CONIC SINGULARITIES, GENERALIZED SCATTERING MATRIX, AND INVERSE SCATTERING ON ASYMPTOTICALLY HYPERBOLIC SURFACES

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Abstract. We consider an inverse problem associated with some 2-dimensional non-compact surfaces with conical singularities, cusps and regular ends. Our motivating example is a Riemann surface $\mathcal{M} = \Gamma\backslash\mathbf{H}^2$ associated with a Fuchsian group of the 1st kind Γ containing parabolic elements. $\mathcal{M}$ is then non-compact, and has a finite number of cusps and elliptic singular points, which is regarded as a hyperbolic orbifold. We introduce a class of Riemannian surfaces with conical singularities on its finite part, having cusps and regular ends at infinity, whose metric is asymptotically hyperbolic. By observing solutions of the Helmholtz equation at the cusp, we define a generalized S-matrix. We then show that this generalized S-matrix determines the Riemannian metric and the structure of conical singularities.

1. Introduction

1.1. **Assumptions on the manifold.** Throughout this paper S^r denotes the circle of radius r, which is identified with $\mathbf{R}/(2\pi r\mathbf{Z})$:

$$S^r = \{(x_1, x_2) \quad ; \quad x_1 + ix_2 = re^{(ix/r)} = (r\cos(\theta),\, r\sin(\theta)),\\ 0 \le x \le 2\pi r,\ 0 \le \theta \le 2\pi\},$$

with an obvious identification of $\theta = 0$ and $\theta = 2\pi$. Thus, to write a function on S^r we would write it as $f(x)$, $x \in S^r$, or $f(\theta)$, $\theta \in [0, 2\pi]$ or $\theta \in \mathbb{R}$, assuming $2\pi-$periodicity.

We consider a 2-dimensional orientable connected C^∞-surface without boundary $\mathcal{M}$, which is written as a union of open sets:

$$\mathcal{M} = \mathcal{K} \cup \mathcal{M}_1 \cup \cdots \cup \mathcal{M}_N \tag{1.1}$$

satisfying the following assumptions.

(A-1) *$\overline{\mathcal{K}}$ is compact, and there exists a finite set $\mathcal{M}_{sing} \subset \mathcal{K}$ such that $\mathcal{M} \setminus \mathcal{M}_{sing}$ is equipped with a C^∞-Riemannian metric g.*

(A-2) *For any $p \in \mathcal{M}_{sing}$, there exist a constant $\epsilon_p > 0$ and local coordinates $(r, \theta) \in (0, \epsilon_p) \times [0, 2\pi]$ around p such that $r = 0$ corresponds to p and the Riemannian metric g takes the form*

$$(ds)^2 = (dr)^2 + C_p r^2(1 + h_p(r, \theta))(d\theta)^2, \tag{1.2}$$

Date: August 3, 2011.

where

(A-2-1) *C_p is a positive constant such that $C_p \neq 1$,*

(A-2-2) $h_p(r,\theta) \in C^\infty((0,\epsilon_p) \times [0,2\pi])$,

(A-2-3) *As $r \to 0$, $h_p(r,\theta) \to 0$ uniformly with respect to $\theta \in [0,2\pi]$.*

(A-3) *There exists $\mu \geq 1$ such that for $1 \leq i \leq \mu$, $\mathcal{M}_i$ is isometric to $S^{r_i} \times (1,\infty)$, $r_i > 0$, equipped with the metric*

$$ds^2 = \frac{(dx)^2 + (dy)^2}{y^2}.$$

(A-4) *For $\mu + 1 \leq i \leq N$, $\mathcal{M}_i$ is diffeomorphic to $S^{r_i} \times (0,1)$, $r_i > 0$, and the metric on $\mathcal{M}_i$ has the following form :*

$$ds^2 = y^{-2}\left((dy)^2 + (dx)^2 + A(x,y,dx,dy)\right),$$

$$A(x,y,dx,dy) = a_1(x,y)(dx)^2 + a_2(x,y)dxdy + a_3(x,y)(dy)^2,$$

where $a_i(x,y)$ $(i = 1,2,3)$ satisfies the following condition

$$|\partial_x^\alpha \left(y\partial_y\right)^n a_i(x,y)| \leq C_{\alpha n}(1 + |\log y|)^{-n-1-\epsilon_0}, \quad \forall \alpha, n,$$

for some $\epsilon_0 > 0$.

We say that under the above assumption (A-2), the metric g has a *conical singularity* at $p \in \mathcal{M}_{sing}$. The part $\mathcal{M}_i$, $1 \leq i \leq \mu$, will be called a *cusp*. (This is a little abuse of the standard terminology). Since $\mu \geq 1$, $\mathcal{M}$ has at least one cusp. If $\mu = N$, all the ends have a cusp. We call $\mathcal{M}_i$, $\mu + 1 \leq i \leq N$, *regular part*. The metric on regular parts are allowed to be different from each other.

In [24], spectral theory for asymptotically hyperbolic manifolds without conical singularities is discussed, and the arguments there can be extended to the above situation. Let Δ_g be the Laplace-Beltami operator for the metric g, and H the Friedrichs extension of $-\Delta_g - 1/4$ associated with the quadratic form $A_g[u,v] = (\nabla u, \nabla v) - \frac{1}{4}(u,v)$ with $u, v \in \mathcal{D}(A_g) = H^1(\mathcal{M})$. It has continuous spectrum $\sigma_c(H) = [0,\infty)$, and the discrete spectrum $\sigma_d(H) \subset (-\infty, 0)$. If at least one of the ends is regular, there is no eigenvalues in $(0,\infty)$. If all the ends are cusps, H may have embedded eigenvalues in $(0,\infty)$, which are discrete with possible accumulation points 0 and ∞.

1.2. **Inverse scattering from regular ends.** An important notion to describe the spectral properties of H is the S-matrix. Usually, it is introduced by observing the asymptotic behavior, as time tends to $\pm\infty$, of solutions to the time-dependent Schrödinger equation or the wave equation on $\mathcal{M}$, i.e. $S = W_+^* W_-$, where $W_\pm = \mathrm{s} - \lim_{t\to\pm\infty} e^{itH} e^{-itH_0}$, or $W_\pm = \mathrm{s} - \lim_{t\to\pm\infty} e^{it\sqrt{H}} e^{-it\sqrt{H_0}}$, where H_0 is the *unperturbed* operator, to which H is asymptotic at infinity. An equivalent way is to observe asymptotic expansions at infinity of *physical* solutions to the Helmholtz equation on $\mathcal{M}$. In the case of our manifold $\mathcal{M}$, by the physical solution u, we roughly mean that u behaves like $O(y^{1/2})$ on each end. The (*physical*) S-matrix $\widehat{S}(k)$, k being the square root of the energy of the system, is an operator valued $N \times N$ matrix, $\widehat{S}(k) = \big(\widehat{S}_{ij}(k)\big)$, where $\widehat{S}_{ij}(k)$ corresponds to the wave coming in from the end $\mathcal{M}_j$ and going out of the end $\mathcal{M}_i$ (see e.g. [8, 9, 12, 18, 19, 20, 27, 28, 44, 46, 47, 53] for variuos related results on the spectral and scattering theory for hyperbolic and asymptocally hyperbolic spaces.) Having $S-$matrix, one can then talk about the inverse problem. Let us consider the case without singular points. Suppose we are given two such manifolds $\mathcal{M}^{(1)}$, $\mathcal{M}^{(2)}$, and assume $\mathcal{M}_1^{(i)}$ is a regular end for $i = 1, 2$. We also assume that for $\mathcal{M}^{(1)}$ and $\mathcal{M}^{(2)}$, the $(1, 1)$ component of the associated S-matrix coincide, i.e. $\widehat{S}_{11}^{(1)}(k) = \widehat{S}_{11}^{(2)}(k)$ for all $k > 0$. If, furthermore, two ends $\mathcal{M}_1^{(1)}$ and $\mathcal{M}_1^{(2)}$ are isometric for large y, these two manifolds $\mathcal{M}^{(1)}$ and $\mathcal{M}^{(2)}$ are shown to be globally isometric (see [24]). Let us note that, when all the ends are regular, Sa Barreto [49], see also [15], proved that, in the framework of scattering metric due to Melrose, two such manifolds are isometric, if the whole scattering matrix for all energies coincide, without assuming that one end is known to be isometric. The related inverse boundary value problems for compact Riemannian manifolds can presently be solved with fixed frequency data in the zero energy case [41, 22], when the metric is real analytic [42, 40], or when the tensor is known to be of appropriate type up to a conformal factor [11, 16, 17]. On review on the positive results and counterexamples for these problems, see [13]. For the resonance problem, another view point for inverse scattering, see e.g. [19] and [8], [10].

1.3. **Main result.** The problem we address here is the case in which we observe the waves coming in and going out of a cusp. Recall that the end $\mathcal{M}_1$ has a cusp at infinity. Since the continuous spectrum due to the cusp is 1-dimensional, the associated S-matrix component $\widehat{S}_{11}(k)$ is a complex number, and it does not have enough information to determine the whole manifold. Therefore, we generalize the notion of the S-matrix. This generalized S-matrix was introduced in [23] in the inverse scattering from a fixed energy for Schrödinger operators on asymptotically hyperbolic manifolds.

The Helmholtz equation has the following form in the cusp $\mathcal{M}_1$

$$-y^2(\partial_y^2 + \partial_x^2)u - \frac{1}{4}u = k^2 u,$$

where $k > 0$. Passing to the Fourier series, we see that all solutions of this equation have the asymptotic expansion

$$\begin{aligned} u(x, y) \simeq a_0\, y^{\frac{1}{2}-ik} &+ \sum_{n\neq 0} a_n \Big(\frac{r_1}{2\pi|n|}\Big)^{1/2} e^{inx/r_1} e^{|n|y/r_1} \\ &+ b_0\, y^{\frac{1}{2}+ik} + \sum_{n\neq 0} b_n \Big(\frac{\pi r_1}{2|n|}\Big)^{1/2} e^{inx/r_1} e^{-|n|y/r_1} \end{aligned}$$

as $y \to \infty$. We call the operator

$$\mathbf{S}_{11}(k) : \{a_n\} \to \{b_n\}$$

the generalized S-matrix, actually its (1,1) component (see §4 for the precise definition). We shall show that this generalized S-matrix determines the whole manifold $\mathcal{M}$. Namely, suppose we are given two manifolds $\mathcal{M}^{(1)}, \mathcal{M}^{(2)}$ satisfying the assumptions (A-1) $\sim$ (A-4). Let $\mathcal{M}^{(i)}_{sing} = \{p^{(i)}_1, \cdots, p^{(i)}_{k_i}\}$ be the set of singular points.

Our main result is the following.

Theorem 1.1. *Suppose we are given two manifolds $\mathcal{M}^{(1)}$ and $\mathcal{M}^{(2)}$ satifying the assumptions* **(A-1)** $\sim$ **(A-4)**. *Let the (1,1) component of the generalized scattering matrix coincide :*

$$\mathbf{S}^{(1)}_{11}(k) = \mathbf{S}^{(2)}_{11}(k), \quad \forall k > 0, \quad k^2 \notin \sigma_p(H^{(1)}) \cup \sigma_p(H^{(2)}),$$

and $r^{(1)}_1 = r^{(2)}_1$. Then there is an isometry between $\mathcal{M}^{(1)}$ and $\mathcal{M}^{(2)}$ in the following sense.
(1) There is a homeomorphism $\Phi : \mathcal{M}^{(1)} \to \mathcal{M}^{(2)}$.
(2) $\Phi(\mathcal{M}^{(1)}_{sing}) = \mathcal{M}^{(2)}_{sing}$.
(3) $\Phi : \mathcal{M}^{(1)} \setminus \mathcal{M}^{(1)}_{sing} \to \mathcal{M}^{(2)} \setminus \mathcal{M}^{(2)}_{sing}$ is a Riemannian isometry.
(4) If $p \in \mathcal{M}^{(1)}_{sing}$, then $C^{(1)}_p = C^{(2)}_{\Phi(p)}$ and there is β such that, in coordinates (A-2), we have $h^{(1)}_p(r,\theta) = h^{(2)}_{\Phi(p)}(r, \widehat{\theta + \beta})$.

Here, for any $\theta \in \mathbb{R}$, $\widehat{\theta} \in [0, 2\pi)$ satisfies $\theta - \widehat{\theta} \in 2\pi\mathbb{Z}$.

As will be seen from the arguments in §2 and §3, we can introduce the *physical* S-matrix for manifolds satisfying (A-1) $\sim$ (A-4) of this paper, and generalize the results on the inverse scattering from regular ends in [24] to our case. Moreover we can also prove the same result for the inverse scattering with respect to the generalized S-matrix. Therefore, Theorem 1.1 combined with [24], implies the following theorem.

Theorem 1.2. *Suppose we are given two manifolds $\mathcal{M}^{(1)}$ and $\mathcal{M}^{(2)}$ satisfying the assumptions (A-1) $\sim$ (A-4). Suppose there exist ν_1 and ν_2 such that the (ν_1, ν_1) and (ν_2, ν_2) components of the generalized S-matrices coincide:*

$$\mathbf{S}^{(1)}_{\nu_1\nu_1}(k) = \mathbf{S}^{(2)}_{\nu_2\nu_2}(k), \quad \forall k > 0, \quad k^2 \notin \sigma_p(H^{(1)}) \cup \sigma_p(H^{(2)}).$$

Assume, furthermore, that their ends $\mathcal{M}^{(1)}_{\nu_1}$ and $\mathcal{M}^{(2)}_{\nu_2}$ are isometric. Then we have the same conclusion as in Theorem 1.1.

A good example of a surface with conical singularities is a 2-dimensional Riemannian orbifold, and classical examples are given by hyperbolic orbifolds with finite elliptic singular points. For example, consider $\mathcal{M} = \Gamma\backslash\mathbf{H}^2$, where Γ is a Fuchsian group. As will be explained in §2, if Γ is a geometrically finite Fuchsian group, $\Gamma\backslash\mathbf{H}^2$ satisfies the assumptions (A-1) $\sim$ (A-4). Therefore, the following theorem holds.

Theorem 1.3. *Given two geometrically finite hyperbolic orbifolds* $\Gamma_1\backslash\mathbf{H}^2$ *and* $\Gamma_2\backslash\mathbf{H}^2$*, suppose there exist* ν_1 *and* ν_2 *such that the* (ν_1,ν_1) *and* (ν_2,ν_2) *components of the generalized S-matrices coincide:*

$$\mathbf{S}^{(1)}_{\nu_1\nu_1}(k) = \mathbf{S}^{(2)}_{\nu_2\nu_2}(k), \quad \forall k > 0, \quad k^2 \notin \sigma_p(H^{(1)}) \cup \sigma_p(H^{(2)}).$$

Assume, furthermore, that their ends $\mathcal{M}^{(1)}_{\nu_1}$ *and* $\mathcal{M}^{(2)}_{\nu_2}$ *are isometric. Then we have the same conclusion as in Theorem 1.1. Moreover,* $\Phi : \mathcal{M}^{(1)} \to \mathcal{M}^{(2)}$ *is an anlytic diffeomorphism, and is lifted to an orbifold isomorphism between* $\mathcal{M}^{(1)}$ *and* $\mathcal{M}^{(2)}$*.*

For the notions of and geometrically finite hyperbolic orbifolds and orbifold isomorphism, see Subsections 2.1 and 2.3.

To prove Theorem 1.1, we need to study it from two sides : the forward problem and the inverse problem. In both issues, the arguments are centered around asymptotically hyperbolic ends and singularities in the finite parts. The main ingredient of the forward problem is the spectral and scattering theory for Laplace-Beltrami operators on asymptotically hyperbolic manifolds, which two of the authors have studied in [24]. Since this part does not depend on the space dimension, we shall state only the results in this paper, leaving the detailed explanations in our paper [26], where we extend the above theorem to the higher dimensional case. Relations to the collapse theory of Riemannian manifolds will be discussed in [38].

The crucial idea for the inverse problem part is the boundary control method. Just like our previous paper for the inverse scattering on manifolds with cylindrical ends [25], we reduce the issue to the inverse boundary value problem from an artificial boundary in the end $\mathcal{M}_1$. The new ingredient in this paper is the argument around conic singularities based on the explicit form of the metric (1.2).

We use a variety of notions from algebra, geometry and analysis in this paper: Fuchsian groups, orbifolds, conical singularities, spectral theory for self-adjoint operators with continuous spectrum, boundary control method. They are not complicated in themselves, however, we shall try to make the paper as readable as possible, by giving detailed explanations for elementary parts, sometimes referring to other papers for precise proofs. In §2, we recall basic facts on the Fuchsian groups, 2-dimensional hyperbolic orbifolds to explain our motivating example, and introduce the manifold with conical singularities. In §3, we study spectral properties of the Laplace-Beltrami operator of our manifold. The generalized S-matrix is defined in §4. We shall prove Theorem 1.1 in §5, and Theorem 1.3 in §6.

The notations used in this paper are standard. For Banach spaces $\mathcal{X}$ and $\mathcal{Y}$, $\mathbf{B}(\mathcal{X};\mathcal{Y})$ denotes the set of all bounded linear operators from $\mathcal{X}$ to $\mathcal{Y}$. For a self-adjoint operator A in a Hilbert space $\mathcal{H}$, $\sigma(A)$, $\sigma_p(A)$, $\sigma_c(A)$, $\sigma_d(A)$, $\sigma_e(A)$, $\sigma_{ac}(A)$ denote its spectrum, point spectrum (the set of all eigenvalues of A), continuous spectrum, discrete spectrum, essential spectrum and absolutely continuous spectrum, respectively, and $\mathcal{H}_{ac}(A)$ and

$\mathcal{H}_{pp}(A)$ are the absolutely continuous subspace for A and the closure of the linear hull of eigenvectors for A, respectively. Generic points on $\mathcal{M}$ are denoted by $p, \dots$, or $X, Y, \dots$, while those in the ends $\mathcal{M}_j$ are often written as (x, y). $\mathbf{N}$ denotes the set of all positive integers. When $\mathbf{h}$, $I \subset \mathbb{R}$ is an interval and $d\mu$ is a measure on I, $L^2(I, \mathbf{h}; d\mu)$ denotes the space of all $\mathbf{h}$-valued L^2-functions on I with respect to $d\mu$.

2. 2-dimensional hyperbolic orbifolds and conical singularties

2.1. **Fuchsian groups.** The upper-half space model of 2-dimensional hyperbolic space $\mathbf{H}^2$ is $\mathbf{C}_+ = \{z = x + iy\,; y > 0\}$ equipped with the metric

$$ds^2 = \frac{(dx)^2 + (dy)^2}{y^2}. \tag{2.1}$$

The infinity of $\mathbf{H}^2$ is

$$\partial\mathbf{C}_+ = \mathbf{R} \cup \infty.$$

$\mathbf{H}^2$ admits an action of $SL(2, \mathbf{R})$ defined by

$$SL(2, \mathbf{R}) \times \mathbf{C}_+ \ni (\gamma, z) \to \gamma \cdot z := \frac{az + b}{cz + d}, \quad \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}. \tag{2.2}$$

The right-hand side, Möbius transformation, is an isometry on $\mathbf{H}^2$. The mapping : $\gamma \to \gamma\cdot$ is 2 to 1, and the corresponding factor group of Möbius transformations is isomorphic to $PSL(2, \mathbf{R}) = SL(2, \mathbf{R})/\{\pm I\}$. For $\gamma \neq \pm I$, the transformation (2.2) is classified into 3 categories :

$$\begin{aligned}
\textit{elliptic} &\Longleftrightarrow \text{there is only one fixed point in } \mathbf{C}_+ \\
&\Longleftrightarrow |\mathrm{tr}\,\gamma| < 2, \\
\textit{parabolic} &\Longleftrightarrow \text{there is only one degenerate fixed point on } \partial\mathbf{C}_+ \\
&\Longleftrightarrow |\mathrm{tr}\,\gamma| = 2, \\
\textit{hyperbolic} &\Longleftrightarrow \text{there are two fixed points on } \partial\mathbf{C}_+ \\
&\Longleftrightarrow |\mathrm{tr}\,\gamma| > 2.
\end{aligned}$$

Let Γ be a discrete subgroup, *Fuchsian group*, of $SL(2, \mathbf{R})$, and $\mathcal{M} = \Gamma\backslash\mathbf{H}^2$ by the action (2.2). Γ is said to be *geometrically finite* if the fundamental domain $\Gamma\backslash\mathbf{H}^2$ is chosen to be a finite-sided convex polygon. The sides are then geodesics of $\mathbf{H}^2$. The geometric finiteness is equivalent to that Γ is finitely generated ([31], p. 104). Let us give two simple but important examples.

2.1.1. *Parabolic cyclic group.* Consider the cyclic group Γ generated by the action $z \to z + \tau$. This is parabolic with fixed point ∞. The associated fundamental domain is then $[-\tau/2, \tau/2] \times (0, \infty)$ with the sides $x = \pm\tau/2$ being geodesics. The Riemann surface $\mathcal{M}$ is then equal to $S^{\tau/2\pi} \times (0, \infty)$, which is a hyperbolic manifold with metric (2.1). It has two infinities : $S^{\tau/2\pi} \times \{0\}$ and ∞. The part $S^{\tau/2\pi} \times (0, 1)$ has an infinite volume. The part $S^{\tau/2\pi} \times (1, \infty)$ has a finite volume, and is called the *cusp*.

2.1.2. *Hyperbolic cyclic group.* Another simple example is the cyclic group generated by the hyperbolic action $z \to \lambda z$, $\lambda > 1$. The sides of the fundamental domain $\{1 \leq |z| \leq \lambda\}$ are semi-circles orthogonal to $\{y = 0\}$, which are geodesics. The quotient manifold is diffeomorphic to $S^{(\log \lambda)/2\pi} \times (-\infty, \infty)$. It is parametrized by (t, r), where $t \in \mathbf{R}/(\log \lambda)\mathbf{Z}$ and r is the signed distance from the segment $\{(0, s)\,; 1 \leq s \leq \lambda\}$. The metric is then written as

$$ds^2 = (dr)^2 + \cosh^2 r\, (dt)^2. \tag{2.3}$$

The part $r > 0$ (or $r < 0$) is called the *funnel.* Letting $y = 2e^{-r}$, $r > 0$, one can rewrite (2.3) as

$$ds^2 = \Big(\frac{dy}{y}\Big)^2 + \Big(\frac{1}{y} + \frac{y}{4}\Big)^2 (dt)^2.$$

Therefore, the funnel is regarded as a perturbation of the infinite volume part $S^{(\log \lambda)/2\pi} \times (0, 1)$ of the fundamental domain for the parabolic cyclic group.

2.2. **Classification of 2-dimensional hyperbolic manifolds.** The set of limit points of a Fuchsian group Γ, denoted by $\Lambda(\Gamma)$, is defined as follows : $w \in \Lambda(\Gamma)$ if there exist $z_0 \in \mathbf{C}_+$ and $\gamma_n \in \Gamma$, $\gamma_n \neq \mathrm{I}$, such that $\gamma_n \cdot z_0 \to w$. Since Γ acts discontinuously on $\mathbf{C}_+$, $\Lambda(\Gamma) \subset \partial\mathbf{H}^2 = \partial\mathbf{C}_+$. There are only 3 possibilities.

- (*Elementary*) : $\Lambda(\Gamma)$ is a finite set.
- (*The 1st kind*) : $\Lambda(\Gamma) = \partial\mathbf{H}^2$.
- (*The 2nd kind*) : $\Lambda(\Gamma)$ is a perfect (i.e. every point in $\Lambda(\Gamma)$ is an accumulation point of $\Lambda(\Gamma)$), nowhere dense set of $\partial\mathbf{H}^2$.

Any elementary group is either cyclic or is conjugate in $PSL(2, \mathbf{R})$ to a group generated by $\gamma \cdot z = \lambda z$, $(\lambda > 1)$, and $\gamma' \cdot z = -1/z$ (see [31], Theorem 2.4.3).

For non-elementary case, we have the following theorem ([8], Theorem 2.13). Although [8] deals with the case without elliptic fixed points, this theorem holds for the case with elliptic fixed points.

