac{r}{2}\right) & \frac{r}{2} \le \alpha \le 1 - \frac{r}{2}, \\ w_2[r + 1 - \sqrt{2r(1-\alpha)}] & 1 - \frac{r}{2} \le \alpha \le 1, \end{cases} \quad (4.3)$$

and

$$C(\beta,w) = C(\beta, w_2, r) = \begin{cases} \frac{w_2}{1-\beta}\left[\frac{-2\beta\sqrt{2r\beta}}{3} + \frac{r+1}{2} + \frac{r^2}{8}\right] & 0 \le \beta \le \frac{r}{2}, \\ w_2[\frac{r}{2} + \frac{(1+\beta)}{2} + \frac{r^2}{6(1-\beta)}] & \frac{r}{2} \le \beta \le 1 - \frac{r}{2}, \\ w_2[r + 1 - \frac{2}{3}\sqrt{2r(1-\beta)}] & 1 - \frac{r}{2} \le \beta \le 1. \end{cases} \quad (4.4)$$

Note that $r \le 1$ when $w_1 \le w_2$. The next lemma shows that the maximum value for the CVaR function can only be attained at points on the VaR boundary (i.e. when the VaR constraint is active)(See Figure 4.2).

**Lemma 4.2** *Consider a portfolio $Z = w_1X_1 + w_2X_2$ where $X_1$ and $X_2$ are independent uniform random variables on $[0,1]$. Then $C(\beta, w)$ given by (4.4) attains a maximum when $V_\alpha(w) = v$.*

**Proof :** If $w = (w_1, w_2)$ is such that $V_\alpha(w) < v$. Let $r = \frac{w_1}{w_2}$. The following cases arise:

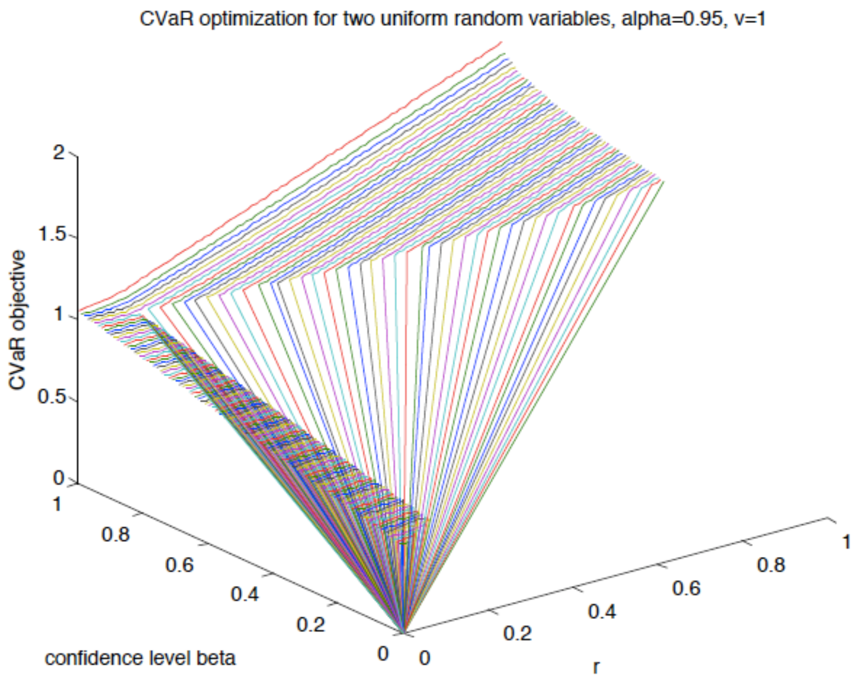


Figure 4.2: VaR for a sum of two uniform random variables

**Case (i)** $[\frac{r}{2} \leq \alpha \leq 1 - \frac{r}{2}]$**:** From (4.3), $V_\alpha(w) = w_2\left(\alpha + \frac{r}{2}\right) < v$. Now consider the point $w'$ where

$$w' \triangleq \left(\frac{rv}{\alpha + \frac{r}{2}}, \frac{v}{\alpha + \frac{r}{2}}\right).$$

Consequently, we have $r = \frac{w_1}{w_2} = \frac{w_1'}{w_2'}$. Therefore, for both $w$ and $w'$, $\beta$ can lie in either $\frac{r}{2} \leq \beta \leq 1 - \frac{r}{2}$ or $1 - \frac{r}{2} \leq \beta \leq 1$. In either case, from (4.4) and since $w_2 < w_2'$, we obtain that $C(\beta, w) < C(\beta, w')$.

**Case (ii)** $[1 - \frac{r}{2} \leq \alpha \leq 1]$ **:** From (4.3), $V_\alpha(w) = w_2[r+1-\sqrt{2r(1-\alpha)}] < v$. Applying a similar logic, we get that for

$$w_2' = \frac{v}{r + 1 - \sqrt{2r(1-\alpha)}}$$

and $w' = (rw_2', w_2')$ , it follows that $C(\beta, w) < C(\beta, w')$. Thus, if $w$ is such that $V_\alpha(w) < v$, we can find a $\hat{w}$ such that $V_\alpha(\hat{w}) = v$ and $C(\beta, w) < C(\beta, \hat{w})$. This shows that for any $\beta$, the maximum value of $C(\beta, w)$ can be attained only at points $w$ satisfying $V_\alpha(w) = v$. ∎

In view of the previous lemma, unless otherwise specified, a point $w$ will be assumed to satisfy $V_\alpha(w) = v$. Also we assume $\alpha$ and $\beta$ are always greater than 0.9, since in practice $\alpha$ is taken to be 0.9 or 0.99 etc. Further, the constraint $V_\alpha(w) = v$ determines $w_2$ as a function of $r$. Therefore in what follows, we consider $C(\beta, w)$ as a function of $\beta$ and $r$ only. Thus we have,

$$w_2 = \begin{cases} \frac{v}{\sqrt{2\alpha r}} & 0 \leq \alpha \leq \frac{r}{2}, \\ \frac{v}{\left(\alpha + \frac{r}{2}\right)} & \frac{r}{2} \leq \alpha \leq 1 - \frac{r}{2}, \\ \frac{v}{[r+1-\sqrt{2r(1-\alpha)}]} & 1 - \frac{r}{2} \leq \alpha \leq 1. \end{cases} \tag{4.5}$$

