Optimising the directional sensitivity of LISA

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It was shown in a previous work that the data combinations canceling laser frequency noise constitute a module - the module of syzygies. The cancellation of laser frequency noise is crucial for obtaining the requisite sensitivity for LISA. In this work we show how the sensitivity of LISA can be optimised for a monochromatic source - a compact binary - whose direction is known, by using appropriate data combinations in the module. A stationary source in the barycentric frame appears to move in the LISA frame and our strategy consists of *coherently tracking* the source by appropriately *switching* the data combinations so that they remain optimal at all times. Assuming that the polarisation of the source is not known, we average the signal over the polarisations. We find that the best statistic is the 'network' statistic, in which case LISA can be construed of as two independent detectors. We compare our results with the Michelson combination, which has been used for obtaining the standard sensitivity curve for LISA, and with the observable obtained by optimally switching the three Michelson combinations. We find that for sources lying in the ecliptic plane the improvement in SNR increases from 34% at low frequencies to nearly 90% at around 20 mHz. Finally we present the signal-to-noise ratios for some known binaries in our galaxy. We also show that, if at low frequencies SNRs of both polarisations can be measured, the inclination angle of the plane of the orbit of the binary can be estimated.

I. INTRODUCTION

The goal of the LISA mission [1] is to detect and analyze low frequency gravitational signals essentially in the range from 0.1 mHz to 0.1 Hz. In this frequency range, sources of gravitational waves are mainly the wide population of galactic compact binaries, and interactions between black holes with a range of complexity. The study of the emission of GW from known binaries could be extremely useful for firstly, direct determination of distances, and secondly, possible small general relativistic effects, if the signal to noise ratio (SNR) is large enough. For this reason, we focus on optimising the sensitivity of LISA for a given astrophysical source with known direction. The sensitivity of LISA can be improved by solving technological problems, but it can also be improved by employing certain optimal data analysis strategies. In this paper, we show how algebraic methods previously developed in [2], henceforth referred to as paper I, can be used to design optimal strategies for combining data with appropriate time-delays. We show in this paper that it is possible to maintain the optimality during the year by continuously updating the parameters of the combination or simply by switching to optimal data combinations as the source appears to move in the LISA frame, as LISA moves in its complex orbit around the sun. The problem of optimisation of SNR, in this context, has been addressed before in [3] and [4]. In [3] optimisation has been carried out before averaging over the directions and polarisations, while in [4], henceforth referred to as paper II, the averaging is done first and then the optimisation. In this paper, the averaging over the polarisations is performed first and then the SNR is optimised for the average signal for a given direction over the relevant data combinations - those data combinations canceling laser frequency noise. Thus in this optimisation, the direction of source is assumed to be known, but not its polarisation. This would be the case for several binaries in our galaxy. We analyse two observables in this context: (i) optimal data combination in the module which yields the maximum SNR for a given direction and (ii) a 'network' observable which is obtained by squaring and adding the SNRs of two independent(orthogonal) data combinations one of them being the optimal combination mentioned in (i) and another orthogonal to it. The network observable in general yields higher SNR. We analyse how these SNRs depend on direction of the source. For an integration time of one year, we compare our results by plotting sensitivity curves with those obtained from, (a) the Michelson combination X [5], which has been used for obtaining the standard sensitivity curve for LISA, and (b) the observable X_{switch} obtained by switching the three Michelson combinations optimally. We find that for sources lying in the ecliptic plane, the network sensitivity at low frequencies is about 34% more than the optimally switched Michelson combinations which rises to nearly 90% at 20 mHz. Finally we compute the optimal SNRs for six known binary systems in our galaxy, whose SNRs are significantly high, for an integration time of one year. We show that if at low frequencies the two SNRs of the orthogonal data combinations can be measured, then it is possible to estimate the inclination angle of the binary's orbit.

FIG. 1: The schematic LISA configuration
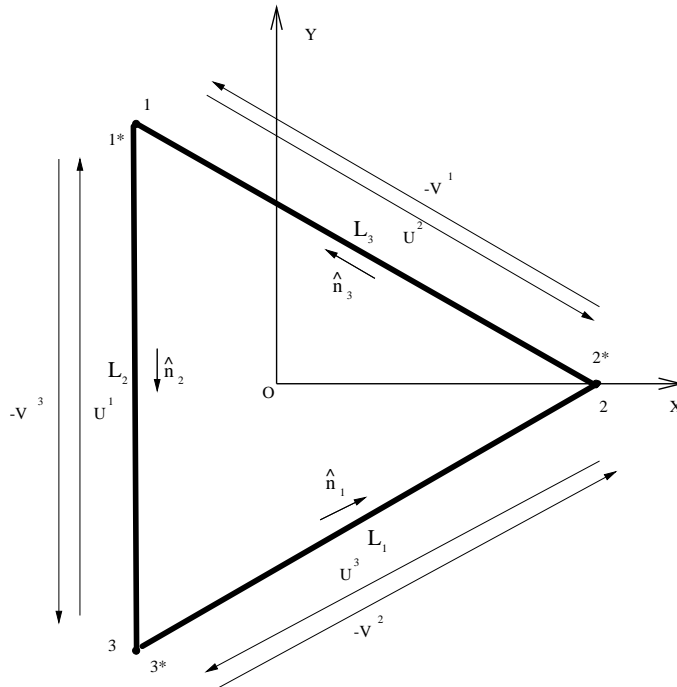
The paper is organised as follows: In section II we briefly describe the formalism involving commutative algebra developed in paper I. We also present a convenient set of generators for the data combinations found in paper II, which consisted of the eigenvectors of the noise-covariance matrix, and in which the computations are simplified. In section III we discuss the gravitational wave (GW) signal from non-spinning compact binaries and average the signal over the polarisations. In section IV we obtain the optimum SNR by the method of Lagrange multipliers, where we must solve an eigenvalue problem. In section V we describe the transformations from the LISA frame to the barycentric frame and vice-versa. In section VI we compute the optimal SNRs and the sensitivity curves and compare them with the standard curve for LISA obtained with the Michelson data combination X and the optimally switched Michelson combination X_{switch}. In this section we also estimate the SNRs for a few known binaries in our galaxy and describe a method to estimate the angle of inclination of the binary's orbit, if the two SNRs of the orthogonal combinations can be measured.

II.   THE MODULE OF SYZYGIES AND ITS GENERATORS

In this section, we briefly review the results from papers I and II which are needed for the analysis that follows. In LISA the six elementary data streams are labeled as U^i and $V^i, i = 1, 2, 3$; if the space-crafts are labeled clockwise, the U^i beams also travel clockwise, while the V^i beams travel counter-clockwise. The beam U^1 travels from space-craft 3 to space-craft 1 along the arm of length L_2 in the direction $-\hat{\mathbf{n}}_2$, while $-V^1$ represents the beam traveling from space-craft 2 to space-craft 1 along the arm of length L_3 in the direction of $\hat{\mathbf{n}}_3$. The remaining 4 beams are described by cyclically permuting the indices. These beams contain the laser frequency noise, other noises such as optical path, acceleration etc. and also the GW signal. For performing the analysis, we choose a frame (x_L, y_L, z_L) tied to the LISA constellation. We choose the centroid of the LISA triangle as the origin, the LISA triangle to lie in the (x_L, y_L) plane and space-craft 2 to lie on the x_L-axis. We will henceforth drop the suffix 'L' from the LISA frame quantities and only introduce it when other frames are being considered in the discussion. A schematic diagram of LISA is shown in Fig.1.

