Existence and uniqueness of classifying spaces for fusion systems over discrete p-toral groups

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ABSTRACT

A major questions in the theory of p-local finite groups was whether any saturated fusion system over a finite p-group admits an associated centric linking system, and when it does, whether it is unique. Both questions were answered in the affirmative by A. Chermak, using the theory of partial groups and localities he developed. Using Chermak's ideas combined with the techniques of obstruction theory, Bob Oliver gave a different proof of Chermak's theorem. In this paper we generalise Oliver's proof to the context of fusion systems over discrete p-toral groups, thus positively resolving the analogous questions in p-local compact group theory.

A p-local compact group is an algebraic object designed to encode in an algebraic setup the p-local homotopy theory of classifying spaces of compact Lie groups and p-compact groups, as well as some other families of a similar nature [**BLO3**]. The theory of p-local compact groups includes, and in many aspects generalises, the earlier theory of p-local finite groups [**BLO2**]. A p-local compact group is thus a triple $(S, \mathcal{F}, \mathcal{L})$, where S is a discrete p-toral group (Definition 1.1(c)), $\mathcal{F}$ is a saturated fusion system over S (Definition 1.4), and $\mathcal{L}$ is a centric linking system associated to $\mathcal{F}$ [**BLO3**, Definition 4.1].

In [**Ch**] A. Chermak showed that for any saturated fusion system $\mathcal{F}$ over a finite p-group S, there exists an associated centric linking system, which is unique up to isomorphism. To do so he used the theory of partial groups and localities, which he developed in order to provide an alternative, more group theoretic approach, to p-local group theory. Armed with Chermak's ideas and techniques of obstruction theory, B. Oliver [**O**, Theorem 3.4] proved that the obstructions to the existence and uniqueness of a centric linking system associated to a saturated fusion system all vanish. In particular, this implies Chermak's theorem.

For a fusion system $\mathcal{F}$ over a discrete p-toral group S, let $\mathcal{O}(\mathcal{F}^c)$ denote the associated orbit category of all $\mathcal{F}$-centric subgroup $P \leq S$, and let $\mathcal{Z} \colon \mathcal{O}(\mathcal{F}^c)^{\mathrm{op}} \to \mathbf{Ab}$ denote the functor which associates with a subgroup its centre, [**BLO3**, Section 7]. Throughout this paper we will write $H^*(\mathcal{C}; F)$ for $\varprojlim_{\mathcal{C}}^* F$ where $F \colon \mathcal{C} \to \mathbf{Ab}$ is a functor from a small category $\mathcal{C}$. The main result of this paper is the following generalisation of [**O**, Theorem 3.4] to saturated fusion systems over discrete p-toral groups.

THEOREM A. *Let $\mathcal{F}$ be a saturated fusion system over a discrete p-toral group S. Then $H^i(\mathcal{O}(\mathcal{F}^c), \mathcal{Z}) = 0$ for all $i > 0$ if p is odd, and for all $i > 1$ if $p = 2$.*

The following result (cf. [**Ch**], and [**O**, Theorem A]) now follows from Proposition 1.7 below.

THEOREM B. *Let $\mathcal{F}$ be a saturated fusion system over a discrete p-toral group. Then there exists a centric linking system associated to $\mathcal{F}$ which is unique up to isomorphism.*

The proof of Theorem A follows very closely Oliver's argument in [**O**], adapting his methods to the infinite case. The main new input in this paper is the re-definition of *best offenders* in the context of discrete p-toral groups (Definition 2.2). Chermak, in his original solution of the existence-uniqueness problem, relies on a paper by Meierfrankenfeld and Stellmacher [**MS**], which in turn depends on the classification theorem of finite simple groups. Oliver's interpretation of Chermak's work, and as a consequence our result, remain dependent on the classification theorem.

The paper is organised as follows. In Section 1 we collect the definitions, notation and background material needed throughout the paper. Section 2 introduces the Thompson subgroups and offenders in the context of discrete p-toral groups, and analyses the properties of these objects along the lines of [**O**]. Finally in Section 3 we prove Theorem A, which will be restated there as Theorem 3.6. In Section 4 we give an outline of Oliver's proof and highlight the changes necessary to adapt it to the infinite case we deal with. Readers who are familiar with [**O**] may find it useful to read this section first.

The crucial observations that led to Definition 2.2, without which this paper could not have been written, were made by Andy Chermak, and we are deeply indebted to him for his interest in these results.

1. *Background*

1.1. *virtually discrete p-toral groups*

Let p be a prime which we fix for the remainder of this paper.

DEFINITION 1.1.   Let $\mathbb{Z}/p^\infty$ denote union of all $\mathbb{Z}/p^r$, $r \geq 1$ under the obvious inclusion. Alternatively, $\mathbb{Z}/p^\infty = \mathbb{Z}[\frac{1}{p}]/\mathbb{Z}$.
(a) A *discrete p-torus* is a group T isomorphic to $(\mathbb{Z}/p^\infty)^r$, for some r called the *rank* of T.
(b) A *virtually discrete p-toral group* is a group Γ which contains a normal discrete p-torus T of finite index.
(c) If Γ is a virtually discrete p-toral group and the index of T in Γ is a power of p, then we say that Γ is *discrete p-toral*.

The subgroup T in Definition 1.1 is, in fact, the maximal subgroup of Γ which is isomorphic to a discrete p-torus, namely it contains any discrete p-torus in Γ. It is also the minimal normal subgroup of Γ of finite index. It is therefore fully characteristic in Γ (namely T is invariant under endomorphisms of Γ). We refer to this subgroup as the *maximal torus* of Γ, or the *identity component* of Γ, and denote it by Γ_0.

The *order* of a virtually discrete p-toral group Γ is the pair $(\mathrm{rk}(\Gamma_0), |\Gamma/\Gamma_0|)$ with the left lexicographic order: $(a, b) \preceq (a', b')$ if $a < a'$ or if $a = a'$ and $b \leq b'$. Any subgroup $\Gamma' \leq \Gamma$ is itself a virtually discrete p-toral (see [**BLO3**, Lemma 1.3]). Moreover, $\mathrm{ord}(\Gamma') \leq \mathrm{ord}(\Gamma)$ and equality holds if and only if $\Gamma' = \Gamma$.

The group Γ contains a maximal normal discrete p-toral subgroup denoted $O_p(\Gamma)$. It is the preimage in Γ of the maximal normal p-subgroup $O_p(\Gamma/\Gamma_0) \trianglelefteq \Gamma/\Gamma_0$, and in particular it contains Γ_0. Also, Γ contains a maximal discrete p-toral subgroup S, given as the preimage in Γ of a Sylow p-subgroup of Γ/Γ_0.

DEFINITION 1.2.   A maximal discrete p-toral subgroup S of a virtually discrete p-toral group Γ is said to be a *Sylow p-subgroup*. The collection of all Sylow p-subgroups of Γ is denoted by $\mathrm{Syl}_p(\Gamma)$.

A Sylow p-subgroup $S \leq \Gamma$ has the property that any discrete p-toral subgroup of Γ is conjugate to a subgroup of S. Hence in particular all Sylow p-subgroups of Γ are conjugate.

LEMMA 1.3 (cf. [**O**, Lem. 1.14]).
(a) *For any pair P, Q of discrete p-toral groups, $P < Q$ implies $P < N_Q(P)$.*
(b) *(Frattini's argument) Suppose that $\Gamma' \trianglelefteq \Gamma$ are virtually discrete p-toral groups, and that $S \in \mathrm{Syl}_p(\Gamma')$. Then $\Gamma = \Gamma' \cdot N_\Gamma(S)$.*

Proof.   Part (a) is [**BLO3**, Lemma 1.8]. For Part (b), the usual argument works: Both Γ' and Γ act transitively by conjugation on $\mathrm{Syl}_p(\Gamma')$, hence $\Gamma/N_\Gamma(S) \cong \Gamma'/N_{\Gamma'}(S)$.   □

1.2. Saturated fusion systems

A *fusion system* $\mathcal{F}$ over a discrete p-toral group S is a category whose objects are the subgroups of S and whose morphisms are group monomorphisms such that the following holds
- $\mathrm{Hom}_S(P,Q) \subseteq \mathrm{Hom}_{\mathcal{F}}(P,Q)$ for all $P, Q \leq S$, where $\mathrm{Hom}_S(P,Q)$ denotes the set of homomorphisms induced by conjugation in S.
- For each $f \in \mathrm{Hom}_{\mathcal{F}}(P,Q)$, $f^{-1} \colon f(P) \to P$ is a morphism in $\mathcal{F}$.

Two subgroups $P, Q \leq S$ are said to be *$\mathcal{F}$-conjugate* if they are isomorphic as objects in $\mathcal{F}$, and we let $P^{\mathcal{F}}$ denote the $\mathcal{F}$-conjugacy class of P. The *orbit category* of $\mathcal{F}$, denoted $\mathcal{O}(\mathcal{F})$, has the same underlying object set, and

$$\mathrm{Mor}_{\mathcal{O}(\mathcal{F})}(P,Q) = \mathrm{Rep}_{\mathcal{F}}(P,Q) \overset{\mathrm{def}}{=} \mathrm{Hom}_{\mathcal{F}}(P,Q)/\mathrm{Inn}(Q).$$

If $P = Q$, we write $\mathrm{Out}_{\mathcal{F}}(P) \overset{\mathrm{def}}{=} \mathrm{Aut}_{\mathcal{O}(\mathcal{F})}(P)$.

DEFINITION 1.4 ([**BLO3**, Def. 2.2]).   Let $\mathcal{F}$ be a fusion system over a discrete p-toral group S. A subgroup $P \leq S$ is *fully centralized* in $\mathcal{F}$ if $\mathrm{ord}(C_S(P)) \geq \mathrm{ord}(C_S(Q))$ for any $Q \in P^{\mathcal{F}}$. It is *fully normalized* in $\mathcal{F}$ if $\mathrm{ord}(N_S(P)) \geq \mathrm{ord}(N_S(Q))$ for any $Q \in P^{\mathcal{F}}$. It is *$\mathcal{F}$-centric* if $C_S(Q) = Z(Q)$ for all $Q \in P^{\mathcal{F}}$.

The fusion system $\mathcal{F}$ is *saturated* if the following three conditions hold
 (I) For each $P \leq S$ which is fully normalized in $\mathcal{F}$, P is fully centralized in $\mathcal{F}$, $\mathrm{Out}_{\mathcal{F}}(P)$ is finite and $\mathrm{Out}_S(P) \in \mathrm{Syl}_p(\mathrm{Out}_{\mathcal{F}}(P))$.
(II) If $P \leq S$ and $\varphi \in \mathrm{Hom}_{\mathcal{F}}(P, S)$ are such that $\varphi(P)$ is fully centralized in $\mathcal{F}$ and if we set

$$N_\varphi = \{g \in N_S(P) | \varphi \circ c_g \circ \varphi^{-1} \in \mathrm{Aut}_S(\varphi(P))\}$$

then there is $\tilde{\varphi} \in \mathrm{Hom}_{\mathcal{F}}(N_\varphi, S)$ such that $\tilde{\varphi}|_P = \varphi$.
(III) If $P_1 \leq P_2 \leq \ldots$ is an increasing sequence of subgroups in S and if $\varphi \colon P_\infty \to S$, where $P_\infty = \cup_n P_n$, is a homomorphism such that $\varphi|_{P_n} \in \mathrm{Hom}_{\mathcal{F}}(P_n, S)$ for all n, then $\varphi \in \mathrm{Hom}_{\mathcal{F}}(P_\infty, S)$.

Let $\mathcal{F}^c$ (resp. $\mathcal{O}(\mathcal{F}^c)$) be the full subcategory of $\mathcal{F}$ (resp. $\mathcal{O}(\mathcal{F})$) whose objects are the $\mathcal{F}$-centric subgroups of S. Note that the collection of $\mathcal{F}$-centric subgroups is closed under overgroups in S, namely if $P \leq Q \leq S$ and $P \in \mathcal{F}^c$ then $Q \in \mathcal{F}^c$.

The next result provides a basic family of examples of saturated fusion systems over discrete p-toral groups S. A partial converse to this statement, known as the "Model Theorem", will be

proven below (Proposition 1.21). Let Γ be a virtually discrete p-toral group and $S \in \mathrm{Syl}_p(\Gamma)$. Let $\mathcal{F}_S(\Gamma)$ be the fusion system over S whose objects are the subgroups of S, and whose morphisms are the homomorphisms $P \to Q$ in $\mathrm{Hom}_\Gamma(P, Q)$, i.e., all homomorphisms induced by conjugation by some $g \in \Gamma$.

PROPOSITION 1.5 (cf. [**BLO2**, Prop. 1.3]). *Let Γ be a virtually discrete p-toral group, and $S \in \mathrm{Syl}_p(\Gamma)$. Then $\mathcal{F}_S(\Gamma)$ is a saturated fusion system over S.*

Proof. All elements of Γ have finite order. Also, it is easy to see that any subgroup and any quotient of a virtually discrete p-toral group is itself virtually discrete p-toral. Finally, if $P_1 \leq P_2 \leq P_3 \leq \ldots$ is an increasing sequence of discrete p-toral subgroups of Γ, then $C_\Gamma(P_1) \geq C_\Gamma(P_2) \geq \ldots$ is a decreasing sequence of virtually discrete p-toral groups which must stabilise, because the sequence $\{\mathrm{ord}(C_\Gamma(P_n))\}_n$ must have a minimum in the well-ordered set $\mathbb{N} \times \mathbb{N}$ equipped with the lexicographical order. The result follows from [**BLO3**, Proposition 8.3]. $\square$

Recall that for any $Q \leq S$, the normaliser fusion system $N_\mathcal{F}(Q)$ is the fusion subsystem of $\mathcal{F}$ defined over $N_S(Q)$, whose morphisms are:

$$\mathrm{Hom}_{N_\mathcal{F}(Q)}(P, P') = \{\varphi \colon P \to P' \mid \exists \psi \in \mathrm{Hom}_\mathcal{F}(PQ, P'Q), \ \psi|_P = \varphi, \ \text{and} \ \psi(Q) = Q\}.$$

PROPOSITION 1.6 ([**BLO6**, Thm 2.3]). *Let $\mathcal{F}$ be a saturated fusion system over a discrete p-toral group S and let $Q \leq S$ be fully normalized in $\mathcal{F}$. Then $N_\mathcal{F}(Q)$ is a saturated fusion system over $N_S(Q)$.*

We say that a subgroup $Q \leq S$ is normal in $\mathcal{F}$, and write $Q \trianglelefteq \mathcal{F}$, if $\mathcal{F} = N_\mathcal{F}(Q)$.

1.3. *Higher limits*

Let $\mathcal{Z} \colon \mathcal{O}(\mathcal{F}^c)^{\mathrm{op}} \to \mathbf{Ab}$ denote the functor which assigns to every $P \in \mathcal{F}^c$ its centre $Z(P)$, and to a morphism $P \xrightarrow{\varphi} P'$ in $\mathcal{O}(\mathcal{F}^c)$ the group monomorphism $Z(P') \xrightarrow{\mathrm{incl}} Z(f(P)) \xrightarrow{f^{-1}} Z(P)$ induced by a representative $f \colon P \to P'$ in $\mathcal{F}^c$ for φ. The following result is an analogue of [**BLO2**, Proposition 3.1]. Since in this paper we will not use centric linking systems in a fundamental way, we refer the reader to [**BLO3**, Section 4] for their definition and properties.