Since $\alpha \leq \beta$, we obtain $C(\beta, r)$ to be given as

$$C(\beta,r) = \begin{cases} \frac{v}{(\alpha+\frac{r}{2})}[\frac{r}{2} + \frac{(1+\beta)}{2} + \frac{r^2}{6(1-\beta)}] & \frac{r}{2} \leq \beta \leq 1-\frac{r}{2}, \frac{r}{2} \leq \alpha \leq 1-\frac{r}{2}, \\ \frac{v}{(\alpha+\frac{r}{2})}[r+1-\frac{2}{3}\sqrt{2r(1-\beta)}] & 1-\frac{r}{2} \leq \beta \leq 1, \frac{r}{2} \leq \alpha \leq 1-\frac{r}{2}, \\ \frac{v}{[r+1-\sqrt{2r(1-\alpha)}]}[r+1-\frac{2}{3}\sqrt{2r(1-\beta)}] & 1-\frac{r}{2} \leq \beta \leq 1, 1-\frac{r}{2} \leq \alpha \leq 1. \end{cases} \tag{4.6}$$

Thus, the Trader's problem in this case reduces to maximizing a piecewise smooth function in the two dimensional space of $(\beta, r)$. Now, we study the behavior of the CVaR function in each of these three kinks and observe that the CVaR function has the following behavior.

**Lemma 4.3** *Consider a portfolio consisting of two independent uniform random variables on* $[0,1]$. *Let* $C(\beta, r)$ *be given by* (4.6). *For* $i = 1, \ldots, 3$, *let* $C(\beta_i, r_i)$ *denote the maximal value of* $C(\beta, r)$ *on the ith kink. Assume that the given VaR level* $\alpha \geq 0.86$. *Then the following hold:*

1. $(\beta_3, r_3) = (1, 1)$ *is the maximizer of* $C(\beta, r)$ *on the third kink.*
2. $(\beta_2, r_2) = (1, 2(1-\alpha))$ *is the maximizer of* $C(\beta, r)$ *on the second kink.*
3. $C(\beta_2, r_2) \leq C(\beta_3, r_3)$.
4. $C(\beta_1, r_1) \leq C(\beta_2, r_2)$.

**Proof :**

1. Consider the third kink where,

$$\begin{aligned} C(\beta,r) &= \frac{v}{[r+1-\sqrt{2r(1-\alpha)}]}[r+1-\frac{2}{3}\sqrt{2r(1-\beta)}] \\ &\leq \frac{v}{[r+1-\sqrt{2r(1-\alpha)}]}[r+1] = g(r). \end{aligned}$$

The function $g(r)$ on the right hand side above is an increasing function of $r$, since

$$\frac{d}{dr}\left(\frac{1}{1-\frac{\sqrt{2r(1-\alpha)}}{r+1}}\right) = \left(\frac{1}{1-\frac{\sqrt{2r(1-\alpha)}}{r+1}}\right)^2 \frac{d}{dr}\left(\frac{\sqrt{2r(1-\alpha)}}{r+1} - 1\right).$$

A further computation and simplification gives that

$$\frac{d}{dr}\left(\frac{1}{1-\frac{\sqrt{2r(1-\alpha)}}{r+1}}\right) = \left(\frac{1}{1-\frac{\sqrt{2r(1-\alpha)}}{r+1}}\right)^2 \left(\frac{\sqrt{2(1-\alpha)}}{r+1}\right)\left(\frac{1-r}{\sqrt{r}(r+1)}\right) > 0,$$

since $r > 2(1-\alpha)$. Thus, the upper bound of the function $g(r)$ is given by $2v/(2-\sqrt{2(1-\alpha)})$ (when $r = 1$). But this value is achieved by $C(\beta, r)$ when $(\beta, r) \equiv (1, 1)$. This implies that the global maximizer of $C(\beta, r)$ over the third kink is $(1, 1)$.

2. On the second kink, for a fixed $r$ we have that $C(\beta, r)$ is an increasing function of $\beta$ since

$$\frac{dC}{d\beta} = \frac{v}{(\alpha + \frac{r}{2})}\left(\frac{2\sqrt{r}}{\sqrt{(1-\beta)}}\right) > 0.$$

Thus, on the second kink, $C(\beta, r)$ is maximized when $\beta = 1$. In this case

$$C(1, r) = \frac{v(r+1)}{\alpha + \frac{r}{2}}.$$

Now, $C(1, r)$ is an increasing function of $r$ since

$$\frac{dC}{dr} = \frac{v}{\alpha + \frac{r}{2}} - \frac{v(r+1)}{2(\alpha + \frac{r}{2})^2} = \frac{2v(\alpha + \frac{r}{2}) - v(r+1)}{2(\alpha + \frac{r}{2})^2} = \frac{v(2\alpha + r - r - 1)}{2(\alpha + \frac{r}{2})^2} \geq 0,$$

since $\alpha \geq 0.5, 2\alpha - 1 \geq 0$ Thus, on the second kink, $C(\beta, r)$ is maximized at $\beta = 1$ and $r = 2(1-\alpha)$(which is the largest value of $r$ in this region).The maximum value is given by

$$C(1, 2(1-\alpha)) = \frac{v}{\alpha + 1 - \alpha}[2(1-\alpha) + 1] = v(1 + 2(1-\alpha)).$$

3. From the proof above we have, $C(\beta_2, r_2) = C(1, 2(1-\alpha)) = v(1+2(1-\alpha))$ and

$$C(\beta_3, r_3) = C(1, 1) = \frac{2v}{2-\sqrt{2(1-\alpha)}}$$

Thus, $C(\beta_2, r_2) \leq C(\beta_3, r_3)$ if

$$v(1 + 2(1-\alpha)) \leq \frac{2v}{2-\sqrt{2(1-\alpha)}}.$$

But this is implied by

$$2-\sqrt{2(1-\alpha)}+2(1-\alpha)[2-\sqrt{2(1-\alpha)}] \leq 2.$$

By cross multiplying, squaring and rearranging terms, we get that we need to show that

$$2(1-\alpha)\left(2-\sqrt{2(1-\alpha)}-\frac{\sqrt{2(1-\alpha)}}{2(1-\alpha)}\right) \leq 0$$