In paper I, we showed that all data combinations canceling laser frequency noise form a module over the ring of polynomials in three indeterminates, these being the three-time delay operators $E_i, i = 1, 2, 3$ of the light travel time along the three arms with lengths L_i respectively. Thus for any data stream $X(t) : E_i X(t) = X(t - L_i)$. For the LISA specifications $L_i \sim 16.7$ sec. corresponding to an armlength of 5 million km. A general data combination is a

linear combination of the elementary data streams U^i, V^i:

$$X(t) = \sum_{i=1}^{3} p_i V^i(t) + q_i U^i(t), \tag{1}$$

where p_i and q_i are polynomials in the time-delay operators $E_i, i = 1, 2, 3$. Thus any data combination can be expressed as a six-tuple polynomial 'vector' (p_i, q_i). For cancellation of laser frequency noise only the polynomial vectors satisfying this constraint are allowed and they form the module of syzygies mentioned above. An important advantage of this formalism is that, given the generators of the module, any other data combination canceling the laser frequency noise is simply a linear combination of the generators with the coefficients being polynomials in the ring.

Several sets of generators have been listed in paper I. Based on the physical application as also on convenience, we may choose one set over another. For the monochromatic sources, as argued in paper II, only three generators are sufficient to generate the relevant data combinations. In general 4 generators are necessary to generate the module but if the source is monochromatic, except at certain frequencies which are solutions of $e^{i(L_1+L_2+L_3)\Omega} = 1$, the fourth generator can be effectively eliminated for the purposes of maximising the SNR. Since this maximisation is possible arbitrarily close to the singular frequencies the singularities do not seem to be important.

We found in paper II that a convenient set of generators for our purpose is the one that diagonalises the noise covariance matrix. These generators are the eigenvectors of the noise covariance matrix. The noise covariance matrix $\mathcal{N}_{(J)}^{(I)}, I, J = 1, 2, 3$ is defined as the outer product $\mathcal{N}_{(J)}^{(I)} = N^{(I)} N_{(J)}^*$ where $N^{(I)}$ is a 12 dimensional complex noise vector of the generator $X^{(I)} = \left(p_i^{(I)}, q_i^{(I)} \right)$ (The $X^{(I)}$ could be any set of generators not necessarily the eigenvectors of the noise covariance matrix $\mathcal{N}_{(J)}^{(I)}$). The noise vector is given by:

$$N^{(I)} = \left(\sqrt{S^{pf}}(2p_i^{(I)} + r_i^{(I)}), \sqrt{S^{pf}}(2q_i^{(I)} + r_i^{(I)}), \sqrt{S^{opt}}p_i^{(I)}, \sqrt{S^{opt}}q_i^{(I)} \right), \tag{2}$$

where the r_i polynomials are defined through the equations $r_1 = -(p_1+E_3 q_2) = -(q_1+E_2 p_3)$ plus cyclic permutations for r_2 and r_3. The $S^{pf} = 2.5 \times 10^{-48}(f/1\mathrm{Hz})^{-2}\mathrm{Hz}^{-1}$ and $S^{opt} = 1.8 \times 10^{-37}(f/1\mathrm{Hz})^2\mathrm{Hz}^{-1}$ are the one-sided power spectral densities (psd) of the proof-mass noise and the optical-path noise respectively [1, 7].

For the purposes of this paper when computing the SNR for any given data combination, the differences in arm-lengths are extremely small compared to the wavelength of the GW we are interested in detecting. Thus we may take the armlengths to be equal, i.e., $E_1 = E_2 = E_3 = E = e^{i\Omega L}$. For the purposes of this analysis it is sufficient to consider LISA as an equilateral triangle; we ignore the deviations arising from this assumption. This simplifies the expressions for the noise and further also for the signal. The eigenvectors now contain overall common factors which are polynomials in E. These common factors make no difference to the computation of SNR as they cancel out from the numerator and denominator which comprise of the signal and the noise respectively. The unnormalised eigenvectors (with common factors canceled) of the noise covariance matrix is the set $\{Y^{(I)}\}$ which we list below:

$$
\begin{aligned}
Y^{(1)} &= (1 - E, 1 + 2E, -2 - E, 1 + 2E, 1 - E, -2 - E), \\
Y^{(2)} &= (-E - 1, 1, E, -1, 1 + E, -E), \\
Y^{(3)} &= (1, 1, 1, 1, 1, 1).
\end{aligned} \tag{3}
$$

In paper II, the $Y^{(I)}$ have been listed with the uncanceled common factors.

We adopt the following terminology: we refer to a single element of the module as a data *combination*; while a function of the elements of the module, such as taking the maximum over several data combinations in the module or squaring and adding data combinations belonging to the module, is called as an *observable*. The important point to note is that the laser frequency noise is also suppressed for the observable although it may not be an element of the module.

III. THE GW SIGNAL FROM BINARIES

A. The waveform

Since binaries will be important sources for LISA the analysis of such sources is relevant. One such class is of massive or super massive binaries whose individual masses could range from $10^3 M_\odot$ to $10^8 M_\odot$ and which could be

around a Gpc away. Another class of interest are known binaries within our own galaxy whose individual masses are of the order of a solar mass but are just at a distance of a few kpc or less. Here we will assume the direction of the source to be known, which is justified for known binaries in our galaxy; but even for the former case of distant binaries, it amounts to 'looking' in a specific direction.

The spacetime metric perturbation for a gravitational wave propagating in a $\hat{w}$ direction can be written as,

$$
\begin{aligned}
h_{ij}(t, \vec{r}) \; = \; & h_+(t - \hat{w} \cdot \vec{r}) \, (\theta_i \theta_j - \phi_i \phi_j) \\
& + h_\times(t - \hat{w} \cdot \vec{r}) \, (\theta_i \phi_j + \theta_j \phi_i) \,,
\end{aligned}
\tag{4}
$$

where the direction to the source is described by direction angles (θ, ϕ), the unit vector pointing to the source is $\hat{w} = (\sin\theta \cos\phi, \sin\theta \sin\phi, \cos\theta)$ and $\hat{\theta}$ and $\hat{\phi}$ are unit vectors in the direction of the coordinates θ and ϕ, such that $\{\hat{w}, \hat{\theta}, \hat{\phi}\}$ form a right-handed triad.

For a binary that is not chirping, and hence monochromatic with frequency Ω, the two polarisation amplitudes $h_+(t)$ and $h_\times(t)$ [2] are given by,

$$
\begin{aligned}
h_+(t) \; = \; & H \left[\frac{1 + \cos^2\epsilon}{2} \cos 2\psi \, \cos \Omega t + \cos\epsilon \, \sin 2\psi \, \sin \Omega t \right] \,, \\
h_\times(t) \; = \; & H \left[-\frac{1 + \cos^2\epsilon}{2} \sin 2\psi \, \cos \Omega t + \cos\epsilon \, \cos 2\psi \, \sin \Omega t \right] \,,
\end{aligned}
\tag{5}
$$

where the angles (ϵ, ψ) describe the orientation of the binary orbit (ϵ, ψ could be the direction angles of the orbital angular momentum vector) and H the overall amplitude which depends on the masses, the distance and the frequency f as given below:

$$
H = 1.188 \times 10^{-22} \cdot \left[\frac{\mathcal{M}}{1000 \, M_\odot} \right]^{\frac{5}{3}} \cdot \left[\frac{R}{1 \, \text{Gpc}} \right]^{-1} \cdot \left[\frac{f}{1 \, \text{mHz}} \right]^{2/3} \,,
\tag{6}
$$

where R is the distance to the source, $f = \Omega/2\pi$ is the GW frequency of the source and $\mathcal{M} = \left(\mu^3 M^2\right)^{\frac{1}{5}}$ the chirp-mass, where μ and M are respectively the reduced and the total mass of the binary. For the typical parameters taken above the frequency evolves very slowly, so much so that the time for the system to coalesce is more than 20 years. For an observational time of the order of an year, the frequency of the binary changes very little so that the source can be practically taken to be monochromatic.