PROPOSITION 1.7. *Let $\mathcal{F}$ be a saturated fusion system over a discrete p-toral group S. An associated centric linking system exists if $H^3(\mathcal{O}(\mathcal{F}^c), \mathcal{Z}) = 0$, and if $H^2(\mathcal{O}(\mathcal{F}^c), \mathcal{Z}) = 0$, then it is unique up to an isomorphism (of linking systems associated to $\mathcal{F}$).*

Proof. By [**BLO3**, Proposition 4.6] there is a one-to-one correspondence between the isomorphism classes of linking systems associated to $\mathcal{F}$ and rigidifications of the functor $\mathbf{B} \colon \mathcal{O}(\mathcal{F}^c) \to \mathrm{hoTop}$, which takes a subgroup P to its classifying space (up to natural homotopy equivalence). Since P is a discrete group, it is elementary to check that $\mathrm{Map}(BP, BP)_{\mathrm{Id}} \simeq BZ(P)$, and therefore the functors $\alpha_i \colon \mathcal{O}(\mathcal{F}^c)^{\mathrm{op}} \to \mathbf{Ab}$, defined in [**BLO3**, Corrolary A.4] by $\alpha_i(P) = \pi_i(\mathrm{Map}(BP, BP)_{\mathrm{Id}})$ have the form $\alpha_1 = \mathcal{Z}$, and $\alpha_n = 0$ for all $n \geq 2$. Hence by [**BLO3**, Corrolary A.4] a rigidification of $\mathbf{B}$ exists if $H^3(\mathcal{O}(\mathcal{F}^c); \mathcal{Z}) = 0$ and it is unique up to natural homotopy if $H^2(\mathcal{O}(\mathcal{F}^c); \mathcal{Z})$. $\square$

Notice that Theorem B follows at once from Theorem A and Proposition 1.7.

We next recall a technique introduced by Jackowski-McClure-Oliver in [**JMO**, Section 5] for calculating $H^*(\mathcal{O}(\mathcal{F}^c), F)$, where $F \colon \mathcal{O}(\mathcal{F}^c)^{\mathrm{op}} \to \mathbb{Z}_{(p)}\text{-}\mathrm{mod}$ is a functor. Let G be a finite group, M be a $\mathbb{Z}_{(p)}[G]$-module, and fix some $S \in \mathrm{Syl}_p(G)$. Let $\mathcal{O}_p(G)$ be the category of transitive G-sets whose isotropy groups are p-groups. It contains a skeletal subcategory $\mathcal{O}_S(G) \subseteq \mathcal{O}_p(G)$ whose objects are the orbits of the form G/P where $P \leq S$. Define a functor $A_M \colon \mathcal{O}_p(G)^{\mathrm{op}} \to \mathbb{Z}_{(p)}\text{-}\mathrm{mod}$, or equivalently $A_M \colon \mathcal{O}_S(G)^{\mathrm{op}} \to \mathbb{Z}_{(p)}\text{-}\mathrm{mod}$, by sending $G/1 \mapsto M$ and sending all other objects to the trivial group. Define

$$\Lambda^*(G; M) = H^*(\mathcal{O}_p(G); A_M).$$

LEMMA 1.8. *Let $\mathcal{F}$ be a saturated fusion system over a discrete p-toral group S, and let $Q \in \mathcal{F}^c$ be fully normalised. Set $\Gamma = \mathrm{Out}_{\mathcal{F}}(Q)$ and $\Sigma = \mathrm{Out}_S(Q) = N_S(Q)/Q$. Let $\mathcal{O}_\Sigma(\Gamma)$ be the associated orbit category. Then there is a functor*

$$\alpha \colon \mathcal{O}_\Sigma(\Gamma) \to \mathcal{O}(\mathcal{F}^c), \tag{1.1}$$

whose image is (isomorphic to) the full subcategory of $\mathcal{O}(N_{\mathcal{F}}(Q))$ spanned by the objects $P \leq N_S(Q)$ which contain Q (and are therefore $\mathcal{F}$-centric). Thus, if $Q \trianglelefteq \mathcal{F}$, then α embeds $\mathcal{O}_\Sigma(\Gamma)$ as the full subcategory of $\mathcal{O}(\mathcal{F}^c)$ on the objects $P \leq S$ which contain Q.

Proof.   The group Γ is finite by [**BLO3**, Proposition 2.3] and Σ is a Sylow p-subgroup of Γ. The objects of $\mathcal{O}_\Sigma(\Gamma)$ have the form $\Gamma/\mathrm{Out}_R(Q)$, where $Q \leq R \leq \Sigma$. Morphisms

$$\Gamma/\mathrm{Out}_{R_1}(Q) \to \Gamma/\mathrm{Out}_{R_2}(Q)$$

are cosets $g\mathrm{Out}_{R_2}(Q)$ in Γ such that $g^{-1}\mathrm{Out}_{R_1}(Q)g \leq \mathrm{Out}_{R_2}(Q)$. Define the functor $\alpha \colon \mathcal{O}_\Sigma(\Gamma) \to \mathcal{O}(\mathcal{F}^c)$ as in [**BLO3**, Proof of Proposition 5.4]. On objects, $\alpha(\Gamma/\mathrm{Out}_R(Q)) \overset{\mathrm{def}}{=} R$. This is well defined since $C_S(Q) \leq Q$. A morphism $\Gamma/\mathrm{Out}_{R_1}(Q) \overset{[\varphi]}{\longrightarrow} \Gamma/\mathrm{Out}_{R_2}(Q)$, represented by $\varphi \in \mathrm{Aut}_{\mathcal{F}}(Q)$, is sent by α to $[\tilde{\varphi}] \in \mathrm{Rep}_{\mathcal{F}}(R_1, R_2)$, where $\tilde{\varphi} \colon R_1 \to R_2$ in $\mathcal{F}$ is an extension of φ, the existence of which is guaranteed by Axiom (II) in Definition 1.4 and the fact that Q is $\mathcal{F}$-centric. The class $[\tilde{\varphi}] \in \mathrm{Rep}_{\mathcal{F}}(R_1, R_2)$ is independent of the choices by [**BLO3**, Proposition 2.8]. The description of the image of α follows at once from the definition.   □

A functor $F \colon \mathcal{O}(\mathcal{F}^c)^{\mathrm{op}} \to \mathbb{Z}_{(p)}\text{-}\mathrm{mod}$ is called *atomic* on the $\mathcal{F}$-conjugacy class of some $Q \in \mathcal{F}^c$, if $F(P) = 0$ for any $P \in \mathcal{F}^c$ not in $Q^{\mathcal{F}}$. Let F be an atomic functor on Q, which may be chosen to be fully normalised. Set $\Gamma = \mathrm{Out}_{\mathcal{F}}(Q)$, $\Sigma = \mathrm{Out}_S(Q)$, and let $\alpha \colon \mathcal{O}_\Sigma(\Gamma) \to \mathcal{O}(\mathcal{F}^c)$ be the functor defined in Lemma 1.8. Clearly $F(Q)$ is an $\mathrm{Out}_{\mathcal{F}}(Q)$-module, and since F is atomic $\alpha^*(F) \overset{\mathrm{def}}{=} F \circ \alpha = A_{F(Q)}$. We obtain a natural map

$$H^*(\mathcal{O}(\mathcal{F}^c); F) \overset{\mathrm{res}}{\longrightarrow} H^*(\mathcal{O}_\Sigma(\Gamma); \alpha^*(F)) \equiv \Lambda^*(\mathrm{Out}_{\mathcal{F}}(Q); F(Q)).$$

Showing that it is an isomorphism is the heart of the following fundamental result.

PROPOSITION 1.9 ([**BLO3**, Prop. 5.4]).   *Let $\mathcal{F}$ be a saturated fusion system over a discrete p-toral group S. Let $F \colon \mathcal{O}(\mathcal{F}^c)^{\mathrm{op}} \to \mathbb{Z}_{(p)}\text{-}\mathrm{mod}$ be a an atomic functor on the $\mathcal{F}$-conjugacy class of some $Q \in \mathcal{F}^c$. Then*

$$H^*(\mathcal{O}(\mathcal{F}^c); F) \cong \Lambda^*(\mathrm{Out}_{\mathcal{F}}(Q); F(Q)).$$

The next proposition gives useful conditions for the vanishing of the Λ-functors. The lemma following it is an easy consequence we will use later.

PROPOSITION 1.10 ([**O**, Prop. 1.11]).   *For any finite group G and $\mathbb{Z}_{(p)}[G]$-module M,*

(a) *If $p \nmid |G|$ then $\Lambda^i(G; M) = \begin{cases} M^G & \text{if } i = 0 \\ 0 & \text{if } i > 0 \end{cases}$*

(b) *Let $H = C_G(M)$ be the kernel of the action of G on M. Then $\Lambda^*(G; M) \cong \Lambda^*(G/H; M)$ if $p \nmid |H|$ and $\Lambda^*(G; M) = 0$ if $p||H|$.*

(c) *If $O_p(G) \neq 1$ then $\Lambda^*(G; M) = 0$.*

(d) *If $0 \to M' \to M \to M'' \to 0$ is a short exact sequence of $\mathbb{Z}_{(p)}[G]$-modules then there is a long exact sequence*

$$0 \to \Lambda^0(G; M') \to \Lambda^0(G; M) \to \Lambda^0(G; M'') \to \dots$$
$$\dots \to \Lambda^{n-1}(G; M'') \to \Lambda^n(G; M') \to \Lambda^n(G; M) \to \dots$$

LEMMA 1.11.   *Let $\pi \colon G \to H$ be an epimorphism of finite groups and let V be a $\mathbb{Z}_{(p)}[H]$ module. If the order of $K = \ker \pi$ is prime to p then $\Lambda^*(G; V) \cong \Lambda^*(H; V)$.*

Proof.   Let L be the image of H in $\mathrm{Aut}(V)$ and note that $p||C_H(V)| \Leftrightarrow p||C_G(V)|$ since $p \nmid |K|$. Now apply Proposition 1.10(b) to the epimorphisms $H \to L$ and $G \to L$. $\square$

1.4. *Intervals*

Let $\mathcal{F}$ be a fusion system over S, and let $\mathcal{C}$ be a collection of subgroups of S. Then we let $\mathcal{F}^{\mathcal{C}}$ and $\mathcal{O}(\mathcal{F}^{\mathcal{C}})$ denote the full subcategories of $\mathcal{F}$ and $\mathcal{O}(\mathcal{F})$ on the object set $\mathcal{C}$. In particular, if $\mathcal{C}$ is the collection of all $\mathcal{F}$-centric subgroups of S, then the corresponding subcategories are $\mathcal{F}^c$ and $\mathcal{O}(\mathcal{F}^c)$.

For any group Γ let $\mathcal{S}_p(\Gamma)$ denote the poset of its discrete p-toral subgroups. Thus, for a fusion system $\mathcal{F}$ over a discrete p-toral group S, the object set of $\mathcal{F}$ is $\mathcal{S}_p(S)$.

DEFINITION 1.12.   Let $\mathcal{C}$ be a poset. An *interval* in $\mathcal{C}$ is a subset $\mathcal{R}$ with the property that for every $X \leq Y \leq Z$ in $\mathcal{C}$, if $X, Z \in \mathcal{R}$ then $Y \in \mathcal{R}$. For an object $X \in \mathcal{C}$ the interval of all $Y \in \mathcal{C}$ such that $Y \geq X$ is denoted $\mathcal{C}_{\geq X}$.

Let $\mathcal{F}$ be a fusion system over a discrete p-toral group S. We say that a collection $\mathcal{R} \subseteq \mathcal{S}_p(S)$ is $\mathcal{F}$-invariant, if whenever $P \in \mathcal{R}$ then $P^{\mathcal{F}} \subseteq \mathcal{R}$. Clearly an interval $\mathcal{R} \subseteq \mathcal{S}_p(S)$ is closed under overgroups if and only if $S \in \mathcal{R}$. Any $\mathcal{F}$-invariant interval $\mathcal{C}$ in $\mathcal{S}_p(S)$ has the form $\mathcal{R} \setminus \mathcal{R}_0$, where $\mathcal{R}_0 \subseteq \mathcal{R}$ are $\mathcal{F}$-invariant intervals closed under overgroups.

DEFINITION 1.13.   Let $\mathcal{F}$ be a fusion system over a discrete p-toral group S, let $\mathcal{C} \subseteq \mathcal{F}^c$ be an $\mathcal{F}$-invariant collection in $\mathcal{S}_p(S)$, and let $\Phi \colon \mathcal{O}(\mathcal{F}^{\mathcal{C}})^{\mathrm{op}} \to \mathbf{Ab}$ be a functor. Let $\mathcal{R}$ be an $\mathcal{F}$-invariant interval in $\mathcal{C}$. Define a functor

$$\Phi^{\mathcal{R}} \colon \mathcal{O}(\mathcal{F}^{\mathcal{C}})^{\mathrm{op}} \to \mathbf{Ab}$$

by $\Phi^{\mathcal{R}}(P) = \Phi(P)$ if $P \in \mathcal{R}$ and $\Phi^{\mathcal{R}}(P) = 0$ if $P \notin \mathcal{R}$.

Note that the functor $\Phi^{\mathcal{R}}$ is a quotient functor of Φ if $\mathcal{R}$ is closed under overgroups in $\mathcal{C}$ (and a subquotient more generally). If $\mathcal{R}_0 \subseteq \mathcal{R}$ are $\mathcal{F}$-invariant intervals in $\mathcal{C}$ such that $P \in \mathcal{R}_0$ and $Q \in \mathcal{R} \setminus \mathcal{R}_0$ implies $P \not\geq Q$, then $\Phi^{\mathcal{R}_0}$ is a subfunctor of $\Phi^{\mathcal{R}}$.

DEFINITION 1.14 (cf. [**O**, Def. 1.5]).   Let $\mathcal{F}$ be a saturated fusion system over a discrete p-toral group S. Let $\mathcal{Z}\colon \mathcal{O}(\mathcal{F}^c)^{\mathrm{op}} \to \mathbf{Ab}$ denote the functor sending a subgroup P to its centre $Z(P)$. Let $\mathcal{C} \subseteq \mathcal{F}^c$ be an $\mathcal{F}$-invariant collection.

(a) Denote the restriction of $\mathcal{Z}$ to $\mathcal{O}(\mathcal{F}^{\mathcal{C}})^{\mathrm{op}}$ by $\mathcal{Z}_{\mathcal{F}^c}$.

(b) For an $\mathcal{F}$-invariant interval $\mathcal{R}$ in $\mathcal{C}$, let $\mathcal{Z}_{\mathcal{F}^c}^{\mathcal{R}}$ denote $(\mathcal{Z}_{\mathcal{F}^c})^{\mathcal{R}}$, as in Definition 1.13.

(c) If $\mathcal{R}$ is an $\mathcal{F}$-invariant interval in $\mathcal{C}$ set

$$L^*(\mathcal{F}^{\mathcal{C}};\mathcal{R}) \stackrel{\mathrm{def}}{=} H^*(\mathcal{O}(\mathcal{F}^{\mathcal{C}}); \mathcal{Z}_{\mathcal{F}^c}^{\mathcal{R}}).$$

Thus, the goal of this paper is to prove that for a saturated fusion system $\mathcal{F}$ over a discrete p-toral group S, $L^i(\mathcal{F}^c; \mathcal{F}^c) = 0$ for all $i \geq 1$ if $p \neq 2$, and for all $i \geq 2$ if $p = 2$.

LEMMA 1.15 (cf. [**O**, Lem. 1.6]).   Let $\mathcal{F}$ be a saturated fusion system over a discrete p-toral group S and let $\mathcal{C} \subseteq \mathcal{F}^c$ be an $\mathcal{F}$-invariant collection viewed as a subposet of $\mathcal{S}_p(S)$. Suppose that $\mathcal{Q}$ is an $\mathcal{F}$-invariant interval in $\mathcal{C}$ such that $S \in \mathcal{Q}$ (hence $S \in \mathcal{C}$).