Since $2(1-\alpha) \geq 0$, the above inequality holds only if

$$2-\sqrt{2(1-\alpha)}-\frac{1}{\sqrt{2(1-\alpha)}} \leq 0.$$

Again, since $\sqrt{2(1-\alpha)} \geq 0$, this holds only if

$$\begin{aligned} 2\sqrt{2(1-\alpha)}-2(1-\alpha)-1 \leq 0 &\implies 2\sqrt{2(1-\alpha)}-(3-2\alpha) \leq 0 \\ &\implies \sqrt{2(1-\alpha)} \leq \frac{3-2\alpha}{2}. \end{aligned}$$

Again, by a further simplification we see that this holds if

$$(1-2\alpha)^2 \geq 0.$$

4. We now show that $C(\beta_1, r_1) \leq C(\beta_2, r-2)$. From the proof above we have,

$$C(\beta_2, r_2) = C(1, 2(1-\alpha)) = v(1+2(1-\alpha)).$$

On the first kink we have, $\frac{r}{2} \leq \beta \leq 1-\frac{r}{2}$, $\frac{r}{2} \leq \alpha \leq 1-\frac{r}{2}$ and

$$C(\beta, r) = \frac{v}{\left(\alpha+\frac{r}{2}\right)}\left[\frac{r}{2}+\frac{(1+\beta)}{2}+\frac{r^2}{6(1-\beta)}\right]$$

For a fixed $r$, on the first kink, we have $C(\beta, r)$ is an increasing function of $\beta$ as

$$\frac{dC}{d\beta} = \frac{v}{\left(\alpha+\frac{r}{2}\right)}\left(\frac{1}{2}+\frac{r^2}{6(1-\beta)^2}\right) > 0.$$

Thus, the maximum value of $C(\beta, r)$ in this region occurs when $\beta =$

$1-\frac{r}{2}$ and the maximum value is

$$\begin{aligned} C\left(1-\frac{r}{2},r\right) &= \frac{v}{\left(\alpha+\frac{r}{2}\right)}\left(\frac{r}{2}+\frac{(1+\beta)}{2}+\frac{r^2}{6(1-\beta)}\right) \\ &= \frac{v}{\left(\alpha+\frac{r}{2}\right)}\left(\frac{r}{2}+\frac{(1+1-\frac{r}{2})}{2}+\frac{r^2}{6(1-1+\frac{r}{2})}\right) \\ &= \frac{v}{\left(\alpha+\frac{r}{2}\right)}\left(\frac{r}{2}+1-\frac{r}{4}+\frac{r}{3}\right) = \frac{v}{\left(\alpha+\frac{r}{2}\right)}\left(\frac{7r}{12}+1\right). \end{aligned}$$

Consider $f(r) = \frac{v}{\left(\alpha+\frac{r}{2}\right)}\left(\frac{7r}{12}+1\right)$. Then

$$\begin{aligned} f'(r) &= \frac{v}{\left(\alpha+\frac{r}{2}\right)}\left(\frac{7}{12}\right) - \left(\frac{7r}{12}+1\right)\frac{v}{2\left(\alpha+\frac{r}{2}\right)^2} \\ &= \frac{\frac{7v}{6}\left(\alpha+\frac{r}{2}\right)-(\frac{7r}{12}+1)v}{2\left(\alpha+\frac{r}{2}\right)^2} = v\frac{\left(\frac{7}{6}\alpha+\frac{7r}{12}-\frac{7r}{12}-1\right)}{2\left(\alpha+\frac{r}{2}\right)^2} \\ &= v\frac{\left(\frac{7}{6}\alpha-1\right)}{2\left(\alpha+\frac{r}{2}\right)^2} \geq 0 \quad \text{if} \quad \alpha \geq \frac{6}{7} = 0.86. \end{aligned}$$

Thus, for $\alpha \geq \frac{6}{7}$, the maximum of $f(r)$ on the region 1 will be when $r$ takes on the maximum possible value in this region i.e. when $r = 2(1-\alpha)$. Thus, on region 1

$$\begin{aligned} C(\beta,r) \leq f(2(1-\alpha)) &= \frac{v}{\left(\alpha+\frac{2(1-\alpha)}{2}\right)}\left(\frac{7.2(1-\alpha)}{12}+1\right) \\ &= v\left(\frac{7(1-\alpha)}{6}+1\right). \end{aligned}$$

Since we have that $v\left(\frac{7(1-\alpha)}{6}+1\right) \leq v\left(1+2(1-\alpha)\right)$, it follows that $C(\beta_1,r_1) \leq C(\beta_2,r_2)$.

∎

Let $U = (U,U)$ denote two independent uniform random variables on $[0,1]$. In this case, the solution to the Trader's problem $Trad(U)$ is characterized in the proposition that follows.

**Proposition 4.4 (Solution for a portfolio with two uniforms on [0,1] )** *Given two independent uniform random variables on* $[0,1]$ *and* $\alpha \geq 0.9$*, the solution to the Trader's problem* $(Trad(U))$ *occurs at points given by* $(\beta^*, w^*)$

*where*

$$\beta^* = 1, \qquad w_1^* = w_2^*, \text{ and } V_\alpha(w^*) = v.$$

**Proof :** By Lemma 4.3(3),(4), we have that $C(\beta_1, r_1) \leq C(\beta_2, r_2) \leq C(\beta_3, r_3)$. Thus, the maximum value of $C(\beta, r)$ is attained on the third region. By Lemma 4.3(1), $(1,1)$ is the global maximizer of $C(\beta, r)$ over the third region. It follows that $C(\beta, r)$ in (4.6) attains the global maximum when $\beta = 1, r = 1$. But $r = 1$ is equivalent to $w_1 = w_2$. Also, the equation (4.6) holds for points with $V_\alpha(w) = v$. This completes the proof of the proposition. ∎

