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Solution of convex conservation laws in a strip

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Abstract. In this paper we consider scalar convex conservation laws in one space variable in a strip $D = \{(x,t): 0 \leqslant x \leqslant 1,\ t > 0\}$ and obtain an explicit formula for the solution of the mixed initial boundary value problem, the boundary data being prescribed in the sense of Bardos-Leroux and Nedelec. We also get an explicit formula for the solution of weighted Burgers equation in a strip.

Keywords. Conservation laws; boundary value problem; explicit formula.

1. Introduction

We consider mixed initial boundary value problem for scalar convex conservation laws of the form

$$u_t + f(u)_x = 0 \tag{1.1}$$

in a strip $D = \{(x,t): 0 \leqslant x \leqslant 1,\ t \geqslant 0\}$ with initial condition

$$u(x,0) = u_0(x). \tag{1.2}$$

The boundary conditions are prescribed in the sense of Bardos *et al* [1]. Let $u_1(t)$ and $u_2(t)$ are any bounded measurable functions, then this condition requires $u(0+,t)$ and $u(1-,t)$ to satisfy the following:

$$\sup_{k \in I(u(0+,t),\, u_1(t))} [\operatorname{sgn}(u(0+,t) - k)(f(u(0+,t)) - f(k))] = 0 \tag*{$(1.3)_0$}$$

$$\inf_{k \in I(u(1-,t),\, u_2(t))} [\operatorname{sgn}(u(1-,t) - k)(f(u(1-,t)) - f(k))] = 0 \tag*{$(1.3)_1$}$$

where for any real numbers a and b, $I(a,b)$ denotes the closed interval $[\min(a,b), \max(a,b)]$.

We assume the flux function $f(u)$ satisfies the following two conditions

$$f''(u) > 0 \tag*{(A_1)}$$

and

$$\lim_{|u| \to \infty} \frac{f(u)}{|u|} = \infty. \tag*{(A_2)}$$

Under the assumption (A_1), it can be easily checked that $(1.3)_0$ and $(1.3)_1$ are equivalent to $(1.3)'_0$ and $(1.3)'_1$ respectively, see Lefloch [3].

$$\left.\begin{array}{l} u(0+,t)=u_1^+(t) \\ \qquad \text{or} \\ f'(u(0+,t))\leqslant 0 \text{ and } f(u(0+,t))\geqslant f(u_1^+(t)) \end{array}\right\} \text{a.e } t>0, \tag{1.3$'_0$}$$

$$\left.\begin{array}{l} u(1-,t)=u_2^-(t) \\ \qquad \text{or} \\ f'(u(1-,t))\geqslant 0 \text{ and } f(u(0-,t))\geqslant f(u_2^-(t)) \end{array}\right\} \text{a.e } t>0, \tag{1.3$'_1$}$$

where

$$\left.\begin{array}{l} u_1^+(t)=\max\{u_1(t),\lambda\}, \\ u_2^-(t)=\min\{u_2(t),\lambda\}. \end{array}\right\}. \tag{1.4}$$

Here λ is the unique solution of the equation $f'(u)=0$. Because of the assumption (A_1) on $f(u)$, this λ satisfies

$$f(\lambda)=\min_{u\in R^1} f(u).$$

In order to have uniqueness of solution for (1.1), it is known that, an additional condition called entropy condition should be imposed. Under the condition (A_1) on $f(u)$ this condition requires $u(x+,t)$ and $u(x-,t)$ to satisfy

$$u(x-,t)\geqslant u(x+,t) \tag{1.5}$$

for every $0<x<1$, $t>0$.

Existence and uniqueness of solution of (1.1), (1.2), $(1.3)'_0$, $(1.3)'_1$, (1.4) and (1.5) follows from the work of Bardos *et al* [1], where they consider a more general problem in several space variable. In this paper we are interested in obtaining an explicit formula for the solution in the case of one space variable and $f(u)$ satisfying conditions (A_1) and (A_2). The formula we derive here is an extension to the mixed initial boundary case of an explicit formula derived by Lax [5], for the pure initial value problem. In the case of one boundary, i.e., when $D=\{x,t):x\geqslant 0,\ t\geqslant 0\}$, this problem was studied by Lefloch [3] and Joseph and Gowda [4]. Hamilton-Jacobi equation with Neumann type boundary condition was studied by Lions [6]. In one space variable they are closely related to conservation laws with Dirichlet boundary condition.

This paper is organized as follows. In § 2, we state the main result: In § 3, we give a detailed proof of the main theorem and in § 4, we study the weighted Burgers equation.

2. Statement of the main theorem

Before the statement of our main theorem we introduce some notations. For each fixed (x,y,t), $0\leqslant x\leqslant 1$, $0\leqslant y\leqslant 1$, $t>0$, $|i-j|\leqslant 1$, $i,j=0,1,2,\ldots$, $\mathscr{C}_{i,j}(x,y,t)$ denotes the following class of paths β in the strip

$$D=\{(z,s):0\leqslant z\leqslant 1,\ s\geqslant 0\}.$$

Each path connects the point $(y,0)$ to (x,t) and is of the form $z=\beta(s)$ where $\beta(s)$ is piecewise linear function which are straight lines in the interior of D, and having i straight line pieces lie on $x=0$ and j of them lie on $x=1$. For the cases $(i,j)=(0,0)$, $(i,j)=(2,1)$, $(i,j)=(1,2)$ see figures 1a, 1b and 1c respectively.

Denote

$$\mathscr{C}(x,y,t)=\bigcup_{\substack{i\geqslant 0,\, j\geqslant 0\\ |i-j|\leqslant 1}}\mathscr{C}_{ij}(x,y,t).$$

Let $f^*(u)$ is the convex dual of $f(u)$.

$$f^*(u)=\max_{\theta}\,[\theta u-f(\theta)]. \tag{2.1}$$

Let $u_0(x)\in L^\infty(0,1)$ and $u_1(t)$ and $u_2(t)$ are continuous bounded functions and let $u_1^+(t)$ and $u_2^-(t)$ be defined by (1.4). Let (x,y,t) be kept fixed. For each $\beta\in\mathscr{C}(x,y,t)$, we define

$$J(\beta)=-\int_{\{s:\beta(s)=0\}} f(u_1^+(s))\mathrm{d}s-\int_{\{s:\beta(s)=1\}} f(u_2^-(s))\mathrm{d}s+\int_{\{s:0<\beta(s)<1\}} f^*\left(\frac{\mathrm{d}\beta}{\mathrm{d}s}\right)\mathrm{d}s. \tag{2.2}$$

We let

$$Q(x,y,t)=\min_{\beta\in\mathscr{C}(x,y,t)} J(\beta). \tag{2.3}$$

We will see later that $Q(x,y,t)$ is Lipschitz continuous w.r.t. (x,y,t). Denote

$$Q_1(x,y,t)=\partial_x Q(x,y,t). \tag{2.4}$$

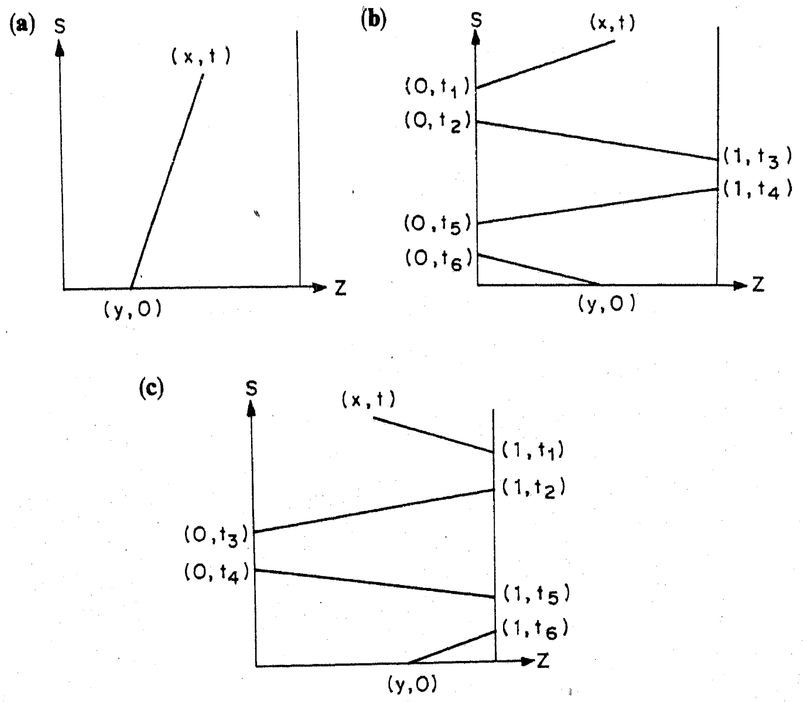

Figure 1(a–c).