(a) Let $F\colon \mathcal{O}(\mathcal{F}^{\mathcal{C}})^{\mathrm{op}} \to \mathbf{Ab}$ be a functor such that $F(P) = 0$ for any $P \in \mathcal{C} \backslash \mathcal{Q}$. Let $F|_{\mathcal{Q}}$ denote its restriction to $\mathcal{O}(\mathcal{F}^{\mathcal{Q}})^{\mathrm{op}}$. Then

$$H^*(\mathcal{O}(\mathcal{F}^{\mathcal{C}}); F) \cong H^*(\mathcal{O}(\mathcal{F}^{\mathcal{Q}}); F|_{\mathcal{Q}}).$$

(b) Suppose that $\mathcal{F} = \mathcal{F}_S(\Gamma)$ for some virtually discrete p-toral group Γ and $S \in \mathrm{Syl}_p(\Gamma)$. Let $\mathcal{Q}$ be the interval $\mathcal{F}_{\geq Y}$, for some discrete p-toral subgroup $Y \trianglelefteq \Gamma$ of finite index, such that $C_\Gamma(Y) \leq Y$. Then

$$L^k(\mathcal{F}^c; \mathcal{Q}) \stackrel{\mathrm{def}}{=} H^k(\mathcal{O}(\mathcal{F}^c); \mathcal{Z}_{\mathcal{F}^c}^{\mathcal{Q}}) \cong \begin{cases} Z(\Gamma) & \text{if } k = 0 \\ 0 & \text{if } k > 0 \end{cases}$$

Proof.   For (a), the proof of the corresponding statement in [**O**, Lemma 1.6], which only uses properties of the bar construction, can be read verbatim with the categories $\mathcal{O}(\mathcal{F}^{\mathcal{Q}}) \subseteq \mathcal{O}(\mathcal{F}^{\mathcal{C}})$ in place of the categories $\mathcal{C}_0 \subseteq \mathcal{C}$ in [**O**]. The hypothesis that $\mathcal{F} = \mathcal{F}_S(\Gamma)$ made in [**O**] is unnecessary. The proof of (b) is also identical to the one in [**O**]. One only needs to observe that $\bar{\Gamma} = \Gamma/Y$ is a finite group by hypothesis, and that for any $P, Q \in \mathcal{Q}$, $Y \leq P$ implies that P is centric in Γ, so $\mathrm{Mor}_{\mathcal{O}(\mathcal{F}^c)}(P, Q) = N_\Gamma(P, Q)/Q$. $\square$

We remark that for a functor $F\colon \mathcal{O}(\mathcal{F}^c)^{\mathrm{op}} \to \mathbf{Ab}$, the limit $\varprojlim F = H^0(\mathcal{O}(\mathcal{F}^c); F)$ is the subgroup of "stable elements", namely the subgroup of all elements $x \in F(S)$ with the property that for any $P \in \mathcal{F}^c$ and any $\varphi \in \mathrm{Hom}_{\mathcal{F}}(P, S)$ we have $F([\varphi])(x) = F([\mathrm{incl}_P^S])(x)$. In fact, one only needs considering $P \in \mathcal{F}^c$ such that $F(P) \neq 0$.

LEMMA 1.16 (cf. [**O**, Lem. 1.7]).   Let $\mathcal{F}$ be a saturated fusion system over a discrete p-toral group S, and let $\mathcal{C} \subseteq \mathcal{F}^c$ be an $\mathcal{F}$-invariant collection which contains S. Let $\mathcal{Q}$ and $\mathcal{R}$ be $\mathcal{F}$-invariant intervals in $\mathcal{C}$ such that

 (i) $\mathcal{Q} \cap \mathcal{R} = \emptyset$.

 (ii) $\mathcal{Q} \cup \mathcal{R}$ is an interval in $\mathcal{C}$.

(iii) If $Q \in \mathcal{Q}$ and $R \in \mathcal{R}$ then $Q \nleq R$.

Then there is a short exact sequence of functors $0 \to \mathcal{Z}_{\mathcal{F}^c}^{\mathcal{R}} \to \mathcal{Z}_{\mathcal{F}^c}^{\mathcal{Q} \cup \mathcal{R}} \to \mathcal{Z}_{\mathcal{F}^c}^{\mathcal{Q}} \to 0$ and a long exact sequence

$$0 \to L^0(\mathcal{F}^{\mathcal{C}};\mathcal{R}) \to L^0(\mathcal{F}^{\mathcal{C}};\mathcal{Q} \cup \mathcal{R}) \to L^0(\mathcal{F}^{\mathcal{C}};\mathcal{Q}) \to$$
$$\cdots \to L^{k-1}(\mathcal{F}^{\mathcal{C}};\mathcal{Q}) \to L^k(\mathcal{F}^{\mathcal{C}};\mathcal{R}) \to L^k(\mathcal{F}^{\mathcal{C}};\mathcal{Q} \cup \mathcal{R}) \to L^k(\mathcal{F}^{\mathcal{C}};\mathcal{Q}) \to \cdots \quad (1.2)$$

In particular

(a) If $L^k(\mathcal{F}^{\mathcal{C}}; \mathcal{R}) = 0 = L^k(\mathcal{F}^{\mathcal{C}}; \mathcal{Q})$ for some $k \geq 0$ then $L^k(\mathcal{F}^{\mathcal{C}}; \mathcal{Q} \cup \mathcal{R}) = 0$.
(b) Assume $\mathcal{F} = \mathcal{F}_S(\Gamma)$, where Γ is a virtually discrete p-toral group, with $S \in \mathrm{Syl}_p(\Gamma)$. Suppose that $Y \trianglelefteq \Gamma$ is a discrete p-toral subgroup of finite index satisfying $C_\Gamma(Y) \leq Y$, and that $\mathcal{C} = \mathcal{F}^c$ and $\mathcal{Q} \cup \mathcal{R} = \mathcal{F}_{\geq Y}$. Then for any $k \geq 2$,

$$L^{k-1}(\mathcal{F}^{\mathcal{C}}; \mathcal{Q}) \cong L^k(\mathcal{F}^{\mathcal{C}}; \mathcal{R}),$$

and there is a short exact sequence

$$0 \to C_{Z(Y)}(\Gamma) \to C_{Z(Y)}(\Gamma^*) \to L^1(\mathcal{F}^c; \mathcal{R}) \to 0$$

where $\Gamma^* = \langle g \in \Gamma \mid gPg^{-1} \in \mathcal{Q} \text{ for some } P \in \mathcal{Q} \rangle$.

Proof. The short exact sequence of functors follows by inspection of Definition 1.14 and the hypotheses on $\mathcal{Q}$ and $\mathcal{R}$. It implies the long exact sequence (1.2) which implies, in turn, point (a).

Point (b) follows by the same argument as in [**O**] from the exact sequence (1.2) and Lemma 1.15(b). The last short exact sequence uses the description of $H^0(\mathcal{O}(\mathcal{F}^c); -)$ in terms of stable elements, recalling that $S \notin \mathcal{R}$ so $\mathcal{Z}^{\mathcal{R}}_{\mathcal{F}^c}(S) = 0$, and that $Y \trianglelefteq \Gamma$ is centric in Γ so $Z(\Gamma) = C_{Z(Y)}(\Gamma)$ and $Z(S) \leq Z(Y)$. $\square$

COROLLARY 1.17 (cf. [**O**, Cor. 1.10]). *Let $\mathcal{F}$ be a saturated fusion system over a discrete p-toral group S and fix some $k \geq 0$. Let $F \colon \mathcal{O}(\mathcal{F}^c)^{\mathrm{op}} \to \mathbb{Z}_{(p)}\text{-}\mathrm{mod}$ be a functor. Then the following properties hold.*
(i) *If $\Lambda^k(\mathrm{Out}_{\mathcal{F}}(P); F(P)) = 0$ for all $P \in \mathcal{F}^c$. Then $H^k(\mathcal{O}(\mathcal{F}^c); F) = 0$.*
(ii) *If $\mathcal{R} \subseteq \mathcal{F}^c$ is an $\mathcal{F}$-invariant collection such that $F(Q) = 0$ for each $Q \in \mathcal{F}^c \backslash \mathcal{R}$, and $\Lambda^k(\mathrm{Out}_{\mathcal{F}}(P); F(P)) = 0$ for all $P \in \mathcal{R}$, then $H^k(\mathcal{O}(\mathcal{F}^c); F) = 0$.*

Proof. For Part (i), the proof of [**BLO3**, Corrolary 5.6] can be read verbatim. For Part (ii), the hypothesis implies that $\Lambda^k(\mathrm{Out}_{\mathcal{F}}(P); F(P)) = 0$ for all $P \in \mathcal{F}^c$ (either $P \in \mathcal{R}$, or $P \notin \mathcal{R}$, in which case $F(P) = 0$). $\square$

LEMMA 1.18. *Let $\mathcal{F}$ be a saturated fusion system over a discrete p-toral group S, and suppose that $Q \in \mathcal{F}^c$ is normal in $\mathcal{F}$. Let $\mathcal{T}$ denote the $\mathcal{F}$-invariant interval $\mathcal{F}_{\geq Q}$ and let $F \colon \mathcal{O}(\mathcal{F}^c)^{\mathrm{op}} \to \mathbb{Z}_{(p)}\text{-}\mathrm{mod}$ be a functor. Let $F^{\mathcal{T}}$ be quotient functor of F defined in 1.13. Then $\eta \colon F \to F^{\mathcal{T}}$ induces an isomorphism $H^*(\mathcal{O}(\mathcal{F}^c); F) \cong H^*(\mathcal{O}(\mathcal{F}^c); F^{\mathcal{T}})$.*

Proof. The kernel $K = \ker(\eta)$ is a functor which vanishes on all $P \in \mathcal{T}$, and it suffices to show that it is acyclic. Let $P \leq S$ be $\mathcal{F}$-centric, such that $P \not\geq Q$. Since Q is normal in $\mathcal{F}$, every automorphism of P in $\mathcal{F}$ extends to an automorphism of QP, and by, Lemma 1.3(a)

$$1 \neq N_{QP}(P)/P \cong \mathrm{Out}_{QP}(P) \trianglelefteq \mathrm{Out}_{\mathcal{F}}(P),$$

so P is not $\mathcal{F}$-radical. Hence, by Proposition 1.10(c), $\Lambda^*(\mathrm{Out}_{\mathcal{F}}(P); K(P)) = 0$. The acyclicity of K now follows from Corollary 1.17(ii). $\square$

The proof of the next lemma is very similar to the corresponding statement in [**O**]. The original proof contains several references, which have to be modified to fit the context. Thus to avoid confusion we reinterpret Oliver's proof here in full detail.

LEMMA 1.19 (cf. [**O**, Lem. 1.12]).   *Let $\mathcal{F}$ be a saturated fusion system over a discrete p-toral group S, and let $Q \in \mathcal{F}^c$ be fully normalized. Set $\mathcal{E} = N_{\mathcal{F}}(Q)$, and let $\mathcal{H}$ denote the full subcategory of $\mathcal{E}^c$, with objects which are also $\mathcal{F}$-centric. Define a collection*

$$\mathcal{T} \overset{\text{def}}{=} \{P \in \mathcal{E}_{\geq Q} \mid R \in Q^{\mathcal{F}} \text{ and } R \trianglelefteq P \text{ imply } R = Q\}.$$

Let $F \colon \mathcal{O}(\mathcal{F}^c)^{\text{op}} \to \mathbb{Z}_{(p)}\text{-}\mathfrak{mod}$ be any functor which vanishes on any subgroup not $\mathcal{F}$-conjugate to a subgroup in $\mathcal{T}$, and let F_0 denote the restriction of F to $\mathcal{O}(\mathcal{H})$. Let $F_1 \colon \mathcal{O}(\mathcal{E}^c)^{\text{op}} \to \mathbb{Z}_{(p)}\text{-}\mathfrak{mod}$ be a functor such that $F_1|_{\mathcal{O}(\mathcal{H})^{\text{op}}} = F_0$, and $F_1(P) = 0$ for all $P \in \mathcal{E}^c \setminus \mathcal{H}$. Then restrictions induce natural isomorphisms

$$H^*(\mathcal{O}(\mathcal{F}^c); F) \overset{R}{\underset{\cong}{\Longrightarrow}} H^*(\mathcal{O}(\mathcal{H}); F_0) \overset{R_1}{\underset{\cong}{\longleftarrow}} H^*(\mathcal{O}(\mathcal{E}^c); F_1).$$

   Proof.   By [**BLO3**, Lamma 2.5], $\text{Out}_{\mathcal{E}}(Q)$ is a finite group, and hence so is $N_S(Q)/Q \in \text{Syl}_p(\text{Out}_{\mathcal{E}}(Q))$. In particular $Q_0 = N_S(Q)_0$, and $\mathcal{E}_{\geq Q}$ is a finite poset (see Section 1.1).
   Clearly the objects of $\mathcal{H}$ form an $\mathcal{E}$-invariant interval in $\mathcal{E}^c$ which contains $N_S(Q)$, so Lemma 1.15(a) implies that R_1 is an isomorphism. It remains to show that R is an isomorphism.
   Since $\mathcal{T} \subseteq \mathcal{E}_{\geq Q}$ is finite, F vanishes except on finitely many $\mathcal{F}$-conjugacy classes of subgroups of S. There is therefore, a finite filtration of F with filtration quotients which are atomic functors on the $\mathcal{F}$-conjugacy classes of the subgroups in $\mathcal{T}$. The five-lemma applied to the long exact sequences in $H^*(\mathcal{O}(\mathcal{F}^c)^{\text{op}}; -)$ and $H^*(\mathcal{O}(\mathcal{H})^{\text{op}}; -)$ reduces the problem to the case that F is atomic on the $\mathcal{F}$-conjugacy class of some $P \in \mathcal{T}$, which we will henceforth fix.
   Let $\varphi \in \text{Hom}_{\mathcal{F}}(P, S)$ be a morphism such that $Q \trianglelefteq \varphi(P)$. Then $\varphi^{-1}(Q) \trianglelefteq P$ so $\varphi^{-1}(Q) = Q$ since $P \in \mathcal{T}$. Thus, $\varphi(Q) = Q$, namely $\varphi \in \mathcal{E}$. Hence

$$\text{Out}_{\mathcal{E}}(P) = \text{Out}_{\mathcal{F}}(P) \qquad \text{and} \qquad P^{\mathcal{E}} = \{P' \in P^{\mathcal{F}} \mid Q \trianglelefteq P'\}.$$

In particular $P^{\mathcal{E}} \subseteq \mathcal{T}$, so we may assume that P is fully normalized in $\mathcal{E}$.
   Notice that $N_S(P) \leq N_S(Q)$, because if $x \in N_S(P)$, then $R = xQx^{-1} \trianglelefteq P$, so by definition of $\mathcal{T}$, $R = Q$, and hence $x \in N_S(Q)$. Set $\Gamma = \text{Out}_{\mathcal{E}}(P)$, and $\Sigma = \text{Out}_S(P) = N_S(P)/P$. Then $\Sigma \in \text{Syl}_p(\Gamma)$ by Axiom (I). Since $\text{Out}_{\mathcal{E}}(P) = \text{Out}_{\mathcal{F}}(P)$ it follows that P is fully normalised in $\mathcal{F}$ as well. From the construction in Lemma 1.8 we obtain functors

$$\alpha_1 \colon \mathcal{O}_{\Sigma}(\Gamma) \to \mathcal{O}(\mathcal{E}^c) \qquad \text{and} \qquad \alpha \colon \mathcal{O}_{\Sigma}(\Gamma) \to \mathcal{O}(\mathcal{F}^c)$$

which factor through a functor $\alpha_0 \colon \mathcal{O}_{\Sigma}(\Gamma) \to \mathcal{O}(\mathcal{H})$ and the inclusions $\mathcal{O}(\mathcal{H}) \subseteq \mathcal{O}(\mathcal{E}^c)$ and $\mathcal{O}(\mathcal{H}) \subseteq \mathcal{O}(\mathcal{F}^c)$. Since $\mathcal{H}$ is an interval in $\mathcal{E}^c$ which contains $N_S(Q)$, we can extend $F_0 = F|_{\mathcal{O}(\mathcal{H})^{\text{op}}}$ to $F_1 \colon \mathcal{O}(\mathcal{E}^c)^{\text{op}} \to \mathbb{Z}_{(p)}\text{-}\mathfrak{mod}$ by assigning $F_1(R) = F_0(R)$ for any $R \in \mathcal{H}$ and $F_1(R) = 0$ for any $R \in \mathcal{E}^c \backslash \mathcal{H}$. Combining all this, we obtain a commutative diagram,

$$H^*(\mathcal{O}(\mathcal{F}^c); F) \xrightarrow{\quad R \quad} H^*(\mathcal{O}(\mathcal{H}); F_0) \xleftarrow[\cong]{\quad R_1 \quad} H^*(\mathcal{O}(\mathcal{E}^c); F_1) \ .$$
$$\Lambda^*(\Gamma; F(P))$$

with maps α^* ($\cong$), α_0^*, α_1^* to $\Lambda^*(\Gamma; F(P))$.

