Rapid dissipation of magnetic fields due to Hall current

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We propose a mechanism for the fast dissipation of magnetic fields which is effective in a stratified medium where ion motions can be neglected. In such a medium, the field is frozen into the electrons and Hall currents prevail. Although Hall currents conserve magnetic energy, in the presence of density gradients, they are able to create current sheets which can be the sites for efficient dissipation of magnetic fields. We recover the frequency, ω_{MH}, for Hall oscillations modified by the presence of density gradients. We show that these oscillations can lead to the exchange of energy between different components of the field. We calculate the time evolution and show that magnetic fields can dissipate on a timescale of order $1/\omega_{MH}$. This mechanism can play an important role for magnetic dissipation in systems with very steep density gradients where the ions are static such as those found in the solid crust of neutron stars.

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I. INTRODUCTION

Rapid dissipation of magnetic fields is currently one of the key problems in astrophysics. On account of the generally large electrical conductivities that obtain in astrophysical settings, the Ohmic dissipation of fields usually takes place on very long time-scales. However, it is quite often observed that astrophysical magnetic fields change topology on a very short time-scale, giving rise to a variety of transient phenomena. An explanation of such fast changes is crucial to the understanding of solar activity, in particular, solar flares [1], and other active phenomena observed in stars [1,2]. It is also known that fast reconnection of magnetic fields is basic to the operation of non-linear dynamos [3].

If a current sheet is formed in a plasma, the reconnection takes place slowly due to the time-scale for the removal of matter from the site of reconnection, as in the Parker-Sweet mechanism [4]. Indeed, if the dissipation in the current sheets were to be fast, with the field moving with Alfvén speed toward the current sheet and getting dissipated there due to reconnection, then the matter that is frozen into the field would also move with the same speed towards the sheet. The plasma would, therefore, accumulate at the sheets and halt the reconnection, unless there is some efficient evacuation process operating at the sheet. The sheets are usually narrow and the outflow is rather inefficient, even if it takes place at the Alfvén speed [5].

We propose a mechanism of fast dissipation of magnetic fields that occurs at modified Hall frequencies. The mechanism is relevant in all situation when Hall currents predominate, and there is a density stratification. In this case, the magnetic field follows the (electric) drift velocity of the electrons. In the presence of density gradients, the profile of magnetic field changes in such a way that it forms a current sheet. This steepening of the front is *not* accompanied by the flow of plasma towards the sheet, the drift velocity being parallel to it. Consequently, the current sheet is formed and the field is efficiently dissipated with no accumulation of material in contrast to the Parker-Sweet reconnection mechanism.

The modified Hall frequency that we recover below, ω_{MH}, occurs in a stratified medium when the ions remain static. Two example of such situations are: penetration waves in low collision plasmas relevant for plasma switches [6], and neutron star crusts. In the case of penetration waves, the ion response time is long compared to the wave timescale and the ions are approximately static. In the case of neutron star crusts, the ions form a very high density lattice of iron rich nuclei with densities varying from $\sim 10^6 \mathrm{g/cm}^{-3}$ to $\sim 10^{11} \mathrm{g/cm}^{-3}$ under 0.8 km [7]. In both cases, the field dynamic is governed by the electron drift motion.

In the following sections, we discuss how Hall currents in a stratified medium can generate fast dissipation in the non-linear regime. In Section II, we describe the linear Hall oscillations and discuss how poloidal and toroidal fields exchange energy during these oscillations. We also recover the modified Hall frequency for a stratified medium. We discuss the limitations of the oscillatory solutions about a stationary configuration in Sec. III. We argue that in general there is no stationary configuration for large scale magnetic fields and that current sheets develop. The oscillations can occur only "locally", i.e., on small scales. In Sec. IV, we show that the magnetic field evolution is governed by a non-linear equation similar to Burgers equation. We solve the evolution for a toroidal field configuration numerically and show that current sheets develop and magnetic dissipation is efficient. The dissipation timescale is $\sim 1/\omega_{MH}$. We describe numerical solutions for two configurations: a toroidal magnetic field of one polarity; and a toroidal magnetic field consisting of two oppositely directed fields.

We show that these fields evolve towards forming current sheets that rapidly dissipate. In Sec. V, we relate the dynamics of toroidal fields with that of poloidal fields, and summarize the different possibilities of the field evolution. We close by discussing the application of this physical mechanism focusing particularly on the case of neutron stars crusts (Sec. VI).

II. LINEAR OSCILLATIONS.

Consider the magnetic field evolution in the case where the motion of ions can be neglected. This is the case for neutron stars' solid crust. We will consider collisional plasma. Then, the field evolution follows from Ohm's law,

$$\nabla \times \mathbf{B} = \frac{4\pi}{c}\sigma(\mathbf{E} + \frac{1}{c}\mathbf{v}_e \times \mathbf{B}), \tag{1}$$

where $\mathbf{v}_e$ is electron velocity. As the conductivity is usually high, the left-hand side of (1) can be neglected, resulting in

$$\mathbf{E} + \frac{1}{c}\mathbf{v}_e \times \mathbf{B} = 0, \tag{2}$$

corresponding to the electric drift of electrons.

