NOTE ON THE NONVANISHING OF $L(1)$

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It is well known that if $\chi(m)$ is a real nonprincipal character (mod k), then

$$L(1) = \sum_{1}^{\infty} \frac{\chi(m)}{m} \neq 0,$$

and many proofs have been found. We give a very simple proof when $k=p$ an odd prime, in which case $\chi(m)=(m/p)$, the Legendre symbol. This makes it possible to simplify the proof that if $p\nmid a$, then there are infinitely many primes congruent to a modulo p. Write

$$\zeta = e^{2\pi i/p}, \qquad P = \frac{\prod_{n} (1-\zeta^{n})}{\prod_{r} (1-\zeta^{r})},$$

where n runs through the quadratic nonresidues of p and r runs

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through the quadratic residues. We prove first that $L(1)\neq 0$ if $P\neq 1$. Since

$$\frac{1}{1-Z}=\exp\left\{\sum_{m=1}^{\infty}\frac{Z^m}{m}\right\}\qquad (|Z|\leqq 1, Z\neq 1)$$

we have

$$P=\exp\left\{\sum_{m=1}^{\infty}\frac{1}{m}\left(\sum_r \zeta^{rm}-\sum_n \zeta^{nm}\right)\right\}$$
$$=\exp\left\{S\sum_{m=1}^{\infty}\frac{1}{m}\left(\frac{m}{p}\right)\right\}=\exp\{SL(1)\},$$

where

$$S=\sum_r \zeta^r-\sum_n \zeta^n=\sum_{m=1}^{p-1}\left(\frac{m}{p}\right)\zeta^m.$$

Hence $L(1)\neq 0$ if $P\neq 1$. Let c be any fixed positive integer which is a quadratic nonresidue of p, e.g., $c=p-1$ if $p\equiv 3 \pmod 4$. Then since $n\equiv cr \pmod p$, the equation $P=1$ can be written as

$$\prod_r\left(\frac{1-\zeta^{cr}}{1-\zeta^r}\right)=1.$$

Then the polynomial

$$\prod_r\left(\frac{1-Z^{cr}}{1-Z^r}\right)-1$$

has a zero ζ which satisfies the irreducible equation $1+Z+Z^2+\cdots+Z^{p-1}=0$. Hence if Z is any variable,

$$\prod_r\left(\frac{1-Z^{cr}}{1-Z^r}\right)-1=f(Z)(1+Z+Z^2+\cdots+Z^{p-1}),$$

where $f(Z)$ is a polynomial in Z with integral coefficients. Put $Z=1$. Then $c^{(p-1)/2}-1\equiv 0 \pmod p$, which is a contradiction, in view of Euler's criterion for quadratic residuacity. This finishes the proof.

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