We will see that for a.e. (x,t) there exists only one $y_0(x,t)$ which minimises

$$\min_{0\leqslant y\leqslant 1}\left[\int_0^y u_0(z)\mathrm{d}z+Q(x,y,t)\right] \tag{2.5}$$

We shall prove the following theorem.

Main theorem. *Let $u(x,t)$ be defined by*

$$u(x,t)=Q_1(x,y_0(x,t),t), \tag{2.6}$$

where $Q_1(x,y,t)$ is defined by (2.4) *and $y_0(x,t)$ minimizes* (2.5). *Then* (i) *$u(x,t)$ satisfies $u_t+f(u)_x=0$, in the sense of distributions and satisfies the initial condition* (1.2) (ii) *$u(0+,t)$ and $u(1-,t)$ exists a.e. and satisfies the boundary conditions* $(1.3)'_0$ *and* $(1.3)'_1$. (iii) *For each fixed $t>0$, $0<x<1$, $u(x\pm 0,t)$ exists and satisfies the entropy condition* (1.5).

3. Proof of the main theorem

The proof of the main theorem is broken up into several steps formulated as Lemmas. First we need some preliminaries. By definition any curve β in $\mathscr{C}_{i,j}(x,y,t)$ starts at $(y,0)$ and ends at (x,t) and is made up of straight lines joined together at point of the boundary: $x=0$ or $x=1$. Let a curve β is given and let $(\beta(t_1),t_1)$, $(\beta(t_2),t_2)\ldots$ be the corners, *i.e.* the point of intersection of two straight lines of β. We assume that $t_1,t_2,t_3\ldots$ are ordered such that

$$t\geqslant t_1>t_2>..>0.$$

Note that $\beta(t_j)$ can take either 0 or 1 only, see figures 1a, 1b and 1c.

If (x,t) is a point on the boundary *i.e.* if $x=0$ or $x=1$, and let $\beta\in\mathscr{C}_{i,j}(x,y,t)$, by convention we take $t_1=t$ iff for some $\varepsilon>0$, $(t-\varepsilon,t)\subset\{s:\beta(s)=x\}$, see figure 2.

For $l=0,1$, we define

$$\mathscr{C}^l_{i,j}(x,y,t)=\{\beta\in\mathscr{C}_{i,j}(x,y,t):\beta(t_1)=l\}.$$

Clearly, for $k=0,1,2,\ldots$

$$\left.\begin{aligned}
&\mathscr{C}^0_{k+1,k}(x,y,t)=\mathscr{C}_{k+1,k}(x,y,t),\ \mathscr{C}^1_{k+1,k}(x,y,t)=\phi,\\
&\mathscr{C}^0_{k,k+1}(x,y,t)=\phi,\ \mathscr{C}^1_{k,k+1}(x,y,t)=\mathscr{C}_{k,k+1}(x,y,t),\\
&\quad\mathscr{C}_{k,k}(x,y,t)=\mathscr{C}^0_{k,k}(x,y,t)\bigcup\mathscr{C}^1_{k,k}(x,y,t),\\
&\quad\mathscr{C}(x,y,t)=\bigcup_{k=0}^{\infty}\{\mathscr{C}^0_{k,k}(x,y,t)\bigcup\mathscr{C}^1_{k,k}(x,y,t)\bigcup\mathscr{C}^1_{k,k+1}(x,y,t)\\
&\qquad\bigcup\mathscr{C}^0_{k+1,k}(x,y,t)\}.
\end{aligned}\right\} \tag{3.2}$$

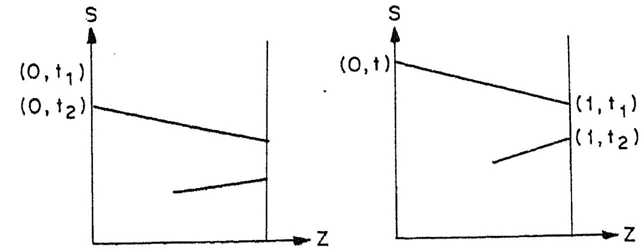

Figure 2.

For $\beta \in \mathscr{C}^0_{k+1,k}(x, y, t)$, we have

$$J(\beta) = J^0_{k+1,k}(x, y, t, t_1, \ldots t_{4k+2}\},$$

where

$$\left.\begin{aligned} J^0_{k+1,k}(x, y, t, t_1, t_2, \ldots t_{4k+2}) &= -\sum_{j=0}^{k}\left\{\int_{t_{4j+3}}^{t_{4j+1}} f(u_1^+(s))\mathrm{d}s\right\} \\ -\sum_{j=0}^{k-1}\left\{\int_{t_{4j+4}}^{t_{4j+3}} f(u_2^-(s))\mathrm{d}s\right\} &+ (t-t_1)f^*\left(\frac{x}{t-t_1}\right) \\ +\sum_{j=1}^{2k}\left\{(t_{2j}-t_{2j+1})f^*\left(\frac{1}{t_{2j}-t_{2j+1}}\right)\right\} &+ t_{4k+2}f^*\left(\frac{-y}{t_{4k+2}}\right). \end{aligned}\right\} \tag{3.3}$$

For $\beta \in \mathscr{C}^1_{k,k+1}(x, y, t)$, we have

$$J(\beta) = J^1_{k,k+1}(x, y, t, t_1, t_2, \ldots t_{4k+2}),$$

where

$$\left.\begin{aligned} J^1_{k,k+1}(x, y, t, t_1, t_2, \ldots t_{4k+2}) &= -\sum_{j=0}^{k-1}\left\{\int_{t_{4j+4}}^{t_{4j+3}} f(u_1^+(s)\mathrm{d}s\right\} \\ -\sum_{j=0}^{k}\left\{\int_{t_{4j+2}}^{t_{4j+1}} f(u_2^-(s))\mathrm{d}s\right\} &+ (t-t_1)f^*\left(\frac{x}{t-t_1}\right) \\ +\sum_{j=1}^{2k}\left\{(t_{2j}-t_{2j+1})f^*\left(\frac{1}{t_{2j}-t_{2j+1}}\right)\right\} &+ t_{4k+2}f^*\left(\frac{-y}{t_{4k+2}}\right). \end{aligned}\right\} \tag{3.4}$$

For $\beta \in \mathscr{C}^1_{k,k}(x, y, t)$, we have

$$J(\beta) = J^1_{k,k}(x, y, t, t_1, t_2, \ldots t_{4k}\},$$

where

$$\left.\begin{aligned} J^1_{k,k}(x, y, t, t_1, t_2, \ldots t_{4k}) &= -\sum_{j=0}^{k-1}\left\{\int_{t_{4j+4}}^{t_{4j+3}} f(u_1^+(s))\mathrm{d}s\right\} \\ -\sum_{j=0}^{k-1}\left\{\int_{t_{4j+2}}^{t_{4j+1}} f(u_2^-(s))\mathrm{d}s\right\} &+ (t-t_1)f^*\left(\frac{x}{t-t_1}\right) \\ +\sum_{j=1}^{2k-1}\left\{(t_{2j}-t_{2j+1})f^*\left(\frac{1}{t_{2j}-t_{2j+1}}\right)\right\} &+ t_{4k}f^*\left(\frac{-y}{t_{4k}}\right). \end{aligned}\right\} \tag{3.5}$$

