The conditional tunneling time for reflection using the WKB wave-function

S. Anantha Ramakrishna[*(a)] and A.M. Jayannavar[(b)†]

[(a)] Raman Research Institute, C.V. Raman Avenue, Bangalore 560 080, India.

[(b)] Institute of Physics, Sachivalaya Marg, Bhubaneswar 751 005, India

We derive an expression for the conditional time for the reflection of a wave from an arbitrary potential barrier using the WKB wavefunction in the barrier region. Our result indicates that the conditional times for transmission and reflection are equal for a symmetric barrier within the validity of the WKB approach.

In this paper, we deal with the dynamical aspect of scattering, namely, the time delay undergone by a wave in the process of being scattered by a potential. Obviously, this time is related to the actual time of sojourn in the scattering region. For the case of a classical particle, both the times will be one and the same. But it is not obvious that the equality would hold for a wave. This time of sojourn during the scattering is of interest for mesoscopic systems, and has a long and controversial history (See Ref. [1–5] for recent reviews). The deformable nature of a wave-packet makes it difficult to accept the Wigner phase (ϕ) delay time [$\tau_w = \hbar(\partial\phi)/(\partial E)$] [6], based on following the motion of a fiducial feature such as the peak on the wave-packet, as causally related to the actual time of sojourn in the spatial region of interest [7,8]. Hence, several authors have made other proposals for identifying a physically *meaningful* timescale of interaction of the quantum particle with the scattering potential. These include the quantum clocks that utilize the co-evolution, in a locally applied infinitesimal field / potential, of an extra degree of freedom (such as the spin [9]) attached to the traversing particle. Even these proposals are not completely free from problems [2,4,10,11].

One strongly debated issue, among others, has been the conditional scattering time, conditional upon the incoming and outgoing channels. Thus, we have the conditional transmission or reflection time in the one-dimensional case. The case of transmission (tunneling for sub-barrier energies) has been dealt with extensively by several researchers using different methods [7,9,12–17]. There is near unanimous agreement that the traversal time for tunneling across a nearly opaque barrier of width L and height V_0 is given by the Büttiker-Landauer time [7]:

$$\tau_{BL} = \frac{mL}{\hbar\kappa}, \tag{1}$$

where m is the mass of the particle and $\hbar\kappa = \sqrt{2m(V_0 - E)}$. This can, of course, be generalized to an arbitrary potential as an integral over infinitesimally wide rectangular barriers, if the law of addition of scattering times for non-intersecting spatial regions holds. But there is no such universal agreement about the time for reflection. In this paper, we will use the WKB wave-function in the barrier region for a tunneling particle to obtain an expression for the conditional time of reflection from an arbitrary potential. In an earlier work [13], this approach was successfully used to determine the traversal time in agreement with the Büttiker-Landauer time. Our current work indicates that the conditional time for Reflection is the same as the conditional traversal time for a symmetric barrier within the limits of the WKB approximation.

Let us consider an arbitrary (static) potential $V(x)$ as shown in Fig. 1, with the wave at an energy E incident from the left. The wave is now partially transmitted to the right with a probability amplitude T, and partially reflected to the left with a probability amplitude R. Within the barrier region the wave function $\psi(x)$ of the particle is a superposition of growing and decaying real functions (which are purely exponential for a rectangular barrier). The current density at any point, $j(x) = (\hbar/2im)[\psi^*(d\psi/dx) - \psi(d\psi^*/dx)]$, is non-zero only due to the complex coefficients differing by a phase, in the superposition of real functions. Now one can associate a total velocity field $v(x)$ with the particles, given by $j(x) = v(x)P(x)$, where $P(x) = \psi^*(x)\psi(x)$ is the probability density. This turns out to be the same the velocity field obtained in the Bohmian view [18,19]. Using the WKB wave-function in the barrier region [20],

$$\psi(x) = \frac{A}{\sqrt{p(x)}} \left[\frac{1}{2} \exp\left(-\int_x^a \frac{p(x')}{\hbar} dx' \right) - i \exp\left(\int_x^a \frac{p(x')}{\hbar} dx' \right) \right], \tag{2}$$

---

where $p(x) = \sqrt{2m(V(x) - E)}$ and A is a complex normalization coefficient, one can evaluate the velocity field $v(x)$:

$$v(x) = \frac{p(x)}{m} \left[\frac{1}{4} \exp\left(-2 \int_x^a \frac{p(x')}{\hbar} dx' \right) + \exp\left(2 \int_x^a \frac{p(x')}{\hbar} dx' \right) \right]^{-1}, \tag{3}$$

$$\simeq \frac{p(x)}{m} \exp\left(-2 \int_x^a \frac{p(x')}{\hbar} dx' \right). \tag{4}$$

