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FAST TRACK COMMUNICATION

Bright and dark periods in the entanglement dynamics of interacting qubits in contact with the environment

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Abstract
Interaction among qubits is the basis to many quantum logic operations. We report how such inter-qubit interactions can lead to new features, in the form of bright and dark periods in the entanglement dynamics of two qubits subject to environmental perturbations. These features are seen to be precursors to the well-known phenomenon of sudden death of entanglement (Yu and Eberly 2004 *Phys. Rev. Lett.* **93** 140404) for noninteracting qubits. Further, we find that the generation of bright and dark periods is generic and occurs for wide varieties of environment models. We present explicit results for two popular models.

(Some figures in this article are in colour only in the electronic version)

One of the prime requirements for quantum computation is designing logic gates that can be used to implement algorithms based on the principles of quantum mechanics [1]. Over the last decade, substantial theoretical understanding and technological advancement have been acquired in this respect [2]. In analogy to fundamental gates like XOR in Boolean logic, it has been shown that the two-qubit logic gate along with single qubit rotation can perform fundamental logic operations (C-NOT) for quantum computation [3, 4]. In many cases, coherent qubit–qubit interactions have been invoked in constructing such quantum logic gate operations. Moreover, such qubit–qubit interactions have become especially important in the context of recent advances in quantum logic gate operations using trapped ions [5, 6] and semiconductor dots [7–10]. Note that an important performance factor for a quantum logic gate is its fidelity, which in turn depends on the entanglement between the two qubits. Sustained entanglement among the qubits is a must for optimized gate operation in the practical implementation of quantum algorithms. Unfortunately, entanglement among quantum systems is extremely fragile and susceptible to decoherence [11], an effect which arises due to unavoidable interaction of the physical system with its environment. Thus, the study of decoherence effects on the entanglement dynamics and ways to suppress it is of utter importance for quantum information science. As an ongoing effort in this respect, an important question to study is: how does the inter-qubit interaction in the presence of environmental perturbations affect the initial entanglement among the qubits and thus its operational capability? In this communication, we investigate this important question for two initially entangled interacting qubits.

We report that in the presence of environmental perturbations the qubit–qubit interaction leads to a new feature in the entanglement dynamics of two qubits. The two initially entangled interacting qubits get repeatedly disentangled and entangled as they dynamically evolve, leading to bright and dark periods in the entanglement. Eventually, for longer times we observe '*entanglement sudden death*' (ESD) [12]. Note that even though ESD has been studied extensively for non-interacting qubits [13–19] in contact with different environments, the case of interacting qubits as considered by us has not been extensively studied. An earlier work [20] had observed and discussed revivals of entanglement due to unitary interactions among the entangled sub-systems.

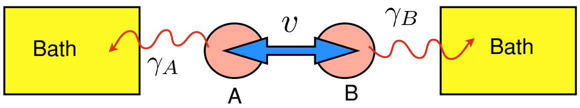

Figure 1. Schematic diagram of two qubits modelled as two two-level atoms coupled to each other by an interaction parameter v. The qubits A and B independently interact with their respective environments (baths), which leads to local decoherence as well as loss of entanglement.

Further, it was shown [21] that in the strong coupling regime of system–reservoir interaction the deterioration of entanglement can be controlled. In our work, we find that the bright and dark periods in the entanglement dynamics are precursors to ESD. Moreover, we show explicitly that the phenomenon of generation of bright and dark periods is quite generic and occurs for different kinds of models of the environment, such as the pure dephasing environment. Further, we also find that these bright and dark periods in entanglement can occur in the case of interacting qubits for states which do not exhibit ESD in the absence of interaction. Note that while we concentrate on qubits, Paz and Roncaglia [22][1] consider the case of continuous variables, i.e. harmonic oscillators, and demonstrate, in a certain parameter domain, such bright and dark periods in entanglement. Our results along with those of [22] would even lead one to think of the existence of such features in entanglement in a much larger class of systems. Further, while we focus on the effect of qubit–qubit interactions on decoherence, some recent works have shown how external coherent fields can be used to control decoherence effects in the entanglement of two qubits [23–25].