By Lemma 1.15(a), R_1 is an isomorphism, and by Proposition 1.9, α^* is an isomorphism. Let $\mathcal{Q}$ denote the interval $\mathcal{E}_{\geq Q}$ in $\mathcal{E}^c$. By Lemma 1.18 the natural transformation $F_1 \to F_1^{\mathcal{Q}}$ induces an isomorphism $H^*(\mathcal{O}(\mathcal{E}^c)^{\text{op}}; F_1) \cong H^*(\mathcal{O}(\mathcal{E}^c)^{\text{op}}; F_1^{\mathcal{Q}})$. Also, $F_1^{\mathcal{Q}}$ is atomic on the $\mathcal{E}$-conjugacy class of P since $P^{\mathcal{E}} = P^{\mathcal{F}} \cap \mathcal{E}_{\geq Q}$ and since F is atomic. Now, α_1^* is an isomorphism by Proposition 1.9, hence so is α_0^* and therefore R as well. $\square$

   LEMMA 1.20 (cf. [**O**, Lem. 1.13]).   *Let G be a virtually discrete p-toral group. Fix $H \trianglelefteq G$ such that $G_0 \leq H$ and fix $S \in \text{Syl}_p(G)$, and set $T = S \cap H \in \text{Syl}_p(H)$. Set $\mathcal{F} = \mathcal{F}_S(G)$ and*

$\mathcal{E} = \mathcal{F}_T(H)$. Assume that there exists $Y \leq T$ such that $Y \trianglelefteq G$ and $C_G(Y) \leq Y$, and fix it once and for all. Let $\mathcal{Q}$ be an $\mathcal{F}$-invariant interval in $\mathcal{S}_p(S)_{\geq Y}$, such that $S \in \mathcal{Q}$, and such that $Q \in \mathcal{Q}$ implies $H \cap Q \in \mathcal{Q}$. Set $\mathcal{Q}_0 = \{Q \in \mathcal{Q} \mid Q \leq H\}$, namely $\mathcal{Q}_0 = \mathcal{Q} \cap \mathcal{E}$. Then restriction induces an injective homomorphism

$$L^1(\mathcal{F}^c; \mathcal{Q}) \xrightarrow{R} L^1(\mathcal{E}^c; \mathcal{Q}_0).$$

Proof.   The proof of [**O**, Lemma 1.13] can be read verbatim, keeping in mind the following comments. The reference to Lemma 1.6 in [**O**] should be replaced by Lemma 1.5 in this paper. The subgroup T contains G_0, so $N_G(T)/T$ is finite and therefore the reference to the Cartan-Eilenberg "stable elements theorem" [**CE**, Theorem XII.10.1] is valid. The use of the Frattini argument in the proof can be read directly, since [**O**, Lemma 1.14(b)] generalises to the context of virtually discrete p-toral groups (Lemma 1.3). Observe also that G/H is finite, since $G_0 \leq H$. For the same reason the subgroup H^*, defined as the subgroup of H generated by all h such that for some $Q \in \mathcal{Q}$, $hQh^{-1} \in \mathcal{Q}$, contains $N_H(T)$ as a subgroup of finite index prime to p, and so the argument involving the trace homomorphism is valid. The rest of the proof does not involve any finiteness considerations. $\square$

1.5.   *The model theorem*

A saturated fusion system $\mathcal{F}$ over a p-toral group S is called *constrained* if there is $Q \leq S$ in $\mathcal{F}^c$ such that $Q \trianglelefteq \mathcal{F}$. A *model* (Γ, S, Q) for $\mathcal{F}$ consists of a virtually discrete p-toral group Γ such that $S \in \mathrm{Syl}_p(\Gamma)$, and $\mathcal{F} = \mathcal{F}_S(\Gamma)$ and $Q \trianglelefteq \Gamma$ is centric in Γ (i.e. $C_\Gamma(Q) \leq Q$). When S is a finite group the existence of models was shown in [**BCGLO**, Proposition C], [**AKO**, Proposition III.5.10]. We will prove the general case originally due to Gonzalez [**G**].

PROPOSITION 1.21.   *Let $\mathcal{F}$ be a constrained fusion system over a discrete p-toral group S with $Q \in \mathcal{F}^c$ normal in $\mathcal{F}$. Then $\mathcal{F}$ has an associated centric linking system $\mathcal{L}$ which is unique up to isomorphism. Furthermore, $G = \mathrm{Aut}_\mathcal{L}(Q)$ is a model for $\mathcal{F}$.*

Proof.   By Proposition 1.7, to prove the existence and uniqueness of $\mathcal{L}$, it suffices to prove that $H^i(\mathcal{O}(\mathcal{F}^c); \mathcal{Z}) = 0$ for $i = 2, 3$. The argument here is basically the same argument as in the proof of [**BCGLO**, Proposition 4.2]. Let $\mathcal{T}$ be the interval $\mathcal{F}^c_{\geq Q}$. By Lemma 1.18, $H^i(\mathcal{O}(\mathcal{F}^c); \mathcal{Z}) \cong H^i(\mathcal{O}(\mathcal{F}^c); \mathcal{Z}^\mathcal{T})$, and by Lemma 1.15(a) restriction induces an isomorphism

$$H^*(\mathcal{O}(\mathcal{F}^c); \mathcal{Z}^\mathcal{T}) \cong H^*(\mathcal{O}(\mathcal{F}^\mathcal{T}); \mathcal{Z}^\mathcal{T}|_{\mathcal{O}(\mathcal{F}^\mathcal{T})}).$$

Set $\Gamma = \mathrm{Out}_\mathcal{F}(Q)$ and $\Sigma = \mathrm{Out}_S(Q) = S/Q$. Then $\Sigma \in \mathrm{Syl}_p(\Gamma)$, and since $Q \trianglelefteq \mathcal{F}$, the functor $\alpha \colon \mathcal{O}_\Sigma(\Gamma) \to \mathcal{O}(\mathcal{F}^c)$ defined in Lemma 1.8 is an isomorphism onto the full subcategory $\mathcal{O}(\mathcal{F}^\mathcal{T})$. Set $M \stackrel{\mathrm{def}}{=} \mathcal{Z}(Q)$. By definition Γ acts on $M = Z(Q)$. Let

$$H^0 M \colon \mathcal{O}_\Sigma(\Gamma)^{\mathrm{op}} \to \mathbb{Z}_{(p)}\text{-}\mathtt{mod}$$

be the functor sending an orbit Γ/R to $\mathrm{Hom}_\Gamma(\mathbb{Z}[\Gamma/R], M) \cong C_M(R)$. Then one easily observes that $\mathcal{Z}^\mathcal{T} \circ \alpha \cong H^0 M$, and by [**JMO**, Proposition 5.2] $H^i(\mathcal{O}_\Sigma(\Gamma); H^0 M) = 0$ for $i \geq 1$. This shows that $H^i(\mathcal{O}(\mathcal{F}^c); \mathcal{Z}) = 0$ for $i \geq 1$, and thus proves the first claim.

Let $\mathcal{L}$ be a centric linking system associated to $\mathcal{F}$, and let $Q \trianglelefteq \mathcal{F}$ be a normal centric subgroup as before. Throughout, we will refer to the notation in [**BLO3**, Definition 4.1] for the definition of linking systems. It remains to show that $G = \mathrm{Aut}_\mathcal{L}(Q)$ is a model for $\mathcal{F}$. This will be done in two steps.

Step 1: The group G is virtually discrete p-toral, since $G/\delta_Q(Q) = \mathrm{Out}_\mathcal{F}(Q)$ is finite by [**BLO3**, Lemma 2.5]. We show that S is a Sylow p-subgroup of G. For any $P \in \mathcal{F}^c$ choose

lifts ι_P^S in $\mathcal{L}$ for the inclusion $P \leq S$. By [**BLO3**, Lemma 4.3(a)] for any inclusion $P \leq R$ in $\mathcal{F}^c$ there is a unique lift ι_P^R in $\mathcal{L}$ such that $\iota_R^S \circ \iota_P^R = \iota_P^S$. Consider an inclusion $Q \leq P$. Since $Q \trianglelefteq \mathcal{F}$, restriction yields a homomorphism $\mathrm{res}_P^Q \colon \mathrm{Aut}_{\mathcal{F}}(P) \to \mathrm{Aut}_{\mathcal{F}}(Q)$. Application of [**BLO3**, Lemma 4.3(a)] again gives a homomorphism $\mathrm{Res}_P^Q \colon \mathrm{Aut}_{\mathcal{L}}(P) \to \mathrm{Aut}_{\mathcal{L}}(Q)$ which lifts res_P^Q, and [**BLO3**, Proposition 2.8] together with axioms (A) and (C) show that Res_P^Q is injective. Thus, $\epsilon \overset{\mathrm{def}}{=} \mathrm{Res}_S^Q \circ \delta_S \colon S \to G$ is a monomorphism, and we may view S as a subgroup of G in this way. For any $P \geq Q$, axiom (C) and [**BLO3**, Lemma 4.3(a)] imply that $\epsilon|_P = \mathrm{Res}_P^Q \circ \delta_P$. In particular $\epsilon|_Q = \delta_Q$, and therefore Q (identified with $\epsilon(Q)$) is normal in G. Axioms (A) and (C) now apply to show that $C_G(Q) = Z(Q)$. Also, for any $x \in S$ one has $\pi(\epsilon(x)) = \mathrm{res}_S^Q(c_x) = c_x$, hence $\pi(\epsilon(S)) = \mathrm{Aut}_S(Q) \in \mathrm{Syl}_p(\mathrm{Aut}_{\mathcal{F}}(Q))$, and it follows that $S \in \mathrm{Syl}_p(G)$.

Step 2: To complete the proof that G is a model, it remains to show that $\mathcal{F} = \mathcal{F}_S(G)$. Both fusion systems are saturated (see Proposition 1.5), and since Q is centric and normal in both, it follows from the argument in Lemma 1.18 that any $P \leq S$ which is centric and radical in either $\mathcal{F}$ or in $\mathcal{F}_S(G)$ must contain Q. By Alperin's fusion theorem [**BLO3**, Theorem 3.6] both fusion systems are generated by automorphisms of overgroups of Q, and so it remains to prove that $\mathrm{Aut}_{\mathcal{F}}(P) = \mathrm{Aut}_{\mathcal{F}_S(G)}(P)$ for all $P \geq Q$.

Consider $P \geq Q$ and some $\varphi \in \mathrm{Aut}_{\mathcal{L}}(P)$ and set $\psi = \mathrm{Res}_P^Q(\varphi)$. For any $x \in P$ axiom (C) implies $\varphi \circ \delta_P(x) \circ \varphi^{-1} = \delta_P(\pi(\varphi)(x))$ in $\mathrm{Aut}_{\mathcal{L}}(P)$. By applying Res_P^Q we get $\psi \circ \epsilon(x) \circ \psi^{-1} = \epsilon(\pi(\varphi)(x))$. Hence $c_{\psi} \in \mathrm{Aut}_G(P)$ is equal to $\pi(\varphi) \in \mathrm{Aut}_{\mathcal{F}}(P)$.

Now, for any $f \in \mathrm{Aut}_{\mathcal{F}}(P)$ choose a lift $\varphi \in \mathrm{Aut}_{\mathcal{L}}(P)$, and set $\psi = \mathrm{Res}_P^Q(\varphi)$. Then $f = c_{\psi} \in \mathrm{Aut}_G(P)$, and hence $\mathrm{Aut}_{\mathcal{F}}(P) \leq \mathrm{Aut}_{\mathcal{F}_S(G)}(P)$. Conversely, consider $\psi \in N_G(P)$ and set $f = \pi(\psi)$. Then for any $x \in P$ there exists $y \in P$ such that $\psi \circ \epsilon(x) \circ \psi^{-1} = \epsilon(y)$. By applying π we obtain $f \circ c_x \circ f^{-1} = c_y \in \mathrm{Aut}_P(Q)$. By the extension axiom (II), and since $C_S(Q) \leq P$, f extends to $h \in \mathrm{Aut}_{\mathcal{F}}(P)$ which lifts to some $\tilde{h} \in \mathrm{Aut}_{\mathcal{L}}(P)$. By axioms (A) and (C) there is $z \in Z(Q)$, such that $\varphi \overset{\mathrm{def}}{=} \tilde{h} \circ \delta_P(z)$ satisfies $\mathrm{Res}_P^Q(\varphi) = \psi$. Since $c_{\psi} = \pi(\varphi) \in \mathrm{Aut}_{\mathcal{F}}(P)$, we deduce that $\mathrm{Aut}_{\mathcal{F}_S(G)}(P) \leq \mathrm{Aut}_{\mathcal{F}}(P)$. Thus $\mathrm{Aut}_{\mathcal{F}_S(G)}(P) = \mathrm{Aut}_{\mathcal{F}}(P)$ for each $P \geq Q$ and the proof is complete. $\qquad\square$

1.6. The subcategory $\mathcal{F}^{\bullet}$

We briefly recall the "bullet" construction from [**BLO3**, Section 3], whose properties are crucial for our purpose in this paper.

Let $\mathcal{F}$ be a saturated fusion over a discrete p-toral group S and let p^m be the order of S/S_0. Set $W = \mathrm{Aut}_{\mathcal{F}}(S_0)$. For any subgroup $D \leq S_0$ set $I(D) \overset{\mathrm{def}}{=} C_{S_0}(C_W(D))$. This is a discrete p-toral group with identity component $I(D)_0$. For any $P \leq S$ let $P^{[m]}$ denote the subgroup generated by g^{p^m} for all $g \in P$. This is a subgroup of S_0 and we set $P^{\bullet} \overset{\mathrm{def}}{=} P \cdot I(P^{[m]})_0$. See [**BLO3**, Def. 3.1] for details. By [**BLO3**, Lemma 3.2, Prop. 3.3] the assignment $P \mapsto P^{\bullet}$ gives rise to an endofunctor $(-)^{\bullet} \colon \mathcal{F} \to \mathcal{F}$, whose image is a full subcategory $\mathcal{F}^{\bullet}$ with finitely many S-conjugacy classes of objects. In fact, the functor $(-)^{\bullet}$ is left adjoint to the inclusion $\mathcal{F}^{\bullet} \subseteq \mathcal{F}$ and it is idempotent in the sense that its restriction to $\mathcal{F}^{\bullet}$ is the identity. We now record further properties of $\mathcal{F}^{\bullet}$ that will be used in this paper.

PROPOSITION 1.22 [**BLO3**, Prop. 5.2].   *Let $\mathcal{F}$ be a saturated fusion system over a discrete p-toral group S. Let $\mathcal{F}^c \subseteq \mathcal{F}$ be the full subcategory on all $\mathcal{F}$-centric objects, and let $\mathcal{F}^{c\bullet}$ denote $\mathcal{F}^c \cap \mathcal{F}^{\bullet}$. Then for any functor $F \colon \mathcal{O}(\mathcal{F}^c)^{\mathrm{op}} \to \mathcal{Z}_{(p)}$ restriction to $\mathcal{F}^{c\bullet}$ induces an isomorphism*

$$H^*(\mathcal{O}(\mathcal{F}^c), F) \cong H^*(\mathcal{O}(\mathcal{F}^{c\bullet}), F|_{\mathcal{O}(\mathcal{F}^{c\bullet})}).$$

LEMMA 1.23. *Let $\mathcal{F}$ be a saturated fusion system over a discrete p-toral group S. Then the collections $\mathrm{Obj}(\mathcal{F}^{\bullet})$ and $\mathrm{Obj}(\mathcal{F}^c \cap \mathcal{F}^{\bullet})$ are $\mathcal{F}$-invariant.*

Proof. If $Q \to P^{\bullet}$ is an $\mathcal{F}$-isomorphism it extends by [**BLO3**, Prop. 3.3 and Lemma 3.2(b)] to a morphism $Q^{\bullet} \to P^{\bullet}$ in $\mathcal{F}$ which must therefore be an isomorphism, hence $Q = Q^{\bullet}$. This shows that the collection $\mathcal{F}^{\bullet}$ is $\mathcal{F}$-invariant and its intersection with the $\mathcal{F}$-invariant collection $\mathcal{F}^c$ must be $\mathcal{F}$-invariant too. $\qquad\square$

LEMMA 1.24. *Let $\mathcal{F}$ be a saturated fusion system over a discrete p-toral group S. Then for any $P \in \mathcal{F}^{\bullet}$, if $Q \leq S$ satisfies $P_0 \leq Q \leq P$ then $Q = Q^{\bullet}$.*

Proof. By [**BLO3**, Definition 3.1], for any $R \leq S$, $R^{\bullet} = R \cdot T$ where T is a discrete p-torus, and $R \leq N_S(T)$. Thus, $R = R^{\bullet}$ if and only if $R_0 = (R^{\bullet})_0$. By assumption, $P_0 \leq Q \leq P$, so $P_0 \leq Q^{\bullet} \leq P^{\bullet} = P$ by [**BLO3**, Lemma 3.2(c)]. Hence $(Q^{\bullet})_0 = P_0 = Q_0$, so $Q = Q^{\bullet}$. $\qquad\square$

2. The Thompson subgroup and offenders

Central to this paper is a suitable generalisation of the notion of "best offenders". The idea for this more general notion is due to Chermak, and we are indebted to him for sharing his insight with us.