Since we deal with essentially monochromatic sources, the Fourier domain is appropriate for further analysis. In the Fourier domain we have,

$$
\begin{aligned}
h_+(\Omega) \; = \; & H \left[\frac{1 + \cos^2\epsilon}{2} \cos 2\psi - i \, \cos\epsilon \, \sin 2\psi \right] \,, \\
h_\times(\Omega) \; = \; & H \left[-\frac{1 + \cos^2\epsilon}{2} \sin 2\psi - i \, \cos\epsilon \, \cos 2\psi \right] \,.
\end{aligned}
\tag{7}
$$

B. The signal matrix

The GW response for a generator $Y^{(I)} = (p_j^{(I)}, q_j^{(I)})$ is (see papers I and II),

$$
h^{(I)}(\Omega) = F_+^{(I)}(\Omega) h_+(\Omega) + F_\times^{(I)}(\Omega) h_\times(\Omega) \,,
\tag{8}
$$

where,

$$
\begin{aligned}
F_+^{(I)}(\Omega) \; = \; & \sum_{i=1}^{3} \left(p_i^{(I)} F_{Vi;+} + q_i^{(I)} F_{Ui;+} \right)(\Omega) \,, \\
F_\times^{(I)}(\Omega) \; = \; & \sum_{i=1}^{3} \left(p_i^{(I)} F_{Vi;\times} + q_i^{(I)} F_{Ui;\times} \right)(\Omega) \,,
\end{aligned}
\tag{9}
$$

where we have expressed the response of the two polarisations in terms of the responses of the elementary data streams for each of the polarisations. Below we state the responses for the $+$ and $\times$ polarisations for the beams U^1 and V^1 only; the responses for the remaining four beams U^2, V^2, U^3, V^3 are obtained from cyclic permutations:

$$
\begin{aligned}
F_{U_1;+,\times} &= \frac{e^{i\Omega(\hat{w}\cdot\vec{r}_3+L_2)}}{2\left(1+\hat{w}\cdot\hat{n}_2\right)}\left(1-e^{-i\Omega L_2(1+\hat{w}\cdot\hat{n}_2)}\right)\xi_{2;+,\times}\,, \\
F_{V_1;+,\times} &= -\frac{e^{i\Omega(\hat{w}\cdot\vec{r}_2+L_3)}}{2\left(1-\hat{w}\cdot\hat{n}_3\right)}\left(1-e^{-i\Omega L_3(1-\hat{w}\cdot\hat{n}_3)}\right)\xi_{3;+,\times}\,,
\end{aligned}
\tag{10}
$$

where,

$$
\xi_{i;+} = (\hat{\theta}\cdot\hat{n}_i)^2 - (\hat{\phi}\cdot\hat{n}_i)^2,\qquad \xi_{i;\times} = 2(\hat{\theta}\cdot\hat{n}_i)(\hat{\phi}\cdot\hat{n}_i)\,.
\tag{11}
$$

We define the signal covariance matrix $h_{(J)}^{(I)}$ in analogous fashion as the noise covariance matrix. It is given as follows:

$$
\begin{aligned}
h_{(J)}^{(I)} &= h^{(I)}h_{(J)}^*, \\
&= \left(F_+^{(I)}h_+ + F_\times^{(I)}h_\times\right)\left(F_{+(J)}h_+ + F_{\times(J)}h_\times\right)^*\,.
\end{aligned}
\tag{12}
$$

In general we may not have any knowledge of the polarisation of the GW binary source. We therefore average over the polarizations and assume that the direction of the orbital angular momentum of the binary is uniformly distributed over the sphere. The orientation of the binary (its orbital angular momentum vector) has been described in terms of the angles ϵ and ψ in Eq.(7). Thus we carry out the averaging of $h_{(J)}^{(I)}$ over (ϵ, ψ) which results in an overall factor of $2/5$. The averaged matrix we denote by $\mathcal{H}_{(J)}^{(I)}$. In the Fourier domain it is given by,

$$
\mathcal{H}_{(J)}^{(I)}(\Omega) = H^2\left(\frac{2}{5}\right)\left(F_+^{(I)}F_{+(J)}^* + F_\times^{(I)}F_{\times(J)}^*\right)(\Omega)\,.
\tag{13}
$$

The signal matrix so averaged has the following properties:

- $\mathcal{H}$ is the sum of outer products of two vectors: $F_+^{(I)}$ with its complex conjugate and $F_\times^{(I)}$ with its complex conjugate. Thus the natural basis for expressing $\mathcal{H}$ consists of the two vectors $F_+^{(I)}$ and $F_\times^{(I)}$. In the analysis that follows we will use this fact.

- Because we average over the polarisations, $\mathcal{H}$ is constructed out of two vectors. Its rank is two, everywhere except on the $\theta = \frac{\pi}{2}$ plane where it is one when $F_\times^{(I)}$ goes identically to zero. In paper II we had obtained a signal matrix of rank three because there we had averaged over the directions as well. While in [3], since optimisation is performed first before averaging, the signal matrix is constituted from a single vector and thus has rank one.

IV. OPTIMIZING SNR

For a generic data combination $\alpha_{(I)}Y^{(I)}$ where $\alpha_{(I)}$ are polynomials in E, the SNR is given by:

$$
SNR^2 = \frac{\alpha_{(I)}\alpha^{(J)*}H_{(J)}^{(I)}}{\alpha_{(I)}\alpha^{(J)*}N_{(J)}^{(I)}}\,.
\tag{14}
$$

If the psd is given in units of Hz^{-1} then the above equation yields the square of the SNR integrated over one second. Because one second is short compared to the various time-scales envisaged in the problem we call this SNR the *instantaneous* SNR. Since we are dealing with monochromatic sources, $E = e^{i\Omega L}$, the coefficients $\alpha_{(I)}$ reduce to just complex numbers. Also in the generating set $\{Y^{(I)}\}$ the noise covariance matrix is diagonal with diagonal elements $n_{(I)}^2$. However, we find $n_{(1)} = n_{(2)}$ in our case, so that the first two eigenvectors correspond to the same eigenvalue. Thus, the denominator of Eq.(14) simplifies to a sum of squares $\left|\alpha_{(1)}\right|^2 n_1^2 + \left|\alpha_{(2)}\right|^2 n_1^2 + \left|\alpha_{(3)}\right|^2 n_3^2$ which we can set

equal to unity because the SNR does not depend on the normalisation of the data combination. Moreover, it is convenient to define coefficients β which are scaled by the noise, $\beta_{(1)} = \alpha_{(1)}n_1$, $\beta_{(2)} = \alpha_{(2)}n_2$, $\beta_{(3)} = \alpha_{(3)}n_3$ so that the $\beta_{(I)}$ satisfy,

$$\left|\beta_{(1)}\right|^2 + \left|\beta_{(2)}\right|^2 + \left|\beta_{(3)}\right|^2 = 1. \tag{15}$$

The expression for SNR simplifies to,

$$SNR^2 = \alpha^T \cdot \mathcal{H} \cdot \alpha = \beta^T \cdot \rho \cdot \beta, \tag{16}$$

where we define a SNR matrix ρ by,

$$\rho_{(J)}^{(I)} = \frac{\mathcal{H}_{(J)}^{(I)}}{n_{(I)}n_{(J)}}. \tag{17}$$

In the above equations the α, β are construed of as 3×1 column matrices and $\mathcal{H}$ and ρ as 3×3 square matrices.

