Model Reduction in Power Systems Using Krylov Subspace Methods

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***Abstract*—This paper describes the use of Krylov subspace methods in the model reduction of power systems. Additionally, a connection between the Krylov subspace model reduction and coherency in power systems is proposed, aiming at retaining some physical relationship between the reduced and the original system.**

***Index Terms*—Coherency, Krylov subspaces, power system dynamics, power system simulation, reduced-order systems.**

I. Introduction

IN power systems, the dimension of the models may easily reach the order of several thousands of state variables in applications like dynamic simulation, trajectory sensitivity analysis, control etc. Therefore, these types of analyses pose a formidable computational burden. Model reduction consists of replacing the original system with one of a much smaller dimension according to the following guidelines.

- The reduced system must be an accurate representation of the original one for the analysis performed.
- The cost of generating the reduced model must be much smaller than the cost of performing the analysis using the original model.

Most methods of model reduction focus on linear systems, which, in many cases, provide accurate descriptions of the physical systems. Depending on the properties of the original system that are retained in the reduced model, there are different model reduction methodologies. Hence, there are techniques based in directly identifying and preserving certain modes of interest (modal model reduction [1], [2]) or based on the singular value decomposition (SVD), such as balanced truncation [3], Hankel norm approximation [4], etc., focusing on the observability and controllability properties of the system. Another family of model reduction techniques, on which this paper builds on, is the moment matching methods [5], [6]. The property of interest here is the leading coefficients of a power series expansion of the transfer function of the reduced system around a user-defined point that have to match those of the original system transfer function.

Modal model reduction has been proven efficient, yet the determination of the full set of truly dominant modes of the system is not a fully resolved question although significant progress has been made [7]–[9]. SVD based methods are also very efficient but computationally intensive. In [10] SVD model reduction is shown to yield excellent global approximation in the frequency domain when compared to moment matching techniques whose performance is good in a limited range of frequencies. Nevertheless, moment matching methods based on Krylov subspaces present less computational effort and less storage requirements, requiring little empirical parameter adjusting.

The bulk of model reduction techniques in power systems are tailored for the tasks of control design and transient/small signal stability analysis. Concepts like *coherency* treated in [11]–[15], *synchrony* introduced in [16] and [17], *singular perturbation analysis* in [18] and [19], and *modal analysis* in [7], [8], [18] and [20] form the basis for a wide variety of model reduction tools developed.

For model reduction, power systems may be partitioned into two areas: the *study area* and the *external area* [13]. The study area contains the variables of interest, and therefore it is modeled in detail. The external area is important only as far as it influences the analysis in the study area and is represented by a linear model for studies such as small signal stability analysis. Furthermore, it is often the case that the external area input/output behavior is of interest only in very low frequencies (less than 2 Hz) depending on the nature of its interconnection to the study area and the level of generator modeling. This characteristic makes moment matching methods suitable for model reduction application on the external area.

In this paper, a moment matching model reduction methodology based on projection on Krylov subspaces is presented. The application of such techniques in moment matching model reduction has been introduced and analyzed in [21]–[29]. Here, a validation of this approach is attempted on power systems by reducing the size of the linearized model of the external area and observing the effect of the approximation by simulating a fault in the study area, which retains a nonlinear representation. The second goal of this paper is to demonstrate that the Krylov subspace bases derived by projecting linear power systems onto them may be used in identifying coherent generators in the external area. This is the first step in the model reduction of nonlinear power systems, as it is observed that coherent generators may be replaced in a straightforward way by an equivalent generator, which reproduces their behavior in simulation studies [12], [13].

The paper is organized as follows: In the first part a brief presentation of the Krylov subspace methodology in the model reduction of linear systems is given. In the second part, the application of the Krylov subspace model reduction on the linearized power system is illustrated via a numerical example. The main

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thrust of the paper is in its third part, where the Krylov subspace associated with the linearized power system is used to determine sets of coherent generators.

II. Krylov Subspace Model Reduction

A. Krylov Subspaces

For a square matrix $\mathbf{A}$ of dimension N and a vector $\mathbf{b}$, the subspace spanned by the vectors $[\mathbf{b}\;\mathbf{Ab}\;\cdots\;\mathbf{A}^{m-1}\mathbf{b}]$ is called a Krylov subspace of dimension m generated by $\{\mathbf{A},\mathbf{b}\}$ denoted as $K_m\{\mathbf{A},\mathbf{b}\}$. Krylov subspaces are characterized by two important properties.

- The Krylov vectors gradually point to the dominant eigenvector of $\mathbf{A}$ as m increases.
- Every vector belonging to the Krylov subspace is a result of an operation of a polynomial of $\mathbf{A}$ on vector $\mathbf{b}$.