Theorem 2.1. *Let $\mathcal{M} = \Gamma\backslash\mathbf{H}^2$ be a non-elementary geometrically finite hyperbolic manifold. Then there exists a compact subset $\mathcal{K}$ such that $\mathcal{M} \setminus \mathcal{K}$ is a finite disjoint union of cusps and funnels.*

The regions mentioned above, i.e. fundamental domains of parabolic cyclic groups, hyperbolic cyclic groups, and non-elementary geometrically finite groups are the models of hyperbolic spaces to be dealt with in this paper.

Other important theorems are the following (see [31], Theorems 4.5.1, 4.5.2 and 4.1.1).

Theorem 2.2. *A Fuchsian group is of the 1st kind if and only if its fundamental domain has a finite area.*

Theorem 2.3. *A Fuchsian group of the 1st kind is geometrically finite.*

For the Fuchsian group of the 1st kind, therefore, the ends of its fundamental domain are always cusps. In this case, usually it is compactified around parabolic fixed points and made to a compact Riemann surface. The automorphic functions associated with this group turn out to be algebraic functions on this Riemann surface (see [43]).

It is well-known that there is a 1 to 1 correspondence between the compact Riemann surfaces and the fields of algebraic functions. This suggests a general idea that a surface will be determined by a set of functions on it. What we pursue in this paper is an analogue of this fact. Asymptotically hyperbolic manifolds, more generally non-compact Riemannian manifolds with good structure at infinity will be determined by the set of solutions to the Helmholtz equation, more precisely, by the asymptotic behavior at infinity of solutions to the Helmholtz equation. Before going into the detail of this issue, we need to recall the notion of orbifolds.

2.3. **Elliptic fixed point and analytic structure of $\Gamma\backslash\mathbf{H}^2$.** Now, we study the analytic structure of $\mathcal{M} = \Gamma\backslash\mathbf{H}^2$, where Γ is a Fuchsian group. Let $\mathcal{M}_{sing}$ be the set of all elliptic fixed points in $\mathcal{M}$. Under the assumption of geometric finiteness, $\mathcal{M}_{sing}$ is a finite set.

Lemma 2.4. *Let $p \in \mathcal{M}_{sing}$, and*

$$\mathcal{I}(p) = \{\gamma \in \Gamma\,;\, \gamma \cdot p = p\}$$

the isotropy group of p. Then, it is a finite cyclic group, and its generator γ_0 satisfies

$$\frac{w-p}{w-\overline{p}} = e^{2\pi i/n}\frac{z-p}{z-\overline{p}}, \quad w = \gamma_0 \cdot z \tag{2.4}$$

for some $n = n_p \in \mathbf{N}$.

Proof. Recall that the cross ratio

$$(z_1, z_2, z_3, z_4) = \frac{z_1 - z_3}{z_1 - z_4} \cdot \frac{z_2 - z_4}{z_2 - z_3}$$

is invariant by the fractional linear transformation $z \to w = \gamma \cdot z = (az + b)(cz + d)^{-1}$. Suppose $p, q \in \mathbf{C}$ are the fixed points of $z \to w$. Then, since $z = \infty$ is mapped to $w = a/c$, we have $(w, a/c, p, q) = (z, \infty, p, q)$, which implies

$$\frac{w-p}{w-q} = \kappa\frac{z-p}{z-q}, \quad \kappa = \frac{a-cp}{a-cq}.$$

For the elliptic case, $|\kappa| = 1$, since $q = \overline{p}$. By the linear fractional transformation $T(z) = (z-p)/(z-\overline{p})$, γ is written as $\gamma = T^{-1}\kappa T$. Therefore, $\mathcal{I}(p)$ is isomorphic to a discrete subgroup of $SO(2)$, which proves (2.4). $\square$

To introduce the analytic structure near p, we let ι be the canonical projection

$$\iota : \mathbf{H}^2 \ni z \to [z] = \{g \cdot z\,;\, g \in \Gamma\} \in \Gamma\backslash\mathbf{H}^2.$$

Using n from (2.4), we introduce the local coordinates $\varphi_p(\iota(z))$ near p by

$$\zeta := \varphi_p(\iota(z)) = T(z)^n = \left(\frac{z-p}{z-\overline{p}}\right)^n, \quad \zeta(p) = 0.$$

Identifying z and $\iota(z)$, we have as $\zeta \to 0$

$$z = \frac{p - \overline{p}\zeta^{1/n}}{1-\zeta^{1/n}} = p + (p-\overline{p})\zeta^{1/n} + \cdots .$$

Therefore,

$$\frac{(dx)^2+(dy)^2}{y^2} = \frac{dz\, d\overline{z}}{(\operatorname{Im} z)^2} = \frac{\left|dz/d\zeta\right|^2}{(\operatorname{Im} z)^2} d\zeta d\overline{\zeta}. \tag{2.5}$$

Direct computation entails

$$\frac{dz}{d\zeta} = \frac{p-\overline{p}}{n}\zeta^{1/n-1}(1-\zeta^{1/n})^{-2},$$

$$\operatorname{Im} z = \frac{p-\overline{p}}{2i}\frac{1-|\zeta^{1/n}|^2}{|1-\zeta^{1/n}|^2}.$$

Therefore, we have

$$\frac{|dz/d\zeta|^2}{(\operatorname{Im} z)^2} = \frac{4}{n^2}|\zeta|^{-\lambda}\left(1-|\zeta|^{2/n}\right)^{-2}, \quad \lambda = 2 - \frac{2}{n}. \tag{2.6}$$

Note that $1 \le \lambda < 2$. The volume element and the Laplace-Beltrami operator are then rewritten as

$$\frac{dx\wedge dy}{y^2} = \frac{i}{2y^2}dz\wedge d\overline{z} = \frac{i\left|dz/d\zeta\right|^2}{2(\operatorname{Im} z)^2}d\zeta\wedge d\overline{\zeta},$$

$$y^2\left(\partial_x^2+\partial_y^2\right) = 4(\operatorname{Im} z)^2\frac{\partial^2}{\partial z\partial\overline{z}} = \frac{4(\operatorname{Im} z)^2}{\left|dz/d\zeta\right|^2}\frac{\partial^2}{\partial\zeta\partial\overline{\zeta}}.$$

Both of them have singularities at p. However, if f, g are C^∞-functions with respect to ζ supported near p, we have

$$\int_{\mathcal{M}} y^2\left(\partial_x^2+\partial_y^2\right)f\cdot\overline{g}\,\frac{dxdy}{y^2} = 2i\int_{|\zeta|<\epsilon}\frac{\partial^2}{\partial\zeta\partial\overline{\zeta}}f\cdot\overline{g}\,d\zeta d\overline{\zeta} = -2i\int_{|\zeta|<\epsilon}\frac{\partial f}{\partial\zeta}\,\overline{\frac{\partial g}{\partial\zeta}}\,d\zeta d\overline{\zeta}.$$

What is important is that the singularity of the volume element and that of the Laplace-Beltrami operator cancel.

Let $\mathcal{M}_{sing} = \{p_1, \cdots, p_L\}$. We take a small open set $U_j \subset \mathcal{M}$ such that $p_j \in U_j$, $U_i \cap U_j = \emptyset$ if $i \neq j$. We construct a smooth partition of unity $\{\chi_j\}_{j=0}^L$ such that $\operatorname{supp}\chi_j \subset U_j$, $j = 1, \cdots, L$, and $\sum_{j=0}^L \chi_j = 1$ on $\mathcal{M}$. We put

$$dV_H^{(j)} = \begin{cases} \dfrac{dx\wedge dy}{y^2} = \dfrac{i}{2}\dfrac{dz\wedge d\overline{z}}{(\operatorname{Im} z)^2} & (j=0), \\ \dfrac{i\left|dz/d\zeta\right|^2}{2(\operatorname{Im} z)^2}d\zeta\wedge d\overline{\zeta} & (j\neq 0), \end{cases} \tag{2.7}$$

$$dV_E^{(j)} = \begin{cases} \frac{i}{2} dz \wedge d\overline{z} & (j=0), \\ \frac{i}{2} d\zeta \wedge d\overline{\zeta} & (j \neq 0), \end{cases} \tag{2.8}$$

and define a quadratic form $Q_{AS}[u,v]$ by

$$Q_{AS}[u,v] = \sum_{j=0}^{L} \int_{\mathcal{M}} \chi_j u \overline{v}\; dV_H^{(j)} + \sum_{j=0}^{L} \int_{\mathcal{M}} \chi_i \nabla u \cdot \nabla \overline{v}\; dV_E^{(j)},$$

where

$$\nabla = \begin{cases} (\partial_x, \partial_y) & (j=0), \\ (\partial_t, \partial_s) & (j \neq 0), \quad (\zeta = t + is). \end{cases}$$

Let $L^2(\mathcal{M})$ be the Hilbert space of L^2-functions on $\mathcal{M}$ with respect to the measure $dxdy/y^2$. As is easily seen, $\sqrt{Q_{AS}(u,u)}$ defines a norm on $C_0^\infty(\mathcal{M} \setminus \mathcal{M}_{sing})$. Let $D(Q_{AS})$ be the completion of $C_0^\infty(\mathcal{M} \setminus \mathcal{M}_{sing})$ with respect to the norm $\sqrt{Q_{AS}[u,u]}$. This is the counterpart of the 1st-order Sobolev space on $\mathcal{M}$.

Lemma 2.5. *Let Γ be a geometrically finite Fuchsian group. Then, for any compact set $K \subset \Gamma\backslash\mathbf{H}^2$, the imbedding*

$$D(Q_{AS}) \ni u \to u\big|_K \in L^2(K)$$

is compact.

Proof. This is obvious if K does not contain elliptic fixed points. Around an elliptic fix point p_j $(1 \leq j \leq L)$, we take local coordinate $\zeta = t + is$ as above, and for a suffiently small $r > 0$, let $B_r = \{(t,s)\,;\, t^2 + s^2 < r^2\}$. Then, by (2.8), if $u \in D(Q_{AS})$ has a support in B_r,

$$\int_{B_r} |u|^2 dtds \leq C \int_{B_r} |u|^2 dV_E^{(j)}, \tag{2.9}$$

with a constant $C > 0$. By the Sobolev imbedding $H^s(\mathbf{R}^d) \subset L^p_{loc}(\mathbf{R}^d)$, where $0 \leq s < d/2$, $p = 2d/(d-2s)$, we have

$$H^1(\mathbf{R}^2) \subset L^p_{loc}(\mathbf{R}^2), \quad \forall p > 2, \tag{2.10}$$

with continuous inclusion.

We take α, β such that $\alpha^{-1} + \beta^{-1} = 1$, $1 < \alpha < 2/\lambda$. Then by (2.6), (2.7), and Hölder's inequality

$$\int_{B_\delta} |u|^2 dV_H^{(j)} \leq C \int_{B_\delta} r^{-\lambda} |u|^2 dtds \leq C \left(\int_{B_\delta} r^{-\lambda\alpha} dtds \right)^{1/\alpha} \left(\int_{B_\delta} |u|^{2\beta} dtds \right)^{1/\beta},$$

where $r = (s^2 + t^2)^{1/2}$. Since $\lambda\alpha < 2$, the 1st term of the most right-hand side tends to 0 when $\delta \to 0$. To the 2nd term of the most right-hand side we apply (2.10). Using (2.9), for any $\epsilon > 0$ there exists $\delta > 0$ such that

$$\int_{B_\delta} |u|^2 dV_H^{(j)} \leq \epsilon \left(\int_{B_\delta} |u|^2 dV_H^{(j)} + \int_{B_{2\delta}} |\nabla u|^2 dV_E^{(j)} \right).$$

Suppose we are given a bouded sequence $\{u_n\}$ in $D(Q_{AS})$. Then the integral of $|u_n|^2$ over B_δ with respect to the measure $dV_H^{(j)}$ can be made small uniformly in n. Outside B_δ, we use the usual Rellich theorem. This proves the lemma. □

Let H_{AS} be the Laplce-Beltrami operator $-\Delta_g - 1/4$ on $\mathcal{M}$, defined through the quadratic form $Q_{AS}[u,v]$. It is well-known that

$$D(H_{AS}) \subset D(H_{AS}^{1/2}) = D(Q_{AS}). \tag{2.11}$$

Corollary 2.6. *$\chi(H_{AS} - z)^{-1}$, $z \notin \mathbf{R}$, is compact on $L^2(\Gamma\backslash\mathbf{H}^2)$ for any $\chi \in C_0^\infty(\Gamma\backslash\mathbf{H}^2)$.*

Proof. This follows from Lemma 2.5 and (2.11). □

Using these facts, one can discuss the forward problem, i.e. the spectral theory for H_{AS} in the same way as in [24]. In order to discuss the inverse problem, however, it is more appropriate to change the differentiable structure around singular points and introduce the notion of conical singularities.

2.4. **Manifolds with conical singularities. Orbifolds.** Let us repeat the definition of a Riemannian surface with conical singularities. We warn the reader not to confuse it with the Riemann surface (the 1-dimensional complex manifold).

Definition 2.7. *A C^∞-surface $\mathcal{M}$ is said to be a Riemannian surface with conical singularities if there exists a discrete subset $\mathcal{M}_{sing}$ of $\mathcal{M}$ such that*
(i) there exists a smooth Riemannian metric g on $\mathcal{M} \setminus \mathcal{M}_{sing}$,
(ii) for each $p \in \mathcal{M}_{sing}$, there is an open neighborhood U_p of p such that the assumption (A-2) is satisfied on U_p.

Let $\mathcal{M}$ be a Riemannian surface with conical singularities. Then, near $p \in \mathcal{M}_{sing}$, letting $x^1 = r\cos\theta$, $x^2 = r\sin\theta$, we see that the metric $g = g_{ij}dx^idx^j$ satisfies

$$C^{-1}I \leq (g_{ij}) \leq CI, \quad C > 1. \tag{2.12}$$

This shows that, although the metric g may be singular at $\mathcal{M}_{sing}$, the H^1-norm

$$\|u\|_{H^1(\mathcal{M})} = \left(\int_M |u|^2\sqrt{g}dx + \int_M g^{ij}\frac{\partial u}{\partial x^i}\frac{\partial \overline{u}}{\partial x^j}\sqrt{g}dx\right)^{1/2} \tag{2.13}$$

can be introduced in the same way as in the case of C^∞-Riemannian manifold. In particular, the following lemma holds.

Lemma 2.8. *Let $\mathcal{M}$ be a surface satisfying (A-1) $\sim$ (A-4), and Δ_g its Laplace-Beltrami operator. Then $-\Delta_g - 1/4$ has a self-adjoint realization through the quadratic form, which is denoted by H. Then, for any $\chi \in C_0^\infty(\mathcal{M})$, $\chi(H-z)^{-1}$, $z \notin \mathbf{R}$, is a compact operator on $L^2(\mathcal{M})$.*

Next we return to $\mathcal{M} = \Gamma\backslash\mathbf{H}^2$, where Γ is a Fuchsian group. We show that (A-2) is satisfied around an elliptic fixed point $p \in \mathcal{M}$. By (2.6), putting $\zeta = \rho e^{i\theta}$, the metric (2.5) takes the form

$$\frac{4}{n^2}|\zeta|^{-\lambda}(1-|\zeta|^{2/n})^{-2}d\zeta d\overline{\zeta} = (1-\rho^{2/n})^{-2}\frac{4}{n^2}\rho^{-\lambda}\left((d\rho)^2 + \rho^2(d\theta)^2\right).$$

Putting $t = 2\rho^{1-\lambda/2} = 2\rho^{1/n}$, we can rewrite it as

$$(1-t^2/4)^{-2}\left((dt)^2 + \frac{t^2}{n^2}(d\theta)^2\right).$$

Solving $dr = (1-t^2/4)^{-1}dt$, we have

$$r = \log\frac{2+t}{2-t} = \log\frac{1+\rho^{1/n}}{1-\rho^{1/n}}.$$

Therefore $\rho^{1/n} = (e^r-1)/(e^r+1)$, and (2.5) takes the form

$$\frac{(dx)^2+(dy)^2}{y^2} = (dr)^2 + \frac{1}{n^2}\sinh^2 r(d\theta)^2. \tag{2.14}$$

This shows that (A-2) is satisfied for any $p \in \mathcal{M}_{sing}$. We cover $\mathcal{M}\backslash\mathcal{M}_{sing}$ by standard local coordinate patches of the quotient Riemannian surface $\Gamma\backslash\mathbf{H}^2$. Therefore, $\mathcal{M}$ is a Riemannian surface with conical singularties. Actually, the structure of conical singularities on $\mathcal{M} = \Gamma\backslash\mathbf{H}^2$ is of a special form, making it a it Riemannian orbifold.

To define an (orientable) 2D-Riemannian orbifold, let $\mathcal{M}$ be a 2D-manifold. Suppose there exists a discrete subset $\mathcal{M}_{sing} \subset \mathcal{M}$ such that $\mathcal{M}\backslash\mathcal{M}_{sing}$ is an orientable Riemannian manifold with a C^∞-Riemannian metric g. We assume that each point $p \in \mathcal{M}_{sing}$ has a neighborhoods for which the following properties hold (see [48], [52]);

- **(B-1)** There exists an open set $\widetilde{U}_p^\varepsilon$ in $\mathbf{R}^2$, containing the origin 0 and equipped with a Riemannian metric $\widetilde{g}_p$, such that, with respect to $\widetilde{g}_p$, $\widetilde{U}_p^\varepsilon$ is the ball of radius ε centered at 0.
- **(B-2)** There is a finite group of rotations $\Gamma_{n_p} \subset SO(2)$ of order $n_p > 1$, so that $\widetilde{g}_p$ is invariant with respect to the action of Γ_{n_p}.
- **(B-3)** $U_p^\varepsilon \sim \widetilde{U}_p^\varepsilon/\Gamma_{n_p}$, where U_p^ε is the ball of radius ε on $\mathcal{M}$, centered at p, and $\sim$ stands for the isometry.

If these assumptions are satisfied, we say that $\mathcal{M}$ is a 2-dimensional Riemannian orbifold. We call n_p the *order* of $p \in \mathcal{M}_{sing}$. For the neighborhoods defined in condition B-3 we denote by $\pi_p : \widetilde{U}_p^\varepsilon \to U_p^\varepsilon$ the associated canonical projections and say that $(\widetilde{U}_p^\varepsilon, \widetilde{g}_p)$ is the uniformizing cover of (U_p^ε, g_p).

A homeomorphism Φ between Riemannian orbifolds $\mathcal{M}^{(1)}$ and $\mathcal{M}^{(2)}$ is said to be an *orbifold isomorphism* if it has the following properties:
(1) $\Phi : \mathcal{M}^{(1)} \setminus \mathcal{M}^{(1)}_{sing} \to \mathcal{M}^{(2)} \setminus \mathcal{M}^{(2)}_{sing}$ is a Riemannian isometry.

(2) For any $p^{(1)} \in \mathcal{M}^{(1)}_{sing}$ and $p^{(2)} = \Phi(p^{(1)})$, $\Phi : U^{\epsilon}_{p^{(1)}} \to U^{\epsilon}_{p^{(2)}}$ is lifted to an isometry between the coverings $\widetilde{\Phi} : \widetilde{U}^{\epsilon}_{p^{(1)}} \to \widetilde{U}^{\epsilon}_{p^{(2)}}$.

To bridge the notion of a surface with conical singularities with that of a 2-dimensional Riemannian orbifold, note that an orbifold singularity is a particular case of a conical singularity characterized by two properties:

Condition 2.9. i. $C_p = \big(1/n_p\big)^2$.

ii. The metric tensor (1.2), rewritten in coordinates $x^1 = r\cos(\theta/n_p)$, $x^2 = r\sin(\theta/n_p)$, being continued periodically onto $\tilde{U}_{\varepsilon}(0) = \{r < \varepsilon\}$ is smooth.

Returning to $\Gamma\backslash\mathbf{H}^2$ and using equation (2.14), straighforward calculations show that each singular point $p \in \Gamma\backslash\mathbf{H}^2$ satisfies conditions **i., ii.**

Let us summarize what we have done in this section. For the Fuchsian group $\Gamma \in SL(2, \mathbf{R})$, $\mathcal{M} = \Gamma\backslash\mathbf{H}^2$ has a structure of a 2D-Riemannian orbifold. It is a Riemann surface, i.e. 1-dimensional complex manifold without singularities. By changing the differentiable structure around $\mathcal{M}_{sing}$ = the set of the elliptic fixed points, $\mathcal{M}$ is regarded as a Riemannian surface with conical singularities. These two local coordinate systems have the following features.

- They coincide except for a small neighborhood of $\mathcal{M}_{sing}$, and give an equivalent C^∞-differentiable structure on $\mathcal{M} \setminus \mathcal{M}_{sing}$.
- They equip $\mathcal{M} \setminus \mathcal{M}_{sing}$ with the hyperbolic metric, which is singular at $\mathcal{M}_{sing}$ in the case of orbifold.
- The associated Laplace-Beltarmi operators are unitarily equivalent.

It follows from these properties that the associated (generalized) S-matrices coincide, since they are defined by the asymptotic behavior at infinity of solutions to the Helmholtz equations.

This new coordinate system resolves the singularities of the hyperbolic metric at elliptic fixed points, which makes the proof of local compactness of the resolvent easier. The merit of introducing the notion of conical singularities is not restricted here, however. It is used effectively in the inverse problem in §5. On the other hand, the original coordinate system is analytic even at elliptic fixed points. This fact will be used in §6 to discuss the orbifold isomorphism.

3. Spectral theory for asymptotically hypebolic manifolds

In [24], for manifolds without conical singularities, we have already studied spectral properties of the Laplace-Beltrami operators on asymptotically hyperbolic manifolds : limiting absorption principle for the resolvent, spectral representations, S-matrices. Thanks to Lemma 2.8, and also to the fact that $\mathcal{M}_{sing}$ is a finite set, the proof of the above facts works well without any change. We shall explain below the basic ideas for this forward problem and summarize the results.

Let A be a self-adjoint operator in a Hilbert space $\mathcal{H}$. If $\lambda \in \sigma(A)$, the limit $\lim_{\epsilon\to 0}(A-\lambda \mp i\epsilon)^{-1}$ does not exist in $\mathbf{B}(\mathcal{H};\mathcal{H})$. However, in some important cases, when $\lambda \in \sigma_c(A)$, it is possible to define $\lim_{\epsilon\to 0}(A-\lambda\mp i\epsilon)^{-1}$. This is achieved by choosing suitable Banach spaces $\mathcal{H}_+, \mathcal{H}_-$ satisfying

$$\mathcal{H}_+ \subset \mathcal{H} \subset \mathcal{H}_-,$$

with continuous injections, so that

$$\lim_{\varepsilon\to 0} (A-\lambda \mp i\varepsilon)^{-1} \in \mathbf{B}(\mathcal{H}_+;\mathcal{H}_-).$$

This fact is usually called the limiting absorption principle. For $A = -\Delta$ in $\mathbf{R}^n$, the best choice of $\mathcal{H}_\pm$ are the Besov type spaces $\mathcal{B}, \mathcal{B}^*$ found by Agmon-Hörmander [1]. We first define a counterpart of $\mathcal{B}, \mathcal{B}^*$ in the case of hyperbolic spaces.