We now discuss our model and show how the interactions between qubits lead to these bright and dark periods in the entanglement. Our model consists of two initially entangled interacting qubits, labelled A and B. Each qubit can be characterized by a two-level system with an excited state $|e\rangle$ and a ground state $|g\rangle$. Further, we assume that the qubits interact independently with their respective environments. This leads to both local decoherence and loss of entanglement of the qubits. The decoherence, for instance, can arise due to spontaneous emission from the excited states. Figure 1 shows a schematic diagram of our model. The Hamiltonian for our model is then given by

$$\mathcal{H} = \hbar\omega_0\left(S_{\rm A}^z + S_{\rm B}^z\right) + \hbar v\left(S_{\rm A}^+ S_{\rm B}^- + S_{\rm B}^+ S_{\rm A}^-\right), \tag{1}$$

where v is the interaction between the two qubits and S_i^z, S_i^+, S_i^- (i = A, B) are the atomic energy, raising and lowering operators, respectively, which obey angular momentum algebra. We use the two-qubit product basis given by

$$\begin{aligned} |1\rangle &= |e\rangle_{\rm A}\otimes|e\rangle_{\rm B}, \qquad & |2\rangle &= |e\rangle_{\rm A}\otimes|g\rangle_{\rm B}, \\ |3\rangle &= |g\rangle_{\rm A}\otimes|e\rangle_{\rm B}, \qquad & |4\rangle &= |g\rangle_{\rm A}\otimes|g\rangle_{\rm B}. \end{aligned} \tag{2}$$

Now as each qubit independently interacts with its respective environment, the dynamics of this interaction can be treated in the general framework of master equations. The time evolution of the density operator ρ which gives us information about the dynamics of the system can then be evaluated from the quantum-Liouville equation of motion,

$$\dot\rho = -\frac{\rm i}{\hbar}[\mathcal{H},\rho] - \sum_{j=A,B}\frac{\gamma_j}{2}\left(S_j^+S_j^-\rho - 2S_j^-\rho S_j^+ + \rho S_j^+S_j^-\right), \tag{3}$$

where $\gamma_{\rm A}(\gamma_{\rm B})$ is the spontaneous decay rate of qubit A (B) to the environment. To investigate the effect of interaction between the two qubits on decoherence, we need to study the dynamics of two-qubit entanglement. The entanglement for any bipartite system is best identified by examining the concurrence [26, 27], an entanglement measure that relates to the density matrix of the system ρ. The concurrence for two qubits is defined as

$$C(t) = \max\{0, \sqrt{\lambda_1} - \sqrt{\lambda_2} - \sqrt{\lambda_3} - \sqrt{\lambda_4}\}, \tag{4}$$

where λ are the eigenvalues of the non-Hermitian matrix $\rho(t)\tilde\rho(t)$ arranged in decreasing order of magnitude. The matrix $\rho(t)$ is the density matrix for the two qubits and the matrix $\tilde\rho(t)$ is defined as

$$\tilde\rho(t) = \left(\sigma_y^{(1)}\otimes\sigma_y^{(2)}\right)\rho^*(t)\left(\sigma_y^{(1)}\otimes\sigma_y^{(2)}\right), \tag{5}$$

where $\rho^*(t)$ is the complex conjugation of $\rho(t)$ and σ_y is the usual Pauli matrix expressed in the basis (2). The concurrence varies from $C = 0$ for a separable state to $C = 1$ for a maximally entangled state. The density matrix needed to evaluate the concurrence for our model should in general have 16 elements. However, following [12] we take it as

$$\rho = \frac{1}{3}\begin{pmatrix} a & 0 & 0 & 0 \\ 0 & b & z & 0 \\ 0 & z^* & c & 0 \\ 0 & 0 & 0 & d \end{pmatrix}, \tag{6}$$

where, unlike [12], we allow the possibility of $z = |z|\,{\rm e}^{{\rm i}\chi}$ to be complex. We have proved from the solution of the quantum-Liouville equation (3) that the *initial density matrix* (6) *preserves its form for all t*. Finally, for our model the concurrence is found to be

$$C(t) = {\rm Max}\{0, \tilde C(t)\}, \tag{7}$$

where $\tilde C(t)$ is given by

$$\tilde C(t) = 2\left\{|\rho_{23}(t)| - \sqrt{\rho_{11}(t)\rho_{44}(t)}\right\}. \tag{8}$$

