

Universal Properties of the Two-Dimensional Kuramoto-Sivashinsky Equation

C. Jayaprakash and F. Hayot

Department of Physics, The Ohio State University, Columbus, Ohio 43210

Rahul Pandit

Department of Physics, Indian Institute of Science, Bangalore 560 012, India

We investigate the properties of the Kuramoto-Sivashinsky equation in two spatial dimensions. We show by an explicit, numerical, coarse-graining procedure that its long-wavelength properties are described by a stochastic, partial differential equation of the Kardar-Parisi-Zhang type. From the computed parameters in our effective, stochastic equation we argue that the length and time scales over which the correlation functions cross over from linear diffusive to those of the full nonlinear equation are very large. The behavior of the three-dimensional equation is also discussed.

In this paper we examine the relationship between the long-wavelength properties of two systems—the deterministic Kuramoto-Sivashinsky (KS) equation [1,2] and the stochastic Kardar-Parisi-Zhang (KPZ) model [3,4]. The KS equation has been used to study diffusive instabilities of wrinkled flame fronts and chemical waves; the KPZ equation is a generic model for the morphology of growing interfaces in a wide variety of physical and chemical situations including thin film growth. The KS equation also serves as an example of an extended, deterministic, dynamical system that exhibits complex spatiotemporal phenomena: chaotic behavior characterized by a nonvanishing density of positive Liapunov exponents and long-wavelength properties that are scale invariant. The general question of the extent to which the long-wavelength properties of such systems can be described by effective (coarse-grained), stochastic models such as the KPZ equation, with the stochastic noise being generated by intrinsic, chaotic fluctuations, has been addressed in the context of the KS equation. The KS equation is given by

$$dh/dt = -\nabla^2 h - \nabla^4 h + (\nabla h)^2, \quad (1)$$

where $h(\mathbf{x}, t)$ can be interpreted as the height of a d -dimensional interface (with no overhangs), embedded in $d+1$ dimensions. In $d=1$ some workers [5-7] have shown that, in the long-wavelength and long-time limit, the KS equation exhibits scale-invariant behavior which is the same as that of the one-dimensional KPZ equation [3], thus confirming a suggestion due to Yakhot [8]. The KPZ equation is given by

$$\partial h / \partial t = \nu \nabla^2 h + \frac{\lambda}{2} (\nabla h)^2 + \eta(\mathbf{x}, t), \quad (2)$$

where η is an external white noise with correlations described by

$$\langle \eta(\mathbf{x}, t) \eta(\mathbf{x}', t') \rangle = 2D \delta^d(\mathbf{x} - \mathbf{x}') \delta(t - t'). \quad (3)$$

However, in dimensions $d > 2$ little is known about the

scale invariance of the asymptotic behavior of the KS equation. The identity of the nonlinear terms in the two equations might lead one to suspect that the KS and KPZ equations exhibit similar infrared behavior. In this Letter we present the results of a coarse-graining procedure for the KS equation in $d=2$, providing numerical evidence to support the hypothesis that the long-wavelength behaviors of the KS and KPZ equations are the same. We determine the effective parameters for the coarse-grained equation and show that for the KS equation, the cross-over length and time scales, at which the asymptotic KPZ behavior sets in, are enormous. We also comment on the behavior of the KS equation in three dimensions. Our results contradict work at $d=2$ where the authors [9] claim to give a proof that the KS and KPZ equations belong in different universality classes. We compare our conclusions and also comment on the differences with more recent work by the same group [10] for $d > 2$.

Let us recapitulate some well-known features of the KS equation: The only parameter in Eq. (1) is the length L , which plays the role of the Reynolds number in fluid turbulence; all other coefficients have been fixed by appropriate rescaling. At the linear level, the Fourier modes with wave vector \mathbf{k} evolve as $e^{\omega_k t}$, where the rate $\omega_k = k^2 - k^4$ and $k = |\mathbf{k}|$; thus long-wavelength disturbances are linearly unstable. The nonlinear term stabilizes the system by converting the growing, small- k modes into decaying, large- k ones.

We solve the KS equation (1) in $d=2$ numerically by a Euler method for temporal updating (with a time discretization of 0.02 and, for some checks, of 0.01), a symmetric discretization of spatial derivatives (with a mesh of unit size) on a square lattice of side L , with L ranging from 256 to 512. We have also checked our results by using a spectral code [11] (which approximates the continuum KS equation better than the finite-difference scheme but is computationally more expensive) on a 308×308 lattice with 384×384 modes.