3.1. **Besov type spaces.** Let $\mathbf{h}$ be a Hilbert space endowed with inner product $(\ ,\ )_{\mathbf{h}}$ and norm $\|\cdot\|_{\mathbf{h}}$. We decompose $(0,\infty)$ into $(0,\infty) = \cup_{k\in\mathbf{Z}} I_k$, where

$$I_k = \begin{cases} \big(\exp(e^{k-1}), \exp(e^k)\big], & k \geq 1, \\ \big(e^{-1}, e\big], & k = 0, \\ \big(\exp(-e^{|k|}), \exp(-e^{|k|-1})\big], & k \leq -1. \end{cases}$$

Let $\mathcal{B} = \mathcal{B}(\mathbf{h})$ be the Banach space of $\mathbf{h}$-valued function on $(0,\infty)$ such that

$$\|f\|_{\mathcal{B}} = \sum_{k\in\mathbf{Z}} e^{|k|/2} \left(\int_{I_k} \|f(y)\|^2_{\mathbf{h}} \frac{dy}{y^2}\right)^{1/2} < \infty. \tag{3.1}$$

The dual space of $\mathcal{B}$ is identified with the space equipped with norm

$$\|u\|_{\mathcal{B}^*} = \left(\sup_{R>e} \frac{1}{\log R} \int_{1/R}^R \|u(y)\|^2_{\mathbf{h}} \frac{dy}{y^2}\right)^{1/2} < \infty. \tag{3.2}$$

For example, for $\phi \in \mathbf{h}$, $y^{1/2}\phi$ belongs to $\mathcal{B}^*$. We also use the following weighted L^2-space: for $s \in \mathbf{R}$,

$$L^{2,s} \ni u \Longleftrightarrow \|u\|_s = \left(\int_0^\infty (1+|\log y|)^{2s} \|u(y)\|^2_{\mathbf{h}} \frac{dy}{y^2}\right)^{1/2} < \infty. \tag{3.3}$$

For $s > 1/2$, the following inclusions hold:

$$L^{2,s} \subset \mathcal{B} \subset L^{2,1/2} \subset L^2 \subset L^{2,-1/2} \subset \mathcal{B}^* \subset L^{2,-s}. \tag{3.4}$$

If $u, v \in \mathcal{B}^*$ satisfy

$$\lim_{R\to\infty} \frac{1}{\log R} \int_{1/R}^{1/2} \|u(y)-v(y)\|^2_{\mathbf{h}} \frac{dy}{y^2} = 0, \quad \lim_{R\to\infty} \frac{1}{\log R} \int_2^R \|u(y)-v(y)\|^2_{\mathbf{h}} \frac{dy}{y^2} = 0, \tag{3.5}$$

we regard that u and v have the same asymptotic behavior at infinities, $y = 0$, and $y = \infty$, correspondingly. We have the following lemma.

Lemma 3.1. *For $u \in \mathcal{B}^*$, the following two assertions are equivalent.*

$$\lim_{R\to\infty} \frac{1}{\log R} \int_{1/R}^{R} \|u(y)\|^2_{\mathbf{h}} \frac{dy}{y^2} = 0. \tag{3.6}$$

$$\lim_{R\to\infty} \frac{1}{\log R} \int_0^\infty \rho\Big(\frac{\log y}{\log R}\Big) \|u(y)\|^2_{\mathbf{h}} \frac{dy}{y^2} = 0, \quad \forall \rho \in C_0^\infty((0,\infty)). \tag{3.7}$$

The proof of the above results are given in [24], Chap. 1, §2.

3.2. **Bessel functions.** We use the following knowledge of Bessel functions. For the details, see [54]. The modified Bessel function (of the 1st kind) $I_\nu(z)$, with parameter $\nu \in \mathbf{C}$, is defined by

$$I_\nu(z) = \Big(\frac{z}{2}\Big)^\nu \sum_{n=0}^{\infty} \frac{(z^2/4)^n}{n!\,\Gamma(\nu+n+1)}, \quad z \in \mathbf{C} \setminus (-\infty, 0]. \tag{3.8}$$

It is related to the Bessel function $J_\nu(z)$ as follows

$$I_\nu(y) = e^{-\nu\pi i/2} J_\nu(iy), \quad y > 0.$$

The following function $K_\nu(z)$ is also called the modified Bessel function, or the K-Bessel function, or sometimes the Macdonald function:

$$K_\nu(z) = \frac{\pi}{2} \frac{I_{-\nu}(z) - I_\nu(z)}{\sin(\nu\pi)}, \quad \nu \notin \mathbf{Z}, \tag{3.9}$$

$$K_n(z) = K_{-n}(z) = \lim_{\nu\to n} K_\nu(z), \quad n \in \mathbf{Z}.$$

These $I_\nu(z), K_\nu(z)$ solve the following equation

$$z^2 u'' + z u' - (z^2 + \nu^2) u = 0, \tag{3.10}$$

and have the following asymptotic expansions as $|z| \to \infty$:

$$I_\nu(z) \sim \frac{e^z}{\sqrt{2\pi z}} + \frac{e^{-z+(\nu+1/2)\pi i}}{\sqrt{2\pi z}}, \quad |z| \to \infty, \quad -\frac{\pi}{2} < \arg z < \frac{\pi}{2}, \tag{3.11}$$

$$K_\nu(z) \sim \sqrt{\frac{\pi}{2z}} e^{-z}, \quad |z| \to \infty, \quad -\pi < \arg z < \pi. \tag{3.12}$$

The asymptotics as $z \to 0$ are as follows:

$$I_\nu(z) \sim \frac{1}{\Gamma(\nu+1)} \Big(\frac{z}{2}\Big)^\nu, \tag{3.13}$$

$$K_\nu(z) \sim \frac{\pi}{2\sin(\nu\pi)} \left(\frac{1}{\Gamma(1-\nu)} \Big(\frac{z}{2}\Big)^{-\nu} - \frac{1}{\Gamma(1+\nu)} \Big(\frac{z}{2}\Big)^{\nu} \right), \quad \nu \notin \mathbf{Z} \tag{3.14}$$

$$K_n(z) \sim \begin{cases} -\log z, & n = 0, \\ 2^{n-1}(n-1)!\, z^{-n}, & n = 0, 1, 2, \ldots. \end{cases}$$

3.3. **Spectral properties of the model space.** By Theorem 2.1, the surfaces whose ends are asymptotically equal to $S\times(0,1)$ or $S\times(1,\infty)$, $S := S^1$, equiped with the metric given by

$$ds^2 = \frac{(dy)^2 + (dx)^2}{y^2}, \quad 0 \le x \le 2\pi, \tag{3.15}$$

form a broad and meaningfull class of 2-dimensional surfaces. In this subsection, we shall introduce a model for such surfaces and study the spectral properties of the Laplace-Beltrami operator on it. Since it is an unperturbed (free) space, we put the subscript *free* for every related object on it. We put $M_{free} = S$ and let ∂_x^2 be the Laplace-Beltrami operator on S. It has eigenvalues and eigenvectors

$$\lambda_n = n^2, \quad \varphi_n(x) = e^{inx}/\sqrt{2\pi}, \quad n \in \mathbf{Z}. \tag{3.16}$$

Let $\mathcal{M}_{free} = M_{free} \times (0,\infty)$ and H_{free} be given by

$$H_{free} = -y^2(\partial_y^2 + \partial_x^2) - \frac{1}{4}. \tag{3.17}$$

$\mathcal{M}_{free}$ has two infinities corresponding to $y = 0$ and $y = \infty$. We call the former the *regular end*, and the latter the *cusp*. In the following, the subscripts c and reg mean the cusp and regular end, respectively.

3.3.1. *Green's operator.* Green's kernel of H_{free} is computed as follows. Consider the 1-dimensional operators

$$L_{free}(\zeta) = y^2(-\partial_y^2 + \zeta^2) - \frac{1}{4}, \quad \zeta \in \mathbf{R}, \tag{3.18}$$

$$(L_{free}(\zeta) + \nu^2)^{-1} =: G_{free}(\zeta, \nu). \tag{3.19}$$

If $\zeta \neq 0$, by (3.9), (3.10), $G_{free}(\zeta,\nu)$ has the following expression (see [24], Chap. 1, §3),

$$\left(G_{free}(\zeta,\nu)\psi\right)(y) = \int_0^\infty G_{free}(y,y';\zeta,\nu)\psi(y')\frac{dy'}{(y')^2}, \tag{3.20}$$

$$G_{free}(y,y';\zeta,\nu) = \begin{cases} \left(yy'\right)^{1/2} K_\nu(\zeta y) I_\nu(\zeta y'), & y > y' > 0, \\ \left(yy'\right)^{1/2} I_\nu(\zeta y) K_\nu(\zeta y'), & y' > y > 0. \end{cases} \tag{3.21}$$

Let us remark that in [24], L_{free}, G_{free} are denoted by L_0, G_0. In what follows, the subscript 0 is, however, reserved to denote the terms associated with the eigenvalue $\lambda_0 = 0$.

When $\zeta = 0$, we have (see [24], Chap. 3, §2),

$$\left(G_{free}(0,\nu)\psi\right)(y) = \int_0^\infty G_{free}(y,y';0,\nu)\psi(y')\frac{dy'}{(y')^2}, \tag{3.22}$$

$$G_{free}(y,y';0,\nu) = \frac{1}{2\nu}\begin{cases} y^{\frac12+\nu}(y')^{\frac12-\nu}, & y' > y > 0, \\ y^{\frac12-\nu}(y')^{\frac12+\nu}, & y > y' > 0. \end{cases} \tag{3.23}$$

We define $\mathcal{B}(\mathbf{C})$ and $\mathcal{B}(\mathbf{C})^*$ by putting $\mathbf{h} = \mathbf{C}$ in Subsection 3.1. Then we have, by [24], Chap. 1, Lemma 3.8,

$$\|G_{free}(\zeta,\nu)\psi\|_{\mathcal{B}(\mathbf{C})^*} \leq C\|\psi\|_{\mathcal{B}(\mathbf{C})}, \tag{3.24}$$

where the constant C is independent of ν, when ν varies over a compact set in $\{\mathrm{Re}\,\nu \geq 0\} \setminus \mathbf{Z}$, and also of ζ, when $\mathrm{Re}\,\zeta > 0$. One can also prove (3.24) for $\zeta = 0$.

Recalling (3.16), we put, for $f(x,y) \in \mathcal{H}_{free} := L^2((0,\infty) : L^2(S); dy/y^2)$,

$$\widehat{f}_n(y) = \int_{M_{free}} f(x,y)\overline{\varphi_n(x)}dx. \tag{3.25}$$

Let $R_{free}(z) = (H_{free} - z)^{-1}$, $z = -\nu^2$. Then

$$\begin{aligned} R_{free}(-\nu^2)f &= \sum_{n\in\mathbf{Z}} \varphi_n(x)\left(\left(L_{free}(|n|) + \nu^2\right)^{-1}\widehat{f}_n(\cdot)\right)(y) \\ &= \sum_{n\in\mathbf{Z}} \varphi_n(x)\left(G_{free}(|n|,\nu)\widehat{f}_n(\cdot)\right)(y). \end{aligned} \tag{3.26}$$

For $0 < a < b$, we put

$$J_\pm = \{z \in \mathbf{C}\ ;\ a \leq \mathrm{Re}\,z \leq b,\ \pm\mathrm{Im}\,z > 0\}. \tag{3.27}$$

The estimate (3.24) then implies

$$\|R_{free}(z)f\|_{\mathcal{B}^*} \leq C\|f\|_{\mathcal{B}}, \tag{3.28}$$

with $\mathcal{B} = \mathcal{B}(L^2(S))$ and $\mathcal{B}^* = \mathcal{B}(L^2(S))^*$, where the constant C is independent of $z \in J_\pm$. This uniform estimate is crucial in proving the limiting absorption principle. In fact, by [24], Chap. 3, Theorems 3.5 and 3.8, the following theorem holds.

Theorem 3.2. *(1)* $\sigma(H_{free}) = [0,\infty)$.
(2) $\sigma_p(H_{free}) = \emptyset$.
(3) For $\lambda > 0, f \in \mathcal{B} = \mathcal{B}(L^2(S))$, *the following limit exists in the weak* $*$*-sense*

$$\lim_{\epsilon\to 0} R_{free}(\lambda \pm i\epsilon)f =: R_{free}(\lambda \pm i0)f,$$

i.e. there exits the limit

$$\lim_{\epsilon\to 0} \left(R_{free}(\lambda \pm i\epsilon)f, g\right), \quad \forall f, g \in \mathcal{B}.$$

Note that, since $\mathrm{Re}\,\nu \geq 0$, we have, letting $\nu = -i(k \pm i\epsilon)$, $k > 0$,

$$R_{free}(k^2 \pm i0)f = \sum_{n\in\mathbf{Z}} \varphi_n(x)\left(G_{free}(|n|, \mp ik)\widehat{f}_n(\cdot)\right)(y). \tag{3.29}$$

3.3.2. *Fourier transform.* Let $f \in C_0^\infty(\mathcal{M}_{free})$, and $k > 0$. For $n \neq 0$, the associated Fourier-Bessel transform is defined by

$$F_{free,n}(k)f = \frac{\big(2k\sinh(k\pi)\big)^{1/2}}{\pi}\int_0^\infty y^{1/2}K_{ik}(|n|y)\widehat{f}_n(y)\frac{dy}{y^2}. \tag{3.30}$$

For $n = 0$, the associated Mellin transform is defined by

$$F^{(\pm)}_{free,0}(k)f = \frac{1}{\sqrt{2\pi}}\int_0^\infty y^{\frac{1}{2}\pm ik}\widehat{f}_0(y)\frac{dy}{y^2}. \tag{3.31}$$

Definition 3.3. We put

$$\mathbf{h} = \mathbf{C} \oplus L^2(S), \quad \widehat{\mathcal{H}} = L^2((0,\infty);\mathbf{h};dk),$$

and define $\mathcal{F}^{(\pm)}_{c,free}(k)$ and $\mathcal{F}^{(\pm)}_{reg,free}(k)$ by

$$\mathcal{F}^{(\pm)}_{c,free}(k)f = F^{(\mp)}_{free,0}(k)f. \tag{3.32}$$

$$\begin{aligned}\left(\mathcal{F}^{(\pm)}_{reg,free}(k)f\right)(x) = {} & C_0^{(\pm}(k)F^{(\pm)}_{free,0}(k)f \\ & + \sum_{n\in\mathbf{Z}\setminus\{0\}} C_n^{(\pm)}(k)\varphi_n(x)F_{free,n}(k)f,\end{aligned} \tag{3.33}$$

$$C_n^{(\pm)}(k) = \begin{cases} \left(\dfrac{n}{2}\right)^{\mp ik} & (n \neq 0), \\ \dfrac{\pm i}{k\omega_\pm(k)}\sqrt{\dfrac{\pi}{2}} & (n = 0), \end{cases} \tag{3.34}$$

$$\omega_\pm(k) = \frac{\pi}{(2k\sinh(k\pi))^{1/2}\Gamma(1\mp ik)}. \tag{3.35}$$

Finally, we define the Fourier transform assocaited with H_{free} by

$$\mathcal{F}^{(\pm)}_{free}(k) = \left(\mathcal{F}^{(\pm)}_{c,free}(k), \mathcal{F}^{(\pm)}_{reg,free}(k)\right).$$

The important step for the spectral representation is the following Parseval's formula

$$\frac{k}{\pi i}\left([R_{free}(k^2+i0) - R_{free}(k^2-i0)]f, f\right) = \|\mathcal{F}^{(\pm)}_{free}(k)f\|^2_{\mathbf{h}}. \tag{3.36}$$

This and the uniform estimate (3.28) imply the following inequality

$$\|\mathcal{F}^{(\pm)}_{free}(k)f\|_{\mathbf{h}} \leq C\|f\|_{\mathcal{B}}. \tag{3.37}$$

Therefore, $\mathcal{F}^{(\pm)}_{free}(k)$ can be extended uniquely on $\mathcal{B}$. For $f \in \mathcal{B}$, we define an $\mathbf{h}$-valued function of $k \in (0,\infty)$ by

$$\left(\mathcal{F}^{(\pm)}_{free}f\right)(k) = \mathcal{F}^{(\pm)}_{free}(k)f.$$

Then, by integrating (3.36) with respect to k over $(0,\infty)$, we see that $\mathcal{F}^{(\pm)}_{free}$ can be extended to an isometry from $\mathcal{H}_{free}$ to $\widehat{\mathcal{H}}$. In fact, it is unitary (see [24], Chap. 3, Theorem 2.5).

Theorem 3.4. *$\mathcal{F}^{(\pm)}_{free}$ is uniquely extended to a unitary operator from $\mathcal{H}_{free}$ to $\widehat{\mathcal{H}}$. Moreover, if $f \in D(H_{free})$*

$$(\mathcal{F}^{(\pm)}_{free} H_{free} f)(k) = k^2 (\mathcal{F}^{(\pm)}_{free} f)(k).$$

The Fourier transform $\mathcal{F}^{(\pm)}_{free}$ is related to the asymptotic expansion of the resolvent at infinity in the following way.

Theorem 3.5. *For $k > 0$ and $f \in \mathcal{B}$, we have*

$$\lim_{R\to\infty} \frac{1}{\log R} \int_{1/R}^{1} \|\big(R_{free}(k^2 \pm i0) f\big)(\cdot, y) - v^{(\pm)}_{reg}(\cdot, y)\|^2_{L^2(S)} \frac{dy}{y^2} = 0, \tag{3.38}$$

$$v^{(\pm)}_{reg}(x, y) = \omega_\pm(k)\, y^{\frac{1}{2} \mp ik} \left(\mathcal{F}^{(\pm)}_{reg,free}(k) f\right)(x),$$

$$\lim_{R\to\infty} \frac{1}{\log R} \int_{1}^{R} \|\big(R_{free}(k^2 \pm i0) f\big)(\cdot, y) - v^{(\pm)}_{c}\|^2_{L^2(S)} \frac{dy}{y^2} = 0, \tag{3.39}$$

$$v^{(\pm)}_{c} = \omega^{(c)}_\pm(k)\, y^{\frac{1}{2} \pm ik} \mathcal{F}^{(\pm)}_{c,free}(k) f,$$

where

$$\omega^{(c)}_\pm(k) = \pm \frac{i}{k} \sqrt{\frac{\pi}{2}}. \tag{3.40}$$

This theorem is proven by comparing the form of Green's function (3.21), (3.23) with the definition of $\mathcal{F}^{(\pm)}_{free}$, and using the asymptotic expansion of Bessel functions. See [24], Chap. 3, Theorem 2.6.

3.4. **Basic spectral properties for asymptotically hyperbolic manifolds.** We turn to the spectral properties of the manifold $\mathcal{M}$ satisfying the assumptions **(A-1)** $\sim$ **(A-4)** in §1. To deal with the Laplace-Beltrami operator $-\Delta_g$ for $\mathcal{M}$, we first pass it to the gauge transformation

$$-\Delta_g - \frac{1}{4} \to -\rho^{1/4} \Delta_g \rho^{-1/4} - \frac{1}{4}.$$

Here $\rho \in C^\infty(\mathcal{M})$ is a positive function such that $\rho = 1$ in a small neighborhood of $\mathcal{M}_{sing}$. On each end $\mathcal{M}_j$,

$$\rho = g_{free(j)}/g,$$

where $g_{free(j)}$ and g define the volume elements, in the $(x, y)-$coordinates, of the unperturbed and perturbed metrics on $\mathcal{M}_j$. Note that $\rho =$ const in each $\mathcal{M}_i$, $i = 1, \dots, \mu$. Let H be the self-adjoint extension of $-\rho^{1/4}\Delta_g \rho^{-1/4} - 1/4$ defined in the same way as in Lemma 2.8. Our first concern is the (non) existence of the embedded eigenvalues in the continuous spectrum.

Theorem 3.6. *(1) $\sigma_e(H) = [0, \infty)$.*
(2) If one of $\mathcal{M}_i$'s is a regular end, then $\sigma_p(H) \cap (0, \infty) = \emptyset$.
(3) If all of the $\mathcal{M}_i$'s have a cusp, then $\sigma_p(H) \cap (0, \infty)$ is discrete with finite multiplicities, whose possible accumulation points are 0 and ∞.

For the proof, see [24], Chap. 3, Theorems 3.2 and 3.5. The assertion (1) is a consquence of Theorem 3.2 (1) and Weyl's theorem on the perturbation of essential spectrum. The main tool for proving the assertion (2) is a theorem on the growth property of solutions to an abstract differential equation with operator-valued coefficients ([24], Chap. 2, Theorem 3.1). The assertion (3) is a standard result which follows from the a-priori estimates for solutions to the reduced wave equation

$$\big(- \Delta_g - \frac{1}{4} - z\big) u = f \tag{3.41}$$

and the short-range perturbation theory for the Schrödinger equation. $\square$

Take $\chi_0 \in C_0^\infty(\mathcal{M})$ such that $\chi_0 = 1$ on $\mathcal{K}$, and put $\chi_i = 1 - \chi_0$ on $\mathcal{M}_i$, $\chi_i = 0$ on $\mathcal{M} \setminus \mathcal{M}_i$. Then $\{\chi_0, \chi_1, \cdots, \chi_N\}$ is a partition of unity on $\mathcal{M}$ subordinated to decomposition (1.1).

We define the Besov space $\mathcal{B}_i$ by $\mathcal{B}_i = \mathcal{B}(\mathbf{C})$, when $\mathcal{M}_i$ has a cusp, and $\mathcal{B}_i = \mathcal{B}(L^2(S^{r_i}))$, when $\mathcal{M}_i$ has a regular infinity. We then put

$$\|f\|_{\mathcal{B}} = \|\chi_0 f\|_{L^2(\mathcal{M})} + \sum_{i=1}^N \|\chi_i f\|_{\mathcal{B}_i},$$

$$\|u\|_{\mathcal{B}^*} = \|\chi_0 u\|_{L^2(\mathcal{M})} + \sum_{i=1}^N \|\chi_i u\|_{\mathcal{B}_i{}^*},$$

which define the Besov type spaces $\mathcal{B}$ and $\mathcal{B}^*$ on $\mathcal{M}$.

Let $R(z) = (H - z)^{-1}$ be the resolvent of H.

Theorem 3.7. *For $\lambda \in (0, \infty) \setminus \sigma_p(H)$, there exists a limit*

$$\lim_{\epsilon \to 0} R(\lambda \pm i\epsilon) \equiv R(\lambda \pm i0) \in \mathbf{B}(\mathcal{B}\, ; \mathcal{B}^*)$$

in the weak $*$-sense. Moreover, for any compact interval $I \subset (0, \infty) \setminus \sigma_p(H)$, there exists a constant $C > 0$ such that

$$\|R(\lambda \pm i0) f\|_{\mathcal{B}^*} \leq C \|f\|_{\mathcal{B}}, \quad \lambda \in I.$$

For $f, g \in \mathcal{B}$, $(R(\lambda \pm i0) f, g)$ is continuous with respect to $\lambda \in (0, \infty) \setminus \sigma_p(H)$.

This theorem is proved in [24], Chap. 3, Theorem 3.8. The proof consists of two main ingredients. We first establish some a-prori estimates for solutions to the reduced wave equation (3.41) by the elementary tool of integration by parts ([24], Chap. 2, Lemmas 2.4 $\sim$ 2.8). This 1st step is essentially the 1-dimensional problem. The proof of Theorem 3.7 is done by the argument of contradiction, using the compactness of the perturbation and reducing the problem to the uniqueness of solutions of the equation (3.41) satisfying the corresponding radiation condition.

The above mentioned radiation condition is as follows. Let

$$\sigma_{\pm}(\lambda) = \frac{1}{2} \mp i\sqrt{\lambda}, \quad \lambda > 0.$$

We say that a solution $u \in \mathcal{B}^*$ of the equation $(-\Delta_g - \frac{1}{4} - \lambda)u = f \in \mathcal{B}$ satisfies the *outgoing radiation condition*, or u is *outgoing*, if

(3.42)
$$\frac{1}{\log R}\int_2^R \|\big(y\partial_y - \sigma_+(\lambda)\big)u(\cdot,y)\|^2_{L^2(S^{r_j})}\frac{dy}{y^2} \to 0, \quad (j = 1, \cdots, \mu),$$
$$\frac{1}{\log R}\int_{1/R}^{1/2} \|\big(y\partial_y - \sigma_+(\lambda)\big)u(\cdot,y)\|^2_{L^2(S^{r_j})}\frac{dy}{y^2} \to 0, \quad (j = \mu+1, \cdots, N)$$

hold as $R \to \infty$. The following theorem follows from [24], Chap. 3, Theorems 3.7 and 3.8.

Theorem 3.8. *Let $\lambda \in (0,\infty) \setminus \sigma_p(H)$.*
(1) If $u \in \mathcal{B}^*$ satisfies $(H-\lambda)u = 0$ and is outgoing, then $u = 0$.
(2) For $f \in \mathcal{B}$, $R(\lambda + i0)f$ is outgoing.

3.5. **Fourier transforms associated with H.** We shall make use of the perturbation method to construct the Fourier transform for H from that of the model space. Let $H_{free(j)}$ be defined by

$$H_{free(j)} = -y^2(\partial_y^2 + \Delta_{M_j}) - \frac{1}{4}, \tag{3.43}$$

where Δ_{M_j} is the Laplace-Beltrami operator of M_j. Let χ_j be the partition of unity as above. We put

$$\widetilde{V}_j = H - H_{free(j)} \quad \text{on} \quad \mathcal{M}_j. \tag{3.44}$$

This is symmetric on $C_0^\infty(\mathcal{M}_j)$, since so are H and $H_{free(j)}$. Observe that

$$(H_{free(j)} - \lambda)\chi_j Q_j(\lambda \pm i0)R(\lambda \pm i0),$$

where

$$Q_j(z) = \chi_j + \Big([H_{free(j)}, \chi_j] - \chi_j\widetilde{V}_j\Big)R(z). \tag{3.45}$$

Therefore, we have the following equality

$$\chi_j R(\lambda \pm i0) = R_{free(j)}(\lambda \pm i0)Q_j(\lambda \pm i0). \tag{3.46}$$

This formula suggests how the generalized Fourier transform is constructed by the perturbation method.

