THE REPRESENTATION OF A LARGE NUMBER AS A SUM OF 'ALMOST EQUAL' CUBES

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1. WRIGHT* has shown that, if $\lambda_1,..., \lambda_s$ are positive numbers whose sum is 1 and if $k \geqslant 3$, $s \geqslant (k-2)2^{k-1}+5$, then every sufficiently large n can be expressed as a sum $n = m_1^k+...+m_s^k$ of s positive kth powers such that

$$\lambda_i n - m_i^k = O(n^{1-\beta}) \qquad (i = 1, 2,..., s)$$

with $0 < \beta < \alpha$ where α is given as a certain function of k and s; for example, if $k = 3$ and $s = 9$, then $\alpha = \frac{1}{51}$.

In this paper we prove the following special results in the same direction.

THEOREM 1. *Every sufficiently large n can be expressed in the form*

$$n = \sum_{i=1}^{8} m_i^3,$$

where $\frac{1}{8}n - m_i^3 = O(n^{1-\beta}) \quad (i = 1, 2,..., 8), \qquad 0 < \beta < \alpha,$

and (i) $\alpha = \frac{1}{1608}$ (*without any hypothesis*);

(ii) $\alpha = \frac{1}{42}$, *if the 'extended Riemann hypothesis' is true*;†

(iii) $\alpha = \frac{1}{24}$, *if hypothesis* (*P*) *is true*.‡

THEOREM 2. *Every sufficiently large n can be expressed in the form*

$$n = m_1^3+m_2^3+...+m_9^3,$$

where $\frac{1}{9}n - m_i^3 = O(n^{1-\beta}) \quad (i = 1, 2,..., 9), \qquad 0 < \beta < \alpha,$

and (i) $\alpha = \frac{1}{42}$, *if the e. R. h. is true*;

(ii) $\alpha = \frac{1}{24}$, *if hypothesis* (*P*) *is true*.

Theorem 2 sharpens a special case of Wright's result (his α is $\frac{1}{51}$), assuming, however, unproved results. Theorem 1 is new.

* *Math. Zeits.* 38 (1934), 730–46.

† 'e. R. h.' = 'extended Riemann hypothesis' in what follows.

‡ Let $(a, b) = 1$. Hypothesis (*P*) is that for any given positive ϵ the number of primes congruent to b (to modulus a) between x and $x+x^\epsilon$ tends to infinity with x.

2. In what follows t is a prime such that $t \equiv 17 \pmod{48}$ and p a prime such that $p \equiv 2 \pmod 3$. We denote by $A(m)$ the assumption* that the number of primes congruent to 2 (to modulus 3) between x and $x+x^{1-1/m+\delta}$ tends to infinity with x. We shall assume that $A(m)$ is true for some fixed m.

We choose t to be the prime such that $t \equiv 17 \pmod{48}$ which is nearest to $N^{1/3(3m+1)}$. Then, since $A(m)$ is true, we can find, for sufficiently large N, at least 10 primes congruent to 2 (to modulus 3), and such that

$$(8t^3+3t^2+3t+2)p^9 < N < (8t^3+4t^2+3t+1)p^9. \tag{1}$$

Since $p^9 < N$ it follows that† at least one of the ten primes p is prime to N (large). For our further argument we therefore suppose that $(p,N) = 1$.

It follows from (1) that

$$p \sim \frac{N^{m/3(3m+1)}}{\sqrt[3]{2}}, \qquad \text{since } t \sim N^{1/3(3m+1)}. \tag{2}$$

Now $p \equiv 2 \pmod 3$, $(p,N) = 1$; hence we can‡ find β such that

$$N-\beta^3 = p^3 M, \tag{3}$$

where, further,
$$tp^3 < \beta < (t+1)p^3. \tag{4}$$

From (1), (3), (4),

$$(7t^3+1)p^6 < M < (7t^3+4t^2+3t+1)p^6. \tag{5}$$

We set
$$M = 6t^3p^6+M_1. \tag{6}$$

Then, from (5),

$$(t^3+1)p^6 < M_1 < (t^3+4t^2+3t+1)p^6. \tag{7}$$

Now, we can§ find γ such that

$$M_1-\gamma^3 = 6M_2, \tag{8}$$

where
$$0 \leqslant \gamma < 96, \qquad M_2 \not\equiv 0, 7, 12, 15 \pmod{16}. \tag{9}$$