It is these two properties that make Krylov subspace methods suitable for eigenvalue calculations and the solution of linear systems of equations [30]. Frequently, one seeks to construct a base for the Krylov subspace $K_m\{\mathbf{L^{-1}A},\mathbf{L^{-1}b}\}$, whose eigenvalue characteristics are more suitable for certain analyses (preconditioning). $\mathbf{L}$ is in general an easily invertible matrix.

The main challenge in constructing a basis for a Krylov subspace is to avoid the ill-conditioning caused by repeated multiplications by the matrix A. Most proposed approaches are variations of the Lanczos and the Arnoldi methods, which build the Krylov base in an iterative manner. Their implementation involves, in general, matrix-vector multiplications, inner vector products and/or the solution of "easy" linear systems of equations. These operations are computationally fast and highly parallelizable; hence the popularity of Krylov based methods when large, sparse linear systems are concerned.

B. Moment Matching and Krylov Subspaces

Consider the linear, single-input single-output, time-invariant system of dimension N

$$\begin{aligned}\dot{\mathbf{x}} &= \mathbf{Ax}+\mathbf{b}u\\ y &= \mathbf{c}^{\mathrm{T}}\mathbf{x}\end{aligned} \tag{1}$$

and the expansion of its transfer function around σ

$$\mathbf{H}(\mathrm{s})=\sum_{j=1}^{\infty}-\mathbf{c}^{\mathrm{T}}\left[(\mathbf{A}-\sigma\mathbf{I})^{-1}\right]^{j}\mathbf{b}\cdot(s-\sigma)^{j-1}.$$

The shifted moments of the system represent the value and the subsequent derivatives of the transfer function around σ and are given by

$$-\mathbf{c}^{\mathrm{T}}\left[(\mathbf{A}-\sigma\mathbf{I})^{-1}\right]^{j}\mathbf{b}.$$

Borrowing the notation from [27], let the *right* and *left* nonsingular transformation matrices decomposed as

$$\mathbf{T}_{right}^{N\times N}=\begin{bmatrix}\mathbf{V}^{N\times n} & \mathbf{V}_{\perp}^{N\times(N-n)}\end{bmatrix}$$

and

$$\mathbf{T}_{left}^{N\times N}=\begin{bmatrix}\mathbf{Z}^{N\times n} & \mathbf{Z}_{\perp}^{N\times(N-n)}\end{bmatrix}$$

where $\mathbf{V}_{\perp}$ and $\mathbf{Z}_{\perp}$ are the complements of $\mathbf{V}$ and $\mathbf{Z}$ respectively. The application of this transformation to (1) yields

$$\begin{aligned}\begin{bmatrix}\mathbf{Z}^{\mathrm{T}}\mathbf{V} & \mathbf{Z}^{\mathrm{T}}\mathbf{V}_{\perp}\\ \mathbf{Z}_{\perp}^{\mathrm{T}}\mathbf{V} & \mathbf{Z}_{\perp}^{\mathrm{T}}\mathbf{V}_{\perp}\end{bmatrix}\begin{bmatrix}\dot{\hat{\mathbf{x}}}\\ \dot{\hat{\mathbf{x}}}_{\perp}\end{bmatrix} &= \begin{bmatrix}\mathbf{Z}^{\mathrm{T}}\mathbf{AV} & \mathbf{Z}^{\mathrm{T}}\mathbf{AV}_{\perp}\\ \mathbf{Z}_{\perp}^{\mathrm{T}}\mathbf{AV} & \mathbf{Z}_{\perp}^{\mathrm{T}}\mathbf{AV}_{\perp}\end{bmatrix}\begin{bmatrix}\hat{\mathbf{x}}\\ \hat{\mathbf{x}}_{\perp}\end{bmatrix}\\ &\quad+\begin{bmatrix}\mathbf{Z}^{\mathrm{T}}\mathbf{b}\\ \mathbf{Z}_{\perp}^{\mathrm{T}}\mathbf{b}\end{bmatrix}u\\ y &= [\mathbf{c}^{\mathrm{T}}\mathbf{V}\quad \mathbf{c}^{\mathrm{T}}\mathbf{V}_{\perp}]\begin{bmatrix}\hat{\mathbf{x}}\\ \hat{\mathbf{x}}_{\perp}\end{bmatrix}\end{aligned} \tag{2}$$

where $\mathbf{x}=\mathbf{V}\hat{\mathbf{x}}+\mathbf{V}_{\perp}\hat{\mathbf{x}}_{\perp}$. Then, assuming $\mathbf{Z}^{\mathrm{T}}\mathbf{V}$ is nonsingular, the leading subsystem is retained to form the reduced model of dimension n

$$\begin{aligned}\dot{\hat{\mathbf{x}}} &= (\mathbf{Z}^{\mathrm{T}}\mathbf{V})^{-1}\mathbf{Z}^{\mathrm{T}}\mathbf{AV}\hat{\mathbf{x}}+(\mathbf{Z}^{\mathrm{T}}\mathbf{V})^{-1}\mathbf{Z}^{\mathrm{T}}\mathbf{b}u\\ y &= \mathbf{c}^{\mathrm{T}}\mathbf{V}\hat{\mathbf{x}}\end{aligned} \tag{3}$$

or

$$\begin{aligned}\dot{\hat{\mathbf{x}}} &= \hat{\mathbf{A}}\hat{\mathbf{x}}+\hat{\mathbf{b}}u\\ y &= \hat{\mathbf{c}}^{\mathrm{T}}\hat{\mathbf{x}}.\end{aligned}$$