Let $\lambda_{j,n} = (n/r_j)^2$, $\varphi_{j,n}(x) = e^{inx/r_j}/\sqrt{2\pi r_j}$ be the eigenvalues and normalized eigenvectors of Δ_{M_j}. We define $\mathcal{F}^{(\pm)}_{c,free(j)}(k)$ by (3.32), and $\mathcal{F}^{(\pm)}_{reg,free(j)}(k)$ by (3.33) with M replaced by M_j, φ_n by $\varphi_{j,n}$, and $C_n^{(\pm)}(k)$ by $C_{j,n}^{(\pm)}(k)$, i.e.

$$C_{j,n}^{(\pm)}(k) = \begin{cases} \left(\dfrac{\sqrt{\lambda_{j,n}}}{2}\right)^{\mp ik}, & (\lambda_{j,n} \neq 0), \\ \dfrac{\pm i}{k\omega_\pm(k)}\sqrt{\dfrac{\pi}{2}}, & (\lambda_{j,n} = 0). \end{cases} \tag{3.47}$$

3.5.1. *Definition of* $\mathcal{F}^{(\pm)}_{free(j)}(k)$. Recall that, for $1 \leq j \leq \mu$, $\mathcal{M}_j$ has a cusp, and, for $\mu+1 \leq j \leq N$, $\mathcal{M}_j$ has a regular infinity.

(i) For $1 \leq j \leq \mu$ (the case of cusp), we define

$$\mathcal{F}^{(\pm)}_{free(j)}(k) = \mathcal{F}^{(\pm)}_{c,free(j)}(k). \tag{3.48}$$

(ii) For $\mu+1 \leq j \leq N$ (the case of regular infinity), we define

$$\mathcal{F}^{(\pm)}_{free(j)}(k) = \mathcal{F}^{(\pm)}_{reg,free(j)}(k), \tag{3.49}$$

3.5.2. *Definition of* $\mathcal{F}^{(\pm)}(k)$. For $1 \leq j \leq N$, we define

$$\mathcal{F}^{(\pm)}_j(k) = \mathcal{F}^{(\pm)}_{free(j)}(k) Q_j(k^2 \pm i0), \tag{3.50}$$

Finally, we define the Fourier transform associated with H by

$$\mathcal{F}^{(\pm)}(k) = \big(\mathcal{F}^{(\pm)}_1(k), \cdots, \mathcal{F}^{(\pm)}_N(k)\big). \tag{3.51}$$

3.5.3. *Eigenfunction expansion theorem.* Let

$$\mathbf{h}_\infty = \oplus_{j=1}^N \mathbf{h}_j, \quad \mathbf{h}_j = \mathbf{C},\ 1 \leq j \leq \mu, \quad \mathbf{h}_j = L^2(M_j),\ \mu+1 \leq j \leq N. \tag{3.52}$$

and for $\varphi, \psi \in \mathbf{h}_\infty$, define the inner product by

$$(\varphi, \psi)_{\mathbf{h}_\infty} = \sum_{j=1}^{\mu} \varphi_j \overline{\psi_j} |M_j| + \sum_{j=\mu+1}^{N} (\varphi_j, \psi_j)_{L^2(M_j)}, \tag{3.53}$$

where $|M_j| = 2\pi r_j$ is the length of M_j. We put

$$\widehat{\mathcal{H}} = L^2((0,\infty); \mathbf{h}_\infty; dk).$$

Theorem 3.9. *We define* $\big(\mathcal{F}^{(\pm)} f\big)(k) = \mathcal{F}^{(\pm)}(k) f$ *for* $f \in \mathcal{B}$. *Then* $\mathcal{F}^{(\pm)}$ *is uniquely extended to a bounded operator from* $L^2(\mathcal{M})$ *to* $\widehat{\mathcal{H}}$ *with the following properties.*
(1) Ran $\mathcal{F}^{(\pm)} = \widehat{\mathcal{H}}$.
(2) $\|f\| = \|\mathcal{F}^{(\pm)} f\|$ *for* $f \in \mathcal{H}_{ac}(H)$.
(3) $\mathcal{F}^{(\pm)} f = 0$ *for* $f \in \mathcal{H}_p(H)$.
(4) $\left(\mathcal{F}^{(\pm)} H f\right)(k) = k^2 \left(\mathcal{F}^{(\pm)} f\right)(k)$ *for* $f \in D(H)$.
(5) $\mathcal{F}^{(\pm)}(k)^* \in \mathbf{B}(\mathbf{h}_\infty; \mathcal{B}^*)$ *and* $(H-k^2)\mathcal{F}^{(\pm)}(k)^* = 0$ *for* $k^2 \in (0,\infty)\setminus\sigma_p(H)$.
(6) For $f \in \mathcal{H}_{ac}(H)$, *the inversion formula holds:*

$$f = \left(\mathcal{F}^{(\pm)}\right)^* \mathcal{F}^{(\pm)} f = \sum_{j=1}^N \int_0^\infty \mathcal{F}^{(\pm)}_j(k)^* \left(\mathcal{F}^{(\pm)}_j f\right)(k) dk.$$

The most important step of the proof of this theorem is Parseval's formula

$$\frac{k}{\pi i}\left(\big[R(k^2+i0) - R(k^2-i0)\big] f, g\right) = \left(\mathcal{F}^{(\pm)}(k) f, \mathcal{F}^{(\pm)}(k) g\right)_{\mathbf{h}_\infty}$$

for $f, g \in \mathcal{B}$, $k^2 \in (0,\infty) \setminus \sigma_p(H)$ ([24], Chap. 3, Lemma 3.11), which is proven by the following Theorem 3.10. The remaining arguments are routine. See [24], Chap. 3, Theorem 3.12 for the details.

Remark 1. The meaning of the integral in (6) is as follows. Let $(0,\infty) \setminus \sigma_p(H) = \cup_{i=1}^{\infty} I_i$, where $I_i = (a_i, b_i)$ are non-overlapping open intervals. For $g(k) \in \widehat{\mathcal{H}}$, we have by (5)

$$\int_{\sqrt{a_i}+\epsilon}^{\sqrt{b_i}-\epsilon} \mathcal{F}_j^{(\pm)}(k)^* g(k) dk \in \mathcal{B}^*.$$

As a matter of fact, it belongs to $L^2(\mathcal{M})$, and

$$\lim_{\epsilon \to 0} \int_{\sqrt{a_i}+\epsilon}^{\sqrt{b_i}-\epsilon} \mathcal{F}_j^{(\pm)}(k)^* g(k) dk \in L^2(\mathcal{M})$$

in the sense of strong convergence in $L^2(\mathcal{M})$. Denoting this limit by

$$\int_{\sqrt{I_i}} \mathcal{F}_j^{(\pm)}(k)^* g(k) dk,$$

we define

$$\int_0^\infty \mathcal{F}_j^{(\pm)}(k)^* g(k) dk = \sum_{i=1}^{\infty} \int_{\sqrt{I_i}} \mathcal{F}_j^{(\pm)}(k)^* g(k) dk.$$

3.5.4. *Asymptotic expansion of the resolvent.* For $f, g \in \mathcal{B}^*$ on $\mathcal{M}$, by $f \simeq g$ we mean that on each end the following relation holds,

$$\lim_{R\to\infty} \frac{1}{\log R} \int_{1/R}^{R} \rho_j(y) \| f(y) - g(y) \|^2_{L^2(M_j)} \frac{dy}{y^2} = 0$$

where $\rho_j(y) = 1$ $(y < 1/2)$, $\rho_j(y) = 0$ $(y > 1)$, when $\mathcal{M}_j$ has a regular infinity, and $\rho_j(y) = 0$ $(y < 1)$, $\rho_j(y) = 1$ $(y > 2)$, when $\mathcal{M}_j$ has a cusp. Theorem 3.5 shows that $\mathcal{F}^{(\pm)}_{free(j)}(k)f$ is computed from the asymptotic expanison of $R_{free(j)}(\lambda \pm i0)f$ at infinity. This, combined with the formula (3.46) and definition (3.50), implies the following theorem (see [24], Chap. 3, Theorem 3.10).

Theorem 3.10. *Let $f \in \mathcal{B}$, $k^2 \in \sigma_e(H) \setminus \sigma_p(H)$, and χ_j be the partition of unity on $\mathcal{M}$. Then we have*

$$\begin{aligned} R(k^2 \pm i0) f \simeq \omega_\pm^{(c)}(k) \sum_{j=1}^{\mu} \chi_j y^{1/2 \pm ik} \mathcal{F}_j^{(\pm)}(k) f \\ + \omega_\pm(k) \sum_{j=\mu+1}^{N} \chi_j y^{1/2 \mp ik} \mathcal{F}_j^{(\pm)}(k) f. \end{aligned}$$

The following theorem is a characterization of the solution space of the Helmholtz equation, and is proved in the same way as in [24], Chap. 2, Theorem 7.8.

Theorem 3.11. *If $k^2 \in (0,\infty) \setminus \sigma_p(H)$, we have*

$$\mathcal{F}^{(\pm)}(k)\mathcal{B} = \mathbf{h}_\infty,$$

$$\{u \in \mathcal{B}^* \,;\, (H-k^2)u = 0\} = \mathcal{F}^{(\pm)}(k)^*\mathbf{h}_\infty.$$

3.6. S **matrix.** We derive an asymptotic expansion of solutions to the Helmholtz equation. Let V_ℓ be the differential operator defined by

$$V_\ell = [H_{free(\ell)}, \chi_\ell] - \chi_\ell \widetilde{V}_\ell \quad (1 \leq \ell \leq N),$$

where $\widetilde{V}_\ell$ is defined by (3.44). We put

$$J_j(k) = \sum_{\lambda_{j,m} \neq 0} \left(\frac{\sqrt{\lambda_{j,m}}}{2}\right)^{-2ik} P_{j,m} = \left(\frac{\sqrt{-\Delta_{M_j}}}{2}\right)^{-2ik} P_j^+, \tag{3.54}$$

where Δ_{M_j} is the Laplace-Beltami operator on M_j and P_j^+ is the projection onto the subspace on which $-\Delta_{M_j} > 0$. For $1 \leq j,\ell \leq N$, we define $\widehat{S}_{j\ell}(k) \in \mathbf{h}_\ell; \mathbf{h}_j$ by

$$\widehat{S}_{j\ell}(k) = \begin{cases} \dfrac{\pi i}{k}\mathcal{F}_j^{(+)}(k)\big(V_\ell\big)^*\left(\mathcal{F}^{(-)}_{free(\ell)}(k)\right)^*, & 1 \leq j \leq \mu, \\ \delta_{j\ell}J_j(k) + \dfrac{\pi i}{k}\mathcal{F}_j^{(+)}(k)\big(V_\ell\big)^*\left(\mathcal{F}^{(-)}_{free(\ell)}(k)\right)^*, & \mu+1 \leq j \leq N. \end{cases} \tag{3.55}$$

We define an operator-valued $N \times N$ matrix $\widehat{S}(k)$ by

$$\widehat{S}(k) = \left(\widehat{S}_{j\ell}(k)\right)_{j,\ell=1}^N, \tag{3.56}$$

and call it *S-matrix*. This is a bounded operator on $\mathbf{h}_\infty$.

Theorem 3.12. *(1) For any $u \in \mathcal{B}^*$ satisfying $(H-k^2)u = 0$, there exists a unique $\psi^{(\pm)} = (\psi_1^{(\pm)}, \cdots, \psi_N^{(\pm)}) \in \mathbf{h}_\infty$ such that*

$$\begin{aligned} u \simeq\, & \omega_-^{(c)}(k)\sum_{j=1}^{\mu}\chi_j\, y^{1/2-ik}\,\psi_j^{(-)} + \omega_-(k)\sum_{j=\mu+1}^{N}\chi_j\, y^{1/2+ik}\,\psi_j^{(-)} \\ & - \omega_+^{(c)}(k)\sum_{j=1}^{\mu}\chi_j\, y^{1/2+ik}\,\psi_j^{(+)} - \omega_+(k)\sum_{j=\mu+1}^{N}\chi_j\, y^{1/2-ik}\,\psi_j^{(+)}. \end{aligned}$$

(2) For any $\psi^{(-)} \in \mathbf{h}_\infty$, there exists a unique $\psi^{(+)} \in \mathbf{h}_\infty$ and $u \in \mathcal{B}^*$ satisfying $(H-k^2)u = 0$, for which the expansion (1) holds. Moreover

$$\psi^{(+)} = \widehat{S}(k)\psi^{(-)}.$$

(3) $\widehat{S}(k)$ is unitary on $\mathbf{h}_\infty$.

For the proof, see [24], Chap. 3, Theorems 3.14, 3.15, 3.16.

3.7. **Helgason's theorem.** Before closing this section, we give some remarks on Theorems 3.11 and 3.12. As the most fundamental example of hyperbolic space, let us consider the Poincaré disc D in $\mathbf{C}$. As is well-known, the Poisson integral

$$u(z) = \frac{1}{2\pi}\int_0^{2\pi}\left(\frac{1-|z|^2}{|e^{i\theta}-z|^2}\right)^s f(\theta)d\theta, \tag{3.57}$$

$f(\theta)$ being a function on the boundary $\partial D = S^1$, gives a solution to the Helmholtz equation in D:

$$(-\Delta_g - E)u = 0, \quad E = 4s(s-1). \tag{3.58}$$

Our solution space $\mathcal{B}^*$, which is associated with the case in which the boundary space is $L^2(S^1)$, has the following feature: Regarding the decay at infinity, which corresponds to the boundary $\partial D = S^1$, of solutions for (3.58), $\mathcal{B}^*$ is the smallest space. In fact, by [24], Chap. 3, Theorem 3.6, if a solution u of the equation (3.58) has a faster decay rate than $\mathcal{B}^*$ at regular infinity, u vanishes identically. The largest solution space for (3.58) was given by Helgason. In [21], he proved that *all* solutions of the Helmholtz equation is written by (3.57), where $f(\theta)$ is Sato's hyperfunction on the boundary. This result was extended to real hyperbolic spaces by [45] and to general symmetric spaces of rank 1 by [33].

Remark 2. Let $A(S^1)$ be the space of functions on S^1 having analytic continuations in a neighborhood of S^1. By the correspondence

$$\mathbf{c} = (c_n)_{n\in\mathbf{Z}} \Longleftrightarrow f_{\mathbf{c}} = \sum_{n\in\mathbf{Z}} c_n e^{inx}, \tag{3.59}$$

$A(S^1)$ is identified with the set of sequences

$$\mathbf{c}: \quad \exists\rho > 1 \quad \text{s.t.} \quad \sum_{n\in\mathbf{Z}} |c_n|\rho^{|n|} < \infty.$$

The dual space of $A(S^1)$, the space of Sato's hyperfunctions on S^1, is identified with the set of sequences

$$\mathbf{d} = (d_n)_{n\in\mathbf{Z}}: \quad 0 < \forall\rho < 1, \quad \sup_{n\in\mathbf{Z}} |d_n|\rho^{|n|} < \infty.$$

Although $\mathcal{B}^*$ is the smallest solution space, it has sufficiently many solutions if one of the ends is regular. In fact, one can determine the whole manifold from the knowledge of a component of the S-matrix associated with regular end, see [24]. It is not the case for the cusp due to the fact that the cusp gives rise only to the 1-dimensional contribution to the continuous spectrum. This requires us to generalize the notion of the S-matrix.

4. Generalized S-matrix

4.1. **Exponentially growing solutions.** In order to enlarge the solution space of the Helmholtz equation, we enlarge the associated space at infinity.

Definition 4.1. We introduce the sequential spaces $l^{2,\pm\infty}$ by

$$l^{2,\infty} \ni \mathbf{a} = (a_n)_{n\in\mathbf{Z}} \Longleftrightarrow \forall \rho > 1, \quad \sum_{n\in\mathbf{Z}} |a_n|^2 \rho^{|n|} < \infty,$$
$$l^{2,-\infty} \ni \mathbf{b} = (b_n)_{n\in\mathbf{Z}} \Longleftrightarrow \exists \rho > 1, \quad \sum_{n\in\mathbf{Z}} |b_n|^2 \rho^{-|n|} < \infty.$$

By the correspondence (3.59), $l^{2,\infty}$ is identified with the space of functions on S^1 having analytic continuations on $\mathbf{C}\setminus\{0\}$, moreover

$$l^{2,\infty} \subset A(S^1), \quad A(S^1)' \subset l^{2,-\infty}.$$

Let $0 \neq k \in \mathbf{R}$. Suppose $u(x,y) \in C^\infty(\mathbf{R}\times(1,\infty))$ is $2\pi r$-periodic in x, $u(x,y) = u(x+2\pi r, y)$, and satisfies there the equation

$$-y^2\big(\partial_x^2 + \partial_y^2\big)u - \frac{1}{4}u = k^2 y. \tag{4.1}$$

Expanding u into the Fourier series

$$u(x,y) = \frac{1}{\sqrt{2\pi r}} \sum_{n\in\mathbf{Z}} e^{inx/r} u_n(y),$$

we have

$$y^2\Big(-\partial_y^2 + \frac{n^2}{r^2}\Big)u_n(y) - \frac{1}{4}u_n(y) = k^2 u_n(y), \quad y > 1.$$

Then u_n is written as

$$u_n(y) = \begin{cases} a_n\, y^{1/2} I_{-ik}(|n|y/r) + b_n\, y^{1/2} K_{ik}(|n|y/r), & (n \neq 0), \\ a_0\, y^{1/2-ik} + b_0\, y^{1/2+ik}, & (n = 0). \end{cases} \tag{4.2}$$

Let us note that $K_{-\nu}(z) = K_\nu(z)$.

Lemma 4.2. *Given $u(x,y) \in C^\infty(\mathbf{R}\times(1,\infty))$, which is $2\pi r-$periodic in x and satisfies (4.1), let $\mathbf{a} = (a_n)_{n\in\mathbf{Z}}$, $\mathbf{b} = (b_n)_{n\in\mathbf{Z}}$ be defined by (4.2). If $\mathbf{a} \in l^{2,\infty}$, then $\mathbf{b} \in l^{2,-\infty}$.*

Proof. Recall the asymptotic expansion of the modified Bessel functions (3.11), (3.12). Since $\mathbf{a} \in l^{2,\infty}$, we have $\sum_n |a_n|^2 \big|I_{-ik}(|n|y/r)\big|^2 < \infty$ for any $y > 1$. By Parseval's formula,

$$\begin{aligned} & y^{-1}\|u(\cdot,y)\|^2_{L^2(0,2\pi r)} \\ & = \sum_{n\neq 0} \big|a_n I_{-ik}(|n|y/r) + b_n K_{ik}(|n|y/r)\big|^2 + |a_0 y^{-ik} + b_0 y^{ik}|^2. \end{aligned}$$

We then have $\sum_{n\neq 0} |b_n|^2 \big|K_{ik}(|n|y/r)\big|^2 < \infty$, $y > 1$, hence $\mathbf{b} \in l^{2,-\infty}$. $\square$

We introduce the *spaces of generalized scattering data at infinity* :

$$\mathbf{A}_{\pm\infty} = \left(\mathop{\oplus}_{j=1}^{\mu} l^{2,\pm\infty}\right) \oplus \left(\mathop{\oplus}_{j=\mu+1}^{N} L^2(M_j)\right), \tag{4.3}$$

M_j being S^{r_j} with metric $ds^2 = (dx)^2$, $x \in [0, 2\pi r_j)$.

We use the following notation. For

$$\psi^{(in)} = (\mathbf{a}_1, \cdots, \mathbf{a}_\mu, \psi^{(in)}_{\mu+1}, \cdots, \psi^{(in)}_N) \in \mathbf{A}_\infty, \tag{4.4}$$

$$\psi^{(out)} = (\mathbf{b_1}, \cdots, \mathbf{b}_\mu, \psi^{(out)}_{\mu+1}, \cdots, \psi^{(out)}_N) \in \mathbf{A}_{-\infty}, \tag{4.5}$$

we let

$$u^{(in)}_j = \begin{cases} a_{j,0}\, y^{1/2-ik} + \sum_{n\neq 0} a_{j,n}\, e^{inx/r_j} y^{1/2} I_{-ik}(|n|y/r_j), & 1 \le j \le \mu \\ \omega_-(k)\, y^{1/2+ik} \psi^{(in)}_j(x), \quad \mu+1 \le j \le N, \end{cases} \tag{4.6}$$

$$u^{(out)}_j = \begin{cases} b_{j,0}\, y^{1/2+ik} + \sum_{n\neq 0} b_{j,n}\, e^{inx/r_j} y^{1/2} K_{ik}(|n|y/r_j), & 1 \le j \le \mu \\ \omega_+(k)\, y^{1/2-ik} \psi^{(out)}_j(x), \quad \mu+1 \le j \le N. \end{cases} \tag{4.7}$$

Here $a_{j,n}, b_{j,n}$ are the n-th components of $\mathbf{a}_j \in l^{2,\infty}, \mathbf{b}_j \in l^{2,-\infty}$. Let $\langle\ ,\ \rangle_j$ be the inner product of $L^2(S^{r_j})$:

$$\langle f, g\rangle_j = \int_{S^{r_j}} f\overline{g}\, dl.$$

Lemma 4.3. *Let $k > 0$ be such that $k^2 \notin \sigma_p(H)$, $\psi^{(in)}$, $u^{(in)}_j$ as in (4.4), (4.6), and $u^{(in)} = \sum_{j=1}^N \chi_j u^{(in)}_j$. Then, there exists a unique solution u such that*

$$(H - k^2)u = 0, \quad u - u^{(in)} \text{ is outgoing}, \tag{4.8}$$

i.e. $u - u^{(in)}$ belongs to $\mathcal{B}^*$ on $\mathcal{M}$, and satisfies (3.42). For this u, there exists $\psi^{(out)} = (\mathbf{b_1}, \cdots, \mathbf{b}_\mu, \psi^{(out)}_{\mu+1}, \cdots, \psi^{(out)}_N) \in \mathbf{A}_{-\infty}$ such that
(1) for $j = 1, \cdots, \mu$,

$$u = u^{(in)}_j - u^{(out)}_j, \quad in \quad \mathcal{M}_j \cap (\mathrm{supp}\,\chi_0)^c, \tag{4.9}$$

(2) for $j = \mu+1, \cdots, N$,

$$u - u^{(in)}_j \simeq -u^{(out)}_j, \quad \text{in} \quad \mathcal{M}_j. \tag{4.10}$$

Explicitly, $\mathbf{b}_j$ and $\psi^{(out)}_j$ are given by

$$b_{j,0} = \frac{1}{2ik\sqrt{2\pi r_j}} \int_0^\infty (y)^{1/2-ik} \widehat{f}_{j,0}(y) \frac{dy}{(y)^2}, \tag{4.11}$$

$$b_{j,n} = \frac{1}{\sqrt{2\pi r_j}} \int_0^\infty y^{1/2} I_{-ik}(|n|y/d_j) \widehat{f}_{j,n}(y) \frac{dy}{y^2}, \quad n \neq 0, \tag{4.12}$$

$$\psi^{(out)}_j = \mathcal{F}^{(+)}_j(k) f, \quad \mu+1 \le j \le N, \tag{4.13}$$

where

$$f = (H - k^2)u^{(in)}, \quad f_j = \chi_j f + [H_{free(j)}, \chi_j] R(k^2 + i0) f, \tag{4.14}$$

$$\widehat{f}_{j,n} = \frac{1}{\sqrt{2\pi r_j}} \langle f_j, e^{inx/r_j} \rangle_j. \tag{4.15}$$

Proof. The uniqueness follows from Theorem 3.8. To show the existence, we represent

$$u = u^{(in)} - R(k^2 + i0)f. \tag{4.16}$$

Then the condition (4.8) is satisfied by Theorem 3.8. By Theorem 3.10, we have

$$R(k^2+i0)f \simeq \omega_+^{(c)}(k) \sum_{j=1}^{\mu} \chi_j y^{1/2+ik} \mathcal{F}_j^{(+)}(k) f + \omega_+(k) \sum_{j=\mu+1}^{N} \chi_j y^{1/2-ik} \mathcal{F}_j^{(+)}(k) f,$$

which proves (4.10) and (4.13).

