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An approximate theoretical analysis and experimental verification of turbulent entrance region flow of drag reducing fluids

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With 3 figures and 1 table

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Nomenclature

A_1	Coefficient in eq. [7]
A	Slope of logarithmic velocity profile
a	Exponent in eq. [10]
B	Intercept function for logarithmic velocity profile
De	Deborah number, $\frac{\theta_{fl} u^{*2}}{\nu}$
f	Friction factor
F	Function, eq. [30]
G	Function given in eq. [11]
$\tilde{p}$	Static pressure, dynes/cm^2
q	Index of power law velocity profile
R	Pipe radius, cm
r	Radial distance, cm
R_δ	Core radius, cm
Re	Reynolds number
$\tilde{u}$	Axial velocity, cm/s
u_c	Core velocity, cm/s
u^+	Dimensionless velocity, eq. [5]
u^*	Friction velocity, $\sqrt{\frac{\tau_w}{\rho}}$, cm/s
$\tilde{v}$	Radial velocity, cm/s
V	Average velocity, cm/s
x	Axial distance, cm
x_e	Entry length, cm
y	Distance from the wall, cm
y^+	Dimensionless distance, eq. [5]
y_l^+	Dimensionless viscous sublayer thickness

Greek symbols

α	coefficient in eq. [17]
β	exponent of Reynolds number in eq. [17]
$\dot{\gamma}$	shear rate, s^{-1}
δ	boundary layer thickness, cm
θ_{fl}	fluid relaxation time, s
μ	fluid viscosity, gm/cm s
ν	kinematic viscosity, cm^2/s
ξ_l	laminar sublayer thickness, dimensionless
ρ	fluid density, gm/cm^3
τ	shear stress, dynes/cm^2
τ_w	shear stress at the wall, dynes/cm^2
$\psi_1, \psi_2, \psi_3, \psi_4$	functions in eq. [27]

Superscripts

~	time averaged quantities
—	dimensionless quantity

1. Introduction

Studies of turbulent drag reduction have assumed great importance in recent years. After the discovery of the phenomenon by *Toms* (10) many investigators have looked at various aspects of drag reduction from both an experimental and theoretical standpoint. However, not much attention has been paid to the phenomenon so important in drag reduction studies, viz. entrance region flow. For reliable pressure drop studies, one needs to know the exact value of the entry length, since the velocity and pressure distributions in the entrance region are significantly different in comparison to that in the fully developed flow. The theoretical analysis and experimental contribution of entrance flow of non-Newtonian inelastic fluids in laminar flow (see e.g. *Mashelkar* (3)) is well known. Also there is some information on non-Newtonian viscoelastic laminar flows (see e.g. *Boger* and *Ramamurthy* (1)). However, no such

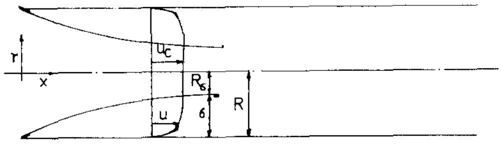

Fig. 1. A schematic diagram of the entrance region flow

information appears to exist on the turbulent entrance region flow of drag reducing fluids. There is a single experimental study by *Seyer* and *Catania* (7) which gives some information on the magnitude of the entry length required for drag reducing fluids. However, no correlations have been provided by these authors to enable information for other fluids or other pipe diameters nor is there an analytical effort made for the prediction of the entry length. It is precisely to fill this gap that the present work was undertaken.

2. Background

In the present work we attempt a theoretical solution of the entrance region flow problem in pipe flows. Due to the nature of the problem, exact calculations are not possible and hence resort must be made to approximate solutions. The problem can be solved by numerical solution of the full time averaged Navier-Stokes equations. The entrance region flow of the Newtonian fluids has been studied in this manner by some investigators. *Richman* and *Azad* (4) solved the problem by finite difference technique. They solved simultaneously the elliptic forms of the vorticity transport and stream function equations. *Ross* and *Whippany* (5) used boundary layer theory together with empirical equations for the shear stress at the wall and shape factors. Their analysis, however, is limited to about 10 pipe diameters downstream. *Bowlus* and *Brighton* (2) used momentum integral technique to solve the problem and they used *Schultz-Grunow* (6) relation for flat plate turbulent skin friction. In order to analyse the entrance region flow problem for a turbulently flowing moderately elastic drag reducing fluid, we shall use the momentum integral technique. In the following we shall systematically outline the salient steps leading to the entry length predictions.