In [27], it is shown that if $\mathbf{V}$ and $\mathbf{Z}$ are chosen to be bases the Krylov subspaces

$$K_J\left\{(\mathbf{A}-\sigma\cdot\mathbf{I})^{-1},(\mathbf{A}-\sigma\cdot\mathbf{I})^{-1}\mathbf{b}\right\}$$

and

$$K_J\left\{(\mathbf{A}-\sigma\cdot\mathbf{I})^{-\mathrm{T}},(\mathbf{A}-\sigma\cdot\mathbf{I})^{-\mathrm{T}}\mathbf{c}\right\}$$

respectively, for an interpolation point σ, then the shifted moments of the reduced system

$$-\hat{\mathbf{c}}^{\mathrm{T}}\left[(\hat{\mathbf{A}}-\sigma\cdot\mathbf{I})^{-1}\right]^{j}\hat{\mathbf{b}}\quad j=1,\ldots,J-1$$

match the first J moments of the original system. This argument is generalized for multiple interpolation points by taking the union of the generated Krylov subspaces for each point (rational interpolation). Consequently, the transfer function of the reduced system (3) is a good approximation to the transfer function of the original system (1) for a range of frequencies corresponding to the chosen interpolation points. The reduced system effectively retains the modes that are most important to the input-output behavior of the system at certain user-defined frequency ranges.

A major problem that arises in many cases of Krylov subspace model reduction concerns the stability of the reduced model. It is necessary, when approximating a stable model, that the reduced model be also stable. Unfortunately, this is not guaranteed using this methodology. There have been successful attempts to obtain stable reduced models via implicit restarts as described in [28] and [29]. When an unstable reduced model is derived it is partitioned into its stable and unstable components and the stable ones are retained [29].

Power systems are rarely modeled as single-input, single-output systems. Then, one has to extend the previous analysis to include multi-input, multi-output systems. The Krylov subspaces are replaced by *block-Krylov subspaces* to account for the multidimensional input and output matrices $\mathbf{B}$ and $\mathbf{C}$. There are a number of well-established block-Krylov methods in the literature, see for example [30]. The consequence of using block Krylov algorithms is that the dimension of the reduced model is significantly larger compared to the single-input single-output case since the block-Krylov subspace has to contain all the information generated by the individual Krylov subspaces corresponding to the columns of $\mathbf{B}$ and $\mathbf{C}$.

p : number of columns of $\mathbf{B}$
J : number of matched moments
$\sigma^{(j)}$: interpolation point at iteration j
$(\mathbf{V},\mathbf{R}) = QRfactor\left(\mathrm{Real}\left\{\left(\mathbf{A}-\sigma^{(j)}\mathbf{I}\right)^{-1}\mathbf{B}\right\}\right)$
for $j = 1:J$
  $\boldsymbol{\omega} = \mathrm{Real}\left\{\left(\mathbf{A}-\sigma^{(j)}\mathbf{I}\right)^{-1}\mathbf{V}_j\right\}$
  for $i = 1: j+p$
    $h(i,j) = \boldsymbol{\omega}^{\mathrm{T}}\cdot\mathbf{V}_i$
    $\boldsymbol{\omega} = \boldsymbol{\omega} - h(i,j)\cdot\mathbf{V}_i$
  end
  if $\|\boldsymbol{\omega}\|_2 \geq 10^{-6}$
    $\mathbf{V}_{j+1} = \boldsymbol{\omega}/\|\boldsymbol{\omega}\|_2$
  else
    $p = p-1$
  end
  if $p = 0$ exit
end

Fig. 1. Block–Arnoldi for multiple interpolation points.

There is a plethora of implementation approaches for the construction of the bases $\mathbf{V}$ and $\mathbf{Z}$ in transformation (2). For a description of a number of algorithms see [30]. The Lanczos method has the advantages of being simple, efficient, and very fast but it suffers from numerical stability issues, as analyzed in [31]–[33]. Because of that, the choice in this paper is the Arnoldi method that is far more stable [30], [34]. Specifically, we use the Rational Krylov method [35]–[37], adjusted for multiple starting vectors. This algorithm, shown in Fig. 1, is equivalent to a shifted-inverted Arnoldi with the addition that the shift is updated when the desired number of iterations has been performed with its current value. Imaginary points are incorporated by restricting the resulting complex vectors to their real parts. Because of the structure of the $\mathbf{A}$ matrix in the examples to follow, operating on the real or the imaginary part of the complex vector makes no significant numerical difference.