Since t is a prime and $t \equiv 2 \pmod 3$ we can‖ also find γ' such that

$$M_1-\gamma'^3 \equiv O(t), \qquad 0 \leqslant \gamma' < t. \tag{10}$$

Since, further, $t \equiv 1 \pmod{16}$, it follows from (8), (9), (10) that we can find δ such that

$$M_1-\delta^3 = 6tM_3, \tag{11}$$

where
$$tp^2-96t \leqslant \delta < tp^2 \tag{12}$$

and
$$M_3 \not\equiv 0, 7, 12, 15 \pmod{16}. \tag{13}$$

* For every positive δ; naturally, $m > 1$.

† See Landau, *Primzahlen*, 555–9, for similar arguments.

‡ See ibid.; *Math. Annalen*, 66 (1909), 102–5.

§ See Landau, *Primzahlen*, 556–7. ‖ As in the proof of (3).

From (7), (11), (12) it follows that

$$\frac{1}{6t}p^6 < M_3 < \frac{4t^2+3t+1}{6t}p^6+O(p^4t^2). \tag{14}$$

From (13) we have $$M_3 = y_1^2+y_2^2+y_3^2, \tag{15}$$

where, from (2) and (14), since $m > 1$,

$$y_1, y_2, y_3 = O(\sqrt{t}p^3). \tag{16}$$

From (3), (6), (11), (15) we obtain

$$N = \beta^3+6t^3p^9+(p\delta)^3+6tp^3(y_1^2+y_2^2+y_3^2) \tag{17}$$

$$= \beta^3+(p\delta)^3+\sum_{s=1}^{3}\{(tp^3+y_s)^3+(tp^3-y_s)^3\} \tag{18}$$

$$= x_1^3+x_2^3+...+x_8^3. \tag{19}$$

From (4), (12), and (16) it now follows that

$$x_1 = tp^3+O(p^3), \qquad x_2 = tp^3+O(pt), \tag{20}$$

$$x_s = tp^3+O(\sqrt{t}p^3) \quad (3 \leqslant s \leqslant 8). \tag{21}$$

Since, for large N, $\sqrt{t}p^3 > p^3 > tp$ $(m > 1)$, it follows that, for $1 \leqslant s \leqslant 8$,

$$x_s^3-t^3p^9 = O(t^{\frac{5}{2}}p^9) \tag{22}$$

$$= O(N^{1-1/(18m+6)}), \tag{23}$$

from (2).

But $$N-8t^3p^9 = O(t^2p^9). \tag{24}$$

From (22), (23), (24), for $1 \leqslant s \leqslant 8$,

$$x_s^3-\tfrac{1}{8}N = O(N^{1-1/(18m+6)}). \tag{25}$$

Theorem 1 is now an immediate consequence of (19) and (25) when we observe that, for arbitrary positive ϵ,

(i) $A(m)$ is true* for $m = 250+\epsilon$;

(ii) $A(m)$ is true† for $m = 2+\epsilon$, if the e. R. h. is true;

(iii) $A(m)$ is true for $m = 1+\epsilon$, if hypothesis (P) is true.

The proof of Theorem 2 is exactly similar. We start with

$$(9t^3+3t^2+3t+2)p^9 < N < (9t^3+4t^2+3t+1)p^9 \tag{26}$$

instead of (1), and obtain

$$N = (tp^3)^3+\beta^3+(p\delta)^3+\sum_{s=1}^{3}\{(tp^3+y_s)^3+(tp^3-y_s)^3\} \tag{27}$$

in place of (18).

Theorem 2 is also easily shown to be a consequence of Theorem 1.

* Heilbronn, *Math. Zeits.* 36 (1933), 394–423.

† This is well known: see, for example, Landau's *Primzahlen.*