If we consider $X_1 \sim U([0,a])$ and $X_2 \sim U([0,b])$ and $Z = w_1X_1 + w_2X_2$, then repeating the calculations as before, we get that the density function of $Z$ is the density function obtained above with $w_1$ replaced by $aw_1$ and $w_2$ replaced by $bw_2$ under the observation that the formulae hold if $aw_1 \leq bw_2$. If $aw_1 \geq bw_2$, then we use the same formulae with $aw_1$ replaced by $bw_2$ and vice-versa, .ie. for $0 \leq aw_1 \leq bw_2$ the density of $Z$ is given by

$$f_{Z(w_1,w_2)}(p) = \begin{cases} \frac{p}{abw_1w_2} & 0 \leq p \leq aw_1, \\ \frac{1}{bw_2} & aw_1 \leq p \leq bw_2, \\ \frac{aw_1+bw_2-p}{abw_1w_2} & bw_2 \leq p \leq aw_1 + bw_2, \\ 0 & \text{otherwise.} \end{cases}$$

This gives us the distribution function of $Z$ as

$$F_{Z(w_1,w_2)}(p) = \begin{cases} 0 & p \leq 0 \\ \frac{p^2}{2abw_1w_2} & 0 \leq p \leq aw_1, \\ \frac{p}{bw_2} - \frac{aw_1}{2bw_2} & aw_1 \leq p \leq bw_2, \\ 1 - \frac{(aw_1+bw_2-p)^2}{2abw_1w_2} & aw_2 \leq p \leq aw_1 + bw_2, \\ 1 & p \geq aw_1 + bw_2. \end{cases}$$

Let $U = (U_1, U_2)$ denote two independent uniform random variables on $[0,a]$ and $[0,b]$ respectively. In this case, the solution to the Trader's problem $Trad(U)$ is characterized in the proposition that follows.

**Proposition 4.5 (Solution for a portfolio with two general uniforms)** *Given two independent uniform random variables on* $[0, a]$ *and* $[0, b]$ *respectively and* $\alpha \geq 0.9$*, the solution to the Trader's problem* (Trad(D)) *occurs at points given by* $(\beta^*, w^*)$ *where*

$$\begin{aligned} \beta^* &= 1 \\ a w_1^* &= b w_2^* \\ V_\alpha(w^*) &= v. \end{aligned}$$

The VaR and CVaR obtained earlier in the case of two uniforms on unit interval, can be modified by replacing $w_1$ by $aw_1$ and $w_2$ by $bw_2$ to get the VaR and CVaR functions for uniforms on $[0, a], [0, b]$. Following an analysis similar to the case of two uniforms on the unit interval, the proof of the above proposition is straightforward and is omitted.

### 4.3.2 Returns with general distributions with bounded support

In this section, we generalize the results obtained for returns with uniform distributions to a regime where returns have general distributions, albeit with bounded support. In this setting, we show that the Trader's problem always has a bounded(finite) solution. In other words, given the VaR constraint imposed by the trading firm, there is a limit to the CVaR risk that the trader can take on. The distributional requirement is formalized through the following assumption.

**Assumption 4.1 (A1)** *Consider a random variable* $Z$ *defined as* $Z = w_1 X_1 + w_2 X_2 + \ldots + w_n X_n$ *with density function and distribution function denoted by* $f(z)$ *and* $F(z)$*, respectively. Suppose* $G(x)$ *is defined such that* $G'(x) = x f(x)$ *and the following hold:*

*(a) The smallest support of the distribution is* $[b, \bar{z}]$*.*

*(b)* $G$ *is continuous at* $\bar{z}$ *or* $\lim_{t \to \bar{z}} G(t) = G(\bar{z})$*.*

Assumption (A1(a)) allows for prescribing distributions with compact supports while (A1(b)) is a continuity requirement that aids in analysis. It should also be noted that $\mathbb{E}[Z] = \lim_{t \to \infty} G(t)$.

**Proposition 4.6 (Distributions with bounded support)** *Under Assumption (A1), the Trader's problem (4.1) has bounded CVaR risk and the maximal CVaR risk is bounded below by $\bar{z}$:*

$$\bar{z} \leq \max_{\beta,w} C(\beta, w) < \infty.$$

**Proof :** Consider the expectation of $Z$, conditional on $Z \geq z$. We denote this (tail) expectation by $\mathbb{E}_T(Z;z)$ and observe that

$$\mathbb{E}_T(Z;z) = \int_z^\infty af(a)da.$$

Clearly, since the distribution has support in $[b, \bar{z}]$, if $z > \bar{z}, \mathbb{E}_T(Z;z) = 0$. Now, if $b \leq z \leq \bar{z}$ then we have

$$\mathbb{E}_T(Z;z) = \int_z^\infty af(a)da = \int_z^{\bar{z}} G'(a)da = G(\bar{z}) - G(z),$$

where the second equality follows from the fundamental theorem of Calculus. Now consider $C(\beta, w)$ which can be simplified as

$$C(\beta, w) = \frac{1}{1-\beta}\int_{z_\beta}^\infty af(a)da = \frac{1}{1-F(z_\beta)}\mathbb{E}_T(Z;z_\beta) = \frac{G(\bar{z}) - G(z_\beta)}{1-F(z_\beta)}.$$

As $\beta \to 1$, the quantile of the distribution will tend to $\bar{z}$ since $[b, \bar{z}]$ is the smallest support of the distribution. Thus, we have that as $\beta \to 1$, $z_\beta \to \bar{z}$ or in other words $z \to \bar{z}$. Consequently, we obtain that

$$\lim_{\beta\to 1} C(\beta, w) = \lim_{z\to\bar{z}} \frac{G(\bar{z}) - G(z)}{1-F(z)}.$$

By assumption $\lim_{t\to\bar{z}} G(t) = G(\bar{z})$. By applying L'Hôpital's rule, we obtain

$$\lim_{\beta\to 1} C(\beta, w) = \lim_{z\to\bar{z}} \frac{-G'(z)}{-F'(z)} = \lim_{z\to\bar{z}} \frac{zf(z)}{f(z)} = \lim_{z\to\bar{z}} z = \bar{z}. \tag{4.7}$$

Thus, as $\beta \to 1$, $C(\beta, w)$ remains bounded. For all other $\beta$, $C(\beta, w) = \frac{G(\bar{z})-G(z_\beta)}{1-\beta}$ is clearly bounded. Thus, the objective of the Trader's problem is always bounded. Further, the feasible region of the Trader's problem is bounded. Thus, we have that for distributions with bounded support the Trader's problem 4.1 always has a bounded solution. In other words, the optimal CVaR risk for Trader's problem is bounded.

