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Large time behaviour of solutions of a system of generalized Burgers equation

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Abstract. In this paper we study the asymptotic behaviour of solutions of a system of N partial differential equations. When $N = 1$ the equation reduces to the Burgers equation and was studied by Hopf. We consider both the inviscid and viscous case and show a new feature in the asymptotic behaviour.

Keywords. Burgers equation; explicit formula; asymptotic behaviour.

1. Introduction

We consider the system of generalized Burgers equations for N unknown variables $u = (u_1, u_2, \ldots, u_N)$,

$$(u_j)_t + \sigma(c,u)(u_j)_x = \frac{\nu}{2}(u_j)_{xx}, \quad j = 1, 2, \ldots, N, \tag{1.1}$$

where $c = (c_1, c_2, \ldots, c_N)$ is a constant vector in R^N and $\sigma(c,u) = \sum_{k=1}^{N} c_k u_k$ is the usual inner product in R^N. We study the solution of (1.1) with initial conditions

$$u_j(x,0) = u_{0j}(x), \quad j = 1, 2, \ldots, N. \tag{1.2}$$

When $\nu > 0$, (1.1) is a system of nonlinear parabolic equation describing the interplay between nonlinearity and diffusion, ν being the viscosity parameter.

When $\nu = 0$, the system (1.1) is hyperbolic with coinciding wave speeds $\sigma(c,u)$, and the nonlinearity and the nonconservative form makes the initial value problem complex. Indeed the system being nonlinear, solution cannot be continued as a smooth solution even when the initial data is smooth. Further if $N > 1$, the product $\sigma(c,u) \cdot (u_j)_x$ is nonconservative and does not make sense in the usual distributional sense. The solution should be understood in a generalized sense.

For the case $\nu > 0$, Joseph [5] used a generalized Hopf–Cole transformation to linearize the system of equations (1.1) and solve (explicitly) with the conditions (1.2) in terms of a family of probability measures $\mathrm{d}\mu^{\nu}_{(x,t)}(y)$. These measures depend on the initial data (1.2) in a nonlinear and nonlocal manner and takes the form

$$\mathrm{d}\mu^{\nu}_{(x,t)}(y) = \frac{\mathrm{e}^{-\frac{1}{\nu}\left[I(y)+\frac{(x-y)^2}{2t}\right]}\mathrm{d}y}{\int_{-\infty}^{\infty} \mathrm{e}^{-\frac{1}{\nu}\left[I(y)+\frac{(x-y)^2}{2t}\right]}\mathrm{d}y}, \tag{1.3}$$

where

$$I(y) = \int_0^y \sigma(c, u_0(z))\mathrm{d}z. \tag{1.4}$$

When $\nu > 0$, the solution of (1.1) and (1.2) was shown to be

$$u_j^\nu(x,t) = \int_{R^1} u_{0j}(y)\mathrm{d}\mu^\nu_{(x,t)}(y), \quad j = 1, 2, 3, \ldots, N. \tag{1.5}$$

Let $u_j^\nu(x,t)$ be the solution of (1.1) and (1.2) given by (1.5). It was shown in [6] that when u_{0j} is Lipschitz continuous, for each $t > 0$, except for a countable x the limits

$$u_j(x,t) = \lim_{\nu\to 0} u_j^\nu(x,t)$$

exist and is given by the formula

$$u_j(x,t) = u_{0j}(y(x,t)), \quad j = 1, 2, \ldots, N \tag{1.6}$$

where $y(x,t)$ is a minimizer of

$$\min_{-\infty<y<\infty} (I(y) + (x-y)^2/2t) \tag{1.7}$$

and $I(x)$ is given by (1.4).

Using some ideas from the earlier works of Joseph [4] and LeFloch [8], $u_j(x,t)_{j=1,2,\ldots,N}$ was shown to be a solution of an inviscid case ($\nu = 0$) in (1.1) with initial data from (1.2), the nonconservative product was justified in the sense of Volpert product [10] (see Dal Maso, LeFloch and Murat [2] for a generalization of Volpert product). In [5] solution for general initial data is constructed in the sense of Colombeau [1].