Let us now consider a class of mixed states [12] with a single parameter a satisfying initially $a \geqslant 0, b = c = |z| = 1$ and $d = 1 - a$. Then ρ has the form $\rho \equiv 1/3(a|e_1e_2\rangle\langle e_1e_2| + d|g_1g_2\rangle\langle g_1g_2| + |\psi\rangle\langle\psi|)$, where $|\psi\rangle \equiv (|e_1g_2\rangle + {\rm e}^{{\rm i}\chi}|g_1e_2\rangle)$. This has the structure of a Werner state [28]. The entanglement part of the state depends on χ. Using the solution of (3) in (8), we obtain one of our key results

$$\begin{aligned} \tilde C(t) = \tfrac{2}{3}\,{\rm e}^{-\gamma t}\big[&(\cos^2\chi + \sin^2\chi\cos^2(2vt))^{1/2} \\ &- \sqrt{a(1 - a + 2w^2 + w^4 a)}\big], \end{aligned} \tag{9}$$

where $w = \sqrt{1 - {\rm e}^{-\gamma t}}$. For simplicity, we have assumed equal decay rates of both the qubits, $\gamma_{\rm A} = \gamma_{\rm B} = \gamma$. One can clearly see the dependence of $\tilde C(t)$ on the interaction v between the

[1] This paper also contains an extensive bibliography on the question of existence of ESD in continuous variables.

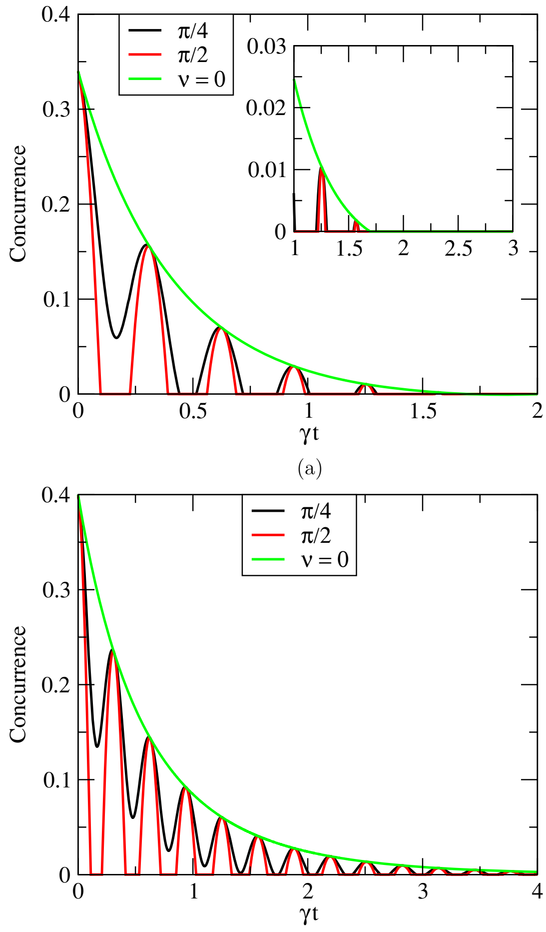

Figure 2. Concurrence as a function of time for two initially entangled, interacting qubits with initial conditions $b = c = |z| = 1.0$ and different initial phases χ. Parts (a) and (b) are for $a = 0.4$ and $a = 0.2$, respectively. The inset in (a) shows the long-time behaviour of concurrence. The red and black curves in both figures are for $v = 5\gamma$.

qubits and the initial phase χ. We see from (7) and (9) that in the absence of the interaction v, the concurrence becomes independent of the initial phase and yields the well-established result of Yu and Eberly [12].