For $\beta \in \mathscr{C}^0_{k,k}(x,y,t)$, we have

$$J(\beta) = J^0_{k,k}(x,y,t,t_1,t_2,\ldots t_{4k}),$$

where

$$\left.\begin{aligned} J^0_{k,k}(x,y,t,t_1,t_2,\ldots t_{4k}) &= -\sum_{j=0}^{k-1}\left\{\int_{t_{4j+2}}^{t_{4j+1}} f(u_1^+(s))\mathrm{d}s\right\} \\ -\sum_{j=0}^{k-1}\left\{\int_{t_{4j+4}}^{t_{4j+3}} f(u_2^-(s))\mathrm{d}s\right\} &+ (t-t_1)f^*\left(\frac{x}{t-t_1}\right) \\ +\sum_{j=1}^{2k-1}\left\{(t_{2j}-t_{2j+1})f^*\left(\frac{1}{t_{2j}-t_{2j+1}}\right)\right\} &+ t_{4k}f^*\left(\frac{-y}{t_{4k}}\right). \end{aligned}\right\} \tag{3.6}$$

For $l = 0, 1$, $|i-j| \leqslant 1$, $i,j = 0,1,2,\ldots$, define

$$A^l_{i,j}(x,y,t) = \min_{\beta \in \mathscr{C}^l_{i,j}(x,y,t)} J(\beta). \tag{3.7}$$

From (3.3)–(3.7) it follows that

$$\left.\begin{aligned} A^0_{k+1,k}(x,y,t) &= \min_{\beta \in \mathscr{C}^0_{k+1,k}(x,y,t)} J(\beta) \\ &= \min_{0<t_{4k+2}<\ldots t_2<t_1\leqslant t} J^0_{k,k}(x,y,t,t_1,t_2,\ldots t_{4k+2}) \\ A^0_{k,k}(x,y,t) &= \min_{\beta \in \mathscr{C}^0_{k,k}(x,y,t)} J(\beta) \\ &= \min_{0<t_{4k}<\ldots t_2<t_1\leqslant t} J^0_{k,k}(x,y,t,t_1,t_2,\ldots t_{4k}) \\ A^1_{k,k+1}(x,y,t) &= \min_{\beta \in \mathscr{C}^1_{k,k+1}(x,y,t)} J(\beta) \\ &= \min_{0<t_{4k+2}<\ldots<t_2<t_1\leqslant t} J^1_{k,k+1}(x,y,t,t_1,t_2,\ldots,t_{4k}) \\ A^1_{k,k}(x,y,t) &= \min_{\beta \in \mathscr{C}^1_{k,k}(x,y,t)} J(\beta) \\ &= \min_{0<t_{4k}<\ldots<t_2<t_1\leqslant t} J^1_{k,k}(x,y,t,t_1,t_2,\ldots t_{4k}) \end{aligned}\right\} \tag{3.8}$$

It follows from (3.2), (3.7), (3.8) and the definition (2.3) of $Q(x,y,t)$ that

$$\begin{aligned} Q(x,y,t) = \inf_{\{k=0,1,2,\ldots\}} [\min\{&A^0_{k,k}(x,y,t),\, A^1_{k,k}(x,y,t), \\ &A^0_{k+1,k}(x,y,t),\, A^1_{k,k+1}(x,y,t)\}]. \end{aligned} \tag{3.9}$$

Since $u_1^+(s)$ and $u_2^-(s)$ are bounded it follows $Q(x,y,t)$ defined by (2.3) is uniformly bounded in (x,t). Hence it follows from the assumption (A_2) on $f(u)$, that in the minimisation of (3.8) it is enough to consider t_j such that $t_{2j}-t_{2j+1} \geqslant C > 0$, where C is a constant depending only on the L^∞ norm of $u_1^+(t)$ and $u_2^-(t)$ and of course on f. The reason for this is that the term $\Sigma(t_{2j}-t_{2j+1})f^*(1/t_{2j}-t_{2j+1}) \to \infty$ if at least one of $t_{2j}-t_{2j+1} \to 0$, because of assumption A_2 on f. From this fact the following Lemma immediately follows.

Lemma 3.1. *Let $T > 0$ be given, then there exists an integer $N(T)$ depending only on T (and of course on $\|u_1^+(t)\|_\infty$, $\|u_2^-(t)\|_\infty$ and $f(u)$) such that*

$$Q(x,y,t) = \min_{k\in\{0,1,2,..N(T)\}} [\min\{A^0_{k,k}(x,y,t),\ A^1_{k,k}(x,y,t),\ A^0_{k+1,k}(x,y,t),\ A^1_{k,k+1}(x,y,t)\}] \tag{3.10}$$

for all $0 \leqslant t \leqslant T$, $0 \leqslant x \leqslant 1$, $0 \leqslant y \leqslant 1$.

Now standard arguments of Conway and Hopf [2] and Lax [5] can be used to show that $A^0_{k+1,k}(x,y,t)$, $A^0_{k,k}(x,y,t)$, $A^1_{k,k+1}(x,y,t)$ and $A^1_{k,k}(x,y,t)$, defined by (3.8) are Lipschitz continuous with respect to (x,y,t). Lemma (3.1) says that $Q(x,y,t)$ is minimum of these Lipschitz continuous functions and hence, we have the following corollary to Lemma (3.1).

COROLLARY 3.2.

$Q(x,y,t)$ is Lipschitz continuous function of (x,y,t).

To proceed further, we need to study more about $A^0_{k+1,k}(x,y,t)$, $A^0_{k,k}(x,y,t)$, $A^1_{k,k+1}(x,y,t)$ and $A^1_{k,k}(x,y,t)$. Let us take the case $A^0_{k+1,k}(x,y,t)$ and the corresponding minimization problem

$$A^0_{k+1,k}(x,y,t) = \min_{t\geqslant t_1 > t_2 > \dots t_{4k+2} > 0} J^0_{k+1,k}(x,y,t,t_1,t_2,\dots t_{4k+2}).$$

Let $(t_1(x,y,t),\ t_2(x,y,t),\dots t_{4k+2}(x,y,t))$ denote a value $(t_1,t_2,\dots t_{4k+2})$ for which minimum is attained. There may be several $(t_1,t_2,\dots t_{4k+2})$ for which this happens. For $j = 1,2,\dots 4k+2$, define

$$\left.\begin{aligned} t_j^+(x,y,t) &= \max\{t_j(x,y,t)\} \\ t_j^-(x,y,t) &= \min\{t_j(x,y,t)\}. \end{aligned}\right\} \tag{3.11}$$

Similar definition can be made for the minimization problem for $J^1_{k,k+1}$, $J^0_{k,k}$ and $J^1_{k,k}$. Let $y_0(x,t)$ denote a value $0 \leqslant y_0 \leqslant 1$, for which minimum is attained in (2.5), let

$$\left.\begin{aligned} y_0^+(x,t) &= \max\{y_0(x,t)\}, \\ y_0^-(x,t) &= \min\{y_0(x,t)\}. \end{aligned}\right\} \tag{3.12}$$

First we shall prove the following Lemma.