Taking $\nabla\times$ of equation (1), we recover the induction equation, with the Hall effect,

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla \times [\mathbf{v}_e \times \mathbf{B}] - \nabla \times \eta \nabla \times \mathbf{B} =$$
$$-\nabla \times \left(\frac{c}{4\pi n e}[\nabla \times \mathbf{B}] \times \mathbf{B}\right) - \nabla \times \eta \nabla \times \mathbf{B}, \tag{3}$$

where $\eta = c^2/4\pi\sigma$. Equation (3) results in the following energy balance,

$$\frac{1}{2}\frac{\partial}{\partial t}\int B^2 dV = -\int \eta (\nabla \times \mathbf{B})^2 dV. \tag{4}$$

In order to describe the evolution of the field in a stratified medium, we consider and axisymmetric magnetic field, which can be expressed as the sum of poloidal and toroidal components,

$$\mathbf{B} = \mathbf{B}_p + \mathbf{B}_t.$$

In order to simplify the geometry we assume that the radius of the star, R, is large compared to the wavelengths involved such that we can work in Cartesian coordinates on the surface of a sphere. We define the x and y axes in the horizontal plane as the latitudinal and azimuthal (longitudinal) directions respectively, while z is the vertical direction. Then, the poloidal field is described by

$$\mathbf{B}_p = \{B_x(x,z), 0, B_z(x,z)\},$$

while the toroidal field is given by

$$\mathbf{B}_t = \{0, B_y(x,z), 0\}.$$

If the resistivity η can be neglected, which is justified in highly conducting media such as neutron star crusts, then the magnetic energy is conserved, according to (4). Therefore, all that happens to the magnetic field are *oscillations* about a stationary configuration.

Consider, for example, an initial poloidal field, $\mathbf{B}_0 = \{B_0, 0, 0\}$, where B_0 is a constant background field. Assuming first that the density is also constant, and considering small perturbations of the magnetic field of the form,

$$\mathbf{b} = \tilde{\mathbf{b}} e^{-i\omega t + ik_x x + ik_z z} \tag{5}$$

($k_y = 0$ because of axial symmetry), we find substituting in 3 for $\eta \to 0$

$$\omega = \omega_H = \frac{|(\mathbf{k}\cdot\boldsymbol{\omega}_e)|c^2 k}{\omega_p^2}, \tag{6}$$

where $\boldsymbol{\omega}_e = e\mathbf{B}_0/mc$ is the electron cyclotron frequency, and ω_p is the plasma frequency. We have thus recovered the well known Hall oscillations or whistlers. Note that even if $b_y = 0$ initially, i.e., the toroidal component is absent, it will be generated reaching the level of the (perturbed) poloidal component; and thus, the energy will be exchanged between the poloidal and toroidal components.

The situation is different if the large scale background field is toroidal, i.e., $\mathbf{B}_0 = \{0, B_0, 0\}$. Then a perturbation of the form (5) would not result in oscillations (6), because $(\mathbf{k}\cdot\boldsymbol{\omega}_e) = 0$.

Let us now recall that the density is not a constant, but, rather, it has a steep dependence on z. By including the spatial dependence of the density in (3), we recover the Hall frequency modified by the presence of density gradients,

$$\omega = \omega_{MH} = \frac{(\mathbf{k}\cdot[\boldsymbol{\omega}_e \times \nabla n])c^2}{\omega_p^2 n}. \tag{7}$$

Note that $\boldsymbol{\omega}_e$ is a pseudo-vector, and therefore the frequency ω_{MH} is a real scalar, as it should be, just as ω_H is a real scalar as well, see (6). The phase velocity, corresponding to (7) can be written as

$$\mathbf{v}_{MH} = -\frac{c^2}{\omega_p^2}\left[\frac{\nabla n}{n} \times \boldsymbol{\omega}_e\right], \tag{8}$$

and $\omega_{MH} = \mathbf{k}\cdot\mathbf{v}_{MH}$. A similar case is known in low collisional plasmas where the corresponding wave is called magnetic penetration wave [6].