3. Momentum integral equations

The time averaged equations of continuity and motion in steady state for axisymmetric flow in cylindrical co-ordinates are given as

$$\frac{\partial \tilde{u}}{\partial x} + \frac{1}{r}\frac{\partial (r\tilde{v})}{\partial r} = 0\,, \qquad [1]$$

$$\tilde{u}\frac{\partial \tilde{u}}{\partial x} + \tilde{v}\frac{\partial \tilde{u}}{\partial r} = -\frac{1}{\rho}\frac{\partial \tilde{P}}{\partial x} - \frac{1}{\rho r}\frac{\partial (r\tau)}{\partial r}\,. \qquad [2]$$

In the above, we have neglected the axial variation of the small normal stress terms, which will arise in the equation of motion.

Following the arguments advanced by the boundary-layer theory (6) the turbulent flow in a pipe can be divided into several zones in which the inertial and the viscous forces assume different significance. Whilst the bulk of the flow in the core of the pipe will be dominated by the inertial force, the constitutive properties of the fluid will gain on importance closer to the wall.

With these assumptions in mind the equation of motion for the core region can be written as

$$\frac{1}{\rho}\frac{\partial \tilde{p}}{\partial x} = -\tilde{u}_c\frac{d\tilde{u}_c}{dx}\,. \qquad [3]$$

Combining eqs. [1] and [2], non dimensionalizing and integrating from centre line to the wall, the momentum equations become:

$$\frac{d}{d\bar{x}}\int_0^1 \bar{u}^2\,\bar{r}\,d\bar{r} = -\frac{\tau_w}{\rho V^2} + \frac{\bar{u}_c}{2}\frac{d\bar{u}_c}{d\bar{x}}\,. \qquad [4]$$

Here, lengths are non-dimensionalized with respect to radius of the tube and velocities with respect to the average velocity in the tube. Solution of eq. [4] needs boundary conditions. We are assuming that at the entrance the velocity profile is flat or that the velocity is uniform with a value given by $\bar{u}_c$ at $\bar{x} = 0$.

In order to make the problem explicit we need expressions in momentum integral equation [4] for both $\bar{u}$ and τ_w. In the following we shall deduce an approximate way of arriving at these for turbulently flowing drag reducing fluids.

4. An explicit expression for wall shear stress

Unlike the laminar profile the turbulent velocity profile cannot be predicted theoretically. This is due to the existence of velocity fluctuations which give rise to additional (turbulent) stresses which cannot be satisfactorily related to the main stream parameters of the flow. Relying essentially on dimensional arguments and on extensive experimental data the turbulent velocity profiles of Newtonian fluids in pipes of circular cross-sectional area are satisfactorily correlated by the following relationship:

$$u^+ = A \ln y^+ + B \qquad [5]$$

in which $u^+ = u/u^*$ and $y^+ = yu^*/\nu$ are the dimensionless velocity and the dimensionless distance from the wall respectively. The term $u^* = \sqrt{\tau_w/\rho}$ which appears frequently in the analyses of turbulent flows and has the dimension of velocity is called the friction velocity or the shear velocity. Eq. [5] is known as the universal velocity distribution and it has been often used to correlate the experimental data both in the wall region as well as in the core. When analysing the different regions separately it can be shown that a relationship

$$u^+ = y^+ \qquad [6]$$

fits well the data in the region close to the wall whilst a power law type of relationship is more adequate for the flow away from the wall

$$u^+ = A_1 (y^+)^q . \qquad [7]$$

Seyer and *Metzner* (8) showed that for drag reducing fluids the parameter B in eq. [5] is no longer a constant but that it becomes a function of the Deborah number. Eq. [5] will then have the form:

$$u^+ = A \ln y^+ + B(\mathrm{De}) \qquad [8]$$

where the value of A is that for Newtonian flows whilst B (De) is a function of dimensionless time De called the 'Deborah number'. They experimentally deduced that in the lower range of De the function B (De) can be expressed approximately as:

$$B(\mathrm{De}) = 5.6 + 1.55\,\mathrm{De} .$$

The Deborah number is given as

$$\mathrm{De} = \theta_{\mathrm{f1}} \frac{u^{*2}}{\nu} . \qquad [9]$$

If $\frac{u^{*2}}{\nu}$ is taken to be approximately proportional to the wall shear rate, then the experimental determination of the fluid relaxation times in the high shear rate range varies with shear rate at the wall as:

$$\theta_{\mathrm{f1}} \propto \dot{\gamma}^{-a} \qquad [10]$$

where the value of the exponent a varies from 0.5 to 1.0. Combining eqs. [9] and [10] we find that Deborah number is approximately constant. This argument which will be consistently made in this paper, is not accurate in detail but it will still be useful for deriving approximate expression for the wall shear stress in turbulent flows of drag reducing fluids.