The aim of the present note is to study the asymptotic behaviour of the solution for the parabolic (viscous) case as well as the hyperbolic (inviscid) case. Study of asymptotic behaviour of solutions is important, see [9] and the references therein for the parabolic case and [7] for the inviscid case. When $N = 1$, this system is the celebrated Burgers equation and explicit solution and its asymptotic behavior as t tends to infinity and diffusion parameter $\nu \to 0$ was studied by Hopf [3]. We show that Hopf's analysis give asymptotic form of the solution for the viscous case. When $N = 1$ and $c_1 \neq 0$, it is well-known from the work of Lax [7] that for solution of inviscid Burgers equation with initial data supported in the compact interval $[-\ell, \ell]$, $\ell > 0$ the solution decays at the rate $O(t^{-\frac{1}{2}})$ and support spreads at a rate $O(t^{\frac{1}{2}})$ for large time. From an explicit computation we will show that the decay rate is not true in general for the present case, but still the support spread at the same rate. We start with the viscous case.

2. Asymptotic behaviour with viscous term

In this section we study the asymptotic behavior of solution of (1.1) and (1.2) when $\nu > 0$ and fixed. On the initial conditions $u_{0j}(x)$, $j = 1, 2, \ldots, N$, assume that $\lim_{x\to\infty} I(x) = I(\infty)$, $\lim_{x\to-\infty} I(x) = I(-\infty)$, $\lim_{x\to\infty} u_{0j}(x) = u_{0j}(\infty)$, $\lim_{x\to-\infty} u_{0j}(x) = u_{0j}(-\infty)$ exists and is finite. With these assumptions, we shall prove the following result.

Theorem 2.1. *The solution* $u^\nu(x,t) = (u_1^\nu(x,t), u_2^\nu(x,t), \dots, u_N^\nu(x,t))$ *of* (1.1) *and* (1.2) *has the following asymptotic behaviour as t tends to infinity*:

$$u_j^\nu(x,t) \approx \frac{u_{0j}(\infty)e^{\frac{-I(\infty)}{\nu}} \int_{-\infty}^{x/\sqrt{(t\nu)}} e^{-\frac{y^2}{2}}\,dy + u_{0j}(-\infty)e^{\frac{-I(-\infty)}{\nu}} \int_{x/\sqrt{(t\nu)}}^{\infty} e^{-\frac{y^2}{2}}\,dy}{e^{\frac{-I(\infty)}{\nu}} \int_{-\infty}^{x/\sqrt{(t\nu)}} e^{-\frac{y^2}{2}}\,dy + e^{\frac{-I(-\infty)}{\nu}} \int_{x/\sqrt{(t\nu)}}^{\infty} e^{-\frac{y^2}{2}}\,dy}. \tag{2.1}$$

Proof. First we note that the solution $u_j^\nu(x,t)$, $j = 1, 2, \dots, N$ of (1.1) and (1.2) is given by (1.5) where the measure $d\mu_{(x,t)}^\nu(y)$ is given by eqs (1.3) and (1.4). Writing explicitly the formula, we get

$$u_j^\nu(x,t) = \frac{\int_{-\infty}^{\infty} u_{0j}(y)e^{-\frac{1}{\nu}\left[I(y)+\frac{(x-y)^2}{2t}\right]}dy}{\int_{-\infty}^{\infty} e^{-\frac{1}{\nu}\left[I(y)+\frac{(x-y)^2}{2t}\right]}dy}.$$