In the long-time limit of our simulation, the system

shows a cellular structure (Fig. 1), with the side of a cell given roughly by $2\pi/k_{\max}$, where k_{\max} is the magnitude of the wave vector of the (linearly) most unstable modes. We have explicitly computed the number of positive Liapunov exponents averaging over at least four initial conditions and evolving in time for up to 400 time units for each initial condition. The number of positive Liapunov exponents increases from 6 for $L = 10$, to 15 for $L = 24$, and up to 25 for $L = 32$. Our results are consistent with the system reaching a steady state with a nonzero density of positive Liapunov exponents corresponding roughly to one positive exponent per cell.

Next we address the issue of scale-invariant behavior. The KPZ equation (2) displays dynamic scaling behavior in the asymptotic, long-wavelength properties of the correlations of h . This scaling behavior is governed by a strong-coupling fixed point for $d < 2$, for all λ and, for $d > 2$, for sufficiently large (bare) values of λ . Our focus will be on two-point correlators and linear response functions; e.g., the interface is rough with a width $w(L, t) = [\langle (h - \langle h \rangle)^2 \rangle]^{1/2}$ that scales with system size L and time t as

$$w(L, t) \sim t^\beta W(t/L^2). \quad (4)$$

Instead of determining such correlation functions directly for the KS equation, we proceed indirectly: We obtain an effective, stochastic equation via a coarse-graining procedure in which short-wavelength degrees of freedom are integrated out, renormalizing the diffusion constant to a positive value, and generating an effective noise term.

$$\partial h / \partial t = -v_\Lambda(k) k^2 h(\mathbf{k}, t) - \sum^< \mathbf{q} \cdot (\mathbf{k} - \mathbf{q}) h(\mathbf{q}, t) h(\mathbf{k} - \mathbf{q}, t) + F_\mathbf{k}(t), \quad (5)$$

where $\sum^<$ denotes a sum over only those \mathbf{q} 's such that all the modes in the sum belong in $S^<$; the additional term $F_\mathbf{k}(t)$ is given by

$$F_\mathbf{k}(t) = [v_\Lambda(k) + 1 - k^2] k^2 h(\mathbf{k}, t) - \sum^> \mathbf{q} \cdot (\mathbf{k} - \mathbf{q}) h(\mathbf{q}, t) h(\mathbf{k} - \mathbf{q}, t), \quad (6)$$

where $\sum^>$ denotes a sum over those \mathbf{q} 's such that at least one of the modes belongs in $S^>$. The cutoff A is chosen to be less than k_{\max} since we expect the effective diffusion constant to be positive only on larger length scales; moreover, we have checked the independence of our results on A for a range of values. We now identify $F_\mathbf{k}(t)$ as the noise term generated by the short-wavelength modes. In order that $F_\mathbf{k}(t)$ be a genuine, stochastic noise we impose, following Zaleski [5], the physical requirement that the height h at an earlier time $(t-s)$ is not correlated with the noise F at a later time t for small but nonzero k , i.e.,

$$C_{Fh}(s) = \langle [F_\mathbf{k}(t) - \langle F_\mathbf{k}(t) \rangle] [h(-\mathbf{k}, t-s) - \langle h(-\mathbf{k}, t-s) \rangle] \rangle \quad (7)$$

vanishes for large s (the angular brackets denote an average over t). If $v_\Lambda(k)$ can be chosen so that this condition is satisfied for large s then $F_\mathbf{k}(t)$ is a stochastic noise term. We can define an s -dependent, effective, diffusion constant $v_\Lambda(k, s)$ in terms of $\langle h(\mathbf{k}, t) h(-\mathbf{k}, t-s) \rangle$ and $\sum^> \mathbf{q} \cdot (\mathbf{k} - \mathbf{q}) \langle h(\mathbf{q}, t) h(\mathbf{k} - \mathbf{q}, t) h(-\mathbf{k}, t-s) \rangle$ from Eqs. (6) and (7) by setting $C_{Fh}(s) = 0$. We find in our numerical simulations (see below) that $v_\Lambda(k, s)$ saturates at a constant, positive value v_B , for large s , and all small k . This saturation of $v_\Lambda(k, s)$ is essential to obtain an effective stochastic equation; it is guaranteed if the two-

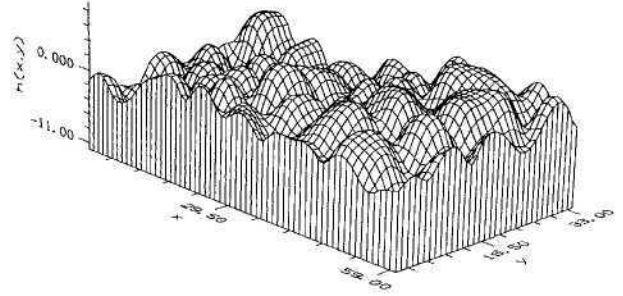


FIG. 1. Cellular structures in the asymptotic, steady state of the KS equation in two dimensions; an instantaneous "snapshot" of the height $h(x, y)$ vs x and y for a small section (60 x 34) of a 384 x 384 lattice.

