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Distribution of the delay time and the dwell time for wave reflection from a long random potential

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Abstract. We re-examine and correct an earlier derivation of the distribution of the Wigner phase delay time for wave reflection from a long one-dimensional disordered conductor treated in the continuum limit. We then numerically compare the distributions of the Wigner phase delay time and the dwell time, the latter being obtained by the use of an infinitesimal imaginary potential as a clock, and investigate the effects of strong disorder and a periodic (discrete) lattice background. We find that the two distributions coincide even for strong disorder, but only for energies well away from the band-edges.

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1 Introduction

The delay time associated with potential scattering is one of the important quantities related efficiently to the dynamical aspect of scattering in quantum mechanics. One of the common measures for this quantity is the Wigner phase (ϕ) delay time ($T_\phi = \hbar(\partial\phi/\partial E)$) [1], which essentially entails following a fiducial feature such as the peak of the wavepacket as it traverses the scattering region. This procedure is, however, rendered meaningless under conditions of strong distortion of the wavepacket by the scattering potential [2,3]. Further, there is the problem of indentifying the position of the particle with the peak of the wavepacket. Several researchers have made other proposals for identifying a physically *meaningful* timescale of interaction of the particle with the scattering potential (see for recent reviews [4,5]). These include the quantum clocks that utilize the co-evolution, in a locally applied infinitesimal field / potential, of an extra degree of freedom (such as the spin [6]) attached to the traversing particle. Even these proposals are not completely free from problems [4,5,7,8]. One related quantity that has, however, remained uncontroversial is the dwell time obtained with the 'non-Unitary' clock [3,7–9] involving absorption/amplification due to a locally applied infinitesimal imaginary potential (V_i)

$$\tau_d^R = \frac{\hbar}{2}\lim_{V_i\to 0}\frac{\partial \ln|R|^2}{\partial V_i}, \qquad (1)$$

for the case of total reflection ($|R|^2 = 1$). In this case, it also turns out to be the average Smith dwell time $\tau_d = (1/j)\int|\psi|^2 \mathrm{d}x$, where j is the incoming flux in the steady state situation, and ψ is the wavefunction in the scattering region of interest. In this paper, we will consider the Wigner phase delay time and the dwell time, given by the non-Unitary clock, for total reflection from a long one-dimensional disordered medium. These times are, however, not self-averaging and one must have their full probability distribution over a statistical ensemble of random samples.

The distribution of these times for the random media has been investigated recently by several workers [9–22]. A delay time distribution that appears universal for wave reflection from a long one-dimensional (one-channel) random system was derived recently by Texier and Comtet [16] in the limit of high energy (E) and weak disorder as

$$P^0_\infty(\tau) = \frac{\alpha}{\tau^2}\exp\left(-\frac{\alpha}{\tau}\right), \qquad (2)$$

where $\alpha = 4(\Delta^2 k)^{-1}$, Δ^2 is the strength of the disorder (see Eq. (6)), and the dimensionless delay time $\tau = ET_\phi/\hbar$. This was later confirmed by Ossipov *et al.* [17] for a discrete random chain.

Earlier, we had derived the distribution of the dwell time for total reflection, *i.e.*, in the insulating limit, using

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the non-Unitary clock for both passive as well as active (absorbing or amplifying) one-dimensional random continuous media [9]. The dwell time distribution obtained by us, under the condition of a random phase approximation which is valid for high energy and weak scattering, coincided exactly with the delay time distribution of Texier and Comtet. However, an earlier calculation by Jayannavar *et al.* [10] for the Wigner phase delay time had obtained a slightly different form from the distribution and it has been speculated [17] that this discrepancy may well be due to the continuum model used by them. In this paper, we first re-examine and correct the calculation of Jayannavar *et al.* for the Wigner phase delay time distribution. We find that the discrepancy noted above arises actually from an inconsistency of the approximations made within the random phase approximation (RPA), and that when the approximations are carried out consistently, their expression reduces to the universal distribution as in equation (2). We then examine numerically the distribution of the delay times and the dwell times for strong disorder using the transfer matrix method for a one-dimensional disordered chain with a one-band tight binding Hamiltonian. We find that the distribution of the Wigner phase delay time and the dwell time, clocked by the non-Unitary clock, agree exactly even in the strong disorder regime, but for energies far away from the band-edge. We further examine the effect of a periodic lattice on the delay time by varying the energy within the band. We find that for strong disorder, and for energies close to a band-edge, the Wigner phase delay time distribution differs considerably from that of the dwell time given by the Non-Unitary clock. The Wigner phase delay time can even become negative under such conditions. The dwell time, however, remains positive as it must for total reflection ($|R| = 1$).