Setting $\xi = x/\sqrt{\nu t}$, and then making a change of variable $z = \frac{\sqrt{\nu t}\xi - y}{\sqrt{\nu t}}$ and renaming z as y, we get

$$u_j^\nu(x,t) = \frac{\int_{-\infty}^{\infty} u_{0j}(\sqrt{\nu t}(\xi - y)e^{-\left[\frac{I(\sqrt{\nu t}(\xi-y)}{\nu}+y^2/2\right]}dy}{\int_{-\infty}^{\infty} e^{-\left[\frac{I(\sqrt{\nu t}(\xi-y))}{\nu}+y^2/2\right]}dy}. \tag{2.2}$$

Now we split the integral in (2.2) in the following manner:

$$\begin{aligned}
&\int_{-\infty}^{\infty} u_{0j}(\sqrt{\nu t}(\xi - y))e^{-\left[\frac{I(\sqrt{\nu t}(\xi-y))}{\nu}+y^2/2\right]}dy \\
&\quad= \int_{-\infty}^{\xi-\delta} u_{0j}(\sqrt{\nu t}(\xi - y))e^{-\left[\frac{I(\sqrt{\nu t}(\xi-y))}{\nu}+y^2/2\right]}dy \\
&\qquad+ \int_{\xi+\delta}^{\infty} u_{0j}(\sqrt{\nu t}(\xi - y))e^{-\left[\frac{I(\sqrt{\nu t}(\xi-y))}{\nu}+y^2/2\right]}dy \\
&\qquad+ \int_{\xi-\delta}^{\xi+\delta} u_{0j}(\sqrt{\nu t}(\xi - y))e^{-\left[\frac{I(\sqrt{\nu t}(\xi-y))}{\nu}+y^2/2\right]}dy.
\end{aligned} \tag{2.3}$$

Now we fix $\delta > 0$ and study each of these integrals as t tends to infinity. We have under the assumptions of the theorem on $I(x)$ and $u_{0j}(x)$, as t tends to infinity:

$$\int_{-\infty}^{\xi-\delta} u_{0j}(\sqrt{\nu t}(\xi - y))e^{-\left[\frac{I(\sqrt{\nu t}(\xi-y))}{\nu}+y^2/2\right]}dy \approx e^{-\frac{I(+\infty)}{\nu}} u_{0j}(\infty) \int_{-\infty}^{\xi-\delta} e^{-y^2/2}dy.$$

$$\int_{\xi+\delta}^{\infty} u_{0j}(\sqrt{\nu t}(\xi - y))e^{-\left[\frac{I(\sqrt{\nu t}(\xi-y))}{\nu}+y^2/2\right]}dy \approx e^{-\frac{I(-\infty)}{\nu}} u_{0j}(-\infty) \int_{\xi+\delta}^{\infty} e^{-y^2/2}dy,$$

$$\limsup_{t\to\infty} \left| \int_{\xi-\delta}^{\xi+\delta} u_{0j}(\sqrt{\nu t}(\xi - y))e^{-\left[\frac{I(\sqrt{\nu t}(\xi-y))}{\nu}+y^2/2\right]}dy \right| = O(\delta).$$

Now let t tend to infinity and then δ tend to 0 in (2.3). We get

$$\int_{-\infty}^{\infty} u_{0j}(\sqrt{\nu t}(\xi - y))\mathrm{e}^{-[I(\sqrt{\nu t}(\xi-y))\nu + y^2/2]}\mathrm{d}y \approx \mathrm{e}^{-\frac{I(+\infty)}{\nu}} u_{0j}(\infty) \int_{-\infty}^{\xi} \mathrm{e}^{-y^2/2}\mathrm{d}y + \mathrm{e}^{-\frac{I(-\infty)}{\nu}} u_{0j}(-\infty) \int_{\xi}^{\infty} \mathrm{e}^{-y^2/2}\mathrm{d}y. \tag{2.4}$$