Such a procedure was explicitly carried out by Zaleski [5] in $d=1$. In other words, we determine the effective equation at length scales such that one is still in the vicinity of the free-field fixed point and show that the effective equation has terms that correspond to those in the KPZ equation. Since in the coarse-graining procedure we remain in the vicinity of the free-field fixed point, we can obtain good statistics on moderate-sized lattices.

The coarse-graining proceeds as follows: The space of Fourier modes is divided into a subset $S^<$ of long-wavelength modes $h(\mathbf{k})$, with $|k_x| < A$, $|k_y| < A$, where A is the cutoff wave vector, and a complementary subset $S^>$. The latter is integrated out to obtain an effective equation for the long-wavelength modes which can be written as follows:

and three-point functions mentioned above decay in the same manner for large s . We have checked that this is indeed the case for the KS equation numerically and via perturbation theory for the KPZ equation. We choose $v_\Lambda(k)$ to be v_B in our effective equation [Eq. (5)]. With this choice of the diffusion constant we then verify that the noise-noise correlation function decays exponentially in time establishing that Eq. (5) is a KPZ equation with a positive diffusion constant whose bare value is v_g and with a noise term which has the appropriate behavior.

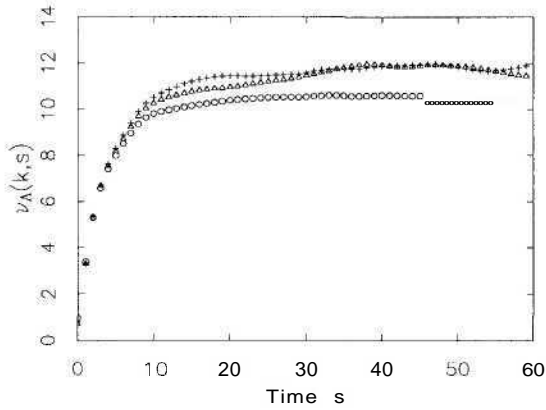


FIG. 2. Plot of $\nu_A(\mathbf{k},s)$ vs s for a lattice of side $L=384$ showing saturation to a constant value ν_B for k values (1,1) (circles), (2,1) (triangles), and (2,2) (crosses) in units of $2\pi/L$. The data were obtained by averaging over 20000 time units with $A=0.30$.

Most of our computations were performed on a 384×384 system, averaging over 20000 to 30000 time units corresponding to over 1000000 updates. We obtained data for ten different initial conditions. Figure 2 shows how $\nu_A(k,s)$ saturates to a constant value $\nu_B \approx 11 \pm 1.5$, with $A=0.30$. An independent check of this value was obtained as follows: The decay time $\tau(\mathbf{k})$ for the height-height correlation function $C_{hh}(\mathbf{k},s)$ diverges as $k \rightarrow 0$ as $\tau(\mathbf{k}) \approx 1/\nu_B k^2$, since we are in the vicinity of the free-field fixed point. We computed $C_{hh}(\mathbf{k},s)$ numerically and estimated the value of ν_B ; this value agrees with the one we obtained above by imposing the requirement that $C_{FF}(s) \rightarrow 0$ for large s .

Having fixed ν_B , we can determine the decay characteristics of the noise-noise correlation function

$$C_{FF}(s) = \langle [F_{\mathbf{k}}(t) - \langle F_{\mathbf{k}}(t) \rangle] \times [F_{-\mathbf{k}}(t+s) - \langle F_{-\mathbf{k}}(t+s) \rangle] \rangle. \quad (8)$$

Figure 3 shows the decay of $C_{FF}(k,s)$ as a function of s . We estimate the decay time to be roughly 3.5 for small values of k ; this value agrees with the inverse of the maximum Liapunov exponent of ≈ 0.23 , which characterizes the chaotic decorrelations, consistent with the noise being generated by the short-wavelength chaotic fluctuations. We have also verified numerically that the probability distribution of $F_{\mathbf{k}}$ is consistent with the assumption of a Gaussian distribution for the external noise in the KPZ equation and the irrelevance of higher moments.