III. Application to Power Systems

The following set of dynamic equations are considered for the study of model reduction of an m-machine n-bus power system:

$$
\begin{aligned}
\dot{\delta}_i &= \omega_i - \omega_s \quad i=1,\dots,m \\
\frac{2H_i}{\omega_s}\dot{\omega}_i &= T_{M_i} - \frac{E_i V_i \sin(\delta_i-\theta_i)}{X'_{d_i}} \quad i=1,\dots,m \\
P_{L_i}(V_i)+P_{G_i} &= \sum_{j=1}^{n+p} V_i V_j B_{ij}\sin(\theta_i-\theta_j) \quad i=1,\dots,n \\
Q_{L_i}(V_i)+Q_{G_i} &= -\sum_{j=1}^{n+p} V_i V_j B_{ij}\cos(\theta_i-\theta_j) \quad i=1,\dots,n \\
P_{G_i} &= \frac{E_i V_i \sin(\delta_i-\theta_i)}{X'_{d_i}} \quad i=1,\dots,m \\
Q_{G_i} &= \frac{-V_i^2 + E_i V_i \cos(\delta_i-\theta_i)}{X'_{d_i}} \quad i=1,\dots,m. \quad (4)
\end{aligned}
$$

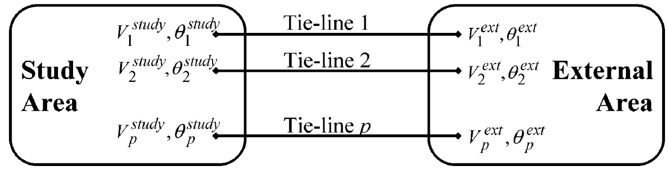

Fig. 2. System configuration of the study and the external areas.

This model represents the external system connected to the study area via p tie-lines. H_i and D_i represent the inertia and damping coefficients, respectively, of machine i, θ_i and V_i are the angle and voltage magnitude of bus i, and each machine is modeled as a voltage of constant magnitude $E_i\langle\delta_i$ behind a transient reactance X'_{d_i}. Loads P_{L_i} and Q_{L_i} are modeled as constant impedances. The reader is referred to [38] for a step-by-step description of how to reach (4) from a detailed model of the system as well as for a discussion of the notation used here. The inputs of the external system, denoted as $\mathbf{u}$, are considered to be the angles and voltages of the p connected buses belonging to the study area. The outputs of the system, denoted as $\mathbf{y}$, are considered to be the angles and the voltages of the p corresponding buses of the external area.

Equation (4) is linearized around an equilibrium point $(\boldsymbol{\delta}^0, \boldsymbol{\omega}^o)$ and the network algebraic equations are eliminated, yielding

$$
\begin{aligned}
&\begin{bmatrix}\Delta\dot{\boldsymbol{\delta}}\\ \Delta\dot{\boldsymbol{\omega}}\end{bmatrix} = \begin{bmatrix}\mathbf{0}_{(m-1)\times(m-1)} & \mathbf{I}_{(m-1)\times(m-1)}\\ \hat{\mathbf{A}}_{(m-1)\times(m-1)} & -\mathbf{M}_{(m-1)\times(m-1)}\end{bmatrix}\begin{bmatrix}\Delta\boldsymbol{\delta}\\ \Delta\boldsymbol{\omega}\end{bmatrix} + \\
&\begin{bmatrix}\mathbf{0}\\ \hat{\mathbf{B}}_{(m-1)\times p}\end{bmatrix}\Delta\mathbf{u} \\
&= \mathbf{A}\begin{bmatrix}\Delta\boldsymbol{\delta}\\ \Delta\boldsymbol{\omega}\end{bmatrix} + \mathbf{B}\Delta\mathbf{u} \\
&\mathbf{y} = \begin{bmatrix}\hat{\mathbf{C}}^{\mathrm{T}}_{(m-1)\times p} & \mathbf{0}^{\mathrm{T}}\end{bmatrix}\begin{bmatrix}\Delta\boldsymbol{\delta}\\ \Delta\boldsymbol{\omega}\end{bmatrix} = \mathbf{C}^{\mathrm{T}}\begin{bmatrix}\Delta\boldsymbol{\delta}\\ \Delta\boldsymbol{\omega}\end{bmatrix}. \quad (5)
\end{aligned}
$$

This formulation allows the external area to be modeled along the lines of (1), and the Krylov subspace model reduction methodology described in Section II can be applied in a straightforward manner.

A. Fifty-Machine System Connected to the Study Area via Three Tie-Lines

For the examples to follow, a 16-machine, 68-bus system taken from [39] is assumed to be the study area. A nonlinear two-axis model is used for each generator with an IEEE Type I exciter, resulting in seven differential equations per machine [38]. This system is connected to a 50-machine system also taken from [39] that serves as the external area as shown in Fig. 2. The external area is represented in its linear form as in (5).