Similarly

$$\int_{-\infty}^{\infty} \mathrm{e}^{-\left[\frac{I(\sqrt{\nu t}(\xi-y))}{\nu} + y^2/2\right]}\mathrm{d}y \approx \mathrm{e}^{\frac{-I(+\infty)}{\nu}} \int_{-\infty}^{\xi} \mathrm{e}^{-y^2/2}\mathrm{d}y + \mathrm{e}^{\frac{-I(-\infty)}{\nu}} \int_{\xi}^{\infty} \mathrm{e}^{-y^2/2}\mathrm{d}y. \tag{2.5}$$

We observe that due to our assumption on $I(x)$, this limit in (2.5) is a postive real number and hence letting t tend to infinity in (2.2) and using (2.4) and (2.5) we get the result (2.1). The proof of the theorem is complete. □

Remark. An interesting case here is when the initial data u_{0j}, $j = 1, 2, \ldots, N$ satisfies the following conditions. $u_{0j}(\infty)$, $u_{0j}(-\infty)$ are nonzero and there is a cancellation in $\sigma(c, u_0)(x) = \sum_1^N c_k u_{0k}(x)$ so that this quantity is integrable.

3. Asymptotic behaviour of solutions of generalized Hopf equation

In this section we study the asymptotic behaviour of vanishing viscosity solutions of

$$(u_j)_t + \sigma(c, u)(u_j)_x = 0 \tag{3.1}$$

with initial data

$$u_j(x, 0) = u_{0j}(x), \tag{3.2}$$

for $j = 1, 2, \ldots, N$. We recall the definition of $\sigma(c, u)$ namely $\sigma(c, u) = \sum_1^N c_k u_k$ where $c = (c_1, c_2, \ldots, c_N)$ a given constant vector and $u = (u_1, u_2, \ldots, u_N)$, the unknown vector variable.

It is easy to see that $u_j^\nu(x, t)$, $j = 1, 2, \ldots, N$ is a solution of (1.1) and (1.2) iff $\sigma^\nu(x, t) := \sigma(c, u^\nu(x, t))$ satisfies the Burgers equation

$$\sigma_t + \left(\frac{\sigma^2}{2}\right)_x = \frac{\nu}{2}\sigma_{xx}$$

with initial condition

$$\sigma(x, 0) = \sigma(c, u_0)(x).$$

By Hopf–Cole transformation, the solution can be written in the form

$$\sigma^\nu(x, t) = \int_{R^1} \sigma(c, u_0(y))\mathrm{d}\mu^\nu_{(x,t)}(y).$$

As in [6] it is easy to see that, when u_{0j} is Lipschitz continuous, for each $t > 0$, except for a countable number of points of x, the limit

$$\sigma(x, t) = \lim_{\nu \to 0} \sigma^{\nu}(x, t)$$

exists and is given by the formula

$$\sigma(x, t) = \sigma(c, u_0(y(x, t))) = \sigma(c, u),$$

where $y(x, t)$ is a minimizer of

$$\min_{-\infty < y < \infty} (I(y) + (x - y)^2/2t)$$

and $I(x)$ is given by (1.4). Here we remark that this formula is slightly different from that of Hopf [3]. Once $\sigma(c, u)$ is known, (3.1) can be treated as N scalar linear equation with discontinuous coefficient. For any function $a(x, t)$ on $[(x, t)\colon -\infty < x < \infty, t > 0]$, $\|a(., t)\|_{\infty}$ denotes the essential supremum of $a(x, t)$ with respect to the space variable x keeping the time variable $t \geq 0$ fixed. We define the left boundary curve $x = s^{-}(t)$ and right boundary curve $x = s^{+}(t)$ of the support of $a(x, t)$ where $s^{-}(t) = \sup[y\colon a(x, t) = 0$ for all $x < y]$ and $s^{+}(t) = \inf[y\colon a(x, t) = 0$ for all $x > y]$. It is well-known from [7] that when the initial data is supported in $[-l, l]$, being solution of the inviscid Burgers equation, $\sigma(c, u)$ have the following estimates. There exist constants $A > 0$ and $C > 0$ which depend on l and $\|u_0\|_{\infty}$, such that

$$\begin{aligned} -l - At^{\frac{1}{2}} \leq s^{-}(t) \leq s^{+}(t) \leq l + At^{\frac{1}{2}}, \\ \|\sigma(., t)\|_{\infty} \leq Ct^{\frac{-1}{2}}. \end{aligned} \tag{3.3}$$

We shall prove that for u_j, $j = 1, 2, \ldots, N$ again the same estimate holds for the spread of support but the decay result is not valid in general. First we have the following theorem.