The wave described by (7,8) corresponds to only toroidal perturbations due to the chosen initial configurations. In this special case there is no poloidal field initially and no energy exchange occur between toroidal and poloidal components. However, in general the two components are present and this exchange does take place. In order to see this, let us return to the large scale poloidal field, $\mathbf{B}_0 = \{B_0, 0, 0\}$, taking into account that the density is a function of z. We look for solutions of the linearized equations in the form,

$$\mathbf{b} = \{\partial_z a(z), b_y(z), -ik_x a(z)\} e^{-i\omega t + ik_x x}, \qquad (9)$$

cf. (5). Then we obtain the following dispersion relation,

$$\omega^2 a = \frac{(\mathbf{k}\cdot\boldsymbol{\omega}_e)^2 c^4}{\omega_p^4}(k_x^2 - \partial_z\partial_z)a. \qquad (10)$$

In order to estimate the frequency consider two zones $z_2 \leq z < z_1$, with density n_2, and $z_1 \leq z \leq 0$ ($z = 0$ is the top of the crust), with density n_1, and $|z_1| = h_1$, and $z_1 - z_2 = h_2$, $h_{1,2}$ being the scale hight in these two zones. Assuming that $n_2 \gg n_1$, and $h_2 \gg h_1$, we obtain

$$\omega = \frac{\pi}{2}\frac{|(\mathbf{k}\cdot\boldsymbol{\omega}_e)|c^2}{\omega_{p_2}^2}|k_x + 1/h_1|, \qquad (11)$$

where ω_{p_2} is the plasma frequency based on the density n_2. Note that if $k_x \ll 1/h_1$ (large horizontal length scale), the frequency is essentially the same as ω_{MH} in (7). This is the main characteristic frequency of magnetic fluctuations in the crust due to the steep density gradient. As seen from (9), this mode does involve both poloidal and toroidal components.

The most general case involves non-linear coupling between the poloidal and toroidal fields. Qualitatively the same situation will take place: if we start with a poloidal field, supported by the currents in the crust, a toroidal field will be generated. The current velocity is toroidal, and, according to (3), the toroidal field is stretched out from the poloidal, analogously to the effect of differential rotation. However, unlike the latter, the Hall current conserves the energy, and therefore the new toroidal field will grow at the expense of the poloidal field. In other words, while the strength of the toroidal field is increasing, that of the poloidal component should decrease.Of course, the toroidal field cannot grow indefinitely under these circumstances, and eventually the field will either reach some steady state, or the poloidal and toroidal fields will exchange their energies, oscillating with frequency ω_{MH}.

Note, however, that including dissipation may drastically change the situation, and, in some cases, discussed below in Sec. III, and IV, the field will rapidly dissipate instead of oscillate.

III. THE PROBLEM OF STATIONARY STATES.

This simple picture of oscillations implicitly assumes that they proceed about some stationary state, which presumably exists. The large scale background field considered above was uniform and trivially stationary. We will show that, in general, the large scale field is not stationary but evolves with time.

It is clear from (3) that the stationary state is possible if, neglecting diffusion,

$$\frac{c}{4\pi ne}[\nabla\times\mathbf{B}]\times\mathbf{B} = \nabla\Phi, \qquad (12)$$

that is, the electric field is potential. We will show that condition (12) does not trivially occur even for extremely simple topologies, due to the gradient of the density. Indeed, consider an initial configuration consisting only of a toroidal field.

Equation (3) for a pure toroidal field can be written as

$$\partial_t B_y + \tilde{v}_x\partial_x B_y + \tilde{v}_z\partial_z B_y = \eta\nabla^2 B_y, \qquad (13)$$

where

$$\tilde{v}_x = \frac{c\partial_z n}{4\pi en^2}B_y - \partial_x\eta, \quad \tilde{v}_z = -\frac{c\partial_x n}{4\pi en^2}B_y - \partial_z\eta. \qquad (14)$$

If we neglect the resistivity in this expression, we recover the penetration wave velocity (8), $\tilde{\mathbf{v}} \to \mathbf{v}_{MH}$ as $\eta \to 0$.

It can be seen from equation (14) that the x-component of the velocity is non-vanishing, due to the vertical gradient of the density. Note that both the density gradient and the gradient of the resistivity η, are negligible in the x direction, and the $\tilde{v}_z$ component defined only by the resistivity gradient is also small. Since any toroidal field should vanish at least at the two poles, there is always a latitudinal dependence of the toroidal field, that is to say that B_y is always a function of x. Hence, according to (13), the toroidal magnetic field can never attain a stationary state. In other words, the electric field cannot be irrotational, as in (12), and its non-potential part results in the time evolution of the magnetic field.

Note that in infinite space equation (13) conserves magnetic flux,

$$\int B_y dxdz = \text{const}, \qquad (15)$$

but the magnetic energy is dissipated according to,

$$\frac{1}{2}\frac{\partial}{\partial t}\int B_y^2 dxdz = -\int \eta(\nabla B_y)^2 dxdz, \qquad (16)$$

which is a particular case of (4).

On the other hand, for a real toroidal field which should vanish at the poles, i.e., at $x = \pm\pi R/2$, the magnetic flux *is not conserved*. Indeed, according to (13),

$$\frac{1}{2}\frac{\partial}{\partial t}\int B_y dxdz = \int \eta\partial_x B_y(x = \pi R/2)dz$$
$$- \int \eta\partial_x B_y(x = -\pi R/2)dz \; . \qquad (17)$$

The right hand side gives considerable contribution when current sheets are formed at $x = \pm\pi R/2$.