Recall that a prime p was fixed throughout this paper. For any abelian group A, let $\Omega_n A$ denote the subgroup of A of the elements whose order divides p^n. It is clearly a fully characteristic subgroup of A, i.e., any endomorphisms of A takes $\Omega_n(A)$ to itself).

LEMMA 2.1. *Let D be an abelian discrete p-toral group. Then the following hold.*
(a) *$D \cong D_0 \times E$ where E is a finite abelian p-group. In particular, $D = \cup_{n=1}^{\infty} \Omega_n D$.*
(b) *The only action of a discrete p-torus T on D is the trivial action.*
(c) *If $D \trianglelefteq \Gamma$, where Γ is a virtually discrete p-toral group, then $\Gamma_0 \leq C_{\Gamma}(D)$.*

Proof. Since D_0 is divisible it is a direct factor of D (see [**Fu**, Theorem 21.2]), proving Part (a). Any non-trivial quotient of T is infinite ([**Fu**, Sec. 20]), so T must act trivially on any of the finite groups $\Omega_n D$. This proves part (b) which implies (c). $\qquad\square$

Recall that for a discrete p-toral group P, the order of P, denoted $\mathrm{ord}(P)$ is the pair (r, n), where r is the rank of P_0 and n is the order of P/P_0 [**BLO3**, Definition 1.1].

DEFINITION 2.2 (cf. [**O**, Def. 2.1]).
(a) Let S be a discrete p-toral group. Set

$$d(S) = \max\{\mathrm{ord}(A) \mid A \text{ is an abelian subgroup of } S\},$$
$$\mathcal{A}(S) = \{A \leq S \mid A \text{ is abelian and } \mathrm{ord}(A) = d(S)\}$$

Define the Thompson subgroup of S by $J(S) \overset{\text{def}}{=} \langle \mathcal{A}(S) \rangle$.
(b) Let G be a finite group which acts faithfully on an abelian discrete p-toral group D. A *best offender* in G on D is an abelian subgroup $A \leq G$ such that
  (i) $D_0 \leq C_D(A)$ and
  (ii) $|A| \cdot |C_D(A)/D_0| \geq |B| \cdot |C_D(B)/D_0|$ for any $B \leq A$.
(c) Let $\mathcal{A}_D(G)$ be the set of best offenders in G on D. Set $J_D(G) = \langle \mathcal{A}_D(G) \rangle$.

(d) Let Γ be a virtually discrete p-toral group which acts on an abelian discrete p-toral group D. Define $J(\Gamma, D)$ as the preimage in Γ of $J_D(\Gamma/C_\Gamma(D))$.

The maximum in Definition 2.2(a) makes sense because S_0 is abelian, and only subgroups A in the finite poset $\mathcal{S}_p(S)_{\geq S_0}$ need to be considered, since if $A \not\geq S_0$ then $A_0 < S_0$, so $\mathrm{ord}(A) < \mathrm{ord}(S_0)$. The definition of $J(\Gamma, D)$ makes sense, because $\Gamma_0 \leq C_\Gamma(D)$ by Lemma 2.1(c), so $\Gamma/C_\Gamma(D)$ is finite and acts faithfully on D.

REMARK 2.3.   Suppose that $\Gamma' \leq \Gamma$ are virtually discrete p-toral groups and that Γ acts on an abelian discrete p-toral group D. Then $J(\Gamma', D) \leq J(\Gamma, D)$, since the inclusion $\Gamma'/C_{\Gamma'}(D) \leq \Gamma/C_\Gamma(D)$ implies $\mathcal{A}_D(\Gamma'/C_{\Gamma'}(D)) \subseteq \mathcal{A}_D(\Gamma/C_\Gamma(D))$. Hence, if $\Gamma \geq \Gamma' \geq J(\Gamma, D)$, then $J(\Gamma', D) = J(\Gamma, D)$. In particular $J(J(\Gamma, D), D) = J(\Gamma, D)$.

REMARK 2.4.   If P is a discrete p-toral group and $Q \leq P$, then $d(Q) \leq d(P)$, and if $d(Q) = d(P)$, then $J(Q) \leq J(P)$. Hence, if $Q \geq J(P)$, then $d(Q) = d(P)$ and $J(Q) = J(P)$. In particular $J(J(P)) = J(P)$. If $d(Q) = d(P)$ and $\mathrm{ord}(J(Q)) = \mathrm{ord}(J(P))$, then $J(Q) = J(P)$. In particular, if $Q \leq P$ and $Q \cong J(P)$, then $Q = J(P)$.

REMARK 2.5.   With the notation of Definition 2.2(a), $J(S) \geq S_0$ since $S_0 \leq A$ for any $A \in \mathcal{A}(S)$. If $x \in C_S(J(S))$ and $A \in \mathcal{A}(S)$, then $x \in C_S(A)$ so $\langle x, A \rangle$ is abelian discrete p-toral (see [**BLO3**, Lemma 1.3]), and the maximality of $\mathrm{ord}(A)$ implies $x \in A$. It follows that $C_S(J(S)) \leq J(S)$; in fact, $C_S(J(S)) = \cap_{A \in \mathcal{A}(S)} A$. In particular $Z(S) \leq J(S)$.

DEFINITION 2.6.   Let G be a finite group, and let V be a $\mathbb{Z}[G]$-module. Let $[g, v] \in V$ denote the element $gv - v$, and let

$$[G, V] \overset{\mathrm{def}}{=} \langle [g, v] \mid g \in G, \ v \in V \rangle \leq V.$$

We say that G acts *quadratically* if $[G, [G, V]] = 0$. If G acts faithfully on an abelian p-toral group V, we say that $A \leq G$ is a *quadratic best offender* in G on V, if it is a best offender, and it acts quadratically on V.

LEMMA 2.7.   *Let A be a finite group acting on an abelian discrete p-toral group V. Then there is some $N > 0$, such that the following statements hold for all $n \geq N$.*
(a) *$V = V_0 + \Omega_n(V)$.*
(b) *$C_A(V) = C_A(\Omega_n V)$.*
If in addition A acts faithfully on V and trivially on V_0, then
(c) *for any $B \leq A$, $|C_{\Omega_n V}(B)| = |C_V(B)/V_0| \cdot |\Omega_n(V_0)|$, and hence*
(d) *any $B \leq A$ is a (quadratic) best offender on V if and only if it is a (quadratic) best offender on $\Omega_n V$.*

Proof.   Lemma 2.1(a) readily implies (a). Also, $C_A(V) = \cap_{n=1}^\infty C_A(\Omega_n V)$, so (b) follows since A is finite.

It remains to prove (c) and (d), under the assumption that A acts faithfully on V and trivially on V_0. By (a) and (b) A acts faithfully on $\Omega_n V$, and $V = V_0 + \Omega_n V$ provided n is sufficiently large. Let $B \leq A$ be any subgroup. Since A acts trivially on V_0, it follows that $[B, V] = [B, \Omega_n V]$, so B acts quadratically on V if and only if it acts quadratically on $\Omega_n V$.

Also,

$$C_V(B) = C_{V_0 + \Omega_n V}(B) = C_{\Omega_n V}(B) + V_0$$

and therefore $|C_V(B)/V_0| = |C_{\Omega_n V}(B)|/|\Omega_n(V_0)|$. This proves (c). Applying this equality to any $B' \leq B$ we deduce that B is a best offender on V if and only if it is a best offender on $\Omega_n V$, which proves (d). $\qquad\square$

LEMMA 2.8 (cf. [**O**, Lem. 2.2]).
(a) *Suppose that a finite group G acts faithfully on an abelian discrete p-toral group D. If $A \leq G$ is a best offender on D, and if $U \leq D$ is A-invariant, then $A/C_A(U)$ is a best offender in $N_G(U)/C_G(U)$ on U.*
(b) *Suppose that S is discrete p-toral, and that $D \trianglelefteq S$ is a normal abelian subgroup. Set $\bar{S} = S/C_S(D)$, and consider some $A \in \mathcal{A}(S)$. Then $\bar{S}$ is a finite p-group, and the image of A in $\bar{S}$ is a best offender on D.*

Proof. (a) By definition of best offenders A acts trivially on D_0, hence trivially on U_0. By Lemma 2.7(b), $C_G(U) = C_G(\Omega_n U)$ for all sufficiently large n, and hence $C_A(U) = C_A(\Omega_n U)$. By Lemma 2.7(d), for all sufficiently large n, A is a best offender on $\Omega_n D$ in G, and since $\Omega_n U$ is an A-invariant subgroup of $\Omega_n D$, Lemma [**O**, Lemma 2.2(a)] implies that $A/C_A(\Omega_n U) = A/C_A(U)$ is a best offender in $N_G(U)/C_G(U) \leq N_G(\Omega_n U)/C_G(\Omega_n U)$ on $\Omega_n U$. Lemma 2.7(d) applied to U and $N_G(U)/C_G(U)$ now shows that $A/C_A(U)$ is a best offender on U.

(b) By [**BLO3**, Lemma 1.3] D is a discrete p-toral group and by Lemma 2.1(c) $S_0 \leq C_S(D)$, hence $\bar{S}$ is a finite p-group. Set $S^* = S/S_0$, and for any subgroup $B \leq S$, let B^* and $\bar{B}$ denote its images in S^* and $\bar{S}$ respectively under the obvious projections. Note that D_0 is the maximal torus of $S_0 \cap D$, and set $d = |S_0 \cap D/D_0|$.

If $A \in \mathcal{A}(S)$, then $S_0 \leq A$ by Remark 2.5, and Lemma 2.1(b) implies that $S_0 \leq C_A(D)$. Any subgroup of $\bar{A} = A/C_A(D)$ has the form $\bar{B}$ where $C_A(D) \leq B \leq A$. It follows that $|\bar{B}| = |B^*|/|C_A(D)^*|$. Also, $C_D(B) \cap S_0 = D \cap S_0$, since B is abelian and contains S_0, and therefore $|C_D(B)/D_0| = d \cdot |C_D(B)^*|$. Since A is abelian and contains S_0, we also deduce that $C_D(B)S_0 \cap B \leq DS_0 \cap A \leq C_D(A)S_0$, hence $|B^* \cap C_D(B)^*| \leq |C_D(A)^*|$. In addition, $BC_D(B)$ is an abelian subgroup of S, and since $A \in \mathcal{A}(S)$ and both contain S_0, it follows that $|BC_D(B)^*| \leq |A^*|$. Hence for any subgroup $\bar{B} \leq \bar{A}$, such that $C_A(D) \leq B \leq A$,

$$|\bar{B}| \cdot |C_D(\bar{B})/D_0| = \frac{|B^*|}{|C_A(D)^*|} \cdot |C_D(B)^*| \cdot d =$$
$$\frac{|BC_D(B)^*| \cdot |B^* \cap C_D(B)^*|}{|C_A(D)^*|} \cdot d \leq \frac{|A^*|}{|C_A(D)^*|} \cdot |C_D(A)^*| \cdot d = |\bar{A}| \cdot |C_D(A)/D_0|.$$

Hence $\bar{A}$ is a best offender on D in $\bar{S}$. $\qquad\square$

COROLLARY 2.9 (cf. [**O**, Cor. 2.3]). *Suppose that Γ is a virtually discrete p-toral group, and $D \trianglelefteq \Gamma$ a normal abelian discrete p-toral subgroup.*
(a) *If $U \leq D$ and $U \trianglelefteq \Gamma$, then $J(\Gamma, U) \geq J(\Gamma, D)$.*
(b) *If Γ is discrete p-toral, then $J(\Gamma) \leq J(\Gamma, D)$.*

Proof. If $A \leq \Gamma/C_\Gamma(D)$ is a best offender on D, then its image in $\Gamma/C_\Gamma(U)$ is a best offender on U, by Lemma 2.8(a). Hence $\Gamma/C_\Gamma(D) \to \Gamma/C_\Gamma(U)$ carries $J_D(\Gamma/C_\Gamma(D))$ to $J_U(\Gamma/C_\Gamma(U))$, and so $J(\Gamma, D) \leq J(\Gamma, U)$, as stated in (a). Part (b) follows similarly, using Lemma 2.8(b). $\square$

LEMMA 2.10 (cf. [**O**, Lem.2.4]).   *Let G be a finite group which acts faithfully on an abelian discrete p-toral group D. Assume that $p \cdot D \leq D_0$ and that G acts trivially on D_0. If G acts quadratically, then G is an elementary abelian p-group.*

Proof.   By Lemma 2.1(a), $D = D_0 \oplus E$ for some $E \leq D$, which must be elementary abelian since $p \cdot D \leq D_0$. Since G acts trivially on D_0 it follows that $C_G(E) = C_G(D) = 1$. We deduce that G acts faithfully and quadratically on $\Omega_1 D$ since $\Omega_1 D \geq E$. The claim now follows from [**O**, Lemma 2.4]. $\square$

The following is a generalisation of Timmesfeld's replacement theorem to the context of discrete p-toral groups.

THEOREM 2.11 (cf. [**O**, Thm. 2.5]).   *Let V be a nontrivial abelian discrete p-toral group, and let A be a nontrivial finite abelian p-group. Suppose that A acts faithfully on V and trivially on V_0, and that it is a best offender on V. Then there exists $1 \neq B \leq A$ such that B is a quadratic best offender on V. In fact, $B = C_A([A, V])$ is such a subgroup, and in this case $|A||C_V(A)/V_0| = |B||C_V(B)/V_0|$ and $C_V(B) = [A, V] + C_V(A) < V$.*

Proof.   By Lemma 2.7 there is some $N > 0$, such that for all $n \geq N$, $V = V_0 + \Omega_n V$, A acts faithfully on $\Omega_n V$, and a subgroup $B \leq A$ is a best offender on V if and only if it is a best offender on $\Omega_n V$. In particular A is a best offender on $\Omega_n V$.

Fix some $n > N$. Then, by [**O**, Theorem 2.5],

$$B \overset{\text{def}}{=} C_A([A, \Omega_n V]) = C_A([A, V]) \neq 1$$

is a quadratic best offender on $\Omega_n V$, and moreover

$$|A||C_{\Omega_n V}(A)| = |B||C_{\Omega_n V}(B)| \qquad \text{and} \qquad C_{\Omega_n(V)}(B) = [A, \Omega_n V] + C_{\Omega_n V}(A) < \Omega_n V.$$

By Lemma 2.7(d), B is a quadratic best offender on V.

Every subgroup $A' \leq A$ acts trivially on V_0, and since $V = V_0 + \Omega_n V$ it follows that $[A', V] = [A', \Omega_n V]$ and $C_V(A') = C_{\Omega_n V}(A') + V_0$, so in particular $|C_V(A')/V_0| = |C_{\Omega_n V}(A')|/|\Omega_n V_0|$. Applying these equalities to the display above when $A' = B$ and $A' = A$ we get that $|A||C_V(A)/V_0| = |B||C_V(B)/V_0|$ and $C_V(B) = [A, V] + C_V(A)V$. The latter group is a proper subgroup of V since $C_{\Omega_n V_0}(B) = \Omega_n V_0$. $\square$

3. *Proof of the main theorem*

In this section we prove Theorem A, restated here as Theorem 3.6. For any prime p set $k(p) = 1$ if p is odd, and $k(2) = 2$.