Theorem 3.1. *Let $u_j(x, t)$, $j = 1, 2, \ldots, N$ be the vanishing viscosity solution of* (3.1) *and* (3.2) *with initial data supported in $[-l, l]$. Let $x = s_j^{-}(t)$ and $x = s_j^{+}(t)$ are the support curves for $u_j(x, t)$. Then there exists a constant $A > 0$ which depend on l and $\|u_0\|_{\infty}$, so that the following estimate holds for $t \gg 1$,*

$$-l - At^{\frac{1}{2}} \leq s_j^{-}(t) \leq s_j^{+}(t) \leq l + At^{\frac{1}{2}}.$$

Proof. The proof easily follows from the fact that the characteristic speed $\sigma(c, u) = 0$ outside the region $-l - At^{\frac{1}{2}} \leq x \leq l + At^{\frac{1}{2}}$. So in this region the characteristics connecting the points (x, t) to a base point $(y, 0)$ are parallel to the t-axis and hence is of the form $x = y$ and since the solution is constant along the characteristics, we have $u(x, t) = u_0(y) = u_0(x)$. When (x, t) is outside $-l - At^{\frac{1}{2}} \leq x \leq l + At^{\frac{1}{2}}$, the x co-ordinate of the base point of the characteristic, y, lies outside $[-l, l]$ where $u_0(y)$ is zero and hence $u(x, t)$ is zero. The proof of the theorem is complete. □

Next we show that solution does not decay in general, by giving an example. Here we construct vanishing viscosity solution of (3.1) with initial data of the special form which is supported in a compact set, namely

$$u_j(x, 0) = \begin{cases} 0, & \text{if } x < -l \\ u_{0j}, & \text{if } -l < x < l \\ 0, & \text{if } x > l \end{cases} \tag{3.4}$$

where u_{0j} is a constant and l is a positive real number. Let $u_j^\nu(x,t)_{j=1,2,\ldots,N}$ be the solution of (1.1) with the initial data (3.4). We shall prove the following.

Theorem 3.2. *Let* $\sigma_0 = \sum_{k=1}^N c_j u_{0j}$, *then* $u_j(x,t) = \lim_{\nu\to 0} u_j^\nu(x,t)$ *exists and takes the following form*:

When $\sigma_0 = 0$,

$$u_j(x,t) = \begin{cases} 0, & \text{if } x < -l \\ u_{0j}, & \text{if } -l < x < l \\ 0, & \text{if } x > l \end{cases}. \tag{3.5}$$

When $\sigma_0 < 0$,

$$u_j(x,t) = \begin{cases} 0, & \text{if } x < \frac{\sigma_0}{2}\cdot t - l,\, t < \frac{-4l}{\sigma_0} \\ u_{0j}, & \text{if } \frac{\sigma_0}{2}t - l < x < \sigma_0 t + l,\, t < \frac{-4l}{\sigma_0} \\ \frac{u_{jl}}{\sigma_0}\cdot\frac{x-l}{t}, & \text{if } \sigma_0 t < x < l,\, t < \frac{-4l}{\sigma_0} \\ 0, & \text{if } x < l - (-4l\sigma_0 t)^{\frac{1}{2}},\, t > \frac{-4l}{\sigma_0} \\ \frac{u_{0j}}{\sigma_0}\cdot\frac{x-l}{t}, & \text{if } l - (-4l\sigma_0 t)^{\frac{1}{2}} < x < l,\, t > \frac{-4l}{\sigma_0} \\ 0, & \text{if } x > l \end{cases}. \tag{3.6}$$