The latter approximation is valid because $\int p(x)/\hbar dx \gg 1$ inside the barrier, which is the essence of the WKB approximation.
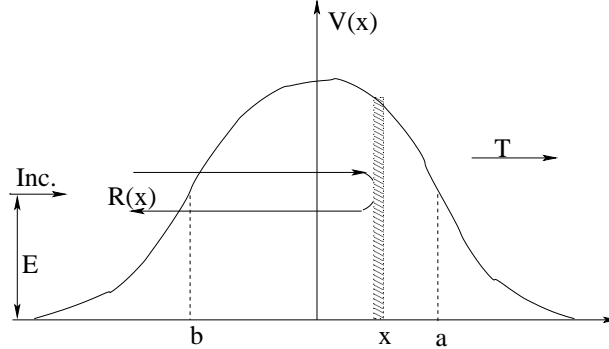

FIG. 1. A schematic of the potential showing the classical turning points and the partial reflections in the WKB approach.

Now at any point, x, inside the barrier, the forward (moving to the right) particles would be reflected backwards (to the left) due to the potential barrier extending from x to a with a probability $\mathcal{R}(x)$ (in this WKB picture there are no multiple scatterings). By noting that the speed at a given point would only depend on the potential at that point, the total velocity field can be decomposed into a forward and a backward velocity field:

$$v(x) = \frac{p(x)}{m} - \mathcal{R}(x)\frac{p(x)}{m} = \mathcal{T}(x)\frac{p(x)}{m}. \tag{5}$$

Now in Eq. (4), the exponential term is nothing but the transmission probability ($\mathcal{T}(x)$) for the barrier extending from x to a in the WKB approximation. Hence the forward velocity was identified as $p(x)/m$, giving the traversal time as

$$\tau^{(T)} = \int_b^a \frac{m}{p(x')} dx'. \tag{6}$$

For a rectangular barrier, this reduces to the Büttiker-Landauer time. It should be noted that such an identification of the forward velocity field holds within the WKB picture and might not possible in general. We also point out that the form of $v(x)$ in Eq. 5 naturally leads to a simple interpretation in the form of a symmetric statistical composition law for the velocity field without any quantum interference terms.

Now we will give an expression for the conditional reflection time within the WKB picture. The reflection time for tunneling can be defined as a properly weighted sum over the transit time of the particles partially reflected from each point within the barrier as

$$\tau^{(R)} = \int_b^a 2 \left[\int_b^x \frac{m \, dx'}{p(x')} \right] \frac{\mathcal{R}(x) \, dx}{\int_b^a \mathcal{R}(x) dx}. \tag{7}$$

In other words, the particle reflected at a point x (see Fig. 1) will take a time of $2 \int_b^x p(x')/m \, dx'$ to go upto the point x and back. $\mathcal{R}(x)$ is the (probability) reflection coefficient of the barrier extending from only x upto a, and in the WKB approximation is given by [18]

$$\mathcal{R}(x) = 1 - \exp\left[-2 \int_x^a p(x'')/\hbar \, dx'' \right]. \tag{8}$$

Now Eqs. (7) and (8) constitute the complete expression for the conditional reflection time.

Next, we will proceed to calculate the reflection time for two symmetric potentials (there exists a centre of reflection), *viz.*, a rectangular potential barrier of height V_r and width L, and a parabolic potential barrier $V(x) = -(1/2)\,\omega^2 x^2$. For the rectangular barrier, we obtain the reflection delay time as

$$\tau^{(R)} = \frac{mL}{p_0 N} - \frac{m\hbar}{p_0^2 N} + \frac{2m}{p_0 N}(\frac{\hbar}{2p_0})^2 \left(1 - e^{-2p_0 L/\hbar}\right),\tag{9}$$

where $N = [1 + \hbar/2p_0 L(1 - e^{-2p_0 L/\hbar})]$ and $p_0 = \sqrt{2m(V_r - E)}$. We note that for a sufficiently wide barrier $p_0 L/\hbar \gg 1$, and the reflection time can be expanded in powers of $\hbar/p_0$. To the zeroeth order, $\tau^{(R)} = \tau^{(T)} = mL/p_0$, *i.e.*, the reflection time is the same as the transmission time. In the case of the parabolic barrier, the transmission time is $\tau^{(T)} = \pi\sqrt{m}/\omega$ and the reflection time is