DEFINITION 3.1 (cf. [**O**, Def. 3.1], [**Ch**, 6.3]).   A *general setup* is a triple (Γ, S, Y) where
- Γ is a virtually discrete p-toral group, $S \in \mathrm{Syl}_p(\Gamma)$, and $Y \leq S$.
- $\Gamma_0 \leq Y \trianglelefteq \Gamma$ and $C_\Gamma(Y) \leq Y$ (Y is centric in Γ).

A *reduced setup* is a general setup (Γ, S, Y), such that $Y = O_p(\Gamma)$, $C_S(Z(Y)) = Y$, and $O_p(\Gamma/C_\Gamma(Z(Y))) = 1$.

Note that $Z(Y)$ is an abelian discrete p-toral group by [**BLO3**, Lemma 1.3], and that $\Gamma/C_\Gamma(Z(Y))$ is finite since $\Gamma_0 \leq Y$. Also, $\mathcal{F}_S(\Gamma)$ is a saturated fusion system by Proposition 1.5. The next result and its proof are essentially the same as [**O**, Proposition 3.2], but some modifications are needed to take into account the more general concept of best offenders.

Despite the similarity to Oliver's proof, it is remarkable that the Meierfrankenfeld-Stellmacher classification of (finite) FF-offenders, exploited through [**O**, Proposition 4.5], is sufficiently strong to prove our more general result involving best offenders on abelian discrete p-toral groups. See the remarks at the end of section 4.1. Due to the importance of [**O**, Proposition 4.5], let us quote it.

PROPOSITION [**O**, Prop. 4.5]. *Let G be a non-trivial finite group with $O_p(G) = 1$ and let V be a faithful $\mathbb{F}_p[G]$-module. Let $\mathcal{U}$ be the set of quadratic best offenders in G on V and assume that $G = \langle \mathcal{U} \rangle$. Set $K = O^p(G)$ and $W = C_V(K)[K, V]/C_V(K)$. Assume, for some p-subgroup $P \leq G$, some $\mathbb{F}_p[N_G(P)/P]$-module $X \leq C_W(P)$, and some $k \geq k(p)$ that $\Lambda^k(N_G(P)/P; X) = 0$. Then each $U \in \mathcal{U}$ is conjugate in G to a subgroup of P.*

PROPOSITION 3.2 ([**O**, Prop. 3.2]). *Let (Γ, S, Y) be a reduced setup, and set $D = Z(Y)$. Assume that the finite group $G = \Gamma/C_\Gamma(D)$ is generated by its quadratic best offenders on D. Set $\mathcal{F} = \mathcal{F}_S(\Gamma)$, and let $\mathcal{R} \subseteq \mathcal{F}^c$ be the set of all $R \in \mathcal{F}^c$, such that $Y \leq R$ and $J(R, D) = Y$. Then $L^{k(p)}(\mathcal{F}^c; \mathcal{R}) = 0$.*

Proof. By definition any best offender in G on D acts trivially on D_0, and since they generate G, it follows that $C_D(\Gamma) \geq D_0$. Set $V = \Omega_1 D$. By Lemma 2.7(b), G acts faithfully on $\Omega_n D$ for some $n \geq 1$. By Definition 3.1, $O_p(G) = 1$, so by [**O**, Lemma 1.15] G acts faithfully on V.

For any $H \leq \Gamma$, let $\bar{H}$ denote its image in G. By definition 3.1, $Y = C_S(D) \in \mathrm{Syl}_p(C_\Gamma(D))$. If $Y \leq P \leq S$, then $P \in \mathrm{Syl}_p(PC_\Gamma(D))$, and Lemma 1.3(b) applied to $PC_\Gamma(D) \trianglelefteq N_\Gamma(PC_\Gamma(D))$ implies that $N_\Gamma(PC_\Gamma(D)) = N_\Gamma(P)C_\Gamma(D)$. Therefore

$$Y \leq P \leq S \quad \Rightarrow \quad N_G(\bar{P}) = \overline{N_\Gamma(P)}, \text{ and } |C_\Gamma(D) : C_P(D)| < \infty \text{ is prime to } p. \qquad (3.1)$$

Note that if $R \in \mathcal{R}$, then $Y \leq R$, and since Y is centric in Γ, so is R, and hence $\mathrm{Out}_\mathcal{F}(R) \cong N_\Gamma(R)/R$. Fix the prime p, and set $k = k(p)$ for short. We will now show that $\Lambda^k(N_\Gamma(R)/R; Z(R)) = 0$ for all $R \in \mathcal{R}$. This will complete the proof by applying Corollary 1.17.

Consider some $R \in \mathcal{R}$. Then $Y \leq R$ and since Y is centric in Γ, so is R. It follows that $Z(R) \leq Z(Y) = D$, and hence $C_D(R) = Z(R)$ since $D \leq R$. Since $D_0 \leq C_D(\Gamma)$, Lemma 2.7(a) implies that $C_D(R) = C_{\Omega_m D}(R) + D_0$ for some sufficiently large m. It follows that $N_\Gamma(R)/R$ acts trivially on $C_D(R)/(C_{\Omega_m D}(R)) \cong D_0/\Omega_m(D_0)$. By Propositions 1.10(a,b), $\Lambda^k(N_\Gamma(R)/R; Z(R)/C_{\Omega_m D}(R)) = 0$ (since $k \geq 1$). Proposition 1.10(d) implies that to complete the proof it suffices to show that

$$\Lambda^k(N_\Gamma(R)/R; C_{\Omega_i D}(R)/C_{\Omega_{i-1} D}(R)) = 0 \qquad \text{for all } 1 \leq i \leq m.$$

Set $V_i = \Omega_i D/\Omega_{i-1} D$. Each V_i can be identified with an $\mathbb{F}_p[G]$-submodule of V via the maps $V_i \xrightarrow{[x] \mapsto p^{i-1}x} \Omega_1(D) = V$. Hence, it remains to show that

$$\Lambda^k(N_\Gamma(R)/R; X) = 0 \qquad \forall R \in \mathcal{R}, \quad \forall N_\Gamma(R)\text{-invariant } X \leq C_V(R).$$

The action of Γ on D, and hence its action on V, factors through G. If $R \in \mathcal{R}$, then by (3.1) $\overline{N_\Gamma(R)} = N_G(\bar{R})$, and the kernel of the epimorphism $N_\Gamma(R)/R \to N_G(\bar{R})/\bar{R}$ is finite of order prime to p. So by Lemma 1.11 we need to show that

$$\Lambda^k(N_G(\bar{R})/\bar{R}; X) = 0 \qquad \forall R \in \mathcal{R}, \quad \forall N_G(\bar{R})\text{-invariant } X \leq C_V(\bar{R}).$$

Set $W_1 = C_V(O^p(G))$, $W_2 = W_1[O^p(G), V]$, and $W = W_2/W_1$. The action of G on W_1 and on V/W_2 factors through the p-group $G/O^p(G)$. For any $R \in \mathcal{R}$, and any $N_G(\bar{R})$-invariant

$W' \leq C_{W_1}(\bar{R})$ and $W'' \leq C_{V/W_2}(\bar{R})$, Propositions 1.10(a,b,c) imply that $\Lambda^k(N_G(\bar{R})/\bar{R}; W') = 0$ and $\Lambda^k(N_G(\bar{R})/\bar{R}; W'') = 0$ (since $k \geq 1$). Now, for any $X \leq C_V(\bar{R})$ the inclusion $(X \cap W_2)/(X \cap W_1) \leq W$ and the exact sequence of Proposition 1.10(d) applied to the short exact sequences $0 \to X \cap W_2 \to X \to XW_2/W_2 \to 0$ and $0 \to X \cap W_1 \to X \cap W_2 \to X \cap W_2/X \cap W_1 \to 0$ reduce the problem to showing that

$$\Lambda^k(N_G(\bar{R})/\bar{R}; X) = 0, \quad \forall R \in \mathcal{R}, \quad \forall N_G(\bar{R})\text{-invariant } X \leq C_W(\bar{R}).$$

Assume this is not the case for some $R \in \mathcal{R}$ and $X \leq C_W(R)$. Proposition 1.10(a) implies that $G \neq 1$. Also, $O_p(G) = 1$ by definition of reduced setups. Now, since G acts faithfully on V, Lemma 2.8(a) implies that (quadratic) best offenders on D are also (quadratic) best offenders on V. Since G is generated by its quadratic best offenders on D, it is generated by its quadratic best offenders on V. Since $G \neq 1$ there must exist a quadratic best offender $1 \neq A \leq G$ on D and therefore on V. By [**O**, Proposition 4.5] A is conjugate in G to a subgroup A' of $\bar{R}$. Then $A' \in \mathcal{A}_D(\bar{R})$ since A is a best offender in G on D. But $J(R, D) = Y$ since $R \in \mathcal{R}$, and $C_R(D) = Y$ since (Γ, S, Y) is a reduced setup, so $J_D(\bar{R}) = 1$. Hence, $A' = 1$, which is a contradiction. $\qquad\square$

The following is a simplified version of [**AKO**, Proposition I.5.4] for virtually discrete p-toral groups.

LEMMA 3.3.   *Let Γ be a virtually discrete p-toral group, with $S \in \mathrm{Syl}_p(\Gamma)$, and set $\mathcal{F} = \mathcal{F}_S(\Gamma)$. Let $Q \leq S$ be fully normalized in $\mathcal{F}$, such that $\Gamma_0 \leq Q$ and $C_\Gamma(Q) \leq Q$. Then $N_S(Q) \in \mathrm{Syl}_p(N_\Gamma(Q))$ and $N_\mathcal{F}(Q) = \mathcal{F}_{N_S(Q)}(N_\Gamma(Q))$.*

Proof.   Since Q is centric in Γ and fully $\mathcal{F}$-normalized, $\mathrm{Out}_S(Q) \cong N_S(Q)/Q$ is a Sylow p-subgroup in $\mathrm{Out}_\mathcal{F}(Q) \cong N_\Gamma(Q)/Q$. Hence $N_S(Q) \in \mathrm{Syl}_p(N_\Gamma(Q))$. Clearly $\mathcal{F}_{N_S(Q)}(N_\Gamma(Q)) \subseteq N_\mathcal{F}(Q)$. Conversely, if $P, P' \leq N_S(Q)$, and $c_g \colon P \to P'$ belongs to $N_\mathcal{F}(Q)$, then c_g extends to a morphism $c_h \colon PQ \to P'Q$, such that $c_h(Q) = Q$. Hence $c_g = c_h|_P$, and $h \in N_\Gamma(Q)$, so the opposite inclusion holds. $\qquad\square$

The statement of the next proposition is identical to that of [**O**, Proposition 3.3]. Its proof requires only minor modifications to the original proof. Those will be spelled out in context.

PROPOSITION 3.4 (cf. [**O**, Prop. 3.3]).   *Let (Γ, S, Y) be a general setup. Set $\mathcal{F} = \mathcal{F}_S(\Gamma)$, $D = Z(Y)$. Let $\mathcal{R} \subseteq \mathcal{S}_p(S)_{\geq Y}$ be an $\mathcal{F}$-invariant interval such that for any $Q \in \mathcal{S}_p(S)_{\geq Y}$, $Q \in \mathcal{R}$ if and only if $J(Q, D) \in \mathcal{R}$. Then $L^k(\mathcal{F}^c, \mathcal{R}) = 0$ for all $k \geq k(p)$.*

Proof.   Notice first that $|\Gamma/Y| < \infty$ and that $\mathcal{R}$ is a finite poset because $\Gamma_0 \leq Y$. Assume the proposition is false. Let $(\Gamma, S, Y, \mathcal{R}, k)$ be a counterexample for which the 4-tuple $(k, \mathrm{ord}(\Gamma), |\Gamma/Y|, |\mathcal{R}|)$ is smallest possible in the lexicographical order. Following the same argument as in [**O**], Step 1 shows that $\mathcal{R} = \{R \leq S | J(R, D) = Y\}$, and Step 2 that $k = k(p)$. Step 3 proves that (Γ, S, Y) is a reduced setup, and in Step 4 one shows that $\Gamma/C_\Gamma(D)$ is generated by quadratic best offenders on D. This is a contradiction to Proposition 3.2, and thus completes the proof. These four steps correspond exactly to the four steps in the proof of [**O**, Proposition 3.3]. The following table gives the necessary changes according to their appearance in each step. The rest of the proof consists of only a few remaining remarks.

Step	Original reference in [O]	Replace with
1	[AKO, Proposition I.5.4]	Lemma 3.3
	Lemma 1.7(a)	Lemma 1.16(a)
	Lemma 1.2	[**BLO6**, Lemma 1.7(c)]
	Lemma 1.12	Lemma 1.19
	Corollary 2.3(a)	Corollary 2.9(a)
	Remark following Definition 2.1	Remark 2.3
2	Lemma 1.7(b)	1.16(b)
3	[AKO, Proposition I.5.4] in paragraph 1	Lemma 3.3
	Lemma 1.14(a)	Lemma 1.3(a)
	Proposition 1.11(b) and (c)	Proposition 1.10(b) and (c)
	Corollary 1.10	Corollary 1.17
	Corollary 2.3(a)	Corollary 2.9
	Lemma 1.12	Lemma 1.19
	Lemma 1.7	Lemma 1.16
4	Theorem 2.5	Theorem 2.11
	Lemma 1.6(b)	Lemma 1.15
	Lemma 1.7(b)	Lemma 1.16
	Lemma 1.13	Lemma 1.20

Step 1: Notice that in the proof in [**O**], $Y_1 \geq Y$, and therefore $\Gamma_0 \leq Y_1$ so (Γ, S, Y_1) is a general setup. Observe also that the notation R_0 for a group in the collection $\mathcal{R}$ is somewhat unfortunate since the subscript 0 should not be confused with our notation in this paper for the identity component of a discrete p-toral group R.

Step 2: Follows verbatim with the given replacement of cross-reference.

Step 3: Note that $C_\Gamma(D) \geq \Gamma_0$ by Lemma 2.1(c), so $\Gamma/C_\Gamma(D)$ is finite and contains $SC_\Gamma(D)/C_\Gamma(D)$ as a Sylow p-subgroup. Also note that Γ_2 defined in [**O**] is discrete toral as a subgroup of one. With respect to the first replacement in this step, notice that Y_2 is fully normalized in $\mathcal{F}$ since it is strongly closed, and $\Gamma_0 \leq Y_2$. The second replacement is appropriate since PY_2 is discrete p-toral by [**BLO3**, Lemma 1.3]. Regarding the third and fourth replacements, note that $G = \mathrm{Out}_\Gamma(P)$ is finite since $P \geq Y$. Finally, with respect to the last replacement, note that $\mathcal{R}_2$ is closed to overgroups in $\mathcal{R}$.

Step 4: With the given replacements, note that $\Gamma_0 \leq Y \leq \Gamma_3$. We also remark that for any $Y \leq R \leq S_3$ we have $R \in \mathcal{R}_3 \Leftrightarrow J(R, D) \in \mathcal{R}_3$ because $\mathcal{R}_3 = \mathcal{R} \cap \mathcal{S}_p(S_3)$ and $J(R, D) \leq R$. $\square$

LEMMA 3.5. *Let $\mathcal{F}$ be a saturated fusion system over a discrete p-toral group S. For any $Q \in \mathcal{F}$ there is $Q' \in Q^{\mathcal{F}}$ such that both Q' and $J(Q')$ are fully normalized.*

Proof. We may assume that Q is fully normalized. By Lemma [**BLO6**, Lemma 1.7] there is an isomorphism $f\colon J(Q) \to Y$ in $\mathcal{F}$, where Y is a fully $\mathcal{F}$-normalized subgroup of S, which extends to $\tilde{f}\colon N_S(J(Q)) \to N_S(Y)$. Set $Q' = \tilde{f}(Q)$ and note that $Y = J(Q')$. Since $N_S(Q) \leq N_S(J(Q))$ and Q is fully normalised, so is Q'. $\square$

We are now ready to state and prove the main result of this paper. The argument is essentially the same as the one used by Oliver in [**O**, Theorem 3.4], but is developed here in the context of fusion systems over discrete p-toral groups, as there are many details that need checking in

the passage from the finite to the infinite case. The main issue we need to address is that $\mathcal{F}^c$ need not have finitely many $\mathcal{F}$-conjugacy classes.