Further, from (4.7), we can conclude

$$\bar{z} \;\leq\; \max_{\beta,w} C(\beta, w) \;<\infty$$

∎

**Remark:** In effect, the maximal CVaR though finite takes on a large value when the support is large although finite.

## 4.4 Asset distributions with unbounded support

In section 4.4.1, we first analyze the Trader's problem in a setting where the portfolio consists of independent Gaussian random variables. Next we extend our findings to the case of general distributions with unbounded support. The assumptions used in analyzing this generalization are captured in Assumption 4.2.

### 4.4.1 Gaussian returns

We begin this section with a brief review of some properties of Gaussian random variables. Recall that a Gaussian random variable with mean zero and variance one is called a *standard Gaussian random variable*. The probability density function of a standard Gaussian random variable is denoted by $\phi$ and is given by

$$\phi(z) \triangleq \frac{1}{\sqrt{2\pi}} \exp\left(\frac{-z^2}{2}\right).$$

The distribution function of a standard Gaussian distribution is denoted by $\Phi$. If $X \sim N(\mu, \sigma)$, then the *density* of $X$ is given by

$$f(x) = \phi\left(\frac{x-\mu}{\sigma}\right) = \frac{1}{\sqrt{2\pi\sigma}} \exp\left(\frac{-1}{2}\left(\frac{x-\mu}{\sigma}\right)^2\right),$$

where $\phi(z)$ is the density of a standard Gaussian random varialbe. The *cumulative distribution function of* $X$ is given by

$$F(x) = \Phi\left(\frac{x-\mu}{\sigma}\right),$$

where $\Phi(z)$ denotes the distribution function of a standard Gaussian random variable. From these equations, the VaR at level $\alpha$ and the CVaR at level $\alpha$ for the random variable $X \sim N(\mu, \sigma)$ are given by

$$\text{VaR}_{\alpha}(X) = \mu + \sigma\Phi^{-1}(\alpha) \quad \text{and} \quad \text{CVaR}_{\alpha}(X) = \mu + \sigma\frac{\phi(z_\alpha)}{1-\Phi(z_\alpha)},$$

where $z_\alpha = \frac{x_\alpha - \mu}{\sigma}$ is the standardized $\alpha$-th quantile of the distribution. If $X_i \sim N(0,1)$, the standard Gaussian distribution for $i \in \{1, 2, \ldots, n\}$, then $Z(w) = \sum_{i=1}^{n} w_i X_i \sim N(0, \sum_{i=1}^{n} w_i^2)$. Thus the Trader's problem (Trad(G)), (where $G$ denotes the standard Gaussian distribution) in this case becomes

$$\begin{aligned} (\text{Trad(G)}) \qquad & \max_{\beta\in[0,1), w\geq 0} && \|w\| \left(\frac{\phi(z_\beta)}{1-\Phi(z_\beta)}\right) \\ & \text{subject to} && \|w\|\ \Phi^{-1}(\alpha) \leq v. \end{aligned}$$

This can be rewritten as

$$\begin{aligned} (\text{Trad(G)}) \qquad & \max_{\beta\in[0,1), w\geq 0} && \|w\| \left(\frac{\phi(z_\beta)}{1-\beta}\right) \\ & \text{subject to} && \|w\| \leq \frac{v}{\Phi^{-1}(\alpha)}. \end{aligned}$$

Substituting $\|w\| = R$ and $\frac{v}{\Phi^{-1}(\alpha)} = c$, we get

$$\begin{aligned} (\text{Trad(G)}) \qquad & \max_{\beta\in[0,1), R\geq 0} && \left(R\frac{\phi(z_\beta)}{1-\beta}\right) \\ & \text{subject to} && R \leq c. \end{aligned}$$

Now, consider the objective function

$$C(\beta, R) = \left(R\frac{\phi(z_\beta)}{1-\beta}\right).$$

For a fixed $\beta$, $C(\beta, R)$ is an increasing function in $R$ since $\frac{\phi(z_\beta)}{1-\beta} \geq 0$. Thus the maximum will occur when $R = c$ and the Trader's problem reduces to

$$\text{(Trad(G))} \qquad \max_{\beta \in [0,1)} \quad c\left(\frac{\phi(z_\beta)}{1-\beta}\right) \tag{4.8}$$

Clearly, the optimal value of this problem is independent of $c$ and depends only on the behavior of the function on $\beta$ which we study next.

**Lemma 4.7** *Suppose $Z \sim N(\mu, \sigma^2)$. Let $\phi$ denote the standard Gaussian density. If $z_\beta$ is the standardized $\beta$-th quantile of $Z$, then*

$$\lim_{\beta \to 1} \quad \frac{\phi(z_\beta)}{1-\beta} = \infty. \tag{4.9}$$

**Proof :** We show that given any $M > 0$ there exists a $\beta \in [0, 1]$ such that $\frac{\phi(z_\beta)}{1-\beta} > M$. By the definition of $\phi$, this is equivalent to requiring that

$$\frac{\frac{1}{\sqrt{2\pi}} e^{\frac{-z_\beta^2}{2}}}{1-\beta} > M.$$

By simplifying and taking logarithms on both sides, we get that this is equivalent to requiring that

$$z_\beta < \sqrt{\log\left(\sqrt{2\pi}M(1-\beta)\right)^{-2}}. \tag{4.10}$$

Since the distribution function $\Phi$ is an increasing function, applying $\Phi$ to both sides and using the fact that $\Phi(z_\beta) = \beta$, we get that (4.10) holds if and only if

$$\beta < \Phi\left(\sqrt{\log\left(\sqrt{2\pi}M(1-\beta)\right)^{-2}}\right).$$

Define a function $g(\beta)$ as follows:

$$g(\beta) \triangleq \Phi\left(\sqrt{\log\left(\sqrt{2\pi}M(1-\beta)\right)^{-2}}\right) - \beta.$$