Note that $\tilde{C}(t)$ can become negative if

$$a(1 - a + 2w^2 + w^4 a) > (1 - \sin^2\chi \sin^2(2vt)), \qquad (10)$$

in which case the concurrence is zero and the qubits get disentangled. To understand how the interaction would effect entanglement, we study the analytical result of equation (9) for different values of parameters a and χ. In figure 2, we show the time dependence of the entanglement for $v = 5\gamma$ and for different values of the initial phase χ. The inset of figure 2(a) shows the long-time behaviour of entanglement for this case. In figure 2(a) we show that for $a = 0.4$, the non-interacting qubits ($v = 0$) exhibit sudden death of entanglement (ESD) (more clearly visible in the inset), whereas when they interact ($v \neq 0$) the concurrence oscillates between zero and non-zero values with diminishing magnitude and eventually shows ESD. Thus, the initially entangled qubits in the presence of interaction v get repeatedly disentangled and entangled before finally becoming completely disentangled. Hence, as a result of interaction between the qubits, the concurrence exhibits *bright and dark periods* in the entanglement. Further we observe that when concurrence becomes zero, it remains zero for a time range before reviving. It is worth mentioning here that such bright and dark periodic behaviour in entanglement has been predicted for qubits undergoing unitary evolution in a lossless cavity [29].[2] This time range is determined by condition (10). In figure 2(b) we plot the concurrence for $a = 0.2$. Note that for $a = 0.2$, no ESD is observed when the qubits are non-interacting and the concurrence monotonically goes to zero as $t \longrightarrow \infty$. For $v \neq 0$, we observe the bright and dark periods in entanglement with diminishing magnitudes and $C(t) \longrightarrow 0$ as $t \longrightarrow \infty$. Figure 3 shows the bright and dark periods in two-qubit entanglement for three different spontaneous decay rates and $a = 0.2$. The initial phase χ is chosen to be $\pi/2$. For this value of a we observe no ESD but only collapse and revival as expected.

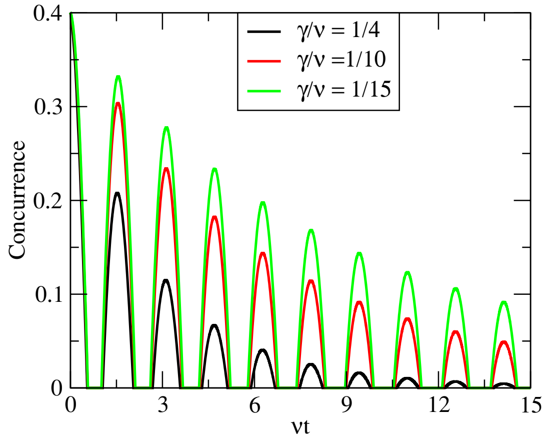

Figure 3. Concurrence as a function of time for different decay rates of two initially entangled qubits with initial conditions $a = 0.2, b = c = |z| = 1, \chi = \pi/2$.

Pure dephasing of qubits due to interaction with the environment

In order to demonstrate the generic nature of our results, we consider other models of the environment. A model which has been successfully used in experiments [30] involves pure dephasing. In this case, the last two terms in the master equation (3) are replaced by

$$-\sum_{i=A,B} \Gamma_i \left(S_i^z S_i^z \rho - 2 S_i^z \rho S_i^z + \rho S_i^z S_i^z \right) \qquad (11)$$

where $2\Gamma_A (2\Gamma_B)$ is the dephasing rate of qubit A (B). Note that in such a model the populations do not decay as a result of the interaction with the environment whereas coherences such as $\rho_{23}(t)$ decay as $\rho_{23}(0)\, e^{-(\Gamma_A+\Gamma_B)t}$. Let us now study the effect of interaction v between the qubits on the dynamics

[2] A recent paper (Yögnac and Eberly 2008 *Opt. Lett.* **33** 270) reports on such bright and dark periods in entanglement for *noninteracting* qubits driven by single-mode quantized fields, which is in a way reminiscent of Jaynes–Cummings dynamics.