2 Distribution of the Wigner phase delay time for total reflection in the RPA

Here we will re-examine the earlier calculation of Jayannavar *et al.* [10,13] for the distribution of the Wigner phase delay time for total reflection from a one-dimensional (one-channel) disordered continuum. Again, we will begin with the invariant imbedding equation for the reflection amplitude $R(L) = \sqrt{r(L)}\exp[\mathrm{i}\phi(L)]$ given by (in the notations of Ref. [10])

$$\frac{\mathrm{d}R(L)}{\mathrm{d}L} = 2\mathrm{i}kR(L) + \frac{\mathrm{i}k}{2}\eta_r(L)\left[1 + R(L)\right]^2, \qquad (3)$$

where $\eta_r(L) = -V_r(L)/E$ is the normalized fluctuating potential. In the limit of large lengths ($L \gg l_\mathrm{c}$, the localization length), the reflection becomes approximately total ($r(L) \simeq 1$), and equation (3) yields an equation for the reflection phase $\phi(L)$ as

$$\frac{\mathrm{d}\phi}{\mathrm{d}L} = 2k + k\eta_r(L)(1 + \cos\phi). \qquad (4)$$

The equation for the phase delay time $T_\phi = \hbar(\mathrm{d}\phi/\mathrm{d}E) = 1/c_g(\mathrm{d}\phi/\mathrm{d}k)$ (where c_g is the group velocity), is obtained by differentiating the above equation for ϕ with respect to k:

$$\frac{\mathrm{d}T_\phi}{\mathrm{d}L} = \frac{1}{c_g}\left[2 + \eta_r(L)\left(1 + \cos\phi - kc_gT_\phi\sin\phi\right)\right]. \qquad (5)$$

As before, we will assume the random refractive index $\eta_r(L)$ to be a Gaussian white noise with a zero mean, *i.e.*,

$$\langle\eta_r(L)\rangle = 0, \quad \langle\eta_r(L)\eta_r(L')\rangle = \Delta^2\delta(L - L'). \qquad (6)$$

Using the Novikov theorem [23], we can now set up a Fokker-Planck equation for the joint probability distribution function $P(T_\phi, \phi; L)$ over the ensemble of $\eta_r(L)$. However, we will be interested in the marginal probability distribution $P(T_\phi; L)$, which can be obtained by integrating over the phase angle ϕ. To this end, we make the random phase approximation (RPA) and set $P(T_\phi, \phi; L) = P(T_\phi; L)/2\pi$, *i.e.*, assume a factored out uniform distribution over the phase angle ϕ. The RPA is a good approximation for high energy and weak disorder [24]. We obtain the equation for $P(T_\phi; L)$ as

$$\frac{\partial P}{\partial l} = \frac{\partial}{\partial T_\phi}\left[\frac{\partial}{\partial T_\phi}\left(\frac{T_\phi^2}{2} + \frac{3}{2c_g^2k^2}\right) + \left(T_\phi - \frac{4}{c_g\Delta^2k^2}\right)\right]P, \qquad (7)$$

where the dimensionless length $l = L/l_\mathrm{c} = 1/2\ \Delta^2k^2L$. In the limit of large lengths, $l \gg 1$, the distribution saturates and we can set $\partial P/\partial l = 0$. Hence, we obtain the solution [10]