Lemma 3.3. *Let $S_k(x,y,t)$ be any set in $\{\mathscr{C}^0_{k,k}(x,y,t),\ \mathscr{C}^1_{k,k}(x,y,t),\ \mathscr{C}^0_{k+1,k}(x,y,t),\ \mathscr{C}^1_{k,k+1}(x,y,t)\}$. Let β_0 achieve minimum for $\min_{\beta\in S_k(x,y,t)} J(\beta)$. Let (x^*,t^*) be a point on β_0 and β_0^* be the restriction of β_0 on $[0,t^*]$ and let $\beta_0^*\in S_{k_0}(x^*,y,t^*)$ for some k_0, where $S_{k_0}(x^*,y,t^*)$ is one of the sets in $\{\mathscr{C}^0_{kk'}(x^*,y,t^*),\ \mathscr{C}^1_{k,k}(x^*,y,t^*),\ \mathscr{C}^0_{k+1,k,}(x^*y,t^*),\ \mathscr{C}^1_{k,k+1}(x^*,y,t^*)\}$. Then β_0^* achieves minimum for $\min_{\beta\in S_{k_0}(x^*,y,t^*)} J(\beta)$. The same result is true if one replaces $S_k(x,y,t)$ by $\mathscr{C}(x,y,t)$ and $S_{k_0}(x^*,y,t^*)$ by $\mathscr{C}(x^*,y,t^*)$.*

Proof. Suppose, on the contrary, there exists $\bar\beta\in S_{k_0}(x^*,y,t^*)$ such that

$$J(\beta_0^*) > J(\bar\beta) \tag{3.13}$$

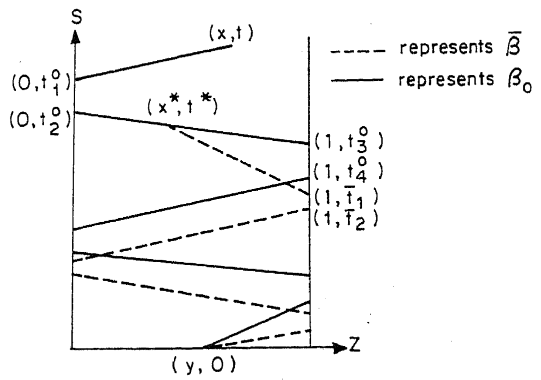

Figure 3.

and $\min_{\beta\in S_{k_0}(x^*,y,t^*)} J(\beta)=J(\bar\beta)$. There are several cases to consider, among them we take a typical case. The other cases can be treated similarly. We take the case when (x^*,t^*) is in the interior of D and is on the straight line joining $(\beta_0(t_i^0),t_j^0)=(0,t_j^0)$ and $(\beta_0(t_{j+1}^0),t_{j+1}^0)=(1,t_{j+1}^0)$ for some j. Here $t>t_1^0>t_2^0>\ldots>t_j^0>t_{j+1}^0>\ldots>t_{4k+2}^0>0$ corresponds to parameters of β_0 when it meets or leaves the boundary $x=0$ or $x=1$, see (3.8). Thus we assume $\beta_0^*\in s_{k_0}(x^*,y,t^*)=\mathscr{C}^1_{k_0,k_0+1}$, for some k_0. Let $t_1^*>t_2^*>,..>t_{4k_0+2}^*$ be the parameters corresponding to β_0^* and $\bar t_1>\bar t_2>\ldots\bar t_{4k_0+2}$ the parameters corresponding to $\bar\beta$. The only interesting case we need to consider is when $t_1^*\neq\bar t_1$, see figure 3 for the case $S_k(x,y,t)=\mathscr{C}^0_{2,2}(x,y,t)$, $S_{k_0}(x,y,t)=\mathscr{C}^0_{1,2}(x,y,t)$.

Now define the curve $\beta_1(s)$ on $[0,t)$ by

$$\beta_1(s)=\begin{cases}\beta_0(s) & \text{for } s\in[t^*,t),\\ \bar\beta(s) & \text{for } s\in[0,t^*]\end{cases}$$

and

$$\beta_2(s)=\begin{cases}\beta_1 & \text{for } s\in[t_j^0,t]\cup[0,\bar t_1]\\ \text{straight line joining } (0,t_j^0) \text{ and } (1,\bar t_1) & \text{on } [\bar t_1,t_j^0].\end{cases}$$

By Jensen's inequality, we obtain

$$\int_{\bar t_1}^{t_j^0} f^*\left(\frac{d\beta_1}{ds}\right)\geqslant(t_j^0-\bar t_1)f^*\left(\frac{1}{t_j^0-\bar t_1}\right)$$
$$=\int_{\bar t_1}^{t_j^0} f^*\left(\frac{d\beta_2}{ds}\right)ds. \tag{3.14}$$

Now from (3.13) and the definition of $\beta_1(s)$, it follows that

$$J(\beta_0)>J(\beta_1). \tag{3.15}$$

Using (3.14) and the definition of β_2 we obtain from (3.15)

$$J(\beta_0)>J(\beta_1)\geqslant J(\beta_2). \tag{3.16}$$

By construction $\beta_2\in S_k(x,y,t)$ and hence (3.16) contradicts the fact that β_0 is a minimizer for $\min_{\beta\in S_k(x,y,t)}J(\beta)$. The proof of lemma is complete.

Lemma 3.4. *Let* β_1, $i=1,2$ *are minimizers for* $\min_{\beta\in\mathscr{C}^0_{k+1,k}(x_i,y,t_i)} J(\beta)$, $t_1 \geqslant t_2$. *Then* β_1 *and* β_2 *cannot cross with different slopes in the interior of* D. *The same result is true when* $\mathscr{C}^0_{k+1,k}(x_i,y,t_i)$ *is replaced by* $\mathscr{C}^1_{k,k+1}(x_i,y,t_i)$ *or* $\mathscr{C}^0_{k,k}(x_i,y,t_i)$ *or* $\mathscr{C}^0_{k,k}(x_i,y,t_i)$ *or* $\mathscr{C}(x_i,y,t_i)$.

Proof. The proof follows from the previous lemma and a construction similar to the one done in the proof of that Lemma and Jensen's inequality, the details are omitted.

In the next Lemma we obtain some useful properties of $t_1^{\pm}(x,y,t)$.

Lemma 3.5. (i) *Let us consider the minimization problem* $A^0_{k+1,k}(x,y,t)=\min_{0<t_{4k+2}<\ldots<t_1} J^0_{k+1,k}(x,y,t,t_1,t_2,\ldots t_{4k+2})$ *or* $A^0_{k,k}(x,y,t)=\min_{0<t_{4k}<\ldots<t_1} J^0_{k,k}(x,y,t,t_1,t_2,\ldots t_{4k})$. *For each fixed* $t>0$, $0\leqslant y\leqslant 1$, $t_1^+(.,y,t)$ *and* $t_1^-(.,y,t)$ *are non-increasing function of* x. $t_1^+(.,y,t)$ *is right continuous and* $t_1^-(.,y,t)$ *is left continuous. The two functions have the same set of points of discontinuity which is countable subset of* $[0,\infty)$ *and except on this countable set* $t_1^+(.,y,t)$ *and* $t_1^-(.,y,t)$ *are equal. Moreover*

$$\left.\begin{aligned} t_1^+(x,y,t)=t_1^+(x+0,y,t)=t_1^-(x+0.y,t)\forall 0\leqslant x<1,\\ t_1^-(x,y,t)=t_1^-(x-0,y,t)=t_1^+(x-0,y,t)\forall 0<x\leqslant 1.\end{aligned}\right\} \tag{3.17}$$

(ii) *Let us consider the minimization problem*

$$A^1_{k,k+1}(x,y,t)=\min_{0<t_{4k+2}<\ldots<t_1} J^1_{k,k+1}(x,y,t,t_1,t_2,\ldots t_{4k+2})$$

or

$$A^1_{k,k}(x,y,t)=\min_{0<t_{4k}<\ldots<t_1} J^1_{k,k}(x,y,t,t_1,t_2,\ldots t_{4k}).$$

For each fixed $t\geqslant 0$, $0\leqslant y\leqslant 1$, $t_1^+(.,y,t)$ *and* $t_1^-(.,y,t)$ *are non-decreasing function of* x. $t_1^+(.,y,t)$ *is left continuous and* $t_1^-(.,y,t)$ *is right continuous. The two functions have the same set of points of discontinuity which is countable subset of* $[0,\infty)$ *and except on this countable set* $t_1^+(.,y,t)$ *and* $t_1^-(.,y,t)$ *are equal. Moreover*

$$\left.\begin{aligned} t_1^+(x,y,t)=t_1^+(x-0,y,t)=t_1^-(x-0,y,t)\forall 0<x\leqslant 1,\\ t_1^-(x,y,t)=t_1^-(x+0,y,t)=t_1^+(x+0,y,t)\forall 0\leqslant x<1.\end{aligned}\right\} \tag{3.18}$$