For $j = 1, \cdots, \mu$, let $H_{free(j)} = -y^2\Delta - 1/4$ on $S^{r_j} \times (0,\infty)$, and put

$$R_{free(j)}(z) = (H_{free(j)} - z)^{-1}.$$

Since

$$(H_{free(j)} - \lambda)\chi_j R(\lambda \pm i0) = \chi_j + [H_{free(j)}, \chi_j] R(\lambda \pm i0),$$

we have

$$\chi_j R(\lambda \pm i0) = R_{free(j)}(\lambda \pm i0)\chi_j + R_{free(j)}(\lambda \pm i0)[H_{free(j)}, \chi_j] R(\lambda \pm i0). \tag{4.17}$$

Note that on $\mathcal{M}_j$, $f = [H, \chi_j] u_j^{(-)}$, and $[H, \chi_j]$ is a 1st-order differential operator with coefficients which are compactly supported in $\mathcal{M}_j$. Therefore, f_j is compactly supported, in particular $f_j = 0$ on $\mathcal{M}_j \cap (\operatorname{supp}\chi_0)^c$, and, by (4.14),

$$\chi_j R(k^2 + i0) f = R_{free(j)}(k^2 + i0) f_j. \tag{4.18}$$

By (3.20), (3.21), (3.26), and taking account of (3.29), we have for large $y > 0$,

$$\begin{aligned} &\frac{1}{\sqrt{2\pi r_j}} \big\langle R_{free(j)}(k^2 + i0) f_j, e^{inx/r_j} \big\rangle_j \\ &= \begin{cases} y^{1/2} K_{-ik}(|n|y/d_j) \displaystyle\int_0^\infty (y')^{1/2} I_{-ik}(|n|y'/d_j) f_{j,n}(y') \frac{dy'}{(y')^2}, & n \neq 0, \\ \dfrac{1}{2ik} y^{1/2+ik} \displaystyle\int_0^\infty (y')^{1/2-ik} f_{j,0}(y') \frac{dy'}{(y')^2}, & n = 0. \end{cases} \end{aligned} \tag{4.19}$$

Using $K_{-ik}(z) = K_{ik}(z)$, and noting that

$$u - u_j^{(in)} = -\sum_n \big\langle R_{free(j)}(k^2 + i0) f_j, \frac{e^{inx/r_j}}{\sqrt{2\pi r_j}} \big\rangle_j \frac{e^{inx/r_j}}{\sqrt{2\pi r_j}},$$

we prove (1). $\square$

Given $u_j^{(in)}$, $j = 1, \cdots, \mu$, one can compute $b_{j,n}$ by observing the asymptotic behavior of $u - u^{(in)}$ in a neighborhood of the cusp. With this in mind, we make the following definition.

Definition 4.4. We call the operator

$$\mathbf{S}(k) : \mathbf{A}_\infty \ni \psi^{(in)} \to \psi^{(out)} \in \mathbf{A}_{-\infty}$$

the *generalized S-matrix.*

4.2. **Splitting the manifold.** We take a compact submanifold, $\Gamma \subset \mathcal{M}$, of codimension 1, and split $\mathcal{M}$ into 2 parts, $\mathcal{M}_{ext}$ and $\mathcal{M}_{int}$, in the following way:

$$\mathcal{M} = \mathcal{M}_{ext} \cup \mathcal{M}_{int}, \quad \mathcal{M}_{ext} \cap \mathcal{M}_{int} = \Gamma.$$

Here $\mathcal{M}_{ext} \setminus \Gamma$ and $\mathcal{M}_{int} \setminus \Gamma$ are assumed to be open submanifolds of $\mathcal{M}$ with boundary Γ inheriting the Riemannian structure of $\mathcal{M}$. Assume also that $\mathcal{M}_{ext}$ is non-compact, has infinity common to $\mathcal{M}_1$ and no other infinity. Recall that the end $\mathcal{M}_1$ has a cusp. We also assume that $\mathcal{M}_{sing}$ is in the interior of $\mathcal{M}_{int}$.

Let $-\Delta_g$ be the Laplace-Beltrami operator on $\mathcal{M}$, H_{ext} and H_{int} be $-\Delta_g - 1/4$ defined on $\mathcal{M}_{ext}$, $\mathcal{M}_{int}$ with Neumann boundary condition on Γ, respectively. If $\mathcal{M}$ has only one end (i.e. N = 1), $\mathcal{M}_{int}$ is a compact manifold, and H_{int} has a discrete spectrum. If $N \geq 2$, both of $\mathcal{M}_{int}$ and $\mathcal{M}_{ext}$ are non-compact, and, although now $\partial\mathcal{M}_{ext} = \partial\mathcal{M}_{int} = \Gamma \neq \emptyset$, the theorems in §3 and §4 also hold for H_{ext}, H_{int}. We denote the inner product of $L^2(\Gamma)$ by

$$\langle f, g\rangle_\Gamma = \int_\Gamma f\overline{g}\, dl.$$

We put

$$\phi_{n,free} = \begin{cases} y^{1/2-ik}, & n = 0, \\ e^{inx/r_1} y^{1/2} I_{-ik}(|n|y/r_1), & n \neq 0, \end{cases}$$

$$g_n = (H - k^2)\chi_1\phi_{n,free} = [H_{free(1)}, \chi_1]\phi_{n,free},$$

$$\phi_n^{(+)} = \chi_1\phi_{n,free} - R(k^2 + i0)g_n. \tag{4.20}$$

Lemma 4.5. *We take $\Gamma = \Gamma_0 = \{y = y_0\} \subset \mathcal{M}_1$, $y_0 > 2$. Let $k > 0$ and $k^2 \notin \sigma_p(H) \cap \sigma_p(H_{int})$. Let $f \in L^2(\Gamma_0)$ satisfy*

$$\langle f, \partial_\nu\phi_n^{(+)}\rangle_{\Gamma_0} = 0 \quad \forall n \in \mathbf{Z}, \tag{4.21}$$

where ν is the unit normal to Γ_0, $\partial_\nu = \partial_y$. Then $f = 0$.

Proof. Note that Γ_0 is naturally identified with S^{r_1}. We define an operator $\delta'_{\Gamma_0} \in \mathbf{B}(H^{-1/2}(\Gamma_0); H^{-2}(\mathcal{M}))$ by

$$(\delta'_{\Gamma_0} v, w) = \langle v, \partial_\nu w\rangle_{\Gamma_0}, \quad \forall v \in H^{-1/2}(\Gamma_0), \quad \forall w \in H^2(\mathcal{M}), \tag{4.22}$$

and define $u = R(k^2 - i0)\delta'_{\Gamma_0} f$ by duality, i.e. for $w \in L^{2,s}$, $s > 1/2$,

$$\begin{aligned}(R(k^2 - i0)\delta'_{\Gamma_0} f, w) &= (\delta'_{\Gamma_0} f, R(k^2 + i0)w) \\ &= \langle f, \partial_y R(k^2 + i0)w\rangle_{\Gamma_0}.\end{aligned}$$

Note that $(H - k^2)u = \delta'_{\Gamma_0} f$ in the sense of distribution, hence, in the classical sense,

$$(H - k^2)u = 0, \quad \mathcal{M} \setminus \Gamma_0. \tag{4.23}$$

Considering $H_{free(1)}$ on $M_1 \times (1, \infty)$, we have

$$R_{free(1)}(k^2 - i0)\delta'_{\Gamma_0} f = \frac{1}{2\pi r_1} \sum_{n \in \mathbf{Z}} A_n(y) \widehat{f}_n e^{inx/r_1}, \quad f_n = \langle f, e^{inx/r_1}\rangle_1. \tag{4.24}$$

Here, taking account of (3.29), for $n \neq 0$,

$$A_n(y) = \begin{cases} \Big(y^{1/2} K_{ik}(|n|y/r_1)\Big)'\Big|_{y=y_0} y^{1/2} I_{ik}(|n|y/r_1), & y < y_0, \\ \Big(y^{1/2} I_{ik}(|n|y/r_1)\Big)'\Big|_{y=y_0} y^{1/2} K_{ik}(|n|y/r_1), & y > y_0, \end{cases} \tag{4.25}$$

and, for $n = 0$,

$$A_0(y) = \begin{cases} \Big(y^{1/2-ik}\Big)' y_0^{1/2+ik}, & y < y_0, \\ \Big(y^{1/2+ik}\Big)' y_0^{1/2-ik}, & y > y_0. \end{cases} \tag{4.26}$$

By (4.17), (4.20) and (4.22), when $y > y_0$ and $n \neq 0$, $u = R(k^2 - i0)\delta'_{\Gamma_0} f$ satisfies

$$\begin{aligned}&2\pi r_1 \langle u(\cdot, y), e^{inx/r_1}\rangle_1 \\ &= y^{1/2} K_{ik}(|n|y/r_1) \int_0^\infty \int_0^{2\pi r_1} e^{-inx/r_1} (y')^{1/2} I_{ik}(|n|y'/r_1) \\ &\qquad\qquad \times \left\{\chi_1 + [H_{free(1)}, \chi_1] R(k^2 - i0)\right\} \delta'_\Gamma f \frac{dx dy'}{(y')^2} \\ &= y^{1/2} K_{ik}(|n|y/r_1) \left(\delta'_\Gamma f, \left\{\chi_1 - R(k^2 + i0)[H_{free(1)}, \chi_1]\right\} \phi_{n,free}\right) \\ &= y^{1/2} K_{ik}(|n|y/r_1) \langle f, \partial_y \phi_n^{(+)}\rangle_{\Gamma_0} = 0.\end{aligned}$$

Similarly, one can show that, for large y,

$$\langle u(\cdot, y), 1\rangle_1 = 0.$$

Therefore, $u = 0$ when y is large enough. Since $(H - k^2)u = 0$ in $\mathcal{M}_{ext}$, the unique continuation theorem imply that $u = 0$ in $\mathcal{M}_{ext}$. Let $\xi(y) \in C_0^\infty(1, \infty)$ have value 1 in a neighborhood of $y = y_0$. Then,

$$R(k^2 - i0)\delta'_{\Gamma_0} f - \xi(y) R_{free(1)}(k^2 - i0)\delta'_{\Gamma_0} f \in C^\infty(\mathcal{M}).$$

Thus, using formulas (4.24) $\sim$ (4.26), we see that $\partial_y R(k^2 - i0)\delta'_{\Gamma_0} f$ is continuous across Γ_0. Therefore, in $\mathcal{M}_{int}$, u satisfies $(H_{int} - k^2)u = 0$ and the

Neumann boundary condition on Γ_0, hence $u = 0$ in $\mathcal{M}_{int}$. This follows from the assumption $k^2 \notin \sigma_p(H_{int})$ when $\mathcal{M}_{int}$ is compact, and from Theorem 3.8 when $\mathcal{M}_{int}$ is non-compact. We then have $u = 0$ in $\mathcal{M}$, which implies $\delta'_{\Gamma_0} f = 0$. Thus, by (4.22), $\langle f, \partial_y w\rangle_{\Gamma_0} = 0$, $\forall w \in H^2(\mathcal{M})$, which proves $f = 0$. $\square$

The generalized S-matrix $\mathbf{S}(k)$ is an operator-valued $N \times N$ matrix. Let $\mathbf{S}_{11}(k)$ be its $(1,1)$ entry. For $\mathbf{a} \in l^{2,\infty}$, we put $\mathbf{b} = \mathbf{S}_{11}(k)\mathbf{a} \in l^{2,-\infty}$, and

$$\Phi = \sum_{n\in\mathbf{Z}} a_n \phi_n^{(+)}.$$

Then, $(H - k^2)\Phi = 0$ and by (4.6) and (4.7) it takes the form

$$\Phi = u_1^{(in)} - u_1^{(out)}$$

In particular, in $\mathcal{M}_1$,

$$u_1^{(in)} = a_0 y^{1/2-ik} + \sum_{n\neq 0} a_n e^{inx/r_1} y^{1/2} I_{-ik}(|n|y/r_1),$$

$$u_1^{(out)} = b_0 y^{1/2+ik} + \sum_{n\neq 0} b_n e^{inx/r_1} y^{1/2} K_{ik}(|n|y/r_1).$$

Therefore, the knowledge of $\mathbf{S}_{11}(k)$ is equivalent to the observation, for any incoming exponentially growing wave $u_1^{(in)}$ at the cusp $\mathcal{M}_1$, the corresponding outgoing exponentially decaying wave $u_1^{(out)}$ at $\mathcal{M}_1$.

4.3. **Gel'fand problem, BSP and N-D map.** Before going to proceed, let us recall the *Gel'fand inverse boundary-spectral problem.* Let Ω be a compact Riemannian manifold with boundary $\partial\Omega$, $\Gamma \subset \partial\Omega$ be an open subset, and $-\Delta_g$ be the associated Laplace-Beltrami operator. Let $0 = \lambda_1 < \lambda_2 < \cdots$ be its Neumann eigenvalues without counting multiplicities, and $\varphi_{n,1}, \cdots, \varphi_{n,m(n)}$ be the orthonormal system of eigenvectors associated with the eigenvalue λ_n. Let us call the set

$$\Big\{(\lambda_n, \varphi_{n,1}\big|_\Gamma, \cdots, \varphi_{n,m(n)}\big|_\Gamma)\Big\}_{n=1}^\infty$$

the boundary spectral data (**BSD**). The problem raised by Gel'fand is :
Do BSD determine the Riemannian manifold Ω? This problem was solved by Belishev-Kurylev [4] using the boundary control method (BC-method) first proposed by Belishev [3] for inverse problems in Euclidean domains. Later, the method has been developed to study inverse problems on compact Riemannian manifolds, [2, 35, 29, 37, 32, 36, 34, 39] and non-compact manifolds [5, 25]. The BC-method combines the control theory obtained from unique continuation results [50, 51] with Blagovestchenskii's identity that gives the inner product of the solutions of the wave equation in terms of the boundary data. This identity was originally used in the study of one-dimensional inverse problems, see [6, 7].

Although it is formulated in terms of BSD, what is actually used in the BC-method is the boundary spectral projection (**BSP**) defined by

$$\Big\{\big(\lambda_n, \sum_{j=1}^{m(n)} \varphi_{n,j}(x)\overline{\varphi_{n,j}(y)}\big|_{(x,y)\in\Gamma\times\Gamma}\big)\Big\}_{n=1}^{\infty}. \tag{4.27}$$

This appears in the kernel of the Neumann to Dirichlet map (N-D map)

$$\Lambda(z) : f \to u, \tag{4.28}$$

where u is the solution to the Neumann problem

$$\begin{cases} (-\Delta_g - z)u = 0 \quad \text{in} \quad \Omega, \\ \partial_\nu u = f \in H^{1/2}(\Gamma), \end{cases} \tag{4.29}$$

ν being the outer unit normal to Γ, $z \notin \sigma(-\Delta_g)$. The N-D map is related to the resolvent $(-\Delta_g - z)^{-1}$ in the following way :

$$\Lambda(z) = \delta_\Gamma^*(-\Delta_g - z)^{-1}\delta_\Gamma, \quad z \notin \sigma(-\Delta_g). \tag{4.30}$$

Here $\delta_\Gamma \in \mathbf{B}\left((H^{1/2}(\Gamma))'; (H^1(\Omega))'\right)$ is the adjoint of the trace operator,

$$r_\Gamma : H^1(\Omega) \ni w \to w\big|_\Gamma \in H^{1/2}(\Gamma),$$

$$(\delta_\Gamma f, w)_{L^2(\Omega)} = (f, r_\Gamma w)_{L^2(\Gamma)}, \quad f \in H^{-1/2}(\Gamma), \quad w \in H^1(\Omega), \tag{4.31}$$

and we denote by $(H^s)'$ the dual to H^s with respect to the L^2-pairing. More precisely, we have

Lemma 4.6. *To give BSP is equivalent to give the N-D map $\Lambda(z)$ for all $z \notin \sigma(-\Delta_g)$.*

We refer for analogous equivalence results for different kind of boundary data to [36, 30].

Let Ω be non-compact with asymptotically hyperbolic ends of the type discussed in this paper. Let $\Gamma \subset \partial\Omega$ be compact, and consider its shifted Laplace-Beltrami operator with Neumann boundary condition, $H = -\Delta_g - \frac{1}{4}$. It has continuous spectrum $\sigma_c(H) = [0, \infty)$, and, furthermore, H has a spectral representation $\mathcal{F}$ like the one discussed in §3. In this case we define the BSP to be the collection

$$\Big\{\delta_\Gamma^*\mathcal{F}(k)^*\mathcal{F}(k)\delta_\Gamma \,;\, k > 0\Big\} \cup \Big\{(\lambda_n, \delta_\Gamma^* P_n \delta_\Gamma)\Big\}_{n=1}^{m}.$$

Here λ_n is the eigenvalue of H, P_n is the associated eigenprojection and m is the number of eigenvalues which, in principle could be infinite, see Theorem 3.6. In this case, we extend the N-D map $\Lambda(z)$ for $z \in \mathbb{C} \setminus \sigma(H)$ by using the solution u of (4.29). Note that we can extend the definition of $\Lambda(z)$ for $z = k^2 \pm i0 \notin \sigma_p(H)$ by using the outgoing or incoming radiation conditions. Then Lemma 4.6 also holds in this case. (See [24], Chap. 5, §3 and §4, [25], Lemma 5.6.)

Denote by $G(z;X,Y)$, $z\in\mathbb{C}\setminus\sigma(H)$, the Schwartz kernel of $(H-z)^{-1}$. Since

$$(H-z)^{-1}=\sum_{n=1}^{m}\frac{1}{\lambda_n-z}P_n+\int_0^\infty\frac{1}{k^2-z}\mathcal{F}(k)^*\mathcal{F}(k)\,dk,$$

we have, in view of (4.30), that

$$G(z;\,\cdot\,,\cdot\,)\Big|_{\Gamma\times\Gamma}=\sum_{n=1}^{m}\frac{1}{\lambda_n-z}\delta_\Gamma^*P_n\delta_\Gamma+\int_0^\infty\frac{1}{k^2-z}\delta_\Gamma^*\mathcal{F}(k)^*\mathcal{F}(k)\delta_\Gamma\,dk. \tag{4.32}$$

Here the left-hand side is understood as the Schwartz kernel of the operator in the right-hand side of the formula.

4.4. **Generalized S-matrix and N-D map.** Returning to our problem concerning 2-dimensional non-compact surfaces with conical singularities, we take $\Omega=\mathcal{M}_{int}$ with $\Gamma_0=\{X\in\mathcal{M}_1\,:\,y=y_0\}$, $y_0>2$. We define the N-D map for $\mathcal{M}_{int}$ by (4.28) and (4.29).

Now suppose we are given two manifolds $\mathcal{M}^{(i)}$, $i=1,2$, satisfying the assumptions (A-1) $\sim$ (A-4) in §1. Let $-\Delta^{(i)}$ be the Laplace-Betrami operator of $\mathcal{M}^{(i)}$. Assume that $\mathcal{M}^{(i)}$ has N_i numbers of ends, and let $\mathbf{S}_{11}^{(i)}(k)$ be the $(1,1)$ entry of the generalized S-matrix for $H^{(i)}=-\Delta^{(i)}-\frac14$.

Assuming that $r_1^{(1)}=r_1^{(2)}$, we can naturally identify $\mathcal{M}_1^{(1)}$ and $\mathcal{M}_1^{(2)}$. Taking Γ_0 as above, we split $\mathcal{M}^{(i)}$ into $\mathcal{M}_{int}^{(i)}\cup\mathcal{M}_{ext}^{(i)}$ by using Γ_0. Let $H_{int}^{(i)}=-\Delta_{int}^{(i)}-\frac14$ be the shifted Laplace-Beltrami operator of $\mathcal{M}_{int}^{(i)}$ with Neumann boundary condition on Γ_0, and define the N-D map $\Lambda^{(i)}(z)$ for $H_{int}^{(i)}$. With this preparation, we can prove the following lemma.

Lemma 4.7. *Let $k>0$, $k^2\notin\sigma_p(H^{(1)})\cup\sigma_p(H^{(2)})\cup\sigma_p(H_{int}^{(1)})\cup\sigma_p(H_{int}^{(2)})$. If $\mathbf{S}_{11}^{(1)}(k)=\mathbf{S}_{11}^{(2)}(k)$, we have $\Lambda^{(1)}(k^2+i0)=\Lambda^{(2)}(k^2+i0)$.*

Proof. For $i=1,2$, we construct $\phi_n^{(i)}$ as in (4.20), and put $u=\phi_n^{(1)}-\phi_n^{(2)}$. Then u satisfies

$$(H^{(i)}-k^2)u=(-\Delta^{(i)}-k^2-\frac14)u=0\quad\text{for } X\in\mathcal{M}_{ext}^{(1)}=\mathcal{M}_{ext}^{(2)},\ y(X)>2,$$

due to $\chi_1=1$ there, see (3.41). Since $\mathbf{S}_{11}^{(1)}(k)=\mathbf{S}_{11}^{(2)}(k)$, by the same argument as in the proof of Lemma 4.5, we have $u=0$ for $X\in\mathcal{M}_{ext}^{(1)}=\mathcal{M}_{ext}^{(2)}$, $y(X)>2$. Hence, $\partial_\nu\phi_n^{(1)}=\partial_\nu\phi_n^{(2)}$ on Γ_0.