The external area has six inputs (angles and voltage magnitudes of study area) and six outputs (angles and voltage magnitudes of the external area). A reduced-order model is constructed as discussed in Section II by interpolating the system at zero frequency ($\sigma = 0$) since the slow inter-area modes are of interest. In order to limit the size of the Krylov subspaces, we consider that the matrix $\mathbf{B}$ in the algorithm in Fig. 1 for building base $\mathbf{V}(\mathbf{Z})$ is the sum of the input (output) matrices

TABLE I
MODES OF THE ORIGINAL AND THE REDUCED SYSTEMS

Frequency (Hz)		Damping (Eigenvalue real part)		
Full System	Reduced System	Full System	Reduced System	Frequency Error (%)
0.2037	0.2037	-0.2857	-0.2857	0.00
0.3177	0.3189	-0.2506	-0.2586	-0.36
0.3839	0.4033	-0.2602	-0.3741	-5.06
0.4871	0.5453	-0.2835	-0.4505	-11.93
0.5822	0.6168	-0.3405	-0.5884	-5.94
0.7133	0.8523	-0.3353	-0.4042	-19.49
0.7852	1.0404	-0.3501	-0.6302	-32.50
0.8307	1.3528	-0.4298	-0.2945	-62.85
0.9271	2.3483	-0.2575	-0.3709	-153.30

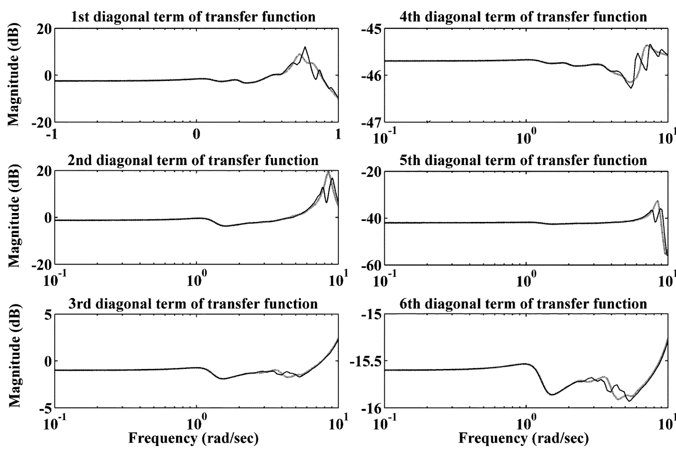

Fig. 3. Frequency response for the linearized system.

corresponding to the angle and voltage inputs (outputs).[1] The size of the reduced model is set at 18×18, which indicates that the reduced model matches the first 6 coefficients of the power series expansion around zero of the transfer function matrix of the original system.

Although eigenvalues are not being explicitly retained, it is interesting to see whether the slowest modes are present in the reduced model. Indeed, from Table I one may see that the two slowest modes are reproduced adequately in the reduced model, with the second and third slowest modes being significantly overestimated in terms of their damping.

The same information is conveyed in Fig. 3, showing the diagonal terms of the frequency response matrix of the unreduced (solid line) and the reduced systems (dotted line). For economy of space, nondiagonal terms of the transfer function matrix are omitted. For frequencies above 3 rad/s the reduced system frequency response is not accurate.

A time domain simulation permits the evaluation of the reduced-order model. A self-clearing fault at bus 24 of the study area, occurring at 0.1 s and cleared at 0.2 s is simulated. Note that the study area retains a nonlinear representation. The outcomes of the simulation using the unreduced (solid lines) and the reduced linear models (dotted lines) for the external area are shown in Fig. 4 in terms of the active power flow on the tie-lines.

[1]In general, such kind of heuristics for economy in the size of the base when a block-Krylov algorithm is used does not work well. However, we expect that the input (output) vectors corresponding to the angle and the voltage magnitudes for the same bus will contain the same information regarding the construction of the subspace.

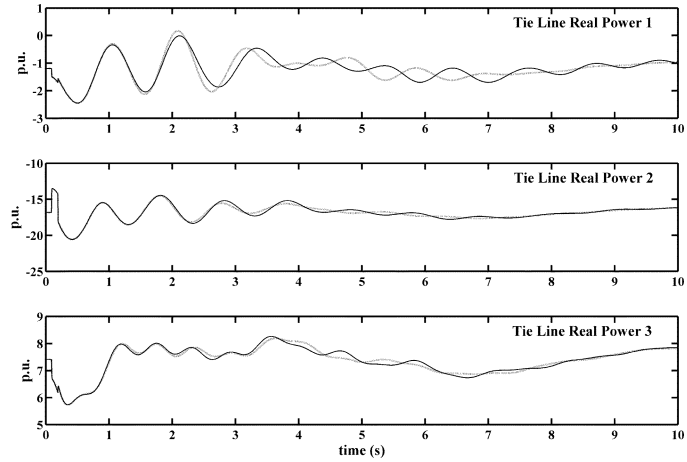

Fig. 4. Real power flows on tie-lines.