Using the fact that $\Phi' = \phi$, the derivative of $g(\beta)$ can easily be computed as

$$g'(\beta) = \frac{M}{\sqrt{\log\left(\sqrt{2\pi}M(1-\beta)\right)^{-2}}} - 1.$$

Thus, the above derivative is positive when

$$\frac{1}{\log\left(\sqrt{2\pi}M(1-\beta)\right)^{-2}} > \frac{1}{M^2}.$$

But the term on the right above is always positive. Thus, we get that $g'(\beta) > 0$ when the denominator on the left is positive. This holds when $\sqrt{2\pi}M(1-\beta) > 1$. In other words when

$$\beta < 1 - \frac{1}{\sqrt{2\pi}M},$$

we have that $g'(\beta) > 0$, implying that the function $g(\beta)$ is an increasing function. Furthermore, since $\Phi(.) \geq 0$ we obtain that $g(0) \geq 0$. Consequently, we may claim that the function $g(\beta)$ is an increasing function of $\beta$ and takes a positve value when $\beta = 0$. Therefore $g(\beta)$ is positive. Thus, given the $M$, there exists a $\beta < 1$ such that

$$\beta < \Phi\left(\sqrt{\log\left(\sqrt{2\pi}M\left(1-\beta\right)\right)^{-2}}\right).$$

In other words, we have shown given any $M > 0$ there exists a $\beta \in [0,1]$ such that

$$\frac{\phi(z_\beta)}{1-\beta} > M.$$

This shows that the limit articulated by (4.9) holds as required. ∎

This is supported by Figure 4.3 where it can be seen that as $\beta \to 1$, the CVaR function becomes unbounded. By Lemma 4.7, the objective in the Trader's problem (Trad(G)) is unbounded in $\beta$. In fact, in the next proposition, we show that a trader can assume infinite CVaR risk.

**Proposition 4.8 (Unbounded CVaR risk for standard Gaussian returns)** *Consider a portfolio consisting of $n$ independent and identically distributed standard Gaussian random variables. Then the solution to the Trader's prob-*

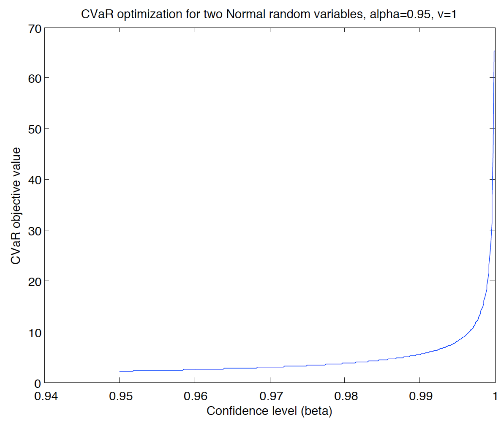


Figure 4.3: CVaR of a sum of two Gaussian random variables

*lem* (Trad(G)) *is given by any feasible $w$ together with $\beta = 1$. Furthermore, the CVaR risk corresponding to every solution of* (Trad(G)) *is infinite.*

**Proof :** From Lemma 4.7, the objective in (Trad(G)) is unbounded in $\beta$. Thus, for any $w$ that satisfies the VaR constraint, as $\beta \to 1$, the objective in (Trad(G)) goes to infinity. Thus the objective function attains its maximum at any feasible $w$ together with $\beta = 1$. Therefore, any feasible $w$ along with $\beta = 1$ is a solution to the Trader's problem for a portfolio consisting of $n$ standard Gaussian random variables. ∎

Next, we generalize the above result to the case of general (rather than standard) independent Gaussian random varaibles.

**Proposition 4.9 (Unbounded CVaR risk for general Gaussian returns)** *Consider a portfolio consisting of $n$ independent (not necessarily standard) Gaussian random variables with parameters $(\mu_1, \sigma_1), (\mu_2, \sigma_2), \ldots, (\mu_n, \sigma_n))$. Then, the solution to the Trader's problem* (Trad(G))*, (where $G = (G_1, G_2, ..., G_n)$) is given by any feasible $w$ together with $\beta = 1$. Furthermore, the CVaR risk corresponding to every solution of* (Trad(G)) *is infinite.*

**Proof :** For $i = 1, \ldots, n$, if $X_i \sim N(\mu_i, \sigma_i^2)$ then $Z \sim N\left(\sum_{i=1}^n \mu_i w_i, \sqrt{(\sum_{i=1}^n \sigma_i^2 w_i^2)}\right)$.

Then the Trader's problem is given by

$$\text{(Trad(G))} \qquad \max_{\beta\in[0,1),w} \quad \left(\sum_{i=1}^{n}\mu_i w_i + \sqrt{\sum_{i=1}^{n}\sigma_i^2 w_i^2}\frac{\phi(z_\beta)}{1-\Phi(z_\beta)}\right)$$
$$\text{subject to} \qquad \sqrt{\sum_{i=1}^{n}\sigma_i^2 w_i^2}\ \Phi^{-1}(\alpha) \le v.$$

Again in this case, the feasible region bounded. However, $\frac{\phi(z_\beta)}{1-\Phi(z_\beta)} \to \infty$ as $\beta \to 1$. Thus any feasible $w$ together with $\beta = 1$ is a solution to the Trader's problem and leads to unbounded CVaR risk. ∎

### 4.4.2 Assets with general distributions with unbounded support

In this section, we generalize the results of the previous section to allow for distributions with unbounded support. We show that under certain assumptions A2 (4.2) on the distributions, the Trader's problem has an unbounded solution. In other words, the trader has the ability to choose portfolios that drive the CVaR risk to infinity while obeying the VaR constraint imposed by the firm. We make the following assumption on the distributions of returns.