On integrating eq. [8] in the manner of *Seyer* and *Metzner* (8), we can show that:

$$\sqrt{\frac{2}{f}} = A(1-\xi_l)^2 \ln \mathrm{Re}\sqrt{f} + (1-\xi_l)^2 \cdot [B(\mathrm{De}) - A \ln 2\sqrt{2}] - G . \qquad [11]$$

The value of G can be effectively considered as constant and equal to 3.0. The appearance of ξ_l shows marked thickening of viscous sublayer and it can be evaluated by noting that, near wall $u^+ = y^+$.

Hence at the end of the viscous sublayer

$$y^+ = A \ln y^+ + B(\mathrm{De}) . \qquad [12]$$

We notice that

$$y_l^+ = \frac{y_l u^*}{\nu} \qquad [13]$$

and

$$u^* = V \sqrt{\frac{f}{2}} , \qquad [14]$$

and since $\xi_l = y_l/R$, we get

$$\xi_l = \frac{y_l^+ 2\sqrt{2}}{\mathrm{Re}\sqrt{f}} . \qquad [15]$$

On substituting this in eq. [11] we get:

$$\sqrt{\frac{2}{f}} = A\left(1 - \frac{y_l^+ 2\sqrt{2}}{\mathrm{Re}\sqrt{f}}\right)^2 \ln \mathrm{Re}\sqrt{f} + \left(1 - \frac{y_l^+ 2\sqrt{2}}{\mathrm{Re}\sqrt{f}}\right)^2 \cdot [B(\mathrm{De}) - A \ln 2\sqrt{2}] - 3.0 . \qquad [16]$$

B (De) was evaluated at a given value of De between 0 and 10 and eq. [16] was solved by iteration, for values of friction factors for various Reynolds numbers. The resulting values of f and Re were fitted as

$$f = \frac{\alpha}{\mathrm{Re}^{\beta}} . \qquad [17]$$

The values of α and β so obtained for various De are given in table 1. Note that the range of De from 0 to 10 is quite adequate for very dilute polymer solutions. Although we have not done calculations in the intermediate region, we have performed a calculation for the asymptotic case when $\mathrm{De} \to \infty$ (viz. for $\alpha = 0.42$ and $\beta = 0.55$, see *Virk* (11)).

Table 1. See text

α	β	De
0.0791	0.25	0
0.0960	0.28	1
0.0869	0.282	2
0.0808	0.285	3
0.0771	0.290	4
0.0752	0.296	5
0.0749	0.304	6
0.0760	0.313	7
0.0783	0.323	8
0.0819	0.334	9
0.0867	0.346	10
0.42	0.55	∞

5. An explicit expression for velocity distribution in the turbulent flow of a drag reducing fluid

The empirical equation to describe the velocity distribution in turbulent flow outside the wall region is

$$u^+ = A_1 (y^+)^q. \qquad [7]$$

Integrating eq. [7] we get the average velocity as

$$\frac{V}{u^*} = A_1 \frac{2}{(q+1)(q+2)} \left(\frac{R u^* \rho}{\mu}\right)^q \qquad [18]$$

and since $f = 2\left(\frac{u^*}{V}\right)^2$ we get, after substituting various quantities,

$$f = \left(\frac{(q+1)(q+2)\sqrt{2}}{2} A_1 8^{2/2}\right)^{\frac{2}{q+1}} (\mathrm{Re})^{-\frac{2q}{q+1}}. \qquad [19]$$

For a Newtonian fluid $A_1 = 8.56$ and $q = \frac{1}{7}$ and hence from eq. [19] we find that $f = 0.0791\,\mathrm{Re}^{-1/4}$, a well known result.