TABLE II
MODES OF THE ORIGINAL AND THE REDUCED SYSTEMS

Frequency (Hz)		Damping (Eigenvalue real part)		
Full System	Reduced System	Full System	Reduced System	Frequency Error (%)
0.2037	0.2037	-0.2857	-0.2857	0.00
0.3177	0.3177	-0.2506	-0.2506	0.00
0.3839	0.3839	-0.2602	-0.2602	0.00
0.4871	0.4871	-0.2835	-0.2835	0.01
0.5822	0.5823	-0.3405	-0.3451	-0.01
0.7133	0.7130	-0.3353	-0.3819	0.04
0.7852	0.7923	-0.3501	-0.3697	-0.90
0.8307	0.9244	-0.4298	-0.3709	-11.28
0.9271	0.9259	-0.2575	-1.5547	0.13
1.0584	1.1618	-0.2551	-0.5175	-9.76
1.0933	1.3642	-0.2500	-0.3631	-24.77
1.1438	2.3641	-0.2504	-0.2624	-106.69

From Fig. 4, one can see that the reduced-order model approximates the external system well only for low frequencies. The overestimation of the damping of the second slowest mode is more evident on the flow on the first tie-line.

There are two ways to improve the quality of the reduced model. Either the dimension of the Krylov subspace is increased by matching additional moments of the transfer function around zero, or by interpolating another point and thus matching the moments of the transfer function around that frequency as well. The second choice is adopted here by interpolating the transfer function at 0 and at j3 (imaginary interpolation point, corresponding at 3 rad/s). Hence, the accuracy above 0.5 Hz is expected to increase. Note that the size of the reduced system is increased to 24×24.

Table II shows that the seven slowest modes are retained in the reduced system among which the faster ones are erroneous as far as their damping is concerned.

Figs. 5 and 6 show the frequency response and the results of the same simulation as before where the improvement in the approximation by the reduced model is clear. It has to be noted that when the size of the reduced model reaches 30, the error in the approximation for these frequencies is negligible.

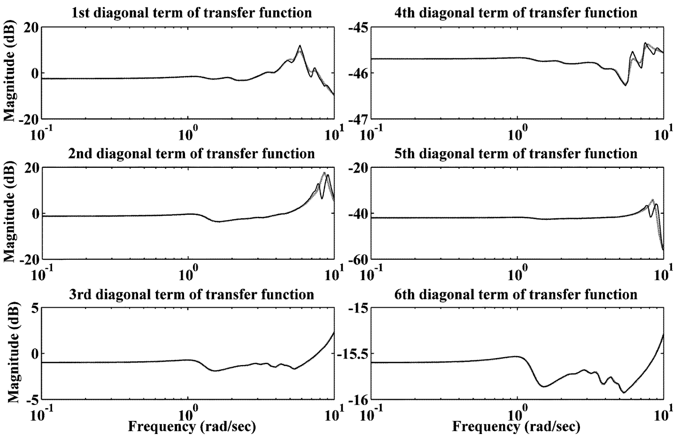

Fig. 5. Frequency response for the linearized system.

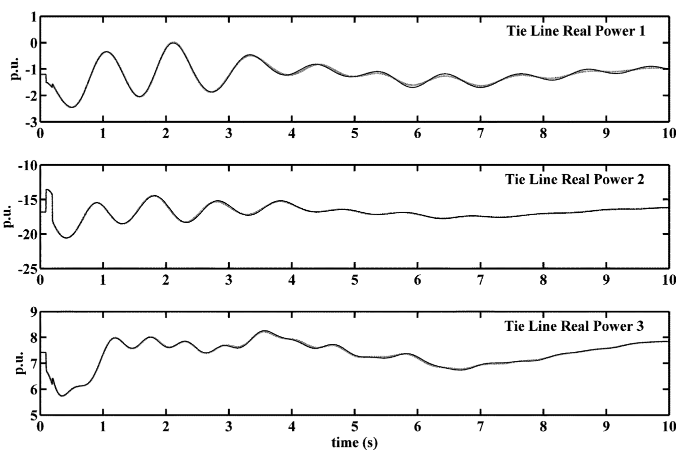

Fig. 6. Real power flows on tie-lines.