In $\mathcal{M}_{int}^{(i)}$, $\phi_n^{(i)}$ is the outgoing solution of the equation $(H_{int}^{(i)}-k^2)\phi_n^{(i)}=0$. Hence, $\partial_\nu\phi_n^{(i)}\big|_{\Gamma_0}=\Lambda^{(i)}(k^2+i0)\phi_n^{(i)}\big|_{\Gamma_0}$, where we again use that $\chi_1=1$ near Γ_0. This implies

$$\Lambda^{(1)}(k^2+i0)\phi_n^{(1)}\big|_{\Gamma_0}=\Lambda^{(2)}(k^2+i0)\phi_n^{(2)}\big|_{\Gamma_0},\quad\forall n. \tag{4.33}$$

Lemma 4.5 implies that the linear span of $\{\partial_\nu\phi_n^{(i)}\big|_{\Gamma_0}\,;\,n\in\mathbf{Z}\}$ is dense in $L^2(\Gamma_0)$. Therefore, by (4.33), $\Lambda^{(1)}(k^2+i0)=\Lambda^{(2)}(k^2+i0)$. $\square$

Corollary 4.8. *Let (a,b) be an interval such that $(a,b)\cap\big(\sigma_p(H^{(1)})\cup\sigma_p(H^{(2)})\big)=\emptyset$, and assume that $\mathbf{S}_{11}^{(1)}(k)=\mathbf{S}_{11}^{(2)}(k)$ for $k^2\in(a,b)$. Then $\Lambda^{(1)}(z)=\Lambda^{(2)}(z)$ if $z\notin\sigma(H_{int}^{(1)})\cup\sigma(H_{int}^{(2)})$. Moreover, BSP's for $H_{int}^{(1)}$ and $H_{int}^{(2)}$ and Green's kernels $G^{(i)}(z;X,Y)$ for $(H^{(i)}-z)^{-1}$, $i=1,2$, coincide on $\Gamma_0\times\Gamma_0$.*

Proof. For $f\in H^{1/2}(\Gamma_0)$, let $F\in H_0^2(\mathcal{M}_{int}^{(i)})$ satisfy

$$\partial_\nu F|_{\Gamma_0}=f,\quad \mathrm{supp}(F)\subset S^{r_1}\times[2,y_0]. \tag{4.34}$$

Then,

$$\Lambda^{(i)}(z)f=r_{\Gamma_0}\left(F-(-\Delta^{(i)}-z)^{-1}w_f(z)\right),$$

where $w_f(z)=-(H_{int}^{(i)}-z)F$ is independent of $i=1,2$, due to (4.34). Note that $\Lambda^{(i)}(z)f$ is analytic, if $z\notin\sigma(H_{int}^{(i)})$, and have a limit, $\Lambda^{(i)}(k^2\pm i0)f$, when $z\to k^2\pm i0$, $k^2\notin\sigma_p(H_{int}^{(i)})$. Using Lemmas 4.7, 4.6 and (4.32) we obtain the result. □

5. Uniqueness of inverse scattering

5.1. **Blagovestchenskii's identity.** To prove the uniqueness of the inverse scattering problem we start with some auxiliary results. Let Ω be a (possibly non-compact) Riemannian surface with conical singularities (and asymptotically hyperbolic ends) and $H=-\Delta_g-\frac{1}{4}$ be the Hamiltonian corresponding to Neumann boundary condition on $\partial\Omega$. We denote by $\mathcal{F}(k)$ the Fourier transform associated to H and by P_j the orthogonal projections corresponding to eigenvalues λ_j of H using the convention that $\mathcal{F}(k)=0$ when Ω is compact. Let $\Gamma\subset\partial\Omega$ be open. Consider the solution $u^f(X,t)$ of the initial boundary value problem

$$\begin{cases}\partial_t^2u-\Delta_gu-\dfrac{1}{4}u=0, & \text{in}\quad \Omega\times\mathbb{R}_+,\\ u\big|_{t=0}=\partial_tu\big|_{t=0}=0, & \text{in}\quad\Omega,\\ \partial_\nu u=f, \quad \text{in}\quad \partial\Omega\times\mathbb{R}_+, & \mathrm{supp}\,f\subset\Gamma\times\mathbb{R}_+.\end{cases} \tag{5.1}$$

Let

$$B(t,\lambda)=\begin{cases}\frac{\sin(\sqrt{\lambda}t)}{\sqrt{\lambda}} & \text{for } \lambda\neq0,\\ t & \text{for } \lambda=0.\end{cases}$$

Lemma 5.1. *Assume that we are given the curve $\Gamma\subset\partial\Omega$, the length element dl on Γ and the BSP of H on Γ. Then, for any given $f,h\in C_0^\infty(\Gamma\times\mathbb{R}_+)$ and $t,s>0$, these data uniquely determine*

$$(u^f(t),u^h(s))_{L^2(\Omega)}=\int_\Omega u^f(X,t)\,\overline{u^h(X,s)}\,dS_X$$

and

$$(u^f(t),1)_{L^2(\Omega)}=\int_\Omega u^f(X,t)\,dS_X.$$

Moreover, the hyperbolic N-D map $R_\Gamma^T : f \mapsto u^f|_{\Gamma\times(0,T)}$ can be written in terms of BSP as

$$\begin{aligned} R_\Gamma^T f(\,\cdot\,,t) &= \int_0^t dt' \Big(\sum_{n=1}^m B(t-t',\lambda_n)\delta_\Gamma^* P_n \delta_\Gamma + \\ &\qquad + \int_0^\infty dk\, B(t-t',k^2)\delta_\Gamma^* \mathcal{F}(k)^*\mathcal{F}(k)\delta_\Gamma \Big) f(\,\cdot\,,t'). \end{aligned} \tag{5.2}$$

Proof. The solution $u^f(t)$ can be written as

$$u^f(X,t) = \int_0^t dt' \Big(\sum_{n=1}^m B(t-t',\lambda_n) P_n \delta_\Gamma + \int_0^\infty dk\, B(t-t',k^2)\mathcal{F}(k)^*\mathcal{F}(k)\delta_\Gamma \Big) f(\,\cdot\,,t').$$

Restricting this equation to Γ, we prove (5.2).

Using the similar decomposition for $u^h(s)$, we obtain the following formula:

$$(u^f(t),u^h(s))_{L^2}^2 = \int_0^t dt' \int_0^s ds' \int_\Gamma dl_X \int_\Gamma dl_Y\, \widetilde{K}(t-t',s-s',X,Y) f(X,t')h(Y,s'). \tag{5.3}$$

Here

$$\begin{aligned} \widetilde{K}(t,s,\cdot,\cdot) = &\sum_{n=1}^m B(t,\lambda_n)B(s,\lambda_n)\delta_\Gamma^* P_n \delta_\Gamma \\ &+ \int_0^\infty dk\, B(t,k^2)B(s,k^2)\delta_\Gamma^* \mathcal{F}(k)^*\mathcal{F}(k)\delta_\Gamma, \end{aligned}$$

where the left-hand side is understood as the Schwartz kernel of the operator in the right-hand side.

Moreover, as $J(t) = (u^f(t),1)_{L^2}$ satisfies the differential equation

$$\partial_t^2 J(t) = (\partial_t^2 u^f(t),1)_{L^2} = (\Delta_g u^f(t),1)_{L^2} = \int_\Gamma f(Y,t) dl_Y$$

and initial conditions $J(0) = \partial_t J(t)|_{t=0} = 0$, we see that

$$(u^f(t),1)_{L^2} = \int_\Gamma dl_Y \int_0^t dt'\, B(t-t',0) f(Y,t').$$

□

Above, the formula (5.3) is a generalization of Blagovestchenskii identity (see [32, Theorem 3.7]) for Riemannian surfaces with conic singularities.

Next we will apply these formulas to compute the area of the domain of influence

$$\Omega(\tilde\Gamma,T) = \{X\in\Omega:\, d_g(X,\tilde\Gamma) \le T\}, \quad \tilde\Gamma\subset\Gamma, \tag{5.4}$$

where d_g denotes the distance in Ω with respect to g. We denote the area of $\Omega(\tilde\Gamma,T)$ by $S_g(\Omega(\tilde\Gamma,T))$.

Lemma 5.2. *Assume that we are given the curve Γ, the length element dl on Γ and the BSP of H on Γ. Then, for any given open set $\tilde\Gamma \subset \Gamma$ and $T > 0$, these data uniquely determine $S_g(\Omega(\tilde\Gamma, T))$.*

Proof. Let $w \in L^2(\Omega)$ be a function such that $w = 1$ in $\Omega(\tilde\Gamma, T)$. For $f \in C_0^\infty(\tilde\Gamma \times (0,T))$, real-valued, we define the quadratic functional

$$I_T(f) \;=\; \|u^f(\cdot,T) - w\|^2_{L^2(\Omega)} - \|w\|^2_{L^2(\Omega)}.$$

Since $\operatorname{supp}(u^f(\cdot,T)) \subset \Omega(\tilde\Gamma,T)$, we have

$$I_T(f) \;=\; \|u^f(\cdot,T)\|^2_{L^2(\Omega)} - 2(u^f(\cdot,T),1)_{L^2(\Omega)}. \tag{5.5}$$

Hence, by Lemma 5.1, we can compute $I_T(f)$ for any $f \in C_0^\infty(\tilde\Gamma \times (0,T))$ uniquely by using BSP and dl on Γ. In the sequel, this is phrased as *we can compute.*

Now we use again the fact that, for $f \in C_0^\infty(\tilde\Gamma \times (0,T))$, $\operatorname{supp}(u^f(\cdot,T)) \subset \Omega(\tilde\Gamma,T)$ so that (5.5) yields that

$$I_T(f) \;=\; \|u^f(\cdot,T) - \chi_{(\tilde\Gamma,T)}\|^2_{L^2(\Omega)} - \|\chi_{(\tilde\Gamma,T)}\|^2_{L^2(\Omega)},$$

where $\chi_{(\tilde\Gamma,T)}$ is the characteristic function of $\Omega(\tilde\Gamma,T)$. Thus,

$$I_T(f) \ge -S_g(\Omega(\tilde\Gamma,T)), \quad \text{for all } f \in C_0^\infty(\tilde\Gamma \times (0,T)). \tag{5.6}$$

By Tataru's controllability theorem, see [50] and e.g. [32], there is a sequence $h_j \in C_0^\infty(\tilde\Gamma \times (0,T))$, such that

$$\lim_{j\to\infty} u^{h_j}(\cdot,T) = \chi_{\Omega(\tilde\Gamma,T)} \quad \text{in } L^2(\Omega).$$

For this sequence,

$$\lim_{j\to\infty} I_T(h_j) = -S_g(\Omega(\tilde\Gamma,T)). \tag{5.7}$$

On the other hand, if $f_j \in C_0^\infty(\tilde\Gamma \times (0,T))$ is a minimizing sequence for I_T, i.e.,

$$\lim_{j\to\infty} I_T(f_j) = m_0 := \inf\{I_T(f);\, f \in C_0^\infty(\tilde\Gamma \times (0,T))\}, \tag{5.8}$$

then, by (5.6) and (5.7),

$$\lim_{j\to\infty} u^{f_j}(\cdot,T) = \chi_{\Omega(\tilde\Gamma,T)} \quad \text{in } L^2(\Omega).$$

Thus, using any sequence (f_j) satisfying (5.8), we can compute

$$S_g(\Omega(\tilde\Gamma,T)) = \lim_{j\to\infty} (u^{f_j}(\cdot,T), u^{f_j}(\cdot,T))_{L^2(\Omega)}.$$

$\square$

5.2. **Reconstruction near Γ_0.** To prove Theorem 1.1 our first aim is to show that $\mathcal{M}^{(1)}_{reg}$ and $\mathcal{M}^{(2)}_{reg}$ are isometric. The proof is based on the procedure of the continuation of Green's functions, $G^{(i)}(z;X,Y)$, $i=1,2$, of the operators $H^{(i)}_{int}$.

We are going to prove the uniqueness for the inverse problem step by step by constructing relatively open subsets $\mathcal{M}^{(1),rec}\subset\mathcal{M}^{(1)}_{int}$ and $\mathcal{M}^{(2),rec}\subset\mathcal{M}^{(2)}_{int}$, which are isometric and enlarge these sets at each step. In the following, when $\mathcal{M}^{(1),rec}\subset\mathcal{M}^{(1)}_{reg}\cap\mathcal{M}^{(1)}_{int}$ and $\mathcal{M}^{(2),rec}\subset\mathcal{M}^{(2)}_{reg}\cap\mathcal{M}^{(2)}_{int}$ are relatively open connected sets and

$$\Phi^{rec}:\,\mathcal{M}^{(1),rec}\to\mathcal{M}^{(2),rec},$$

is a diffeomorphism, we say that the triple $(\mathcal{M}^{(1),rec},\mathcal{M}^{(2),rec},\Phi^{rec})$ is *admissible* if $\Phi^{rec}:\,\mathcal{M}^{(1),rec}\to\mathcal{M}^{(2),rec}$ is an isometry, that is, $(\Phi^{rec})_*g^{(1)}=g^{(2)}$ and the values of Green's functions $G^{(i)}(z,X,Y)$ on $\mathcal{M}^{(i),rec}$ satisfy, for $X,Y\in\mathcal{M}^{(1),rec}$, the relation

$$G^{(2)}(z;\Phi^{rec}(X),\Phi^{rec}(Y))=G^{(1)}(z;X,Y),\quad\text{for}\quad z\in\mathbb{C}\setminus\mathbb{R}. \tag{5.9}$$

First we consider Green's functions in the set

$$N=\Gamma_0\times(2,y_0]\subset\mathcal{M}_1.$$

Lemma 5.3. *When N is considered both as a subset $\mathcal{M}^{(1)}_{int}$ and $\mathcal{M}^{(2)}_{int}$ and $I:N\to N$ is the identity map, then the triple (N,N,I) is admissible.*

Proof. By the assumpton (A-3) and $r^{(1)}_1=r^{(2)}_1$, the map $I:(N,g^{(1)})\to(N,g^{(2)})$ is an isometry. By Corollary 4.8, we know that

$$G^{(1)}(z;X,Y)=G^{(2)}(z;X,Y),\quad z\in\mathbb{C}\setminus\mathbb{R}_+,\ X,Y\in\Gamma_0.$$

Let $z\in\mathbb{C}\setminus\mathbb{R}$. Since Green's function $G^{(i)}(z;X,Y)$ satisfies the elliptic equation,

$$\begin{aligned}(-\Delta^{(i)}-\frac{1}{4}-z)G^{(i)}(z;\cdot,Y)&=\delta_Y,\quad\text{on }\mathcal{M}^{(i)}_{int},\\ \partial_\nu G^{(i)}(z;\cdot,Y)|_{\Gamma_0}&=0,\end{aligned} \tag{5.10}$$

and $g^{(1)}(X)=g^{(2)}(X)$, for $X\in N$, we can use the principle of unique continuation with respect to X to show that $G^{(1)}(z;X,Y)=G^{(2)}(z;X,Y)$ if $X\in N$, $Y\in\Gamma_0$. Moreover, as $G^{(i)}(z;X,Y)=\overline{G^{(i)}(\overline{z};Y,X)}$, Green's function satisfies an elliptic equation analogous to (5.10) also in the Y variable. Thus, using the principle of unique continuation with respect to Y, we see that $G^{(1)}(z;X,Y)=G^{(2)}(z;X,Y)$ for $X,Y\in N$ and $z\in\mathbb{C}\setminus\mathbb{R}$. □

5.3. **Continuation by Green's functions.** To reconstruct subsets of manifolds $\mathcal{M}_{int}^{(i)}$, $i=1,2$, by continuing Green's function, we need the the following result telling that the values of Green's functions identify the points of the manifold.

Lemma 5.4. *Let $X_1, X_2 \in \mathcal{M}_{int}^{(i)}$ be such that*

$$G^{(i)}(z, X_1, Y) = G^{(i)}(z, X_2, Y) \tag{5.11}$$

for all $Y \in \Gamma_0$ and some $z \in \mathbb{C} \setminus \mathbb{R}$. Then $X_1 = X_2$.

Proof. Using the unique continuation principle for the solutions of elliptic equations as above after (5.10), we see that (5.11) implies that $G^{(i)}(z, X_1, Y) = G^{(i)}(z, X_2, Y)$, for all $Y \in \mathcal{M}_{int}^{(i)} \setminus \{X_1, X_2\}$. As the map $Y \mapsto G^{(i)}(z, X, Y)$ is bounded in the compact subsets of $\mathcal{M}_{int}^{(i)} \setminus \{X\}$ and tends to infinity as Y approaches X, this proves that $X_1 = X_2$. $\square$

Remark 5.5. Lemma 5.4 has the following important consequence: If the triples $(N_1^{(1)}, N_1^{(2)}, \Phi_1)$ and $(N_2^{(1)}, N_2^{(2)}, \Phi_2)$ are admissible and $N_1^{(1)} \cap N_2^{(1)} \neq \emptyset$, then, by Lemma 5.4, the maps $\Phi_1(x)$ and $\Phi_2(x)$ have to coincide in $N_1^{(1)} \cap N_2^{(1)}$. Moreover, if $N_3^{(i)} = N_1^{(i)} \cup N_2^{(i)}$, $i = 1, 2$, and

$$\Phi_3(x) = \begin{cases} \Phi_1(x), & \text{for } x \in N_1^{(1)}, \\ \Phi_2(x), & \text{for } x \in N_2^{(1)}, \end{cases} \tag{5.12}$$

then, by Lemma 5.4, the map $\Phi_3 : N_3^{(1)} \to N_3^{(2)}$ is bijective and hence a diffeomorphims. This implies that the triple $(N_3^{(1)}, N_3^{(2)}, \Phi_3)$ is admissible

The procedure of constructing the isometry between $\mathcal{M}_{int}^{(1)}$ and $\mathcal{M}_{int}^{(2)}$ consists of extending the admissible triple $(\mathcal{M}^{(1),rec}, \mathcal{M}^{(2),rec}, \Phi^{rec})$. In the first step, we apply Lemma 5.3 to the triple (N, N, I). In the subsequent steps we always assume that $N \subset \mathcal{M}^{(i),rec}$.

Let $q_i \in \mathcal{M}^{(i),rec}$, $i = 1, 2$,

$$\Phi^{rec}(q_1) = q_2, \quad d^{(i)}(q_i, \Gamma_0) > (y_0 - 2)/2, \tag{5.13}$$

where $d^{(i)}$ denotes the distance on $\mathcal{M}^{(i)}$. Let $R = R(q) > 0$ be sufficiently small so that $R < (y_0 - 2)/4$ and the Riemannian normal coordinates, centered at q_i, are well defined in $B^{(i)}(q_i, 2R)$, i.e. the ball of the radius $2R$ with respect to the distance $d^{(i)}$. Assume also that R is so small that $\mathcal{O}^{(i)} = B^{(i)}(q_i, R)$ satisfy

$$\overline{\mathcal{O}}^{(i)} \subset \mathcal{M}^{(i),rec} \setminus \Gamma_0. \tag{5.14}$$

Then $\Phi^{rec}(\mathcal{O}^{(1)}) = \mathcal{O}^{(2)}$, $\mathcal{M}^{(i),rec} \setminus \overline{\mathcal{O}}^{(i)}$ are connected and $\mathcal{O}^{(i)}$ has smooth boundary.

Denote $\Omega_{\mathcal{O}}^{(i)} = \mathcal{M}_{int}^{(i)} \setminus \mathcal{O}^{(i)}$. We put $H_{\mathcal{O}}^{(i)} = -\Delta^{(i)} - \frac{1}{4}$ in $\Omega_{\mathcal{O}}^{(i)}$ endowed with the Neumann boundary condition:

$$\partial_\nu v = 0 \quad \text{on} \quad \partial\Omega_{\mathcal{O}}^{(i)}, \tag{5.15}$$

ν being the unit normal to the boundary.

Let $z \in \mathbb{C} \setminus \mathbb{R}$ and consider the Schwartz kernel $G_{\mathcal{O}}^{(i)}(z; X, Y)$ of the operator $(H_{\mathcal{O}}^{(i)} - z)^{-1}$. It satisfies the equation

$$\left(-\Delta^{(i)} - \frac{1}{4} - z\right) G_{\mathcal{O}^{(i)}}(z; \cdot, Y) = \delta_Y, \quad Y \in \Omega_{\mathcal{O}}^{(i)},$$
$$\partial_\nu G_{\mathcal{O}}^{(i)}(z; \cdot, Y)|_{\Gamma_0 \cup \partial\mathcal{O}^{(i)}} = 0. \tag{5.16}$$

Let $\mathcal{O}^{(i)} \subset \mathcal{M}^{(i),rec}$, $i = 1, 2$ be relatively compact subsets with smooth boundaries (which later will be chosen to be the balls described earlier). Let $\Phi : \partial\mathcal{O}^{(1)} \to \partial\mathcal{O}^{(2)}$ be a diffeomorphism. Let $(\delta_{\mathcal{O}^{(i)}}^* \mathcal{F}^{(i)}(k)^* \mathcal{F}^{(i)}(k) \delta_{\mathcal{O}^{(i)}})_{k \in \mathbb{R}_+}$ and $(\lambda_n^{(i)})_{n=1}^{m_i}$ and $(P_n^{(i)})_{n=1}^{m_i}$ be the BSP related to operator $H_{\mathcal{O}}^{(i)}$ on $\partial\mathcal{O}^{(i)}$, $i = 1, 2$. We say that the BSP related to operators $H_{\mathcal{O}^{(1)}}$ on $\partial\mathcal{O}^{(1)}$ and $H_{\mathcal{O}^{(2)}}$ on $\partial\mathcal{O}^{(2)}$ are Φ-*related* if $m_1 = m_2$ and, for all $h \in C^\infty(\partial\mathcal{O}^{(2)})$, $k > 0$, and $j = 1, 2, \ldots, m_1$, we have

$$\delta_{\mathcal{O}^{(1)}}^* \mathcal{F}^{(1)}(k)^* \mathcal{F}^{(1)}(k)((\Phi^* h)\delta_{\mathcal{O}^{(1)}})\Phi^* = \Phi^* \left(\delta_{\mathcal{O}^{(2)}}^* \mathcal{F}^{(2)}(k)^* \mathcal{F}^{(2)}(k)(h\delta_{\mathcal{O}^{(2)}})\right),$$

$$\lambda_n^{(1)} = \lambda_n^{(2)}, \quad \delta_{\mathcal{O}^{(1)}}^* P_n^{(1)}((\Phi^* h)\delta_{\mathcal{O}^{(1)}})\Phi^* = \Phi^* \left(\delta_{\mathcal{O}^{(2)}}^* P_n^{(2)}(h\delta_{\mathcal{O}^{(2)}})\right).$$

Note that Φ^* induces a bounded operator : $H^s(\partial\mathcal{O}^{(2)}) \to H^s(\partial\mathcal{O}^{(1)})$, which is denoted by Φ^* again.

Lemma 5.6. *Let $(\mathcal{M}^{(1),rec}, \mathcal{M}^{(2),rec}, \Phi^{rec})$ be an admissible triple and $\mathcal{O}^{(i)}$, $i = 1, 2$ be relatively compact subsets of $\mathcal{M}^{(i),rec}$ such that $\mathcal{O}^{(2)} = \Phi^{rec}(\mathcal{O}^{(1)})$ and $\mathcal{M}^{(i),rec} \setminus \overline{\mathcal{O}}^{(i)}$ are connected. Let $G_{\mathcal{O}}^{(i)}(z; X, Y)$, $z \in \mathbb{C} \setminus \mathbb{R}$ be the Schwartz kernels of $(H_{\mathcal{O}}^{(i)} - z)^{-1}$. Then*

$$G_{\mathcal{O}}^{(1)}(z; X, Y) = G_{\mathcal{O}}^{(2)}(z; \Phi^{rec}(X), \Phi^{rec}(Y)), \quad X, Y \in \mathcal{M}^{(1),rec}. \tag{5.17}$$

Moreover, the BSPs related to operators $H_{\mathcal{O}}^{(1)}$ on $\partial\mathcal{O}^{(1)}$ and $H_{\mathcal{O}}^{(2)}$ on $\partial\mathcal{O}^{(2)}$ are Φ^{rec}-related.

Proof. Skipping for a while the superscript $^{(i)}$, we start the proof by assuming that we are given $G(z; X, Y)$ for $X, Y \in \mathcal{M}^{rec} \subset \mathcal{M}_{int}$ and $z \in \mathbb{C} \setminus \mathbb{R}$ and showing that if $\mathcal{O}$ is a relatively compact subset with a smooth boundary of the open set $\mathcal{M}^{rec}$ such that $\mathcal{M}^{rec} \setminus \overline{\mathcal{O}}$ is connected, then we can determine $G_{\mathcal{O}}(z; X, Y)$ for $X, Y \in \mathcal{M}^{rec} \setminus \overline{\mathcal{O}}$ and $z \in \mathbb{C} \setminus \mathbb{R}$.

To show this, let us denote by $G_{\mathcal{O}}^{ext}(z; X, Y)$ some smooth extension of $X \mapsto G_{\mathcal{O}}(z; X, Y)$ into $\mathcal{O}$, where $Y \in \mathcal{M}^{rec} \setminus \overline{\mathcal{O}}$. Then

$$\left(-\Delta - \frac{1}{4} - z\right) G_{\mathcal{O}}^{ext}(z; \cdot, Y) - \delta(\cdot, Y) = F(\cdot, Y) \in C^\infty(\mathcal{M}^{rec}),$$

where $\operatorname{supp} F(\cdot, Y) \subset \overline{\mathcal{O}}$ is fixed. Therefore,

$$G_{\mathcal{O}}(z; X, Y) = G(z; X, Y) + \int_{\mathcal{O}} G(z; X, Y') F(Y', Y) dS_{Y'}.$$

In particular, due to boundary condition (5.16), if $X \in \partial\mathcal{O}$,

$$\partial_{\nu(X)} G(z; X, Y) + \int_{\mathcal{O}} \partial_{\nu(X)} G(z; X, Y') F(Y', Y) dS_{Y'} = 0, \tag{5.18}$$

where $\nu(X)$ is the unit normal to $\mathcal{O}$ at X. On the other hand, if $F(\cdot\,, Y) \in C^\infty(\mathcal{M}^{rec})$, $\operatorname{supp} F(\cdot\,, Y) \subset \overline{\mathcal{O}}$, satisfies the equation (5.18), then the function

$$G(z; X, Y) + \int_{\mathcal{O}} G(z; X, Y') F(Y', Y) dS_{Y'}, \quad X, Y \in \mathcal{M}^{rec} \setminus \mathcal{O}, \tag{5.19}$$

is equal to $G_{\mathcal{O}}(z; X, Y)$. As we have in our disposal $G(z; X, Y)$ for $X, Y \in \mathcal{M}^{rec}$, we can verify, for any given function F, if it satisfies the equation (5.18) or not. As the equation (5.18) has, for every $Y \in \partial\mathcal{O}$, at least one solution, this implies that we can find some solution F for the equation (5.18) and thus determine the values of $G_{\mathcal{O}}(z; X, Y)$ for $X, Y \in \partial\mathcal{O}$ and $z \in \mathbb{C} \setminus \mathbb{R}$.

