Rheol. Acta 14, 631–640 (1975)

Department of Chemical Engineering, University of Salford, Salford M5 4WT, Lancashire (England)

Rotational viscoelastic laminar boundary layer flow around a rotating disc

D. D. Kale, R. A. Mashelkar, and *J. Ulbrecht*

With 4 figures and 2 tables

(Received December 2, 1974)

Nomenclature

A	material parameter $\frac{\text{dyne}}{\text{cm}^2}$ sec^{-b}
A_1	*Rivlin-Ericksen* acceleration tensor of first order
A_2	*Rivlin-Ericksen* acceleration tensor of second order
b	material parameter, dimensionless
C_1, C_2	the constants defined by eqs. [37] and [38] respectively
C_M	moment coefficient, dimensionless
$C_{M\text{inel}}$	Moment coefficient for inelastic fluids, dimensionless
$C_{M\text{viscoel}}$	moment coefficient for viscoelastic fluids dimensionless
$\boldsymbol{D}$	rate of deformation tensor
$\dot{\boldsymbol{D}}$	material time derivative of rate of deformation tensor
F'	function of ζ, defined by eq. [15]
$f(n)$	function of n, defined by eq. [45]
G'	function of ζ, defined by eq. [16]
H'	function of ζ, defined by eq. [18]
K	material parameter $\frac{\text{dyne}}{\text{cm}^2}$ (sec)^n
M	moment on a disc, dyne-cm
n	material parameter, dimensionless
P	material parameter $\frac{\text{dyne}}{\text{cm}^2}$ (sec)^q
p	isotropic pressure dyne/cm^2
q	material parameter dimensionless
r	radial coordinate, cm
R	radius of the disc, cm
Re_{0w}	*Reynolds* number
T	stress tensor
$T_{rr}, T_{\theta\theta}, T_{zz}, T_{r\theta}, T_{rz}, T_{\theta z}$	components of stress tensor dyne/cm^2
$\boldsymbol{v}$	velocity vector
v_r, v_θ, v_z	components of velocity vector cm/sec
W	vorticity tensor
Wi_1, Wi_2	*Weissenberg* numbers defined by eqs. [24] and [25] respectively
z	axial coordinate, cm

Greek symbols

$\beta(n)$	function of n defined by eq. [40]
$\dot{\gamma}$	shear rate, sec^{-1}
δ	*Kronecker* delta in eq. [1]
δ'	boundary layer thickness cm, (eq. [14])
Δ	boundary layer thickness at $r = R$, cm (eq. [28])
ζ	dimensionless variable z/δ, (eq. [14])
η	dimensionless variable r/R, (eq. [14])
$\mu(\tilde{II})$	scalar function, defined by eq. [11]
$\lambda(\tilde{II})$	scalar function, defined by eq. [13]
λ	relaxation time of the fluid, sec
∇	nabla operator $\frac{\partial}{\partial r} + \frac{1}{r}\frac{\partial}{\partial \theta} + \frac{\partial}{\partial z}$
ρ	density of the fluid gm/c.c.
$\omega(\tilde{II})$	scalar function defined by eq. [12]
Ω	rotational speed of the disc rad/sec
θ	circumferential coordinate

Following relationships between *Rivlin-Ericksen* tensors and rate of deformation tensor are used:

$$A_1 = 2\boldsymbol{D}$$

$$A_2 = 2\dot{\boldsymbol{D}} + 4\boldsymbol{D}^2$$

where

$$\dot{\boldsymbol{D}} = \frac{\partial D}{\partial t} + (\boldsymbol{v}\cdot\nabla)\boldsymbol{D} + \boldsymbol{W}\cdot\boldsymbol{D} - \boldsymbol{D}\cdot\boldsymbol{W}.$$

An investigation of the flow behaviour of viscoelastic fluids in complex flow situations is of obvious pragmatic importance. An improvement in our understanding of the curious and at times, bizarre, flow behaviour of such fluids can be brought about only by a rational analysis of the equations governing the motion of fluids, supported of course, by relevant experimental data. The complexity of the governing equations of motion increases further due to the complicated constitutive relationships which are necessary to portray the flow behaviour in a realistic manner. One hence prefers to analyse the flow phenomena in asymptotic limits so that suitable simplifying assumptions could be made to make the task at hand easier. Constitutive relationships may then be suitably chosen so that they are sufficiently good for the anticipated kinematic conditions. Under laminar flow conditions viscoelastic flow phenomena in the asymptotic limits of low *Reynolds* number (creeping flow) and large *Reynolds* number (boundary layer flow) have attracted a good deal of attention but the existing literature on these subjects appears to be far from conclusive. In this paper we have tried to look at the problem of viscoelastic rotational boundary layer flow. The possibility of obtaining true similarity solutions is examined first and appropriate equations are derived to make the added complications over the most widely studied two dimensional viscoelastic boundary layers quite clear. Experimental data are then presented which support the conclusions drawn on the basis of inspectional analysis.

Previous work

Apart from the academic challenge that the boundary layer flows offer to a rheologist or an engineer, what appears to be even more fascinating is the observation in the past (1, 2) that no general conclusions about the behaviour of viscoelastic fluids in laminar boundary layer flows can be drawn. *Denn* (1) for instance, has clearly shown how a given viscoelastic fluid with fixed material properties can show surprisingly different behaviour in different geometries. He has also shown, how, for a given geometry the changes in the fluid property parameters (namely, the power law indices for the shear stress and primary normal stress difference functions) can bring about a rather surprising increase or decrease of the drag coefficient. These observations are obviously of great significance, because they lead us to the conclusion that by appropriately modifying the fluid properties, we will be in a position to obtain certain benefits such as friction reduction even under laminar boundary layer flow conditions. This interesting possibility certainly merits a further investigation.

Laminar boundary layer flow past a flat plate with zero angle of incidence has been the most widely studied boundary layer flow problem both for *Newton*ian and non-*Newton*ian fluids. The study of this flow for purely viscous non-*Newton*ian fluids poses no special problems and an exact similarity solution and a momentum integral solution can be easily obtained (see 3, 4, 5, 6). For an elastic fluid, however, no similarity transformation is available.