IV. Krylov Subspace Model Reduction and Coherency

The mechanics of the Krylov subspace model reduction dictate that the states of the unreduced system are confined to the subspace $\mathbf{V}$ as deduced from (1) and (3), i.e.,

$$\mathbf{x} = \mathbf{V}\hat{\mathbf{x}}. \tag{6}$$

Therefore, the rows of $\mathbf{V}$ may be a good indication about the relative movement of the states of the unreduced system. Note, however, that (6) does not imply that the actual trajectories lie on subspace $\mathbf{V}$, but rather that if the trajectories lie on this subspace, then the output of the reduced system matches the output of the unreduced system. From (6)

$$\mathbf{x}_i - \mathbf{x}_j = (\mathbf{V}(i,:) - \mathbf{V}(j,:))\,\hat{\mathbf{x}}. \tag{7}$$

According to [13], the states i and j are coherent if the difference between $\mathbf{x}_i$ and $\mathbf{x}_j$ in (7) is small. A possible criterion to determine the "closeness" of $\mathbf{x}_i$ and $\mathbf{x}_j$ is the angle φ between rows $\mathbf{V}(i,:)$ and $\mathbf{V}(j,:)$ defined as

$$\cos\varphi = \frac{\mathbf{V}(j,:)\mathbf{V}(i,:)^T}{\|\mathbf{V}(i,:)\|_2\,\|\mathbf{V}(j,:)\|_2}. \tag{8}$$

Therefore, one can term $\mathbf{x}_i$ and $\mathbf{x}_j$ as coherent if the angle φ is less than a pre-specified tolerance. Let $\mathbf{H}$ be the matrix

$$\mathbf{H}(i,j) = \arccos\left(\frac{\mathbf{V}(j,:)\mathbf{V}(i,:)^T}{\|\mathbf{V}(i,:)\|_2\,\|\mathbf{V}(j,:)\|_2}\right). \tag{9}$$

TABLE III
Coherent Groups for the 50-Machine System

Coherent Groups	Generator Number
Group 1	2,8,11
Group 2	3,14,36
Group 3	4,5
Group 4	6,25
Group 5	7,10,15,16,17,20,22,23,26,18,21,28,13,27
Group 6	30,31,32

Thus, the entries of $\mathbf{H}$ determine whether or not two states are coherent. Equation (9) does not distinguish between the modes of the reduced system; it assumes that all modes may be excited. It is possible to eliminate certain high-frequency modes that are of no interest and correspond to parasitic modes observed in the last rows of Tables I and II, by ignoring the states of the reduced system that have a relatively large participation on these modes. The small dimension of the reduced system allows a complete eigenanalysis in a robust and efficient way. Therefore, the contribution of each state to the modes of the reduced system can be identified through the use of participation factors [40]. Let $\mathbf{u}$ and $\mathbf{w}$ be the right and left eigenvectors of the reduced system corresponding to an eigenvalue λ. For complex vectors, the magnitudes of each entry of $\mathbf{u}$ and $\mathbf{w}$ are considered. Then, the participation factor of state i to the eigenvalue λ is defined as

$$p = \frac{\mathrm{u}_i \mathrm{w}_i}{\mathbf{u}^{\mathrm{T}}\mathbf{w}}. \tag{10}$$

The participation factors can be normalized so that the largest one is 1. The modes of interest are isolated (for example, the ones corresponding to the low frequencies) and the participation of each state of the reduced system on a certain mode can be computed from (10). The states that contribute the least to the modes of interest are discarded. This is equivalent to ignoring the corresponding columns of $\mathbf{V}$ in (9). Then, $\mathbf{H}$ is constructed using the reduced $\mathbf{V}$.

To illustrate the method, the Krylov base of dimension 24×24 used in the simulations of Section III is taken for building matrix $\mathbf{H}$ in (9), tabulating the resulting coherent generators in Table III. The modes retained are the ones with imaginary part less than 5 and real part greater than -1, that is, the low-frequency modes that mostly affect the steady state solution. States with participation factors less than 0.7 in those modes are omitted.

In order to validate these results, a simulation using the full-order linear model for the external area and a nonlinear model for the study area is performed. A fault at bus 24 of the study area, occurring at 0.1 s and cleared at 0.2 s is simulated. Fig. 7 shows the simulation results and it verifies that the generators belonging to the same group according to Table I are indeed coherent.

From Fig. 7 it is evident that this approach yields conservative results. For example, one may suggest that generators in groups 2, 4 and 5 seem to be coherent even though our method groups them separately. Thus, in order to test the validity of the method we perform the same simulation considering the fault to occur in bus 19 of the study area. The coherent groups should remain the same despite the different location of the fault. Indeed, as observed in Fig. 8, generators in the same group remain coherent with respect to the low frequencies. Furthermore, the responses for generators in groups 2, 4 and 5 are no longer similar. Extensive simulations