The extremisation of SNR is now carried out with the method of Lagrange multipliers because of the normalisation constraint on β. This procedure yields an eigenvalue equation with the Lagrange multiplier appearing as an eigenvalue:

$$\rho \cdot \beta = \lambda \beta. \tag{18}$$

Since $\mathcal{H}$ has atmost rank 2, one eigenvalue is necessarily zero. We now proceed to compute the other two eigenvalues. The analysis is simplified if we go to the basis consisting of the two vectors $\vec{f}_+$ and $\vec{f}_\times$:

$$f_+^{(I)} = h_0 \frac{F_+^{(I)}}{n_{(I)}}, \quad f_\times^{(I)} = h_0 \frac{F_\times^{(I)}}{n_{(I)}}, \tag{19}$$

where $h_0 = (\sqrt{2/5})H$ is the amplitude of the GW averaged over the polarisation states at frequency Ω. We can then write the matrix ρ as a tensor product in terms of these two vectors:

$$\rho = \vec{f}_+ \otimes \vec{f}_+^{\,*} + \vec{f}_\times \otimes \vec{f}_\times^{\,*}. \tag{20}$$

In general the vectors $\vec{f}_+$ and $\vec{f}_\times$ are not orthogonal.

The action of the matrix ρ on any vector $\vec{v}$ is given by,

$$\rho.\vec{v} = (\vec{f}_+ \otimes \vec{f}_+^* + \vec{f}_\times \otimes \vec{f}_\times^*) \cdot \vec{v} = (\vec{f}_+^* \cdot \vec{v})\vec{f}_+ + (\vec{f}_\times^* \cdot \vec{v})\vec{f}_\times. \tag{21}$$

For the eigenvalue problem, we have the eigenvalue equation,

$$(\vec{f}_+^* \cdot \vec{v})\vec{f}_+ + (\vec{f}_\times^* \cdot \vec{v})\vec{f}_\times = \lambda \vec{v}. \tag{22}$$

Expressing the eigenvector $\vec{v}$ as a linear combination of $\vec{f}_+$ and $\vec{f}_\times$, $\vec{v} = c_+ \vec{f}_+ + c_\times \vec{f}_\times$, and from the linear independence of $\vec{f}_+$ and $\vec{f}_\times$ (they are linearly independent in general) we obtain the system of equations for c_+ and $c_\times$ as:

$$\begin{cases} (|\vec{f}_+|^2 - \lambda)c_+ + (\vec{f}_+^* \cdot \vec{f}_\times)c_\times = 0 \\ (\vec{f}_+ \cdot \vec{f}_\times^*)c_+ + (|\vec{f}_\times|^2 - \lambda)c_\times = 0. \end{cases} \tag{23}$$

Setting the determinant of this system to zero, we obtain eigenvalue equation:

$$\lambda^2 - (|\vec{f}_+|^2 + |\vec{f}_\times|^2)\lambda + |\vec{f}_+ \times \vec{f}_\times|^2 = 0, \tag{24}$$

which can be solved to yield the eigenvalues:

$$\lambda_\pm = \frac{1}{2}(|\vec{f}_+|^2 + |\vec{f}_\times|^2 \pm \sqrt{\Delta}), \tag{25}$$

where,

$$\begin{aligned} \Delta &= (|\vec{f}_+|^2 + |\vec{f}_\times|^2)^2 - 4|\vec{f}_+ \times \vec{f}_\times|^2 \\ &= (|\vec{f}_+|^2 - |\vec{f}_\times|^2)^2 + 4|\vec{f}_+ \cdot \vec{f}_\times^{\,*}|^2. \end{aligned} \tag{26}$$

In the low frequency limit as we shall find $\vec{f_+} \cdot \vec{f_\times}^* \approx 0$ so that the positive square root of Δ is just $|\vec{f_+}|^2 - |\vec{f_\times}|^2$ and the eigenvalues are just $|\vec{f_+}|^2$ and $|\vec{f_\times}|^2$ (infact the eigenvalue equation factorises into linear factors) with $\vec{f_+}$ and $\vec{f_\times}$ as eigenvectors respectively. This can also be directly inferred from the structure of ρ. So it may be appropriate to call λ_- as $\lambda_\times$ even in the general case. In the general case the eigenvectors can be easily determined from Eq.(23):

$$\begin{aligned}
\vec{v}_+ &= (\lambda_+ - |\vec{f_\times}|^2)\vec{f_+} + (\vec{f_\times}^* \cdot \vec{f_+})\vec{f_\times}, \\
\vec{v}_\times &= (\vec{f_\times} \cdot \vec{f_+}^*)\vec{f_+} + (\lambda_\times - |\vec{f_+}|^2)\vec{f_\times},
\end{aligned} \tag{27}$$

where $\vec{v}_{+,\times}$ are eigenvectors belonging to the eigenvalues $\lambda_{+,\times}$ respectively. Since ρ is Hermitian the eigenvectors are orthogonal, $\vec{v}_+ \cdot \vec{v}_\times^* = 0$ as can be verified also directly from the expressions of the eigenvectors. These eigenvectors are not normalised.

The eigenvalues $\lambda_{+,\times}$ are the squares of the instantaneous SNRs for the two data combinations described by the two corresponding eigenvectors. The data combinations are $\vec{v}_{+(I)}Y^{(I)}$ and $\vec{v}_{\times(I)}Y^{(I)}$, which we will call eigen-combinations or alternatively eigen-observables, and which give the instantaneous SNRs: $SNR^2_{+,\times} \equiv \lambda_{+,\times}$ respectively. For a given direction (θ, ϕ) in the LISA frame, SNR_+ is the maximum instantaneous SNR among all data combinations. While $SNR_\times$ is the minimum instantaneous SNR among data combinations that are linear combinations of $\vec{v}_+$ and $\vec{v}_\times$; those which lie in the 'plane' of $\vec{v}_+$ and $\vec{v}_\times$. However, $SNR_\times$ is not zero in general and therefore not the absolute minimum; the absolute minimum SNR is zero corresponding to the third eigenvalue which is zero. Moreover, the eigenvectors are orthogonal, which means they yield statistically independent observables. Thus these observables can be combined in quadratures to form a network observable with instantaneous $SNR_{network}$ given by:

$$\begin{aligned}
SNR^2_{network} &= SNR^2_+ + SNR^2_\times, \\
&= \lambda_+ + \lambda_\times = |\vec{f_+}|^2 + |\vec{f_\times}|^2.
\end{aligned} \tag{28}$$

The third eigenvector is $\vec{f_+}^* \times \vec{f_\times}^*$; it is orthogonal to $\vec{f_+}$ and $\vec{f_\times}$ with eigenvalue zero [11]. This means that the data combination corresponding to this vector gives zero response in that particular direction, which may be important if one wishes to 'switch off' the GW coming from that direction.

V. TRACKING A GW SOURCE WITH LISA

In general the amplitude of the GW source will be small and it would be necessary to track or follow the source for a considerable period of time in order to accumulate an adequate SNR. This period of time could range from few days to a year or even years - the life of the LISA experiment. The LISA configuration performs a complex motion due to which the source will appear to move in the LISA frame even if it is stationary in the barycentric frame. For the purposes of this analysis, we will take the observation time to be a year and integrate the SNR for this period of time. It is possible that some GW sources may be sufficiently powerful that SNR integration for an year is not required. In that case it is easily possible to limit the integration to the required period of observation and obtain useful results. We will also assume that the source is stationary in the barycentric frame, that is, its direction remains constant during the observation period. In this section we will present the transformations connecting the barycentric frame and the LISA frame and hence obtain the apparent motion in the LISA frame for a GW source fixed in the barycentric frame.