When we compare eq. [19] with eq. [17] we find that:

$$q = \frac{\beta}{2-\beta}. \qquad [20]$$

Thus, we find that the velocity distribution can be represented by:

$$u^+ = A_1 (y^+)^{\frac{\beta}{2-\beta}}. \qquad [21]$$

6. Derivation of a general wall shear stress expression for a turbulently flowing drag reducing fluid

It can be shown by following arguments similar to *Skelland* (9) that the wall shear stress is given by:

$$\tau_w = \rho V^2 \frac{\alpha}{2} (\mathrm{Re})^{-\beta} (1 - \bar{R}_\delta)^{-\beta}. \qquad [22]$$

We further assume the velocity distribution in the boundary layer in dimensionless form is given as

$$\bar{u} = \bar{u}_c \left(\frac{\bar{y}}{\bar{\delta}}\right)^q \qquad [23]$$

where q is given by eq. [20].

This implies that we have arrived at the proper expressions for τ_w and $\bar{u}$ as functions of the system variable and these can now be readily substituted in eq. [4].

7. Solution of the momentum integral equation

Substituting eq. [23] in eq. [4] we get

$$\frac{d}{d\bar{x}}\left\{\int_0^{1-\bar{R}_\delta} \bar{u}_c^2 \left(\frac{\bar{y}}{1-\bar{R}_\delta}\right)^{2q} (1-\bar{y})\,d\bar{y} + \int_{1-\bar{R}_\delta}^{1} \bar{u}_c^2 (1-\bar{y})\,d\bar{y}\right\} = \frac{\bar{u}_c}{2}\frac{d\bar{u}_c}{d\bar{x}} - \frac{\tau_w}{\rho V^2} \qquad [24]$$

where we note that $\bar{r} = 1 - \bar{y}$.

Similarly from equation of continuity we get

$$\frac{d}{d\bar{x}}\left\{\int_0^{1-\bar{R}_\delta} \bar{u}_c \left(\frac{\bar{y}}{1-\bar{R}_\delta}\right)^{q} (1-\bar{y})\,d\bar{y} + \int_{1-\bar{R}_\delta}^{1} \bar{u}_c (1-\bar{y})\,d\bar{y}\right\} = 0. \qquad [25]$$

Solving eq. [25] and integrating we get

$$\bar{u}_c = \frac{(q+1)(q+2)}{q\bar{R}_\delta[\bar{R}_\delta(q+1)+2]+2}. \qquad [26]$$

(Note that when $\bar{R}_\delta = 1$ we get $\bar{u}_c = 1$.)

Solving eq. [24], substituting for $\bar{u}_c$ and $\frac{d\bar{u}_c}{d\bar{x}}$ from eq. [26] and re-arranging we get a differential equation for $\bar{R}_\delta$ as

$$\psi_1(\bar{R}_\delta)\frac{d\bar{R}_\delta}{d\bar{x}} + \psi_2(\bar{R}_\delta)\psi_3'(\bar{R}_\delta)\frac{d\bar{R}_\delta}{d\bar{x}} = \psi_4(\bar{R}_\delta)\psi_3'(\bar{R}_\delta)\frac{d\bar{R}_\delta}{d\bar{x}} - \frac{\alpha}{2}(\mathrm{Re})^{-\beta}(1-\bar{R}_\delta)^{-\beta} \qquad [27]$$

where

$$\psi_3(\bar{R}_\delta) = \bar{u}_c = \frac{(q+1)(q+2)}{q\bar{R}_\delta[\bar{R}_\delta(q+1)+2]+2},$$

$$\psi_4(\bar{R}_\delta) = \frac{\bar{u}_c}{2} = \frac{\psi_3}{2}(\bar{R}_\delta)$$

and

$$\psi'_3(\bar{R}_\delta) = \frac{d\bar{u}_c}{d\bar{R}_\delta}. \quad [28]$$

$\psi_1(\bar{R}_\delta)$ and $\psi_2(\bar{R}_\delta)$ are functions of $\bar{R}_\delta$ only and arise after substituting eq. [26] for $\bar{u}_c$.