(iii) *Let us consider any of the following four minimization problems,*

$$\min_{0<t_{4k+2}<\ldots<t_1} J^0_{k+1,k}(x,y,t,t_1,t_2,\ldots t_{4k+2}),$$

$$\min_{0<t_{4k}<\ldots<t_1} J^0_{k,k}(x,y,t,t_1,t_2,\ldots t_{4k}),$$

$$\min_{0<t_{4k+2}<\ldots<t_1} J^1_{k,k+1}(x,y,t,t_1,t_2,\ldots t_{4k+2}),$$

or

$$\min_{0<t_{4k}<\ldots<t_1} J^1_{k,k}(x,y,t,t_1,t_2,\ldots t_{4k}).$$

For each fixed $0\leqslant x\leqslant 1$, $0\leqslant y\leqslant 1$, $t_1^+(x,y,.)$ *and* $t_1(\bar{x},y,.)$ *are non-decreasing function of* t *and* $t_1^+(x,y,.)$ *is left continuous and* $t_1^-(x,y,.)$ *is right continuous. The two functions* *have the same set of points of discontinuity which is countable subset of* $[0,\infty)$ *and except on this countable set* $t_1^+(x,y,.)$ *and* $t_1^-(x,y,.)$ *are equal. Moreover*

$$\left.\begin{array}{l} t_1^+(x,y,t)=t_1^+(x,y,t-0)=t_1^-(x,y,t-0)\forall t>0,\\ t_1^-(x,y,t)=t_1^-(x,y,t+0)=t_1^+(x,y,t+0)\forall t>0.\end{array}\right\} \tag{3.19}$$

(iv) *For each fixed* $0\leqslant y\leqslant 1$, $t_1^+(x,y,t)=t_1^-(x,y,t)$ *a.e.* (x,t).

Proof. We shall prove (i). From the definition of $t_1^\pm(x,y,t)$ and using the fact that two minimizers cannot cross, in the interior of D, see Lemma 3.4, we get if $x_1<x_2$

$$t_1^-(x_2,y,t)\leqslant t_1^+(x_2,y,t)\leqslant t_1^-(x_1,y,t)\leqslant t_1^+(x_1,y,t). \tag{3.20}$$

This inequality shows that $t_1^-(.,y,t)$ is non-increasing and hence $t_1^-(.,y,t)$ have atmost a countable number of discontinuity points. By the continuity of $A^0_{k+1,k}$, (3.20) implies (3.17) also.

Proofs of (ii) and (iii) are similar and (iv) follows from (i), (ii) and (iii). Proof of the Lemma is complete.

Now let us compute the left and right derivatives of $A^0_{k,k}(x,y,t)$, $A^0_{k+1,k}(x,y,t)$, $A^1_{k,k}(x,y,t)$ and $A^1_{k,k+1}(x,y,t)$ with respect to x and t, for each fixed $0\leqslant y\leqslant 1$. Denote

$$\frac{\partial A^\pm}{\partial x}=\lim_{\substack{h\to 0\\ h>0}}\left[\frac{A(x\pm h,y,t)-A(x,y,t)}{h}\right]$$

$$\frac{\partial A^\pm}{\partial t}(x,y,t)=\lim_{\substack{h\to 0\\ h>0}}\left[\frac{A(x,y,t\pm h)-A(x,y,t)}{h}\right]$$

We shall prove the following Lemma.

Lemma 3.6. *For* $k=1,2,\dots$

(i) $$\frac{\partial (A^0_{k,k})^+}{\partial x}(x,y,t)=(f^*)'\left(\frac{x}{t-t_1^+(x,y,t)}\right)\forall 0\leqslant x<1,$$

(ii) $$\frac{\partial (A^0_{k,k})^-}{\partial x}(x,y,t)=(f^*)'\left(\frac{x}{t-t_1^-(x,y,t)}\right)\forall 0<x\leqslant 1,$$

(iii) $$\frac{\partial (A^0_{k+1,k})^+}{\partial x}(x,y,t)=(f^*)'\left(\frac{x}{t-t_1^+(x,y,t)}\right)\forall 0\leqslant x<1,$$

(iv) $$\frac{\partial (A^0_{k+1,k})^-}{\partial x}(x,y,t)=(f^*)'\left(\frac{x}{t-t_1^-(x,y,t)}\right)\forall 0<x\leqslant 1,$$

(v) $$\frac{\partial (A^1_{k,k})^+}{\partial x}(x,y,t)=(f^*)'\left(\frac{x}{t-t_1^-(x,y,t)}\right)\forall 0\leqslant x<1,$$

(vi) $$\frac{\partial (A^1_{k,k})^-}{\partial x}(x,y,t)=(f^*)'\left(\frac{x}{t-t_1^+(x,y,t)}\right)\forall 0<x\leqslant 1,$$

(vii) $$\frac{\partial (A^1_{k,k+1})^+}{\partial x}(x,y,t)=(f^*)'\left(\frac{x}{t-t_1^-(x,y,t)}\right)\forall 0\leqslant x<1,$$

(viii) $\dfrac{\partial(A^1_{k,k+1})^-}{\partial x}(x,y,t)=(f^*)'\left(\dfrac{x}{t-t_1^+(x,y,t)}\right)\forall 0<x\leqslant 1,$

(ix) $\dfrac{\partial(A^0_{k,k})^{\pm}}{\partial t}(x,y,t)=-f\left[(f^*)'\left(\dfrac{t}{t-t_1^{\mp}(x,y,t)}\right)\right]\forall t>0,$

(x) $\dfrac{\partial(A^0_{k+1,k})^{\pm}}{\partial t}(x,y,t)=-f\left[(f^*)'\left(\dfrac{t}{t-t_1^{\mp}(x,y,t)}\right)\right]\forall t>0,$

(xi) $\dfrac{\partial(A^1_{k,k})^{\pm}}{\partial t}(x,y,t)=-f\left[(f^*)'\left(\dfrac{t}{t-t_1^{\mp}(x,y,t)}\right)\right]\forall t>0,$

(xii) $\dfrac{\partial(A^1_{k+1,k})^{\pm}}{\partial t}(x,y,t)=-f\left[(f^*)'\left(\dfrac{t}{t-t_1^{\mp}(x,y,t)}\right)\right]\forall t>0,$

Proof. We shall prove (i). By the definition of $A^0_{k,k}(x,y,t)$, we have

$$\begin{aligned}A^0_{k,k}(x+h,y,t)\leqslant &-\sum_{j=0}^{k-1}\int_{t_{4j+2}}^{t_{4j+1}(x,y,t)}f(u_1^+(s))\mathrm{d}s-\sum_{j=0}^{k-1}\int_{t_{4j+4}}^{t_{4j+3}(x,y,t)}f(u_2^-(s))\mathrm{d}s\\&+t_{4k}(x,y,t)f^*\left(\frac{-y}{t_{4k}(x,y,t)}\right)\\&+\sum_{j=1}^{2k-1}t_{2j}(x,y,t)-t_{2j+1}(x,y,t)f^*\left(\frac{1}{t_{2j}(x,y,t)-t_{2j+1}(x,y,t)}\right)\\&+(t-t_1^+(x,y,t))f^*\left(\frac{x+h}{t-t_1^+(x,y,t)}\right)\end{aligned}$$

and

$$\begin{aligned}A^0_{k,k}(x,y,t)= &-\sum_{j=0}^{k-1}\int_{t_{4j+2}}^{t_{4j+1}(x,y,t)}f(u_1^+(s))\mathrm{d}s-\sum_{j=0}^{k-1}\int_{t_{4j+4}}^{t_{4j+3}(x,y,t)}f(u_2^-(s))\mathrm{d}s\\&+t_{4k}(x,y,t)f^*\left(\frac{-y}{t_{4k}(x,y,t)}\right)\\&+\sum_{j=1}^{2k-1}(t_{2j}(x,y,t)-t_{2j+1}(x,y,t))f^*\left(\frac{1}{t_{2j}(x,y,t)-t_{2j+1}(x,y,t)}\right)\\&+(t-t_1^+(x,y,t))f^*\left(\frac{x}{t-t_1^+(x,y,t)}\right)\end{aligned}$$

so that

$$\begin{aligned}&\frac{A^0_{k,k}(x+h,y,t)-A^0_{k,k}(x,y,t)}{h}\\&\leqslant(t-t_1^+(x,y,t))\left[f^*\left(\frac{x+h}{t-t_1^+(x,y,t)}\right)-f^*\left(\frac{x}{t-t_1^+(x,y,t)}\right)\right]\\&\leqslant(f^*)'\left(\frac{x+\eta(h)}{t-t_1^+(x,y,t)}\right),\ 0<\eta(h)<h.\end{aligned}$$