THEOREM 3.6.   *Let $\mathcal{F}$ be a saturated fusion system over a discrete p-toral group S. Then* $H^k(\mathcal{O}(\mathcal{F}^c)^{\mathrm{op}}; \mathcal{Z}_{\mathcal{F}^c}) = 0$ *for all $k \geq k(p)$.*

Proof.   Let $\mathcal{C}$ denote the collection of all $\mathcal{F}$-centric $P \leq S$ such that $P = P^\bullet$, namely $\mathcal{C} = \mathcal{F}^c \cap \mathcal{F}^\bullet$. Then $\mathcal{C}$ is $\mathcal{F}$-invariant by Lemma 1.23, and has finitely many S-conjugacy classes by [**BLO3**, Lemma 3.2].

We choose inductively subgroups $X_0, X_1, \ldots, X_N \in \mathcal{C}$, and $\mathcal{F}$-invariant intervals in $\mathcal{C}$

$$\emptyset = \mathcal{Q}_{-1} \subseteq \mathcal{Q}_0 \subset \mathcal{Q}_1 \subset \mathcal{Q}_2 \subset \cdots \subset \mathcal{Q}_N = \mathcal{C}$$

which are closed under overgroups in $\mathcal{C}$ as follows. Assume that $X_0, \ldots, X_{n-1}$ and $\mathcal{Q}_{n-1}$ for $n \geq 0$ have been defined, and that $\mathcal{Q}_{n-1} \neq \mathcal{C}$. To define X_n and $\mathcal{Q}_n$, consider the following collections in $\mathcal{C}$ (recall Definition 2.2)

$$\mathcal{U}_{n,1} = \{P \in \mathcal{C} \backslash \mathcal{Q}_{n-1} \mid d(P) \text{ is maximal}\}$$
$$\mathcal{U}_{n,2} = \{P \in \mathcal{U}_{n,1} \mid \mathrm{ord}(J(P)) \text{ is maximal}\}$$
$$\mathcal{U}_{n,3} = \{P \in \mathcal{U}_{n,2} \mid J(P) \in \mathcal{F}^c\}$$
$$\mathcal{U}_{n,4} = \begin{cases} \{P \in \mathcal{U}_{n,3} \mid \mathrm{ord}(P) \text{ minimal}\} & \text{if } \mathcal{U}_{n,3} \neq \emptyset \\ \{P \in \mathcal{U}_{n,2} \mid \mathrm{ord}(P) \text{ maximal}\} & \text{if } \mathcal{U}_{n,3} = \emptyset \end{cases}$$

Note that since $\mathcal{C}$ has finitely many S-conjugacy classes, $\mathcal{U}_{n,1}$ and $\mathcal{U}_{n,2}$ are well defined, and since every subset of $\mathcal{C}$ has elements of maximal order, $\mathcal{U}_{n,4}$ is well defined. It is also clear that $\mathcal{U}_{n,1}, \ldots, \mathcal{U}_{n,4}$ are $\mathcal{F}$-invariant since $\mathcal{C}$ and $\mathcal{Q}_{n-1}$ are. Choose X_n to be any subgroup in $\mathcal{U}_{n,4}$, such that X_n and $J(X_n)$ are fully normalized in $\mathcal{F}$ (this is possible by Lemma 3.5). Set

$$\mathcal{Q}_n = \mathcal{Q}_{n-1} \cup \{P \in \mathcal{C} \mid P \geq \varphi(X_n) \text{ for some } \varphi \in \mathrm{Hom}_\mathcal{F}(X_n, S)\},$$
$$\mathcal{R}_n = \mathcal{Q}_n \backslash \mathcal{Q}_{n-1}.$$

We make the following four observations.

(a) $\mathcal{Q}_n$ is an interval in $\mathcal{C}$ which is closed under overgroups since it is the union of two intervals with this property.

(b) Since $\mathcal{C}$ has finitely many $\mathcal{F}$-conjugacy classes ([**BLO3**, Lem. 3.2(a)]), $\mathcal{Q}_N = \mathcal{C}$ for some N.

(c) By construction, if $P \in \mathcal{R}_n$ then $P \geq Q$ for some $Q \in X_n^\mathcal{F}$. By Remark 2.4, $d(X_n) \leq d(P)$ and since $X_n \in \mathcal{U}_{n,1}$ equality must hold and $P \in \mathcal{U}_{n,1}$. Remark 2.4 also implies that $J(Q) \leq J(P)$ and since $J(Q) \cong J(X_n)$ and $X_n \in \mathcal{U}_{n,2}$, it follows that $J(Q) = J(P)$ and $P \in \mathcal{U}_{n,2}$. Thus,

$$P \in \mathcal{R}_n \quad \Rightarrow \quad P \in \mathcal{U}_{n,2} \text{ and } d(P) = d(X_n) \text{ and } J(P) \in J(X_n)^\mathcal{F}. \qquad (3.2)$$

Now suppose that $J(X_n)$ is $\mathcal{F}$-centric, so $X_n \in \mathcal{U}_{n,3}$. By Lemma 1.24, $J(X_n) \in \mathcal{F}^c \cap \mathcal{F}^\bullet = \mathcal{C}$. Since $\mathcal{Q}_{n-1}$ is closed to overgroups in $\mathcal{C}$ and $X_n \notin \mathcal{Q}_{n-1}$, also $J(X_n) \notin \mathcal{Q}_{n-1}$. Since $d(J(X_n)) = d(X_n)$ and $J(J(X_n)) = J(X_n)$ by Remark 2.4 and since $X_n \in \mathcal{U}_{n,2}$, we must have $J(X_n) \in \mathcal{U}_{n,3}$. The minimality of $\mathrm{ord}(X_n)$ now implies

$$J(X_n) \in \mathcal{F}^c \quad \Rightarrow \quad X_n = J(X_n) \qquad (3.3)$$

(d) By Proposition 1.22, it suffices to prove that $H^k(\mathcal{O}(\mathcal{F}^\mathcal{C}); \mathcal{Z}_{\mathcal{F}^c}) = 0$ for all $k \geq k(p)$, where $\mathcal{Z}_{\mathcal{F}^\mathcal{C}}$ is the restriction of $\mathcal{Z}_{\mathcal{F}^c}$ to $\mathcal{O}(\mathcal{F}^\mathcal{C})$. We will show that for every $0 \leq n \leq N$,

$$L^k(\mathcal{F}^\mathcal{C}; \mathcal{R}_n) = 0 \quad \text{for all } k \geq k(p). \qquad (3.4)$$

The theorem then follows by induction on n from Lemma 1.16(a) by showing that $H^k(\mathcal{O}(\mathcal{F}^C); \mathcal{Z}^{\mathcal{Q}_n}_{\mathcal{F}^C}) = 0$.

Thus, in the remainder of the proof we will show that (3.4) holds.

Case 1: $J(X_n) \notin \mathcal{F}^c$. In this case $\mathcal{U}_{n,3}$ must be empty. Any $P \in \mathcal{R}_n$ contains some $Q \in X_n^{\mathcal{F}}$ and we have seen in (3.2) that $P \in \mathcal{U}_{n,2}$. The maximality of $\mathrm{ord}(X_n)$ in $\mathcal{U}_{n,2}$ shows that $Q = P$. It follows that $\mathcal{R}_n$ is the $\mathcal{F}$-conjugacy class of X_n denoted $(X_n)^{\mathcal{F}}$.

By Remark 2.5, $J(X_n)$ is centric in X_n, and by the choice of X_n, $J(X_n)$ is fully normalized in $\mathcal{F}$, and hence it is fully centralized. Since $J(X_n)$ is not $\mathcal{F}$-centric, it follows that $X_n C_S(J(X_n)) > X_n$. Lemma 1.3(a) implies $N_{X_n C_S(J(X_n))}(X_n) > X_n$, so we get an element $g \in N_S(X_n) \backslash X_n$ such that $[g, J(X_n)] = 1$, and by Remark 2.5 also $[g, Z(X_n)] = 1$. Now, $\mathrm{Out}_S(X_n) \cong N_S(X_n)/X_n$ since $X_n \in \mathcal{F}^c$, hence $[c_g] \in \mathrm{Out}_{\mathcal{F}}(X_n)$ is a non-trivial element of p-power order in $C_{\mathrm{Out}_{\mathcal{F}}(X_n)}(Z(X_n))$. We deduce from Proposition 1.10(b) that $\Lambda^*(\mathrm{Out}_{\mathcal{F}}(X_n); Z(X_n)) = 0$ so $L^*(\mathcal{F}^c; \mathcal{R}_n) = 0$ by Corollary 1.17, and therefore (3.4) holds by Proposition 1.22.

Case 2: $J(X_n) \in \mathcal{F}^c$. We have seen in (3.3) that $J(X_n) = X_n$. It now follows from (3.2) and Remark 2.4 that

$$P \in \mathcal{R}_n \Rightarrow \begin{array}{l} d(P) = d(X_n) \text{ and,} \\ \text{if } Q \leq P \text{ and } Q \in (X_n)^{\mathcal{F}} \text{ then } Q = J(P). \end{array} \tag{3.5}$$

Clearly $\mathcal{R}_n = \mathcal{Q}_n \backslash \mathcal{Q}_{n-1}$ is an $\mathcal{F}$-invariant interval in $\mathcal{C}$. We claim that it is, in fact, an interval in $\mathcal{F}^c$. Suppose that $P \leq Q \leq R$ in $\mathcal{F}^c$ and that $P, R \in \mathcal{R}_n$. Then $d(P) \leq d(Q) \leq d(R)$, and (3.2) implies equality. In particular $P_0 = Q_0 = R_0$, and since $R = R^\bullet$, Lemma 1.24 implies that $Q = Q^\bullet$, so $Q \in \mathcal{C}$. Since $\mathcal{R}_n$ is an interval in $\mathcal{C}$, $Q \in \mathcal{R}_n$.

Define the following collection in $\mathcal{F}^c$

$$\tilde{\mathcal{R}}_n = \{P \leq S \mid d(P) = d(X_n), \; J(P) \in (X_n)^{\mathcal{F}}, \text{ and } J(P, Z(J(P))) \in \mathcal{R}_n\}. \tag{3.6}$$

We show that

$$\tilde{\mathcal{R}}_n \text{ is an } \mathcal{F}\text{-invariant interval in } \mathcal{F}^c. \tag{3.7}$$

First, $\tilde{\mathcal{R}}_n$ is $\mathcal{F}$-invariant since if $P \in \tilde{\mathcal{R}}_n$ and if $P' \in P^{\mathcal{F}}$, then $J(P'), J(P) \in X_n^{\mathcal{F}}$, and $J(P', Z(J(P'))) \in \mathcal{R}_n$ because it is $\mathcal{F}$-conjugate to $J(P, Z(J(P)))$ and $\mathcal{R}_n$ is $\mathcal{F}$-invariant. To show that $\tilde{\mathcal{R}}_n$ is also an interval in $\mathcal{F}^c$, suppose that $P \leq Q \leq R$, and $P, R \in \tilde{\mathcal{R}}_n$. Then $d(P) \leq d(Q) \leq d(R)$, and hence they are all equal to $d(X_n)$. Thus $J(P) \leq J(Q) \leq J(R)$, and since $J(P), J(R) \in (X_n)^{\mathcal{F}}$ equality must hold and $J(Q) \in (X_n)^{\mathcal{F}}$. Set $D = Z(J(P))$. By assumption $J(P, D), J(R, D) \in \mathcal{R}_n$. Remark 2.3 applies to $P \leq Q \leq R$, and since $\mathcal{R}_n$ is an interval in $\mathcal{F}^c$, it follows that $J(Q, D) \in \mathcal{R}_n$, and hence $Q \in \tilde{\mathcal{R}}_n$.

Next, we claim that

$$\mathcal{R}_n = \tilde{\mathcal{R}}_n \cap \mathcal{C}. \tag{3.8}$$

Suppose that $P \in \mathcal{R}_n$, and set $D = Z(J(P))$. Then $P \geq Q$ for some $Q \in X_n^{\mathcal{F}}$ by definition of $\mathcal{R}_n$ (in fact, of $\mathcal{Q}_n$) so (3.5) implies that $J(P) \in X_n^{\mathcal{F}}$. By Corollary 2.9(b), $J(P) \leq J(P, D) \leq P$, so $J(P, D) \in \mathcal{R}_n$ since $\mathcal{R}_n$ is an $\mathcal{F}$-invariant interval and $X_n \in \mathcal{R}_n$. Hence $P \in \tilde{\mathcal{R}}_n$ and we deduce that $\mathcal{R}_n \subseteq \tilde{\mathcal{R}}_n$. Next we assume the existence of some $P \in (\tilde{\mathcal{R}}_n \cap \mathcal{C}) \setminus \mathcal{R}_n$ and derive a contradiction. Since $P \in \tilde{\mathcal{R}}_n$ it contains an $\mathcal{F}$-conjugate of X_n, and since $P \in \mathcal{C}$ it follows from the definition of $\mathcal{Q}_n$ that $P \in \mathcal{Q}_n$. But $P \notin \mathcal{R}_n$ so $P \in \mathcal{Q}_{n-1}$ and therefore $P \in \mathcal{Q}_m \backslash \mathcal{Q}_{m-1}$ for some $0 \leq m \leq n-1$. From (3.2), $P \in \mathcal{U}_{m,2}$, $d(P) = d(X_m)$ and $J(P) \in J(X_m)^{\mathcal{F}}$. Also, $J(P)$ is $\mathcal{F}$-conjugate to X_n since $P \in \tilde{\mathcal{R}}_n$, so $J(P) \in \mathcal{F}^c$. Therefore $\mathcal{U}_{m,3} \neq \emptyset$ which implies that $J(X_m) \in \mathcal{F}^c$ since X_m must also belong to $\mathcal{U}_{m,3}$. From (3.3) we deduce that $X_m = J(X_m)$ is $\mathcal{F}$-conjugate to $J(P)$ and hence to X_n, which is an absurd since $m < n$.

Set $\mathcal{T}_n = (\tilde{\mathcal{R}}_n)_{\geq X_n}$. If $P \in \tilde{\mathcal{R}}_n$ then $d(P) = d(X_n)$ and $J(P) \in (X_n)^{\mathcal{F}}$. Since X_n is fully normalised there is an isomorphism $\varphi \in \mathrm{Hom}_{\mathcal{F}}(N_S(J(P)), N_S(X_n))$ such that $\varphi(J(P)) = X_n$

(see [**BLO6**, Lemma 1.7]). Set $P' = \varphi(P)$, and notice that $X_n \leq P' \in \tilde{R}_n$, since $\tilde{R}_n$ is $\mathcal{F}$-invariant. Thus we have shown that every $P \in \mathcal{R}_n$ is $\mathcal{F}$-conjugate to some $P' \in \mathcal{T}_n$. Also, Remark 2.4 shows that every $P \in \mathcal{T}_n$ satisfies $J(P) = X_n$. Hence (cf. (3.5))

$$P \in \mathcal{T}_n \quad \Rightarrow \quad \begin{array}{l} \text{(i) } J(P) = X_n \text{ and,} \\ \text{(ii) if } R \leq P \text{ and } X_n \in R^{\mathcal{F}} \text{ then } R = X_n. \end{array} \tag{3.9}$$

Set $T = N_S(X_n)$, and $\mathcal{E} = N_\mathcal{F}(X_n)$. The fusion systems $\mathcal{E}$ is a saturated fusion system over T by Proposition 1.6. Since X_n is $\mathcal{F}$-centric, it is also $\mathcal{E}$-centric, so Proposition 1.21 applies to $\mathcal{E}$, and there exists a model (Γ, T, X_n) for $\mathcal{E}$, namely Γ is a virtually discrete p-toral group, with $T \in \mathrm{Syl}_p(\Gamma)$, $X_n \trianglelefteq \Gamma$, and $C_\Gamma(X_n) \leq X_n$, such that $\mathcal{E} = \mathcal{F}_T(\Gamma)$. Note that $\Gamma_0 \leq X_n$ since $\mathrm{Out}_\mathcal{F}(X_n) \cong Ga/X_n$ is finite. Thus,

$$(\Gamma, T, X_n) \text{ is a general setup.}$$

Also note that $\mathcal{T}_n \subseteq \mathcal{S}_p(T)$ by (3.9), so $\mathcal{T}_n \subseteq \mathcal{E}^c$.