$$P_\infty(\tau_1) = \frac{\lambda\mathrm{e}^{\lambda\tan^{-1}\tau_1}}{(\mathrm{e}^{\lambda\pi/2} - 1)(1 + \tau_1^2)}, \qquad (8)$$

where $\lambda = 8/\sqrt{3}\Delta^2k$ and the dimensionless time $\tau_1 = c_gkT_\phi/\sqrt{3}$. This expression does yield the τ_1^{-2} *universal tail* behaviour for $\tau_1 \to \infty$, but differs from the distribution of dwell times given by equation (2) at short times. The main difference appears at $\tau = 0$, where this expression for $P_\infty(\tau)$ yields a finite value in contrast to equation (2) which gives $P_\infty^0(\tau) = 0$ because of the essential singularity at $\tau = 0$.

This difference is readily traced to the fact that the RPA is good only in the high energy, weak disorder limit. Indeed, if we consistently take the high energy, weak disorder limit in equation (7) (or in Eq. (8)), *i.e.*, by demanding $c_g^2k^2 \to \infty$ and $\Delta^2/c_g \to 0$ with the product $(\Delta^2/c_g)(c_g^2k^2) = 4/\alpha$ a constant, we obtain the solution as

$$P_\infty(\tau) = \frac{\alpha}{\tau^2}\exp\left(-\frac{\alpha}{\tau}\right), \qquad (9)$$

where $\tau = ET_\phi/\hbar$. This is exactly the full universal distribution of delay times obtained in equation (2) [16] for the case of a free electron with $c_g = \hbar k/m$, thus reconfirming again the universal delay time distribution. We note that the above approximations have to be carried out consistently specially for a large group velocity c_g, which is the case for energies far away from the band edges. This also suggests that the condition of weak disorder $\Delta^2 k \ll 1$ for the one-parameter scaling, which assumes a uniform distribution of the phase (RPA), may have to be modified to $\Delta^2 k/(c_g/c_\phi) \ll 1$, where c_ϕ is the phase velocity.

3 Strong disorder and a periodic background: numerical results

The probability distribution of dwell times in reference [9] was derived for a continuum model in the limit of weak disorder and high energy when the RPA is valid. In this section, we will examine these limitations numerically. In particular, we investigate the distributions of dwell times for the case of strong disorder and compare the distributions of the dwell times and the Wigner phase delay times. We will simulate a disordered lattice, instead of a continuum. The underlying lattice will also provide a discrete periodic background in the system, as distinct from a uniform continuum, whose effect on the delay times will be investigated.

In order to go beyond the RPA, we will use the transfer matrix method involving the products of random transfer matrices [25] to simulate the one-dimensional random medium using the one-band tight binding Hamiltonian (the Anderson Hamiltonian) with diagonal disorder [26]. The Hamiltonian describing the motion of a particle on the random lattice can be written as

$$\mathcal{H} = \sum_n [\epsilon_n |n\rangle\ \langle n|\ +\ V(|n\rangle\ \langle n+1| + |n+1\rangle\ \langle n|)] \quad (10)$$

where $|n\rangle$, ϵ_n and V denote, respectively, the non-degenerate Wannier orbital at the nth site, the site energy at the nth site and the hopping matrix element connecting the nearest neighbours separated by a unit lattice spacing. The site energies ϵ_n can be written explicitly in the form of $\epsilon_n - \mathrm{i}\eta$, with the real parts of the site energies assumed to be independent random variables distributed uniformly over the range $[-W/2, W/2]$ for $1 < n < N$ and zero otherwise. This is so that the disordered chain of N sites is embedded in an otherwise infinitely long ordered lattice. The imaginary part in the site energy $(-\mathrm{i}\eta)$ makes the Hamiltonian non-Hermitian and causes the particles to be formally coherently absorbed or emitted depending on the sign of η, which is taken to be constant and non-zero (though infinitesimally small) over the disordered segment $1 < n < N$ and zero elsewhere. Since all the energies can be scaled with respect to V, we will set V to unity.