We can now determine the remaining "bare" coefficients (labeled with a subscript B) for our effective KPZ equation. We use the $k \rightarrow 0$ limit of $C_{FF}(k,t)$ to obtain $\tilde{D}_B \approx 12 \pm 0.5$ [cf. Eq. (8)]. Note that, because the correlation function decays exponentially with a decay time T , $D_B = 2\tau \tilde{D}_B$. Furthermore, Eqs. (2) and (5) yield $\lambda_B = 2$ for this effective equation. We recall [3] that the (cross-

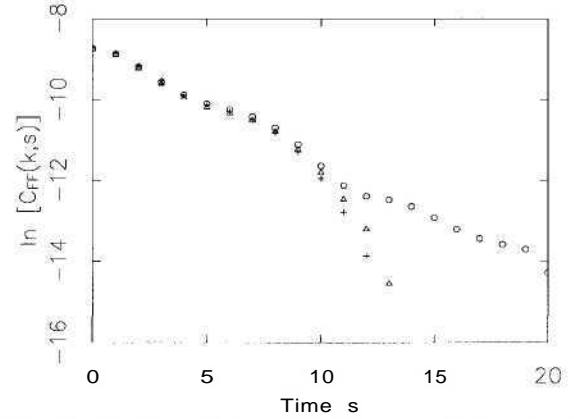


FIG. 3. A plot of the noise-noise correlation function $C_{FF}(k,s)$ [see Eqs. (6) and (8)] showing its decay as a function of time s for the same k values as in Fig. 2. The decay time is roughly 3.5.

over) length scale L_c at which the (long-wavelength) properties of the KPZ equation cross over from the (intermediate) free-field scaling behavior to that of the (asymptotic) strong-coupling fixed point is governed by the KPZ coupling constant $g = \lambda_B^2 \tilde{D}_B / \nu_B^2$ which is dimensionless in $d=2$. From our bare phenomenological parameters we estimate the value of g for the effective KPZ equation to be roughly 0.26. We found comparable results, $g \approx 0.4 \pm 0.2$, on lattices with $L=256$ and $L=512$. The value estimated from our spectral code is approximately $g=0.40$. Since $d=2$ is the marginal dimension [3,12], the crossover length scale L_c displays an exponential dependence on g : $L_c \sim e^{8\pi/g}$. For the KS equation the small value of g leads to an extraordinarily large crossover length scale and similarly a large crossover time scale. This renders it well-nigh impossible to establish the nature of the asymptotic, scaling behavior of the pristine KS equation by direct numerical simulations. However, the result of our coarse-graining procedure allows us to conclude that the long-wavelength behaviors of the KS and KPZ equations are the same [13]. If one considers the KS equation in the presence of external noise, we [14] can perform a dynamic, renormalization-group analysis and obtain a similar result: One can argue, for a restricted range of initial parameter values, that the equation scales to a stochastic KPZ equation. Note that short-distance properties such as local probability distribution functions will not be the same for the two equations.

We believe that an identical coarse-graining procedure can be carried out in $d=3$, with the same result, namely, that the coarse-grained KS equation is the same as the KPZ equation at long length scales. For sufficiently small values of the corresponding (dimensionless) coupling constant g in $d=3$, the KPZ equation exhibits free-field behavior [3,4]. Therefore, the asymptotic behavior of the KS equation will be determined by the precise values of the phenomenological parameters in the effec-

tive, stochastic equation.

We now compare our results with those of Procaccia and co-workers [9,10]. They identify a "nonlocal" theory whose asymptotic behavior differs from that of the strong-coupling KPZ theory. They then proceed to show that both the KS and KPZ equations cannot simultaneously exhibit this behavior. We point out that the infrared behavior of the "nonlocal" theory [10] in $d > 2$ is that of free-field theory that describes a smooth interface in that both the two-point correlation and response functions are the same for both theories.

$d=3$.—In Ref. [10] different behaviors are suggested for the two models in $d=3$. However, it is known [3,4] that the KPZ equation can exhibit either free-field or strong-coupling behavior depending on the bare value of the KPZ parameter in $d=3$. Therefore, we believe that the long-wavelength behavior of the KS equation in $d=3$ is determined by the value of the nonlinear coupling constant in the effective equation; the method proposed by Zaleski in $d=1$ and implemented here in $d=2$ provides a scheme for determining this value and thus deciding whether free-field or strong-coupling behavior obtains.