We obtain the following isomorphisms, where the first uses Proposition 1.22 and (3.8), and the second and third follow from Lemmas 1.19 and 1.18 respectively.

$$L^*(\mathcal{F}^\mathcal{C}; \mathcal{R}_n) \cong L^*(\mathcal{F}^c; \tilde{\mathcal{R}}_n) \cong L^*(\mathcal{E}^c; \tilde{\mathcal{R}}_n \cap \mathcal{E}^c) \cong L^*(\mathcal{E}^c; \mathcal{T}_n), \tag{3.10}$$

Set $D = Z(X_n)$. We now show that for any $P \leq T$ such that $X_n \leq P$,

$$P \in \mathcal{T}_n \quad \Leftrightarrow \quad J(P, D) \in \mathcal{T}_n. \tag{3.11}$$

If $P \in \mathcal{T}_n$ then $X_n = J(P)$, and $J(P, D) \in \mathcal{R}_n$ since $P \in \tilde{\mathcal{R}}_n$. By Corollary 2.9(b) $J(P) \leq J(P, D)$, and we have seen that $\mathcal{R}_n \subseteq \tilde{\mathcal{R}}_n$, hence $J(P, D) \in (\tilde{\mathcal{R}}_n)_{\geq X_n} = \mathcal{T}_n$.

Conversely, suppose that $J(P, D) \in \mathcal{T}_n$. Set $Q = J(P, D)$. Then by (3.9), $d(Q) = d(X_n)$ and $J(Q) = X_n$. Also, $J(P) \leq Q$ by Corollary 2.9(b), and since clearly $Q \leq P$, Remark 2.4 shows that $J(P) = J(Q) = X_n$. Since $Q \in \mathcal{T}_n$, by definition $J(Q, Z(J(Q))) \in \mathcal{R}_n$. But $Z(J(Q)) = D$, so $J(P, D) = J(Q, D) \in \mathcal{R}_n$ by Remark 2.3. Since $D = Z(J(P))$, it follows that $P \in \mathcal{T}_n$.

Finally, Propositions 3.4 applies to (Γ, T, X_n) and $\mathcal{T}_n$ and we deduce that $L^k(\mathcal{E}^c; \mathcal{T}_n) = 0$ for all $k \geq k(p)$. Together with (3.10), this finishes the proof of (3.4). $\qquad\square$

REMARK.   We note that the filtration $\mathcal{Q}_0 \subset \mathcal{Q}_1 \subset \cdots \subset \mathcal{Q}_N$ in the proof of Theorem 3.6 has the property that for any $k \leq \mathrm{rk}(S)$ there is some $m \geq 0$ such that $\mathcal{Q}_m$ contains all the subgroups $P \in \mathcal{C} = \mathcal{F}^c \cap \mathcal{F}^\bullet$ with $\mathrm{rk}(P) \geq k$. That is, first all the subgroups of S belonging to $\mathcal{C}$ of maximal rank are filtered, next are the subgroups of the next lower rank etc. This is clear from the definition of the collections $\mathcal{U}_{n,1}$.

We also point out that the intervals $\tilde{\mathcal{R}}_n$ in $\mathcal{F}^c$ which extend the intervals $\mathcal{R}_n$ in $\mathcal{C}$ have to be defined very carefully as we point out in section 4.2.

4.   Comparison with Oliver's argument

As we remarked before, our paper is an adaptation of Oliver's [**O**] to the context of saturated fusion systems over discrete p-toral groups. When generalising statements in the theory of finite fusion systems to the infinite context, there are some delicate issues one has to handle. In this final section we aim to point out what these issues were, and summarise our approach to resolve them. We start with a brief outline of Oliver's proof of the main result in his paper.

Let $\mathcal{F}$ be a saturated fusion system over a finite p-group S. The proof of [**O**, Theorem 3.4] is based on a cleverly chosen filtration of the functor $\mathcal{Z} \colon \mathcal{O}(\mathcal{F}^c)^{\mathrm{op}} \to \mathbf{Ab}$, all of whose quotients have their higher limits vanish above degree $k(p)$ where $k(p) = 1$ if $p > 2$ and $k(p) = 2$ if $p = 2$. It is obtained by filtering $\mathrm{Obj}(\mathcal{F}^c)$ by means of $\mathcal{F}$-invariant collections $\mathcal{Q}_1 \subset \cdots \subset \mathcal{Q}_N = \mathrm{Obj}(\mathcal{F}^c)$, each one of them is closed under formation of overgroups in S. These collections were

first constructed by Chermak [**Ch**]. Each $\mathcal{Q}_n$ gives a quotient functor $\mathcal{Z}^{\mathcal{Q}_n}$ of $\mathcal{Z}$ ([**O**, Definition 1.5]) and the filtration quotients are the subquotients $\mathcal{Z}^{\mathcal{R}_n}$ of $\mathcal{Z}$ supported on the intervals $\mathcal{R}_n = \mathcal{Q}_n \backslash \mathcal{Q}_{n-1}$ ([**O**, Definition 1.4], Definition 1.12). Each $\mathcal{Q}_n$ is obtained from $\mathcal{Q}_{n-1}$ by adding all the subgroups of S which contain an $\mathcal{F}$-conjugate of a certain fully normalised and $\mathcal{F}$-centric $X_n \leq S$ not contained in $\mathcal{Q}_{n-1}$. The subgroup X_n is chosen by a procedure which uses as its input the orders of the subgroup in $\mathcal{F}^c \backslash \mathcal{Q}_{n-1}$ and, most importantly, the order of their Thompson subgroups ([**O**, Definition 2.1]).

The conditions for the choice of the subgroup X_n are such that the intervals $\mathcal{R}_n$ have two possible "types" covered by Cases 1 and 2 in the proof of [**O**, Thm. 3.4]. Intervals $\mathcal{R}_n$ of "the first type" (Case 1) are the $\mathcal{F}$-conjugacy class of X_n, and $\mathrm{Out}_{\mathcal{F}}(X_n)$ contains an element of order p which acts trivially on $Z(X_n)$. This implies the acyclicity of $\mathcal{Z}^{\mathcal{R}_n}$ using the Λ-functor technology ([**BLO2**, Sec. 3], [**O**, Proposition 1.11]). For the "second type" of intervals $\mathcal{R}_n$, the higher limits of $\mathcal{Z}^{\mathcal{R}_n}$ are shown to be the same as $\varprojlim^*_{\mathcal{O}(\mathcal{E}^c)} \mathcal{Z}^{\mathcal{R}}$, where $\mathcal{E} = N_{\mathcal{F}}(X_n)$ is the normaliser fusion system on $N_S(X_n)$, and $\mathcal{R}$ is an interval in $\mathcal{E}$ of overgroups R of X_n characterised by $R \in \mathcal{R} \Leftrightarrow J(R, Z(X_n)) \in \mathcal{R}$. This is where best offenders [**O**, Definition 2.1] get into the picture and their relationship with the Thompson group plays a crucial role. Also, since $\mathcal{E}$ is a constrained fusion system, the "model theorem" [**AKO**, Proposition III.5.10] reduces the calculation of $\varprojlim^*_{\mathcal{O}(\mathcal{E}^c)} \mathcal{Z}^{\mathcal{R}}$ to a question on finite groups. Showing that these higher limit groups vanish above degree $k(p)$ is the content of [**O**, Proposition 3.3] whose proof is inductive: One chooses a minimal counter-example with respect to the degree where the higher limit does not vanish, the order of the model Γ for the fusion system $\mathcal{E}$, the index of X_n in Γ and the size of the collection $\mathcal{R}$. Such a minimal counter-example must have very special properties giving rise to the notion of "reduced setups" ([**O**, Definition 3.1]) and to the conditions of [**O**, Proposition 3.2] which is the key step in Oliver's proof. A minimal counter-example to [**O**, Proposition 3.3] contradicts [**O**, Proposition 3.2], thus completing the proof. The proof of Proposition [**O**, Proposition 3.2] is where the classification of FF-offenders by Meierfrankenfeld and Stellmacher [**MS**] is required, and since this result depends on the classification of finite simple groups, so does the main theorem in Oliver's paper.

To extend Oliver's result to saturated fusion systems $\mathcal{F}$ over discrete p-toral groups S, a number of points had to be addressed:

(1) Define the Thompson subgroups of discrete p-toral groups so that Oliver's arguments go through.
(2) Define best offenders in the context of finite groups acting on abelian discrete p-toral groups.
(3) Extend Meierfrankenfeld-Stellmacher's classification of FF-offenders [**MS**] to the context of action on discrete p-toral group (probably not a viable option) or, preferably, find a way to reduce to the finite case.
(4) Find a way to address the problem arising from the fact that there are infinitely many $\mathcal{F}$-conjugacy classes of $\mathcal{F}$-centric subgroups of S and hence, the filtration as defined in the proof of [**O**, Theorem 3.4] may not be finite.

4.1. Thompson group and best offenders on discrete p-toral groups

Extending the definition of the Thompson subgroup ([**O**, Definition 2.1]) is straightforward. One only needs to replace the notion of the order of finite groups with the order of discrete p-toral groups (Definition 2.2(a)).

Extending the definition of best offenders seems more subtle. At first, the requirement $D_0 \subseteq C_D(A)$ in Definition 2.2(b) of best offenders may seem contrived. But it is, in fact, a very natural requirement. The definition [**O**, Definition 2.1] of when a finite group A is a best offender on

a finite abelian p-group D can be rephrased as the requirement that for any $B \leq A$,

$$\text{ord}(A/B) \geq \text{ord}(C_D(B)/C_D(A)).$$

If D is replaced with a an abelian discrete p-toral group, this inequality still makes sense (with the more general concept of "order"). Setting $B = 1$, and since A is finite, we see that $\text{rk}(D) = \text{rk}(C_D(A))$, hence $D_0 \subseteq C_D(A)$.

Still, it may be surprising that this strong condition we impose on best offenders is not too strong in order to deduce Proposition 3.4 (cf. [**O**, Proposition 3.3]) from Proposition 3.2 (cf. [**O**, Proposition 3.2]). Indeed, the hypotheses of Proposition 3.2 imply that the reduced setup (Γ, S, Y) has the property that $D_0 \leq Z(\Gamma)$, which seems like a very strong restriction. There is no good intuitive reason we are able to give to explain why our simple minded definition of best offenders is not too restrictive. However, the reason why this still works lies in the structure of Oliver's proof. Recall that $J(G, D)$ is defined as the subgroup in G which is the preimage of the subgroup of $G/C_G(D)$ generated by its best offenders on D. Let $J_q(G, D)$ be defined similarly using quadratic best offenders. The point is that modulo [**O**, Proposition 3.2] and independent group theoretic arguments, the proof of the main result [**O**, Theorem 3.4] is deduced formally only from the following formal properties of $J(G, D)$:
  (1) If $G \leq G_1$ and $D \leq G$ is normal in both, then $J(G, D) \leq J(G_1, D)$.
  (2) If $D \trianglelefteq G$ and if $U \leq D$ is normal in G then $J(G, U) \geq J(G, D)$ ([**O**, Cor. 2.3](a))
  (3) If G is a p-group and $D \trianglelefteq G$ is abelian then $J(G) \leq J(G, D)$ ([**O**, Cor. 2.3](b))
  (4) If $D \trianglelefteq G$ is an abelian p-group and $J(G, D) \gneq C_G(D)$ then $J_q(G, D) \gneq C_G(D)$ (Timmesfeld replacement).
Thus, no matter what the definition for (quadratic) best offenders on abelian discrete p-toral groups is, as long as it leads to these properties of $J(G, D)$, is sufficient to prove Theorem 3.6. Our definition of best offenders has these properties. This may explain the mystery behind how Proposition 3.2 implies Proposition 3.4 despite its seemingly unreasonably strong hypotheses. The strength of Oliver's proof is that it is only [**O**, Proposition 3.2] where the actual definition of best offenders and their classification in [**MS**] play a role.

The requirement that $D_0 \subseteq C_D(A)$ when A is a best offender on D is crucial for the proof of Proposition 3.2, that is our analogue of [**O**, Proposition 3.2]. This condition implies that for some sufficiently large $n \geq 1$, the action of A on $D/\Omega_n(D)$ is trivial, where $\Omega_n(D)$ is the set of elements $x \in D$ whose order divides p^n. This makes it possible to only use the obvious finite filtration of $\Omega_n D$ via $\{\Omega_k(D)\}_{k=1}^n$ in order to prove the vanishing of the higher limit groups. Even more importantly, it allows us, via Lemma 2.7 to go back and forth between best offenders on infinite discrete p-toral groups D and best offenders on the finite groups $\Omega_n(D)$. In this way we were able to take advantage of Meierfrankenfeld-Stellmacher's classification of best offenders through Oliver's [**O**, Proposition 4.5]. This is a somewhat surprising aspect of this paper – that the classification of best offenders on finite groups is enough to deal with the infinite case.

4.2.   *Reduction to finitely many $\mathcal{F}$-conjugacy classes*

The filtration in the proof of Theorem 3.6 may end up being infinite if S is discrete p-toral. To overcome this problem one replaces the category $\mathcal{O}(\mathcal{F}^c)$ with the full subcategory $\mathcal{O}(\mathcal{F}^{c\bullet})$ described in section 1.6. This is a standard procedure, but in our application not quite hassle-free. The problem is that the arguments in Proposition 3.4 (cf. [**O**, Proposition 3.3]) uses normaliser fusion sub-systems which don't behave well with respect to the collection $\mathcal{F}^\bullet$ – it is not even clear that $X_n \in \mathcal{F}^\bullet$ implies that $N_S(X_n) \in \mathcal{F}^\bullet$. So we need to extend the intervals $\mathcal{R}_n$ in $\mathcal{F}^c \cap \mathcal{F}^\bullet$ to intervals $\tilde{\mathcal{R}}_n$ in $\mathcal{F}^c$ as defined in (3.6). We point out that $\tilde{\mathcal{R}}_n$ is not a "naïve" extension of $\mathcal{R}_n$: if we took the smallest interval in $\mathcal{F}^c$ containing $\mathcal{R}_n \subseteq \mathcal{F}^{c\bullet}$ we would end up with an interval which is too small for the argument to continue; The largest interval in $\mathcal{F}^c$

containing all the overgroups of subgroup $R \in \mathcal{R}_n$ which are not elements in the previous step of the filtration is too big.

References

AKO M. Aschbacher, R. Kessar, B. Oliver, Fusion systems in algebra and topology, Cambridge Univ. Press (2011)

BCGLO C. Broto, N. Castellana, J. Grodal, R. Levi, B. Oliver, *Subgroup families controlling p-local finite groups*. Proc. London Math. Soc. (3) 91 (2005), no. 2, 325–354.

BLO2 C. Broto, R. Levi, B. Oliver, *The homotopy theory of fusion systems* J. Amer. Math. Soc. 16 (2003), 779–856

BLO3 C. Broto, R. Levi, B. Oliver, *Discrete models for the p-local homotopy theory of compact Lie groups and p-compact groups*, Geometry & Topology 11 (2007) 315–427

BLO6 C. Broto, R. Levi, B. Oliver, *An algebraic model for finite loop spaces*, to appear.

CE H. Cartan, S. Eilenberg, Homological algebra. Princeton University Press, Princeton, N. J., 1956.

Ch A. Chermak, *Fusion systems and localities*, Acta Math. 211 (2013), no. 1, 47–139.

Fu L. Fuchs, Infinite abelian groups. Vol. I. Pure and Applied Mathematics, Vol. 36 Academic Press, New York-London 1970

G A. Gonzalez, *The structure of p-local compact groups*, PhD thesis, Universitat Autonoma de Barcelona, 2010.

JMO S. Jackowski, J. McClure, B. Oliver, *Homotopy classification of self-maps of BG via G-actions. II.* Ann. of Math. (2) 135 (1992), no. 2, 227–270.

MS U. Meierfrankenfeld, B. Stellmacher, *The general FF-module theorem*, J. Algebra 351 (2012), 1–63.

O B. Oliver, *Existence and uniqueness of linking systems: Chermak's proof via obstruction theory*, Acta Math. 211 (2013), no. 1, 141–175.

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