Letting $h \to 0$, we get

$$\frac{\partial A^0_{k,k}}{\partial x}(x,y,t) \leqslant (f^*)'\left(\frac{x}{t - t_1^+(x,y,t)}\right). \tag{3.21}$$

In a similar way, we get

$$\frac{A^0_{k,k}(x+h,y,t) - A^0_{k,k}(x,y,t)}{h} \geqslant (f^*)'\left(\frac{x+\eta(h)}{t - t_1^+(x+h,y,t)}\right), \quad 0 < \eta(h) < h.$$

Letting $h \to 0$ and using right continuity of $t_1^+(.,y,t)$ we get

$$\frac{\partial (A^0_{k,k})^+}{\partial x}(x,y,t) \geqslant (f^*)'\left(\frac{x}{t - t_1^+(x,y,t)}\right). \tag{3.22}$$

From (3.21) and (3.22) we get (i). The proof of (i) is complete. Similar argument can be used to prove (ii)–(viii). The details are omitted.

Next we compute the right derivative of $A^0_{k,k}(x,y,t)$ with respect to t. As before

$$\frac{A^0_{k,k}(x,y,t+h) - A^0_{k,k}(x,y,t)}{h}$$

$$\leqslant \frac{(t+h-t_1^-(x,y,t))f^*\left(\frac{x}{t+h-t_1^-(x,y,t}\right) - (t - t_1^-(x,y,t))f^*\left(\frac{x}{t-t_1^-(x,y,t)}\right)}{h}.$$

Letting $h \to 0$, we get

$$\frac{\partial (A^0_{k,k})^+}{\partial t}(x,y,t) \leqslant f^*\left(\frac{x}{t-t_1^-(x,y,t)}\right) - \left(\frac{x}{t-t_1^-(x,y,t)}\right)(f^*)'\left(\frac{x}{t-t_1^-(x,y,t)}\right).$$

As in (3.22), using the right continuity of $t_1^-(.,y,t)$ we can show

$$\frac{\partial (A^0_{k,k})^+}{\partial t}(x,y,t) \geqslant f^*\left(\frac{x}{t-t_1^-(x,y,t)}\right) - \left(\frac{x}{t-t_1^-(x,y,t)}\right)(f^*)'\left(\frac{x}{t-t_1^-(x,y,t)}\right)$$

and hence

$$\frac{\partial (A^0_{k,k})^+}{\partial t}(x,y,t) = f^*\left(\frac{x}{t-t_1^-(x,y,t)}\right) - \left(\frac{x}{t-t_1^-(x,y,t)}\right)(f^*)'\left(\frac{x}{t-\bar{t}_1(x,y,t)}\right). \tag{3.23}$$

For convex functions, the following identity is true, namely,

$$f[(f^*)'(s)] = s(f^*)'(s) - f^*(s).$$

Using this in (3.23) we get

$$\frac{\partial (A^0_{k,k})^+}{\partial t}(x,y,t) = -f\left[(f^*)'\left(\frac{x}{t-\bar{t}_1(x,y,t)}\right)\right].$$

The same method can be used to prove (x) — (xii). The proof of Lemma is complete. Next we shall prove the following Lemma.

Lemma 3.7. *For each fixed* $y > 0$,

$$Q_t + f(Q_x) = 0 \quad \text{a.e } (x, t).$$

Proof. By definition

$$A^0_{0,0}(x, y, t) = t\, f^*\left(\frac{x-y}{t}\right)$$

and hence

$$\frac{\partial(A^0_{0,0})}{\partial x}(x, y, t) = (f^*)'\left(\frac{x-y}{t}\right)$$

and

$$\frac{\partial(A^0_{0,0})}{\partial t}(x, y, t) = f^*\left(\frac{x-y}{t}\right) - (f^*)'\left(\frac{x-y}{t}\right)\cdot\left(\frac{x-y}{t}\right)$$

$$= -f(f^*)'\left(\frac{x-y}{t}\right).$$

From these, it follows that $A^0_{0,0}(x, y, t)$ satisfies

$$\frac{\partial v}{\partial t} + f\left(\frac{\partial v}{\partial x}\right) = 0 \tag{3.24}$$

for each fixed $0 \leqslant y \leqslant 1$. It follows from Lemma (3.6) that $A^0_{k,k}(x, y, t)$, $A^1_{k,k}(x, y, t)$, $A^1_{k,k+1}(x, y, t)$ and $A^0_{k+1,k}(x, y, t)$ satisfies (3.24) a.e. (x,t), for each fixed $0 \leqslant y \leqslant 1$. Also for each fixed T, $Q(x, y, t)$ is the minimum of a finite number of functions which satisfies (3.24), in $0 < x < 1$, $0 < t < T$, see Lemma (3.2). Now recall the following result of Conway and Hopf [2]: If $\{v^i(x, t)\colon i = 1, 2, .. N\}$ solves (3.24), so does $v(x, t)$ defined by

$$v(x, t) = \min_{i=1,2,..N} v^i(x, t).$$

Using this fact we get, for fixed $0 \leqslant y \leqslant 1$

$$Q_t + f(Q_x) = 0 \quad \text{a.e. } (x, t),\ 0 < x < 1,\ 0 < t < T.$$

But since T is arbitrary, Lemma follows.

Let (x, y, t) be fixed and let $\beta \in \mathscr{C}(x, y, t)$. Define

$$H(x, y, t, \beta) = \int_0^y u_0(z) + J(\beta), \tag{3.25}$$

then (2.5) can be rewritten in the following way.

Let $\bar{\beta} \in \mathscr{C}(x, y, t)$ and $y_0(x, t)$ be such that

$$\min_{\substack{\beta \in \mathscr{C}(x, y, t)\\ y \geqslant 0}} [H(x, y, t, \beta)] = H(x, y_0(x, t), t, \bar{\beta})). \tag{3.25}$$

Note that RHS of (3.25) is nothing but

$$\int_0^{y_0(x,t)} u_0(z) + Q(x, y_0(x,t), t),$$

we call it $U(x,t)$ i.e.,

$$U(x,t) = \int_0^{y_0(x,t)} u_0(z) + Q(x, y_0(x,t), t).$$

Let $y_0^+(x,t)$ and $y_0^-(x,t)$ be defined by (3.12). We have the following lemma.