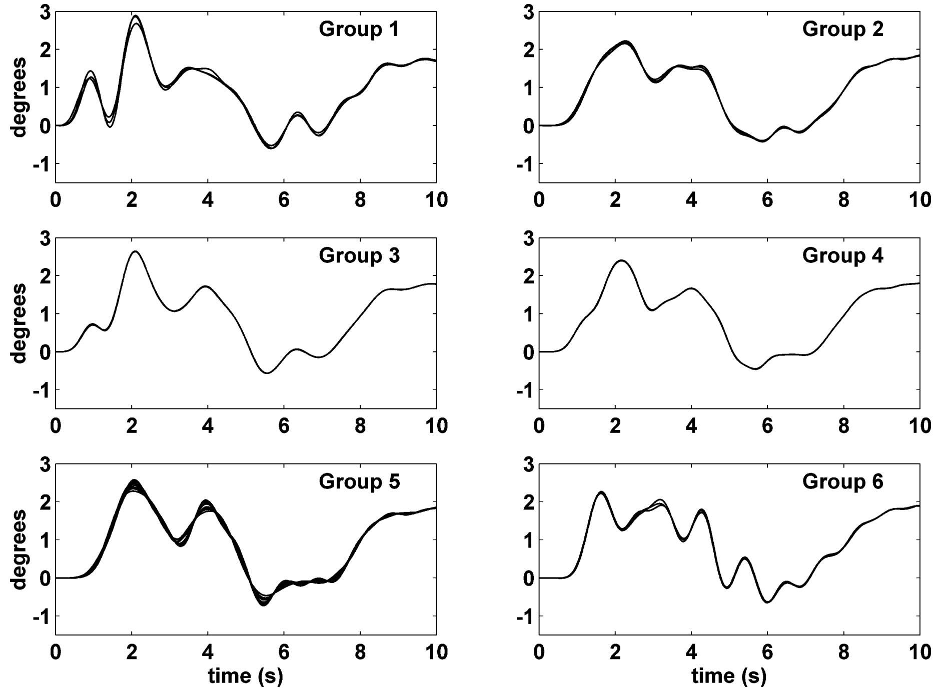

Fig. 7. Angle curves for generators in all coherency groups.

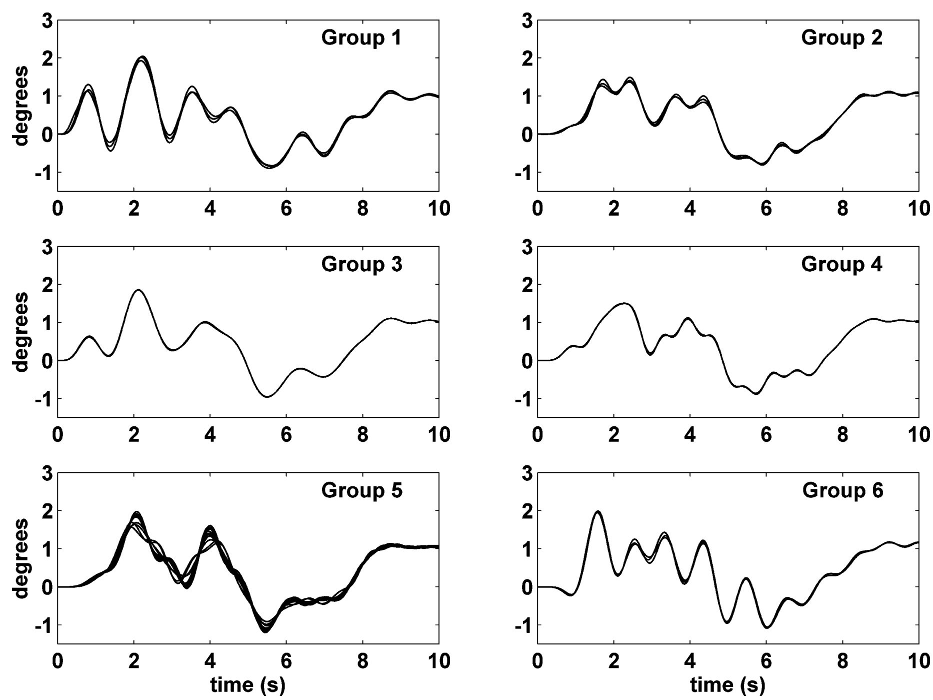

Fig. 8. Angle curves for generators in all coherency groups.

on this particular system for different fault locations confirm this result.

V. Conclusions

In this paper, an approach to model reduction in power systems based on Krylov subspaces has been presented. The reduction process considers the external area of the power system as an input-output system, whose behavior is approximated by the reduced model by matching the leading coefficients of a power series expansion of the transfer function around pre-specified frequencies. The method was illustrated by performing time domain simulations, and it was shown that the trajectories of the reduced model match these of the original one for sufficiently large Krylov subspaces. Therefore, without performing the potentially expensive eigenvalue decomposition, this method seems to adequately reproduce the dominant modes of the original system guided by their influence in its input-output behavior. The method was tested on a 50-machine power system considered as the external area.

The Krylov subspace model reduction in power systems was shown to be able to provide information useful in identifying coherency as well. The merits of this method lie in its ability to determine coherency with respect to any subset of the dominant modes of the system. Therefore, it requires no a priori knowledge of the system and reduces the need for adjustments based on empirical knowledge of the system. Application of this methodology on the 50-machine system has yielded very good results.

Krylov subspace techniques seem to be very promising when applied to power system model reduction. Future research would be focused on explicitly testing these methods on large systems and dealing with the problem of unstable reduced models.

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