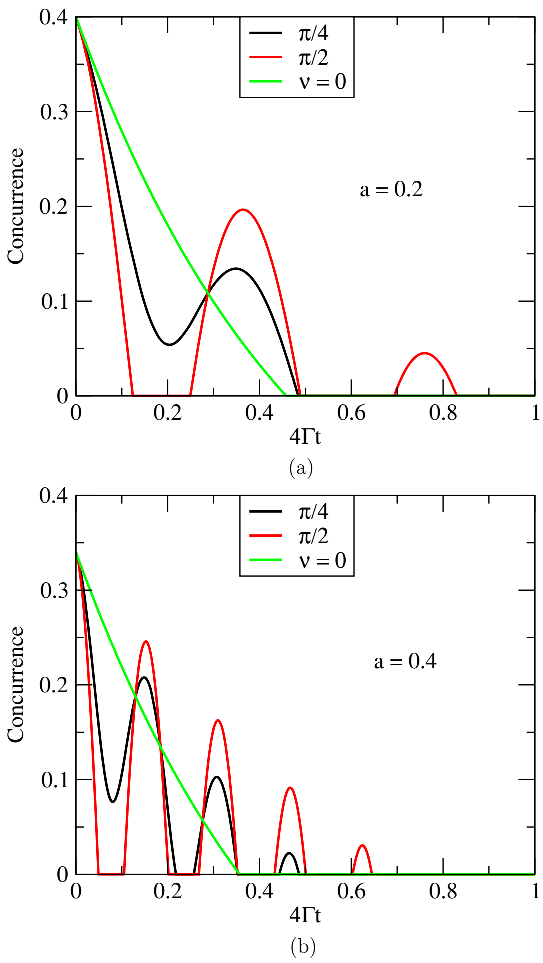

Figure 4. Concurrence as a function of time with initial conditions $b = c = |z| = 1$ and different values of phase χ for the dephasing model. The red and black curves in part (a) are for $v/4\Gamma = 4$ and in (b) for $v/4\Gamma = 10$.

of entanglement. We assume the same initial density matrix of equation (6) with the initial conditions $d = 1 - a$, $b = c = |z| = 1$ and $a \geqslant 0$ to calculate the concurrence. Under pure dephasing, the *form of matrix in (6) is preserved for all time*. Using (7), (8) and (11) we get

$$\tilde{C}_D(t) = \frac{2}{3}\left[e^{-\tau}\left\{e^{-2\tau}\cos^2\chi + \sin^2\chi\left\{\cos(\Omega'\tau) - \frac{1}{\Omega'}\sin(\Omega'\tau)\right\}^2\right\}^{1/2} - \sqrt{a(1-a)}\right], \qquad (12)$$

where we assume $\Gamma_A = \Gamma_B = \Gamma$. Here $\tau = 4\Gamma t$ and $\Omega' = \sqrt{(2v/4\Gamma)^2 - 1}$. For $v = 0$ we get $\tilde{C}_D(t) = 2/3[e^{-2\tau} - \sqrt{a(1-a)}]$, which is independent of the initial phase χ. We find *death of entanglement* for $\tau > -1/2 \ln \sqrt{a(1-a)}$. Note that Yu and Eberly [14] have considered this case earlier but for $a = 1$ only, in which case there is no ESD. In figure 4, we show the time dependence of entanglement for a purely dephasing model, for different values of a and initial coherences governed by the phase χ. From the figures, it is seen that for $v \neq 0$ the two-qubit entanglement exhibits bright and dark periods. Further, we also see that for $v \neq 0$ entanglement exhibits this feature even beyond the time when ESD occurs for noninteracting qubits. Moreover, figure 4(b) shows that the frequency of this periodic feature increases with increase in strength of the interaction v. The dark period between two consecutive bright periods arises as a result of $\tilde{C}_D(t) < 0$, for some time range. This physically means that the two qubits remain disentangled during this time range.

This new feature of bright and dark periods in entanglement should have direct consequences for microscopic systems like ion traps and quantum dots which are currently the forerunners in the implementation of quantum logic gates. The interaction between qubits considered in this communication is inherently present in these systems. In quantum dots for example, $\gamma^{-1} \sim$ few nanoseconds and one can get a very large range of the parameter Γ^{-1} (1–100s ps) [31]. Further, the interaction strength v can have a range between 1μ eV and 1 meV depending on gate biasing [9, 10, 32]. An earlier study [33] reports $\gamma \sim 40$–100μ eV and coupling strength of ~ 100–400μ eV, thereby making $v/\gamma \sim 1$–10 for quantum dot molecules. Thus, the experimental parameters are in the range we used for our numerical calculation.

To summarize, we have shown how the interaction between qubits can effect the entangled dynamics of an initially entangled two-qubit system. The interaction leads to the formation of bright and dark periods in entanglement. We find this feature for different models of the environment. The frequency of bright and dark periods was found to depend on the strength of interaction between the qubits. Further, we find that for noninteracting qubits, even when sudden death of entanglement does not occur, entanglement can exhibit these bright and dark periods when the qubits interact. As a future perspective, we can investigate the entanglement dynamics for qubits in contact with several different environments.

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