Eq. [27] on re-arrangement gives

$$\frac{d\bar{R}_\delta}{d\bar{x}} = \frac{-\frac{a}{2}(\mathrm{Re})^{-\beta}(1-\bar{R}_\delta)^{-\beta}}{\psi_1(\bar{R}_\delta) + (\psi_2(\bar{R}_\delta) - \psi_4(\bar{R}_\delta))\,\psi'_3(\bar{R}_\delta)}. \quad [29]$$

Thus, eq. [29] can be written simply as

$$\frac{d\bar{R}_\delta}{d\bar{x}} = -F(\bar{R}_\delta). \quad [30]$$

From eq. [30] we get

$$d\bar{x} = -\frac{d\bar{R}_\delta}{F(\bar{R}_\delta)}. \quad [31]$$

Integrating from $\bar{x} = 0$ to $\bar{x} = \bar{x}_e$ delivers

$$\bar{x}_e = \int_0^1 \frac{d\bar{R}_\delta}{F(\bar{R}_\delta)}. \quad [32]$$

Eq. [32] is then solved numerically by nine point quadrature method on an ICL 1904 S computer. Here entry length $\bar{x}_e$ is defined as the length where core radius $\bar{R}_\delta$ vanishes. It should be emphasized that the definitions of $\bar{x}_e$ can be various and indeed the absolute value of the entry length can differ substantially depending upon whether the considerations are for velocity distribution or the pressure distribution to develop fully. Furthermore, whether a calculation for 95% development or 99% development is taken can also make the precise values differ significantly, because the growth of the boundary layer is asymptotic. However arbitrary the present definition may seem, it does give a good relative measure of the entry length required when compared with the corresponding Newtonian values.

Figure 2 shows a plot of the theoretically calculated entry lengths versus Reynolds numbers for various Deborah numbers. Plotted in the same figure is the result for the Newtonian case as well as that for the maximum drag reduction case. The entry lengths are somewhat insensitive to Reynolds number but quite sensitive to Deborah number. Thus, for example, at a Reynolds number of 30,000, the entry length of a fluid with De = 5 is 63.14 pipe radii, whilst at

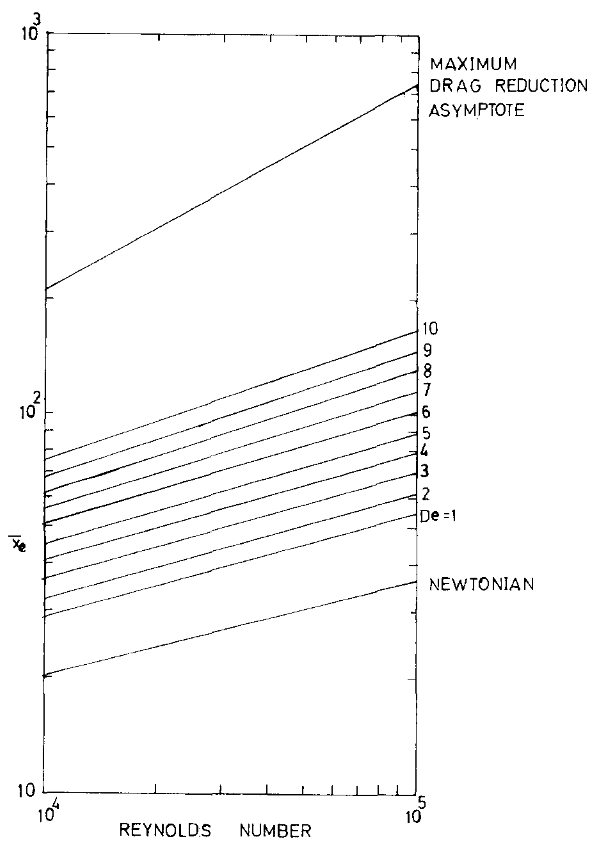

Fig. 2. Entry lengths as a function of Reynolds number and Deborah number

the same Reynolds number, the entry length for a Newtonian fluid is only 26.87 pipe radii. Also, the slope of the graph increases as the Deborah number increases. Thus, for the Newtonian fluids, the slope is 0.25 whilst that for the maximum drag reducers is 0.55, the slopes at intermediate Deborah numbers being in between these. Thus, at De = 5, the slope is 0.296. This can be explained readily, since drag reducing fluids thicken the viscous sublayer thereby suppressing turbulent boundary layer. This results in the velocity profile being more steep resulting in large entry lengths. If thickening of the viscous sublayer was to continue indefinitely, it would result in complete laminarization of the flow and then the entry length would be directly proportional to the Reynolds number.