Let $G_{\mathcal{O}}(z; X, Y)$ be the Schwartz kernel of the operator $\delta^*_{\partial\mathcal{O}}(H_{\mathcal{O}} - z)^{-1}\delta_{\partial\mathcal{O}}$. We have

$$\delta^*_{\partial\mathcal{O}}(H_{\mathcal{O}} - z)^{-1}\delta_{\partial\mathcal{O}} = \int_0^\infty (\lambda - z)^{-1} \delta^*_{\partial\mathcal{O}} \mathcal{F}_{\mathcal{O}}(\lambda)^* \mathcal{F}_{\mathcal{O}}(\lambda) \delta_{\partial\mathcal{O}} d\lambda + \sum_{n=1}^m (\lambda_n - z)^{-1} \delta^*_{\partial\mathcal{O}} P_n \delta_{\partial\mathcal{O}}.$$

Using this we see that, for $\lambda > 0$,

$$\delta^*_{\partial\mathcal{O}} \mathcal{F}_{\mathcal{O}}(\lambda)^* \mathcal{F}_{\mathcal{O}}(\lambda) \delta_{\partial\mathcal{O}} = \frac{1}{2\pi i} \lim_{\varepsilon \to 0+} \left(\delta^*_{\partial\mathcal{O}}(H_{\mathcal{O}} - \lambda - i\varepsilon)^{-1}\delta_{\partial\mathcal{O}} - \delta^*_{\partial\mathcal{O}}(H_{\mathcal{O}} - \lambda + i\varepsilon)^{-1}\delta_{\partial\mathcal{O}} \right) \tag{5.20}$$

and that λ_n are the poles of the meromorphic function $\delta^*_{\partial\mathcal{O}}(H_{\mathcal{O}} - z)^{-1}\delta_{\partial\mathcal{O}}$ in $\mathbb{C}$. Its residues satisfy

$$\operatorname{res}_{z=\lambda_n} \delta^*_{\partial\mathcal{O}}(H_{\mathcal{O}} - z)^{-1}\delta_{\partial\mathcal{O}} = -\delta^*_{\partial\mathcal{O}} P_n \delta_{\partial\mathcal{O}}. \tag{5.21}$$

Summarizing the above, we have shown the set $\mathcal{M}^{rec}$ with its metric and values of Green's function $G_{\mathcal{O}}(z; X, Y)$ for $X, Y \in \partial\mathcal{O}$ and $z \in \mathbb{C} \setminus \mathbb{R}$ determine the BSP on $\partial\mathcal{O}$.

As $(\mathcal{M}^{(1),rec}, \mathcal{M}^{(2),rec}, \Phi^{rec})$ is admissible, we see that $F^{(2)}$ solves equation (5.18) on $\mathcal{M}^{(2),rec}$ if and only if $F^{(1)} = (\Phi^{rec})^* F^{(2)}$ solves equation (5.18) on $\mathcal{M}^{(1),rec}$. Substituting these solutions in (5.19) and using (5.9) we see that $G_{\mathcal{O}^{(i)}}(z; X, Y)$, $i = 1, 2$ satisfy (5.17). Moreover, as the poles of $z \mapsto G^{(i)}_{\mathcal{O}}(z; X, Y)$ in $\mathbb{C}$, that is the eigenvalues of $H_{\mathcal{O}^{(i)}}$, coincide for $i = 1, 2$, we see, using equations (5.20) and (5.21), that BSP related to operators $H^{(1)}_{\mathcal{O}}$ on $\partial\mathcal{O}^{(1)}$ and $H^{(2)}_{\mathcal{O}}$ on $\partial\mathcal{O}^{(2)}$ are Φ^{rec}-related. $\square$

5.4. **BSP for subdomains of $\mathcal{M}_{int}$ and recognition of singular points.** When $\Gamma \subset \partial\mathcal{O}^{(i)}$ and $s>0$, we denote the domain of influence by

$$\Omega^{(i)}_{\mathcal{O}}(\Gamma,s)=\{X\in\Omega^{(i)}_{\mathcal{O}};\ \tilde d^{(i)}(X,\Gamma)<s\}.$$

where $\tilde d^{(i)}$ now is the distance in $\Omega^{(i)}_{\mathcal{O}}$.

Theorem 5.7. *Let $(\mathcal{M}^{(1),rec},\mathcal{M}^{(2),rec},\Phi^{rec})$ be an admissible triple and $\mathcal{O}^{(i)}=B^{(i)}(q_i,R)\subset\mathcal{M}^{(i),rec}$, $i=1,2$, be a ball centered at q_i and radius R satisfying (5.13) and (5.14). Denote*

$$s^{(i)}(q_i)=\min(d^{(i)}(q_i,\mathcal{M}^{(i)}_{sing}),(y_0-2)/4). \tag{5.22}$$

Then $s^{(1)}(q_1)=s^{(2)}(q_2)$. Using the notation $s=s^{(1)}(q_1)$, then, for

$$\widetilde{\mathcal{M}}^{(i),rec}=\mathcal{M}^{(i),rec}\cup\Omega^{(i)}_{\mathcal{O}}(\partial\mathcal{O}^{(i)},s-R),\quad i=1,2,$$

there is a map $\widetilde\Phi^{rec}:\widetilde{\mathcal{M}}^{(1),rec}\to\widetilde{\mathcal{M}}^{(2),rec}$ which is an extension of Φ^{rec}. Moreover, the triple $(\widetilde{\mathcal{M}}^{(1),rec},\widetilde{\mathcal{M}}^{(2),rec},\widetilde\Phi^{rec})$ is admissible.

Note that $B^{(i)}(q_i,s)=\Omega^{(i)}_{\mathcal{O}}(\partial\mathcal{O}^{(i)},s-R)\cup B^{(i)}(q_i,R)$ and that, $d^{(i)}(X,\partial\mathcal{O}^{(i)})=\tilde d^{(i)}(X,\partial\mathcal{O}^{(i)})$ for $X\in\mathcal{M}^{(i),rec}\setminus\mathcal{O}^{(i)}$.

Proof. Assume opposite to the claim that we would have $s^{(1)}(q_1)>s^{(2)}(q_2)$. Let

$$a=s^{(1)}(q_1)-R,\quad b=s^{(2)}(q_2)-R,\quad 0<c<b<a. \tag{5.23}$$

Then, by (5.2), the BSP of the operator $H^{(i)}_{\mathcal{O}}$ on $\partial\mathcal{O}^{(i)}$ determines, on $\partial\mathcal{O}^{(i)}$, the hyperbolic N-to-D map $R^{(i),T}:=R^T_{\partial\mathcal{O}^{(i)}}$ of the Riemannian surface $\Omega^{(i)}_{\mathcal{O}}$, $i=1,2$. This and Lemma 5.6 yield that these maps satisfy

$$(R^{(1),T}(h\circ\Phi^{rec}))(X)=(R^{(2),T}(h))(\Phi^{rec}(X)),\quad X\in\partial\mathcal{O}^{(1)}, \tag{5.24}$$

for all $h\in C^\infty_0(\partial\mathcal{O}^{(2)}\times\mathbb{R}_+)$.

Let us deform the surfaces $\Omega^{(i)}_{\mathcal{O}}$ replacing the metric with a smooth metric in the $(b-c)/2-$neighborhood of the conic points and replacing the ends of the manifolds with compact surfaces. We can do this by smoothly pinching the first end-cylinder, $S^{r_1}\times(3/4y_0+1/2,y_0)\subset\mathcal{M}_1$, to a semisphere $S^2_+(r_1)$ and the parts of the other ends, $\mathcal{M}^{(i)}_j$, $j>1$, which lie outside $\Omega^{(i)}_{\mathcal{O}}(\partial\mathcal{O}^{(i)},a)$, also to appropriate semispheres. These give rise to two smooth compact Riemannian surfaces $\mathcal{N}^{(i)}, i=1,2$, with $\Gamma^{(i)}:=\partial\mathcal{N}^{(i)}=\partial\mathcal{O}^{(i)}$. Then the c-neighborhoods of $\partial\mathcal{N}^{(i)}$ in $\mathcal{N}^{(i)}$, denoted by $\mathcal{N}^{(i)}(\Gamma^{(i)},c)$, are isometric to $\Omega^{(i)}_{\mathcal{O}}(\partial\mathcal{O}^{(i)},c)$. By the finite velocity of the wave propagation, which is equal to one with respect to the underlying metric, the above isometry implies that the N-to-D map $R^{(i),T}_{\partial\mathcal{N}^{(i)}}$ on $\partial\mathcal{N}^{(i)}$ corresponding to manifold $\mathcal{N}^{(i)}$ coincide with the N-to-D map on $\partial\mathcal{O}^{(i)}$ corresponding to manifold $\Omega^{(i)}_{\mathcal{O}}$ for $T<2c$. Together with (5.24), this implies that the inverse of the N-to-D maps $R^{(i),T}_{\partial\mathcal{N}^{(i)}}$, called the hyperbolic D-to-N maps, satisfy the equation similar to (5.24). By [32, Lemma 4.24 and p. 200], the D-to-N maps with time $T < 2c$, determine uniquely the manifolds $\mathcal{N}^{(i)}(\partial\mathcal{N}^{(i)}, c)$, implying that there exists an isometry,

$$\widetilde{\Phi}_c : \mathcal{N}^{(1)}(\partial\mathcal{N}^{(1)}, c) \to \mathcal{N}^{(2)}(\partial\mathcal{N}^{(2)}, c).$$

Note that, if we identify $\partial\mathcal{O}^{(1)}$ with $\partial\mathcal{O}^{(2)}$, then the representation of this map in the boundary normal coordinates, see e.g. [32], is the identity map. As $\mathcal{N}^{(i)}(\partial\mathcal{N}^{(i)}, c)$ is isometric to $\Omega^{(i)}_{\mathcal{O}}(\partial\mathcal{O}^{(i)}, c)$ and above $c < b$ is arbitrary, this implies that there is an isometry

$$\widetilde{\Phi} : \Omega^{(1)}_{\mathcal{O}}(\partial\mathcal{O}^{(1)}, b) \to \Omega^{(2)}_{\mathcal{O}}(\partial\mathcal{O}^{(2)}, b), \tag{5.25}$$

By the conditions of Theorem, if $b' < b$ is so small that $\Omega^{(i)}_{\mathcal{O}}(\partial\mathcal{O}^{(i)}), b') \subset \mathcal{M}^{(i),rec}$, then

$$\widetilde{\Phi}(X) = \Phi^{rec}(X), \quad X \in \Omega^{(1)}_{\mathcal{O}}(\partial\mathcal{O}^{(1)}), b'). \tag{5.26}$$

As Green's functions $G^{(i)}(z, X, Y)$, $i = 1, 2$, satisfy relation (5.9) for $X, Y \in \Omega^{(i)}_{\mathcal{O}}(\partial\mathcal{O}^{(i)}, b')$, we see, using the unique continuation in X and Y variables as in the proof of Lemma 5.3, that

$$\begin{aligned} &G^{(2)}(z; \widetilde{\Phi}(X), \widetilde{\Phi}(Y)) = G^{(1)}(z; X, Y), \\ &for\ z \in \mathbb{C} \setminus \mathbb{R}, \quad X, Y \in \Omega^{(1)}_{\mathcal{O}}(\partial\mathcal{O}^{(1)}, b). \end{aligned} \tag{5.27}$$

Thus $(\Omega^{(1)}_{\mathcal{O}}(\partial\mathcal{O}^{(1)}, b), \Omega^{(2)}_{\mathcal{O}}(\partial\mathcal{O}^{(2)}, b), \widetilde{\Phi})$ is admissible. Using (5.25), (5.27), it follows from Remark 5.5 that Φ^{rec} can be extended by $\widetilde{\Phi}$ as $\widetilde{\Phi}^{rec}$,

$$\begin{aligned} &\widetilde{\Phi}^{rec} : \widetilde{\mathcal{M}}^{(1),rec} \to \widetilde{\mathcal{M}}^{(2),rec}; \\ &\widetilde{\mathcal{M}}^{(i),rec} = \mathcal{M}^{(i),rec} \cup \Omega^{(i)}_{\mathcal{O}}(\partial\mathcal{O}^{(i)}, b) \end{aligned} \tag{5.28}$$

Recall that, by our assumption, $a > b$. Due to (5.13), (5.22), this implies that $\mathcal{M}^{(2)}_{sing} \cap \partial\Omega^{(2)}_{\mathcal{O}}(\partial\mathcal{O}^{(2)}, b) \neq \emptyset$. Next we show that this is not possible.

For $Y \in \partial\mathcal{O}^{(i)}$ we define the boundary-cut-locus distance

$$\tau_{\mathcal{O}^{(i)}}(Y) = \inf\{t > 0;\ \gamma^{(i)}_{Y,\nu}(t) \in \mathcal{M}^{(i)}_{sing} \text{ or } \tilde{d}^{(i)}(\gamma^{(i)}_{Y,\nu}(t), \partial\mathcal{O}^{(i)}) < t\},$$

where $\nu \in T_Y\mathcal{M}^{(i)}$ is the exterior unit normal vector to $\partial\mathcal{O}^{(i)}$ and $\gamma^{(i)}_{Y,\nu}(t)$ is the geodesic on $\mathcal{M}^{(i)}_{int}$.

As the mapping (5.25) is an isometry between $\Omega^{(1)}_{\mathcal{O}}(\partial\mathcal{O}^{(1)}, b)$ and $\Omega^{(2)}_{\mathcal{O}}(\partial\mathcal{O}^{(2)}, b)$, we see that, for all $Y \in \partial\mathcal{O}^{(1)}$,

$$\min(\tau_{\mathcal{O}^{(1)}}(Y), b) = \min(\tau_{\mathcal{O}^{(2)}}(\Phi^{rec}(Y)), b).$$

Next, any point $p^{(i)} \in \Omega^{(i)}_{\mathcal{O}}(\partial\mathcal{O}^{(i)}, b)$ can be written in the form $\gamma^{(i)}_{Y,\nu}(t)$ where $Y \in \partial\mathcal{O}^{(i)}$ and $t \leq \min(\tau_{\mathcal{O}^{(i)}}(Y), b)$. Moreover, if

$$p^{(i)} \in \partial\Omega^{(i)}_{\mathcal{O}}(\partial\mathcal{O}^{(i)}, b) \setminus \partial\mathcal{O}^{(i)}, \tag{5.29}$$

then $d^{(i)}(p^{(i)},\partial\mathcal{O}^{(i)})=b$ and there is

$$Y^{(i)}\in\partial\mathcal{O}^{(i)} \quad \text{such that} \quad b\le\tau_{\mathcal{O}^{(i)}}(Y^{(i)}), \quad p^{(i)}=\gamma^{(i)}_{Y^{(i)},\nu}(b). \tag{5.30}$$

Let

$$p^{(2)}\in\mathcal{M}^{(2)}_{sing}\cap\partial\Omega^{(2)}_{\mathcal{O}}(\partial\mathcal{O}^{(2)},b). \tag{5.31}$$

By definition (5.22), (5.23), $p^{(2)}$ satisfies (5.29), (5.30). By the above, there is a point $Y^{(2)}\in\partial\mathcal{O}^{(2)}$ such that $p^{(2)}=\gamma^{(2)}_{Y^{(2)},\nu}(b)$. Let $Y^{(1)}=(\Phi^{rec})^{-1}(Y^{(2)})$ and consider $p^{(1)}=\gamma^{(1)}_{Y^{(1)},\nu}(b)$. Since $\Omega^{(1)}_{\mathcal{O}}(\partial\mathcal{O}^{(1)},b)$ and $\Omega^{(2)}_{\mathcal{O}}(\partial\mathcal{O}^{(2)},b)$ are isometric, $p^{(1)}$ satisfies (5.29), (5.30). Moreover, since $a<b$, $p^{(1)}\notin\mathcal{M}^{(1)}_{sing}$.

Let

$$p^{(i)}_\varepsilon:=\gamma_{Y^{(i)},\nu}(b-2\varepsilon), \quad \varepsilon>0,\ i=1,2.$$

For $\varepsilon<b/8$, denote by $\widetilde{O}^{(i)}_\varepsilon=B^{(i)}(p^{(i)}_\varepsilon,\varepsilon)$ the metric ball in $\Omega^{(i)}_{\mathcal{O}}$ of radius ε. By using (5.28) and choosing $\varepsilon>0$ to be small, $\widetilde{\mathcal{O}}^{(i)}_\varepsilon$ satisfy the conditions of Lemma 5.6 with $\widetilde{\mathcal{M}}^{(i)}$ instead of $\mathcal{M}^{(i)}$, $\widetilde{\Phi}^{rec}$ instead of Φ^{rec} and $\widetilde{\mathcal{O}}^{(i)}_\varepsilon$ instead of $\mathcal{O}^{(i)}$.

Then, Lemma 5.6 implies that

$$\text{BSP} \quad \text{for } H^{(i)}_\varepsilon \text{ on } \widetilde{\mathcal{O}}^{(i)}_\varepsilon,\ i=1,2, \text{ are } \widetilde{\Phi}^{reg}\text{-related.} \tag{5.32}$$

Here $H^{(i)}_\varepsilon=H^{(i)}_{\widetilde{\mathcal{O}}_\varepsilon}$, i.e. is the Laplace operator associated with $\widetilde{\Omega}^{(i)}_\varepsilon=\mathcal{M}^{(i)}_{int}\setminus\widetilde{\mathcal{O}}^{(i)}_\varepsilon$. Equation (5.32) together with Lemma 5.2 imply that

$$S^{(1)}(\widetilde{\Omega}^{(1)}_\varepsilon(\partial\widetilde{\mathcal{O}}^{(1)}_\varepsilon,r-\varepsilon))=S^{(2)}(\widetilde{\Omega}^{(2)}_\varepsilon(\partial\widetilde{\mathcal{O}}^{(2)}_\varepsilon,r-\varepsilon))$$

when $r>0$.

Since $\widetilde{\Phi}^{rec}$ is an isometry, we also have

$$S^{(1)}(\widetilde{\mathcal{O}}^{(1)}_\varepsilon)=S^{(2)}(\widetilde{\mathcal{O}}^{(2)}_\varepsilon).$$

On the other hand, when $\varepsilon>0$ is small enough,

$$S^{(i)}(B^{(i)}(p^{(i)}_\varepsilon,r))=S^{(i)}(\widetilde{\mathcal{O}}^{(i)}_\varepsilon)+S^{(i)}(\widetilde{\Omega}^{(i)}_\varepsilon(\partial\widetilde{\mathcal{O}}^{(i)}_\varepsilon,r-\varepsilon)).$$

Therefore, the above two equations imply that

$$S^{(1)}(B^{(1)}(p^{(1)}_\varepsilon,r))=S^{(2)}(B^{(2)}(p^{(2)}_\varepsilon,r)). \tag{5.33}$$

Next, we observe that as $d^{(i)}(p^{(i)}_\varepsilon,p^{(i)})\le2\varepsilon$, we have

$$B^{(i)}(p^{(i)},r-2\varepsilon)\subset B^{(i)}(p^{(i)}_\varepsilon,r)\subset B^{(i)}(p^{(i)},r+2\varepsilon), \quad \text{for } r>2\varepsilon.$$

Thus, by the continuity of the area,

$$S^{(i)}(B^{(i)}(p^{(i)},r))=\lim_{\varepsilon\to0}S^{(i)}(B^{(i)}(p^{(i)}_\varepsilon,r)).$$

Together with (5.33), this implies that, for $r>0$,

$$S^{(1)}(B^{(1)}(p^{(1)},r))=S^{(2)}(B^{(2)}(p^{(2)},r)). \tag{5.34}$$

Let us now consider the polar coordinates of $\mathcal{M}^{(i)}$ near $p^{(i)}$ where we note that, due to $d^{(i)}(\mathcal{O}^{(i)}, \Gamma_0) > (y_0 - 2)/2$, we have $p^{(i)} \notin \Gamma_0$. In these coordinates,

$$(ds)^2 = (dr)^2 + C^{(i)} r^2 \left(1 + h^{(i)}(r, \theta)\right) (d\theta)^2,$$

cf. (1.2). It then follows from (5.34), that

$$C^{(i)} = C^{(i)}(p^{(i)}) = \lim_{r \to 0} \frac{1}{\pi r^2} \left[S^{(i)}(B^{(i)}(p^{(i)}, r))\right]$$

satisfy

$$C^{(1)}(p^{(1)}) = C^{(2)}(p^{(2)}). \tag{5.35}$$

Note that, if $p^{(i)} \in \mathcal{M}^{(i)}_{sing}$ we have $C^{(i)} \neq 1$ and if $p^{(i)} \in \mathcal{M}^{(i)}_{reg}$ then $C^{(i)} = 1$. As we assume that $a = s^{(1)}(q_1) - R > b = s^{(2)}(q_2) - R$, we have $p^{(1)} \in \mathcal{M}^{(1)}_{reg}$ and thus $C^{(1)} = 1$. Hence, we also have $C^{(2)} = 1$, and thus $p^{(2)} \in \mathcal{M}^{(2)}_{reg}$, contradicting (5.31). This implies that $a \le b$ which is in contradiction with our assumption that we would have $s^{(1)}(q_1) > s^{(2)}(q_2)$. This shows that we must have

$$s^{(1)}(q_1) = s^{(2)}(q_2).$$

This equation together with (5.28) prove the theorem. □

Let $\mathcal{A}$ be the collection of admissible triples $(\mathcal{W}^{(1)}, \mathcal{W}^{(2)}, \Phi)$ such that $N \subset \mathcal{W}^{(1)}$, $i = 1, 2$. We define a partial order on $\mathcal{A}$ by setting $(\mathcal{W}^{(1)}, \mathcal{W}^{(2)}, \Phi) \le (\widetilde{\mathcal{W}}^{(1)}, \widetilde{\mathcal{W}}^{(2)}, \widetilde{\Phi})$ if $\mathcal{W}^{(1)} \subset \widetilde{\mathcal{W}}^{(1)}$ and $\Phi = \widetilde{\Phi}|_{\mathcal{W}^{(1)}}$.

Note that, by Remark 5.5, if $(\mathcal{W}^{(1)}, \mathcal{W}^{(2)}, \Phi)$ and $(\widetilde{\mathcal{W}}^{(1)}, \widetilde{\mathcal{W}}^{(2)}, \widetilde{\Phi})$ are admissible triples, then $(\mathcal{W}^{(1),en}, \mathcal{W}^{(2),en}, \Phi^{en})$, where

$$\begin{aligned} &\mathcal{W}^{(i),en} = \mathcal{W}^{(i)} \cup \widetilde{\mathcal{W}}^{(i)}, \\ &\Phi^{en}|_{\mathcal{W}^{(1)}} = \Phi, \quad \Phi^{en}|_{\widetilde{\mathcal{W}}^{(1)}} = \widetilde{\Phi}, \end{aligned}$$

is also an admissible triple. Therefore, by Zorn's lemma, there exists a maximal element $(\mathcal{W}^{(1)}_m, \mathcal{W}^{(2)}_m, \Phi_m) \in \mathcal{A}$.