Fredrickson (7) has discussed the conceptual difficulties involved in obtaining a similarity transformation for the boundary layer flow of a viscoelastic fluid. A large number of studies in the literature (e.g. 1, 2, 8, 9) have concerned themselves with the solution of the boundary layer problem past a flat plate for a second order fluid. Although this model does describe most real fluids in some finite range and is the exact lower asymptotic limit of the simple fluid theory, its validity at high enough flow rates to make the boundary layer flow approximations meaningful is in some doubt. The next widely studied boundary layer flows for viscoelastic fluids are stagnation flow (10, 11) and wedge flows (1, 9, 12), respectively. *Denn* (1) has obtained the conditions under which a true similarity solution can be obtained for both these flows. Once again, a large number of publications have dealt with the solution for a second order fluid and the comments in the foregoing apply equally well here. There does appear to exist a controversy over the influence of elasticity on the drag coefficient under such conditions, but this appears to be only apparent rather than real (13).

The next class of boundary layer flow of great pragmatic importance is the rotational flow. The possibility of an exact boundary layer solution for the case of a rotating disc has attracted the attention of a large number of research workers. The flow around a rotating disc for *Newton*ian fluid was first solved by *von Karman* (14) using momentum integral equations. *Cochran* (15) later on improved upon the accuracy of the solution by using numerical techniques. *Mitschka* and *Ulbrecht* (16) analysed the flow of a *Ostwaald de-Waele* power law fluid around a rotating disc and obtained suitable similarity transformations. They solved the resultant set of ordinary differential equations numerically and obtained results to give the velocity distribution and torque.

There have been a few efforts to solve the problem of boundary layer flow of elastic liquids around a rotating disc. *Jain* (17), *Srivastava* (18), *Balaram* et al. (19) and *Kato* et al. (20) have considered the flow for a *Reiner-Rivlin* fluid and investigated the influence of cross viscosity on flow patterns and shear stress at the surface of the disc. *Rathna* (21) and *Elliott* (22) have considered the flow for a second order fluid. *Subba Raju* (23) has investigated the flow for a three constant *Oldroyd* model. An analysis by *Tomita* and *Mochimaru* (24), which has appeared recently, attempts to solve the problem with a restricted form of the *Denn* model, but ultimately succeeds in obtaining only a perturbation solution of the particular limiting case of a second order fluid, with results which are in contradiction with the analogous analysis by *Elliott*. The other theoretical investigations, although useful from a qualitative viewpoint, have little predictive utility. The *Reiner-Rivlin* fluid is thermodynamically inconsistent and seldom will the rheological data fit the predictions of this model. The limitations of the second order fluid have been already mentioned and much the same objections could be laid against *Oldroyd* three constant model. Besides, these investigations do not appear to throw any light on the possibility of obtaining true similarity solutions when a reasonably reliable constitutive equation is used. Generally speaking, there also appears to be a dearth of experimental data on well defined laminar boundary layer flows of viscoelastic fluids. This work is hence intended to contribute in this area.

Theory

The choice of the constitutive equation used to describe the flow behaviour is to be governed by its appropriateness for the rapid external flow which a laminar boundary layer flow around a rotating disc is. We choose the following constitutive equation

$$\tau = -p\delta + \mu(\tilde{II})A_1 + \omega(\tilde{II})A_1^2 - \lambda(\tilde{II})A_2\,, \qquad [1]$$

where τ is the total stress tensor, δ is the *Kronecker* delta, A_1 and A_2 are *Rivlin-Ericksen* tensors of the first and second order. (See Nomenclature for the exact definitions.) The coefficients μ, ω and λ are scalar functions of the second invariant of the rate of strain tensor. This constitutive equation may be looked upon as an approximation to a higher order *Rivlin-Ericksen* expansion in which the effect of *Rivlin-Ericksen* acceleration tensors of order greater than two is lumped into a set of experimentally determinable variable coefficients. Although the strict validity of this procedure may be open to question, in view of the fact that the constant material coefficients of higher order acceleration tensors in *Rivlin-Ericksen* expansion cannot be measured, this procedure does appear to simplify the matter considerably. The functional forms of $\mu(\tilde{II})$, $\omega(\tilde{II})$ and $\lambda(\tilde{II})$ have been discussed by *Soylu* et al. (25) on the basis of viscometric experiments. In general terms it is found to be preferable to express the functional forms of $\mu(\tilde{II})$, $\omega(\tilde{II})$ and $\lambda(\tilde{II})$ as

$$\mu(\tilde{II}) = K[\tfrac{1}{2}(II)]^{\frac{n-1}{2}} \qquad [2]$$

$$\omega(\tilde{II}) = P[\tfrac{1}{2}(II)]^{\frac{q-2}{2}} \qquad [3]$$

$$\lambda(\tilde{II}) = A[\tfrac{1}{2}(II)]^{\frac{b-2}{2}}. \qquad [4]$$

For $n = 1$ and $A = P = 0$, eq. [1] reduces to its *Newton*ian limit. For $n = 1$ and $q = b = 2$, eq. [1] reduces to the second order approximation. For finite values of n but with $A = P = 0$, we have the purely viscous behaviour portrayed by a power-law model. Thus the limiting cases of this constitutive equation may be easily looked into in our analysis of the disc flow problem and a comparison with the analyses in the literature becomes fairly straight forward.

The system of equations of motion can be written down in the usual manner for the boundary layer flow around the disc as

$$v_r \frac{\partial v_r}{\partial r} - \frac{v_\theta^2}{r} + v_z \frac{\partial v_r}{\partial z} = \frac{1}{\rho}\left[-\frac{\partial p}{\partial r} + \frac{\partial T_{rr}}{\partial r} + \frac{T_{rr} - T_{\theta\theta}}{r} + \frac{\partial T_{rz}}{\partial z}\right] \qquad [5]$$

$$v_r \frac{\partial v_\theta}{\partial r} + \frac{v_r v_\theta}{r} + v_z \frac{\partial v_\theta}{\partial z} = \frac{1}{\rho}\left[\frac{1}{r^2}\frac{\partial}{\partial r}(r^2 T_{r\theta}) + \frac{\partial T_{\theta z}}{\partial z}\right] \qquad [6]$$

$$v_r \frac{\partial v_z}{\partial r} + v_z \frac{\partial v_z}{\partial z} = \frac{1}{\rho}\left[-\frac{\partial p}{\partial z} + \frac{1}{r}\frac{\partial}{\partial r}(r T_{rz}) + \frac{\partial}{\partial z}(T_{zz})\right]. \qquad [7]$$