IV. TOROIDAL MAGNETIC FIELD EVOLUTION: FORMATION OF CURRENT SHEETS.

A. Analytical and Numerical Solutions.

In order to study the evolution of the field, according to (13), we reduce this equation to (neglecting the resistivity gradient, and resistive diffusion in z-direction)

$$\partial_t b + b\partial_x b = \eta \partial_x \partial_x b, \tag{18}$$

where

$$b = B_y p, \qquad p = \frac{c\partial_z n}{4\pi e n^2}. \tag{19}$$

This is, in fact, the Burgers equation, the exact solution of which is well known, see, e.g., [8]. First, let us illustrate a solution in the form of a traveling shock wave,

$$b = b_0 \left(1 - \tanh\left\{\frac{(x - b_0 t)b_0}{2\eta}\right\}\right), \tag{20}$$

where b_0 is a constant, cf. e.g. [6,9]. The penetration wave (20) does not decay because the magnetic field is pumped into the system from $-\infty$. Therefore, it is more appropriate for our purposes to use the general exact solution, which we recover by using the transformation

$$b = -2\eta \partial_x \ln \xi, \tag{21}$$

to get,

$$\xi = \int_{-\infty}^{\infty} \frac{1}{(4\pi\eta t)^{1/2}} e^{-(x-x')^2/(4\eta t)} \xi(t=0, x', z) dx' \ . \tag{22}$$

Generally, the toroidal field B_y is a function of both x and z, and, since the z-dependence enters only parametrically into (18,19), the solution (21,22) can be used for each level $z = const$.

To illustrate the time evolution of the magnetic field, we demonstrate the following two cases. In the simplest case, we assume that the toroidal magnetic field does not change sign. Then the horizontal velocity $\tilde{v}_x$ is expected to drive the field to one of the poles, either to the South, or to the North, depending on the sign of the field. The gradient of the field steepens, as in a shock wave, thus forming a current sheet, where the magnetic field is finally dissipated. In the second case, consider two toroidal fields with opposite polarities in the two hemispheres. The toroidal field vanishes at the equator. We then expect that the two toroidal fields can be driven by the latitudinal velocity $\tilde{v}_x$ towards the equator, where the current sheet is formed, and the fields are efficiently destroyed.

The integral in (22) was calculated numerically, and then the distribution of magnetic field B_y was recovered from equations (19) and (21). Let us discuss the first case where the toroidal magnetic field does not change sign. Its evolution is depicted in Fig. 1, where the initial field distribution is indicated by the dashed line. The field profile starts to steepen in very few turn-over time steps, and moves towards the polar region. Note that the magnetic field in Fig. 1 is not pumped into the system, cf. (20). Therefore, unlike the traveling wave (20), as the magnetic field spreads its amplitude decreases keeping the magnetic flux conserved and, thus, the same area under each curve, see (15). As a result of decreasing magnetic field, the process slows down, because the penetration velocity (8) is proportional to $\mathbf{B}$, and therefore it decreases as well.

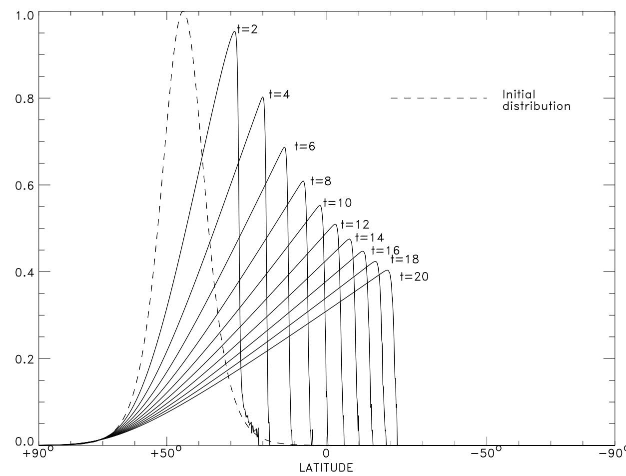

FIG. 1. Evolution of magnetic field of single polarity in the crustal region of a neutron star, as it approaches a polar region. The equator is located at zero latitude, and the time t is expressed in units of turn-over time t_0, given in (27).

In infinite space, both the shock wave (20) and the solution depicted in Fig. 1 do not result in dissipation of magnetic field and the magnetic flux is conserved according to (15). The field is only spread out. However, for a finite case such as that of a star, the boundary conditions at the poles forces the field to go to zero. When the shock wave reaches the pole, a current sheet is formed and the field starts to dissipate according to (17). Eventually, the magnetic flux goes to zero.