When $\sigma_0 > 0$,

$$\lim_{\nu\to 0} u_j^\nu(x,t) = \begin{cases} 0, & \text{if } x > \frac{\sigma_0}{2}\cdot t + l,\, t < \frac{4l}{\sigma_0} \\ u_{0j}, & \text{if } \sigma_0 t - l < x < \frac{\sigma_0}{2}t + l,\, t < \frac{4l}{\sigma_0} \\ \frac{u_{0j}}{\sigma_0}\cdot\frac{x+l}{t}, & \text{if } -l < x < \sigma_0 t - l,\, t < \frac{4l}{\sigma_0} \\ 0, & \text{if } x > l + (4l\sigma_0 t)^{\frac{1}{2}},\, t > \frac{4l}{\sigma_0} \\ \frac{u_{0j}}{\sigma_0}\cdot\frac{x+l}{t}, & \text{if } -l < x < l + (4l\sigma_0 t)^{\frac{1}{2}},\, t > \frac{4l}{\sigma_0} \\ 0, & \text{if } x < -l \end{cases}. \tag{3.7}$$

Proof. To prove (3.5)–(3.7) we use the formula (1.5) to get explicit solution of (1.1) and (3.4) in the form

$$u_j^\nu(x,t) = \frac{u_{j0}\int_{-l}^{l} \mathrm{e}^{-\frac{1}{\nu}\left[\frac{(x-y)^2}{2t}+\sigma_0 y\right]}\mathrm{d}y}{\int_{-\infty}^{-l}\mathrm{e}^{-\frac{(x-y)^2}{2t\nu}}\mathrm{d}y + \int_{-l}^{l}\mathrm{e}^{-\frac{1}{\nu}\left[\frac{(x-y)^2}{2t}+\sigma_0 y\right]}\mathrm{d}y + \int_{l}^{\infty}\mathrm{e}^{-\frac{(x-y)^2}{2t\nu}}\mathrm{d}y} \tag{3.8}$$

which can be written in the form

$$u_j^\nu(x,t) = \frac{u_{j0}\int_{-l}^{l} \mathrm{e}^{-\frac{1}{\nu}\left[\frac{(x-y)^2}{2t}+\sigma_0 y\right]}\mathrm{d}y}{(2t\pi\nu)^{\frac{1}{2}} + \int_{-l}^{l}\mathrm{e}^{-\frac{1}{\nu}\left[\frac{(x-y)^2}{2t}+\sigma_0 y\right]}\mathrm{d}y + \int_{-l}^{l}\mathrm{e}^{-\frac{(x-y)^2}{2t\nu}}\mathrm{d}y}. \tag{3.9}$$

To study the limit, we rewrite this formula in a convenient form by introducing the functions

$$A_{l,\sigma_0}^\nu(x,t) = (2t\nu)^{\frac{1}{2}}\mathrm{e}^{\frac{\sigma_0^2 t}{2\nu}-\frac{\sigma_0 x}{\nu}} erfc\left(\frac{t\sigma_0 - x - l}{(2t\nu)^{\frac{1}{2}}}\right) \tag{3.10}$$

and

$$B^{\nu}_{l,\sigma_0}(x,t) = (2t\nu)^{\frac{1}{2}} e^{\frac{\sigma_0^2 t}{2\nu} - \frac{\sigma_0 x}{\nu}} erfc\left(\frac{t\sigma_0 - x + l}{(2t\nu)^{\frac{1}{2}}}\right), \tag{3.11}$$

where

$$erfc(y) = \int_y^{\infty} e^{-y^2} dy. \tag{3.12}$$

We can rewrite (3.9) as

$$u^{\nu}_j(x,t) = \frac{u_{0j}(A^{\nu}_{l,\sigma_0}(x,t) - B^{\nu}_{l,\sigma_0}(x,t))}{(2t\pi\nu)^{\frac{1}{2}} + A^{\nu}_{l,\sigma_0}(x,t) - B^{\nu}_{l,\sigma_0}(x,t) + A^{\nu}_{l,0}(x,t) - B^{\nu}_{l,0}(x,t)}. \tag{3.13}$$