The LISA constellation trails the Earth in its orbit by $20°$ around the sun. The plane of LISA makes an angle of $60°$ with the plane of the ecliptic and the LISA triangle rotates in its own plane completing one rotation in a year. We describe the barycentric frame by the Cartesian coordinates $\{x_B, y_B, z_B\}$. The $x_B - y_B$ plane coincides with the orbital plane of LISA. The z_B-axis is orthogonal to this plane forming a right-handed coordinate system. The transformation from the barycentric frame to LISA frame is given as follows [1]: A vector $\mathbf{r_B} = (x_B, y_B, z_B)^T$ is transformed to $\mathbf{r_L} = (x_L, y_L, z_L)^T$ by the matrix $\mathcal{R}$ as,

$$\mathbf{r_L} = \mathcal{R} \cdot \mathbf{r_B}, \tag{29}$$

where $\mathcal{R}$ is a product of the three matrices containing Euler angles:

$$\mathcal{R} = \mathcal{C} \cdot \mathcal{B} \cdot \mathcal{A}, \tag{30}$$

where the matrices $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$ are given by,

$$\mathcal{A} = \begin{pmatrix} \cos\psi_a & \sin\psi_a & 0 \\ -\sin\psi_a & \cos\psi_a & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

$$\mathcal{B} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & \frac{\sqrt{3}}{2} \\ 0 & -\frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix},$$

$$\mathcal{C} = \begin{pmatrix} \cos\psi_c & \sin\psi_c & 0 \\ -\sin\psi_c & \cos\psi_c & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{31}$$

Here $\psi_a = \omega t + \alpha_0$, $\psi_c = -\omega t + \beta_0$ and $\omega = 2\pi/T_\odot$. $T_\odot$ is the orbital period of LISA which we take to be of one year duration. The α_0 and β_0 are constants fixing initial conditions when the observation begins. The matrix $\mathcal{R}$ is time-dependent and is a product of two time-dependent matrices.

The unit vector $\hat{w}_B$ of the source direction described by the angles (θ_B, ϕ_B) in the barycentric frame is given by $\hat{w}_B = (\sin\theta_B \cos\phi_B, \sin\theta_B \sin\phi_B, \cos\theta_B)$. The corresponding vector $\hat{w}_L$ in the LISA frame is then, $\hat{w}_L = \mathcal{R}(t) \cdot \hat{w}_B$. From this vector we can explicitly work out the angles (θ_L, ϕ_L):

$$\theta_L = \cos^{-1}\left(\frac{1}{2}\cos\theta_B - \frac{\sqrt{3}}{2}\sin\theta_B \sin(\phi_B - \psi_a) \right),$$

$$\phi_L = \tan^{-1}\left(\frac{w_L^2}{w_L^1} \right), \tag{32}$$

where,

$$\begin{aligned} w_L^1 &= \cos\psi_c \sin\theta_B \cos(\phi_B - \psi_a) \\ &\quad + \frac{1}{2}\sin\psi_c \sin\theta_B \sin(\phi_B - \psi_a) \\ &\quad + \frac{\sqrt{3}}{2}\sin\psi_c \cos\theta_B, \\ w_L^2 &= -\sin\psi_c \sin\theta_B \cos(\phi_B - \psi_a) \\ &\quad + \frac{1}{2}\cos\psi_c \sin\theta_B \sin(\phi_B - \psi_a) \\ &\quad + \frac{\sqrt{3}}{2}\cos\psi_c \cos\theta_B. \end{aligned} \tag{33}$$

For a fixed source direction (θ_B, ϕ_B) in the barycentric frame, the apparent source direction depends on time in the LISA frame; θ_L, ϕ_L are functions of time; the source appears to move in the LISA frame. Thus a given data combination even if it is optimal initially, will not continue to remain optimal subsequently. Our strategy is then to *switch* the data combinations continuously so that the SNR remains optimal at all times. In this way we can optimally track the source and accumulate maximum SNR. The following figures show the apparent trajectory of the source for one year in the LISA frame when (i) the source lies at the pole of the barycentric frame - it is just a circle in the LISA frame: $\theta_L = \pi/3$ (See Fig.2) and (ii) the source lies in plane of the LISA orbit - a figure of 8 is described by the source (See Fig.3).

VI.   OPTIMAL SENSITIVITIES

For optimal tracking of the source, we switch the data combinations so that the SNRs yielded are optimal at all times. Three SNRs are of interest; $SNR_{+,\times}$ and $\text{SNR}_{\text{network}}$, that is, the two eigenvalues and their sum. The eigenvalues are functions of θ_L, ϕ_L which are in turn functions of t as the GW source describes its apparent motion in the LISA frame. The three integrated SNRs are given by:

$$\text{SNR}_{+,\times}^2(\hat{w}_B) = \int_0^T \lambda_{+,\times}(\theta_L(t; \hat{w}_B), \phi_L(t; \hat{w}_B))dt,$$

$$\text{SNR}_{\text{network}}^2(\hat{w}_B) = \text{SNR}_+^2(\hat{w}_B) + \text{SNR}_\times^2(\hat{w}_B), \tag{34}$$

FIG. 2: Apparent position of the source in the sky as seen from LISA frame for ($\theta_B = 0$, $\phi_B = 0$). The track of the source for a period of one year is shown on the unit sphere in the LISA frame.
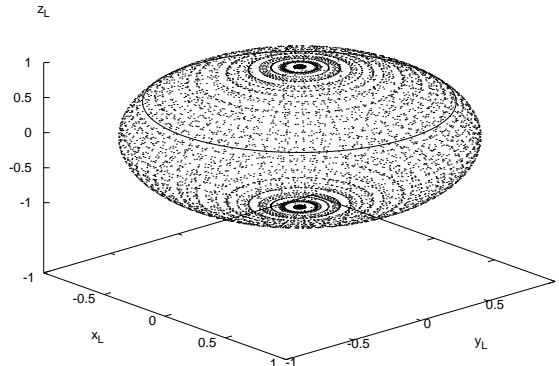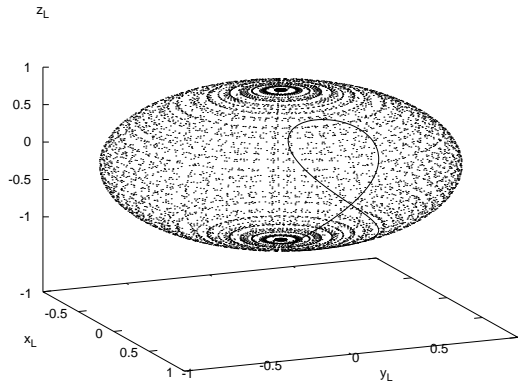

FIG. 3: Apparent position of the source in the sky as seen from LISA frame for ($\theta_B = \frac{\pi}{2}$, $\phi_B = 0$). The track of the source for a period of one year is shown on the unit sphere in the LISA frame.

where we have taken the total observation time to be T and the initial time of observation to be zero. If we integrate for a complete year, we can set the initial observation time to be zero without loss of generality and at the same time set the constants α_0, β_0 to be zero. However, if we make observations for times that are not integral number of years, the constants α_0, β_0 must be chosen appropriately in the transformation matrix $\mathcal{R}$.

An important point is that, since LISA moves in an heliocentric orbit, a Doppler phase depending on the position vector of the LISA projected on to the direction to the source $\hat{w}_B$ and the GW frequency will be added to the phase of the GW signal. This Doppler phase will be a function of time which will be added to the monochromatic part of the phase Ωt of the signal. We assume in this analysis that this phase has been accounted for when integrating the signal. One way is to 'stretch' the time-coordinate so that the signal appears monochromatic (a technique well-known to radio astronomers)[9, 10].

A. The Low Frequency Limit

An important case arises when we consider GW of low frequencies, say below 3 mHz. For this case it is possible to obtain analytical expressions for the optimal SNRs. A large fraction of the sources for LISA fall into this category, for example, massive/supermassive blackhole binaries, several galactic and extragalactic binaries which contribute to the 'confusion noise'.

We need to compute the $\vec{f}_+$ and $\vec{f}_\times$ in order to compute the SNR matrix ρ and then obtain its eigenvalues.