In order to see this dissipation at a zero-point, we proceed to the second example. Namely, consider the toroidal field changing sign at $x = 0$. It is straightforward to construct a solution, analogous to the traveling wave (20),

$$b = -b_0 \tanh\left\{\frac{x b_0}{2\eta}\right\}. \tag{23}$$

Similarly to the Parker-Sweet solution [4], the magnetic field of opposite polarities is transported from $x \to \pm\infty$ with "velocity" $\pm b_0$, and is dissipated at $x = 0$. The solution is stationary because the boundary conditions are: $B_y(x \to \pm\infty) \to \pm b_0/p$, see (19).

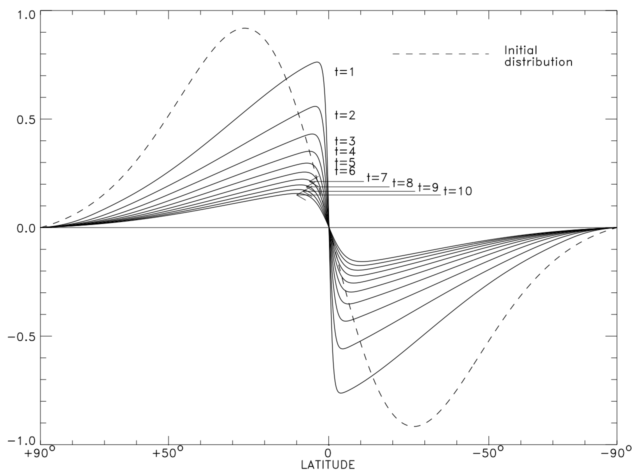

FIG. 2. The evolution of magnetic fields of opposite polarities in the two hemispheres. In this case, the two fields are approaching each other to form a current sheet at the equator, where the magnetic energy is efficiently dissipated.

In Fig. 2 we display the evolution of a toroidal field with opposite polarities in each hemispheres and $B_y(x = 0) = 0$. The field also vanishes at the poles, $B_y(x = \pm\pi R/2) = 0$, which makes the solution evolve in time (again, in contrast to the infinite space case). Here, the two fields get compressed into each other and form a sharp gradient of magnetic field at the equator. Note that the total magnetic flux is zero and, of course, trivially conserved. As to the magnetic energy, it decreases dramatically because of the very efficient Ohmic dissipation at the equatorial region. In this region, a current sheet is formed in practically only one turn-over time and the field dissipates in the same time scale. An analytical estimate of the dissipation time is given in the next subsection; it illustrates why the dissipation observed numerically is so efficient.

B. Physical Interpretation of the Mechanism.

In order to develop a physical interpretation of the solutions above, we draw a few analogies. The equation for the magnetic field (3) resembles the vorticity equation for incompressible hydrodynamics with high Reynolds numbers. Therefore, the modified Hall drift should lead to a situation analogous to a magnetic turbulent state [10]. Another interpretation of our solutions is the nonlinear interaction of different wavenumber Hall oscillations resulting in an energy cascade to small scales [11]. As a result, the field gradients steepen and we get an enhanced local rate of Ohmic dissipation which provides an effective mechanism for the dissipation of magnetic energy.

These analogies, although helpful, cannot be taken completely due to the topological constraints on magnetic fields. For instance, since the magnetic structures are frozen into the electron fluid, they are generally more persistent than vortices in hydrodynamics. The topology of magnetic field cannot be easily changed, a situation similarto what one obtains in MHD [3,12]. The magnetic structures we considered are non-stationary due to global effects (like the boundary conditions at the poles) and not locally unstable like the case of vorticity and magnetic turbulence. Besides, in the case of magnetic turbulence, the characteristic frequency coincides with the Hall oscillations frequency (6), which is smaller than ω_{MH} from (7) for situations with sharp density gradients. Therefore, our mechanism is more efficient.

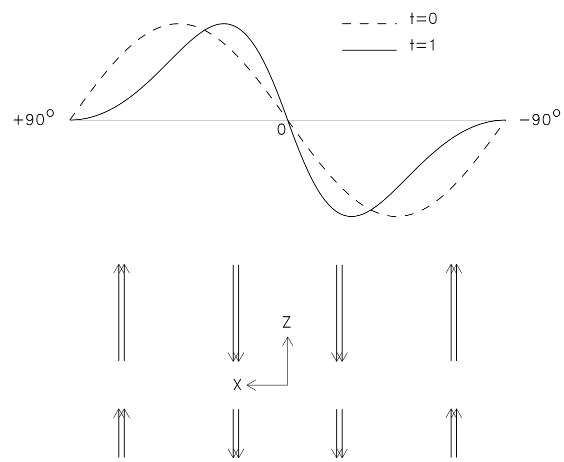

FIG. 3. Origin of the penetration velocity. The magnetic field evolution is depicted on the top of the figure. The direction of the field is out of the page. It can be seen that the field profile steepens during the evolution due to the change in drift velocity (double arrows). The drift proceeds in the vertical direction, and, therefore, there is no accumulation of matter at the current sheet.