Lemma 5.8. *The maximal element $(\mathcal{W}^{(1)}_m, \mathcal{W}^{(2)}_m, \Phi_m)$ of $\mathcal{A}$ satisfies*

$$\mathcal{W}^{(1)}_m = \mathcal{M}^{(1)}_{reg}. \tag{5.36}$$

Proof. If the claim is not true, there exists $X^{(1)}_0 \in M^{(1)}_{reg} \cap \partial \mathcal{W}^{(1)}_m$. Let $\mu([0,1])$ be a smooth path from $\mu(0) = Z = (x, y)$, $x \in \Gamma_0$, $y = 2/3 + y_0/3$ to $\mu(1) = X^{(1)}_0$, such that

$$\mu([0,1)) \subset \mathcal{M}^{(1)}_{reg}, \quad \mu \cap \left(\Gamma_0 \times (\frac{2 + y_0}{2}, y_0)\right) = \emptyset.$$

Then $d_0 = d^{(1)}(\mu, \mathcal{M}^{(1)}_{sing}) > 0$. Let $c = \min(\frac{y_0-2}{4}, \frac{d_0}{2})$. We can cover $\mu([0,1])$ by a finite number of balls $B^{(1)}_j = B^{(1)}(X^{(1)}_j, c/2) \subset \mathcal{M}^{(1)}_{reg}$ so that

$$\overline{B}^{(1)}_j \subset \mathcal{W}^{(1)}_m,\ B^{(1)}_j \cap \Gamma_0 = \emptyset,\ X^{(1)}_{j+1} \in B^{(1)}_j, \tag{5.37}$$

where we order them so that $X^{(1)}_0 \in B^{(1)}_1$. Let $\mathcal{O}^{(1)}_1 = B^{(1)}(X^{(1)}_1, R)$ be a small ball such that $0 < R < c/2$ satisfies (5.13), (5.14), and $\mathcal{O}^{(1)}_1 \subset \mathcal{W}^{(1)}_m$. As $d^{(1)}(X^{(1)}_1, \mathcal{M}^{(1)}_{sing}) > \frac{d_0}{2}$, Theorem 5.7 yields that we can extend the admissible triple $(\mathcal{W}^{(1)}_m, \mathcal{W}^{(2)}_m, \Phi_m)$ onto

$$\widetilde{\mathcal{W}}^{(i)} = \mathcal{W}^{(i)}_m \cup B^{(i)}(X^{(i)}_1, c), \quad X^{(2)}_1 = \Phi_m(X^{(1)}_1).$$

As $X^{(1)}_0 \in B(X^{(1)}_1, c)$, this contradicts the fact that $(\mathcal{W}^{(1)}_m, \mathcal{W}^{(2)}_m, \Phi_m)$ is a maximal element of $\mathcal{A}$, which completes the proof of (5.36). $\square$

Lemma 5.8 proves that there is a diffeomorphism

$$\Phi_m : \mathcal{M}^{(1)}_{reg} \to \mathcal{W}^{(2)}_m, \quad \mathcal{W}^{(2)}_m = \Phi_m(\mathcal{M}^{(1)}_{reg}) \subset \mathcal{M}^{(2)}_{reg},$$

which is a Riemannian isometry. Changing the role of indexes 1 and 2, we see that there is also a diffeomorphism

$$\widetilde{\Phi}_m : \mathcal{M}^{(2)}_{reg} \to \widetilde{\mathcal{W}}^{(1)}_m, \quad \widetilde{\mathcal{W}}^{(1)}_m \subset \mathcal{M}^{(1)}_{reg}$$

which is a Riemannian isometry. Moreover, using Lemma 5.3 we see that $\widetilde{\Phi}_m$ and Φ_m coincide with the identity map on Γ_0.

Using (5.9) we see that for all $z \in \mathbb{C} \setminus \mathbb{R}$, $X \in \mathcal{M}^{(2)}_{reg}$ and $Y \in \Gamma_0$.

$$G^{(1)}(z; \Phi_m(\widetilde{\Phi}_m(X)), Y) = G^{(2)}(z; X, Y).$$

By Lemma 5.4, this implies that $\Phi_m(\widetilde{\Phi}_m(X)) = X$, that is, $\Phi_m \circ \widetilde{\Phi}_m = I$ on $\mathcal{M}^{(1)}_{reg}$. Similarly, we see that $\widetilde{\Phi}_m \circ \Phi_m = I$ on $\mathcal{M}^{(2)}_{reg}$ and hence

$$\mathcal{W}^{(2)}_m = \mathcal{M}^{(2)}_{reg},\ \mathcal{W}^{(1)}_m = \mathcal{M}^{(1)}_{reg}, \quad \text{and } \widetilde{\Phi}_m = \Phi_m^{-1}.$$

Summarizing, we have shown that there is a diffeomorphism

$$\Phi_m : (\mathcal{M}^{(1)}_{reg}, g^{(1)}) \to (\mathcal{M}^{(2)}_{reg}, g^{(2)}),$$

which is a Riemannian isometry.

Skipping again the superscript $^{(i)}$, we show next that

$$d(X,Y) = d_{reg}(X,Y), \quad \text{for any } X, Y \in \mathcal{M}_{reg}, \tag{5.38}$$

where d_{reg} is the distance on $(\mathcal{M}_{reg}, g)$ defined as the infimum of the length of rectifiable paths connecting X to Y. As $\mathcal{M}_{reg} \subset \mathcal{M}$, we have $d(X,Y) \leq d_{reg}(X,Y)$. On the other hand, let $X, Y \in \mathcal{M}_{reg}$ and consider a rectifiable path $\mu : [0,\ell] \to \mathcal{M}$ from X to Y, parametrized by the arc-length. As we consider infimum of the length of paths, we can assume that μ is one-to-one. As $\mathcal{M}_{sing}$ is discrete, μ can intersect it only finite many times. If $p = \mu(t_0) \in \mathcal{M}_{sing}$, let us consider the coordinates $X : U \to [0, \varepsilon_p) \times [0, 2\pi]$ near p defined in (A-2). Let $\varepsilon > 0$ be small enough and $t_-, t_+ \in (0, \ell)$,

$t_- < t_0 < t_+$ be such that $X(\mu(t_\pm)) = (\varepsilon, \theta_\pm)$. If we then modify the path μ by replacing $\mu([t_-, t_+])$ by a segment on the circle, that is, the path $X^{-1}(\varepsilon, J)$ where $J \subset [0, \pi]$ in the interval connecting θ_- to θ_+, the length of μ is increased by $O(\varepsilon)$. By choosing ε small enough and modifying the path μ in the above way in all points where μ intersects $\mathcal{M}_{sing}$, we see that near μ there is a path in $\mathcal{M}_{reg}$ which length is arbitrarily close to the length of μ. This shows that $d(X,Y) \geq d_{reg}(X,Y)$ proving (5.38).

The identity (5.38) implies that $(\mathcal{M}_{int}, d)$, considered as a metric space, is isometric to the completion of the metric space $(\mathcal{M}_{reg},\, d_{reg})$. Thus, we can uniquely extend Φ_m to a metric isometry

$$\Phi : (\mathcal{M}^{(1)}_{int}, d^{(1)}) \to (\mathcal{M}^{(2)}_{int}, d^{(2)}), \tag{5.39}$$

Again, taking into account that the number of singular points is finite, we see that Φ maps singular points to singular points. Let us numerate the singular points on $\mathcal{M}^{(1)}$ and $\mathcal{M}^{(2)}$ as $p^{(1)}_l \in \mathcal{M}^{(1)}$, $l = 1, 2, \dots, L$ and $p^{(2)}_l \in \mathcal{M}^{(2)}$, $l = 1, 2, \dots, L$ so that $p^{(2)}_l = \Phi_m(p^{(1)}_l)$.

The map Φ defined above satisfies conditions (1)–(3) of Theorem 1.1. We prove (4) we use the following Lemma:

Lemma 5.9. *Let*

$$\Phi : \mathcal{M}^{(1)} \to \mathcal{M}^{(2)} \tag{5.40}$$

satisfy the conditions (1)–(3) of Theorem 1.1. Then Φ satisfies the condition (4).

Proof. Let $p^{(1)}_l \in \mathcal{M}^{(1)}_{sing}$, $p^{(2)}_l = \Phi(p^{(1)}_l) \in \mathcal{M}^{(2)}_{sing}$ and $\varepsilon_0 > 0$ be so small that polar coordinates (A-2) centered at $p^{(i)}_l$ are well defined in the ball $B^{(i)}(p^{(i)}_l, \varepsilon_0)$ for $i = 1, 2$. We denote these coordinates by $\psi^{(i)} : B^{(i)}(p^{(i)}_l, \varepsilon_0) \to [0, \varepsilon_0) \times [0, 2\pi)$. Below, we skip the subindex l.

First, using a point $q_1 \in \mathcal{M}^{(1)}_{reg}$ such that $p^{(1)}$ is the unique closest singular point of $\mathcal{M}^{(1)}$ to q_1, the proof of Theorem 5.7, see (5.35), shows that

$$C_1 = C^{(1)}(p^{(1)}) = C^{(2)}(p^{(2)}) = C_2 := C. \tag{5.41}$$

Let us consider a distance minimizing curve in $B^{(i)}(p^{(i)}, \varepsilon_0)$ emanating from the point $p^{(i)}$. We call such curve a radial geodesics and denote it by $\gamma^{(i)}(s)$ where s is the arclength from $p^{(i)}$.

By (1.2) the radial geodesic $\gamma^{(i)} = \gamma^{(i)}([0, \varepsilon_0))$ is given in normal coordinates by $\psi^{(i)}(\gamma^{(i)}) = \{(\theta, r);\ \theta = \alpha_0,\ 0 \leq r < \varepsilon_0\}$ where $\alpha_0 \in [0, 2\pi)$ is a parameter associated to $\gamma^{(i)}$, and we denote below $\alpha^{(i)}(\gamma^{(i)}) = \alpha_0$.

Since Φ is an isometry, it maps any radial geodesic $\gamma^{(1)}$ emanating from $p^{(1)}$ to some radial geodesic $\gamma^{(2)}$ emanating from $p^{(2)}$. When $\gamma^{(i)}_0$ is the geodesic satisfying $\alpha^{(i)}(\gamma^{(i)}_0) = 0$, the parameter $\alpha^{(i)}(\gamma^{(i)})$ associated to $\gamma^{(i)}(s)$

satisfies

$$\alpha^{(i)}(\gamma^{(i)}) = \lim_{\varepsilon \to 0} \frac{\ell_\varepsilon^{(i)}(\gamma^{(i)}(\varepsilon),\, \gamma_0^{(i)}(\varepsilon))}{C_i \varepsilon},$$

where $\ell_\varepsilon^{(i)}(\gamma^{(i)}(\varepsilon),\, \gamma_0^{(i)}(\varepsilon))$ is the arc length of the counter-clockwise oriented path connecting $\gamma_0^{(i)}(\varepsilon)$ and $\gamma^{(i)}(\varepsilon)$ along the circle $S_\varepsilon^{(i)} = \{X \in \mathcal{M}^{(i)};\, d^{(i)}(X, p^{(i)}) = \varepsilon\}$. Let $\beta = a^{(1)}(\gamma^{(1)}) - \alpha(\gamma^{(2)}) \in (-2\pi, 2\pi)$. As Φ is an isometry,

$$h_l^{(1)}(r, \theta) = h_l^{(2)}(r, \widehat{\theta + \beta}), \tag{5.42}$$

where, for $\theta \in \mathbb{R}$, $\widehat{\theta} \in [0, 2\pi)$ satisfies $\theta - \widehat{\theta} \in 2\pi\mathbb{Z}$.

This completes the proof of Lemma 5.9 and Theorem 1.1. □

6. Orbifold isomorphism for $\Gamma\backslash\mathbf{H}^2$

We shall prove Theorem 1.3. Let $\mathcal{M}^{(i)} = \Gamma_i\backslash\mathbf{H}^2$ and $\mathcal{M}_{sing}^{(i)}$ the set of elliptic singular points in $\mathcal{M}^{(i)}$. We have already constructed a hyperbolic isometry $\Phi : \mathcal{M}^{(1)} \setminus \mathcal{M}_{sing}^{(1)} \to \mathcal{M}^{(2)} \setminus \mathcal{M}_{sing}^{(2)}$ in §5. Since the hyperbolic metric is conformal to the Euclidean metric, Φ is conformal. As $\mathcal{M}^{(i)}$ is orientable, we can assume $\Phi : \mathcal{M}^{(1)} \setminus \mathcal{M}_{sing}^{(1)} \to \mathcal{M}^{(2)} \setminus \mathcal{M}_{sing}^{(2)}$ to be analytic. Take $p^{(1)} \in \mathcal{M}_{sing}^{(1)}$ and a small disc $B^{(1)}(p, \epsilon)$ centered at $p^{(1)}$. Since Φ maps $p^{(1)}$ to $p^{(2)} := \Phi(p^{(1)}) \in \mathcal{M}_{sing}^{(2)}$, $p^{(1)}$ is a removable singularity for Φ. Hence Φ is analytic also at $p^{(1)}$. Let $(\widetilde{B}^{(i)}(P^{(i)}, \epsilon), \widetilde{g}^{(i)})$ be the uniformizing covers of $(B^{(i)}(p^{(i)}, \epsilon), g^{(i)})$, $i = 1, 2$. Then Φ can be lifted to an analytic map between the coverings except for the center

$$\widetilde{\Phi} : \widetilde{B}^{(1)}(P^{(1)}, \epsilon) \setminus \{P^{(1)}\} \to \widetilde{B}^{(2)}(P^{(2)}, \epsilon \setminus \{P^{(2)}\}.$$

This implies that $P^{(1)}$ is a removable singularity of $\widetilde{\Phi}$, hence $\widetilde{\Phi}$ is analytic on $\widetilde{B}^{(1)}(P^{(1)}, \epsilon)$.

It follows from (5.42) that $p^{(1)}$ is a singular point of the orbifold if and only if $p^{(2)}$ is a singular point. Moreover, the map

$$r \mapsto r, \quad \frac{\theta}{n} \mapsto \frac{\widehat{\theta + \beta}}{n}, \; n = C^{-1/2},$$

extends to the isometry between $\widetilde{B}^{(i)}(P^{(i)}, \epsilon)$, $i = 1, 2$. This completes the proof of Theorem 1.3. □

By the suitable linear transformation $\gamma \in SL(2, \mathbf{R})$, $\mathcal{M}_{\nu_1}^{(1)}$ is mapped to $\mathcal{M}_{\nu_2}^{(2)}$ conformally. Identifying them, we see that Φ, constructed above, is the identity on $\mathcal{M}_{\nu_1}^{(1)}$, hence is equal to the identity on all of $\mathcal{M}^{(1)}$. This implies that $\gamma\Gamma_1\gamma^{-1}\backslash\mathbf{H}^2 = \Gamma_2\backslash\mathbf{H}^2$, hence $\gamma\Gamma_1\gamma^{-1} = \Gamma_2$. Therefore, the generalized S-matrix determines the conjugate class of geometrically finite Fuchsian groups.

Remark 6.1. The technique used in this paper can be easily extended to consider the case when $\partial\mathcal{M} \neq \emptyset$. In this case we should require, in addition to **(A-1)**, **(A-2)**, that each end is diffeomorphic to either a cylinder or a strip $(0, \ell_i) \times (1, \infty)$ with the metric satisfying **(A-3)**, **(A-4)** where, in the case of a strip, $0 \leq x \leq \ell_i$.

Acknowledgments. The research of Y.K. is partially supported by EPSRC Grant EP/F034016/1 and Mathematical Sciences Research Institute (MSRI). The research of M.L. was financially supported by the Academy of Finland Center of Excellence programme 213476 and MSRI. The research of H.I. was supported by Gakusin Grant in Aid No. 21340028.

References

[1] S. Agmon and L. Hörmander, *Asymptotic properties of solutions of differential equations with simple characteristics*, J. d'Anal. Math. **30** (1976), 1-30.

[2] M. Anderson, A. Katsuda, Y. Kurylev, M. Lassas, M. Taylor, *Boundary regularity for the Ricci equation, geometric convergence, and Gelfand's inverse boundary problem.* Invent. Math. **158** (2004), 261–321.

[3] M. Belishev, *An approach to multidimensional inverse problems for the wave equation*, Dokl. Akad. Nauk SSSR **297** (1987), 524-527 (Engl. transl. Soviet Math. Dokl. **36** (1988), 481-484.

[4] M. Belishev and V. Kurylev, *To the reconstruction of a Riemannian manifold via its spectral data (BC-method)*, Comm. in P. D. E. **17** (1992), 767-804.

[5] K. Bingham, Y. Kurylev, M. Lassas and S. Siltanen, *Iterative time reversal control for inverse problems.* Inverse Problems and Imaging 2 (2008), 63-81.

[6] A. Blagoveščenskii, *A one-dimensional inverse boundary value problem for a second order hyperbolic equation.* (Russian) Zap. Nauchn. Sem. LOMI, **15** (1969), 85–90.

[7] A. Blagoveščenskii, *Inverse boundary problem for the wave propagation in an anisotropic medium.* (Russian) Trudy Mat. Inst. Steklova, **65** (1971), 39–56.

[8] D. Borthwick, *Spectral Theory of Infinite-Area Hyperbolic Surfaces*, Progress in Mathematics **256**, Birkhäuser, Boston-Basel-Berlin, (2007).

[9] D. Borthwick, C. Judge, P. Perry, *Sel'berg's zeta function and the spectral geometry of geometrically finite hyperbolic surfaces.* Comment. Math. Helv. **80** (2005), 483-515.

[10] D. Borthwick, P. Perry, *Inverse scattering results for manifolds hyperbolic near infinity.* J. Geom. Anal. **21** (2011), 305-333.

[11] D. Dos Santos Ferreira, C. Kenig, M. Salo, G. Uhlmann, *Limiting Carleman weights and anisotropic inverse problems.* Invent. Math. **178** (2009), 119-171.

[12] L. Faddeev, Expansion in eigenfunctions of the Laplace operator in the fundamental domain of a discrete group on the Lobacevskii plane, Trudy Moscov. Mat. 17 (1967), 323-350; English transl. in Trans. Moscow Math. Soc. 17 (1967), 357-386.

[13] A. Greenleaf, Y. Kurylev, M. Lassas, and G. Uhlmann, *Invisibility and inverse problems.* Bull. Amer. Math. **46** (2009), 55–97.

[14] C. Guillarmou, R. Mazzeo, *Spectral analysis of the Laplacian on geometrically finite hyperbolic manifolds*, Inventiones Math, to appear.

[15] C. Guillarmou, A. Sa Barreto, *Scattering and inverse scattering on ACH manifolds.* J. Reine Angew. Math. **622** (2008), 1–55.

[16] C. Guillarmou, L. Tzou, *Identification of a connection from Cauchy data space on a Riemann surface with boundary*, Geom. Funct. Anal., to appear.

[17] C. Guillarmou, L. Tzou, *Calderon inverse problem with partial data on Riemann surfaces.* Duke Math. J., to appear

[18] C. Guillarmou, *Meromorphic properties of the resolvent on asymptotically hyperbolic manifolds*, Duke Math. Journal **129** (2005), 1-37.

[19] L. Guillopé and M. Zworski, *Scattering asymptotics for Riemann surfaces*, Ann. of Math. **145** (1997), 597-660.

[20] L. Guillopé and M. Zworski, *Polynomial bounds of the number of resonances for some complex spaces of constant negative curvature near infinity*, Asymp.Anal. **11** (1995), 1-22.

[21] S. Helgason, *A duality for symmetric spaces with applications to group representations*, Adv. in Math. **5** (1970), 1-154.

[22] G. Henkin, V. Michel, *On the explicit reconstruction of a Riemann surface from its Dirichlet-Neumann operator*. Geom. Funct. Anal. **17** (2007), 116-155.

[23] H. Isozaki, *Inverse spectral problems on hyperbolic manifolds and their applications to inverse boundary value problems in Euclidean space*, American J. of Math. **126** (2004), 1261-1313.

[24] H. Isozaki and Y. Kurylev, *Introduction to spectral theory and inverse problems on asymptotically hyperbolic manifolds*, arXiv:1102.5382.

[25] H. Isozaki, Y. Kurylev and M. Lassas, *Forward and inverse scattering on manifolds with asymptotically cylindrical ends*, J. Funct. Anal. **258** (2010), 2060-2118.

[26] H. Isozaki, Y. Kurylev and M. Lassas, *Conic singularities, generalized scattering matrix, and inverse scattering on asymptotically hyperbolic manifolds*, in preparartion.

[27] M. Joshi, A. Sa Barreto, *Recovering asymptotics of metrics from fixed energy scattering data*. Invent. Math. 137 (1999), 127-143.

[28] M. Joshi, A. Sa Barreto, *Inverse scattering on asymptotically hyperbolic manifolds.* Acta Math. 184 (2000), 41-86.

[29] A. Katchalov, Y. Kurylev, *Multidimensional inverse problem with incomplete boundary spectral data.* Comm. Part. Diff. Equat, **23** (1998), 55-95.

[30] Katchalov, A.; Kurylev, Y.; Lassas, M.; Mandache, N. *Equivalence of time-domain inverse problems and boundary spectral problems.* Inverse Problems **20** (2004), 419–436.

[31] S. Katok, *Fuchsian Groups*, University of Chicago Press, Chicago, IL, (1992).

[32] A. Katchalov, Y. Kurylev and M. Lassas, *Inverse Boundary Spectral Problems*, Chapman and Hall/CRC, Monographs and Surveys in Pure and Applied Mathematics, **123** (2001).

[33] M. Kashiwara, A. Kowata, K. Minemura, K. Okamoto, T. Oshima, M. Tanaka, *Eigenfunctions of invariant differential operators on a symmetric space*, Annal. Math. **107** (1978), 1-39.

[34] K. Krupchyk, Y. Kurylev and M. Lassas, *Inverse spectral problems on a closed manifold*. J. Math. Pures Appl. **90** (2008), 42–59.

[35] Y. Kurylev, *A multidimensional Gelfand-Levitan inverse boundary problem.* in: Diff. Equat. and Mathem. Phys. (Birmingham, AL, 1994), 117-131, Int. Press, Boston, MA, 1995,

[36] Y. Kurylev and M. Lassas, *Hyperbolic inverse boundary-value problem and time-continuation of the non-stationary Dirichlet-to-Neumann map.* Proc. Roy. Soc. Edinburgh Sect. A **132** (2002), 931–949.

[37] Y. Kurylev and M. Lassas, *Gelfand inverse problem for a quadratic operator pencil.* J. Funct. Anal. **176** (2000), 247–263.

[38] Y. Kurylev, M. Lassas and T. Yamaguchi, *Inverse spectral problems on an orbifold. I: Uniqueness*, in preparation.

[39] M. Lassas, L. Oksanen: *Inverse problem for wave equation with sources and observations on disjoint set. Inverse Problems*, **26** (2010), 085012.

[40] M. Lassas, M. Taylor and G. Uhlmann, *The Dirichlet-to-Neumann map for complete Riemannian manifolds with boundary.* Comm. Anal. Geom. **11** (2003), no. 2, 207–221

[41] M. Lassas and G. Uhlmann, *On determining a Riemannian manifold from the Dirichlet-to-Neumann map.* Ann. Sci. Ecole Norm. Sup. **34** (2001), 771–787.

[42] J. Lee, G. Uhlmann, *Determining anisotropic real-analytic conductivities by boundary measurements.* Comm. Pure Appl. Math. **42** (1989), 1097-1112.

[43] J. Lehner, *A Short Course in Automorphic Functions*, Holt, Rinhart and Winston, Inc. (1966)

[44] R. Mazzeo, R. Melrose, *Meromorphic extension of the resolvent on comlete spaces with asymptotically constant negative curvature*, J. Funct. Anal. **75** (1987), 260-310.

[45] K. Minemura, *Eigenfunctions of the Laplacian on a real hyperbolic space,*. J. Math. Soc. Japan **27** (1975), 82-105

[46] P. Perry, *The Laplace operator on a hyperbolic manifold. I. Spectral and scattering theory.* J. Funct. Anal. 75 (1987), no. 1, 161-187.

[47] P. Perry, *The Laplace operator on a hyperbolic manifold. II. Eisenstein series and the scattering matrix.* J. Reine Angew. Math. 398 (1989), 67-91.

[48] I. Satake, *The Gauss-Bonnet theorem for V-manifolds*, J. Math. Soc. Japan, (1957), 464-492.

[49] A. Sá Barreto, *Radiation fields, scattering and inverse scatering on asymototically hyperbolic manifolds*, Duke Math. J. **129** (2005), 407-480.

[50] D. Tataru, *Unique continuation for solutions to PDEs, between Hörmander's theorem and Holmgren's theorem*, Comm. Part. Diff. Equations **20** (1995), 855–884.

[51] D. Tataru, *Unique continuation for operators with partially analytic coefficients*, J. Math. Pures Appl. **78** (1999), 505–521.

[52] W. Thurston, *The Geometry and Topology of Three-Manifolds*, http://www.msri.org/publications/books/gt3m

[53] A. Vasy, *Microlocal analysis of asymptotically hyperbolic spaces and high energy resolvent estimates*, preprint arXiv:1104.1376v2 (2011).

[54] G. N. Watson, *A Treatise on the Theory of Bessel Functions*, Cambridge University Press, (1962).

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