Lemma 3.8. *Let* $t > 0$, *be fixed,*

(i) $y_0^+(x,t)$ *and* $y_0^-(x,t)$ *are non-decreasing function of* $x, y_0^+(x,t)$ *is right contiuous and* $y_0^-(x,t)$ *is left continuous. The two functions have the same set of point of discontinuity and except at these countably many points, the two functions are equal. Moreover,*

$$\left.\begin{aligned} y_0^+(x,t) &= y_0^+(x+0,t) = y_0^-(x+0,t), \\ y_0^-(x,t) &= y_0^-(x-0,t) = y_0^+(x-0,t). \end{aligned}\right\} \tag{3.26}$$

(ii) *Suppose the minimum in* (3.25) *for* $H(\bar{x}, y, t, \beta)$ *is attained for some* $\bar{\beta} \in \mathscr{C}^0_{k,k}(\bar{x}, y_0 \cdot (\bar{x}, t), t)$ $(\mathscr{C}^0_{k+1,k}(\bar{x}, y_0(\bar{x}, t), t))$. *Let* $x^* < \bar{x}$, *and let* β^* *attain minimum for* $H(x^*, y, t, \beta)$ *then* $\beta^* \in \mathscr{C}^0_{k,k}(x^*, y_0(x^*, t)t)$. *Moreover, for* $0 \leqslant x^* < \bar{x}$

$$\begin{aligned} t_j^\pm(\bar{x}, y_0^\pm(\bar{x}, t), t) &= t_j^\pm(x^*, y_0^\pm(x^*, t), t), \; j \geqslant 2, \\ y_0^-(\bar{x}, t) &= y_0^+(x^*, t) = y_0^-(x^*, t). \end{aligned} \tag{3.27}$$

(iii) *Suppose the minimum in* (3.25) *for* $H(\bar{x}, y, t, \beta)$ *is attained for some* $\bar{\beta} \in \mathscr{C}^1_{k,k}(\bar{x}, y_0 \cdot (\bar{x}, t), t)$ $(\mathscr{C}^1_{k,k+1}(x, y_0(\bar{x}, t), t))$ *let* $x^* > \bar{x}$, *and let* β^* *attain minimum in* (3.25) *for* $H(x^*, y, t, \beta)$ *then* $\beta^* \in \mathscr{C}^1_{k,k}(x^*, y_0(x, t), t)$ $(\mathscr{C}^1_{k,k+1}(x^*, y_0(x^*, t), t))$. *Moreover,* $\forall \bar{x} < x^* \leqslant 1$

$$\begin{aligned} t_j^\pm(\bar{x}, y_0^+(\bar{x}, t), t) &= t_j^\pm(x^*, y_0^\pm(x^*, t, t)), \; j \geqslant 2 \\ y_1^+(\bar{x}, t) &= y_0^+(x^*, t) = y_0^-(x^*, t). \end{aligned} \tag{3.28}$$

Proof. Proof of (i) is exactly the same as the proof of Lemma (3.5). We shall prove (ii). Let us take the case where $\bar{\beta} \in \mathscr{C}^0_{k,k}(\bar{x}, y_0(\bar{x}, t), t)$, $k \geqslant 1$. Since $\bar{\beta}$ and β^* cannot interest in the interior of D, see Lemma (3.4), it follows that $\beta^* \in \mathscr{C}^0_{k,k}(x^*, y_0(x^*, t), t)$ and

$$t_j^\pm(\bar{x}, y_0^\pm(\bar{x}, t), t) = t_j^\pm(x^*, y_0^\pm(x^*, t), t) \; j \geqslant 2$$

and

$$\bar{y}_0(\bar{x}, t) = y_0^-(x^*, t).$$

Now using part (i), we get $y^+(x^*, t) = y^{-1}(x^*, t)$. Proof of (iii) is similar. This completeness the proof of lemma.

Now we shall prove the main theorem.

Proof of main theorem. Following Lax [5], we introduce

$$\left.\begin{aligned}
u_N(x,t) &= \frac{\displaystyle\int_0^1 Q_1(x,y,t)\exp\left\{-N\left[\int_0^y u_0(z)\mathrm{d}z + Q(x,y,t)\right]\right\}\mathrm{d}y}{\displaystyle\int_0^1 \exp\left\{-N\left[\int_0^y u_0(z) + Q(x,y,t)\right]\right\}\mathrm{d}y}\\
f_N(x,t) &= \frac{\displaystyle\int_0^1 f(Q_1(x,y,t))\exp\left\{-N\left[\int_0^y u(z) + Q(x,y,t)\right]\right\}\mathrm{d}y}{\displaystyle\int_0^1 \exp\left\{-N\left[\int_0^y u_0(z)\mathrm{d}z + Q(x,y,t)\right]\right\}\mathrm{d}y}\\
V_N(x,t) &= \int_0^1 \exp\left\{-N\left[\int_0^y u_0(z)\mathrm{d}z + Q(x,y,t)\right]\right\}\mathrm{d}y\\
U_N(x,t) &= \frac{1}{N}\log V_N.
\end{aligned}\right\} \tag{3.29}$$

As in Lax [5], it follows that,

$$\lim_{N\to\infty} u_N(x,t) = Q_1(x, y_0(x,t), t)$$

$$\lim_{N\to\infty} f_N(x,t) = f(Q_1(x, y_0(x,t), t))$$

and

$$\lim_{N\to\infty} U_N(x,t) = U(x,t) = \int_0^{y_0(x,t)} u_0(z)\mathrm{d}z + Q(x, y_0(x,t), t) \tag{3.30}$$

where $y_0(x,t)$ minimizes (2.5).

Also

$$u_N(x,t) = -\frac{1}{N}\frac{(V_N)_x}{V_N} = (U_N)_x. \tag{3.31}$$

It follows from (3.29) and (3.30) that

$$\frac{\partial U}{\partial x}(x,t) = Q_1(x, y_0(x,t), t).$$

Next we shall show that

$$(U_N)_t = -f_N. \tag{3.32}$$

We consider

$$(U_N)_t = -\frac{1}{N}\frac{(V_N)_t}{V_N}$$

$$= \frac{\int_0^1 \frac{\partial Q}{\partial t}(x,y,t)\exp\left\{-N\left[\int_0^y u_0(z)\mathrm{d}z + Q(x,y,t)\right]\right\}\mathrm{d}y}{\int_0^1 \exp\left\{-N\left[\int_0^y u_0(z)\mathrm{d}z + Q(x,y,t)\right]\mathrm{d}y\right.}$$

$$= \frac{\int_0^1 f(Q_1(x,y,t))\exp\left\{-N\left[\int_0^y u_0(z)\mathrm{d}z + Q(x,y,t)\right]\right\}\mathrm{d}y}{\int_0^1 \exp\left\{-N\left[\int_0^y u_0(z)\mathrm{d}z + Q(x,y,t)\right]\right\}\mathrm{d}y}$$

To obtain the last equality we used Lemma (3.7) and Lemma (3.8). Now (3.32) follows from the definition of f_N. From (3.31) and (3.32) we get

$$(u_N)_t + (f_N)_x = 0.$$

Hence for all test functions $\varphi(x,t)\in C_0^\infty[(0,\infty)\times(0,\infty)]$ we get

$$\int_0^\infty \int_0^1 (u_N\varphi_t + f_N\varphi_x)\mathrm{d}x\,\mathrm{d}t = 0. \tag{3.33}$$

Let $N\to\infty$, in (3.33) and use (3.30) to get

$$\int_0^\infty \int_0^1 (u\varphi_t + f(u)\varphi_x)\mathrm{d}x\,\mathrm{d}t = 0.$$

Now we shall show that $u(x,t)$ satisfies the initial condition. By the argument similar to be Lemma (3.1) we get given $\varepsilon > 0$, $\exists\delta > 0$ such that for all $\varepsilon \leqslant x \leqslant 1-\varepsilon$, $t \leqslant \delta$,

$$u(x,t) = (f^*)'\left(\frac{x - y_0(x,t)}{t}\right)$$

where $y_0(x,t)$ minimizes

$$\min_{0\leqslant y\leqslant 1}\left[\int_0^y u_0(z)\mathrm{d}z + tf^*\left(\frac{x-y}{t}\right)\right].$$

But then Lax's argument [5] can be used to show

$$\lim_{t\to 0} u(x,t) = u_0(x) \quad \text{a.e. } \varepsilon \leqslant x \leqslant 1-\varepsilon.$$

Since $\varepsilon > 0$, is arbitrary, it follows that,

$$\lim_{t\to 0} u(x,t) = u_0(x) \quad \text{a.e. } 0 \leqslant x \leqslant 1.$$

Next we show that $u(x, t)$ satisfy the entropy condition (1.5). Because of Lemmas (3.5), (3.6) and (3.8) and the definition of Q_1, Lemma (3.6) it is clear that $u(x \pm 0, t)$ exists for all $0 < x < 1$. In fact

$$u(x \pm 0, t) = Q_1(x, y_0^{\mp}(x, t), t).$$

Now the entropy condition (1.5) follows from the definition of Q_1, Lemma (3.6) and the increasing nature of $(f^*)'$.