8. Experimental

The aim of the experiment was an exploratory one. The experimental setup consisted of a recirculating loop, containing a reservoir, a gear pump, a rotameter,

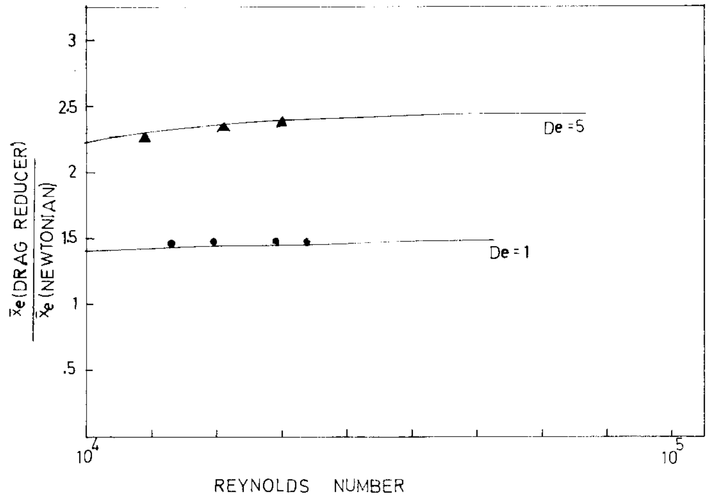

Fig. 3. Comparison of experimental values of entry lengths and that from theoretical predictions on a relative basis
● Experimental data from the present work
▲ Experimental data from *Seyer* and *Catania*'s work (7)
— Theoretical predictions

a test section preceded by a calming section. The test section was only up to about 50 pipe diameters. The ratio of pressure drop in the entrance region to that in the fully developed region for the same pipe of the same length was used to calculate the entry length. This ratio was empirically correlated with length to diameter ratio and entry length was defined as the length where the above ratio is equal to 1.0. Water was used as a Newtonian fluid and dilute solutions of the polymers Polyacrylamide (Separan AP 30) and Polyethyleneoxide (WSR 301) were used as the drag reducing fluids. The concentrations of the polymers were from 50 wppm to 250 wppm and hence the viscosities of the solutions were assumed to be constant because of the high shear rates associated with turbulent flows. The experimental results clearly showed that these solutions required higher entry lengths. We shall later compare the relative increase in entrance length with the theoretically determined one.

As stated previously, the exact values of entry lengths change according to definition and hence comparison with theoretical results is attempted on a relative basis. The ratio of the entry length for the polymer solution to that for the Newtonian fluid is compared using *Seyer* and *Metzner*'s (8) experimental relaxation time data for the 100 wppm PAA solution. We find that when the Deborah number is of the order of one the theoretical ratio of the entry lengths is about 1.47 (this ratio does change with Reynolds number, but this change is not very significant) whilst the present experiments indicate a ratio of about 1.52. The ratio for 50 wppm and 250 wppm solutions is of the same order.

When *Seyer* and *Catania*'s (7) results are examined the same trend is found. Although they found that 100 wppm PAA solution gave about the same entry lengths as that given by Newtonian fluids, there is a scatter in the data. Their results show higher entry lengths for 0.2% PAA solution. The Deborah number was estimated approximately to be about 5 for this solution and when the above indicated ratio of the entry lengths is compared to that of the theory, a good agreement is found.

9. Conclusions

An approximate momentum integral technique has been used to find the entry lengths for moderately drag reducing fluids in turbulent flow in pipe. It is found that fluid elasticity significantly increases the entry length, sometimes even by an order of magnitude. Although the results of the present work are only approximate, in view of the fact that there are no prior studies on the title problem, the present predictions could be used for designing pipe lines transporting drag reducing fluids or even in planning proper experiments with these fluids.

Some experimental measurements on entry lengths using drag reducing fluids made in this work and also in the work of *Seyer* and *Catania* (7) show fair agreement with the theoretical predictions.

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Summary

Entry lengths for pipe flows of moderately drag reducing fluids are determined using momentum integral technique. It is shown theoretically that the entry lengths for drag reducing fluids could be signi-

ficantly larger than the Newtonian fluids flowing turbulently under otherwise identical conditions. The experimental data from the literature bear out the theoretical calculations.

Zusammenfassung

Mit Hilfe der Impuls-Methode wird die Einlauflänge in einer Rohrströmung für Flüssigkeiten mit mäßig starker Widerstandsverminderung berechnet. Es wird vorausgesagt, daß die Einlauflänge für derartige Flüssigkeiten erheblich größer sein kann als für newtonsche Flüssigkeiten unter sonst identischen Bedingungen. Aus der Literatur entnommene experimentelle Daten bestätigen diese theoretischen Berechnungen.

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