The equation of continuity is

$$\frac{\partial v_r}{\partial r} + \frac{v_r}{r} + \frac{\partial v_z}{\partial z} = 0. \qquad [8]$$

We have chosen here $z = 0$ as the plane of rotation and $r = 0$ as the axis of rotation. The boundary conditions for the solution of this equation are

$$z = 0, \; v_r = v_z = 0 \quad \text{and} \quad v_\theta = r\Omega \qquad [9]$$

and

$$z \to \infty, \; v_r = v_\theta = 0. \qquad [10]$$

Substitution of various stress components in eqs. [5], [6] and [7], based on the constitutive relationship given in eq. [1], in conjunction with the definitions of variable coefficients given by eqs. [2], [3] and [4] essentially completes the statement of the problem. Further simplifications will be achieved when we perform the usual ordering arguments concerning the relative magnitude of terms within the small region in which most significant changes in velocity are occurring. Following the previous workers, in the first instance, these ordering arguments are the same as those done for *Newton*ian fluids, which are also applicable for purely viscous fluids. The variable material coefficients $\mu(\tilde{II}), \omega(\tilde{II})$ and $\lambda(\tilde{II})$ can then be shown to reduce to

$$\mu(\tilde{II}) = K\left[\left(\frac{\partial v_\theta}{\partial r}\right)^2 + \left(\frac{\partial v_r}{\partial z}\right)^2\right]^{\frac{n-1}{2}} \qquad [11]$$

$$\omega(\tilde{II}) = P\left[\left(\frac{\partial v_\theta}{\partial z}\right)^2 + \left(\frac{\partial v_r}{\partial z}\right)^2\right]^{\frac{q-2}{2}} \qquad [12]$$

and

$$\lambda(\tilde{II}) = A\left[\left(\frac{\partial v_\theta}{\partial z}\right)^2 + \left(\frac{\partial v_r}{\partial z}\right)^2\right]^{\frac{b-2}{2}}. \qquad [13]$$

The search for a true similarity solution of eqs. [5], [6] and [7] can be done by the use of Group Theory methods (26) but we find it easier to approach the same problem in an alternative way. Eqs. [5], [6] and [7] will be inspected after transformation of the dependent and the independent variable on the same basis as that for the purely viscous case treated by *Mitschka* and *Ulbrecht* (16). We thus transform the equations by using

$$r/R = \eta \quad \text{and} \quad \frac{z}{\delta'} = \zeta,$$

where $\delta' = \left[\frac{K/\rho}{R^{1-n}\Omega^{2-n}}\right]^{\frac{1}{1+n}} \eta^{\frac{n-1}{n+1}}$, [14]

δ' is proportional to the local boundary layer thickness and the expression for δ' can be obtained by using the balance of forces of inertial and viscous nature in the boundary layer. Of course, only purely viscous considerations are involved while evaluating this and it may not be fundamentally correct since the elasticity is likely to affect the boundary layer thickness. But there is no *a priori* way of accounting for this and we will hence proceed with the transformations suggested in eq. [14]. The velocity components are then transformed as

$$v_r = r\Omega F' \quad [15]$$

$$V_r = r\Omega G' \quad [16]$$

and

$$v_z = \delta'\Omega H' \quad [17]$$

where

$$H' = \left[-\left(\frac{3n-1}{n+1}\right)F - \frac{1-n}{1+n}F'\zeta\right]. \quad [18]$$

The transformed boundary conditions are now given by

$$\zeta = 0,\ F' = H' = 0 \quad \text{and} \quad G' = 1 \quad [19]$$

$$\zeta \to \infty,\ F' = G' = 0. \quad [20]$$

The exact algebraical details are rather tedious and have been reported in (27). The first two components of equation of motion (eqs. [5] and [6]) are required for the evaluation of velocity distribution. The transformed r component is given by

$$\begin{aligned}
F'^2 - G'^2 + \left(H' + \frac{1-n}{1+n}\zeta F'\right)F'' &= \frac{d}{d\zeta}\left[(F''^2 + G''^2)^{\frac{n-1}{2}} F''\right] \\
&+ \frac{Wi_2}{(\mathrm{Re}_{0w})^{1/1+n}}\eta^{\frac{2(q-n-1)}{1+n}} \frac{d}{d\zeta}\left[(F''^2 + G''^2)^{\frac{q-2}{2}}\left(-2F'F'' + \zeta G''^2\frac{1-n}{1+n}\right)\right] \\
&- \frac{Wi_1}{(\mathrm{Re}_{0w})^{1/1+n}}\eta^{\frac{2(b-n-1)}{1+n}} \frac{d}{d\zeta}\left\{(F''^2 + G''^2)^{\frac{b-2}{2}}\left[-2F'F'' + \frac{F'}{1+n}\right.\right. \\
&\left.\left.\times (2F'' + \zeta F'''\cdot(1-n)) + H'F'' + G''\left(G' + 2\zeta G''\frac{1-n}{1+n}\right) + F''\left(3F' + 2\zeta F''\frac{1-n}{1+n}\right)\right]\right\} \\
&+ \frac{Wi_2}{(\mathrm{Re}_{0w})^{1/1+n}}\eta^{\frac{2(q-n-1)}{1+n}}\left\{(F''^2 + G''^2)^{\frac{q-2}{2}}\left(\frac{2q+1+n}{1+n}F''^2 - G''^2\right)\right. \\
&\left.+ \frac{1-n}{1+n}\zeta\frac{d}{d\zeta}\left[(F''^2 + G''^2)^{\frac{q-2}{2}}(F'')^2\right]\right\}
\end{aligned} \quad [21]$$