Integrating the eigenvalues for the period of observation will yield the relevant SNRs. To this end, we expand $F_{Ui;+,\times}$ and $F_{Vi;+,\times}$ to the lowest order in the dimensionless frequency ΩL:

$$
\begin{aligned}
F_{U1;+,\times} &= i\frac{\Omega L}{2}\left(1 + i\Omega L\,\tau_2\right)\xi_{2;+,\times}, \\
F_{V1;+,\times} &= -i\frac{\Omega L}{2}\left(1 + i\Omega L\,\tau_3\right)\xi_{3;+\times},
\end{aligned}
\tag{35}
$$

where,

$$
\begin{aligned}
\tau_m &= \frac{1}{2}\left(1 - \frac{\hat{w}\cdot\hat{r}_m}{\sqrt{3}}\right), \\
\xi_{m;+} &= \frac{(1+\cos^2\theta)}{2}\cos\left(2\phi - (2m-1)\frac{\pi}{3}\right) - \frac{1}{2}\sin^2\theta, \\
\xi_{m;\times} &= -\cos\theta\sin\left(2\phi - (2m-1)\frac{\pi}{3}\right),
\end{aligned}
\tag{36}
$$

where, $\hat{r}_m$ is the unit vector in the direction of m-th spacecraft and $m = 1, 2, 3$ and the angles θ, ϕ refer to the LISA frame (as before we have dropped the subscript 'L'). The transfer functions for the four other elementary data streams are obtained by cyclic permutations. In order to get the $\vec{f}_+, \vec{f}_\times$ we must operate on the U^i, V^i with the polynomial operators $p_i^{(I)}, q_i^{(I)}$ given in Eq.(4) and then scale them by the averaged signal divided by the noise. Thus, we obtain after some algebra,

$$
\begin{aligned}
\vec{f}_+ &= \rho_0\frac{(1+\cos^2\theta)}{2}(-\sin\Phi, \cos\Phi, 0), \\
\vec{f}_\times &= -\rho_0\cos\theta(\cos\Phi, \sin\Phi, 0),
\end{aligned}
\tag{37}
$$

where,

$$
\rho_0 = \frac{3}{\sqrt{5}}\frac{H}{n_1}(\Omega L)^2, \qquad \Phi = 2\phi + \frac{\pi}{3}.
\tag{38}
$$

The vectors $\vec{f}_+$ and $\vec{f}_\times$ are expressed in the $Y^{(I)}$ basis and in this basis they have real components. We observe the following properties of the vectors:

- The vectors $\vec{f}_+$ and $\vec{f}_\times$ lie in the $(Y^{(1)}, Y^{(2)})$ plane and the $Y^{(3)}$ component is zero for both vectors. This can be understood if we recall that $Y^{(3)}$ is proportional to symmetric Sagnac combination [4] which is insensitive to GW at low frequency.

- The apparent motion of a GW source in the LISA frame can be optimally and continuously tracked by $\hat{f}_+$ and $\hat{f}_\times$ by rotating the pair $(\hat{f}_+, \hat{f}_\times)$ by Φ at each instant of time as follows:

$$
\begin{pmatrix} \hat{f}_\times \\ \hat{f}_+ \end{pmatrix} = \begin{pmatrix} \cos\Phi & \sin\Phi \\ -\sin\Phi & \cos\Phi \end{pmatrix}\begin{pmatrix} Y^{(1)} \\ Y^{(2)} \end{pmatrix}.
\tag{39}
$$

  Here $\hat{f}_+$ and $\hat{f}_\times$ are unit vectors in the directions of $\vec{f}_+$ and $\vec{f}_\times$ respectively. Thus, optimally tracking a source amounts to orienting the data combinations along $\vec{f}_+$ and $\vec{f}_\times$.

- Moreover, the vectors are orthogonal: $\vec{f}_+ \cdot \vec{f}_\times^* = 0$ in this low frequency limit. The orthogonality implies that these eigen-observables $\vec{f}_+$ and $\vec{f}_\times$ have zero response to $\times$ and $+$ polarizations of GW respectively. We use this fact to estimate the polarisation angles; namely (ϵ, ψ) at the end of this section.

The eigenvalues of the eigen-vectors $\vec{f}_+$ and $\vec{f}_\times$ are just $|\vec{f}_+|^2$ and $|\vec{f}_\times|^2$ and are explicitly given by,

$$
\begin{aligned}
\lambda_+ &= \rho_0^2\left(\frac{1+\cos^2\theta}{2}\right)^2, \\
\lambda_\times &= \rho_0^2\cos^2\theta.
\end{aligned}
\tag{40}
$$

The eigenvalues are the squares of the instantaneous SNRs. We notice that ρ_0 is the maximum instantaneous SNR obtained when the source lies at the poles $\theta = 0$ or π in the LISA frame. If the source is observed over a period of an year, the eigenvalues must be integrated over this length of time. We notice that the eigenvalues do not depend on ϕ. Thus our next task of integrating the SNR becomes somewhat simplified.

FIG. 4: Instantaneous $SNR_{+,\times}$ and $SNR_{network}$ as functions of time for the source direction ($\theta_B = \frac{\pi}{2}, \phi_B = 0$) at the GW frequency of $f = 1$ mHz in units of ρ_0.
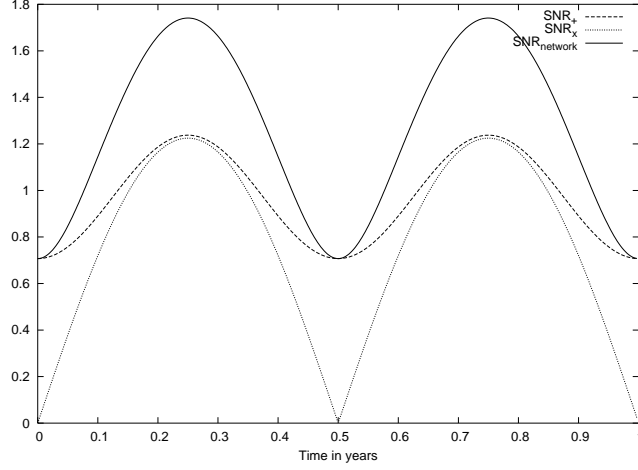
1. The integrated SNR

For a given source direction (θ_B, ϕ_B) the corresponding track $(\theta_L(t), \phi_L(t))$ of the source in LISA frame is given by Eq.(32). We may substitute these values into the integrals for the SNRs and the integrals yield simple analytical expressions for the SNRs if the integration is over an integral number of years. The instantaneous SNRs however change with time. In the Fig.4 we show how the SNRs change with time as LISA orbits the sun during the course of a year. The various SNRs are shown for a source lying in the ecliptic plane, $(\theta_B = \pi/2, \phi_B = 0)$ for the GW frequency of 1 mHz. We have chosen this direction because the SNRs show considerable variations during the course of a year.