$d=2$.—Reference [9] claims to give "a computer-assisted proof that the long-wavelength behavior of the KS equation is different from that of KPZ in two spatial dimensions, in contrast to our results. We note, however, that in Ref. [10] the numerical results of Ref. [9] are used only to suggest this conclusion. The result of Ref. [9] is based on an analytic argument that in the steady state the squared width depends on system size L as $w^2(L) \sim \ln[\ln(L)]$ in the nonlocal theory combined with numerical evidence for such a behavior in the KS equation. This dependence is in contrast to free-field behavior: $w^2(L) \sim \ln(L)$. We believe that the numerical evidence is insufficient to distinguish between these two forms: We studied the temporal behavior of $w^2(L, t)$, for lattices of sizes up to 1024×1024 , and found, not surprisingly, that $w^2(L, t)$ can be fitted equally well by $\ln t$ and $\ln[\ln(t)]$. Let us, however, ignore the fact that the asymptotic behavior of the KS equation is numerically inaccessible, and accept the numerical result ($w(L) \sim \ln[\ln(L)]$) of Ref. [9]. It should then be noted that precisely this behavior has been shown to emerge from a one-loop calculation for the KPZ equation by Tang, Nattermann, and Forrest [12]; moreover, they have observed this behavior in numerical simulations of $w^2(t)$ at intermediate times [15]. Thus the numerical results of Ref. [9] are actually consistent with the intermediate-time behavior of the KPZ equation.

In conclusion, the coarse graining that we have performed provides persuasive numerical support for the claim that, in the long-wavelength limit, the KS equation (1) in two spatial dimensions behaves effectively like the stochastic, KPZ equation (2).

This work was supported by the U.S. Department of Energy (Contract No. DE-F-G02-88ER13916A000) and benefited from computer time on the Cray-YMP provided by The Ohio Supercomputer Center. C.J. is grateful to T. Bohr and J. Krug for useful discussions. We are especially grateful to G. Grinstein for valuable discussions and comments on the manuscript. R.P. would like to thank The Ohio State University for hospitality during the time this work was done.

-
- [1] Y. Kuramoto and T. Tsuzuki, *Prog. Theor. Phys.* 55, 356 (1976); G. I. Sivashinsky, *Acta Astronaut.* 6, 569 (1979).
 - [2] P. Manneville, *Dissipative Structures and Weak Turbulence* (Academic, Boston, 1990).
 - [3] M. Kardar, G. Parisi, and Y. C. Zhang, *Phys. Rev. Lett.* 56, 889 (1986).
 - [4] J. Krug and H. Spohn, in *Solids Far From Equilibrium*, edited by C. Godreche (Cambridge Univ. Press, Cambridge, 1992), pp. 479-582.
 - [5] S. Zaleski, *Physica* (Amsterdam) 34D, 427 (1989).
 - [6] F. Hayot, C. Jayaprakash, and Ch. Josserand, *Phys. Rev.* E 47, 911 (1993).
 - [7] K. Sneppen, J. Krug, M. H. Jensen, C. Jayaprakash, and T. Bohr, *Phys. Rev. A* 46, R7351 (1992).
 - [8] V. Yakhot, *Phys. Rev. A* 24, 642 (1981).
 - [9] I. Procaccia, M. H. Jensen, V. S. L'vov, K. Sneppen, and R. Zeitak, *Phys. Rev. A* 46, 3220 (1992).
 - [10] V. S. L'vov and I. Procaccia, *Phys. Rev. Lett.* 69, 3543 (1992).
 - [11] For details see J. M. Hyman, B. Nicolaenko, and S. Zaleski, *Physica* (Amsterdam) 23D, 265 (1986), and Appendix A of Ref. [5].
 - [12] L-H. Tang, T. Nattermann, and B. M. Forrest, *Phys. Rev. Lett.* 65, 2422 (1990).
 - [13] Our calculations deal directly with the long-wavelength properties of two-point correlation functions and linear response functions only.
 - [14] G. Grinstein and C. Jayaprakash (unpublished).
 - [15] The marginality of the theory in $d=2$ causes the same singularity to appear both in the infrared [12] and the ultraviolet (nonlocal theory) at the one-loop level. This is in accord with the fact that the "nonlocal" theory describes free-field behavior for $d > 2$. By continuity we would expect the "nonlocal" theory to describe free-field behavior precisely at $d=2$ and thus be marginally unstable.