and the θ component is given by

$$\begin{aligned}
2F'G' + \left[H' + \zeta F'\frac{(1-n)}{1+n}\right]G'' &= \frac{d}{d\zeta}\left[(F''^2 + G''^2)^{\frac{n-1}{2}} G''\right] \\
&+ \frac{Wi_2}{(\mathrm{Re}_{0w})^{1/1+n}}\eta^{\frac{2(q-n-1)}{1+n}} \frac{d}{d\zeta}\left\{(F''^2 + G''^2)^{\frac{q-2}{2}}\left[-G''\left(F' + \zeta F''\frac{1-n}{1+n}\right) + F''\left(G''\zeta\frac{1-n}{1+n}\right)\right]\right\} \\
&- \frac{Wi_1}{(\mathrm{Re}_{0w})^{1/1+n}}\eta^{\frac{2(b-n-1)}{1+n}} \frac{d}{d\zeta}\left\{(F''^2 + G''^2)^{\frac{b-2}{2}}\right. \\
&\left.\times\left[G''F''\zeta\left(\frac{1-n}{1+n}\right) + \frac{F'}{1+n}(2G'' + G'''\zeta\cdot(1-n)) + H'G''' + G''(3F' + H'') - G'F''\right]\right\} \\
&+ \frac{Wi_2}{(\mathrm{Re}_{0w})^{1/1+n}}\eta^{\frac{2(q-n-1)}{1+n}}\left\{2(F''^2 + G''^2)^{\frac{q-2}{2}}\frac{q+1+n}{1+n}G''F''\right. \\
&\left.+ \zeta\left(\frac{1-n}{1+n}\right)\frac{d}{d\zeta}\left[(F''^2 + G''^2)^{\frac{q-2}{2}}(G''F'')\right]\right\}.
\end{aligned} \quad [22]$$

where

$$\mathrm{Re}_{0w} = \frac{R^2 \Omega^{2-n} \rho}{K} \quad [23]$$

$$Wi_1 = \frac{A}{K} \left(\frac{R^2 \Omega^3 \rho}{K} \right)^{\frac{b-n}{1+n}} \quad [24]$$

$$Wi_2 = \frac{P}{K} \left(\frac{R^2 \Omega^3 \rho}{K} \right)^{\frac{q-n}{1+n}}. \quad [25]$$

It is readily seen that the dimensionless numbers Wi_1 and Wi_2 appear in the equations. These have the same significance as in the previous boundary layer analyses performed with similar constitutive equations, in that they represent the ratio of elastic to viscous forces. A further explanation of these numbers has been given later. It is important to note that we get two such groups. Wi_1 is to be evaluated on the basis of the parameters involved in the function $\lambda(\tilde{II})$, whereas Wi_2 is to be evaluated on the basis of the parameters involved in the function $\omega(\tilde{II})$. The boundary layer analysis for flat plate mentioned previously does not include the terms involving the function $\omega(\tilde{II})$, since the terms containing $\omega(\tilde{II})$ identically vanish in such two dimensional flows. The limiting form of the eqs. [21] and [22] can be readily examined. It is clearly seen that when $Wi_1 = Wi_2 = 0$, we have the purely viscous behaviour portrayed by an *Ostwaald–de Waele* power-law fluid. These equations are then identical to those obtained by *Mitschka* and *Ulbrecht* (17). For $K = \mu$ and $n = 1$, the equations reduce to those reported by *Schlichting* (28).

The search for a true similarity solution (independent of η) is now made. The transformations employed in eqs. [14]–[17] are clearly unsatisfactory for providing a true similarity solution for all values of material parameters. But it is interesting to observe that for a special case when $b = q = n + 1$, we do have a true similarity solution. This observation is rather akin to the one obtained by *Denn* (1) who observed that only for a two dimensional stagnation flow a similarity solution is possible when $b = n + 1$. For $n = 1$ (and $b = q = 2$), it corresponds to the second order fluid approximation. It does appear that even when $n \neq 1$, it is possible to satisfy the condition $b = n + 1$ for a number of dilute polymer solutions, with moderate shear thinning and moderate departure of normal stress difference functions from $\dot{\gamma}^2$ dependence. The solutions obtained by integrating the set of ordinary differential eqs. [21] and [22] may thus be of some pragmatic significance. If $b \neq n + 1$ and $q \neq n + 1$, then it is possible to integrate the eqs. [21] and [22] as such by direct numerical integration by using the techniques developed in recent years for integrating nonsimilar boundary layer flow eqs. [29].

The expression for torque on the disc can be obtained by carrying out an inspectional analysis. This is obtained by integrating the local shear stress $\tau_{\theta z}$ on the surface of the disc

$$M = -2 \int_0^R \tau_{\theta z} \Big|_{z=0} 2\pi r^2 \, dr . \quad [26]$$

The component $\tau_{\theta z}$ with the assumed constitutive eq. [1] can be obtained in the dimensionless form as

$$\begin{aligned} \tau_{\theta z} = {} & K\left(\frac{R\Omega}{\Delta}\right)^n (F''^2 + G''^2)^{\frac{n-1}{2}} \eta^{\frac{2n}{1+n}} G'' \\ & + P\left(\frac{R\Omega}{\Delta}\right)^{q-1} (F'' + G''^2)^{\frac{q-2}{2}} \eta^{\frac{2(q-1)}{1+n}} \\ & \cdot \Omega \left\{ -G'' \left[F' + \zeta F'' \frac{(1-n)}{1+n} \right] \right. \\ & \left. + F'' G'' \zeta \frac{(1-n)}{1+n} \right\} \\ & - A\left(\frac{R\Omega}{\Delta}\right)^{b-1} (F''^2 + G''^2)^{\frac{b-2}{2}} \eta^{\frac{2(b-1)}{1+n}} \\ & \cdot \Omega \left\{ G'' F'' \zeta \frac{1-n}{1+n} + H' G'' \right. \\ & + \frac{F'}{1+n} [2G'' + (1-n) G'' \zeta] \\ & \left. + G''(3F' + H'') - G' F'' \right\}, \end{aligned} \quad [27]$$

where

$$\Delta = \left[\frac{K/\rho}{R^{1-n} \Omega^{2-n}} \right]^{\frac{1}{1+n}}. \quad [28]$$

Using B. C. [19] we have $F'(0) = H'(0) = 0$ and hence $\tau_{\theta z}$ at $\zeta = 0$ reduces to

$$\begin{aligned} \tau_{\theta z}|_{\zeta=0} = {} & K\left(\frac{R\Omega}{\Delta}\right)^n (F''(0)^2 + G''(0)^2)^{\frac{n-1}{2}} \eta^{\frac{2n}{1+n}} G''(0) \\ & + A\left(\frac{R\Omega}{\Delta}\right)^{b-1} (F''(0)^2 \\ & + G''(0)^2)^{\frac{b-2}{2}} \eta^{\frac{2(b-1)}{1+n}} \Omega F''(0) . \end{aligned} \quad [29]$$