The reflection (R) and the transmission(T) amplitudes can now be calculated using the transfer matrix method [25]. In order to calculate the Wigner phase delay time, the reflection amplitude $R(E) = \sqrt{r(E)}\exp[-\mathrm{i}\phi(E)]$ is computed at two slightly differing values of the incident wave energy, $E = E_0$ and $E = E_0 + \delta E$, for a conservative chain $(\eta = 0)$. The Wigner phase delay time is then calculated as $T_\phi = \hbar(\mathrm{d}\phi/\mathrm{d}E) = \hbar[\phi(E_0+\delta E) - \phi(E_0)]/\delta E$. Similarly, to calculate the dwell time by applying the imaginary potential, the reflection amplitude is computed at two values of the imaginary site energy ($\eta = 0$ and $\eta = \delta\eta$). Now the dwell time is given by $\tau_d = \hbar/2(\mathrm{d}|R|^2/\mathrm{d}\eta) = \hbar/2[|R(E,\eta=\delta\eta)|^2 - |R(E,\eta = 0)|^2]/\delta\eta$. Typically the values of δE and $\delta\eta$ are 10^{-6} and the stability of the results have been checked for their choice within the range $10^{-5} < \delta E,\ \delta\eta < 10^{-7}$. (We will deal with the delay/dwell time in a dimensionless form by multiplying it by V and setting $\hbar = 1$.) For the calculation of the averages and the distributions, we have typically used 10^5 configurations of the disorder. We will present our results for a long sample ($L \gg l_\mathrm{c}$, *i.e.*, lengths much greater than a localization length).

We will first examine the case of wave energies far away from a band edge ($E = 0.0,\ 1.0$). In Figure 1, we show the distribution of the Wigner phase delay time τ_w and the dwell time τ_d for reflection from a long sample for different values of the disorder strengths ($W = 0.1,\ 2$). For weak disorder ($W = 0.1$), the distributions are identical to each other and also correspond exactly to the universal distribution given by equation (2). It should be noted that the RPA is not valid for exactly the band centre ($E = 0$) due to a well-known anomaly, although it is valid for a generic value of energy within the band [27]. The two distributions also coincide for higher disorder strengths [28]. It is interesting to note that equation (2) still describes the distribution reasonably well for moderately large disorder ($W = 2.0$, see Fig. 1b), though the RPA under which the expression was derived is not valid for these cases. The case of $E = 1$ shows similar behaviour, though the peak occurs at a different value, reflecting the smaller group velocity. In Figure 1d, we plot the distributions of dwell and delay times for cases of symmetric disorder (the random site energy is chosen from the range symmetric about zero $[-W/2, W/2]$) and asymmetric disorder (the site energy can only be positive $[0, W]$ or negative $[-W, 0]$). The distributions for the positive and the negative one-sided, asymmetric disorder appear to be the same, regardless of the sign as expected, of course (we have included these two only as a check on our numerics). These, however, are different from the distribution for the symmetric case. The contribution of the prompt part of the reflection arising from the average potential mismatch at the boundary is clearly seen for the cases of asymmetric disorder in that the peak of the distribution occurs slightly earlier, and there is more weight at early times. The asymmetric disorder model changes on an average the lattice potential locally only over the disordered segment (sample) and *not* globally over the sample and the leads. This gives rise to a potential mismatch at the ends of the disordered chain and cannot be absorbed by a mere shift of the incident energy. More formally, in the case of the symmetric disorder the average of the S-matrix, $\langle S\rangle = 0$, while in the case of the asymmetric disorder, $\langle S\rangle \neq 0$.

Now, we will examine the case of wave energies close to the edge of the band ($E = 1.9,\ 1.99$). In Figure 2, we show distributions of the delay time and dwell times. For the case of weak disorder, again the Wigner delay time distributions and the dwell time distribution coincide.