In magnetohydrodynamics, fast reconnection encounters difficulties [1] because the rapid transport of magnetic fields towards the current sheet, where the energy is dissipated, is accompanied by a plasma movement in the same direction. The evacuation of matter from the current sheet limits the rate of reconnection: as matter accumulates and the pressure increases eventually halting the movement of the magnetic field towards the current sheet, thus preventing further reconnection. The speed of the evacuation is limited by the Alfvén velocity in a narrow current sheet. In our mechanism, we do not encounter this difficulty, because $\tilde{v}_x$ does not transport the mass. Indeed, according to (18,19), there is no outflow from the current sheet: the magnetic field is transported only to the sheet by the penetration velocity $\tilde{v}_x$. This is evident from the exact solution (23).

In order to understand why the modified Hall drift or penetration velocity does not transport any mass, we first note that, generally, the penetration velocity (8) is *not parallel* to the electric drift velocity (2). So the question arises, why the magnetic field is moving in the direction of the penetration velocity in the first place? The answer is illustrated in Fig. 3. For simplicity, we choose to illustrate a field that depends only on the x-coordinate (as in (23)). The electric drift of electrons (2) can be easily found from Amper's law,

$$\mathbf{v}_e = -\frac{c}{4\pi n e}\nabla\times\mathbf{B}, \tag{24}$$

and, clearly, it proceeds in the *vertical* direction (depicted by double arrows in Fig. 3). It follows from (24) that

$$\nabla\cdot n\mathbf{v}_e = 0. \tag{25}$$

Due to the density gradient, the plasma gets compressed as it moves down and the descending motion decelerates, as follows from (25). As a result, the magnetic field amplitude increases. On the other hand, the ascending motion is accompanied by the decompression of plasma, and correspondingly, the field amplitude decreases. As a result, the field profile steepens, as if there were a motion of plasma toward the current sheet, that is in the *horizontal* direction. However, as mentioned, the real drift motion proceeds parallel to the sheet (in the vertical direction), and therefore there is no accumulation of matter in the sheet. These circumstances make the fast dissipation of magnetic field possible.

As mentioned above, the penetration wave (7,8) is known to propagate as a shock wave in low collisional plasma [6,9]. Due to the conservation of the magnetic flux, (15), either the magnetic field is only transported by the shock wave (20) or it just disperses to infinity. Only the presence of zero-points result in the destruction of magnetic flux. This happens either at the poles or between two shock waves with opposite magnetic fields that collide. The collisional front width of the shock wave coincides with current sheet thickness. Indeed, the balance between the convective and resistive terms in equation (13) appears in a current sheet of thickness,

$$\delta = \frac{\eta}{\tilde{v}_x}, \tag{26}$$

coinciding with characteristic length of both (20) and (23). Recall that the penetration velocity $\tilde{v}_x$ is defined in (14), and, in a more specific way, in (8).

It is known that the time scale of magnetic dissipation, t_0, is entirely defined by the velocity with which the magnetic field is moving to the current sheet [5]. As seen from the exact stationary solution (23), this speed is in effect the penetration velocity $\tilde{v}_x$. Therefore,

$$t_0 = L/\tilde{v}_x, \tag{27}$$

L being the macroscopic latitudinal scale. It is useful to confirm this dissipation rate estimate analyzing the energy dissipation directly from (16). Namely, we estimate ∇B_y in (16) as B_y/δ, and the area occupied by the current sheet is $S_\delta = \delta L$. Then, we get from (16),

$$\frac{B_y^2}{t_0}L^2 \approx \eta\frac{B_y^2}{\delta^2}S_\delta, \tag{28}$$

from which t_0 is recovered as in (27). Note that the dissipation time (27) is independent of the resistivity η, and therefore the process considered here is fast. This explains why the dissipation is so efficient in the numerical results shown in Fig. 2.

We finally note that the efficient dissipation depicted in Fig. 2 proceeds when the penetration velocities of the two toroidal fields point to each other, and therefore they collide. If we change sign of magnetic fields, then, according to (8), the penetration velocity changes its direction, and, as a result, the two toroidal fields would not collide, but instead drift to the polar regions, where they will eventually decay. As mentioned above in Sec. IVA, the latter process is much less efficient because the magnetic field strength decreases as the fields move to the poles, as seen from Fig. 1, and therefore the penetration velocity decreases.

V. EVOLUTION OF THE POLOIDAL FIELD.

In Figs. 1 and 2, we have shown the solution of our numerical calculationsfor the evolution of toroidal fields with different initial profiles. Since the toroidal and poloidal fields are coupled through non-linear oscillations, we expect the dissipation of toroidal fields to cause the eventual decay of the poloidal field. The exact evolution of the poloidal field is a harder problem to solve at this stage and we leave it for future studies. Below, we only discuss the expected qualitative behavior of the poloidal field.