It is thus interesting to see that the dimensionless groups arising from the coefficient $\omega(\tilde{\Pi})$ (see eq. [12]), do not appear in this expression. Carrying out the integration in eq. [26], we get

$$M = -4\pi R^3 K\left(\frac{R\Omega}{\Delta}\right)^n \int_0^1 \eta^2 \Big\{(F''(0)^2 + G''(0)^2)^{\frac{n-1}{2}} \eta^{\frac{2n}{1+n}} G''(0) + \frac{A}{K}\left(\frac{R\Omega}{\Delta}\right)^{b-n-1} \Omega \eta^{\frac{2(b-1)}{n+1}} (F''(0)^2 + G''(0)^2)^{\frac{b-2}{2}} F''(0)\Big\} d\eta\,. \quad [30]$$

Before expressing the above result in the dimensionless form, it is important to examine the significance of some of the terms appearing in the above expression. It can readily be seen that Δ is the value of δ', the boundary layer thickness at the edge of the disc ($r = R$). Since $R\Omega$ could be taken as a characteristic velocity, $\frac{R\Omega}{\Delta}$ serves as a characteristic shear rate on the surface of the disc. Further, under simple shear flow conditions, we have the shear stress and primary normal stress difference functions predicted as

$$\tau_{12} = K(\dot{\gamma})^n \quad [31]$$

and

$$\tau_{11} - \tau_{22} = A(\dot{\gamma})^b\,. \quad [32]$$

A variable relaxation time can now be defined as (see 30, 31),

$$\lambda = \frac{\tau_{11} - \tau_{22}}{\tau_{12}\dot{\gamma}} = \frac{A}{K}\dot{\gamma}^{b-n-1}\,. \quad [33]$$

On substitution of our characteristic shear rate $\dot{\gamma} = \frac{R\Omega}{\Delta}$, we have the variable relaxation time definition changed to

$$\lambda = \frac{A}{K}\left(\frac{R\Omega}{\Delta}\right)^{b-n-1}\,. \quad [34]$$

This may be taken as the appropriate fluid characteristic time. The search for a characteristic process time may be done in a similar way and $\frac{\Delta}{R\Omega}$ turns out to be the characteristic process time for the boundary layer flow around the disc. We may thus define the ratio of a characteristic fluid and process time as the *Weissenberg* number. Thus we have

$$Wi = \frac{A}{K}\left(\frac{R\Omega}{\Delta}\right)^{b-n}\,. \quad [35]$$

Noting further that $\frac{\Delta}{R} = (\mathrm{Re}_{0w})^{-\frac{1}{1+n}}$ and also defining the dimensionless moment coefficient C_M as $\frac{M}{1/2\rho R^5\Omega^2}$ we get from eq. [30]

$$C_M = \frac{8\pi}{(\mathrm{Re}_{0w})^{1/1+n}}\left[C_1 + C_2\frac{Wi}{(\mathrm{Re}_{0w})^{1/1+n}}\right]. \quad [36]$$

The constants C_1 and C_2 are functions of the indices in the shear stress and normal stress difference functions n, b and q and are given explicitly by

$$C_1 = -\int_0^1 \{[F''(0)^2 + G''(0)^2]^{\frac{n-1}{2}} \eta^{\frac{2(2n+1)}{1+n}} G''(0)\} d\eta = C_1(n, b, q) \quad [37]$$

and

$$C_2 = \int_0^1 \{[F''(0)^2 + G''(0)^2]^{\frac{b-2}{2}} \eta^{\frac{2(b+n)}{1+n}} F''(0)\, d\eta = C_2(n, b, q)\,. \quad [38]$$

The sign and magnitude of C_1 and C_2 will have to be determined either from the theoretical considerations (numerical solution of eqs. [21] and [22]) or from experimentation (by doing torque measurements for elastic liquids and correlating the data). It can, however, be deduced directly that the constant C_1 has to be negative. It has been shown by *Mitschka* and *Ulbrecht* (16) that the moment coefficient for purely viscous *Ostwald-deWaele* power law model is given by

$$C_{M_{\mathrm{inel}}} = \frac{8\pi\beta(n)}{(\mathrm{Re}_{0w})^{1/1+n}} \quad [39]$$

where

$$\beta(n) = 0.1539 \times (6.13)^{\frac{1-n}{2(1+n)}}\,. \quad [40]$$

Hence eq. [36] may be rewritten as

$$C_M = \frac{8\pi\beta(n)}{(\mathrm{Re}_{0w})^{1/1+n}}\left[\frac{C_1}{\beta(n)} + \frac{C_2}{\beta(n)}\frac{Wi}{(\mathrm{Re}_{0w})^{1/1+n}}\right] \quad [41]$$

or as

$$C_{M_{\mathrm{viscoel}}} = C_{M_{\mathrm{inel}}}\left[\frac{C_1}{\beta(n)} + \frac{C_2}{\beta(n)}\frac{Wi}{(\mathrm{Re}_{0w})^{1/1+n}}\right]. \quad [42]$$

Inspection of eq. [42] shows clearly that the possibility of torque suppression for viscoelastic fluids will depend on the change in the magnitude of C_1 brought about through elasticity and also the change in sign and magnitude of C_2 brought about through elasticity. It is not possible to predict on the basis of such inspectional analysis alone whether one would have a torque suppression or a torque increase. It is interesting to recall here the observations of *Denn* (1), who, for the case of laminar boundary layer flow past a flat plate showed how the fluid property parameters b and n influence the drag coefficient and how depending upon their magnitudes the drag coefficient may in fact reduce or increase compared to its value given by purely viscous considerations. The possibility of observing similar behaviour in the case of the disc cannot be outruled.

Experimental

The experimental procedure consisted of measuring the torques experienced by discs of radii 3.75, 5 and 7.5 cm rotating at different speeds in different fluids. The range of speeds covered was from 600 to 1800 rev/min.

The experimental technique used for the measurement of torque was the same as used previously (32). The dynamometer essentially measured the twist in the torsion bar in conjunction with two photocells and an electronic digital counter timer. The details of the dynamometer could be found in ref. (24).