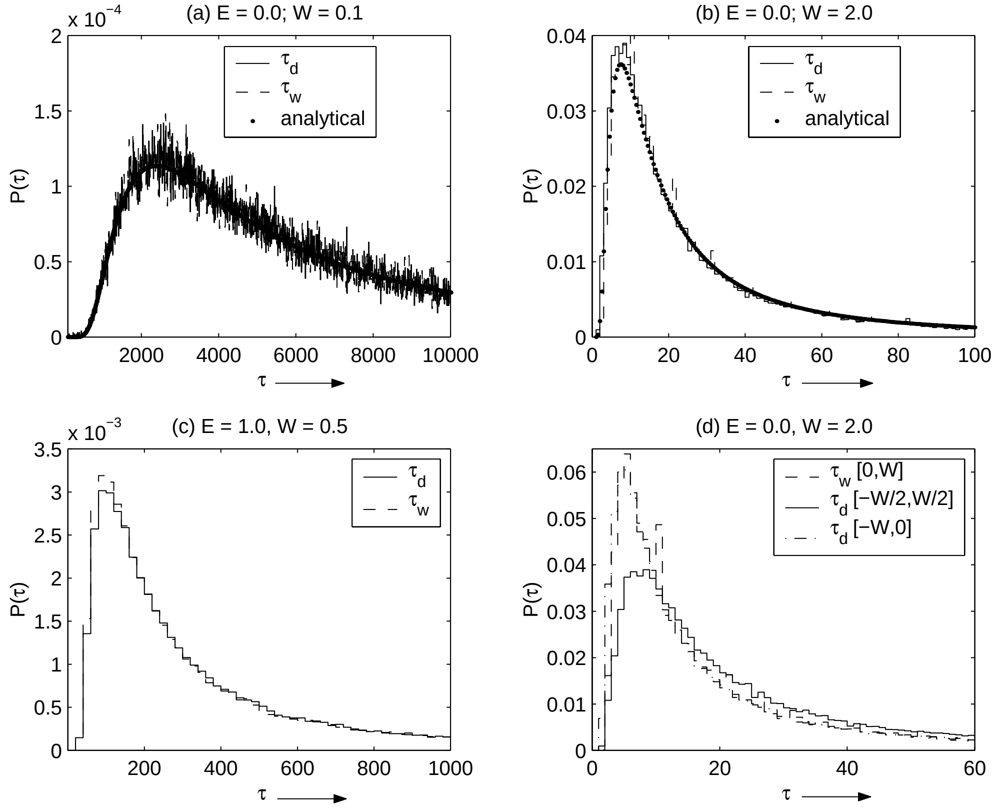

Fig. 1. The distribution of delay and dwell times for reflection from a long disordered passive medium for wave energy at the middle of the band ($E = 0.0,\ 1.0$).