$$\tau^{(R)} = \frac{\pi\sqrt{m}}{\omega} + \frac{2\sqrt{m}}{\omega} \int_{-2E/\sqrt{\omega}}^{2E/\sqrt{\omega}} \frac{\sin^{-1}\left(\frac{\omega x'}{2E}\right) \mathcal{R}(x')\, dx'}{\int_{-2E/\sqrt{\omega}}^{2E/\sqrt{\omega}} \mathcal{R}(x')\, dx'}\tag{10}$$

where

$$\mathcal{R}(x) = 1 - \exp\left(\frac{-\pi E\sqrt{m}}{\hbar\omega}\right) \exp\left[\frac{2E\sqrt{m}}{\hbar\omega}\left(\frac{\omega x}{\sqrt{2E}}\sqrt{1 - \omega^2 x^2/2E} + \sin^{-1}(\omega x/\sqrt{2E})\right)\right]\tag{11}$$

$\mathcal{R}(x) \sim 1$ for a reasonably small ω or a broad potential. Then the second part of the expression (10) for $\tau^{(R)}$ is negligible, giving $\tau^{(R)} = \tau^{(T)}$. For $\mathcal{R}(x) < 1$, the reflection time is slightly lesser than the transmission time. Thus within the validity of the WKB approach, the reflection and the transmission times are roughly equal in this case also.

We now discuss the significance of our result for the reflection time for a symmetric barrier by comparing with other known timescales. The Wigner phase delay time is the same for reflection or traversal for a symmetric barrier, and does not distinguish between reflected and transmitted particles. The conditional reflection and traversal times calculated by using a local infinitesimal imaginary potential as a clock [5,10,11,21] can also be shown to be equal for any arbitrary symmetric potential. The spin rotation time (τ_z) defined by Büttiker [9], however, yields different reflection and transmission times in general for any potential (symmetric or otherwise), although the spin precession times τ_y for reflection and transmission are equal. But it should be noted that there is a relation between the z-components of the spin of the transmitted and the reflected waves due to conservation of angular momentum. This makes it difficult to define separate conditional reflection and transmission times using the spin-rotation (τ_z).

In conclusion, we have given an expression for the conditional reflection time for barrier tunneling using the WKB wave-function in the barrier region. The conditional reflection time appears to be the same as the conditional traversal time for a symmetric barrier within the validity of the WKB picture.

[1] R.Y. Chiao and W. Steinberg, in *Progress in Optics*, Vol. 37, edited by E. Wolf (Elsevier, Amsterdam, 1997).
[2] R. Landauer and Th. Martin, Rev. Mod. Phys. **66**, 217 (1994).
[3] R. Landauer, Ber. Bunsenges. Phys. Chem. **95**, 404 (1991).
[4] E.H. Hauge and J.A. Stϕvneng, Rev. Mod. Phys. **61**, 917 (1989).
[5] M. Büttiker, *Electronic properties of multilayers and low-dimensional semiconductor structures*, edited by J.M. Chamberlain et al. (Plenum Press, New York, 1990).
[6] E.P.Wigner, Phys. Rev. **98**, 145 (1955).
[7] M. Büttiker and R. Landauer, Phys. Rev. Lett. **49**, 1739 (1982).
[8] R. Landauer and Th. Martin, Solid State Commun. **84**, 115 (1992).
[9] M. Büttiker, Phys. Rev. B **27**, 6178 (1983).
[10] R. Golub, S. Felber, R. Gähler and E. Gutsmiedl, Phys. Lett. A **148**, 27 (1990).
[11] See, however, S. Anantha Ramakrishna and N. Kumar, cond-mat/0009269.
[12] R. Bruinsma and P. Bak, Phys. Rev. Lett. **56**, 420 (1986).
[13] A.M. Jayannavar, Pramana J. Phys. **29**, 341 (1987).

[14] C.R. Leavens and G.C. Aers, Phys. Rev B **39**, 1202 (1989).

[15] D. Sokolovski and L.M. Baskin, Phys. Rev. a **36**, 4604 (1987); D. Sokolovski and J.N.L. Connor, Phys. Rev. A **44** 1500 (1990).

[16] M. Büttiker and R. Landauer, Phys. Scr. **32**, 429 (1985).

[17] Th. Martin and R. Landauer, Phys. Rev. A **47**, 2023 (1993).

[18] D. Bohm *Quantum theory* (Prentice Hall, New Jersey, 1951).

[19] T.P. Spiller, T.D. Clark, R.J. Prance and H. Prance, Europhys. Lett. **12**, 1 (1990).

[20] Landau and Lifschitz, *Quantum Mechanics (Non-relativistic theory)*, 3rd Ed. (Pergamon press, Oxford).

[21] S. Anantha Ramakrishna and N. Kumar, Phys. Rev. B **61**, 3163 (2000).