The liquids used were aqueous solutions of Sodium Carboxymethyl cellulose (CMC (Edifas "*B*" ICI)), aqueous solutions of Polyacrylamide (PAA) (Separan AP30, Dow Chemicals), Kaolin suspension made in mixture of Glycerol and Water and also a mixture of Polyacrylamide, Glycerol and Water. Glycerol was used as a *Newton*ian fluid. Table 1 lists the properties of the fluids used. The rheological data were obtained on a *Weissenberg* Rheogoniometer (model R18). Both shear stress – shear rate data were obtained and were correlated on the basis of the predictions in eqs. [31] and [32], respectively. The material parameters K and n and A and b have been listed in table 1. It can be readily seen that due to the absence of any measurable normal stress difference CMC and Kaolin solutions served essentially as inelastic liquids, whereas the Polyacrylamide solutions were significantly elastic.

Results and discussion

The predictions of eq. [36] were tested by plotting the torque R.P.M. data in a suitable dimensionless form by using the fluid property parameters listed in table 1. The accuracy of the data obtained was first tested by plotting the data for *Newton*ian Glycerol solutions. The data for non-*Newton*ian inelastic fluids ($A = 0$) was also plotted to check the theoretical relationship given by eq. [39], and fig. 1 shows the result. It is clearly seen that the agreement between the theory and the experimental data is excellent. In order to see if there is any difference between the behaviour of inelastic and viscoelastic fluids, the data for the latter were also plotted on the basis of eq. [39]. Fig. 2 shows the results. It is seen that the data for viscoelastic polymer solutions are significantly lower than those for inelastic polymer solutions and suspensions. Since all the influence of the shear thinning character has been explicitly taken into account by a relationship of the type [39] the observed difference must be attributed to the presence of elasticity.

The validity of the relationship obtained on the basis of inspectional analysis (eq. [42]) was hence tested. The observed reduction could be accounted for by the changes in the constant C_1 (for inelastic fluids $C_1 = \beta(n)$ and $C_2 = 0$) or both by the contribution through changes in C_1 and a finite value of C_2. C_1 and C_2 will of

Table 1. Properties of the fluids used

Fluid	K $\frac{\text{dyne}}{\text{cm}^2}\,\text{sec}^n$	n	A $\frac{\text{dyne}}{\text{cm}^2}\,\text{sec}^b$	b	ρ gm/c.c.
Glycerine	8.427	1.0	–	–	1.25
Glycerine	1.048	1.0	–	–	1.24
Kaolin	3.7	0.935	–	–	1.20
CMC	70	0.4675	–	–	1.05
0.5% PAA	14	0.4225	26.5	0.713	1.00
1.0% PAA	36	0.373	75.0	0.675	1.005
1.5% PAA	107	0.331	170.0	0.710	1.010
2.0%	250	0.2643	350	0.671	1.015
0.53% PAA in 45% Glycerine	26	0.517	70.7	0.875	1.15

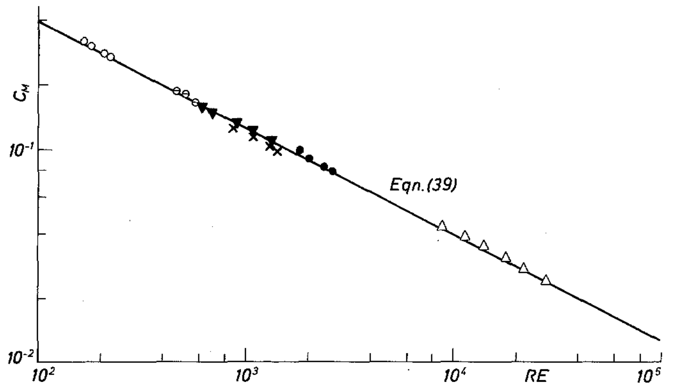

Fig. 1. Verification of eq. [39] for *Newton*ian and inelastic non-*Newton*ian fluids

○	$R = 3.75$ cm	Glycerol 95%
⊖	$R = 5.00$ cm	Glycerol 95%
●	$R = 5.00$ cm	Glycerol 85%
▼	$R = 3.75$ cm	Kaolin
×	$R = 5.00$ cm	Kaolin
△	$R = 7.50$ cm	CMC

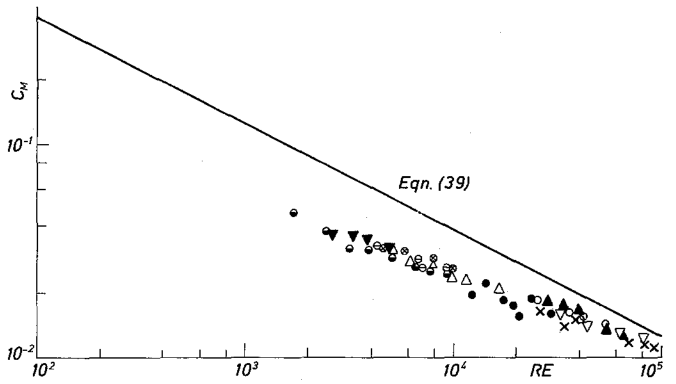

Fig. 2. Torque suppression for viscoelastic fluids under rotational laminar boundary layer flow conditions

○	$R = 3.75$ cm	0.5% PAA
×	$R = 5.00$ cm	0.5% PAA
▽	$R = 7.50$ cm	0.5% PAA
●	$R = 3.75$ cm	1.0% PAA
▲	$R = 5.00$ cm	1.0% PAA
⊖	$R = 3.75$ cm	1.5% PAA
△	$R = 5.00$ cm	1.5% PAA
▼	$R = 3.75$ cm	2.00% PAA
◒	$R = 5.00$ cm	2.00% PAA
⊗	$R = 5.00$ cm	0.53% PAA in 54% Glycerol

course be functions of n, b and q. To examine whether the entire torque suppression could be correlated through C_1 alone or whether the presence and contribution of C_2 had to be taken into account as well, we plotted the data for a given fluid but with different disc dimensions. There was a systematic shift in the data with the disc dimensions. Since only the term $\frac{C_2\,Wi}{\beta(n)(\mathrm{Re}_{0w})^{1/1+n}}$ was capable of correlating such a shift, the presence of this term was found necessary. It was further found that the observed extent of torque suppression reduced with higher *Reynolds* number. This implies that C_2 is negative. The exact functional relationship of $C_1(n, b, q)$ and $C_2(n, b, q)$, however could not be deduced, because the variation in n and b was only between 0.26 to 0.51 and 0.67 to 0.87; respectively. Further q was not measured. Hence the values of C_1 and C_2 were taken to be approximately constant in this work and are reported in table 2. The goodness of prediction of eq. [42] with these values of C_1 and C_2 is shown in fig. 3, where the experimental data are compared with the predictions. The agreement appears to be reasonably sound.