Using the asymptotic expansions of the *erfc*, namely,

$$erfc(y) = \left(\frac{1}{2y} - \frac{1}{4y^3} + o\left(\frac{1}{y^3}\right)\right) e^{-y^2}, \quad y \to \infty$$

and

$$erfc(-y) = (\pi)^{\frac{1}{2}} - \left(\frac{1}{2y} - \frac{1}{4y^3} + o\left(\frac{1}{y^3}\right)\right) e^{-y^2}, \quad y \to \infty$$

in (3.10) and (3.11) we have the following as $\nu \to 0$:

$$A^{\nu}_{l,\sigma_0}(x,t) \approx \begin{cases} \frac{(t\nu)}{-l-x+\sigma_0 t} e^{-\frac{x^2}{2\nu t}}, & \text{if } -l-x+\sigma_0 t > 0 \\ (\frac{\pi t\nu}{2})^{\frac{1}{2}} e^{\frac{\sigma_0^2 t}{2\nu} - \frac{\sigma_0 x}{\nu}}, & \text{if } -l-x+\sigma_0 t = 0 \cdot \\ (2\pi t\nu)^{\frac{1}{2}} e^{\frac{\sigma_0^2 t}{2\nu} - \frac{\sigma_0 x}{\nu}} + \frac{(t\nu)}{-l-x+\sigma_0 t} e^{-\frac{x^2}{2\nu t}}, & \text{if } -l-x+\sigma_0 t < 0 \end{cases} \tag{3.14}$$

$$B^{\nu}_{l,\sigma_0}(x,t) \approx \begin{cases} \frac{(t\nu)}{l-x+\sigma_0 t} e^{-\frac{x^2}{2\nu t}}, & \text{if } l-x+\sigma_0 t > 0 \\ (\frac{\pi t\nu}{2})^{\frac{1}{2}} e^{\frac{\sigma_0^2 t}{2\nu} - \frac{\sigma_0 x}{\nu}}, & \text{if } l-x+\sigma_0 t = 0 \cdot \\ (2\pi t\nu)^{\frac{1}{2}} e^{\frac{\sigma_0^2 t}{2\nu} - \frac{\sigma_0 x}{\nu}} + \frac{(t\nu)}{l-x+\sigma_0 t} e^{-\frac{x^2}{2\nu t}}, & \text{if } l-x+\sigma_0 t < 0 \end{cases} \tag{3.15}$$

It is straightforward to check the formulas (3.5)–(3.7) using (3.14) and (3.15) in (3.13). When $\sigma_0 = 0$, (3.13) becomes

$$u^{\nu}_j(x,t) = \frac{u_{0j}(A^{\nu}_{l,0}(x,t) - B^{\nu}_{l,0}(x,t))}{(2t\pi\nu)^{\frac{1}{2}}}. \tag{3.16}$$

Now take the region $x < -l$. Then $-l-x > 0$ and $l-x > 0$, and using (3.14) and (3.15) in (3.16) we get

$$u^{\nu}_j(x,t) \approx u_{0j} \frac{\frac{t\nu}{-l-x} e^{\frac{-x^2}{2t\nu}} - \frac{t\nu}{l-x} e^{\frac{-x^2}{2t\nu}}}{(2\pi t\nu)^{\frac{1}{2}}}$$

and hence we have

$$\lim_{\nu\to 0} u_j^\nu(x,t) = 0, \quad x < -l.$$

In the region $-l < x < l$, $-l - x < 0$, $l - x > 0$,

$$u_j^\nu(x,t) \approx u_{0j}\frac{(2\pi t\nu)^{\frac{1}{2}} + \frac{t\nu}{-l-x}\mathrm{e}^{\frac{-x^2}{2t\nu}} - \frac{t\nu}{l-x}\mathrm{e}^{\frac{-x^2}{2t\nu}}}{(2\pi t\nu)^{\frac{1}{2}}}.$$