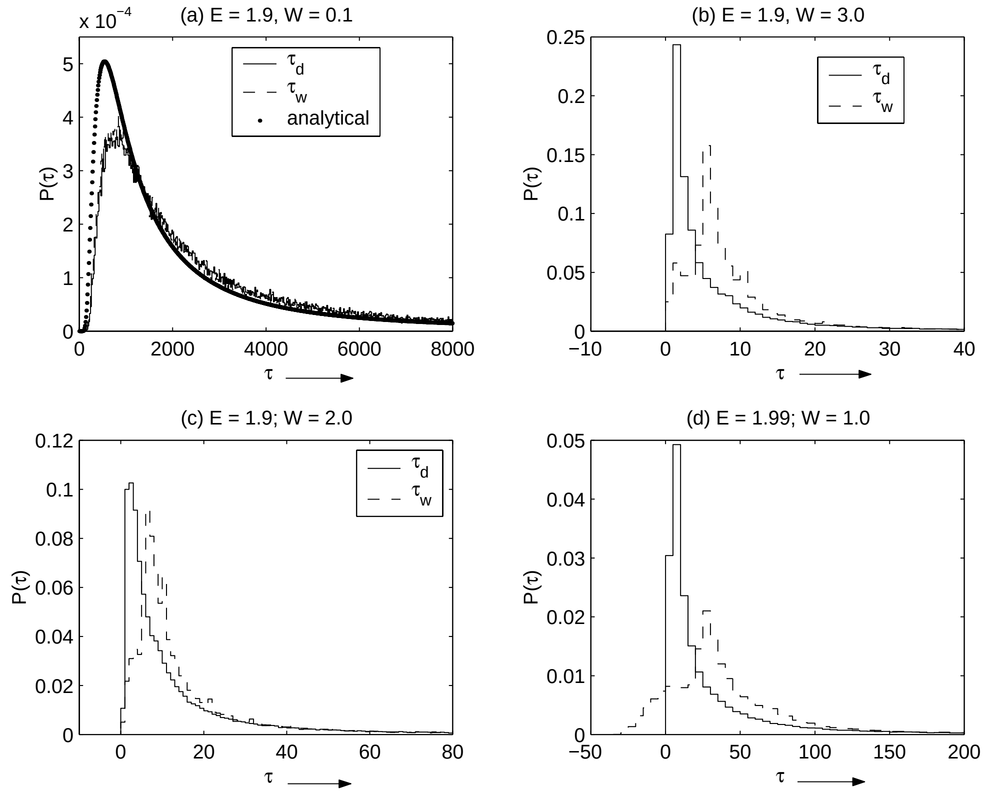

Fig. 2. The distribution of delay and dwell times for reflection from a long disordered passive medium for wave energy close to the edge of the band ($E = 1.9,\ 1.99$).

There is, however, considerable discrepancy from equation (2), as can be seen. We have explicitly verified that the RPA is valid for this case of weak disorder by calculating the distribution of the phase. Thus, the discrepancy cannot be an artefact of the RPA. The most probable reason, perhaps, is that near the band edge, the wave does not penetrate deep enough to fully sample the randomness, before getting reflected promptly. This would call into question the factorization of the joint probability distribution of the phase and its energy derivative, particularly for short times.

For intermediate and strong disorder ($W = 1,\ 2,\ 3$), a more interesting effect occurs. The two distributions, *i.e.*, the Wigner delay time distribution and the dwell time distributions no longer coincide. The difference between the distributions increase with the disorder strength and with their proximity to the band-edge. The Wigner delay time distribution appears quite different from the universal distribution at $E = 0$. Near the band-edge, in fact, for $E = 1.99$ and $W = 1$, the Wigner delay time distribution is non-zero for even negative times. This is, presumably, due to the strong deformation of the wavepacket caused by the strong dispersion near the band-edge. The dwell time distribution given by the 'non-Unitary' clock, however, remains non-zero only for positive times. We also note that the Universal τ^{-2} tail at long times ($\tau \to \infty$) remains unaffected.

4 Conclusions

In conclusion, we have studied the distribution of the delay and the dwell times for reflection from a disordered medium in the limit of total reflection ($|R| = 1$). We have revisited the original calculation of Jayannavar *et al.* [10] for the distribution of the Wigner delay time. We show that, by taking the high-energy limit consistently within the RPA, the correct universal distribution of delay times is reproduced. In the course of the derivation, we note that the single-parameter scaling ansatz for the RPA seems consistent under the condition $\Delta^2 k/(c_g/c_\phi) \ll 1$ (Δ^2 – the disorder strength, c_g – the group velocity and c_ϕ-the phase velocity), instead of $\Delta^2 k \ll 1$ which does not account for the effects of the group velocity. This is in accord with the recent results of reference [17,29]. We have also investigated the distribution of delay times numerically and find the distributions of the Wigner delay time and the dwell time to coincide for energies far away from a band-edge for all disorder strengths. This, however, breaks down for energies close to the band-edge and strong disorder, when the dispersive effects of the band structure deform the wavepacket so much so as to render the description in terms of the motion of a wavepacket meaningless. The concept of a dwell time, clocked by a counter such as the imaginary potential, however, remains meaningful even under such circumstances.