Table 2. Values of constants C_1 and C_2

Fluid	n	b	C_1	C_2
0.5% PAA	0.4225	0.713	0.2165	0.0814
1.0% PAA	0.3730	0.675	0.2222	0.0830
1.5% PAA	0.3312	0.710	0.2362	0.0895
2.0% PAA	0.2643	0.670	0.2640	0.0982
0.53% PAA in 54% Glycerine	0.5170	0.875	0.2100	0.0746

Appropriate comments need to be made about the region preceding the laminar boundary layer regime (creeping flow) and the region after the laminar boundary layer flow (transition to turbulent regime).

Kelkar et al. (31) have clearly shown that in the creeping flow regime, the modifications in the torque are not detectable and the data could be correlated satisfactorily for inelastic and viscoelastic fluids through the considerations of shear thinning viscosity alone. Inspectional analysis similar to the one performed here was

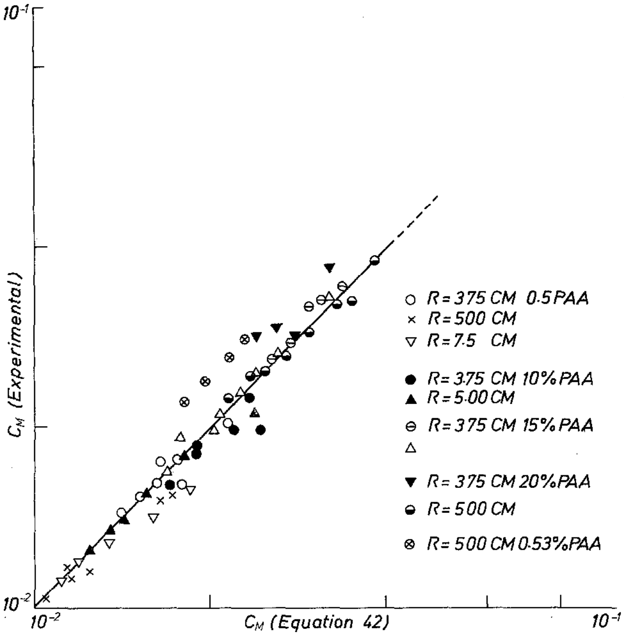

Fig. 3. Comparison of the experimental and predicted values of the moment coefficients for viscoelastic liquids
Notation same as in fig. 2

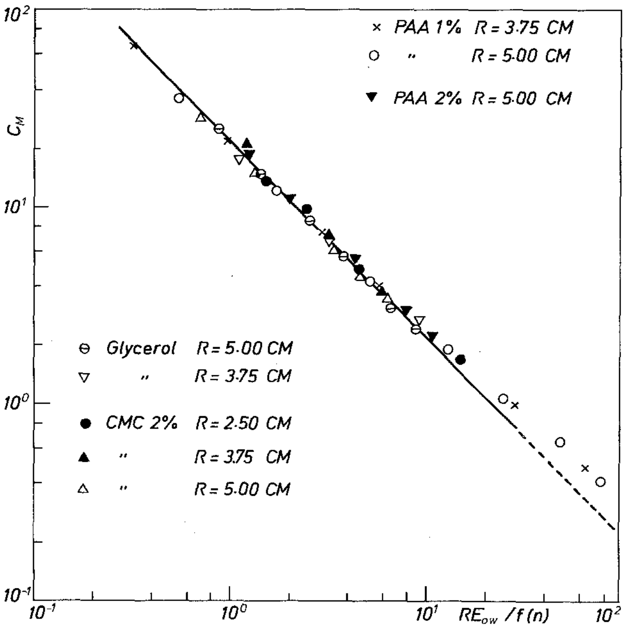

Fig. 4. Moment coefficients of inelastic and viscoelastic fluids under creeping flow conditions
▽ $R = 3.75$ cm Glycerine 95%
⊖ $R = 5.00$ cm Glycerine 95%
● $R = 2.50$ cm CMC 2%
▲ $R = 3.75$ cm CMC 2%
△ $R = 5.00$ cm CMC 2%
× $R = 3.75$ cm PAA 1%
○ $R = 5.00$ cm PAA 1%
▼ $R = 5.00$ cm PAA 2%

done in the creeping flow regime (which is confirmed from the analysis of *Wichterle* and *Ulbrecht* (33) as well) and the following relationship between C_M and Re_{0w} was predicted.

$$C_M = \frac{f(n)}{\mathrm{Re}_{0w}}. \qquad [44]$$

In the absence of any theoretical investigation of flow of power law fluids around a rotating disc in the creeping flow regime, we obtained this relationship by performing an analysis of the experimental data. The following relationship was found to fit the data

$$f(n) = \frac{32}{3}\left(\frac{7n+1}{3n+1}\right). \qquad [45]$$

Fig. 4 shows that data for both inelastic and viscoelastic solutions are equally well fitted through this equation. This shows that the influence of elasticity on the modification of torque is negligible at very low *Reynolds* number. It is only at fairly large *Reynolds* numbers that one has an appreciable influence of elasticity.

The resulting torque suppression does not, however, appear to persist for large *Reynolds* number and fig. 2 clearly shows that the viscoelastic polymer solution data once again tends to join the inelastic line given by eq. [39]. *Kale* et al. (34) have shown that as one goes further in the turbulent regime, there is a significant torque suppression. Thus the rotational viscoelastic flows appear to offer a very interesting range of phenomena depending upon the range of *Reynolds* number. It is interesting to compare here another interesting flow situation, which is well understood and studied in the literature. It has been thus shown that for low *Reynolds* number flow (creeping flow) of a viscoelastic fluid past a sphere or a cylinder there is some reduction in the drag coefficient. On the other hand in stagnation flows there is an enhancement in drag coefficient for viscoelastic polymer solutions. We thus come to a very interesting and important conclusion that the existence and extent of drag reduction under laminar flow conditions may depend upon the particular geometry used the range of *Reynolds* number as well as the values of the material parameters. Our study has clearly shown the range of conditions under which torque suppression under laminar flow conditions can be obtained for a rotating disc, but this study cannot be claimed to be complete in a sense that very little variation in material parameters was used. However, it is hoped, that the exploratory theoretical and experimental analysis of this work may lead to a better understanding of the behaviour of viscoelastic fluids in rotational laminar boundary layer flows. The attempted analysis of the governing equations and the evidence presented for the existence of a true similarity solution should help considerably in this respect.