Integration over a period of $T = T_\odot$ leads to the following results:

$$\begin{aligned}
SNR_+(\theta_B, \phi_B) &= \text{SNR}_0 \; g_+(\cos \theta_B), \\
SNR_\times(\theta_B, \phi_B) &= \text{SNR}_0 \; g_\times(\cos \theta_B),
\end{aligned} \tag{41}$$

where,

$$\text{SNR}_0 = \rho_0 \sqrt{T_\odot} = \frac{3}{\sqrt{5}} \frac{H}{n_1} (\Omega L)^2 \sqrt{T_\odot}, \tag{42}$$

and

$$\begin{aligned}
g_+^2(x) &= \frac{1}{T_\odot} \int_0^{T_\odot} \left(\frac{1 + \cos^2 \theta_L}{2} \right)^2 dt \\
&= \frac{1}{4} \left(1 + \frac{x^2}{4} \right)^2 \\
&\quad + \frac{3}{16} (1 - x^2) \left(1 + \frac{3}{4} x^2 + \frac{9}{32} (1 - x^2) \right), \\
g_\times^2(x) &= \frac{1}{T_\odot} \int_0^{T_\odot} \cos^2 \theta_L dt \\
&= \frac{1}{4} \left[x^2 + \frac{3}{2} (1 - x^2) \right].
\end{aligned} \tag{43}$$

We have purposely not 'simplified' the formulae in powers of x^2 because in this form it is easy to see the limits $x = \pm 1, 0$ corresponding to $\theta_B = 0, \pi, \pi/2$ respectively. Infact since only x^2 occurs in the expressions of $g_{+,\times}$ there is symmetry about the ecliptic plane. The network SNR is just the root mean square of the two SNRs:

$$\text{SNR}_{network}(\theta_B, \phi_B) = \text{SNR}_0 \; g_{network}(\cos \theta_B), \tag{44}$$

FIG. 5: The functions $g_{+,\times}$ and g_{network} as functions of θ_B
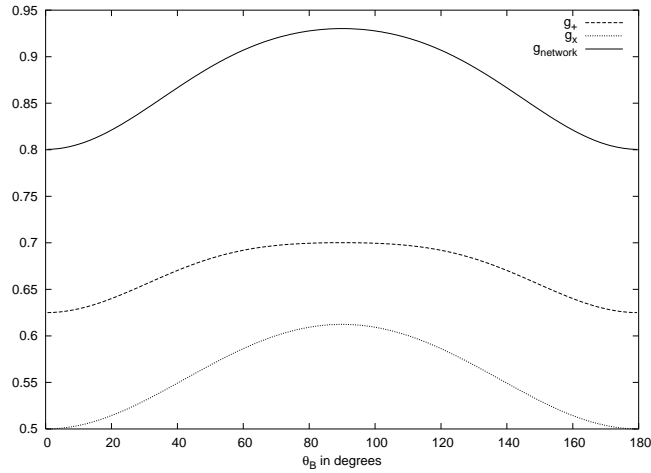
where, $g^2_{\text{network}}(x) = g^2_+(x) + g^2_\times(x)$. In Fig.5 we plot the functions $g_{+,\times}$ and g_{network} as functions of θ_B between $0^o \le \theta_B \le 180^o$. The factors g are of the order of unity and the SNR_0 gives essentially the integrated SNR of a GW source. Recall that this is an SNR averaged over the polarisations.

We observe from Fig.5 that the maximum integrated SNR is obtained for sources lying in the ecliptic plane ($\theta_B = 90°$). This can be readily explained from Fig.3 where the trajectory of such a source is plotted. One observes that a large fraction of the orbit in the LISA frame is *away* from the LISA plane ($\theta_L = 90°$). As seen from Eq.(40) the sensitivity of LISA increases as we go away from the plane of LISA, that is, towards the poles. We also observe that the variations in the three curves are small; among these the network SNR shows the highest variation of $\approx 15\%$. This shows that LISA has a more or less uniform average response over the year as it moves in its orbit. The above formulae are also more generally valid if the integration time T is taken to be in integral multiples of $T_\odot$; then the $T_\odot$ in Eq.(42) should be replaced by T.

We compute the SNRs of six binaries in our galaxy which give high average SNR (averaged over polarisations) if we integrate the SNR optimally over the period of a year. The following table lists six binary systems which give the network SNRs ranging from about 3 to over 100. The SNRs have been computed assuming circular orbits for the binaries. The information about the binaries has been obtained from [8]. We observe that the binary masses are small; $\sim .5 M_\odot$ and companion mass $\sim .02 M_\odot$. The reason for such small masses is that the orbital periods of the binaries must be short, in this case ranging from about .01 to .03 of a day. The GW quadrupole frequency f_{gw} is related to the orbital period by the equation:

$$f_{gw} = \frac{2}{P_{orb}} = 2.3 \left(\frac{P_{orb}}{.01 \text{day}} \right)^{-1} \text{mHz}. \tag{45}$$

Thus the sources radiate GW at frequencies ~ 1 or 2 mHz, the band in which LISA has maximum sensitivity. This yields the high SNRs. They are also close by; $R \sim 100$ or 200 pc which tends to increase the raw gravitational wave amplitude H. Note that the SNRs listed in the table are average and actual observations can yield different values depending on the orientation of the binary orbit. Also the observables used are optimal in the average sense. If the orientation of the binary is known then in general a better observable can be found.

2. Estimation of inclination angle of the orbital plane

We showed in previous sections that if we do not have any prior information of the inclination angle ϵ of the binary orbit and the the angle ψ, then tracking the source with $\hat{f}_+$ and $\hat{f}_\times$ are optimal in the average sense. Typically, the orbital inclination is difficult to estimate from other astrophysical observational means. However, for binaries with known masses and distances (e.g. binaries in table-I), we can estimate (ϵ, ψ) from the output of eigen-observables $\vec{f}_+$ and $\vec{f}_\times$.

TABLE I: In this table we list six binaries in our galaxy whose parameters are known. For these we compute the optimum SNRs where the optimisation has been performed after averaging over the polarisations.

Name	P_{orb} in days	m_1	m_2	$\log(H)$	θ_B°	R in pc	SNR_+	$SNR_\times$	$SNR_{network}$
AM CVn	0.011907	0.5	0.03	-21.4	124.46	100	89.9	75.6	117.4
CP Eri	0.019950	0.6	0.02	-22.0	116.43	200	8.9	7.4	11.4
CR Boo	0.017029	0.6	0.02	-21.7	72.103	100	26.3	22.7	34.7
GP Com	0.032310	0.5	0.02	-22.2	66.997	200	2.2	1.8	2.8
HP Lib	0.012950	0.6	0.03	-21.4	85.040	100	77.6	67.8	103.0
V803 Cen	0.018650	0.6	0.02	-21.7	120.32	100	20.6	17.5	27.0

The integrated SNRs over the period of one year of $\vec{f}_+$ and $\vec{f}_\times$, *without averaging over polarisations* are given by

$$SNR_+(\theta_B, \phi_B) = \sqrt{\frac{5}{2}}\, \text{SNR}_0\, g_+(\cos\theta_B)\, a_+ ,$$

$$SNR_\times(\theta_B, \phi_B) = \sqrt{\frac{5}{2}}\, \text{SNR}_0\, g_\times(\cos\theta_B)\, a_\times , \tag{46}$$

where $a_+ = |h_+(\Omega)/H|$ and $a_\times = |h_\times(\Omega)/H|$ (see Eq. (7)). We note that here the factor $\sqrt{5/2}$ appears, since there is no averaging over the polarisations. The $a_{+,\times}$ can be estimated, if the SNRs appearing on the LHS of Eq. (46) can be measured. Further, straightforward algebra shows that

$$a_+^2 + a_\times^2 = \left(\frac{1+\cos^2\epsilon}{2}\right)^2 + \cos^2\epsilon . \tag{47}$$

From the above equation, we can estimate ϵ. Substituting back in a_+, one can also estimate ψ if needed. This exercise can be carried out for the binaries listed in the table I.