**Assumption 4.2 (A2)** *Consider a random variable $Z$ with unbounded support defined as $Z = w_1X_1 + w_2X_2 + \ldots + w_nX_n$. Further, suppose there exists a function $G(x)$ such that $G'(x) = xf(x)$ such that*

$$\lim_{t\to\infty} G(t) = L; -\infty < L < \infty$$

*where $f(z)$ is the density function of $Z$.*

From Assumption (A2), by unbounded support of $Z$ we have, $\lim_{\beta\to 1} z_\beta = \infty$;

Further note that, if we assume $|\mathbb{E}(Z)| < \infty$ then taking $G(x) = \int_{-\infty}^{x} xf(x)dx$, we get that $G'(x) = xf(x)$ and $\lim_{t\to\infty} G(t) = \mathbb{E}(Z) < \infty$. Thus, assumption (A2) holds for distributions with finite expectation.

**Proposition 4.10 (Returns with unbounded support)**

*Suppose (A2) holds for the random variable* $Z = w_1X_1 + w_2X_2 + \ldots w_nX_n$. *Then the Trader's problem* (4.1) *has unbounded CVaR risk.*

**Proof :** Consider the expectation of $Z$ conditional on $Z \geq z$ and denote this (tail) expectation by $\mathbb{E}_T(Z; z)$. This may be expressed as follows:

$$\mathbb{E}_T(Z; z) = \int_z^\infty af(a)da \;=\; \lim_{t\to\infty}\int_z^t G'(a)da \;=\; \lim_{t\to\infty}\left(G(t) - G(z)\right).$$

By (A2) since $\lim_{t\to\infty} G(t) = L$, we get that the expectation of the tail is

$$\mathbb{E}_T(Z; z) = L - G(z).$$

Now, consider

$$C(\beta, w) = \frac{1}{1-\beta}\int_{z_\beta}^\infty af(a)da \;=\; \frac{1}{1-F(z_\beta)}\mathbb{E}_T(Z; z_\beta) \;=\; \frac{L - G(z_\beta)}{1-F(z_\beta)}.$$

By (A2), as $\beta \to 1$, we have that $z_\beta \to \infty$ or in other words $z \to \infty$. Thus we obtain that

$$\lim_{\beta\to 1} C(\beta, w) = \lim_{z\to\infty}\frac{L - G(z)}{1 - F(z)}.$$

Again by (A2), $\lim_{t\to\infty} G(t) = L$. By applying L'Hôpital's rule we get

$$\lim_{\beta\to 1} C(\beta, w) = \lim_{z\to\infty}\frac{-G'(z)}{-F'(z)} \;=\; \lim_{z\to\infty}\frac{zf(z)}{f(z)} \;=\; \lim_{z\to\infty} z = \infty.$$

As a consequence, for distributions with unbounded support, the Trader's problem 4.1 leads to solutions with unbounded CVaR risk. ∎

As a consequence of the above proposition, we have the following corollaries which follow from verifying that the distributions satisfy (A2).

**Corollary 4.11 (Gaussian distribution)** *If* $X_1, X_2, \ldots, X_n$ *have i.i.d. Gaussian loss distributions then the Trader's problem has unbounded CVaR risk.*

**Proof :** Since $X_1, X_2, \ldots, X_n$ have Gaussian loss distributions, then the random variable $Z = \sum_{i=1}^n w_iX_i$ as in the definition of the Trader's problem

4.1 is also a Gaussian random variable. Without loss of generality, we assume $Z$ is a standard Gaussian random variable with density

$$f(x) = \frac{1}{\sqrt{2\pi}} \exp\left(\frac{-x^2}{2}\right),$$

We now show that $Z$ satisfies assumption (A2). We define $G(x)$ as

$$G(x) = \frac{-1}{\sqrt{2\pi}} \exp\frac{-x^2}{2}.$$

Then $G'(x) = xf(x)$ and clearly $\lim_{t\to\infty} G(t) = 0$. Thus, in the case of Gaussian random variables, (A2) is satisfied. By proposition 4.10, it follows that CVaR risk is unbounded. ∎

**Corollary 4.12 (Exponential distribution)** *If the random variable* $Z = \sum_{i=1}^{n} w_i X_i$ *is an exponential random variable, then the Trader's problem has unbounded CVaR risk.*

**Proof :** For the exponential random variable $Z$ the density is given by

$$f(x) = \begin{cases} \lambda \exp^{-\lambda x} & x \geq 0 \\ 0 & x < 0. \end{cases}$$

Consider

$$G(x) = \begin{cases} \frac{1}{\lambda}\left(-\lambda x \exp^{-\lambda x} - \exp^{-\lambda x}\right) & x \geq 0 \\ 0 & x < 0. \end{cases}$$

Then $G'(x) = \lambda x \exp^{-\lambda x} = xf(x)$. Further, by L'Hôpital's rule the first term in $G(x)$ goes to zero as $x \to \infty$ and the second term clearly goes to zero. Thus,$\lim_{t\to\infty} G(t) = 0$. Thus, for the case of exponential random variables, (A2) is satisfied. By proposition 4.10, it follows that for exponential random variables, the CVaR risk is unbounded. ∎

It can also be shown that such results hold for fat-tailed distributions, defined as follows.

**Definition 4.2 (Fat-tailed distribution)** *The distribution of a random* *variable $X$ is said to exhibit a fat tail if*

$$\lim_{x\to\infty} \mathbb{P}[X \;>\; x] \sim x^{-\alpha}, \qquad \textit{for some } \alpha > 0.$$

**Corollary 4.13 (Fat-tailed distribution)** *If the random variable $Z = \sum_{i=1}^{n} w_i X_i$ has a fat-tailed distribution, then the Trader's problem has unbounded CVaR risk.*

**Proof :** Since $Z$ has a fat-tail, as $x \to \infty$ the density is given by

$$f(x) = x^{-(1+\alpha)} \qquad \text{as } x \to \infty, \quad \alpha > 1.$$

For large $x$, taking $G(x) = \frac{x^{1-\alpha}}{1-\alpha}$ we get

$$G'(x) = \frac{1}{x^{\alpha}} = \frac{x}{x^{1+\alpha}} = xf(x).$$

Since $\alpha > 1$, $\lim_{t\to\infty} G(t) = 0$. Therefore, in the case of fat-tailed distributions, (A2) is satisfied. By proposition 4.10 we get that for fat-tailed distributions, the CVaR risk is unbounded. ∎

# CHAPTER 5

# CONCLUDING REMARKS AND SUMMARY

In the final chapter of this thesis, we present some concluding remarks for each Chapter.