Summary

The equations of motion for the laminar boundary layer flow over a rotating disc have been derived for a fluid which obeys a *Rivlin-Ericksen* type of constitutive equation and whose material parameters are assumed to be arbitrary functions of the second invariant of the rate of deformation tensor. The analysis establishes the conditions under which a true similarity solution is possible. An inspectional analysis yields a relationship between the moment coefficient, a generalized *Reynolds* number and a modified *Weissenberg* number which incorporates a variable relaxation time with a process time characteristic of the boundary layer flow on the disc. Experimental data obtained are analysed in terms of the derived relationship and the agreement between the two, after the determination of the unknown constants, is found to be quite sound. A brief discussion follows which emphasizes the role of geometry, regime of flow and viscoelastic material parameters in giving a wide variety of flow phenomena.

Zusammenfassung

Die Bewegungsgleichungen für die laminare Grenzschichtströmung um eine rotierende Scheibe wurden für eine *Rivlin-Ericksen*-Flüssigkeit abgeleitet. Die Materialparameter in dieser Zustandsgleichung wurden als beliebige Funktionen der zweiten Invarianten des Deformationsgeschwindigkeitstensors gesehen. Die Bedingungen wurden gegeben, unter denen eine echte Ähnlichkeits-Lösung existiert. Die Inspektionsanalyse wurde dann benutzt, eine Gleichung zwischen dem Widerstandskoeffizienten und der *Reynolds*-Zahl abzuleiten, die auch eine *Weissenberg*-Zahl mit einer variablen Relaxationszeit und einer charakteristischen aus der Grenzschichtströmung abgeleiteten Prozeßzeit enthält. Die Versuchsdaten wurden mit Hilfe der Theorie analysiert, und eine gute Übereinstimmung wurde gefunden. Die Arbeit wird mit einer kurzen Diskussion beendet, in der Rolle der Geometrie des laminaren bzw. turbulenten Strömungsbereiches und der viskoelastischen Stoffparameter herausgestellt wird.

References

1) *Denn, M. M.*, Chem. Eng. Sci. **22**, 395 (1967).
2) *White, J. L.*, Amer. Inst. Chem. Eng. J. **12**, 1019 (1966).
3) *Acrivos, A., M. J. Shah*, and *E. E. Petersen*, Amer. Inst. Chem. Eng. J. **6**, 312 (1960).
4) *Schowalter, W. R.*, Amer. Inst. Chem. Eng. J. **6**, 24 (1960).
5) *Defrawi, M. E.* and *B. A. Finlayson*, Amer. Inst. Chem. Eng. J. **18**, 251 (1972).
6) *Fox, V. G., L. E. Erickson*, and *L. T. Fan*, Amer. Inst. Chem. Eng. J. **15**, 327 (1969).
7) *Frederickson, A. G.*, Principles and Applications of Rheology (Englewood Cliffs, N.J. 1964).
8) *Davis, R. T.*, Proc. 10th Midwestern Mechanics Conf., p. 1145 (1967).
9) *Peddieson jr., J.*, Proc. 12th Midwestern Mechanics Conf., p. 153 (1971).
10) *Beard, D. W.* and *K. Walters*, Proc. Camb. Phil. Soc. **60**, 667 (1964).
11) *Rajeshwari, G. K.* and *S. L. Rathna*, Z. angew. Math. Phys. **13**, 43 (1962).
12) *Peddieson jr., J.*, Amer. Inst. Chem. Eng. J. **19**, 377 (1973).
13) *Davies, M. H.*, Z. angew. Math. Phys. **17**, 189 (1966).
14) *Von Karman*, Z. angew. Math. Mech. **1**, 244 (1921).
15) *Cochran, W. G.*, Proc. Camb. Phil. Soc. **30**, 365 (1934).
16) *Mitschka, P.* and *J. Ulbrecht*, Appl. Sci. Res. (A) **15**, 345 (1965).
17) *Jain, M. K.*, Appl. Sci. Res. (A) **10**, 410 (1962).
18) *Srivastava, A. C.*, Bull. Cal. Math. Soc. **50**, 57 (1958).
19) *Balaram, M.* and *K. S. Sastri*, Arch. Mech. Stosowaneji **3**, 359 (1965).
20) *Kato, H., K. Watanabe*, and *K. Ueda*, Bull. J.S.M.E. **15**, 1185 (1972).
21) *Rathna, S. L.*, Z. angew. Math. Mech. **42**, 231 (1962).
22) *Elliot, L.*, Phys. Fluids **14**, 1086 (1971).
23) *Subba Raju, P. V.*, Appl. Sci. Techn. Mech. Appl. Tome **13**, 831 (1968).
24) *Tomita, Y.* and *Y. Mochimaru*, Bull. J.S.M.E. **16**, 291 (1973).
25) *Soylu, M., R. A. Mashelkar*, and *J. Ulbrecht*, Rheol. Acta **13**, 216 (1974).
26) *Ames, W. F.*, Non-linear Partial Differential Equations in Engineering (New York 1965).
27) *Kale, D. D.*, PhD Thesis, Univ. of Salford (1973).
28) *Schlichting, H.*, Boundary Layer Theory (New York 1955).
29) *Parr, W.*, U.S. Naval Ordnance Report NOLTR 63–261 (1964).
30) *Truesdell, C.*, Phys. Fluids. **7**, 1134 (1964).
31) *Kelkar, J. V., R. A. Mashelkar*, and *J. Ulbrecht*, Trans. Instn. Chem. Engrs. **50**, 343 (1972).
32) *Mashelkar, R. A., D. D. Kale, J. V. Kelkar*, and *J. Ulbrecht*, Chem. Eng. Sci. **27**, 973 (1972).
33) *Ulbrecht, J.* and *K. Wichterle*, Chemie. Ingr. Tech. **39**, 656 (1967).
34) *Kale, D. D., R. A. Mashelkar*, and *J. Ulbrecht*, Trans. Inst. Chem. Engrs, (in press).

Authors' address:

D. D. Kale, R. A. Mashelkar, and *J. Ulbrecht*
Department of Chemical Engineering
University of Salford
Salford M5 4WT, Lancashire (England)