As we saw in Sec. II, the linear oscillations exchange energy between the poloidal and toroidal components. In other words, an initial poloidal field would generate a toroidal one. The generated toroidal field could have the nonlinear evolution depicted on either Fig. 1 or Fig. 2. In the case that the generated toroidal field is in the configuration of Fig. 1, it would slowly drift to the poles. We expect that, in this case, one would observe oscillations, because the dissipation is inefficient.

Consider now the case when the toroidal fields are generated with the configuration as in Fig. 2. In order to follow this generation in the nonlinear case, we write, according to (3),

$$\frac{\partial}{\partial t}B_y = -\frac{\partial}{\partial z}\left(\frac{c}{4\pi ne}j_y B_z\right) - \frac{\partial}{\partial x}\left(\frac{c}{4\pi ne}j_y B_x\right), \quad (29)$$

where

$$j_y = \frac{\partial}{\partial z}B_x - \frac{\partial}{\partial x}B_z, \quad (30)$$

is the azimuthal current. Note that, for neutron star crusts, the density gradient in the radial direction is extremely steep in the crustal regions, spanning some nine orders of magnitude over a distance of a few hundred meters below the surface. The variations of other quantities in equation (29) can therefore be neglected to get

$$\frac{\partial}{\partial t}B_y = -j_y B_z \frac{\partial}{\partial z}\frac{c}{4\pi ne} = \frac{c\partial_z n}{4\pi e n^2}j_y B_z. \quad (31)$$

It is evident from (31) that the toroidal field can always be generated due to the sharp density gradient, unless the poloidal field in the crustal region is current free. If, indeed, the field is anchored in the core (meaning that the currents supporting the field are confined there), then there is no Hall current present. At present, the locus where the field is anchored in neutron stars is a matter of debate with no clear resolution (see, e.g., [7,13]).

If we assume that part of the current supporting the field is present in the crustal layers, then the toroidal field will be generated from the poloidal field by the process outlined above in (31). On the other hand, the total magnetic energy is essentially conserved (apart from weak Joule dissipation), which means that the newly generated toroidal field would result in a back reaction on the poloidal field in such a way that the energy of the latter is decreased. In the absence of Ohmic dissipation, the toroidal field would grow to a certain level, and then start to decrease, thus presenting an oscillatory behavior, as described earlier at the end of Sec. II. If we incorporate Ohmic diffusion in the nonlinear case the both the toroidal and the poloidal fields should eventually decay.

Again, in the case of Fig. 1, the toroidal fields would slowly drift to the poles and we expect to observe oscillations, because the dissipation is inefficient. In the case of Fig. 2, the toroidal field is efficiently dissipated and consequently, according to equation (4), both the poloidal and toroidal fields decay. Indeed, the Ohmic dissipation is now increased due to the presence of current sheets, so that the equation (4) can be written in the form

$$\frac{1}{2}\frac{\partial}{\partial t}\int B^2 dV = -\frac{1}{t_0}\int B_y^2 dV. \quad (32)$$

In order to follow the evolution of the poloidal field, we introduce efficient dissipation discussed above into equation (31), to get

$$\frac{\partial}{\partial t}B_y = \omega_{MH}B_p - \frac{B_y}{t_0}, \quad (33)$$

and, because the Hall current conserves the total energy, $\int(B_p^2 + B_y^2)dV$, the back reaction of the toroidal magnetic field on the poloidal component can be expressed analogously,

$$\frac{\partial}{\partial t}B_p = -\omega_{MH}B_y. \quad (34)$$

Seeking solutions $\sim e^{\gamma t}$, we find a dispersion relation,

$$\gamma = -\frac{1}{2t_0} \pm \sqrt{\left(\frac{1}{2t_0}\right)^2 - \omega_{MH}^2}. \quad (35)$$

It can be seen from equation (35) that the decay time for the poloidal component is also of the order of t_0.

To summarize, we can delineate three regimes for magnetic fields in the crusts of neutron stars:

(1) The currents supporting the fields in the crust are anchored in the core, i.e., no currents in the crust. Then, there is no Hall current (by definition), and no evolution of the fields related to the processes we described here.

(2) The currents or part of the current are situated in the crust. Then, the poloidal field inevitably generates toroidal fields, which, depending on the sign of the initial poloidal field, may result in either penetration velocity pushing these toroidal fields apart or pushing them together. In the first case, the dissipation is less efficient, being limited to slow decay at the poles (Fig. 1). Although slower than the equatorial case, the decay at the poles is still faster than the general Ohmic decay.

(3) In the second case, when the toroidal fields are pushed together as in Fig. 2, we expect both toroidal and poloidal fields to decay according to (35).

Note that in any of the above cases there would be oscillations on scales small compared with the radius of the star (i.e., in the geometric optics limit), with frequency ω_{MH}. These oscillations in general will decay with Ohmic decay time.