References

1. E.P. Wigner, Phys. Rev. **98**, 145 (1955).
2. M. Büttiker, R. Landauer, Phys. Rev. Lett. **49**, 1739 (1982).
3. M. Büttiker, *Electronic properties of multilayers and low-dimensional semiconductor structures*, edited by J.M. Chamberlain *et al.* (Plenum Press, New York, 1990).
4. E.H. Hauge, J.A. Støvneng, Rev. Mod. Phys. **61**, 917 (1989).
5. Th. Martin, R. Landauer, Rev. Mod. Phys. **66**, 217 (1994).
6. M. Büttiker, Phys. Rev. B **27**, 6178 (1983).
7. Golub, S. Felber, R. Gähler, E. Gutsmiedl, Phys. Lett. A **148**, 27 (1990).
8. See, however, S. Anantha Ramakrishna, N. Kumar, `cond-mat/0009269` (unpublished).
9. S. Anantha Ramakrishna, N. Kumar, Phys. Rev. B **61**, 3163 (2000).
10. A.M. Jayannavar, G.V. Vijayagovindan, N. Kumar, Z. Phys. B **75**, 77(1989).
11. J. Heinrichs, J. Phys. Cond. Matt. **2**, 1559 (1990).
12. V. Gasparian, M. Pollak, Phys. Rev. B **47**, 2038 (1995).
13. S.K. Joshi, A.M. Jayannavar, Solid State Commun. **106**, 363 (1999).
14. S.K. Joshi, A.K. Gupta, A.M. Jayannavar, Phys. Rev. B **58**, 1092 (1998).
15. S.K. Joshi, A.M. Jayannavar, Solid State Commun. **111**, 547 (1999).
16. C. Texier, A. Comtet, Phys. Rev. Lett. **83**, 4220 (1999); A. Comtet, C. Texier, J. Phys. A **30**, 8017 (1997).
17. A. Ossipov, T. Kottos, T. Geisel, Phys. Rev. B **61**, 11411 (2000); F. Steinbach, A. Ossipov, T. Kottos, T. Geisel, Phys. Rev. Lett. **85**, 4426 (2000).
18. B.A. van Tiggelen, P. Sebbah, M. Stoytchev, A.Z. Genack, Phys. Rev. E **59**, 7166 (1999).
19. C.W.J. Beenakker, K.J.H. van Bemmel and P.W. Brouwer, Phys. Rev. E **60**, R6313 (1999).
20. C.W.J. Beenakker, in *Photonics crystals and light localization*, edited by C.M. Soukolis, NATO Science Series (Kluwer, Dordrecht, 2001) (`cond-mat/0009061`).
21. H. Schomerus, K.J.H. van Bemmel, C.W.J. Beenakker, Europhys. Lett. **52**, 518 (2000) (`cond-mat/0004049`).
22. J. Heinrichs, `cond-mat/0106432` (unpublished).
23. E. Novikov, Sov. Phys. JETP **20**, 1290 (1965).
24. C. Barnes, J.M. Luck, J. Phys. A **23**, 1717 (1990); P. Pradhan, `cond-mat/9703255` (unpublished).
25. See, *e.g.*, F.J. Dyson, Phys. Rev. **92**, 1331 (1953); H. Schmidt, Phys. Rev. **105**, 425 (1957); A.K. Gupta, A.M. Jayannavar, Phys. Rev. B **52**, 4156 (1995).
26. E.N. Economou, *Green's functions in quantum physics* (Springer-Verlag, Berlin, 1979).
27. M. Kappus, F. Wegner, Z. Phys. **45**, 15 (1981); B. Derrida, E.J. Gardner, J. Phys. France **45**, 1283 (1984); A.D. Stone, D.C. Allan, J.D. Joannopoulos, Phys. Rev. B **27**, 836 (1983).
28. Note that though the two distributions over the ensemble might coincide, the phase delay time and the dwell time for a given configuration need not be equal and, in fact, are not equal.
29. L.I. Deych, A.A. Lisyansky, B.L. Altshuler, Phys. Rev. Lett. **84**, 2678 (2000).