Thus we get

$$\lim_{\nu\to 0} u_j^\nu(x,t) = u_{0j}, \quad -l < x < l.$$

In the region $x > l$, we have $-l - x < 0$, $l - x < 0$ and

$$u_j^\nu(x,t) \approx u_{0j}\frac{(2\pi t\nu)^{\frac{1}{2}} + \frac{t\nu}{-l-x}\mathrm{e}^{\frac{-x^2}{2t\nu}} - (2\pi t\nu)^{\frac{1}{2}} - \frac{t\nu}{l-x}\mathrm{e}^{\frac{-x^2}{2t\nu}}}{(2\pi t\nu)^{\frac{1}{2}}}$$

and hence it follows that

$$\lim_{\nu\to 0} u_j^\nu(x,t) = 0, \quad x > l.$$

This completes the proof of the theorem for $\sigma_0 = 0$. The case $\sigma \neq 0$ is similar and is omitted. □

Remark. Thus if we take the initial data (3.4) which has compact support the decay of the vanishing viscosity solution $u_j(x,t)$, $j = 1, 2, \ldots, N$ depends on the initial speed $\sigma_0 = \sum_1^N c_k u_{0k}$. If $\sigma_0 = 0$, the vanishing viscosity solution does not decay. Indeed

$$\sup_{x\in R^1} |u_j(x,t)| = |u_0 j|.$$

On the other hand, from the above theorem it follows that for the case $\sigma_0 \neq 0$, the solution decays, namely

$$\sup_{x\in R^1} |u_j(x,t)| = O\left(t^{-\frac{1}{2}}\right).$$

We conclude with a remark on the solution for (3.1) with the Riemann type initial data

$$(u_j)(x,0) = \begin{cases} u_{jL}, & \text{if } x < 0 \\ u_{jR}, & \text{if } x > 0, \end{cases} \tag{3.17}$$

where u_{jL} and u_{jR} are constants for $j = 1, 2, \ldots, n$.

Let $\sigma_L = \sum_{k=1}^n c_k u_{jL}$ and $\sigma_R = \sum_{k=1}^n c_k u_{kR}$. Then the vanishing viscosity solution for the Riemann problem (3.1) and (3.17) takes the following form [5].

When $\sigma_L < \sigma_R$,

$$u_j(x,t) = \begin{cases} u_{jL}, & \text{if } x \le \sigma_L t \\ \frac{u_{jR}-u_{jL}}{\sigma_R-\sigma_L}\cdot\frac{x}{t} + \frac{u_{jL}\sigma_R - u_{jR}\sigma_L}{\sigma_R-\sigma_L}, & \text{if } \sigma_L t < x < \sigma_R t. \\ u_{jR}, & \text{if } x \ge \sigma_R t \end{cases}$$

When $\sigma_L = \sigma_R = \sigma$,

$$u_j(x,t) = \begin{cases} u_{jL}, & \text{if } x < \sigma \cdot t \\ u_{jR}, & \text{if } x > \sigma \cdot t. \end{cases}$$

When $\sigma_L > \sigma_R$,

$$u_j(x,t) = \begin{cases} u_{jL}, & \text{if } x < \frac{\sigma_L+\sigma_R}{2} \cdot t \\ \frac{u_{jL}+u_{jR}}{2} & \text{if } x = \frac{\sigma_L+\sigma_R}{2} \cdot t \\ u_{jR}, & \text{if } x > \frac{\sigma_L+\sigma_R}{2} \cdot t \end{cases}.$$

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