VI. DISCUSSION

Magnetic fields are an important feature of neutron stars since, together with the rapid rotation of the star, they determine the characteristics of the pulsar emission. The source of a wide range of magnetic field strengths ($\sim 10^8 - 10^{15}$G) associated with neutron stars is yet to be well understood. The seven orders of magnitude span may be attributed to the different environments in which neutron stars are present: from isolated objects to accriting members of binary systems. This range could also be the result of different conditions at the time of birth of neutron stars, such as the gravitational collapse of the progenitor massive star or the accretion-induced collapse of a white dwarf [13]. In any case, the very high electrical conductivity renders the Ohmic decay inefficient with typical accreting of the order of billions of years. If neutron star fields decay over their observable lifetime, an alternative decay mechanism is necessary to explain this behavior.

One of the uncertainties concerning the evolution of neutron star magnetic fields is their location in the stellar interior. Should the field be a fossil remnant left over from the progenitor star, it could permeate the whole body of the neutron star. On the other hand, if the magnetic field is generated after a neutron star is born via a battery effect or a dynamo process [14], it is likely to be confined to its crustal layers. As we discussed in the previous section, the exact location of the currents will determine if the mechanism proposed here is operating in neutron stars or not.

For instance, the interaction between differential rotation and magnetic field during the first few seconds of a nascent neutron star's life would generate strong toroidal magnetic fields in the subsurface layers of the star. With the rapid cooling of the star, the crust solidifies with the ions forming a lattice in the presence of relativistic electrons. Some fraction of the toroidal field will have different signs in the Northern and Southern hemispheres, like the one illustrated in Fig. 2. Under these conditions the magnetic field is frozen into the electron gas and Hall currents in the crustal layers can arise and our mechanism will be effective. In contrast, the Ohmic dissipation in the crustal layers takes place on a very long time scale.

If part of the currents supporting the fields are situated in the crust, we can use our mechanism to estimate the timescale for rapid dissipation to occur. Taking typical numbers for the crustal layers of a neutron star, at density scale height h of $10^4 cm$, $n = 10^{34} cm^{-3}$ and the magnetic field is $10^{12} G$, then $\tilde{v}_x \approx 10^{-8} cm/sec$. The corresponding time scale $t_0 = L/\tilde{v}_x$, where L is horizontal scale of the magnetic field, is $t_0 = 10^{14} sec \approx 3$ million years, assuming $L = 10^6 cm$. On the other hand, for a density scale height of $3 \cdot 10^3 cm$, then $n = 10^{32} cm^{-3}$, we have for $B = 10^{12}$G,$\tilde{v}_x \approx 10^{-5} cm/sec$, therefore, $t_0 = 10^{11} sec \approx 3000$ years. Finally, if we take a scale height of $10^3 cm$, then the density is $n = 10^{30}$, and $\tilde{v}_x \approx 10^{-3} cm/sec$, and $t_0 = 30$ years. In a real neutron star, all of these time scales are present if currents occur throughout the crust.

Indeed, due to the sharp gradient of electrical conductivity in the crustal region, we can consider the depth $10^3 cm$ as a boundary between two layers, with different $\sigma_{1,2}$, index 1 corresponding to the upper layer, and index 2 to the lower layer, with $\sigma_2 \gg \sigma_1$. Then, the tangential component of the electric field is continuous, resulting in [15]

$$\frac{(\nabla \times \mathbf{B})_1}{\sigma_1} = \frac{(\nabla \times \mathbf{B})_2}{\sigma_2}. \qquad (36)$$

It follows from (36) that the currents are much stronger in the inner layer in a quasi-steady state. Another way to see that the currents are pumped down the crustal area, is directly from (14): $\tilde{v}_x = -\partial_x \eta \sim \partial_x \sigma / \sigma^2$. As the conductivity increases inwards, this part of velocity results in pushing down the magnetic flux. Therefore, if the initial currents are evenly distributed in the crustal area, the upper currents dissipate in short time scale (30 years), currents in deeper layers dissipate over longer time scales ($\sim$ 3000 years), while the whole crustal field lasts for a few million years.

As we mentioned before, it is not known which part of the currents supporting the poloidal field is situated in the crust [13,16]. In any event, that part of the crustal currents can dissipate via our mechanism in a very short time scale while the field anchored in the core may remain for time scale comparable with the age of the universe. It is possible that pulsars with relatively low observed magnetic fields indicate a core component of $\sim 10^8$ G, while pulsars with fields of order 10^{12} G are younger and have not had time to lose their crustal field component. As isolated pulsars lose their crustal magnetic field due to rapid decay, they also slow down, in the process crossing the death line to become unobservable. We suggest that as neutron stars in binary systems lose their crustal magnetic fields, they permit an increased rate of accretion that spins them up to give rise to the millisecond pulsar population.

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