## 5.1 Concluding remarks on Chapter 2

In Chapter 2, we examine a class of stochastic Nash games where players are constrained by continuous strategy sets with the overarching goal of characterizing the solution sets of the resulting games. Additionally, we allow for the coupling of strategy sets through possibly stochastic constraints and consider regimes where player payoff functions may be nonsmooth in nature. A corresponding analysis of solution sets of deterministic Nash games may be obtained through an examination of the sufficient equilibrium conditions. However, when the player objectives contain expectations, such an avenue is impeded by the generally intractable nature of the gradient maps. Instead, we consider whether almost-sure sufficiency conditions may be developed that are distinguished by their tractability and verifiability. Notably, these conditions, in turn, guarantee the existence of an equilibrium to a suitably defined scenario-based Nash game, whose equilibrium conditions are given by deterministic scalar variational inequality.

We begin by showing that when player payoffs are smooth, then a satisfaction of a coercivity condition in an almost-sure sense allows one to claim the existence of a Nash equilibrium. A corresponding uniqueness relationship is somewhat weaker and follows if a strict monotonicity of the mapping holds with finite probability. Extending the existence results to the nonsmooth regime, while not immediate, is provided and leads to a set-valued coercivity condition that is required to hold in an almost-sure sense. In both smooth and nonsmooth regimes, monotonicity of the mappings allows one to claim existence under markedly weaker requirements. When strategy sets are coupled by stochastic shared constraints, the associated mappings are, at best, monotone in the primal-dual space, suggesting that uniqueness of equilibria may be difficult to guarantee. Yet, by employing a suitable regularity condition, we prove that an equilibrium in the primal-dual space exists and is unique.

The application of the sufficiency conditions is generally not always immediate and requires analyzing the associated scenario-based Nash games. We apply these conditions to the study of Nash-Cournot games in risk-averse and coupled constraint settings and derive a host of characterization statements in these regimes.

This chapter has assumed convex objectives and strategy sets in a game theoretic regime. When strategy sets are nonconvex, there has been markedly less effort on the analysis of the associated equilibrium problem. One idea relies on leveraging the fact that an equilibrium of the game is a fixed point of the reaction map. This approach relies less on convexity and more on existence of fixed points of a map. Future work in this area would consider relaxing the convexity assumption and explore the use of nonconvex fixed point theory in such regimes to claim existence statements for equilibria.

## 5.2 Concluding remarks on Chapter 3

Finite-dimensional variational inequality and complementarity problems have proved to be extraordinarily useful tools for modeling a range of equilibrium problems in engineering, economics, and finance. This avenue of study is facilitated by the presence of a comprehensive theory for the solvability of variational inequality problems and their variants. When such problems are complicated by uncertainty, a subclass of models lead to variational problems whose maps contain expectations. A direct application of available theory requires access to analytical forms of such integrals and their derivatives, severely limiting the utility of existing sufficiency conditions for solvability.

To resolve this gap, we provide a set of integration-free sufficiency conditions for the existence of solutions to variational inequality problems, quasi-variational generalizations, and complementarity problems in settings where the maps are either single-valued or multi-valued. These conditions find utility in the existence of equilibria in the context of generalized nonsmooth stochastic Nash-Cournot games and strategic problems in power markets. We believe that these statements are but a first step in examining a range of problems in stochastic regimes. These include the development of stability and sensitivity statements as well as the consideration of broader mathematical objects such as stochastic differential variational inequality problems.

More generally, the question of the stability of stochastic problems under perturbations of the probability measure itself as well as under perturbations of the underlying data is an interesting, challenging, important and a practically relevant problem. Once again, the nonlinearity of the expectation has proved to be a challenge in addressing this problem. However, this problem seems amenable to being tackled using a modification of the framework that has been developed in this thesis.

As mentioned earlier in this chapter 5.1, relaxing the convexity assumption is a future research direction. A possible approach would be to use the framework with a fixed point approach instead of a variational one to yield solutions.

## 5.3 Concluding remarks on Chapter 4

Through the analysis provided in Chapter 4, we observe that VaR-based risk management may be inadequate in financial risk management when the associated traders are risk-seeking. By considering a setting where traders employ a CVaR measure, we make two sets of contributions. First, we show that when the underlying asset returns have compact support, the associated maximal CVaR risk can be precisely characterized and is of the order of the upper bound of the support. Furthermore, when the distributions have unbounded support, a trader may take on unlimited CVaR risk while maintaining a VaR threshold imposed by the firm. This suggests that VaR-based risk management tools need reassessment and calls for risk management tools that guard against assuming excessive risk across a family of risk measures. future work.

As a follow up to this work, it may be interesting to see whether the property of unboundedness of risk measure in a VaR constrained setting persists for more general coherent risk measures or convex risk measures.

Obviously, this would be analytically intractable. However, it seems possible to consider the representation of coherent risk measures as the supremum of the expected loss over a convex set of probability measures and glean some information from there. This research would reveal properties required by risk metrics used in financial settings where the incentive to undertake risk is to be managed with reward to the risk-seeking trader and the interests of stockholders as well.

This work in risk management has close relation to risk preferences and incentives in the principal-agent problem in economics. Another approach to this problem could be achieved by looking at this problem through a new lens of the principal-agent problem.

## 5.4 Summary

The dissertation can be viewed as amongst *the first attempts* to provide a tractable and verifiable framework to address the fundamental question of ascertaining existence and uniqueness for a broad class of stochastic Nash games, stochastic variational inequalities. In convex regimes, the framework can been extended to accommodate important and practical generalizations of SVIPs. The important feature of this framework is that it *does not require the evaluation of expectation* and is applicable to a wide range of practical equilibrium problems.

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