B. The general case

In this section we relax the condition of dealing only with low frequencies and consider the entire band-width of LISA. We compare the sensitivities for LISA obtained by using the optimum SNR with the Michelson data combination usually denoted by X. As a polynomial vector (p_i, q_i) it is given by:

$$X = (1 - E_2^2, 0, E_2(E_3^2 - 1), 1 - E_3^2, E_3(E_2^2 - 1), 0). \tag{48}$$

When we set all the E_i to be equal, the factor $(1 - E^2)$ factors out and one is left with a simple polynomial vector $(1, 0, -E, 1, -E, 0)$. This combination has been used to plot the standard LISA sensitivity curve. Two other Michelson observables are obtained by cyclic permutations called Y and Z in the literature [6]. In this section we will compare the sensitivities obtained by using the eigen-combination $\vec{v}_+$ and the network observable with the Michelson combination X and the observable X_{switch} obtained by 'switching' the Michelson's X, Y, Z optimally, that is, $SNR_{X_{\text{switch}}} = \max(SNR_X, SNR_Y, SNR_Z)$. Extending the definition of sensitivity of a data combination from earlier literature [1] to an observable, we define the sensitivity of an observable W as,

$$\text{Sensitivity}_W(f) = \frac{5}{SNR_W(f)} , \tag{49}$$

where $SNR_W(f)$ is the integrated SNR over a observation period T. The number 5 has been chosen following earlier literature. Eq. (49) for a fixed data combination W_0 reduces to the standard one in literature:

$$\text{Sensitivity}_{W_0}(f) = 5\frac{\sqrt{S_{W_0}(f)B}}{|h_{W_0}|} , \tag{50}$$

where $B = 1/T$. As before we take $T = T_\odot$ for plotting the sensitivity curves in the figures below. The results of our findings are displayed in the plots. Fig.6 displays the sensitivity curves for the observables (a) Michelson - X (dotted curve), (b) Switched Michelson X_{switch} (dash-dotted curve), (c) Eigen-combination $\vec{v}_+$ (dashed curve) and

FIG. 6: Sensitivity curves for the observables: Michelson, Switched Michelson, $\vec{v}_+$ and network for the source direction $\theta_B = 90°, \phi_B = 0°$.
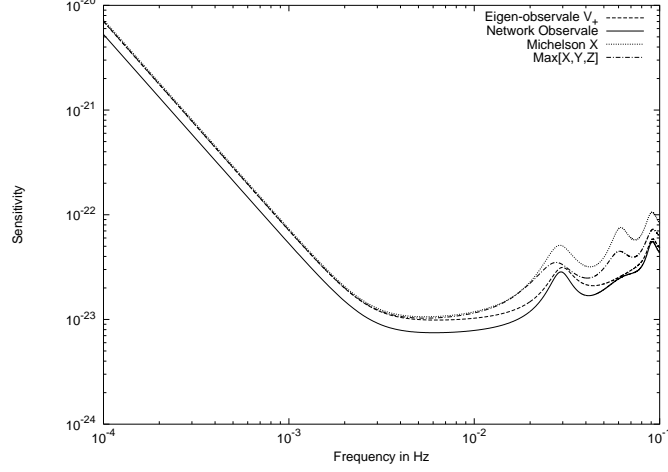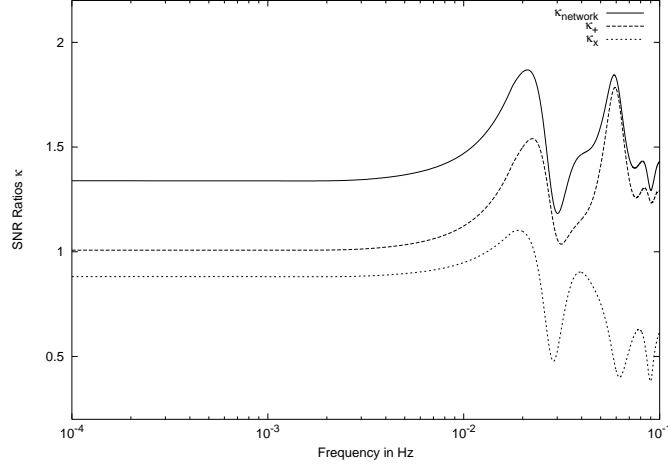

FIG. 7: Ratios of the sensitivities of the observables network, $\vec{v}_{+,\times}$ with X_{switch} for the source direction $\theta_B = 90°, \phi_B = 0°$.


(d) network observable (solid curve). We observe that the sensitivity over the band-width of LISA increases as we go from (a) to (d). We observe that the X_{switch} does not do much better than X. This is because for the source direction chosen X is reasonably well oriented and there is not much switching to Y and Z combinations. However, the network and $\vec{v}_+$ observables show significant improvement in sensitivity over both X and X_{switch}. The sensitivity curves (except X) do not show much variations for other source directions and the plots are similar. The quantitative comparison of sensitivities is shown in Fig.7.

Here we have compared the network, and the eigen-combinations $\vec{v}_{+,\times}$ with X_{switch}. Defining:

$$\kappa_a(f) = \frac{\text{SNR}_a(f)}{\text{SNR}_{X_{\text{switch}}}(f)}, \tag{51}$$

where the subscript a stands for network or $+, \times$ and $\text{SNR}_{X_{\text{switch}}}$ the SNR of the observable X_{switch}, we plot these ratios of sensitivities over the LISA band-width. We notice from the κ_{network} curve that the improvement in sensitivity for the network observable is about 34% at low frequencies and rises to nearly 90 % at about 20 mHz, while at the same time the $\vec{v}_+$ combination shows improvement of 12 % at low frequencies rising to over 50 % at 20 mHz. Finally, Fig.8 exhibits the sensitivities of the network observable over various source directions. Since the sensitivity of this observable is independent of ϕ_B, we plot the curves for several values of $\theta_B = 0, 30, 60, 90$ degrees. Also since the network observable possesses reflection symmetry about the ecliptic plane $\theta_B = 90°$, we do not need to plot the curves θ_B between 90° and 180°. The important observation from this figure is that not much variation in sensitivity is seen as all source directions are scanned. Thus the network observable integrated over the year has essentially uniform

FIG. 8: The sensitivity curves for the network observable for $\theta_B = 0°, 30°, 60°, 90°$
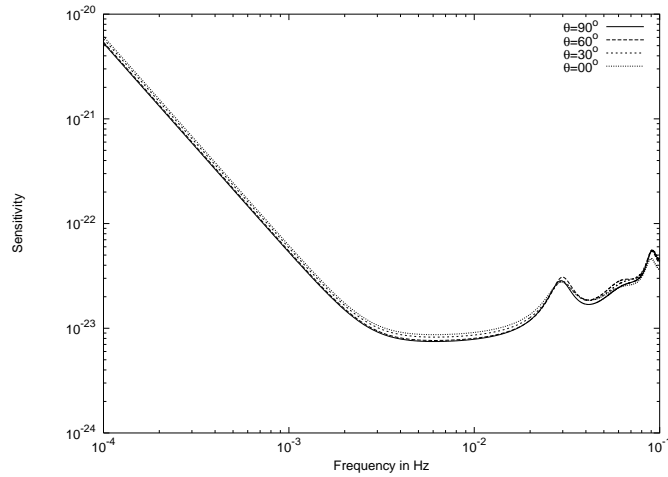
sensitivity to all source directions over the frequency range $10^{-4} - 10^{-1}$ Hz.

VII. CONCLUSION

We have shown how the SNR can be optimised for a GW source with known direction but with unknown polarisation. While obtaining the SNR the signal is averaged over the polarisations and then optimised. Because of this procedure we lose out to some extent on the SNR but on the otherhand it leads to robust results. The optimisation methods are algebraic in that one must solve an eigenvalue equation to determine the optimum SNRs. We separately deal with the low frequency case as it is of considerable astrophysical importance - a large fraction of GW sources are expected to be of this category. Secondly, it lends itself to simple analytical approximations which throw light on the results obtained. Lastly, we deal with the general case covering the full band-width of LISA. We have compared the sensitivities obtained with our strategy to those obtained in the standard way. We find that the improvement in sensitivity of the network observable over the X or X_{switch} ranges from about 34% to nearly 90% over the bandwidth of LISA for a source lying the ecliptic plane. Finally we present a list of few binaries in our galaxy for which the optimal SNRs and the network SNRs have been computed. We also describe a method of extracting information about the inclination angle of the orbit of the binary if $SNR_{+,\times}$ can be measured.

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