Lastly we show that $u(x, t)$ satisfies the boundary condition $(1.3)'_0$ and $(1.3)'_1$. Here again the existence of $u(0+, t)$ and $u(1-, t)$ follows as before. We shall verify that $u(0+, t)$ satisfies $(1.3)'_0$. The proof of $(1.3)'_1$ is similar and hence omitted.

To verify $(1.3)'_0$, at $t = t_0$, first notice that $u(x, t)$ defined by (2.6) satisfies the semigroup property i.e.

$$u(x, t_0) = \bar{u}(x, t_0)$$

where $\bar{u}(x, t)$ is defined by (2.6) with initial data is prescribed at time t_1, $0 < t_1 < t_0$:

$$\bar{u}(x, t_1) = u(x, t_1).$$

Because of this semigroup property and the condition (A_2), an argument similar to the proof of Lemma (3.1) can be used to show that

$$u(x, t) = Q_1(x, y_0(x, t), t) \forall t_0 - \varepsilon \leqslant t \leqslant t_0,$$

for some $\varepsilon > 0$ sufficiently small. Here $y_0(x, t)$ minimizes

$$\min_{0 \leqslant y \leqslant 1} \left[\int_0^y u(z, t_0 - \varepsilon) dz + Q^0(x, y, t) \right]$$

and

$$Q^0(x, y, t) = \min\{A^0_{1,0}(x, y, t), A^0_{0,0}(x, y, t), A^0_{0,1}(x, y, t)\}$$

where

$$A^0_{1,0}(x, y, t) = \min_{t_0 - \varepsilon < t_2 < t_1 < t_0} \left[-\int_{t_2}^{t_1} f(u_1^+(s)) ds + (t_0 - t_1) f^*\left(\frac{x}{t_0 - t_1}\right) + (t_2 - t_0 + \varepsilon) f^*\left(\frac{-y}{t_2 - t_0 + \varepsilon}\right) \right]$$

$$A^0_{0,0}(x, y, t) = (t_0 - (t_0 - \varepsilon)) f^*\left(\frac{x - y}{t_0 - t_1 + \varepsilon}\right) = \varepsilon f^*\left(\frac{x - y}{\varepsilon}\right)$$

$$A^0_{0,1}(x, y, t) = \min_{t_0 - \varepsilon < t_2 < t_1 < t_0} \left[-\int_{t_2}^{t_1} f(u_2^-(s)) ds + (t_0 - t_1) f^*\left(\frac{x - 1}{t_0 - t_1}\right) + (t_2 - t_0 + \varepsilon) f^*\left(\frac{-y}{t_2 - t_0 + \varepsilon}\right) \right].$$

Again because of the condition (A_2) it follows that if $x \leqslant \delta$, $(t_0 - t_1) f^*(x - 1/t_0 - t_1) \to \infty$ as $t_0 - t_1 \to 0$. Hence for $\varepsilon > 0$ sufficiently small one has

$$u(x, t) = Q_1(x, y_0(x, t), t) \forall t_0 - \varepsilon < t \leqslant t_0, \ 0 \leqslant x < \delta$$

where $y_0(x,t)$ minimizes

$$\min_{0\leqslant y\leqslant 1}\left[\int_0^y u(z,t_0-\varepsilon)\mathrm{d}z + Q^0(x,y,t)\right]$$

and

$$Q^0(x,y,t)=\min[A^0_{1,0}(x,y,t),A^0_{0,0}(x,y,t)].$$

In this case $u(x,t)$ satisfies $(1.3)'_0$ was proved in [4]. The proof of theorem is complete.

4. Weighted Burger's Equation

Equations of the type

$$(x^\alpha u)_t+\left(x^\alpha\frac{u^2}{2}\right)_x=0,\ \alpha>-2 \tag{4.1}$$

are interesting, because such kind of equations appear in fluid dynamics with spherical and cylindrical symmetry and is studied by Lefloch [3] in the quarter plane $x>0$, $t>0$.

As is observed in [3], a change of variable

$$v(y,t)=\left(\frac{\alpha}{2}+1\right)x^{\alpha/2}u(x,t),\ y=x^{\alpha/2+1} \tag{4.2}$$

transforms (4.1) into the Burgers equation.

$$v_t+\left(\frac{v^2}{2}\right)_x=0. \tag{4.3}$$

Thus $u(x,t)$ is a solution of (4.1) iff v is a solution of (4.3). From the Bardos *et al* [1] formulation of the initial boundary value problem one easily gets the following formulation of the initial and boundary condition for (4.1) in $D=\{(x,t):0\leqslant x\leqslant 1, t>0\}$.

Initial data for (4.1):

$$u(x,0)=u_0(x)\ 0\leqslant x\leqslant 1. \tag{4.4}$$

Boundary condition at $x=0$.

$$\left.\begin{array}{l}\lim\limits_{x\to 0}[x^{\alpha/2}u(x,t)]=u_1^+(t)\\ \text{or}\\ \lim\limits_{x\to 0}[x^{\alpha/2}u(x,t)]\leqslant 0 \text{ and } \lim\limits_{x\to 0}x^\alpha u^2(0+t_1)\geqslant u_1^+(t)^2\end{array}\right\}\text{a.e. } t>0. \qquad (4.5)_0$$

Boundary condition at $x=1$:

$$\left.\begin{array}{l}u(1-,t)=u_2^-(t)\\ \text{or}\\ u(1-,t)\geqslant 0 \text{ and } u^2(1-,t)\geqslant u_2^-(t)^2\end{array}\right\}\text{a.e. } t>0. \qquad (4.5)_1$$

Here we used the notation $u_1^+(t) = \max(u_1(t), 0)$, $u_2^-(t) = \min(u_2(t), 0)$.

Entropy condition:

$$u(x+,t) \leqslant u(x-t) \tag{4.6}$$

From the main theorem we get the following explicit formula for $u(x,t)$, the solution of the problem (4.1), (4.4) $(4.5)_0$, $(4.5)_1$ and (4.6).

$$u(x,t) = \left(\frac{2}{\alpha+2}\right) x^{-\alpha/2} v(x^{\alpha+2/2}, t)$$

where $v(y,t)$ is given by

$$v(y,t) = Q_1(y, z_0(y,t), t), \tag{4.7}$$

and $z_0(y,t)$ minimizes

$$\min_{\substack{\beta \in \mathscr{C}(y,z,t) \\ 1 \geqslant z \geqslant 0}} \left[\int_0^z \left(\frac{\alpha}{2}+1\right) z_1^{\alpha/\alpha+2} u_0(z_1^{2/\alpha+2}) \mathrm{d}z_1 - \frac{(\alpha+2)^2}{8} \int_{\{s:\beta(s)=0\}} (u_1^+(s))^2 \mathrm{d}s \right.$$
$$\left. \frac{(\alpha+2)^2}{8} \int_{\{s:\beta(s)=1\}} (u_2^-(s))^2 \mathrm{d}s + \frac{1}{2} \int_{\{s:0<\beta<1\}} \left(\frac{\mathrm{d}\beta}{\mathrm{d}s}\right)^2 \mathrm{d}s \right].$$

In (4.7), $Q_1(y, z_0(y,t), t)$ is defined by (2.2), (2.3) and (2.4) with $f(u) = u^2/2$.

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